Basics

Well Ordering Principle: Every nonempty set of positive integers contains a smallest member. (*Take is as an axiom*)

Theorem 0.1 (Division Algorithm): Let a and b be integers with $b > 0$. Then there exist unique integers q and r with the property that $a = bq + r$, where $0 \leq r < b$.

Proof: We begin with the existence portion of the theorem. Consider the set $S = \{a - bk \mid k \text{ is an integer and } a - bk \geq 0\}$. If $0 \in S$, then b divides a and we may obtain the desired result with $q = a/b$ and $r = 0$. Now assume $0 \notin S$. Since S is nonempty [if $a > 0$, $a - b \,.\, 0 \in S$; if $a < 0$, $a - b(2a) = a(1 - 2b) \in S$; $a \neq 0$ since $0 \notin S$], we may apply the Well Ordering Principle to conclude that S has a smallest member, say $r = a - bq$. Then $a = bq + r$ and $r \geq 0$, so all that remains to be proved is that $r < b$. If $r \geq b$, then $a - b(q + 1) = a - bq - b = r - b \geq 0$, so that $a - b(q + 1) \in S$. But $a - b(q + 1) < a - bq$, and $a - bq$ is the smallest member of S. So, $r < b$. To establish the uniqueness of q and r, let us suppose that there are integers q, q', r, and r' such that $a = bq + r$, $0 \leq r < b$, and $a = bq' + r'$, $0 \leq r' < b$. For convenience, we may also suppose that $r' \geq r$. Then $bq + r = bq' + r'$ and $b(q - q') = r' - r$. So, b divides $r' - r$ and $0 \leq r' - r \leq r' < b$. It follows that $r' - r = 0$, and therefore $r' = r$ and $q' = q$. ■

The greatest common divisor of two nonzero integers a and b is the largest of all common divisors of a and b. We denote this integer by gcd(a, b). When gcd(a, b) = 1, we say a and b are relatively prime.

Theorem 0.2 (GCD is a linear combination): For any nonzero integers a and b, there exist integers s and t such that $\gcd(a, b) = as + bt$. Moreover, gcd(a, b) is the smallest positive integer of the form $as + bt$.

Proof: Consider the set $S = \{am + bn \mid m, n \text{ are integers and } am + bn > 0\}$. Since S is obviously nonempty (if some choice of m and n makes $am + bn < 0$, then replace m and n by $-m$ and $-n$), the Well Ordering Principle asserts that S has a smallest member, say, $d = as + bt$. We claim that $d = \gcd(a, b)$. To verify this claim, use the division algorithm to write $a = dq + r$, where $0 \leq r < d$. If $r > 0$, then $r = a - dq = a - (as + bt)q = a - asq - btq = a(1 - sq) + b(-tq) \in S$, contradicting the fact that d is the smallest member of S *(Note that we wanted to show $a - dq < d$ which is obviously true as $r < d$.).* So, $r = 0$ and d divides a. Analogously (or, better yet, by symmetry), d divides b as well.

This proves that d is a common divisor of a and b. Now suppose d' is another common divisor of a and b and write a = d'h and b = d'k. Then d = as + bt = (d'h)s + (d'k)t = d'(hs + kt), so that d' is a divisor of d. Thus, among all common divisors of a and b, d is the greatest.

Corollary: If a and b are relatively prime, then there exist integers s and t such that as + bt = 1.

Theorem 0.3 (Euclid's Lemma): If p is a prime that divides ab, then p divides a or p divides b.

Proof: Suppose p is a prime that divides ab but does not divide a. We must show that p divides b. Since p does not divide a, there are integers s and t such that 1 = as + pt. Then b = abs + ptb, and since p divides the right-hand side of this equation, p also divides b. ■

Fundamental theorem of arithmetic (*to be proved later*): Every integer greater than 1 is a prime or a product of primes. This product is unique, except for the order in which the factors appear.

The least common multiple of two nonzero integers a and b is the smallest positive integer that is a multiple of both a and b.

-2 mod 15 = 13 since -2 = (-1)15 + 13.

In general, if a and b are integers and n is a positive integer, then a mod n = b mod n if and only if n divides a - b

Example: Consider the statement, "The sum of the cubes of any three consecutive integers is divisible by 9." This statement is equivalent to checking that the equation $(n^3 + (n+1)^3 + (n+2)^3)$ mod 9 = 0 is true for all integers n. Because of properties of modular arithmetic, to prove this, all we need do is check the validity of the equation for $n = 0, 1, \ldots, 8$.

Example: We use mod 3 arithmetic to show that there are no integers a and b such that a^2 - 6b = 2. To see this, suppose that there are such integers. Then, taking both sides modulo 3, there is an integer solution to a^2 mod 3 = 2. But trying a = 0, 1, and 2 we obtain a contradiction.

First principle of mathematical induction: Let S be a set of integers containing a. Suppose S has the property that whenever some integer n ≥ a belongs to S, then the integer n + 1 also belongs to S. Then, S contains every integer greater than or equal to a.

Second principle of mathematical induction: Let S be a set of integers containing a. Suppose S has the property that n belongs to S whenever every integer less than n and greater than or equal to a belongs to S. Then, S contains every integer greater than or equal to a.

Example: We will use the Second Principle of Mathematical Induction with a = 2 to prove the existence portion of the Fundamental Theorem of Arithmetic. Let S be the set of integers greater than 1 that are primes or products of primes.

Clearly, $2 \in S$. Now we assume that for some integer n, S contains all integers k with $2 \leq k < n$. We must show that $n \in S$. If n is a prime, then $n \in S$ by definition. If n is not a prime, then n can be written in the form ab, where $1 < a < n$ and $1 < b < n$. Since we are assuming that both a and b belong to S, we know that each of them is a prime or a product of primes. Thus, n is also a product of primes. This completes the proof. ■

Example: The Quakertown Poker Club plays with blue chips worth $5.00 and red chips worth $8.00. What is the largest bet that cannot be made?

To gain insight into this problem, we try various combinations of blue and red chips and obtain 5, 8, 10, 13, 15, 16, 18, 20, 21, 23, 24, 25, 26, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40. It appears that the answer is 27. But how can we be sure? Well, we need only prove that every integer greater than 27 can be written in the form a * 5 + b * 8, where a and b are nonnegative integers. This will solve the problem, since a represents the number of blue chips and b the number of red chips needed to make a bet of a * 5 + b * 8. For the purpose of contrast, we will give two proofs—one using the First Principle of Mathematical Induction and one using the Second Principle. Let S be the set of all integers greater than or equal to 28 of the form a * 5 + b * 8, where a and b are nonnegative. Obviously, $28 \in S$. Now assume that some integer $n \in S$, say, n = a * 5 + b * 8. We must show that $n + 1 \in S$. First, note that since $n \geq 28$, we cannot have both a and b less than 3. If $a \geq 3$, then n + 1 = (a * 5 + b * 8) + (-3 * 5 + 2 * 8) = (a - 3) * 5 + (b + 2) * 8. (Regarding chips, this last equation says that we may increase a bet from n to n 1 1 by removing three blue chips from the pot and adding two red chips.) If $b \geq 3$, then n + 1 = (a * 5 + b * 8) + (5 * 5 - 3 * 8) = (a + 5) * 5 + (b - 3) * 8. (The bet can be increased by 1 by removing three red chips and adding five blue chips.) This completes the proof. To prove the same statement by the Second Principle, we note that each of the integers 28, 29, 30, 31, and 32 is in S. Now assume that for some integer $n > 32$, S contains all integers k with $28 \leq k < n$. We must show that $n \in S$. Since $n - 5 \in S$, there are nonnegative integers a and b such that n - 5 = a * 5 + b * 8. But then n = (a + 1) * 5 + b * 8. Thus n is in S.

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Definition: A partition of a set S is a collection of nonempty disjoint subsets of S whose union is S

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Definition: A function (or mapping) ϕ from a set A to a set B is a rule that assigns to each element a of A exactly one element b of B.

Definition: Let f: A $\rightarrow$ B and c: B $\rightarrow$ C. The composition cf is the mapping from A to C defined by (cf)(a) = c(f(a)) for all a in A

Definition: A function f from a set A is called one-to-one if for every a1, a2 $\in$

A, f(a1) = f(a2) implies a1 = a2.

Alternatively f is one-to-one if a1 ≠ a2 implies f(a1) ≠ f(a2)

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Note that 2nd and 3rd are very easy and straight forward to proof so don't worry