Lecture 7: Repeated Games

School of Management and Economics, CUHK (Shenzhen)

I. Introduction

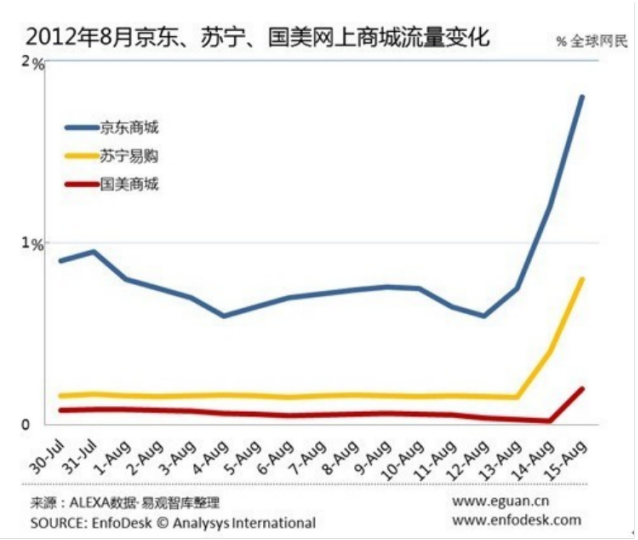
[What are repeated games and what do we need them for?]

Online retailers, Aug 2012

A somehow famous price war

- Reported "largest" online price war in years: JD vowed 0–margin on appliances; hired "spies" and promised refunds if not 10% cheaper than rivals
- Suning: all products priced below JD; issued 8bn RMB bonds to fund war
- Gome: online prices 5% below JD
- If all such claims were fully enforced...

Online retailers, Aug 2012



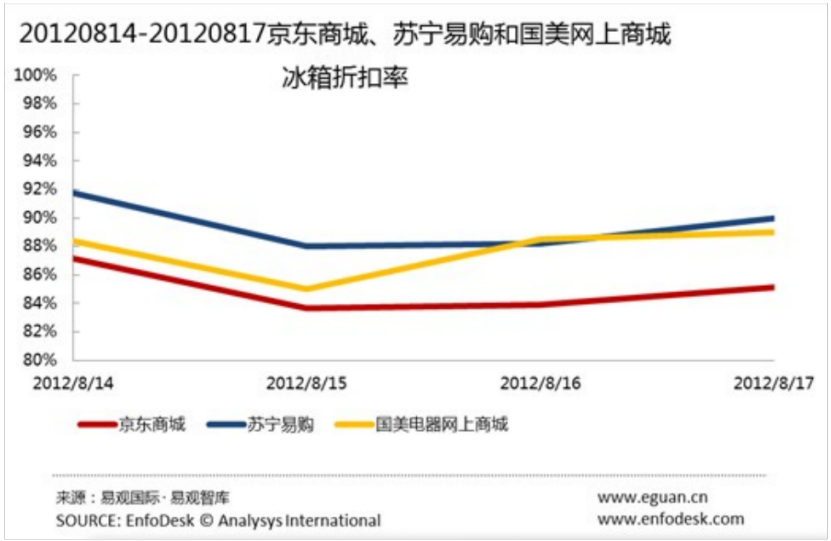


Figure: Actual discount records

Contrasting phenomena

Figure: Businesses with limited price wars

Contrasting phenomena

Figure: Different patterns of competition even within the same product category

Observations

Who engages in price-wars and who doesn't?

- Young, fragile markets (e.g., early online retailers) $\Rightarrow$ short horizons $\Rightarrow$ play stage-game NE ("myopic").
- Mature oligopolies (e.g., Apple vs Samsung): long horizons $\Rightarrow$ scope for tacit cooperation.
- Pattern of competition can change even within an industry.

Framework of repeated games

Conceptually

- Same game played for multiple times
- If the whole game *must* end before some finite number of times, it is finitely repeated
- If there is always a *probability* that the game will continue, it is infinitely repeated
- Main question: can players cooperate instead of simply playing Nash equilibrium repeatedly?
- Based on previous discussion, cooperation is more likely, if ever possible, when interaction is long-term or players are patient

Framework of repeated games

Technically

- Timeline: $t = 1, \ldots, T$ (finite) or $t = 1, 2, \ldots$ (infinite).
- Stage game repeated each period with same players/actions.
- Discount factor $\delta \in (0, 1]$; each maximizes discounted sum.
- Solution concept: subgame-perfect Nash equilibrium (SPNE).

Framework of repeated games

Complex histories & simplifications

- Strategies map *every* history to an action
- Deviations can occur on or off path.
- Finitely repeated: utilize stage-game NE
- Infinitely repeated: history classes, one-step deviation

II. Finitely repeated games

[What are outstanding features of finite repetition?]

Finitely repeated PD

Model

- Time line: $t = 1, \ldots, T$
- Stage game: prisoner's dilemma

		2	
		C(ooperate)	D(efect)
1	C	$1, 1$	$-1, 2$
	D	$2, -1$	$0, 0$

- Discount: $\delta = 1$
- i's payoff: $\sum_t u_i(t)$

Finitely repeated PD

Backward induction

- Number of subgames is large
- 4 possible outcomes (CC, CD, DC, DD) in the first period
- 4×4 possible outcomes in the second period
- 4^{T-1} possible outcomes in the $(T-1)$-th period
- The number of subgames is then $1 + 4 + 4^2 + ... + 4^{T-1}$
- Can't use the tree diagram $\rightarrow$ back to definition of backward induction

Finitely repeated PD

Period T

- Consider a subgame that starts from the last period, period T
- Suppose that an arbitrary history took place from period 1 to period T-1
- By the definition of subgame perfect Nash equilibrium, the players must play a Nash equilibrium in this smallest subgame
- The Nash equilibrium is unique: (D, D)

Finitely repeated PD

Period $T-1$

- Given that players will play (D, D) in period T, what is the Nash equilibrium for a subgame that starts from period $T-1$?
- Again, consider an arbitrary history that took place from period 1 to period $T-2$
- Since they will play (D, D) in period T for sure, the game is equivalent to a one-period Prisoners' Dilemma, and the Nash equilibrium is "each player plays D in period $T-1$; no matter what the outcome of period $T-1$ is, she plays D in period T"

Finitely repeated PD

Unraveling

- Given that they will play D in all subsequent periods no matter what, the subgame that starts from period $m < T - 1$ is no different from a one-shot game
- Hence, "both playing D in period m, and always playing D in every subsequent period" is the unique Nash equilibrium that can occur in the subgame perfect Nash equilibrium
- Conclusion: the SPNE of the whole game is "each player always plays D"

Generalization

Theorem

If the stage game has a unique NE, then in any finitely repeated version, the unique SPNE is to play that NE every period.

- The result holds for any T and δ
- Proof: analogy of previous backward induction
- There can still be many NE of the whole game (example?)

Generalization

Theorem

If players commit to playing a (possibly different) stage-game NE in each specified period, the resulting profile is an SPNE.

- E.g., in $T = 10$ Battle of the Sexes, "(A, A) in odd periods; (B, B) in even periods" is SPNE
- Proof?

Cautionary example

Playing *some* NE in each period is *not* sufficient for SPNE

- For example, consider the following stage game played twice

		2: A	2: B
1	A	$1, 1$	$0, 0$
	B	$0, 0$	$10, 10$

- Is "play (A, A) in period 1; if (A, A) was played in period 1, play (A, A) in period 2, otherwise play (B, B) in period 2" an SPNE?

Sustaining cooperation

Necessary conditions for sustaining cooperation

- Unique NE implies no cooperation $\rightarrow$ at least two NE
- Backward induction $\rightarrow$ game must end with NE
- Cooperation is non-NE behavior $\rightarrow$ one NE as reward and another as punishment, for each player

Sustaining cooperation

		Player 2		
		Cooperate (C)	Defect (D)	Punish (P)
Player 1	Cooperate (C)	1, 1	-1, 2	-3, -3
	Defect (D)	2, -1	0, 0	-3, -3
	Punish (P)	-3, -3	-3, -3	-3, -3

Figure: Variation of PD with two NE

Sustaining cooperation

Two periods

- Period 1: both play C
- Period 2: if both played C in period 1, play (D, D); else play (P, P)
- No need to check incentives for period 2 (why?)
- BR in period 1 requires $1 \geq 2 - 3\delta \iff \delta \geq 1/3$, i.e. sufficient patience

Sustaining cooperation

T periods: cooperate for m periods

- (C, C) in periods $1, \ldots, m$
- After any deviation switch forever to (P, P)
- Otherwise from period $m + 1$ onward, play (D, D)
- "Good" NE as "carrot" and "bad" NE as "stick"

Sustaining cooperation

Analyzing incentives

- *Off-path*: (P, P) forever—NE each period
- *On-path after m*: (D, D) with (P, P) punishments–no profitable deviation
- *On-path at $n \leq m$:* no deviation iff

$$1 + 3\delta + 3\delta^2 + \cdots + 3\delta^{T-n} \geq 2,$$

which is tightest at $n = m$:

$$1 + 3\delta + 3\delta^2 + \cdots + 3\delta^{T-m} \geq 2.$$

LHS $\uparrow$ in δ and T, $\downarrow$ in $m \Rightarrow$ more patience/longer horizon implies more possible cooperation

Sustaining cooperation

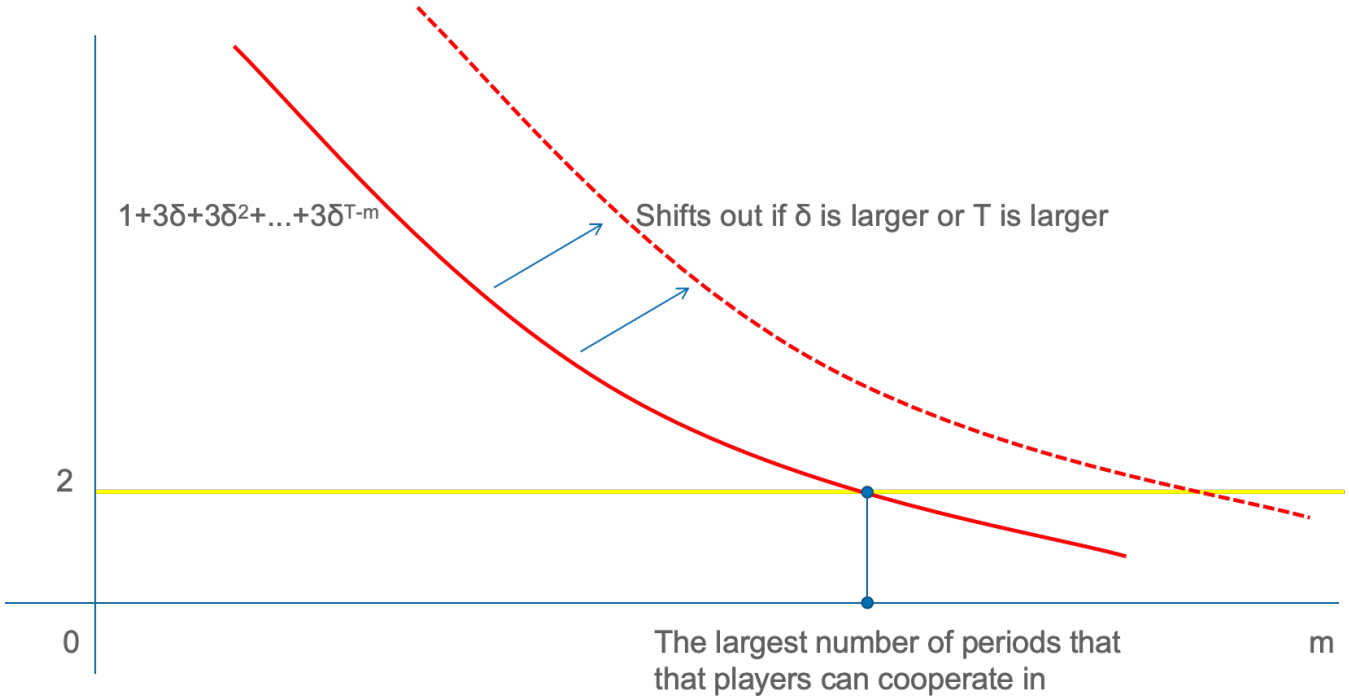


Figure: Graphical intuition of maximum sustainable cooperation

III. Infinitely repeated games

[Why are infinitely repeated games worth studying separately?]

Motivating examples

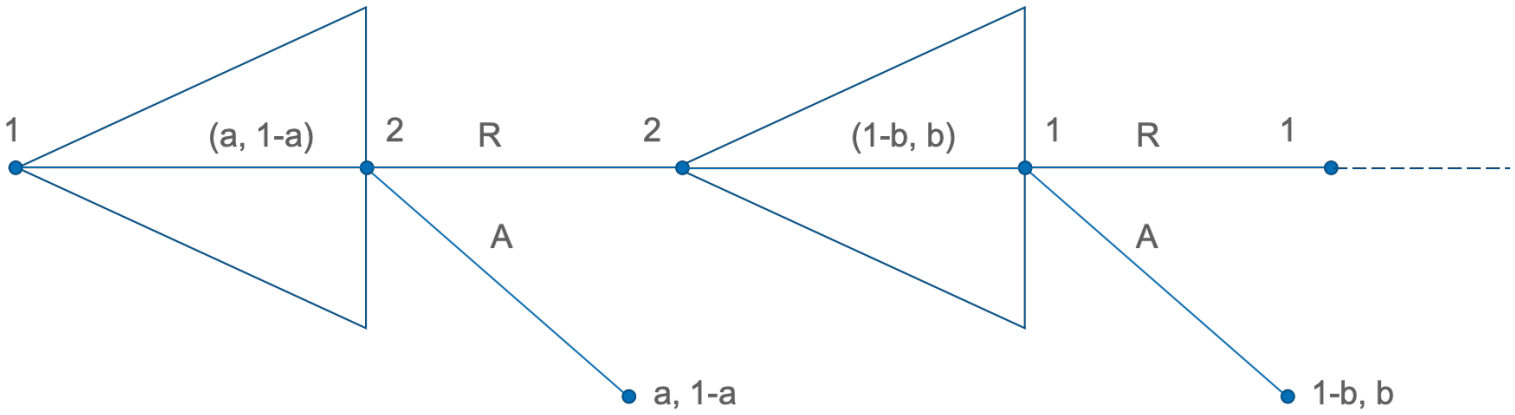


Figure: Infinite bargaining

Who has more bargaining power now? When will there be agreement if ever?

Motivating examples

Infinitely repeated PD

- Is "always defect" still a SPNE?
- Can we sustain (C, C)?
- If yes, for how long?

Specialty about infinite repetition

Similarities to finite repetition

- Subgames defined by histories
- Equilibrium concept is SPNE

Differences

- No terminal node $\Rightarrow$ backward induction inapplicable
- Infinite histories/strategies, hence infinite deviations

Specialty about infinite repetition

Two interpretations of discount factor

- Deterministic infinite horizon with discounting of future utility
- Random horizon: each period continues with probability δ
- Note that payoffs in infinitely repeated games are well-defined only with discounting $\delta \in (0, 1)$

Specialty about infinite repetition

"Descriptive" but not "exhaustive" strategy

- Impossible to enumerate all histories with infinite repetition
- A strategy should categorize history and respective action

Example of strategy in infinitely repeated PD

- Always play (D, D), the NE of stage game
- "Grim trigger": start with (C, C), if both have only played C before, then continue to play (C, C); otherwise, play (D, D) forever
- "Tit-for-Tat": start with (C, C) and mimick the opponent's strategy in the previous period

Result extended from finite repetition

Theorem

In an infinitely repeated game, playing the same Nash equilibrium of the stage game in the same period constitutes a SPNE for any δ.

- Fix any history
- Opponents' subsequent actions are a period-by-period NE plan
- Using corresponding NE action each period must be BR $\Rightarrow$ SPNE

Result extended from finite repetition

Example: Battle of the Sexes

- Consider "play (A, A) in odd periods and (B, B) in even periods"
- Given any history, each player knows the opponent's schedule (odd $\rightarrow A$, even $\rightarrow B$)
- Hence one BR is following the same schedule $\Rightarrow$ SPNE

Our focus for infinite repetition

It is easy to find *some* SPNE

- Prisoner's dilemma, Battle of sexes, Matching pennies...in their infinitely repeated version
- Basically, when the stage game has at least one NE

However, the more interesting aspect of infinite repetition lies in

- Sustaining non-NE behavior
- Range of achievable payoffs

One-step deviation principle

Key challenge for finding (interesting) SPNE

- Checking for incentives is easy if players' actions are fixed for every period regardless of history (as in the theorem)
- However, there are many other strategies
- For instance, "Tit-for-Tat" in infinitely repeated PD: start with (C, C) and mimick the opponent's strategy in the previous period
- If a player deviates (either on the equilibrium path or off the equilibrium path), her opponent's action (still using Tit-for-Tat) is going to change accordingly

The big question: how to identify the "best deviation"?

One-step deviation principle

Answer: we don't have to.

One-step deviation principle. A strategy profile is SPNE if, after every possible history, no player has a profitable one-shot deviation.
The principle applies to (including but not limited to)

- Games that end before a fixed time period
- Infinitely repeated games with < 1 discounting

One-step deviation principle

General procedure of applying the principle

- Propose a strategy profile as equilibrium candidate
- Categorize possible histories
- Use the principle to check for profitable deviation in each category of history

Example: Tit-for-Tat (TFT)

Recall the stage game

		2	
		C(ooperate)	D(efect)
1	C	$1, 1$	$-1, 2$
	D	$2, -1$	$0, 0$

Histories fall into four types depending on last-period outcome: (C, C), (C, D), (D, C), (D, D)

Example: Tit-for-Tat (TFT)

After (C, C), for either player

- A one-step deviation means to defect now and return to TFT
- (D, C) now, (C, D) next, and so on
- No profitable one-step deviation $\Leftrightarrow \frac{1}{1-\delta} \geq \frac{2}{1-\delta^2} - \frac{\delta}{1-\delta^2} \Leftrightarrow \delta \geq \frac{1}{2}$

Example: Tit-for-Tat (TFT)

For other categories of history (exercise)

Example: Tit-for-Tat (TFT)

In conclusion: TFT is a SPNE iff $\delta = 1/2$

- If δ is too small, a player deviates when C is prescribed
- If δ is too large, a player deviates when D is prescribed

TFT doesn't seem a good candidate for sustaining cooperation.

Example: Grim Trigger (GT)

The strategy profile

- Start with (C, C)
- If D was never played: continue with (C, C)
- Otherwise play (D, D)

Example: Grim Trigger (GT)

Checking for incentives

- Off path: (D, D) forever is itself a stage-game NE each period $\Rightarrow$ no profitable deviation
- On path: cooperation payoff $1/(1-\delta)$. One-step deviation gives 2 now, 0 thereafter
- Condition for GT to be SPNE:

$$\frac{1}{1-\delta} \geq 2 \Leftrightarrow \delta \geq \frac{1}{2}$$

A reasonably good strategy for sustaining cooperation

T-period Grim Trigger (TGT)

Cooperation phase

- Play (C, C) if no one has played D in the phase
- If someone deviates: enter the punishment phase

Punishment phase

- Play (D, D) for T periods no matter what (that is, even if someone played C in the punishment phase, it will not affect the length of the phase)
- When the punishment phase ends, go back to the cooperation phase

Idea: allow for mistakes but need to punish before going back on cooperation

T-period Grim Trigger (TGT)

Checking for incentives

- During punishment, deviating to C only hurts today $\Rightarrow$ no profitable (one-shot) deviation
- During cooperation: deviation changes payoff by $1 - \delta - \cdots - \delta^T = 1 - \frac{\delta(1-\delta^T)}{1-\delta}$
- For δ close to 1, a large enough T suffices for SPNE

An arguably even better candidate for sustaining cooperation

IV. Folk Theorem

[What is the most general result on infinitely repeated games?]

Sustainable payoffs

With credible punishments and patient players, one can sustain any feasible payoff vector that dominates the punishment payoff.

Theorem

In infinitely repeated PD, for any $a, b > 0$ that is a convex combination of the four pure-outcome payoffs, there exists a SPNE achieving approximately per-period payoffs (a, b) when δ is close enough to 1.

Proof sketch (in class)

Sustainable payoffs

Implementations

- Need approximation of linear combination, i.e. distribution of action profile
- Public correlation device
- Block strategies (e.g., alternate C and D to approximate target averages)

Sustainable payoffs

Non-Nash punishment (briefly; in class)

V. Discussion

[Should we believe in repeated games?]

Why (not) believe repeated-game predictions?

Reasons to be supportive

- Tacit collusion (e.g. OPEC)
- Nuclear deterrence
- Relationship dynamics (TGT?)

Why (not) believe repeated-game predictions?

Reasons to be skeptical

- Coordinating on a particular equilibrium is nontrivial
- Action sets evolve
- Present vs future selves may have inconsistent preferences

Lecture 8: Application of Repeated Games – Collusion and Bargaining

School of Management and Economics, CUHK (Shenzhen)

I. Introduction

[What are some typical scenarios for repeated games?]

Explicit and implicit cooperation

Figure: Typical "markets" with repeated interaction

Ways to resolve conflicts

Authority

Property rights

Credit scores

Figure: Institutions for maintaining cooperation

Key questions in practice

For players

- How to identify short-term/long-term optimum?
- How to utilize information?

For regulators

- What policies to implement?
- At what cost?

II. Collusion and anti-trust laws

[How do we provide a theoretical foundation for market regulation?]

Collusion and anti-trust laws

Battle against centralized market power

- Four forms of explicit market collusion: *cartel*, *syndicate*, *trust*, *concern*
- Main idea: jointly determine production to maximize total profit
- Core statutes: Sherman Act (1890), Clayton Act (1914), Federal Trade Commission Act (1914), Anti-Monopoly Law of PRC (2008).

What's wrong with collusion?

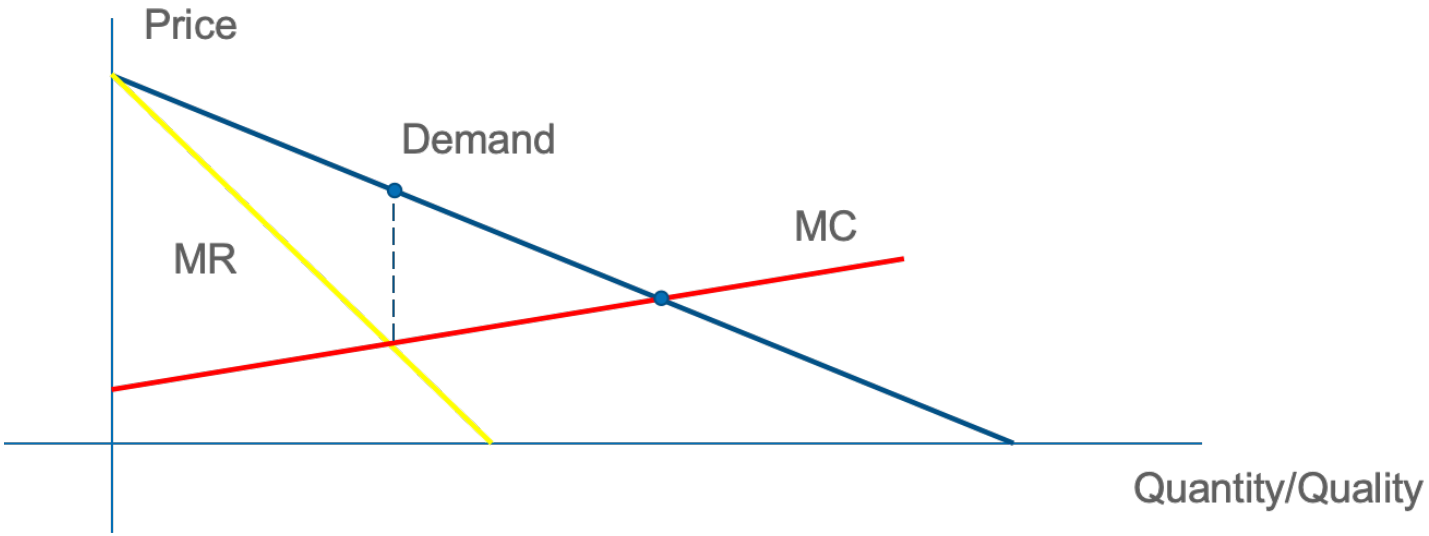


Figure: The conventional view

What's wrong with collusion?

In general, collusion and monopoly may

- Raises prices, reduces output; innovation effects are debated
- influence politics and law
- Enable implicit autocracy and durable market power
- Some interesting references: *Black Mirror*, *Lord of War*, *Too Big to Fail*...

Cournot baseline (two firms)

Inverse demand $P = a - Q$ with $Q = q_1 + q_2$, identical marginal cost c

- Best response of firm i: $q_i = \frac{a-c-q_j}{2}$
- Monopoly split (collusion): each produces $\frac{a-c}{4}$
- Profitable deviation: best response to $\frac{a-c}{4}$ is $\frac{3(a-c)}{8}$

Cournot baseline (two firms)

Static and finite repetition

- In one shot, collusion is not a Nash equilibrium
- In any finite horizon, backward induction kills cooperation
- Result holds for all discount factor
- Number of firms doesn't matter either

Infinitely repeated Cournot

Intuition along the line of infinitely repeated PD

- "Cooperate" is to produce a fraction of the monopoly quantity, maximizing joint profit
- "Defect" is to best respond to the other firm's production, maximizing own profit
- Start with cooperation; punish with NE
- Intuitively, "when can cooperation be sustained" should depends on the set of parameters (a, c, δ)

Infinitely repeated Cournot

Candidate for SPNE with collusion

- On path: each produces $(a - c)/4$
- If any deviation is observed: play the stage-game NE forever, $(a - c)/3$ each
- (Best) One-step deviation: produce $3(a - c)/8$ now, then $(a - c)/3$ thereafter

Infinitely repeated Cournot

Checking incentive constraints

- Payoff from cooperation each period: $\pi^C = \frac{(a-c)^2}{8} \rightarrow$ total: $\frac{(a-c)^2}{8(1-\delta)}$
- Payoff from deviation: current $\pi_0^D = \frac{9(a-c)^2}{64}$; future flow $\pi^{NE} = \frac{(a-c)^2}{9}$ each period $\rightarrow$ total: $\frac{9(a-c)^2}{64} + \delta\frac{(a-c)^2}{9(1-\delta)}$
- Condition for sustaining permanent cooperation:

$$\frac{(a-c)^2}{8(1-\delta)} \geq \frac{9(a-c)^2}{64} + \delta\frac{(a-c)^2}{9(1-\delta)} \Leftrightarrow \delta \geq \frac{9}{17}$$

Infinitely repeated Cournot

Remarks

- Cooperation requires patience (δ high)
- Finite punishments and public correlation devices can restore cooperation after off-path histories
- Threshold δ is independent of a, c under linear demand/cost

Infinitely repeated Cournot

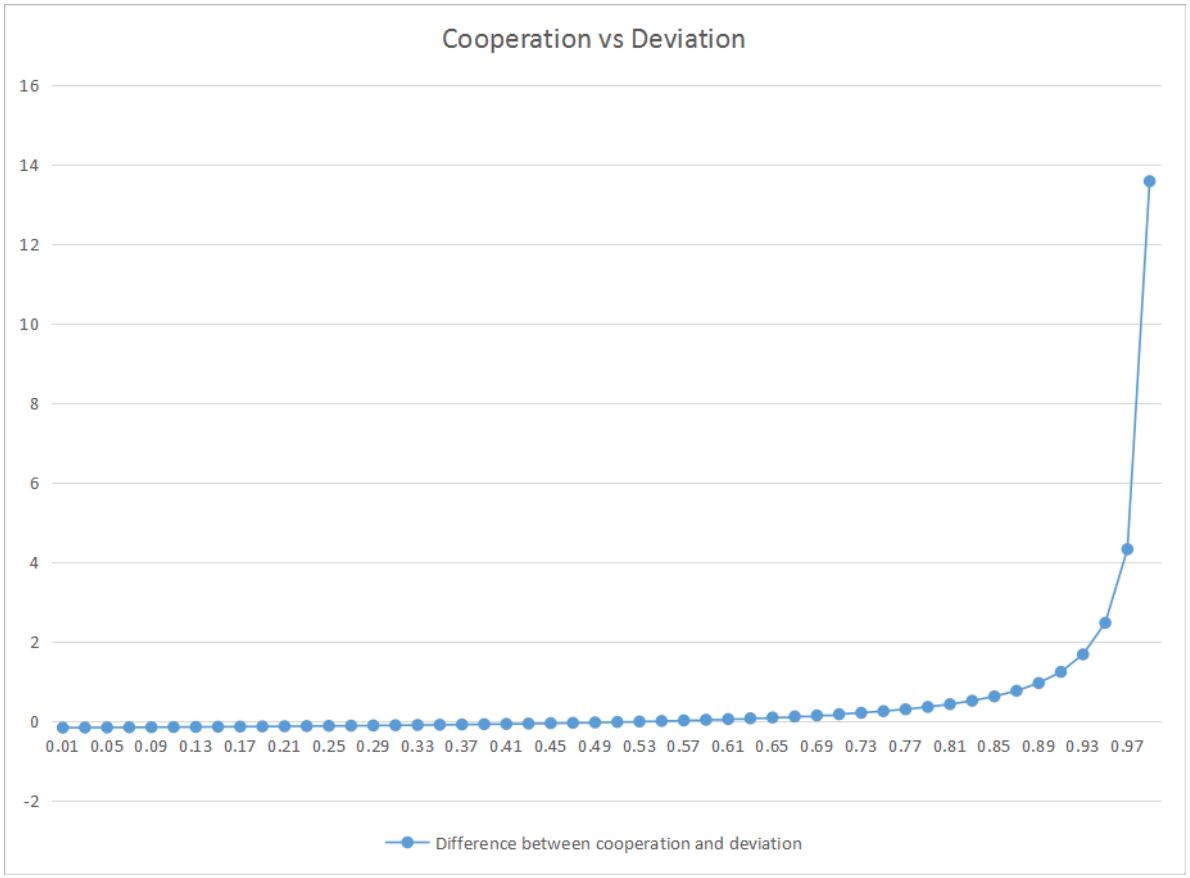


Figure: Net benefit of cooperation as $\delta \uparrow$

Infinitely repeated Cournot

Back to our collusion story

- Under the framework of infinitely repeated games, collusion is sustainable in SPNE
- This justifies the existence of anti-trust laws
- Multiple ways of enforcing anti-trust laws, e.g. lower the barrier of market entrance to bring competition, or supervise the market and punish the firms if collusion is found
- How will these policies deter collusion?

Case: Standard Oil

The company

- Founded by the legendary John Rockefeller in 1870, once the largest oil refiner in the world
- Became a trust by horizontal (purchased competitors) and vertical (purchased retailers) integration
- In 1911, the US Supreme Court ruled that Standard Oil was an illegal monopoly
- As a result, the company was broken down into a number of smaller firms

Case: Standard Oil

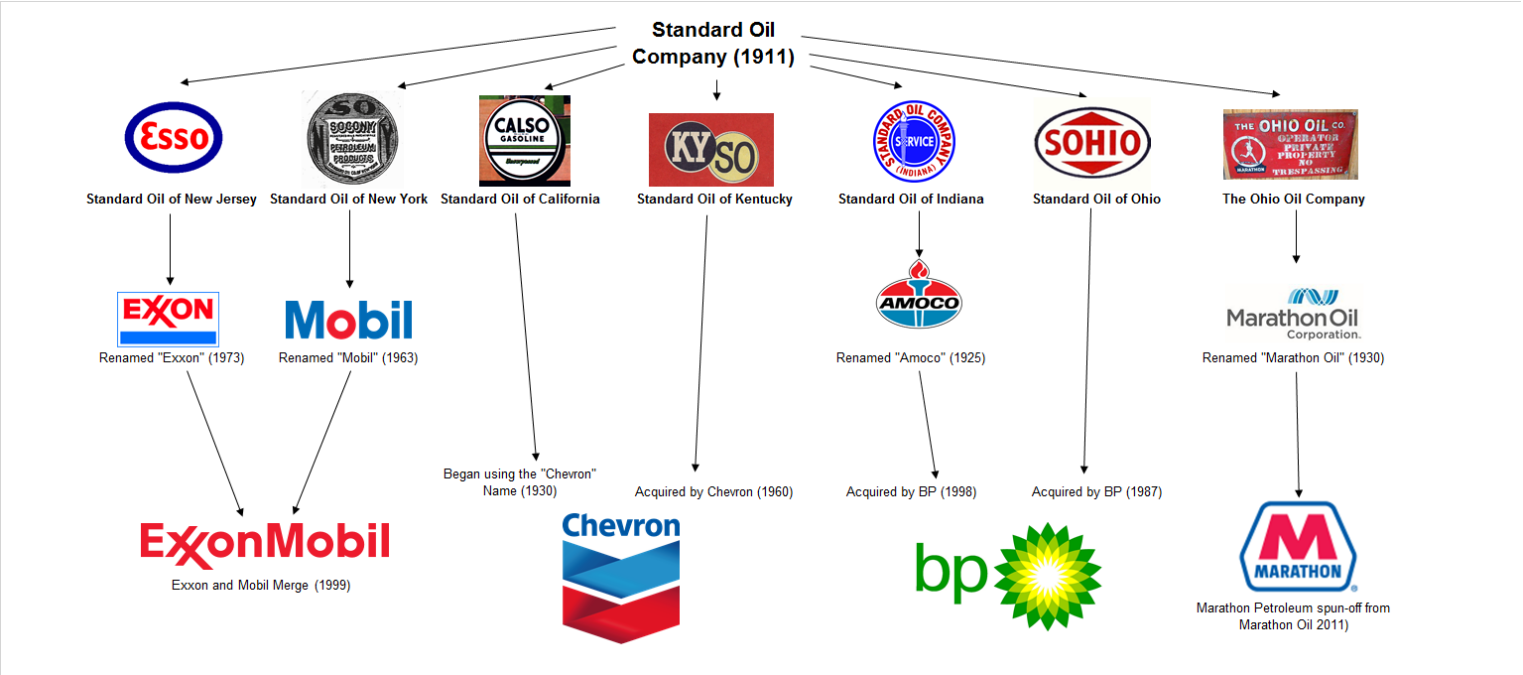


Figure: Standard Oil dismantled

Case: Standard Oil

After the verdict: cournot with n firms

- Collusion: each produces $(a-c)/(2n)$
- Best response when others collude: $q_i = \frac{(a-c)(n+1)}{4n}$
- NE output: $q_i^{NE} = \frac{a-c}{n+1}$

Case: Standard Oil

Grim-trigger with n firms

- Collude on-path, punish with NE
- On-path payoff: $\frac{(a-c)^2}{4n(1-\delta)}$
- Deviation payoff now: $\frac{(a-c)^2(n+1)^2}{(4n)^2}$; future NE flow: $\frac{(a-c)^2}{(n+1)^2}$
- Sustainability requires

$$\delta \geq \frac{(n+1)^2}{(n+1)^2+4n},$$

which increases with $n \Rightarrow$ more competitors $\Rightarrow$ harder to collude

Case: Standard Oil

Intuition for larger $n \to$ harder for collusion

- Imagine $n \to \infty$
- Stay in collusion: each firm earns ≈ 0
- Deviate: in current period, as if BR to a monopoly $\to$ earns > 0

Case: Standard Oil

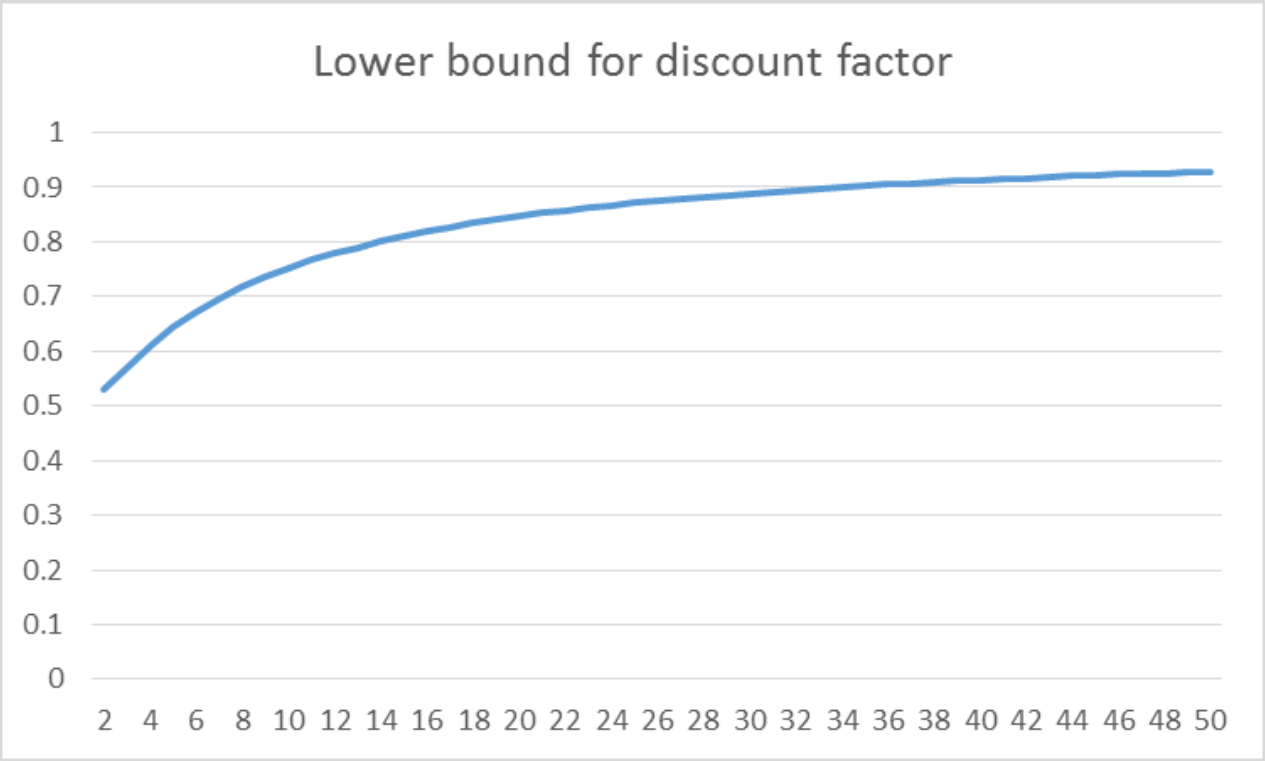


Figure: Lower bound of δ to sustain cooperation

Case: OPEC

Back in the 1980s

- Demand fell; Saudi Arabia-led production cuts tried to prop up price.
- Hidden/lagged actions and noisy prices undermined enforcement
- Cheating eroded the cartel; prices collapsed

Case: OPEC

Interpretation in our framework

- Increased production can only be reflected in oil prices some time afterwards
- Fluctuation in prices can also be explained by exogenous factors, such as risky demand
- Hence, when actions are hidden or noisy, it is harder for players to implement punishment, hence less likely to collude successfully
- Limiting communication among players may serve as a measure for preventing collusion

Other forms of collusion

Bertrand model

- Collusion: price $(a+c)/2$, quantity $(a-c)/4$ each, profit $(a-c)^2/8$.
- Deviation: slightly undercut, get $\approx (a-c)^2/4$ now; punishment NE: price $= c$, profit 0
- Sustainability:
$$\frac{(a-c)^2}{8(1-\delta)} \geq \frac{(a-c)^2}{4} \quad \Leftrightarrow \quad \delta \geq \tfrac{1}{2}.$$
- With n firms: need $\delta \geq 1 - 1/n$.

Other forms of collusion

Hotelling location game

- Two firms on $[0, 1]$, unit demand, individual valuation $1 -$ distance
- Collusive locations yield profit $7/16$ each; NE profit $3/8$
- Best one-shot deviation in the repeated game can reach $79/160 > 7/16$; derive δ threshold analogously

Other forms of collusion

Tragedy of the commons

- Mass a of users choose Highway (benefit $b - cx$) or Local (benefit d)
- Social planner maximizes $(b - cx)x + d(a - x) \Rightarrow$ efficient $x^* = \frac{b-d}{2c}$, whereas Nash gives $x^{NE} = \frac{b-d}{c}$
- "Cooperation" (coordination/taxation) can improve welfare even if it resembles "monopoly"

Remarks

Main takeaways from studying collusion as infinite repeated games

- Infinite repetition sustains collusion via credible punishment
- Higher n and noisier monitoring make collusion harder
- Policy levers: entry/split, monitoring, communication limits

III. Repeated bargaining

[How does the analysis change when the game may actually end some time?]

Repeated games beyond the standard setup

Differences in key features

- Standard infinitely repeated games: simultaneous stage game; continuation probability > 0 regardless of actions
- Other environments: stage game may be *sequential*; whether the game continues may depend on actions
- Examples: bargaining, market entry, social interaction, etc.

Example 1: Infinitely repeated chain-store game

Stage game (entrant vs. incumbent):

	Fight	**Accommodate**
Stay out	$(0, 10)$	$(0, 10)$
Enter	$(-5, 5)$	$(3, 7)$

Can you recover the extensive form?

Example 1: Infinitely repeated chain-store game

When the type of entrants differ:

- *Myopic/different entrant each period:* game replicates the stage game; unique SPNE each period: Enter–Accommodate.
- *Same forward-looking entrant:* staying out can be absorbing $\Rightarrow$ periods are not independent; folk-theorem logic breaks, need dynamic programming / value functions.

Example 2: Infinitely repeated centipede

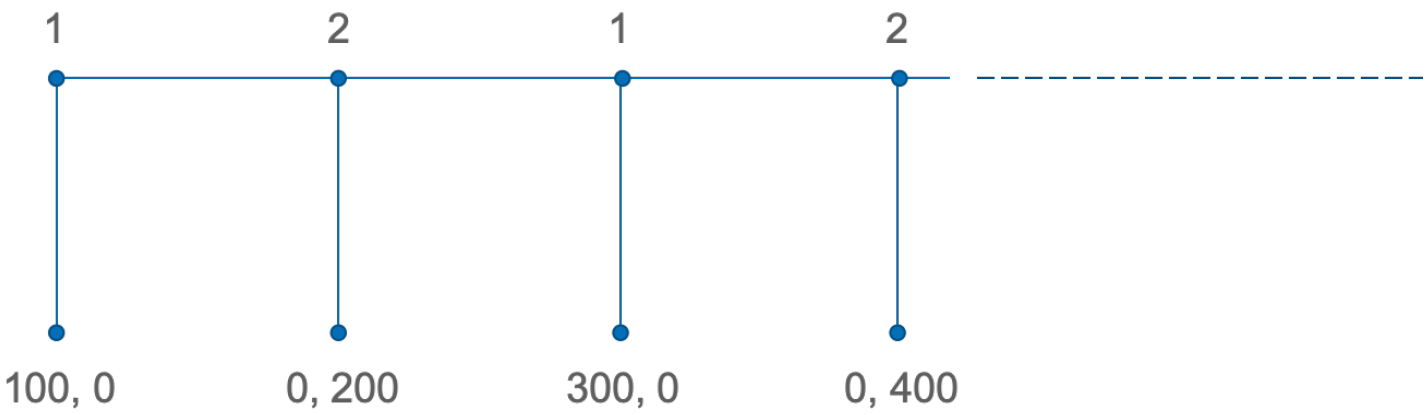


Figure: Centipede game extended to infinity – when should a player stop?

Example 2: Infinitely repeated centipede

Trivial SPNE

- "Always take now": each round is a best response given the other takes next round
- Holds for any δ

"Cooperative thresholds"

- One player waits until m, the other waits until n
- Not SPNE – whoever is supposed to take later deviates to take earlier

Example 2: Infinitely repeated centipede

Strange equilibria

- Suppose the payoffs rise over time
- If the opponent never takes and δ not tiny, BR can be to never take either

Takeaway: infinite-horizon sequential games can be subtle.

Bargaining

Two players bargain over \$1 forever if no agreement.

- Period t: proposer offers $(1 - a_t, a_t)$ to the receiver (a_t is receiver's share).
- Common discount factor $\delta \in (0, 1)$ applied per period to future payoffs.
- If an offer is accepted, game ends; otherwise roles alternate.

Bargaining

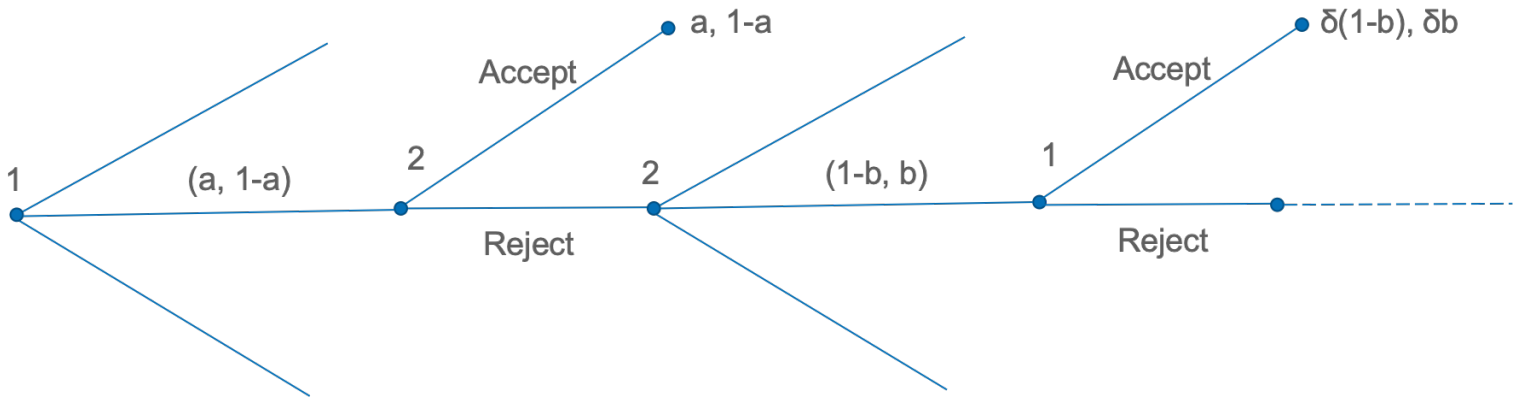


Figure: In(de)finite bargaining – backward induction doesn't apply

Bargaining

Intuition for possible equilibria

- Game is essentially the same at every possible round
- Unlike the infinitely repeated centipede, the players actually have some incentive to "cooperate"
- In centipede, "cooperation" means to wait, but each player has incentive to stop waiting and act prematurely
- In bargaining, cooperation means to reach agreement early, and this is in each player's benefit

Bargaining

Natural candidate: stationary threshold strategies

- Proposer offers a to opponent every time
- Receiver accepts iff offered at least b (reject otherwise)
- Necessary condition from proposer optimality: offer at threshold $\Rightarrow a = b$

Bargaining

Testing proposer's deviation

- Offer $< b$ today $\Rightarrow$ rejection
- Next period as receiver accepts $b \Rightarrow$ discounted payoff δb
- Offering b now yields $1 - b$. No deviation iff

$$1 - b \geq \delta b \quad \Leftrightarrow \quad b \leq \frac{1}{1+\delta}.$$

Bargaining

Testing receiver's deviation

- When offered $\geq b$: accept now vs reject and get $\delta(1-b)$ next period $\Rightarrow$ need $b \geq \delta(1-b)$.
- When offered $< b$: reject (get $\delta(1-b)$) vs accept $< b$ now $\Rightarrow$ need $b \leq \delta(1-b)$ Together $\Rightarrow$ $b = \dfrac{\delta}{1+\delta}$, which also satisfies $b \leq \frac{1}{1+\delta}$

Bargaining

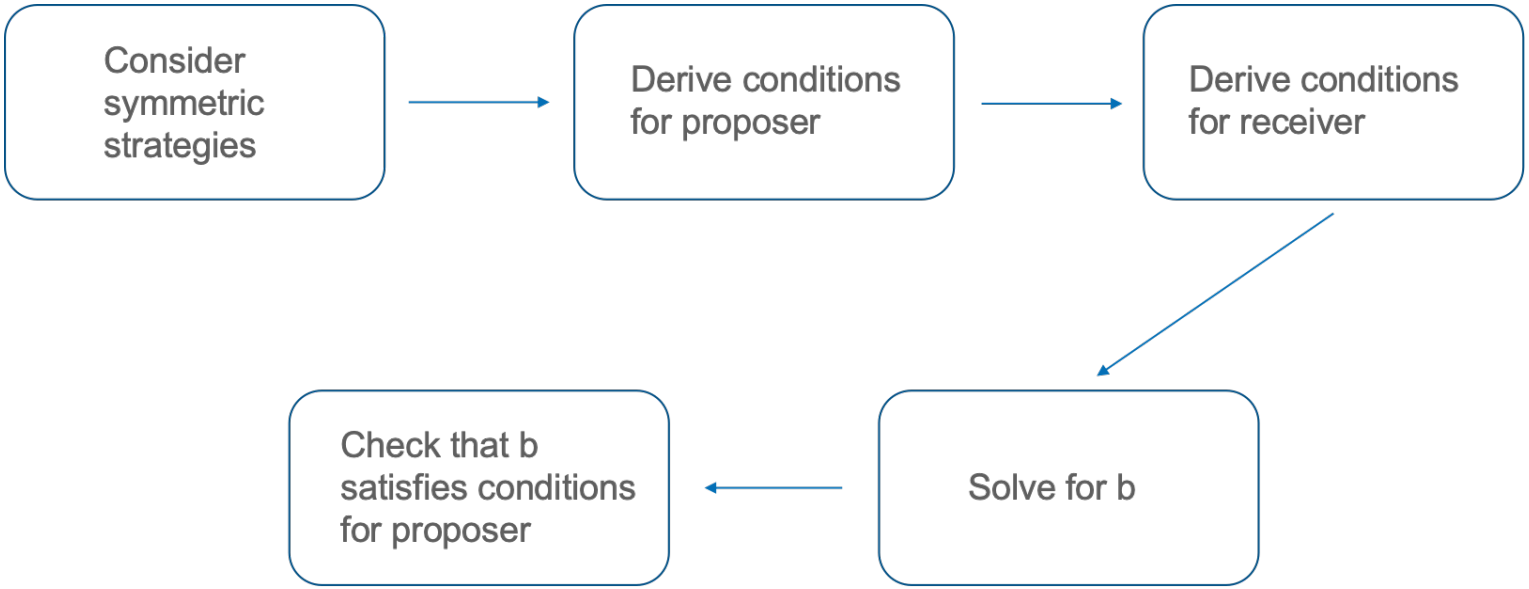


Figure: Flowchart of deriving SPNE

Bargaining

Equilibrium with equal δ

- SPNE: In any period, proposer offers $a = b = \dfrac{\delta}{1+\delta}$ to the opponent; receiver accepts iff offered $\geq b$
- Outcome: Agreement at $t = 1$. Player 1 (first proposer) gets $\dfrac{1}{1+\delta}$; Player 2 gets $\dfrac{\delta}{1+\delta}$ (first-mover advantage)

Bargaining

Stationary thresholds with $\delta_1 \neq \delta_2$

- Player 1 offer a to 2 and accept iff $\geq b$
- Player 2 offer c to 1 and accept iff $\geq d$
- In SPNE, acceptance on path requires $a = d$ and $c = b$ (why?)
- Focus on (a, b)

Bargaining

Player 1:

Offer optimality: $1 - a \geq \delta_1 b$, Acceptance threshold: $b = \delta_1(1 - a)$.

Player 2:

$$1 - b \geq \delta_2 a, \qquad a = \delta_2(1 - b).$$

Solving:

$$a^* = \frac{\delta_2(1 - \delta_1)}{1 - \delta_1\delta_2}, \qquad b^* = \frac{\delta_1(1 - \delta_2)}{1 - \delta_1\delta_2}.$$

Agreement occurs at $t = 1$ with payoffs

$$\left(\frac{1 - \delta_2}{1 - \delta_1\delta_2}, \frac{\delta_2(1 - \delta_1)}{1 - \delta_1\delta_2}\right).$$

Higher $\delta_i \Rightarrow$ higher equilibrium payoff for player i.

Bargaining

Alternative approach: aximatic model

- Feasible utility set $S \subset \mathbb{R}^2$ is compact and convex
- Axiom I: Pareto efficiency
- Axiom II: consistency – $S \subset T$ and outcome given T in $S \Rightarrow$ outcome given S is the same
- Axiom III: fairness – S is symmetric $\Rightarrow$ players have same utility

Nash's theorem: the unique solution satisfying the axioms maximizes $u_1 u_2$ (or $u_1^\alpha u_2^\beta$ for weights).

IV. Discussion

[What are some practical issues to note for policies?]

Why real bargaining takes time

Theory predicts immediate agreement under complete information and common knowledge of δ; however

- Private information about reserve prices
- Evolving outside options and timing frictions
- Psychological/payoff non-linearities
- Deviations hint at hidden frictions

Recap

Infinite-horizon sequential games

- May not fit "folk theorem" molds
- Repeated bargaining admits clean SPNE via stationary thresholds
- Patience (δ) is bargaining power; differing δ_i skew shares
- Nash bargaining provides a crisp axiomatic benchmark

Lecture 9: Bayesian Games

School of Management and Economics, CUHK (Shenzhen)

I. Information

[How do economic theorists understand the word "information"?]

Defining "information"

Some common understanding of information

- News, e.g. most updated stock prices
- Data, e.g. typical consumer profiles on e-trading platforms
- Prediction, e.g. whether a war is likely to break out between countries
- Consequence, e.g. punishment for plagiarism

Defining "information"

As game theorists, information describes the *game*:

- Players: numbers and identities
- Strategies: available actions; what each player can/cannot observe
- Payoffs: outcome values for each strategy profile

Incomplete information

A player does not know exactly *which game* is being played. Examples:

- Centipede: do you know opponent's altruism?
- Cournot: do you know rival's marginal cost?
- Auctions: do you know others' valuations?

Incomplete information

Figure: More practical examples

Incomplete vs imperfect information

Incomplete information means players do not have common knowledge about what game they are playing

- "What game" typically means the relation between action and payoff
- Not sure about own payoff: an employer hires a worker who could be either diligent or lazy
- Not sure about opponent's payoff: compete with a rival firm with either high or low cost

Incomplete vs imperfect information

Imperfect information means that players do not fully observe what happened in the game before

- For instance, repeated Prisoners' Dilemma with some periods' history of play covered and not available to all players
- But such a game is still a game with complete information
- On the other hand, the employer-worker story above is a game with incomplete but perfect information (because whether or not the worker is hired is observable for all)

Incomplete vs imperfect information

To unify incomplete and imperfect information

- "Incomplete" means not knowing the game
- "Imperfect" means not knowing a player's action
- Hence, we can think of the game as some player's action (Nature/God)

II. Basic framework of Bayesian games

[How do we incorporate information into a strategic scenario?]

Example: market for lemons

Used-car market with buyers and sellers

- Two qualities with identical looks
- Good car: buyer value 1000, seller value 800.
- Lemon: buyer value 200, seller value 100.

Example: market for lemons

A buyer and a seller meet and simultaneously decide whether to trade with price p

- If trade: buyer payoff $=$ value $- p -$ benefit depends on quality
- Seller payoff $= p -$ value $-$ opportunity cost depends on quality
- Our focus: whether there exists p to always foster efficient trade

Example: market for lemons

Complete information benchmark

- If quality is observed, two prices (p_g, p_b) can be set
- Buying is weakly dominant for buyer if $p_g < 1000$ and $p_b < 200$
- Selling is weakly dominant for seller if $p_g > 800$ and $p_b > 100$
- Hence any $p_g \in (800, 1000)$ and $p_b \in (100, 200)$ supports a NE with *trade always*

Example: market for lemons

Symmetric incomplete information

- Quality unknown to both sides; prior 1/2 for each type
- Buyer: $\mathbb{E}[\text{value}] = \frac{1}{2}(1000 + 200) = 600 \Rightarrow$ buying weakly dominant if $p < 600$
- Seller: $\mathbb{E}[\text{value}] = \frac{1}{2}(800 + 100) = 450 \Rightarrow$ selling weakly dominant if $p > 450$
- Thus any $p \in (450, 600)$ yields a NE with *trade always*

Example: market for lemons

Asymmetric incomplete information (seller knows quality)

- $p \geq 800$: if trade, buyer's expected value $\leq 600 < p \Rightarrow$ buyer won't buy.
- $p < 100$: sellers (even lemons) won't sell.
- $p \in (200, 800)$: good sellers refuse (< 800), buyers infer lemons and won't buy.
- *Partial trade:* For $p \in [100, 200]$, lemons' sellers sell (good sellers do not); buyers buy—only lemons trade (adverse selection)

Example: market for lemons

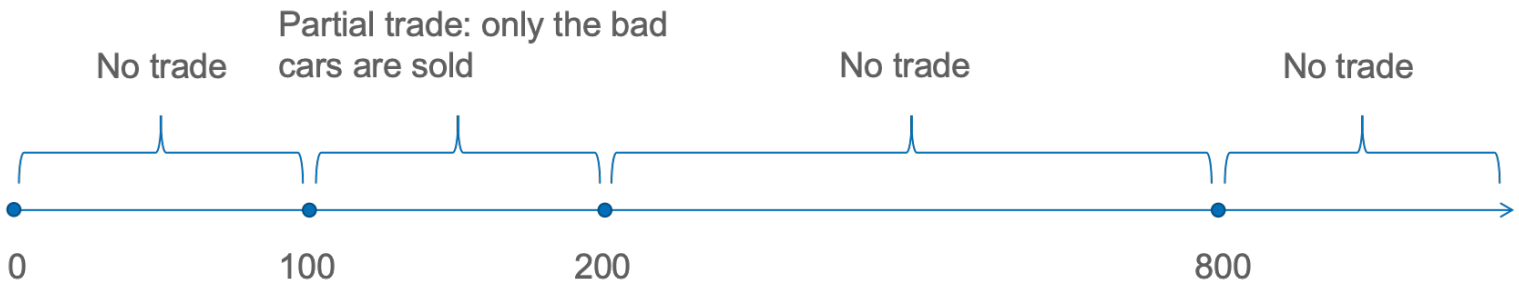


Figure: No-trade zones and partial-trade zone

Example: market for lemons

Suppose buyer posts a price instead

- A buyer never offers > 600 (expected value)
- No offer can screen in only good cars: good sellers accept only $p \geq 800$, lemons accept $p \geq 100$
- Hence only lemons trade at low prices $\Rightarrow$ adverse selection

Observations

Towards a general model

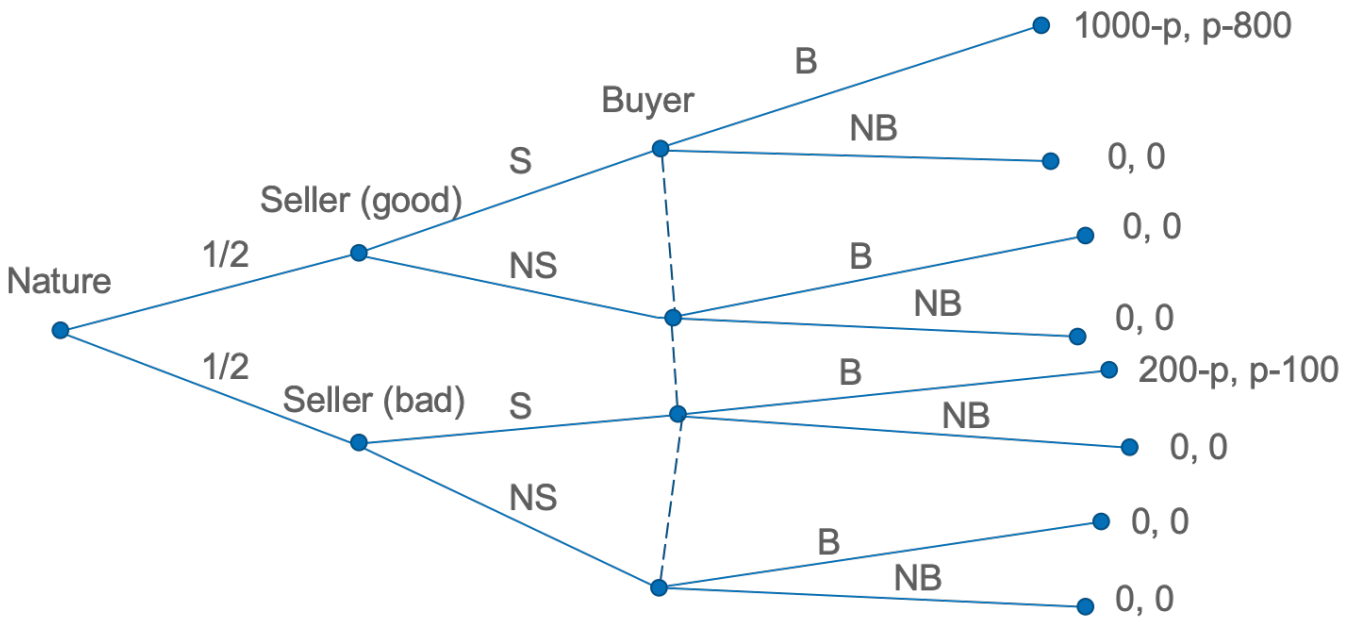
- Game rules: nothing special
- Strategy: mapping from "what I know" to "what I can do"
- Payoff: function of strategy profile *and* information (Nature's choice)

Bayesian game

Components of a Bayesian game

- Players $i \in I$
- Types $t_i \in T_i$ drawn by Nature with distribution $p_i(t_i)$
- Actions A_i
- A (pure) strategy is a function $s_i : T_i \to A_i$
- Payoffs $u_i(a, t)$ may depend on both actions and types

Tree form (market for lemons)



Figure

Matrix form (market for lemons)

		Seller (good)	
		S	NS
Buyer	B	1000-p, p-800	0, 0
	NB	0, 0	0, 0

		Seller (bad)	
		S	NS
Buyer	B	200-p, p-100	0, 0
	NB	0, 0	0, 0

Figure: Each matrix played with probability 0.5

Expected payoff and BR

Type t_i's expected payoff from action a_i given others' strategies s_{-i} is

$$\mathbb{E}_{t_{-i}}[u_i(a_i, s_{-i}(t_{-i}), t_i, t_{-i})]$$

Best response: a_i maximizes this expectation

- Conditional on what i knows (t_i)
- Take expectation over what i doesn't know (t_{-i})

BNE

Definition

A strategy profile is a Bayesian Nash equilibrium if each type t_i of each player i is playing a best response (in expectation) to the strategies of the others.

- Not much different from the standard Nash equilibrium – just treat every type as a separate player
- For instance, in the market for lemons, all buyers buy and all sellers sell at price 300 is not a BNE: the sellers of good cars will deviate
- All buyers buy and only the sellers of bad cars sell at price 300 is not a BNE: the buyers will deviate
- All buyers buy and only the sellers of bad cars sell at price 150 is a BNE

Example

Two workers compete for a promotion

- Each worker chooses whether to exert effort (E) or to slack off (S) simultaneously
- If a worker slacks off, she will never be promoted
- If only one worker exerts effort, she will be promoted
- If both workers exert effort, the more talented one will be promoted
- Effort bears a cost of 1, and promotion brings a benefit of 2

Example

Two workers compete for a promotion (continued)

- Worker 1 has intermediate talent
- Worker 2 is strong (better than intermediate) with probability 0.6 and weak (worse than intermediate) with probability 0.4
- The above distribution is common knowledge

Example

Game matrix when worker 2 is strong

		2: E	2: S
1	E	-1,1	1,0
	S	0,1	0,0

Game matrix when worker 2 is weak

		2: E	2: S
1	E	1,-1	1,0
	S	0,1	0,0

Example

Solving for BNE

- It is dominant for strong worker 2 to choose E
- For worker 1, choosing E yields expected payoff of $-0.2 < 0$ regardless of weak worker 2's choice $\rightarrow$ worker 1 chooses S
- Weak worker 2 best responds by choosing E
- BNE: worker 1 chooses S; worker 2 chooses E regardless of her type

III. Applications

[How do classical games differ in a Bayesian context?]

Incomplete information: a summary

Key remarks for Bayesian games

- "Nature" draws types using a mixed strategy; types determine payoffs.
- Treat different *types* of the same player as distinct decision makers (or maximize ex-ante expected payoff).
- Solution concept: *Bayesian Nash Equilibrium (BNE)* – a profile of (possibly type-contingent) strategies where each type best responds in expectation given beliefs.

Distinction: beliefs and expectations

Suppose that you play a simultaneous game with two other players, A and B

- At the beginning, there are \$100 on the table
- Each player can choose between action g and action b
- For A: $g \rightarrow$ nothing happens; $b \rightarrow$ take \$20 from the table
- For B: $g \rightarrow$ take out \$30 from her pocket and put it on the table, and receive a \$40 reward; $b \rightarrow$ nothing happens
- For you: $g \rightarrow$ take all the money from the table (given A's and B's choices); $b \rightarrow$ take \$100
- Clearly, the game has a unique Nash equilibrium: you choose g, A chooses b, B chooses g

Distinction: beliefs and expectations

Suppose that you are playing the same game with only one other player

- With probability p, your opponent is the same as player A
- With probability $1 - p$, your opponent is the same as player B
- $p < \frac{3}{5}$: the unique BNE is $g/b/g$ (you/A/B)
- $p > \frac{3}{5}$: the unique BNE is $b/b/g$
- $p = \frac{3}{5}$: there are two BNE, $g/b/g$ and $b/b/g$

Cournot with uncertain cost

Environment

- Linear demand $P = 100 - Q$, $Q = q_1 + q_2$
- Costs: $c_1 = 10$; firm 2 has type H with $c_{2H} = 20$ or type L with $c_{2L} = 0$, each with prob. $\frac{1}{2}$.
- Firm 1 does not observe the type; firm 2 knows its own type.

Cournot with uncertain cost

Strategy profile

- Firm 1's strategy space: $[0, \infty)$ – knows nothing $\rightarrow$ chooses a number
- Firm 2's strategy space: $[0, \infty) \times [0, \infty)$ – choose one number for high type and another for low type
- A strategy profile is a vector in $[0, \infty) \times [0, \infty) \times [0, \infty)$

Cournot with uncertain cost

Let $(q_1,\ q_{2H},\ q_{2L})$ be a strategy profile. Firm 1 solves

$$\max_{q_1}\left(a - q_1 - \tfrac{q_{2H}+q_{2L}}{2} - c_1\right) q_1 \Rightarrow q_1 = \frac{a - c_1}{2} - \frac{q_{2H} + q_{2L}}{4}$$

- As if firm 1 faces one firm producing $\frac{q_{2H}+q_{2L}}{2}$, or
- Facing two firms, one producing $\frac{q_{2H}}{2}$ and the other $\frac{q_{2L}}{2}$

Cournot with uncertain cost

Given q_1, type H of firm 2 solves

$$\max_{q_{2H}} \left(a - q_1 - q_{2H} - c_{2H}\right) q_{2H} \Rightarrow q_{2H} = \frac{a - q_1 - c_{2H}}{2}$$

Likewise for type L: $q_{2L} = \frac{a - q_1 - c_{2L}}{2}$.

- Firm 2 has complete information
- Its BR is (more) similar to the standard Cournot

Cournot with uncertain cost

Solving for BNE

- Linear equation system: 3 equations and 3 unknowns
- Unique BNE: $(q_1, q_{2H}, q_{2L}) = (30, 25, 35)$
- Total output is 55 with prob. $\frac{1}{2}$ and 65 with prob. $\frac{1}{2}$

Cournot with uncertain cost

Two-sided uncertainty

- Both firms may be H ($c_H = 20$) or L ($c_L = 0$), i.i.d. with prob. $\frac{1}{2}$
- Consider symmetric strategy (q_H, q_L)
- Best responses reduce to

$$q_H = \frac{a - c_H}{2} - \frac{q_H + q_L}{4}, \qquad q_L = \frac{a - c_L}{2} - \frac{q_H + q_L}{4}.$$

Cournot with uncertain cost

BNE with symmetric uncertainty

- Unique solution: $(q_H, q_L) = (25, 35)$
- Note $(q_H + q_L)/2 = 30$
- Matching firm 1's output in one-sided case
- Fundamental reason: $c_1 = \frac{c_{2H}+c_{2L}}{2}$

Belief updating

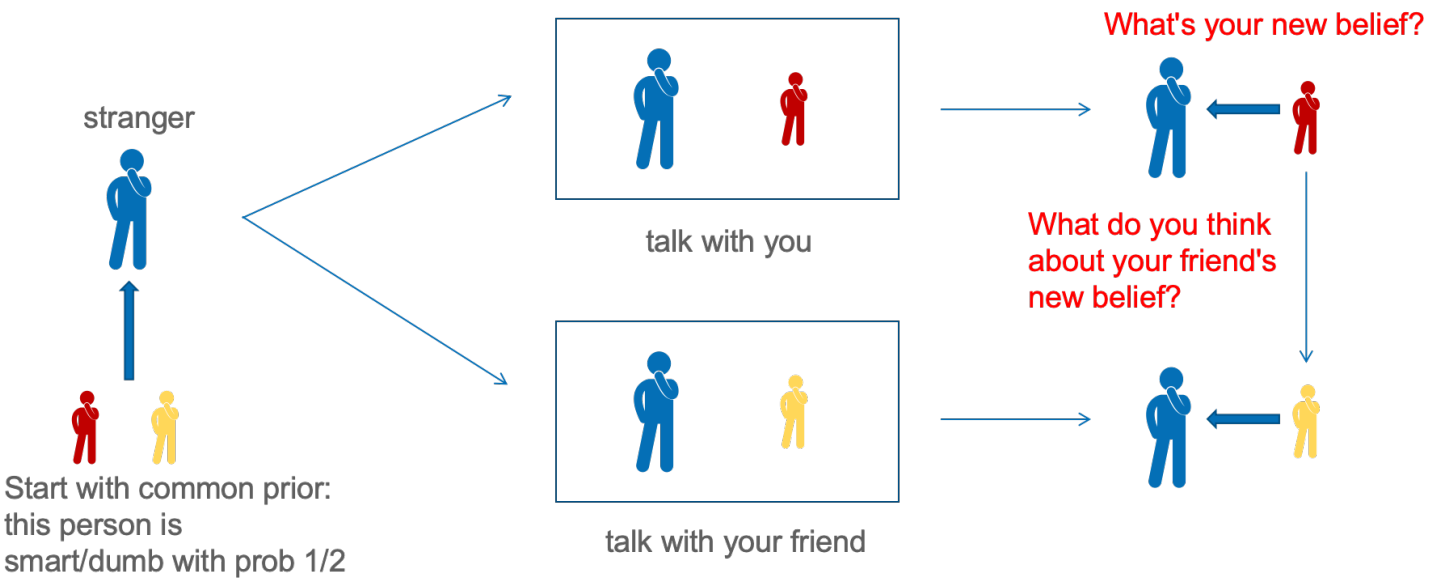


Figure: Example – talking to a stranger

Belief updating

Informative but noisy message

- Common prior: stranger is smart/dumb with prob. 0.5/0.5
- If smart, speaks smart with prob. 0.8
- If dumb, speaks smart with prob. 0.2

Belief updating

Bayes' update

- Your belief about stranger's type after hearing them speak

$$\Pr(\text{smart} \mid S) = \frac{0.8 \cdot 0.5}{0.8 \cdot 0.5 + 0.2 \cdot 0.5} = 0.8, \quad \Pr(\text{smart} \mid D) = 0.2.$$

- Your belief about your friend's posterior after you hear S is mixed: with prob. 0.68 your friend also heard S (posterior 0.8), and with prob. 0.32 heard D (posterior 0.2)

Jury's problem

Setup

- Two jurors $i \in \{1, 2\}$.
- Defendant guilty/innocent with prob. $\frac{1}{2}$
- Independent private signals G/I with accuracy 0.75
- Actions C (convict) or A (acquit), chosen simultaneously; convict iff both choose C
- Payoffs if convict: $+3$ when guilty, -2 when innocent; 0 if acquit

Jury's problem

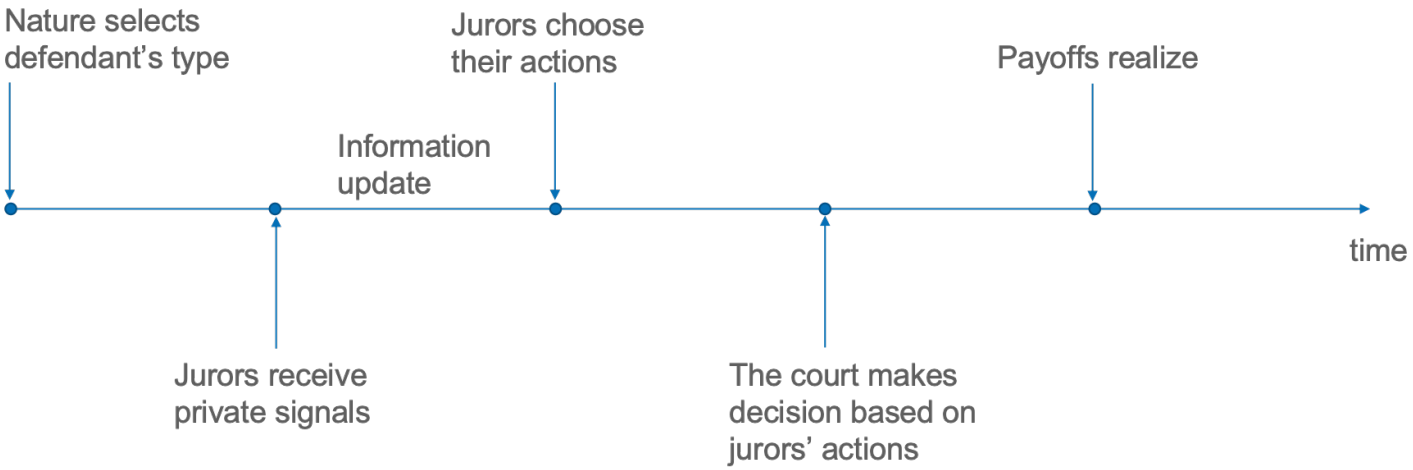


Figure: Timeline of the game

Jury's problem

Is truth-telling a BNE?

- Posterior beliefs: $\Pr(g \mid G) = 0.75$, $\Pr(g \mid I) = 0.25$
- If both use truth-telling $(C|G,\ A|I)$, the court convicts guilty with prob. $(0.75)^2 = 9/16$ and acquits innocent with prob. $1 - (0.25)^2 = 15/16$
- When juror 1 sees I, expected payoff from choosing C is

$$0.25 \cdot 0.75 \cdot 3 \ + \ 0.75 \cdot 0.25 \cdot (-2) = \frac{3}{16} > 0,$$

so C strictly dominates A at signal I. Truth-telling is not a BNE.

Jury's problem

A BNE example

- One equilibrium: juror 1 always plays C; juror 2 plays $C|G,\ A|I$
- Check: For juror 2, $\mathbb{E}[C \mid G] = 0.75 \cdot 3 + 0.25 \cdot (-2) = 1.75 > 0$, and $\mathbb{E}[C \mid I] = 0.25 \cdot 3 + 0.75 \cdot (-2) = -0.75 < 0$
- Equilibrium implications: conviction of the guilty with prob. 0.75; acquittal of the innocent with prob. 0.75 (lower than truth-telling benchmark)

From priors to posteriors

A sketch of knowledge theory (in class)

Takeaways

Dealing with more subtle Bayesian games

- Update from exogenous signals
- Conditional expected payoffs
- Check for incentive for each type
- Update from endogenous behavior (later)

IV. Discussion

[What are some still-remaining challenges regarding information?]

Existence of aliens

Bayesian sanity check

- If a highly credible source tells the truth with prob. 0.99 and prior $p = \Pr(\text{aliens exist})$, then

$$\Pr(\text{aliens} \mid \text{claim}) = \frac{0.99p}{0.99p + 0.01(1-p)} = \frac{99p}{1+98p}$$

- Even strong testimony leaves posterior small when p is tiny

Monty Hall problem

Classical puzzle for information updating

- Three doors, prize behind one and only one
- You pick one first; host reveal one between the rest which is empty
- Will you switch to the remaining door?

Lecture 10: Auctions and Sequential Bayesian Games

School of Management and Economics, CUHK (Shenzhen)

I. Second-Price auction

[What special properties do second-price auctions offer?]

Previously on auctions

Second price auction

- Bidding own valuation is weakly dominant
- Other equilibria where a low-value bidder wins

First price auction

- All bidders have the same valuation $\rightarrow$ equilibrium bid is close to the valuation when bids can only be integers, and is equal to the valuation when bids can be any real number
- Bidders have different valuation $\rightarrow$ equilibrium bid may range from the second highest valuation to the highest valuation

How may the results change when valuation is not common knowledge?

Conjectures

Form of auction still seems to matter a lot

- Would the dominance of bidding own value still hold for second price auction?
- Would bidders still bid close to their valuation in a first price auction?
- Would some previous equivalence relations (SPA $\leftrightarrow$ English, FPA $\leftrightarrow$ Dutch) still hold?

Conjectures

More reasons to be conservative in bidding

- Consider for example a first price auction
- Previously with complete information, bidding (close to) own valuation is because there is no chance of winning otherwise
- Now if every bidder bids own valuation, there is still a chance to win even if bidding lower
- Incentive to bid below own valuation

Conjectures

Equilibrium seems to be affected by distribution of valuation

- Again consider a first price auction
- Opponents are more likely to have higher valuation $\rightarrow$ more incentive to bid higher, and vice versa
- Things probably get more complex when valuation is continuously distributed

SPA: binary valuations

Setting

- Two bidders
- Value is H w.p. p and L w.p. $1 - p$ (private)
- Bids are nonnegative real numbers
- Ties broken at random

SPA: binary valuations

Let V be your value and B the rival bid

- $V > B$: any $V' > B$ wins and pays B; any $V' \leq B$ loses. Better to win $\Rightarrow$ bidding V is BR
- $V < B$: any $V' \geq B$ buys at a loss; any $V' < B$ loses. Better to lose $\Rightarrow$ bidding V is BR
- If $V = B$: indifferent

Conclusion: bidding own value remains weakly dominant under incomplete information.

SPA: binary valuations

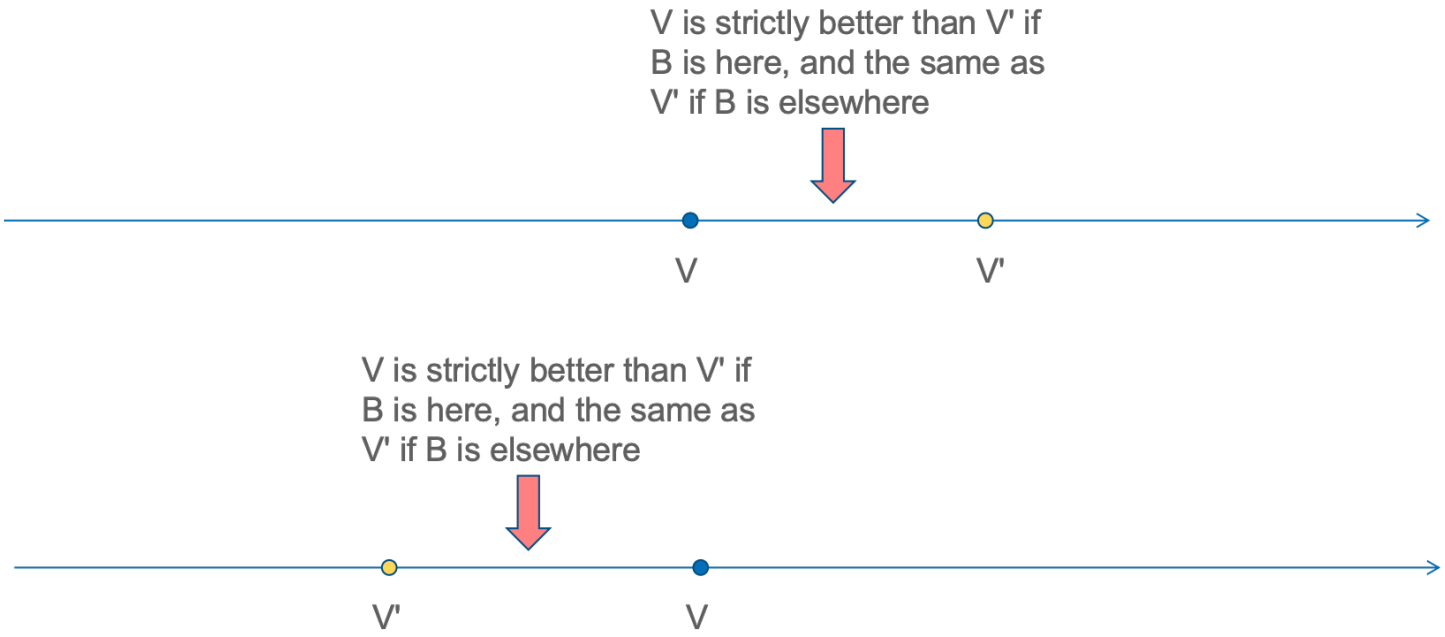


Figure: Above argument carries over to *continuous types*

SPA: binary valuations

Difference made by incomplete information

- Consider the strategy profile that a high-value bidder bids $c < L$ and a low-value bidder bids $d > H$
- This is a NE between a high-value bidder and a low-value one when they know each other's valuation
- However, this strategy profile is not a BNE (why?)
- Incomplete information eliminates some "weird" equilibria

II. First-Price auction: some special cases

[Why is FPA harder to analyze than SPA?]

FPA: binary valuations

Complete information

- Same valuation
  - Unique NE is that both bidders bid their valuation
  - Same as Bertrand competition
- Different valuation
  - Equilibrium bid can range from L to H
  - Low-value bidder bids slightly lower than high-value bidder
  - Again, same as Bertrand competition with different costs
  - Unlike SPA, bidding own valuation is no longer a NE; but the high-value bidder may still be willing to bid H

FPA: binary valuations

Incomplete information

- Consider a symmetric strategy profile: bid only depends on type, not identity of player
- Let h be the bid of a high-value bidder and l be the bid of a low-value bidder
- Low-value bidder: never bid $> L$ in such an equilibrium (why?)
- High-value bidder: never bid $\geq H$ in such an equilibrium (why?)

FPA: binary valuations

Let $k \equiv \max\{h, l\} < H$

- If a high-value bidder bids k in equilibrium: By bidding a bit higher than k, she can significantly increase her payoff via a sure win
- If a high-value bidder doesn't bid k in equilibrium: by bidding a bit higher than her current bid, she can significantly increase her payoff via surely defeating a high-value opponent
- Conclusion: no pure-strategy BNE!

FPA: binary valuations

Finding mixed-strategy BNE

- A naturally that in BNE a high-value bidder always bids higher than a low-value bidder
- A low-value bidder must bid exactly L: what if she bids $< L$ with some probability?
- A high-value bidder uses a mixed strategy: Let x and y denote the lower limit and the upper limit of her bids
- $x \geq L$ by the above

We may identify a number of other necessary conditions in equilibrium.

FPA: binary valuations

Condition 1: $y < H$

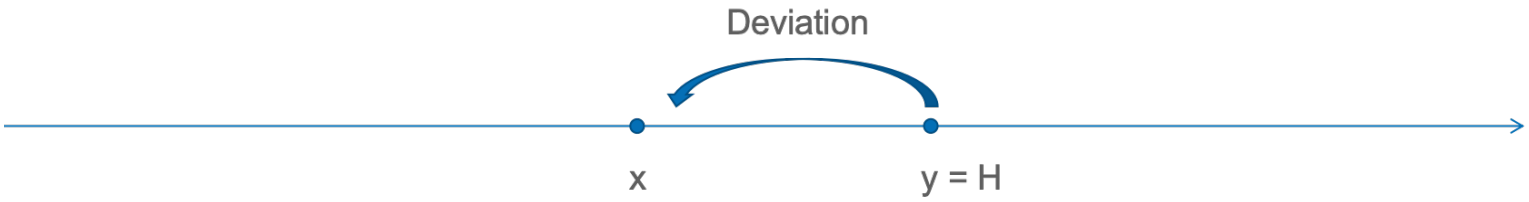


Figure: Profitable deviation if $y = H$

FPA: binary valuations

Condition 2: distribution must be continuous

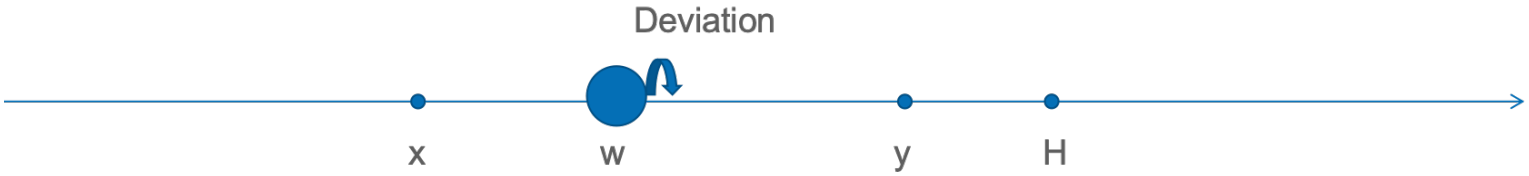


Figure: Profitable deviation if there is a probability "atom"

FPA: binary valuations

Condition 3: $x = L$ (lower limit) but high-value bidder never bids L

Figure: Profitable deviation if $x > L$

FPA: binary valuations

Condition 4: no gap between x and y

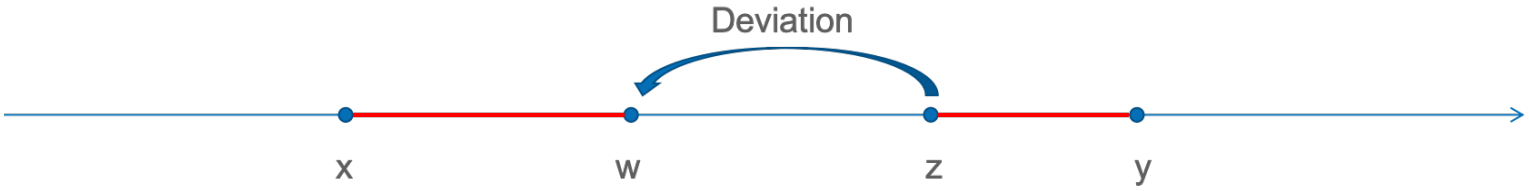


Figure: Profitable deviation if a gap exists between x and y

FPA: binary valuations

Solve for BNE

- L-types bid exactly L
- H-types use a continuous CDF $F(b)$ on $(L, y]$ (or (L, y)) with no atoms; $y < H$
- Apply indifference principle: $\big((1-p) + pF(b)\big)(H - b)$ is a constant

FPA: binary valuations

Solve for BNE (cont'd)

- Finding the constant: payoff $\approx (1-p)(H-L)$ when bidding $\approx L$
- Indifference for H-types:

$$(1-p)(H-L) = \big((1-p)+pF(b)\big)(H-b) \quad \Rightarrow \quad F(b) = \frac{(1-p)(b-L)}{p(H-b)}$$

- $F(y) = 1$ pins down $y = pH + (1-p)L$

FPA: uniform valuations

Suppose now that valuations are i.i.d. and uniform on $[0, 10]$

- CDF $\frac{x}{10}$ and pdf $\frac{1}{10}$
- Bidding own valuation v_i is clearly not a BNE
- Again focus (first) on symmetric equilibrium candidate: $b(v_i)$ (instead of $b_i(v_i)$)
- Still difficult to characterize without further qualifications

FPA: uniform valuations

Conjecture: $b(v_i) = kv_i$

- k is a positive constant
- Rationale: equilibrium bid must relate to distribution + uniform distribution has a linear CDF
- Our goal now: find k so that $b(v_i) = kv_i$ best responds to $b(v_j) = kv_j$

FPA: uniform valuations

Bidder i's problem:

$$\max_{b_i}(v_i - b_i)\frac{b_i}{10k}$$

- $v_i - b_i$ is her payoff upon winning
- $\frac{b_i}{10k}$ is the probability that she wins given the opponent uses $b(v_j) = kv_j$, i.e.

$$\Pr[kv_j < b_i] = \Pr[v_j < \frac{b_i}{k}] = \frac{b_i}{10k}$$

  under this particular uniform distribution

FPA: uniform valuations

Solve the problem

- FOC implies $b_i = \frac{1}{2} v_i$
- Meaning: as long as the opponent uses a linear strategy, i best responds by $b_i = \frac{1}{2} v_i$
- Hence one BNE is characterized by $k = \frac{1}{2}$, i.e. $b(v_i) = \frac{1}{2} v_i$

What if FOC implies a non-linear optimum? Can we generalize this result and claim that every such FPA has a BNE with similar structure to the CDF of valuation?

III. First-Price auction: general case

[How to find BNE without making bold assumptions?]

Model

FPA with general value distribution

- Two bidders $i \in \{1, 2\}$
- Private values $v_i \sim F$
- F i.i.d. on $[0, V]$ with CDF F and pdf f
- Examples: uniform, truncated normal, exponential, etc.

Model

Key difference from before

- Cannot assume specific form of equilibrium bid
- Again start with symmetric candidate $B(v_i)$
- Intuitive assumptions: increasing and continuous (since f exists)

Payoff and best response

Standard problem for bidder i

- With monotone B, bidder i wins iff $v_i > v_j$
- Expected payoff of type v under B:

$$\Pi(v) = F(v)\big(v - B(v)\big).$$

- This problem (choose b against B) is messy due to B^{-1}:

$$\max_b \; F(B^{-1}(b))\,(v - b).$$

Payoff and best response

Induced problem

- Think of B as a machine: you *report* w and it bids $B(w)$ for you
- If B is an equilibrium, truthful reporting $w = v$ must be optimal when others are truthful
- Induced problem for type v:

$$\max_{w \in [0,V]} \; F(w)\big(v - B(w)\big) = vF(w) \; - \; F(w)B(w)$$

Revelation principle

Illustration of the machine story (draw in class)

Revelation principle

The revelation principle in mechanism design

- Recall that auctions constitute a category of mechanism
- A mechanism is a way to an end: designer has a desired outcome (efficiency, profit, etc.) and designs a game (mechanism) trying to achieve this goal among strategic agents
- For instance, an auctioneer designs an auction to maximize her revenue
- If the outcome is achieved in an equilibrium of the game, we say that the mechanism *implements* the outcome

Revelation principle

An important thought in mechanism design

- Every mechanism can be replaced with a mechanism with machines: agent tells machine their type (with possible lies) and machine plays the actual strategy
- A mechanism with machines is called a *direct* mechanism
- *Revelation principle*: every outcome implemented by the original mechanism can also be implemented in a corresponding direct mechanism.

Solve induced problem

FOC gives

$$\left(F(w)B(w)\right)' = vf(w)$$

FOC must hold at $w = v$ for truth telling to be an equilibrium:

$$\left(F(v)B(v)\right)' = vf(v)$$

Remark: which v is given and which is endogenous?

Solve induced problem

Integrate the differential equation with boundary $B(0) = 0$:

$$F(v)B(v) = \int_0^v tf(t)\,dt \quad \Rightarrow \quad \boxed{B(v) = \frac{\int_0^v tf(t)\,dt}{F(v)} = v - \frac{\int_0^v F(t)dt}{F(v)}}.$$

This B is indeed increasing and continuous.

Previous uniform example on $[0, 10]$*:* $F(v) = v/10$, $f(v) = 1/10$ gives

$$B(v) = \frac{\int_0^v t(1/10)\,dt}{v/10} = \frac{v}{2}$$

Revenue equivalence

From the auctioneer's point of view

- An auctioneer is a so-called "mechanism designer"
- If an auctioneer is also trying to maximize her expected payoff, i.e. the expected value of the winning bid, she needs to decide which type of auction to use
- In SPA, bidders bid their valuation but they pay less than what they bid
- In FPA, bidders bid less than their valuation but they pay what they bid

Which type of auction will generate a higher expected payoff for the auctioneer?

Revenue equivalence

Uniform distribution on $[0, 10]$, SPA

- Auctioneer gets the lower valuation, $v_{min} = \min\{v_1, v_2\}$
- v_{min} is a random variable:

$$\begin{aligned}\Pr[v_{min} < v] &= \Pr[v_1 < v \text{ or } v_2 < v] = 1 - \Pr[v_1 \geq v \text{ and } v_2 \geq v] \\ &= 1 - (1 - \frac{v}{10})^2\end{aligned}$$

- This is the CDF of vmin
- Hence the expected payoff of the auctioneer is $\frac{10}{3}$

Revenue equivalence

Uniform distribution on $[0, 10]$, FPA

- In a first price auction, bidder i's bidding function is $B(v_i) = \frac{v_i}{2}$
- The auctioneer gets the higher bid, $\frac{v_{max}}{2} = \frac{\max\{v_1, v_2\}}{2}$
- v_{max} is a random variable:

$$\Pr[v_{max} < v] = \Pr[v_1 < v \text{ and } v_2 < v] = (\frac{v}{10})^2$$

- This is the CDF of v_{max}
- Hence the expected payoff of the auctioneer is also $\frac{10}{3}$

Revenue equivalence

The result holds for general $F(v)$ (revenue equivalence theorem)

- The auctioneer earns the same expected payoff in SPA and FPA, and thus is indifferent between the two forms of auctions
- Auctioneer's expected payoff = bidder 1's expected payment + bidder 2's expected payment
- The bidders are symmetric, so bidder 1's expected payment = bidder 2's expected payment in every auction
- Revenue equivalence implies payment equivalence: the bidders are indifferent between the two forms of auctions as well

IV. Sequential Bayesian games

[How do players resolve the information issue in a dynamic environment?]

Example: observational learning

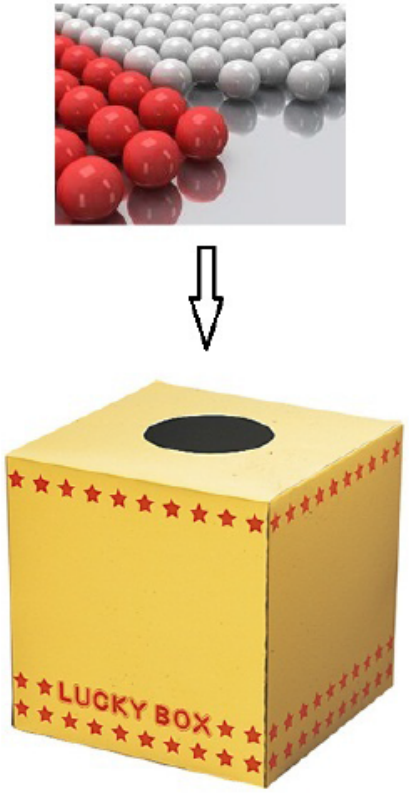


Figure: Simple illustration of information cascade

Example: observational learning

Intuitive equilibrium behavior

- First player: picks observed color
- Second player: definitely picks observed color if it matches first player's choice; indifferent otherwise
- Suppose that second player randomizes when indifferent
- How about third player?

Example: observational learning

Implications

- Herding occurs if first two players make identical choices
- A significant probability of wrong herd, even if population could make correct choice asymptotically should signals be public
- In sequential Bayesian games, players can make further updates upon observing others' actions
- Similar to NE $\rightarrow$ SPNE, BNE $\rightarrow$?

Perfect Bayesian equilibrium (PBE)

An equilibrium specifying BR based on (updated) beliefs

- A PBE consists of a strategy profile and a set of beliefs over the nodes at all information sets
- Note 1: the two elements – strategy profile and set of beliefs – are both necessary for a well-definded PBE
- A PBE needs to satisfy two requirements:
  - At each information set, given the beliefs and the other players' strategies, the strategy for each player is her best response (similar to the definition of SPNE)
  - Given the action profile, the beliefs follow the Bayes' rule whenever possible
- Note 2: for the information sets that are not reached according to the strategy profile, we need to assign beliefs for them, which do not have to follow the Bayes' rule

Example

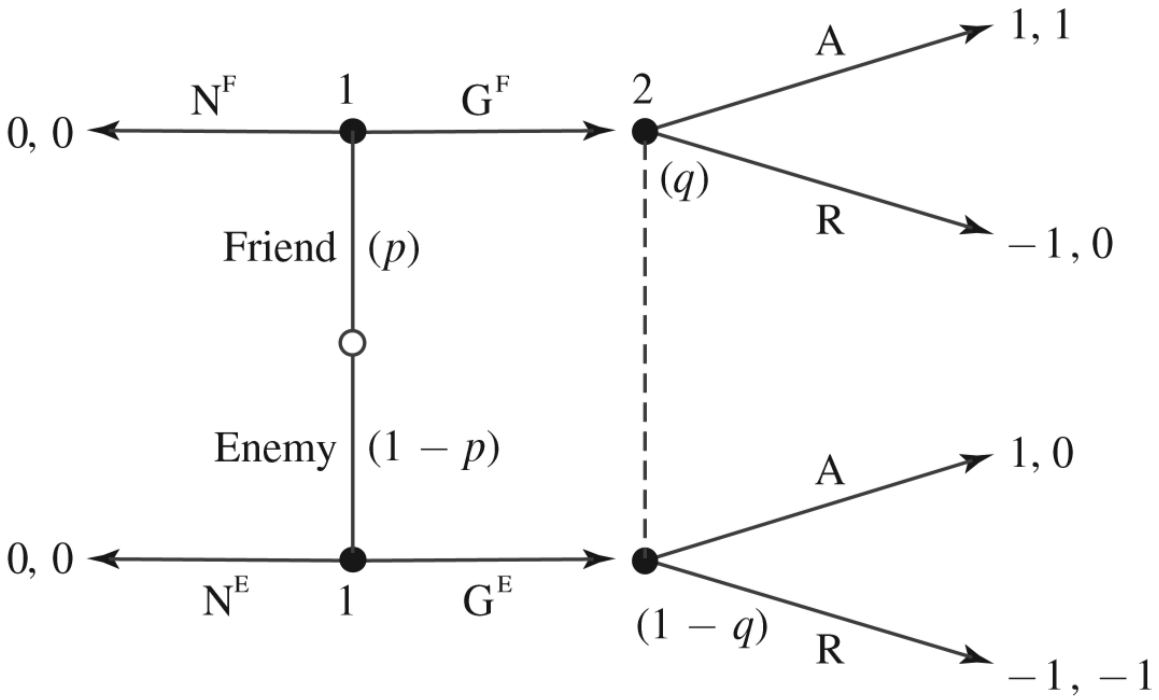


Figure: Example for finding PBE

- Can ("$N|F, N|E$", R) occur in any PBE? If yes, with what beliefs?
- Can ("$G|F, G|E$", A) occur in any PBE? If yes, with what beliefs?

Remarks

Complete information: NE

- No uncertainty about the game, players form beliefs about others' strategies
- The definition of NE implies correct beliefs
- No need for separate beliefs

Complete information: SPNE

- There is no uncertainty about the game played, only need NE in every subgame, not requiring rationality at every information set
- Essentially a sequence of NEs
- Again no need for separate beliefs

Remarks

Incomplete information: BNE

- There is uncertainty about the game played, but BNE (like NE) only requires (common knowledge of) rationality for the whole game (i.e. on path)
- On-path beliefs are automatically Bayesian, and correct in equilibrium
- Again no need for separate beliefs

Incomplete information: PBE

- There is uncertainty about the game played, plus need rationality at every information set
- Some information set may not be reached, hence Bayes' rule cannot pin down beliefs there
- Need separate specification of beliefs

V. Discussion

[What is the practical role of information?]

Clearly, more information could be better...

40

Figure: Cases where information helps

...but not always

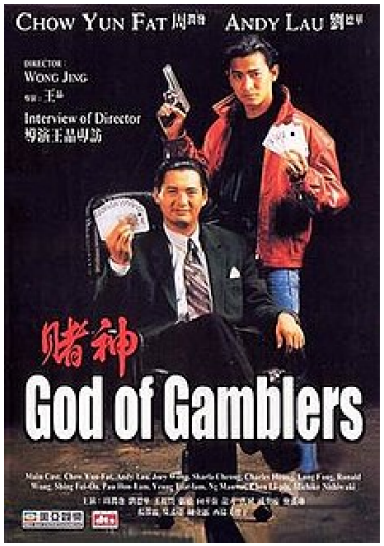


Figure: Cases where information may hurt

Lecture 11: Applications of Bayesian Games – Updating from Various Sources

School of Management and Economics, CUHK (Shenzhen)

I. Introduction

[What questions do we typically ask regarding information in reality?]

Information in practical scenarios

Information is useful in many ways

- Value: private/common/partially common
- Signal: coordination device
- Design: experiment/persuasion

Information in practical scenarios

Positive problems of interest

- How do rational agents react to information?
- Does information aggravate or lessen conflicts of interest? Does it improve or destruct coordination?
- Will incomplete information disappear in the long run?
- What is the optimal way of disclosing information?

Information in practical scenarios

Normative problems of interest

- Is it true that the more information, the better?
- Will information be used/produced efficiently in equilibrium? If not, can we quantify the difference with efficiency?

Information in practical scenarios

This lecture: some classical models regarding information and dynamics

- Cheap talk: information disclosure and conflict of interest
- Herding: (in)efficient information aggregation
- Global game: coordination under heterogeneous beliefs

II. Cheap talk

- Vincent P. Crawford and Joel Sobel, "Strategic information transmission," ECMA 1982

"Cheap talk" = costless message

Cheap and malicious talk

- Pyramid scheme
- Fake news
- (Some) political propaganda

Cheap yet not-always-malicious talk

- Gossip
- Comment/Advice/Suggestion
- (Some) articles and reports

Why we care

Talk is important...

- Transmit information
- Facilitate coordination

...but can be problematic when it's cheap.

- Conflict of interests
- Incentive to lie/mislead

Question: will people lie? How often? Can it be avoided?

Simplest scenario

A player who talks

- Has private information about payoff-relevant state
- Sends message

A player who acts

- Has no information
- Receives message and choose action

Analyst and investor; teacher and student; coach and athlete; CEO and board, etc.

Model

Sender S

- Observes state $m \in [0, 1]$
- m is random with distribution $F(m)$ and density $f(m)$
- Payoff $U^S(y, m, b)$
  - b measures the conflict of interests (see below)

Receiver R

- Take action y
- Payoff $U^R(y, m)$

Model

Timeline

- S sees m
- S sends message $n \in [0, 1]$ to R
- R receives n and takes action $y \in \mathbb{R}$
- Payoffs realize

Analysis

To fix ideas, consider the following stylized utility functions

- $U^S(y, m, b) = -(y - m - b)^2$, $b > 0$
- $U^R(y, m) = -(y - m)^2$

They represent utility structures in which

- Each utility function is "single peaked"
- S and R have different ideals

Analysis

Is truth telling possible in equilibrium?

- How will R respond to a truth-telling S?
- How will S respond to R who *thinks* that S is truth-telling?

Analysis

In fact, there is always a "babbling" equilibrium.

- S sends the same signal regardless of m
- R always takes the optimal action conditional on prior $F(m)$

Now our question becomes: what is the most informative equilibrium?

Result

Theorem

There exists a positive integer $N(b)$ such that, for every $N \in [1, N(b)]$, there exists at least one PBE where

- $[0,1] = \cup_{i=0}^{N-1}[a_i, a_{i+1}]$, $0 = a_0 < a_1 < \cdots < a_N = 1$
- *S sends a uniformly random message on $[a_i, a_{i+1}]$ when $m \in [a_i, a_{i+1}]$*
- *Upon receiving $n \in [a_i, a_{i+1}]$, R takes optimal action conditional on posterior $F(m|m \in [a_i, a_{i+1}])$*

Further, any equilibrium is equivalent to one in this class.

Proof sketch

Key 1: why there is a maximum $N(b)$ (instead of infinity) (Diagram)

- $N(b) = \infty$ means that some truth-telling messages are very close
- However, b is bounded away from 0
- Sender has incentive to deviate

Proof sketch

Key 2: How to determine $N(b)$

- Let $y_{i,i+1}$ denote R's optimal action conditional on posterior $F(m|m \in [a_i, a_{i+1}])$
- When the state is at some boundary a_i, $i \in [1, N-1]$, S must be indifferent between $y_{i,i+1}$ and $y_{i-1,i}$:

$$U^R(y_{i,i+1}, a_i, b) = U^R(y_{i-1,i}, a_i, b)$$

Implications

Lying cannot be eliminated, but can be contained

- Cost of lying?
- Commitment power?
- Repeated games?

III. Herding and information cascade

- Bikhchandani et al., "A theory of fads, fashion, custom, and cultural change as informational cascades," JPE 1992
- Acemoglu et al., "Bayesian learning in social networks," RES 2011

Examples

An information cascade/herding occurs when people...

- Dress as their idols
- Follow investment recommendations
- Choose hot majors in university
- Get married as required
- Vote for the leading candidate

...**without their own thinking**.

Question

Will herding occur among **rational people**?

- Different individuals may possess different information
- Such information may not be observable, but the individuals' actions are
- If actions always reflect information, efficiency is achieved; otherwise herding may arise

Model

A sequence of agents $1, 2, 3, \cdots$

- Each agent needs to take a binary action, 0 or 1

An unknown **state** $\omega \in \{0, 1\}$

- 0 and 1 realize with equal probabilities
- If an agent's action matches ω, she gets payoff $\bar{u}$; otherwise $\underline{u} < \bar{u}$

Model

Before action, an agent observes an i.i.d. private signal

- Signal range $S \subset [0, 1]$
- $\omega = 1$ (0) $\rightarrow$ signal more likely to be large (small)
- Signal is **noisy** but **informative**

She also observes some actions from her predecessors, for instance

- the whole action history
- only the immediate predecessor
- only the first agent

Question (reinterpreted)

Will agent n "almost surely" take the correct action when $n \to \infty$?

- If the signals are publicly observable, the answer will be trivially yes
- Otherwise, the answer hinges on whether equilibrium action is always information-dependent

Simple scenario

Consider binary signal + observation of full history

- Signal is either 1 (good) or 0 (bad)
- First agent: follow signal
- Second agent: if seeing action 1 from first agent
  - Choose 1 if her signal is 1
  - Indifferent if her signal is 0
  - Suppose that she randomizes given signal 0
- Third agent: always choose 1 if seeing 1 from both predecessors

General scenario

Continuous signal distribution

- State-dependent signal distribution F_ω on $[0, 1]$
- Assume **monotone likelihood ratio property (MLRP)**: $\frac{f_1(s)}{f_0(s)}$ increases in s

Observation structure

- Agent n observes the actions of a subset of predecessors, $O_n \subset \{1, \cdots, n-1\}$

Analysis

Two important conditions to prevent herding

- Unbounded private beliefs: signal can be arbitrarily precise

$$\lim_{s \to 1} \frac{f_1(s)}{f_0(s)} = \lim_{s \to 0} \frac{f_0(s)}{f_1(s)} = \infty$$

- Expanding observations: always observe some close predecessor

$$\liminf \max_{k \in O_n} k = \infty$$

Result

Theorem

The following are true:

- *With unbounded private beliefs and expanding observations, the agents will always take the correct action at the limit*
- *Without unbounded private beliefs, the agents will not always take the correct action for typical observation structures*
- *Without expanding observations, the agents will not always take the correct action.*

Result

Proof sketch of first claim

- For expositional convenience, WLOG consider $O_n = \{n-1\}$
- Probability of correct action must increase in n (why?)
- If the probability does not converge to 1, it converge to some $p < 1$
- For large n and extreme signals (close to 1 or 0), a rational agent will act according to signal $\rightarrow$ probability of correct action $> p$, a contradiction

Result

Example for second claim

- $O_n = \{n-1\}$: probability of correct action will not exceed that given the most extreme signal
- Other examples: full history; most recent M predecessors, etc.

Example for third claim

- $O_n = \{m\}$ for a fixed m and an infinite sub-sequence of agents

Herding among rational agents

Rational agents do not always aggregate information efficiently

- Even infinitely many rational players can only take the *ex post* optimal path under limited conditions.
- (Somehow surprisingly) more restricted observation and/or costly observation help reduce herding

IV. Global games

- Stephen Morris and Hyun Song Shin, "Global games: theory and applications," 2001

A two-player model

	Invest	Not invest
Invest	θ, θ	$\theta - 1, 0$
Not invest	$0, \theta - 1$	$0, 0$

- $\theta > 1$: investing is dominant
- $\theta \in [0, 1]$: two NE
- $\theta < 0$: not investing is dominant

Non-common knowledge of belief on θ

θ is random and unobserved

- Prior: uniform on $\mathbb{R}$ (improper prior)
- Player i observes private signal $x_i = \theta + \epsilon_i$
- ϵ_i is independently normal: $N(0, \sigma^2)$

Non-common knowledge of belief on θ

When a player sees signal x, her perceived pdf of θ:

$$f(\theta|x) = \frac{\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\theta)^2}{2\sigma^2}}}{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\theta')^2}{2\sigma^2}} d\theta'} = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\theta)^2}{2\sigma^2}}$$

Hence her belief on θ is $N(x, \sigma^2)$.

Non-common knowledge of belief on θ

When a player sees signal x, her perceived pdf of the other player's signal:

$$f(y|x) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-\theta)^2}{2\sigma^2}} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\theta-x)^2}{2\sigma^2}} d\theta$$

- First term: pdf of y given θ
- Second term: pdf of θ given x

Non-common knowledge of belief on θ

$$
\begin{aligned}
(y-\theta)^2+(\theta-x)^2 &= 2\theta^2-2(x+y)\theta+x^2+y^2 \\
&= 2(\theta^2-(x+y)\theta+(\frac{x+y}{2})^2)+x^2+y^2-\frac{(x+y)^2}{2} \\
&= 2(\theta-\frac{x+y}{2})^2+\frac{1}{2}(x-y)^2
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(y-\theta)^2}{2\sigma^2}}\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(\theta-x)^2}{2\sigma^2}} &= \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{2(\theta-\frac{x+y}{2})^2}{2\sigma^2}}\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{\frac{1}{2}(x-y)^2}{2\sigma^2}} \\
&= \frac{1}{\sqrt{\pi\sigma^2}}e^{-\frac{(\theta-\frac{x+y}{2})^2}{\sigma^2}}\frac{1}{\sqrt{4\pi\sigma^2}}e^{-\frac{(x-y)^2}{4\sigma^2}} \\
&= \frac{1}{\sqrt{2\pi(\frac{\sqrt{2}\sigma}{2})^2}}e^{-\frac{(\theta-\frac{x+y}{2})^2}{2(\frac{\sqrt{2}\sigma}{2})^2}}\frac{1}{\sqrt{2\pi(\sqrt{2}\sigma)^2}}e^{-\frac{(x-y)^2}{2(\sqrt{2}\sigma)^2}}
\end{aligned}
$$

Non-common knowledge of belief on θ

$$f(y|x) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(\frac{\sqrt{2}\sigma}{2})^2}} e^{-\frac{(\theta - \frac{x+y}{2})^2}{2(\frac{\sqrt{2}\sigma}{2})^2}} \frac{1}{\sqrt{2\pi(\sqrt{2}\sigma)^2}} e^{-\frac{(x-y)^2}{2(\sqrt{2}\sigma)^2}} d\theta$$

$$= \frac{1}{\sqrt{2\pi(\sqrt{2}\sigma)^2}} e^{-\frac{(x-y)^2}{2(\sqrt{2}\sigma)^2}}$$

Hence, a player's belief about the other's signal is $N(x, \sqrt{2}\sigma)$.

Iterated solution for equilibrium

Round 1

- Suppose a player observes $x > 1$
- The other invests $\rightarrow$ expected payoff is $x > 1 > 0$
- The other does not invest $\rightarrow$ expected payoff is $x - 1 > 0$
- Investing is dominant
- Likewise, not investing is dominant when $x < 0$

Iterated solution for equilibrium

Round 2

- Suppose your opponent will invest for sure when her signal is > 1 and not invest for sure when her signal is < 0
- Investing is dominant if your signal x is such that, even if the opponent will not invest when her signal is ≤ 1, investing still yields expected payoff > 0
- The condition is characterized by

$$x - \Phi(\frac{1-x}{\sqrt{2}\sigma}) > 0$$

  - Φ is the cdf of standard normal distribution, $N(0,1)$
  - $Prob(y \leq 1|x) = Prob(\frac{y-x}{\sqrt{2}\sigma} \leq \frac{1-x}{\sqrt{2}\sigma}|x)$

Iterated solution for equilibrium

Round 2 (continued)

- Not investing is dominant if your signal x is such that, even if the opponent will invest when her signal is ≥ 0, investing still yields expected payoff < 0
- The condition is characterized by $x - \Phi(\frac{0-x}{\sqrt{2}\sigma}) < 0$
- Note that $x - \Phi(\frac{k-x}{\sqrt{2}\sigma})$ is strictly increasing in x
- Hence, investing is dominant when $x > b(1)$ and not investing is dominant when $x < b(0)$, where $b(k)$ is the x such that

$$x - \Phi(\frac{k-x}{\sqrt{2}\sigma}) = 0$$

Iterated solution for equilibrium

Round $n > 2$

- Suppose your opponent will invest for sure when her signal is $> b^{n-2}(1)$ and not invest for sure when her signal is $< b^{n-2}(0)$
- Apply previous argument
- Investing is dominant when $x > b^{n-1}(1)$ and not investing is dominant when $x < b^{n-1}(0)$

Iterated solution for equilibrium

Round $n \to \infty$?

- Suppose limits $\bar{b} = b^{\infty}(1)$ and $\underline{b} = b^{\infty}(0)$ exist
- At the limit, we must have $\bar{b} = b(\bar{b})$ and $\underline{b} = b(\underline{b})$
- This is a problem about *fixed points*

Iterated solution for equilibrium

Investigate $b(k)$, characterized by $x - \Phi(\frac{k-x}{\sqrt{2}\sigma}) = 0$

- Strictly increasing in k
- $b(k) = k \Leftrightarrow k - \Phi(\frac{k-k}{\sqrt{2}\sigma}) = 0 \Leftrightarrow k = \frac{1}{2}$
- $b(k)$ lies above (below) the 45 degree line when $k < (>) \frac{1}{2}$
- Therefore $\bar{b} = \underline{b} = \frac{1}{2}$

Iterated solution for equilibrium

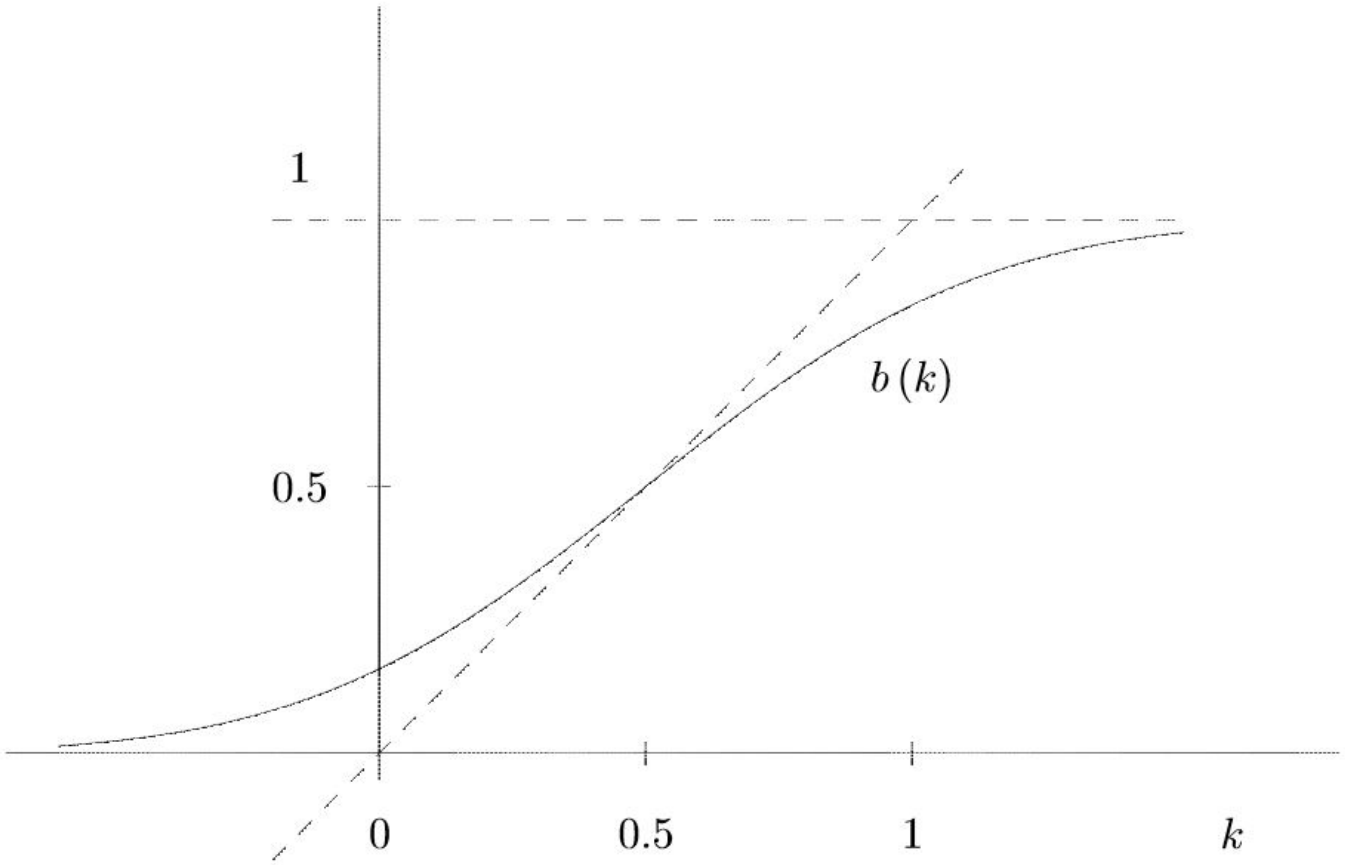


Figure 2.1: Function $b(k)$

What is striking about this model?

Unique switching equilibrium independent of σ

- Suppose you observe private signal 0.49
- Suppose that $\sigma \approx 0$
- You are very close to certain that θ is positive
- You are also very close to certain that the other player is very close to certain that θ is positive
- However, in equilibrium you will surely not invest!

Why does it actually make sense?

Rationalizability

- To make you invest at $x = 0.49$, you need the other player to invest when her signal $\geq$ a bit lower than 0.49, say 0.48
- The other player then needs you to invest when your signal $\geq$ a bit lower than 0.48, say 0.47
- $\ldots$
- Ultimately one of you needs the other to invest when her signal < 0, contradiction

Why does it actually make sense?

The issue goes away when knowledge is common

- Suppose that both players receive private signal 0.01
- Suppose they tell each other their signal values
- Information about θ is still incomplete
- However, coordinating on investing is now an equilibrium

How far we can go

The model can be extended to

- Many players
- Common values with general utility function
- General prior
- Asymmetric payoffs

V. Discussion

[What can be done by further manipulation of information?]

Manipulation of beliefs

From BNE and PBE we observe that

- Strategic players best respond to beliefs
- Hence it'd be great if a designer (government/market maker/media) could manipulate the beliefs
- However, this is only a constrained operation (how?)

Essentially, *information design* is a constrained optimization problem where the choice variable is the distribution of beliefs.

Example (Kamenica and Gentzkow, 2011)

A court trial

- The defendant is guilty with probability 0.3
- The judge has utility 1 from just action (convict if guilty and acquit if innocent) and 0 otherwise
- The prosecutor knows the truth (whether defendant is guilty) and wants the judge to convict

Example (Kamenica and Gentzkow, 2011)

The prosecutor chooses $q(I|i) \in [0,1]$ and $q(G|g) \in [0,1]$

- Can guarantee conviction probability 0.3 by telling truth when guilty, i.e. $q(G|g) = 1$
- Can further increase the probability by choosing some $q(G|i) > 0$
- $\frac{0.3}{0.3+0.7q(G|i)} \geq 0.5 \Rightarrow q(G|i) \leq \frac{3}{7}$
- At optimum, the defendant is convicted with probability 0.6!

Example (Kamenica and Gentzkow, 2011)

The Bayesian constraint

- The prosecutor induces one belief (on defendant being guilty) μ_G by sending G and another belief μ_I by sending I
- Let p_G be the respective (ex ante) probabilities to send G
- The overall probability of defendant being guilty must equal the prior:

$$p_G \mu_G + (1 - p_G)\mu_I = 0.3$$

Remarks

The commitment issue

- Information design requires the designer to commit to the signal structure
- Therefore one can also regard it as "committed talk", as opposed to cheap talk
- The main criticism

Recent development

- Many senders/receivers
- Integration with frameworks in mechanism design, IO, contracts, etc.
- Application in other disciplines

Lecture 12: Contracts

School of Management and Economics, CUHK (Shenzhen)

I. Introduction

[What is a contract in economics and how is it related to our course?]

What is a contract?

To a general audience, a contract is a set of binding agreements

- "Buy one, get one free"
- "Merchandise delivery within 3 days of confirmed payment"
- "3-year employment term with annual salary of $50,000"

A contract specifies available *options* and/or enforceable *outcomes* to relevant parties.

What is a contract?

To an economist, a contract is an induced game

- What type of consumers will buy the commodity in the promotion period?
- How large is the risk of default?
- Will the employee seek for another job?

A contract essentially implies *payoffs* following every action profile of players.

Why do we need contracts?

Are contracts important in the following scenario?

- You are the manager of some elite club
- You can tell each customer's willingness to purchase membership from their appearance
- You are so good at bargaining that every customer ends up paying an amount equal to their willingness

Why do we need contracts?

Are contracts important in the following scenario?

- You are the owner of a factory
- You have surveillance cameras everywhere to monitor your workers' behavior
- There is no minimum wage or any other form of labor protection

Why do we need contracts?

The above examples suggest that we may need contracts when there is

- Hidden information: the characteristics of one (strategic) party is not know to the other
- Hidden action: the action of one party cannot be fully monitored or enforced

How do we study contracts?

Principal-Agent model

- A *principal* who faces hidden information/action but can offer contracts
- An *agent* who strategically chooses available option according to contracts
- Principal maximizes payoff given agent's equilibrium decisions

What do we look for in contracts?

We typically focus on the following:

- First best: principal's optimal contract if she has full information/monitoring
- Second best: principal's optimal contract in the case of hidden information/action
- Rent and efficiency

II. Hidden information: screening

[How does a deliberately designed contract deal with hidden information?]

Adverse selection

Consider a seller (principal) and a buyer (agent)

- The buyer has a private type $\theta \in \{\theta_H, \theta_L\}$, with $\theta_H > \theta_L$
- Prior probability of θ_L is $\beta \in (0, 1)$
- The utility from buying q units of product from the seller is

$$\theta v(q) - T,$$

  where T is the total payment
- Assume that $v(0) = 0$, $v'(q) > 0$, and $v''(q) < 0$

Adverse selection

The seller's problem

- The seller's profit is $T - cq$ where c is the (constant) marginal cost
- Seller may offer contracts to buyer
  - Single price
  - Two-part tariff
  - Non-linear pricing
- Seller seeks to maximize her expected profit

Adverse selection

First best scenario

- Suppose that the seller can observe θ
- She solves for $i = H, L$

$$\max_{T_i, q_i} T_i - cq_i$$

  subject to $\theta_i v(q_i) - T_i \geq 0$
- The inequality is *individual rationality (IR)* constraint

Adverse selection

First best scenario (continued)

- IR must be binding at optimum (why?)
- The seller's problem then becomes

$$\max_{T_i, q_i} \theta_i v(q_i) - cq_i$$

- FOC: $\theta_i v'(q_i) = c$
- The optimum is then given by (T_i, q_i) such that $\theta_i v'(q_i) = c$ and $T_i = \theta_i v(q_i)$
- One implementation is a two-part tariff with price c

Adverse selection

What if the seller offers the first-best contracts under hidden information?

- θ_H buyer will choose (T_L, q_L) instead due to a strictly positive payoff
- Hence the name "adverse selection"
- Other examples
  - Market for lemons
  - Insurance
  - Gresham's law

Second best solution

Linear pricing

- Suppose that the seller only specifies a price P
- Buyer of type $i = H, L$ solves

$$\max_q \theta_i v(q) - Pq$$

- FOC: $\theta_i v'(q) = P \Rightarrow$ demand function $q_i = D_i(P)$

Second best solution

Linear pricing (continued)

- Let $D(P) := \beta D_L(P) + (1-\beta) D_H(P)$
- The seller chooses P to solve

$$\max_P (P-c)D(P)$$

- The optimal price is $P = c - \frac{D(P)}{D'(P)}$

Second best solution

Linear pricing compared with first best

- First best: $\theta_i v'(q_i) = c$ (efficient)
- Linear pricing: $\theta_i v'(q_i) = P = c - \frac{D(P)}{D'(P)} > c$ (inefficiently low consumption)
- Reason: profit loss with lower price

Second best solution

Remarks about linear pricing

- Seller may not actually serve both markets
  - $\theta_L << \theta_H$
  - β very small
- Sometimes linear pricing is already the best option for seller
  - For instance when buyer can resell in some secondary market

Second best solution

Two-part tariff

- Suppose that the seller now offers (Z, P)
  - Z: fixed fee
  - P: unit price
- Let $S_i(P) := \theta_i v(D_i(P)) - PD_i(P)$ be type θ_i buyer's surplus before the fee
- If seller wants to serve two markets, she will set $Z = S_L(P)$ (why?)

Second best solution

Two-part tariff (continued)

- Note that Z is sunk cost to buyer
- Hence seller solves

$$\max_P S_L(P) + (P - c)D(P)$$

- FOC: $S_L'(P) + D(P) + (P - c)D'(P) = 0$
- The optimal price is $P = c - \frac{D(P)+S_L'(P)}{D'(P)}$

Second best solution

Two-part tariff (continued)

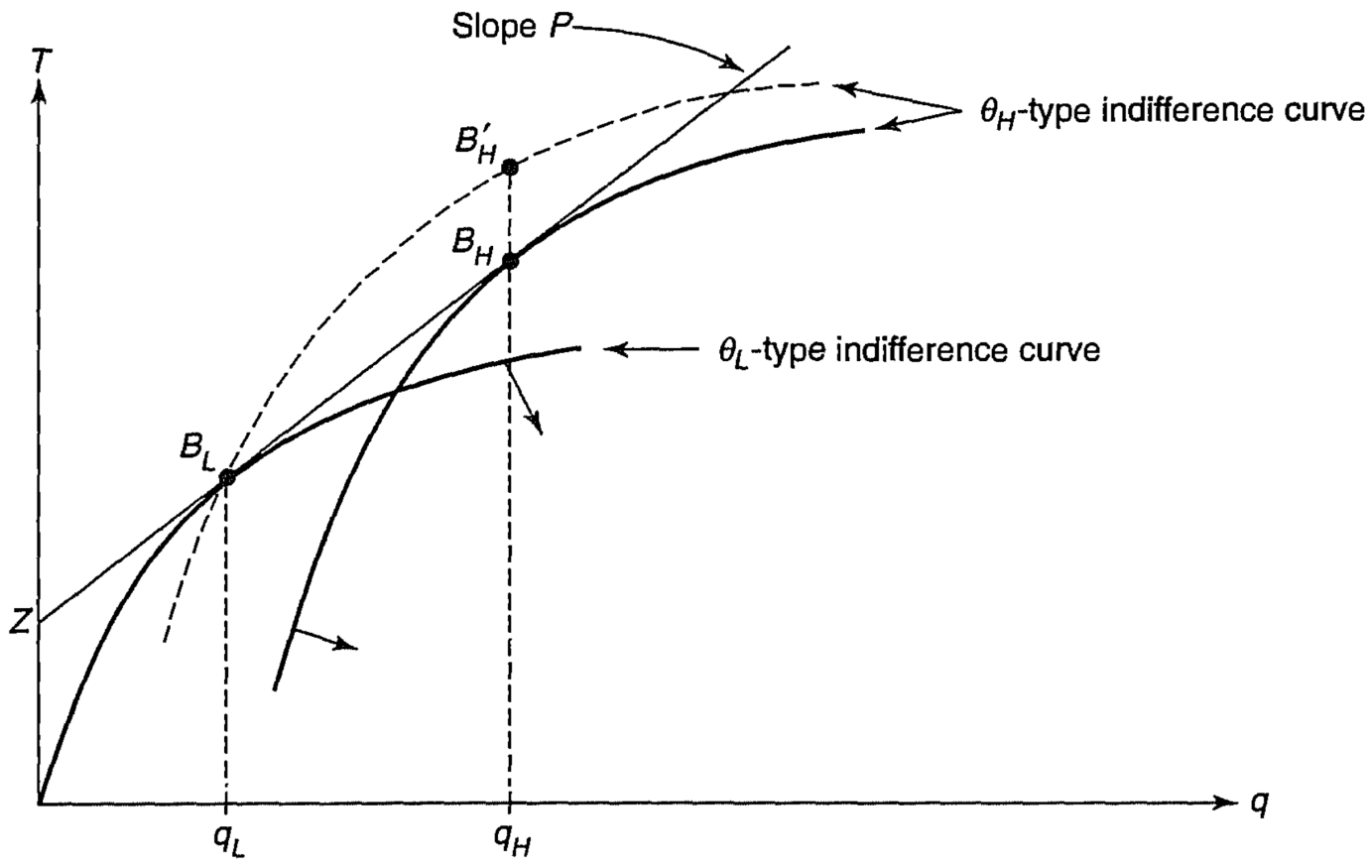
- Recall that $S_L(P) := \theta_L v(D_L(P)) - PD_L(P)$
- $D_L(P)$ is the optimal consumption level of θ_L buyer
- By the Envelope Theorem, $S_L'(P) = -D_L(P) > -D(P)$

Second best solution

Two-part tariff compared with first best

- First best: $\theta_i v'(q_i) = c$ (efficient)
- Two-part tariff: $\theta_i v'(q_i) = P = c - \frac{D(P)+S_L'(P)}{D'(P)} > c$ (inefficiently low consumption, but better than linear pricing)
- Reason: still profit loss with lower price when facing θ_H buyer

Second best solution



Second best solution

Non-linear pricing

- Suppose the seller offers two contracts, (T_H, q_H) and (T_L, q_L)
- The seller's problem becomes

$$\max_{T_i, q_i} \beta(T_L - cq_L) + (1-\beta)(T_H - cq_H)$$

subject to

$$\begin{aligned} \theta_H v(q_H) - T_H &\geq \theta_H v(q_L) - T_L \text{ (ICH)} \\ \theta_L v(q_L) - T_L &\geq \theta_L v(q_H) - T_H \text{ (ICL)} \\ \theta_H v(q_H) - T_H &\geq 0 \text{ (IRH)} \\ \theta_L v(q_L) - T_L &\geq 0 \text{ (IRL)} \end{aligned}$$

- The first two inequalities are *incentive compatibility* constraints

Second best solution

Non-linear pricing (continued)

- Solution step 1: note that IRL + ICH implies IRH

$$\begin{aligned}\theta_H v(q_H) - T_H &\geq \theta_H v(q_L) - T_L \\ &\geq \theta_L v(q_L) - T_L \geq 0\end{aligned}$$

- Hence IRH is automatically satisfied and can be removed from the constraints

Second best solution

Non-linear pricing (continued)

- Solution step 2: ignore ICL for the moment
- ICL is the IC that will not be violated in first best scenario
- Intuitively, θ_L with lower marginal utility is less likely to pretend to be θ_H and purchase more with higher payment
- Reflecting Spence-Mirrlees single-crossing condition:

$$\frac{\partial}{\partial \theta}[-\frac{\partial u/\partial q}{\partial u/\partial T}] > 0$$

Second best solution

Non-linear pricing (continued)

- Solution step 3: focus on binding constraints
- The seller's problem is now

$$\max_{T_i, q_i} \beta(T_L - cq_L) + (1-\beta)(T_H - cq_H)$$

subject to

$$\theta_H v(q_H) - T_H \geq \theta_H v(q_L) - T_L \text{ (ICH)}$$
$$\theta_L v(q_L) - T_L \geq 0 \text{ (IRL)}$$

- If IRL is not binding, seller can raise T_L
- If ICH is not binding, seller can raise T_H

Second best solution

Non-linear pricing (continued)

- Solution step 4: unconstrained optimization
- Further rewrite the seller's problem:

$$\max_{q_L, q_H} \beta(\theta_L v(q_L) - cq_L) + (1-\beta)(\theta_H v(q_H) - cq_H - (\theta_H - \theta_L)v(q_L))$$

- Simple FOC approach yields

$$\theta_H v'(q_H) = c$$
$$\theta_L v'(q_L) = \frac{c}{1 - \frac{1-\beta}{\beta}\frac{\theta_H - \theta_L}{\theta_L}}$$

- Check that ICL is indeed satisfied

Second best solution

Non-linear pricing compared with first best

- θ_H buyer gets efficient consumption
- θ_L buyer gets zero surplus
- Reason: non-linearity allows full surplus extraction from θ_L + cut back θ_L's consumption to maintain θ_H's incentive

Similar properties hold in the case of > 2 or continuum of types.

III. Hidden information: signaling

[How does a passive market react to hidden information?]

The opposite of screening

Same problem of hidden information

- Screening: principal moves first, agent best responds accordingly
- Signaling: agent moves first, principal/market updates and responds
- An example for both: education and job market

Signaling with education

A worker and a firm

- Worker has productivity type i, $i = H, L$ with productivity $r_H > r_L > 0$
- Firm has prior belief $\beta \in (0, 1)$ that worker is of type L
- Worker is willing to work for any positive wage, and firm is willing to hire worker for a wage less than expected productivity

Signaling with education

Education

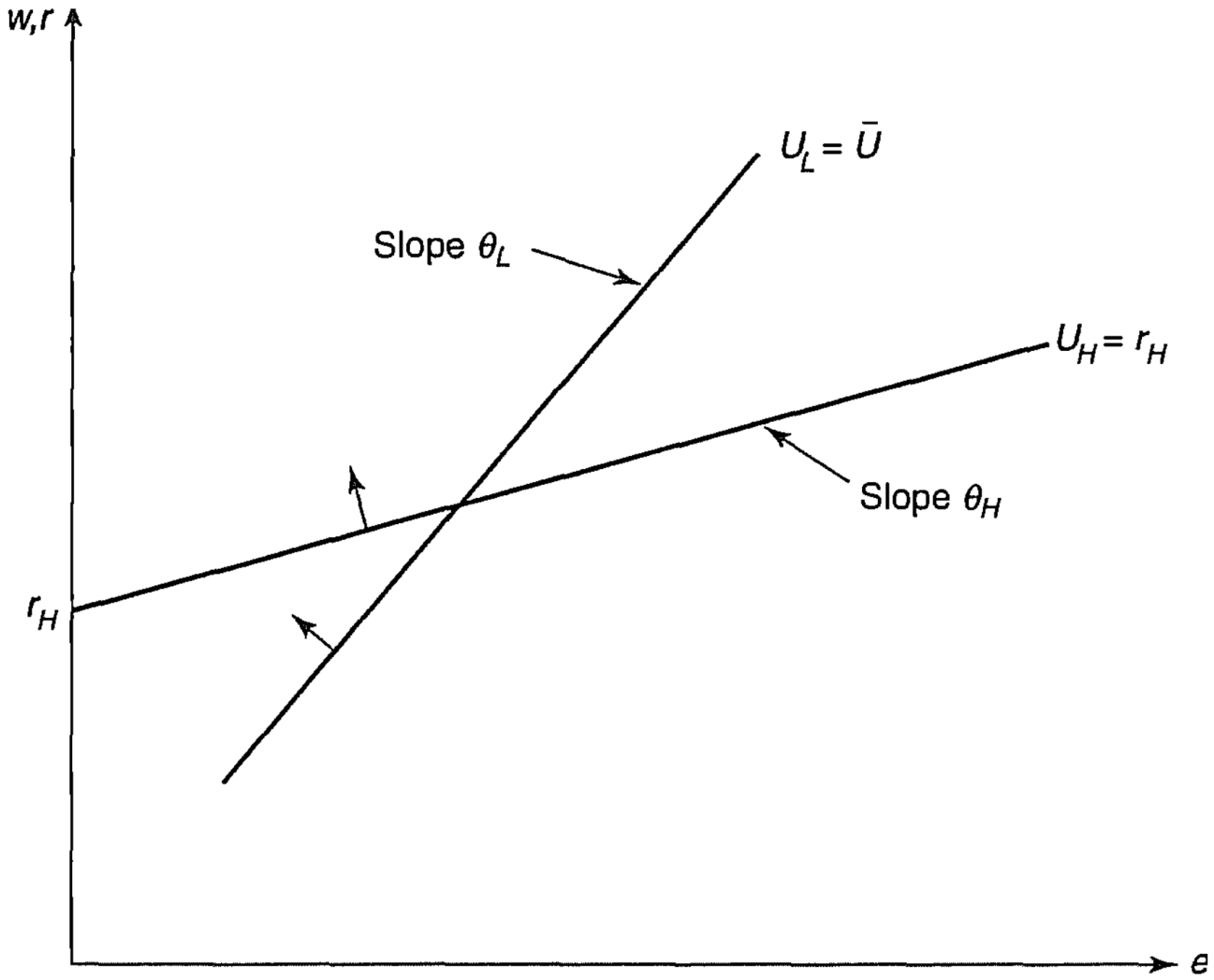
- Worker can get e years of education with cost of $\theta_i e$
- Assume $\theta_H < \theta_L$: easier for high type to finish school
- Education is "wasteful": pure signaling effect

Signaling with education

Timeline

- Stage 1: worker chooses education
- Stage 2: wage determined through bargaining
- Assume worker has full bargaining power: wage equals (conditional) expected productivity

Signaling with education



Signaling with education

First best scenario

- Suppose that worker's type is observable to the market
- Regardless of education, type i worker gets wage r_i
- Type i worker will then (efficiently) always choose $e = 0$

Signaling with education

Candidate for PBE in second best scenario

- Let $\beta(\theta_L|e)$ denote the market's belief given education level
- Worker's wage is then $w(e) = \beta(\theta_L|e)r_L + (1 - \beta(\theta_L|e))r_H$
- Two candidates for (pure strategy) PBE
  - Separating: type H and type L workers get different education $e_H \neq e_L$
  - Pooling: both types get identical education $e_H = e_L = e_p$

Signaling with education

Separating PBE

- $e_L = 0$ for sure (why?)
- e_H satisfies

$$r_H - \theta_H e_H \geq r_L$$
$$r_L \geq r_H - \theta_L e_H$$

- $e_H \in [\frac{r_H - r_L}{\theta_L}, \frac{r_H - r_L}{\theta_H}]$ (but NOT a distribution)
- Feasible off-path belief: $\beta(\theta_L|e) = 1 \ \forall e \neq e_H, e_L$

Signaling with education

Pooling PBE

- In a pooling PBE, $w(e_p) = \beta r_L + (1-\beta) r_H$
- e_p must satisfy $w(e_p) - \theta_i e_p \geq r_L$ for $i = H, L$
- $e_p \in [0, \frac{(1-\beta)(r_H - r_L)}{\theta_L}]$ (again NOT a distribution)
- Feasible off-path belief: $\beta(\theta_L | e) = 1 \ \forall e \neq e_H, e_L$

Signaling with education

Cho-Kreps intuitive criterion

- Let u_i^* denote the equilibrium payoff of type i worker. The criterion requires $\beta(\theta_L|e) = 0 \ \forall e \neq e_H, e_L$ such that $r_H - \theta_L e < u_L^*$ and $r_H - \theta_H e \geq u_H^*$
- Applied to pooling PBE $\Rightarrow e \in (e_p + \frac{r_H - w(e_p)}{\theta_L}, e_p + \frac{r_H - w(e_p)}{\theta_H}] \Rightarrow$ all pooling PBE fail the criterion
- Applied to separating PBE $\Rightarrow e \in (\frac{r_H - r_L}{\theta_L}, e_H] \Rightarrow$ only the least-cost separating PBE ($e_H = \frac{r_H - r_L}{\theta_L}$) satisfies the criterion

IV. Hidden action: moral hazard

[How does hidden action differ from hidden information?]

Effort-based performance

A principal delegates a task to an agent

- The agent exerts costly effort
- Conditional on the effort, outcome of the task is randomly distributed
- Higher effort level $\rightarrow$ higher probability of a better outcome

Principal's problem: offer contract to induce desired effort (where does the name "moral hazard" come from?)

Two performance outcomes

Consider a binary outcome scenario:

- Outcome $q \in \{0, 1\}$ (failure or success)
- Agent exerts effort $a \in \mathbb{R}^+$
- Probability of success $p(a) := \text{Prob}(q = 1|a)$, assume that $p(a)$ is increasing and concave, $p(0) = 0$, $p(\infty) = 1$, $p'(0) > 1$
- Principal offers outcome-based wage $(w_0, w_1) \in \mathbb{R} \times \mathbb{R}$

Two performance outcomes

Payoffs

- Principal's utility function: $V(q - w)$
  - $V'(\cdot) > 0$ and $V''(\cdot) \leq 0$: weakly risk averse
- Agent's utility function: $u(w) - \Psi(a)$
  - $u'(\cdot) > 0$ and $u''(\cdot) \leq 0$: weakly risk averse
  - $\Psi'(\cdot) > 0$ and $\Psi''(\cdot) \leq 0$: weakly convex cost
  - For our analysis, assume that $\Psi(a) = a$

First best contracts

Suppose that effort is observable and verifiable

- Principal's problem:

$$\max_{a,w_i} p(a)V(1-w_1)+(1-p(a))V(-w_0)$$

subject to $p(a)u(w_1)+(1-p(a))u(w_0)-a \geq 0$ (IR)

- The optimum is given by

$$\frac{V'(1-w_1)}{u'(w_1)} = \lambda = \frac{V'(-w_0)}{u'(w_0)} \text{ (Borch rule)}$$

$$p'(a)[V(1-w_1)-V(-w_0)]+\lambda p'(a)[u(w_1)-u(w_0)]-\lambda = 0$$

together with IR.

First best contracts

Risk neutral principal ($V(x) = x$)

- $w_1 = w_0 = w^*$
- $u(w^*) = a$ and $p'(a) = \frac{1}{u'(w^*)}$
- Agent is fully insured

Risk neutral agent ($u(x) = x$)

- $w_1 - w_0 = 1$ and $p'(a) = 1$
- Principal is fully insured

In both cases, marginal productivity is equal to marginal cost (for the principal)

Second best contracts

Now suppose that effort is not observable

- Principal's problem:

$$\max_{a,w_i} p(a)V(1-w_1)+(1-p(a))V(-w_0)$$

subject to

$$p(a)u(w_1)+(1-p(a))u(w_0)-a \geq 0 \text{ (IR)}$$
$$a \in \arg\max_{\hat{a}} p(\hat{a})u(w_1)+(1-p(\hat{a}))u(w_0)-\hat{a} \text{ (IC)}$$

Second best contracts

Reformulate the problem

- IC implies that the agent optimizes:

$$p'(a)(u(w_1) - u(w_0)) = 1 \text{ (IC')}$$

- Letting λ and μ denote the Lagrange multipliers of IR and IC', we have

$$\frac{V'(1-w_1)}{u'(w_1)} = \lambda + \mu \frac{p'(a)}{p(a)}$$
$$\frac{V'(-w_0)}{u'(w_0)} = \lambda - \mu \frac{p'(a)}{p(a)}$$

Second best contracts

Implications

- In general, μ is often positive
  - Imagine the right-hand side of IC' increases from 1
  - To maintain IC', either effort is more productive or agent is less risk averse
  - Both are advantageous for principal
- At optimum, agent is rewarded for better outcome/punished for worse outcome compared to first best

V. Discussion: actual contracts

[Where does our theory still face challenges?]

What is (or should be) the main concern in the following contract?

An AP's contract at a university

- Wage
- Expiration date
- Research, teaching and service
- (Implicit) Bar of tenure

What is (or should be) the main concern in the following contract?

A prenuptial agreement

- List of prenuptial properties
- Division of income and property after marriage, and upon divorce
- Terms of inheritance

What is (or should be) the main concern in the following contract?

A contract for product development

- Expected output
- Delivery date
- Incentive scheme