Lecture 1: Introduction and Basics

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I. How most game theory classes begin

[What are typical examples of games, and what does a game-theoretic problem feature?]

A classic example

Two students caught cheating in exam

- Reporting it to school is troublesome
- Need solid evidence, formal procedure, etc.
- Best way is to make them *confess*
- But how?

A classic example

Solution by a game theory professor

- Students in separate rooms, no communication
- Each offered a simple choice: confess or deny
- Both confess: they both fail the course. (medium punishment)
- A confesses and B denies: A gets 20 points deducted for the exam (very slight punishment) and B gets academic probation because A provides evidence for cheating (severe punishment)
- both deny: they both get 0 for the exam (slight punishment)

A classic example

As smart students, they wonder...

- If the other student confesses
  - Confess: medium punishment
  - Deny: severe punishment
  - Confession is better
- If the other student denies:
  - Confess: very slight punishment
  - Deny: slight punishment
  - Confession is again better

A classic example

The example represents a class of games called "Prisoner's Dilemma".

- If both students are smart and only think of themselves, they will both confess
- However, "both confess" is worse than "both deny" for the students
- The students, their available choices and the possible results constitute a game
- The formal analysis of a game is a theory

Outreach

People use PD to explain various social and economic phenomena

- Arms race in nuclear weapons
- Over-pollution of environment
- Price war
- Involution
- ...

Outreach

In general, game theory allows us to study narrowly defined games...

Outreach

... and so much more.

Questions

- What do the previous scenarios have in common?
- What is the main difference between them and a typical problem in your other economics courses?
- Why is PD a *bad* example of game theory?

II. What to expect in a game

[What natural principles should we follow when analyzing a game?]

A beautiful mind

Figure: *A Beautiful Mind* (2001)

A beautiful mind

To quote "John Nash" in the movie:

"If we all go for the blonde and block each other, not a single one of us is going to get her. So then we go for her friends, but they will all give us the cold shoulder because no on likes to be second choice. But what if none of us goes for the blonde? We won't get in each other's way and we won't insult the other girls. It's the only way to win. It's the only way we all get laid."

A beautiful mind

Suppose each guy went for a brunette

- The blonde would be left alone
- At least one guy would have incentive to talk to the blonde
- Not a "stable" outcome

Suppose that one and only one guy went for the blonde

- That guy would not go for brunette instead
- Every other guy would not try talking to blonde
- "Stable" outcome

A beautiful mind

Principle 1: strategic interaction and individual incentives

- Everyone looking for their best action
- What is "best" depends on others' choices
- A good prediction should leave people no incentive to change behavior

Social distancing


UK politics & policy + Add to myFT

Boris Johnson to end almost all Covid restrictions in England on July 19

Mandatory mask wearing, limits on indoor gatherings and working from home among measures to be scrapped

Social distancing

Rationale behind different observations during epidemic

- China: community-based quarantine, mandatory mask wearing, mass vaccination
- Some other countries: less centralized control, voluntary vaccination, optional work from home
- People “coordinate” in diffent ways

Social distancing

An analogy

- Suppose you're considering whether to voluntarily practice social distancing
- What would you tend to do if knowing all your friends are going out?
- What if all of them are staying home?

Social distancing

Principle 2: payoff structures and possibly multiple solutions

- People respond to incentives in various ways
- Good predictions may not be unique
- Mis-coordination is possible

International relations

How likely will reciprocal tariffs last?

Figure: Tariff war

International relations

China, EU, Japan, India...

- How to respond to prohibitively high tariffs
- How to respond to low tariffs

The US (Donald Trump and his cabin)

- Anticipate trade policies from above countries
- Decide what tariffs to impose

International relations

Principle 3: making inference based on observation and anticipation

- Many (even most) games are dynamic in nature
- Form reasonable expectation of subsequent players' moves
- Especially important for evaluating policies and mechanisms

International relations

A relevant example

Figure: The “empty fort strategy”

"Axioms" in game theory

Strategic interaction

- A "classical" economic problem: Alice has \$100 in her pocket. She goes to the market to buy apples (\$2 each) and oranges (\$1 each). Alice has a utility function $x^{0.3}y^{0.7}$, where x and y are quantities of apple and orange respectively. What is Alice's optimal consumption bundle?
- A game-theoretic problem: Alice and Bob have \$100 each. They would both like to buy the last pack of apples from a seller and decide to auction for it. Alice has a reservation value of \$10 for the pack and Bob \$15. Who will get it in the end, and with what price?

"Axioms" in game theory

Strategic interaction

- The consequence of one player's action relies on the others' actions
- Players take this into account and seek their optimal decisions

"Axioms" in game theory

Rationality

- Each player has an objective function to maximize
- Individual-based vs. group-based
- Static vs. sequential

"Axioms" in game theory

Knowledge of rationality

- Consider the "ultimatum game": There is $1000 on the table. Person A proposes how to split, and person B decides whether to agree. If B agrees, they split the money as proposed. If not, no one gets anything.
- How should A think about B's criterion of acceptance?
- How should B think about A's belief about B's criterion?
- We will typically assume "common knowledge" of rationality. But is this realistic?

III. Essential components of a game

[What do games have in common?]

Components

Towards a theory: what can we abstract from these games?

- Video games
- Sports games
- Auctions
- Social interactions
- Political economy
- ...

Components

Players: individuals who make strategic decisions

- Gamers/Sports players
- Bidders
- Organizations
- Nations

Components

Payoffs: possible results of gameplay

- Winning or losing (e.g. sports, chess)
- Monetary reward (e.g. gambling, bargaining)
- Various forms of interest (e.g. auctions, political campaigns)

Components

Strategies: how a player plans their gameplay

- A *complete* plan
- Assigns an action to *every* contingency the player may face
- Analogous to a “manual” or “user guide”

Components

Example: rock-paper-scissors

- Players: the two opponents
- Strategy set of each player: $\{R, P, S\}$
- Payoffs: 1 (win), -1 (lose) or 0 (tie)

Components

Exercise 1: online auction

Figure: Online auction

Components

Exercise 2: outcry auction

Figure: Outcry auction

IV. Forms to represent a game

[How to analyze a game on a piece of paper?]

Nature and form

Nature: how the game actually proceeds

- *Simultaneous* if players move without observation of others' moves
- *Sequential* if some moves can be observed by some players

Form: how the game is represented

- Normal: players + strategies + payoffs
- Extensive: players + histories + actions + payoffs

Normal form

$G \equiv (N, (S_i)_{i \in N}, (u_i)_{i \in N})$

- N: set of players
- S_i: set of strategies of player i
- $u_i : \times_{j \in N} \to \mathbb{R}$: payoff (utility) function of player i

Normal form

Player 2 ("column player")

Player 1 ("row player")		Rock	Paper	Scissors
	Rock	0, 0	1, 0	0, 1
	Paper	1, 0	0, 0	0, 1
	Scissors	0, 1	1, 0	0, 0

first number:
row player's payoff

second number:
column player's payoff

Figure: Normal form of rock-paper-scissors

Normal form

Prisoner's Dilemma (general version)

		2	
		Confess	Deny
1	Confess	a, a	d, c
	Deny	c, d	b, b

- $a > c$: when the other player confesses, confession is better
- $d > b$: when the other player denies, confession is better
- $b > a$: "both deny" is better than "both confess" (that's why it's a dilemma)

Normal form

Battle of Sexes (general version)

		2: Option A	2: Option B
1	Option A	a, b	c, c
1	Option B	c, c	b, a

- $a, b > c$: both players prefer having the same option to having different options
- $a > b$: player 1 prefers A and player 2 prefers B (hence it's a battle)

Normal form

Exercise: *The Gift of the Magi*

Della and Jim were thinking of what to get for each other for Christmas. Della could sell her hair to Madame Sofronie, a nearby hairdresser, in order to buy Jim a platinum chain for his watch. At the same time, Jim could sell his watch to get Della a set of ornamental combs. For both of them, if they did not sell their valuable, they could not afford any nice gifts.

Madame Sofronie would pay \$20 for Della's hair, while the chain sold for \$21. Della valued her hair at \$$x$, and would enjoy happiness of \$$y$ from the combs if she had her hair. Likewise, Jim could sell his watch for \$30 to buy the combs at \$30. He valued the watch at \$$w$, and would enjoy happiness of \$$z$ from the chain if he had his watch.

Extensive form

$$G \equiv (N, H, P, (V_i)_{i\in N})$$

- N: set of players
- H: set of possible histories
  - Z: set of *terminal* histories
  - $H \setminus Z$: set of non-terminal histories
- $P : H/Z \to N$: function to identify mover at every history
- $V_i : Z \to \mathbb{R}$: payoff (utility) function of player i following every terminal history

Extensive form

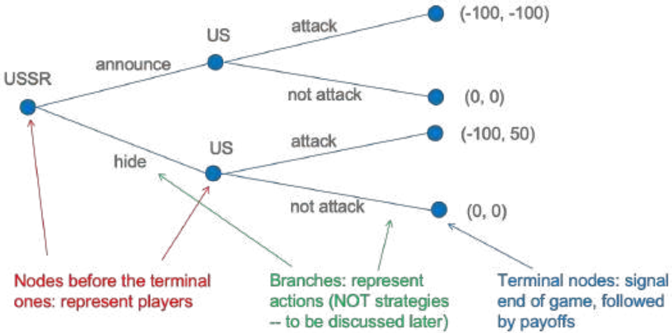

Figure: Extensive form of “doomsday machine”

Extensive form

How to count strategies in an extensive-form game? (in class)

Extensive form

Exercise: *The Lord of the Rings*

Aragorn, leader of the Army of the West, needed to decide whether to attack the Black Gate of Mordor. If he did not attack, the dark lord Sauron would go after Frodo and Sam for his lost ring. If he did, Sauron would have to choose between fighting Aragorn and searching for the ring.

If Sauron went for the ring, he would have a 0.8 probability of stopping Frodo and Sam from destroying it. If he did not, the two Hobbits could successfully destroy the ring with probability 0.3. If the ring was destroyed, Sauron would fall for sure.

On the other hand, if Sauron fought Aragorn at the Gate, he could defeat Aragorn for sure (while his fate still hinged on the ring). Otherwise if the Gate was breached, he could only win with probability 0.8 even if the ring was not destroyed.

Whichever side wins – the force of Middle Earth or Sauron – takes over the world, which we simply assume to bear value \$1.

Transformation

From extensive form to normal form: history dependent actions $\rightarrow$ strategy (in class)

Transformation

From normal form to extensive form: introduction of information sets (in class)

V. Discussion

[What are some limitations of our framework?]

Capability constraint in calculation

Zermelo's Theorem in Chess: one of the following three must be true

- First mover has a must-win strategy
- Second mover has a must-win strategy
- First mover has a must-draw strategy

Behavioral rules in human decisions

Typical “non-econ” findings

- Framing: how information is presented matters
- Anchoring: privilege the first information encountered
- Prospect theory: convex loss and concave gain w.r.t. reference point

Limited usage of simple forms

Matrix/tree not applicable when

- Actions are continuous
- Time is continuous or infinite
- Players are numerous

What's next

Now that we have a basic framework:

- What is an appropriate solution to a game?
- Does that solution always exist?
- How precise a prediction does it make?

Lecture 2: Best Response and Strategic Dominance

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I. Best response

[What is the most natural thing to do in a strategic scenario?]

Beauty contest

Imagine the following game among many players:

- Each player chooses a number from $\{0, 1, ..., 10\}$
- Numbers from all players are collected and the average is calculated
- Whoever's number is closest to $1/2$ of the average wins a prize
- In the case of a tie, the winners split the prize

Beauty contest

A game theorist's thoughts

- All others choose 10 $\rightarrow$ I should choose 5
- All others choose 6 $\rightarrow$ I should choose 3
- All others just randomize $\rightarrow$ the average is around 5 $\rightarrow$ I should choose 2 or 3

Key idea: *adjust* your strategy according to *others'* strategies

Best response in practical scenarios

Figure: An actual beauty contest

Best response in practical scenarios

Figure: Job market

Definition

Recall: normal form game

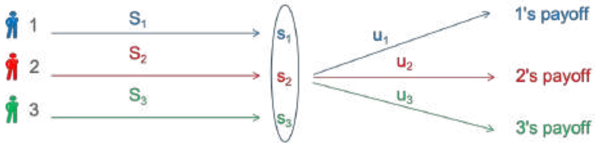

Figure: Illustration of normal form game

Definition

Definition

$s_i \in S_i$ is a best response to $s_{-i} \in S_{-i}$ if $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$ $\forall s_i' \in S_i$.

Let $BR_i(s_{-i}) \subset S_i$ be the set of i's BR actions against $s_{-i} \in S_{-i}$.

- $BR_i(s_{-i})$ is not a function, but a **correspondence**
- It maps a set (S_{-i}) to a set of sets (2^{S_i})
- Let $BR_i = \{BR_i(s_{-i}) : s_{-i} \in S_{-i}\}$

Examples

Coordination

		2	
		B	C
1	B	1,1	0,0
	C	0,0	1,1

- $BR_1(B) = BR_2(B) = \{B\}$
- $BR_1(C) = BR_2(C) = \{C\}$
- $BR_1 = BR_2 = \{B, C\}$
- What if the payoff vector following (C, B) is $(1, 0)$?

Examples

Prisoner's dilemma

		2	
		D	C
1	D	-3,-3	0,-5
	C	-5,0	-1,-1

- $BR_1(C) = BR_2(C) = BR_1(D) = BR_2(D) = \{D\}$
- $BR_1 = BR_2 = \{D\}$

Insight

In an actual strategic scenario, what is a player *really* best responding to?

II. Ways to identify best response

[What are some usual tool kits for finding BR?]

Matrix-representable games

The underlining method

		Player 2	
		Confess	Deny
Player 1	Confess	<u>5</u>, 5	10, 0
	Deny	0, 10	8, 8

Figure: Step 1 – pick a player and fix all the others' strategy, underline larger payoff

Matrix-representable games

The underlining method

		Player 2	
		Confess	Deny
Player 1	Confess	$\underline{5}$, 5	$\underline{10}$, 0
	Deny	0, 10	8, 8

Figure: Step 2 – do the same thing for the same player but another strategy profile of the others

Matrix-representable games

The underlining method

		Player 2	
		Confess	Deny
Player 1	Confess	5, 5	10, 0
	Deny	0, 10	8, 8

Figure: Step 3 – do the same thing for the other player(s). Compare column (row) numbers for row (column) player

Matrix-representable games

More than 2 players

Player 3's strategies

Player 2's strategies

	C	D
A	8, 0, 0	0, 1, 2
B	0, 5, 0	0, 2, 1
	M1	

	C	D
A	4, 2, 0	0, 2, 2
B	0, 1, 1	4, 0, 3
	M2	

Player 1's strategies

Figure: Matrices for a three-player game

Matrix-representable games

More than 2 players

	C	D			C	D
A	8, 0, 0	0, 1, 2		A	4, 2, 0	0, 2, 2
B	0, 5, 0	0, 2, 1		B	0, 1, 1	4, 0, 3
	M1				M2	

Figure: Underlining still applicable, but more tedious

Games without matrices

Cournot competition

- Two firms, quantities chosen simultaneously
- Demand function: $P = 10 - Q$, constant marginal cost of 1
- $BR_i(a_j) = \frac{9-a_j}{2}$ if $a_j \leq 9$, and 0 otherwise
- $BR_i = [0, \frac{9}{2}]$

BRs may go in “circle”

Matching pennies

		2	
		Head	Tail
1	Head	1,-1	-1,1
	Tail	-1,1	1,-1

Shortcoming of BR #1: no stable prediction

III. Best response to mixed strategies

[How to understand best responses in games like Matching Pennies?]

Rationale for mixture

A tennis game between the world's greatest players

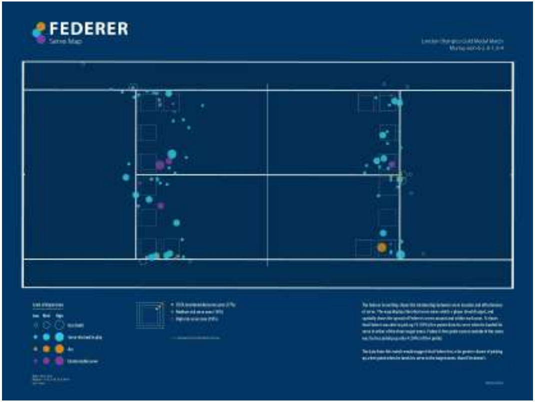

Figure: Serve map of Roger Federer

Rationale for mixture

In a game where BRs may go in circles (Matching pennies, Rock-paper-scissors, most sports games...)

- How often would you think "the opponent must be doing something"?
- Most likely, you'd think "they have some chance of choosing A, and some chance of choosing B"...
- Now, you are best responding not to a single action, but to a *distribution*
- Likewise, you will benefit from some randomization as well

Mixed strategy

		2 (focus on)	
		Left	Right
1 (serve to)	Left	-1,1	1,-1
	Right	1,-1	-1,1

- A strategy like “Left” or “Right” (meaning “always left/right”) is called a pure strategy
- A strategy like “Left with probability 0.4, right with probability 0.6” (denoted 0.4*Left + 0.6*Right) is called a mixed strategy
- A pure strategy is a special case of a mixed strategy

BR to mixed strategy

Let p denote player 1's probability of choosing Left

- If player 2 chooses Left, their payoff is $p + (-1)(1-p) = 2p - 1$
- If player 2 chooses Right, their payoff is $(-1)p + (1-p) = 1 - 2p$
- Hence, Left is their best response if and only if $p \geq 0.5$

BR to mixed strategy

We write $BR_2(p)$ as

- $BR_2(p) = \{Left\}$ if $p > 0.5$
- $BR_2(p) = \{Right\}$ if $p < 0.5$
- $BR_2(p) = \{q \times Left + (1-q) \times Right : q \in [0,1]\}$ if $p = 0.5$

Hence BR_2 is the set of all possible mixed strategies of player 2

Shortcoming of BR #2: sometimes arbitrary

BR to mixed strategy

Exercise: what is BR_1?

	C	D			C	D
A	8, 0, 0	0, 1, 2		A	4, 2, 0	0, 2, 2
B	0, 5, 0	0, 2, 1		B	0, 1, 1	4, 0, 3
	M1				M2	

Figure: Identifying mixed-strategy BR in three-player game

IV. Strategic dominance

[Can we make a sharper prediction based on BR?]

Dealing with BR's shortcomings

Problems with BR as a solution

- Dependent on (belief of) others' actions
- Set of BR may be too large
- How about the contrary, i.e. find the strategies that are *never* a BR?
- One special case is captured by *strategic dominance*

Example of strategic dominance

Figure: Menu of Wing Stop

Definition

Definition

$s_i \in S_i$ in G is strictly dominated if s_i is strictly dominated by some mixed strategy $\alpha_i \in \Delta(S_i)$, i.e. $u_i(\alpha_i, s_{-i}) > u_i(s_i, s_{-i})\ \forall s_{-i} \in S_{-i}$.

- Meaning: α_i is always a better response than s_i regardless of s_{-i}
- In previous example: 2 orders of 10-piece package is always strictly better than 1 order of 20-piece package

Properties of strict dominance

Remarks from the definition

- s_i could also be a mixed strategy
- s_{-i} only needs to be pure strategies (why?)
- Strict dominance is transitive (why?)

Properties of strict dominance

Example: simple 2×2 game

		2	
		D	C
1	D	-3,-3	0,-5
	C	-5,0	x,-1

- $x < 0$: C is strictly dominated by D for player 1
- $x = 0$: C is weakly dominated by D for player 1 (next class)
- $x > 0$: there is no dominated strategy for player 1

Properties of strict dominance

Example: dominated by mixed strategy

		2	
		A	B
1	A	3,3	0,0
	B	0,0	5,5
	C	1,1	1,1

- For player 1, C is not strictly dominated by A or B
- However, C is strictly dominated by a mixture between A and B (how?)

Properties of strict dominance

Example: continuous version of beauty contest

- Many players, each writes a number $a \in [0, 5]$ simultaneously
- Payoff of each player is $-(a - 0.5\bar{a})^2$, where $\bar{a}$ is the average number by all players
- In other words, each player wants to minimize the distance between her number and $0.5\bar{a}$

Properties of strict dominance

Finding dominated strategies without a matrix

- The average number cannot exceed 5, i.e. $\bar{a} \leq 5$
- Hence $0.5\bar{a} \leq 2.5$
- For any number $a' \in (2.5, 5]$, $a' - 0.5\bar{a} > 2.5 - 0.5\bar{a} \geq 0$
- Therefore, $-(a' - 0.5\bar{a})^2 < -(2.5 - 0.5\bar{a})^2$, meaning that a' is strictly dominated by 2.5

Dominance and BR

A strategy is (strictly) dominated $\rightarrow$ It is *never* a best response

- Suppose that s_i^1 strictly dominates s_i^2
- For every pure strategy profile of other players, s_{-i}, we have $u_i(s_i^1, s_{-i}) > u_i(s_i^2, s_{-i})$
- Hence, s_i^2 is never a best response to any s_{-i}
- And s_i^2 is also never a best response to any mixed strategy profile of others

Dominance and BR

Definition

$s_i \in S_i$ in G is a never-best response if $\forall \mu_i \in \Delta(S_{-i})$, $\exists$ some $s_i' \in S_i$ s.t. $u_i(a_i', \mu_i) > u_i(s_i, \mu_i)$.

- A sufficient condition for NBR is being strictly dominated
- It'd be great if we can eliminate all NBRs for each player
- However, finding NBR is not very convenient – in principle we need to find BR first

Dominance and BR

As it turns out, NBR and being strictly dominated are equivalent when correlated actions are allowed.

Theorem

For a finite strategic game G, $a_i \in A_i$ is NBR iff it is strictly dominated.

Dominance and BR

Uncorrelated strategies

- Imagine that players 1 tosses a coin and player 2 tosses another, no one sees the other's coin
- Player 1 will play A if her coin shows head and player 2 will play C if her coin shows head
- Their strategies are independent because the coins are independent

Correlated strategies

- Now imagine that before they toss their coins, God tosses a bigger coin and both players see it (in real life, this can be anything random but observable, e.g. tomorrow's weather)
- They can then play a strategy profile like "1 plays A and 2 plays C if God's coin shows head"
- Correlated strategies include uncorrelated ones (because they can always ignore God's coin)

Example

Not strictly dominated $\leftrightarrow$ BR to some s_{-i}

M_1

	C	D
A	8	0
B	0	0

M_2

	C	D
A	4	0
B	0	4

M_3

	C	D
A	0	0
B	0	8

M_4

	C	D
A	3	3
B	3	3

Figure: Example – matrix player's strategies and payoffs

- Consider strategy M_2
- To dominate M_2, M_4 cannot be used with positive probability
- However, a mixture of only M_1 and M_3 can never dominate M_2
- Hence M_2 is not strictly dominated

Example

Not strictly dominated $\leftrightarrow$ BR to some s_{-i}

M_1:

	C	D
A	8	0
B	0	0

M_2:

	C	D
A	4	0
B	0	4

M_3:

	C	D
A	0	0
B	0	8

M_4:

	C	D
A	3	3
B	3	3

Figure: Example – matrix player's strategies and payoffs

Suppose that player 1 (row) plays A with probability p, and player 2 (column) plays C with probability q

Strategy	M_1	M_2	M_3	M_4
Exp. payoff	$8pq$	$4pq + 4(1-p)(1-q)$	$8(1-p)(1-q)$	3

Example

Strategy	M_1	M_2	M_3	M_4
Exp. payoff	$8pq$	$4pq + 4(1-p)(1-q)$	$8(1-p)(1-q)$	3

To make M2 a best response, the following conditions must be satisfied:

- $4pq + 4(1-p)(1-q) \geq 8pq \Rightarrow p + q \leq 1$
- $4pq + 4(1-p)(1-q) \geq 8(1-p)(1-q) \Rightarrow p + q \geq 1$
- The above two conditions imply that $p + q = 1$
- Finally, we must have $4pq + 4(1-p)(1-q) \geq 3$, with $p + q = 1$ it becomes $8p(1-p) \geq 3$

Example

Strategy	M_1	M_2	M_3	M_4
Exp. payoff	$8pq$	$4pq + 4(1-p)(1-q)$	$8(1-p)(1-q)$	3

Is there a p such that $8p(1-p) \geq 3$?

- This is equivalent to: is the maximum of $8p(1-p)$on $[0,1]$ weakly larger than 3?
- We know that $8p(1-p)$ is largest when $p = 0.5$ (by first order condition, or just by formula)
- Then we have $8p(1-p) \leq 8 \times 0.5 \times 0.5 = 2 < 3$
- Therefore, M2 is never a best response *when others' strategies are uncorrelated*

Example

Not strictly dominated $\leftrightarrow$ BR to some s_{-i}

$$
M_1: \begin{array}{c|c|c|} & C & D \\ \hline A & 8 & 0 \\ \hline B & 0 & 0 \\ \hline \end{array}
\quad
M_2: \begin{array}{c|c|c|} & C & D \\ \hline A & 4 & 0 \\ \hline B & 0 & 4 \\ \hline \end{array}
\quad
M_3: \begin{array}{c|c|c|} & C & D \\ \hline A & 0 & 0 \\ \hline B & 0 & 8 \\ \hline \end{array}
\quad
M_4: \begin{array}{c|c|c|} & C & D \\ \hline A & 3 & 3 \\ \hline B & 3 & 3 \\ \hline \end{array}
$$

Figure: Example – matrix player's strategies and payoffs

With correlated strategies:

- Consider the correlated strategies "$(A, C) \times 0.5 + (B, D) \times 0.5$"

Strategy	M_1	M_2	M_3	M_4
Exp. payoff	$8 \times 0.5 = 4$	$4 \times 0.5 + 4 \times 0.5 = 4$	$8 \times 0.5 = 4$	3

- M_2 is indeed a best response

Dominant strategy

		2: A	B
1	A	5,0	3,2
	B	1,3	2,0
	C	2,4	1,6

- In this game, A dominates both B and C for player 1
- If a player has a strategy that dominates all her other strategies, then that strategy is called a *(strictly) dominant strategy*

Dominant strategy

Definition

$s_i^* \in S_i$ is a strictly dominant strategy for player i if $u_i(s_i^*, s_{-i}) > u_i(s_i, s_{-i})$ $\forall s_i \neq s_i^*$, $\forall s_{-i} \in S_{-i}$.

- s_i^* is strictly dominant $\Leftrightarrow$ s_i^* strictly dominates i's every other strategy
- s_i^* is strictly dominant $\Leftrightarrow$ s_i^* is always the only BR regardless of s_{-i}
- A strictly dominant strategy can never be a mixed strategy
- A player can never have two strictly dominant strategies

V. Discussion

[How confident can we be about today's conclusions?]

Robustness of BR

What will you do in the following game?

		Opponent: a	Opponent: b
You	A	1,0	-1000,-0.1
	B	0.9,0	0.9,-0.1

Non-common knowledge

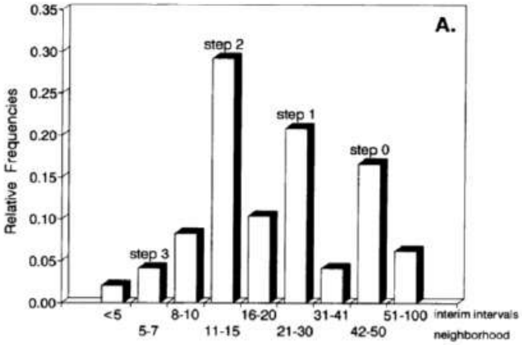

Figure: "Unravaling in Guessing Games: An Experimental Study" by Rosemarie Nagel, AER 1995

Dominated strategies in practice

Figure: Do people actually use dominated strategies?

Lecture 3: Rationalizability and Nash Equilibrium

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I. Rationalizability

[How far can we go based on strategic dominance?]

A typical mindset in gameplay

How do experienced players play Xiangqi?

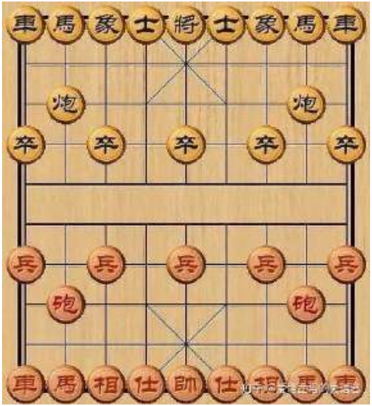

Figure: Chinese chess board

A typical mindset in gameplay

In many games there are "formulas" for the first few moves

- "Dangtoupao" in Xiangqi
- "The Queen's Gambit" in chess
- Rules to follow as a new influencer
- ...

A typical mindset in gameplay

Such "formulas" are the logical result of multi-round strategic thinking

- There are things each player will definitely not do
- Given the above, there are further things each player will definitely not do
- Given the above, ...
- Finally, a set of "very reasonable" strategies survive

This is a refinement of deleting *strictly dominated strategies*

A typical mindset in gameplay

A reverse way of justifying the mindset

- Player 1 plays $A \leftarrow$ 1 thinks 2 will play B
- 1 thinks 2 will play $B \leftarrow$ 1 thinks 2 thinks 1 will play C
- ...
- A strategy is justifiable if the above chain is infinitely long

Each "thinks" must be supported by BR

A typical mindset in gameplay

Recall the continuous version of Beauty Contest

- Many players, each writes a number $a \in [0, 5]$ simultaneously
- Payoff of each player is $-(a - 0.5\bar{a})^2$, where $\bar{a}$ is the average number by all players
- In other words, each player wants to minimize the distance between her number and $0.5\bar{a}$

What number remains after we adopt the mindset?

Definition

Definition

$s_i \in S_i$ is rationalizable if $\exists Z_j \subset S_j$, $j \in N$ s.t.
(1) $s_i \in Z_i$
(2) each $s_j \in Z_j$ is a BR to some belief $\mu_j \in \Delta(Z_{-j})$

- Each action in Z_i can be justified within $\prod_j Z_j$
- We allow **correlation** in strategies

IESDS

One way to obtain a superset of rationalizable strategies

- Delete strictly dominated strategies as they are NBR
- Rationalizable strategies can only appear in the resulting new game
- Delete strictly dominate actions in the new game as they are NBR
- ...

Every strategy thus deleted cannot be rationalizable

IESDS

			Player 2	
		L	M	R
	U	5, 4	7, 2	4, 3
Player 1	C	3, 2	12, 0	7, 3
	D	8, 6	6, 5	5, 7

Figure: Example of IESDS

IESDS

Definition

$X = \prod_{j \in N} X_j \subset S$ in G survives iterated elimination of strictly dominated strategies (IESDS) if $\exists$ finite sequence of sets $\prod_{j \in N} X_j^t$, $t = 0, 1, ..., T \subset S$ such that $\forall j \in N$,
(1) $X_j^0 = S_j$ and $X_j^T = X_j$.
(2) every $s_j \in X_j^t / x^{t+1}$ is strictly dominated in a finite strategic form game $(N, (X_i^t), (u_i))$ for $t = 0, 1, ..., T - 1$.
(3) No action in X_j^T is strictly dominated in $(N, (X_i^T), (u_i))$.

IESDS

As it turns out, surviving IESDS is also sufficient for rationalizability.

Theorem

For a finite game G, $X = \prod_{j \in N} X_j \subset S$ survives iterated elimination of strictly dominated actions iff X_j is the set of all rationalizable actions $\forall j \in N$.

Proof (exercise)

Traveler's Dilemma

Suppose that two travellers' luggage is lost during flight

- The travellers simultaneously report loss, integer from 0 to 300
- Airline company pays each traveller the smaller amount between reported numbers
- Whoever reports a strictly larger number pays 5 to the other

Traveller's Dilemma

Which actions are rationalizable?

- For instance, consider reporting 100
- For 100 to be rationalizable, the player must believe that the other player plays $a > 100$ with positive probability (Why?)
- Hence \$$a$ must also be rationalizable. Then $b > a$, $c > b$, $\cdots$
- However, 300 is not rationalizable
- The only ratioanlizable action is 0

II. Weak dominance

[What if we push strategic dominance even further?]

Weak dominance

Definition

$s_i \in S_i$ is weakly dominated if s_i is weakly dominated by some mixed strategy $\alpha_i \in \Delta(S_i)$, i.e. $u_i(\alpha_i, s_{-i}) \geq u_i(s_i, s_{-i})$ $\forall s_{-i} \in S_{-i}$ with strict inequality for some $s_{-i} \in S_{-i}$.

"Some other (mixed) action is always weakly better than s_i"

Definition

$s_i^* \in S_i$ is a weakly dominant action for player i if $u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i})$ $\forall s_i \in S_i$, $\forall s_{-i} \in A_{-1}$, with strict inequality for some $s_{-i} \in S_{-i}$ and for every $s_i \neq s_i^*$.

"Always a BR, and better than every other action in some case"

Examples

Alternative prisoner's dilemma

		2: D	2: C
1	D	-3,-3	0,-5
	C	-3,0	-1,-1

D weakly dominates C for player 1, and strictly dominates C for player 2.

Examples

Referendum

Figure: The referendum for Brexit

"Voting" weakly dominates "not voting" if voting costs are negligible.

IEWDS

			2	
			C	D
	A		2,1	3,0
1	B		1,1	3,2
	C		3,1	0,2

- B is deleted in the first round
- However, (B, D) is a rationalizable strategy profile
- Problem 1: may delete reasonable strategies

IEWDS

	C	D
A	1,0	0,1
B	0,0	0,1

- Delete B first $\rightarrow$ delete C $\rightarrow$ (A, D) survives
- Delete C first $\rightarrow$ both (A, D) and (B, D) survive
- Problem 2: order matters

Summary

Our solution approaches so far

- Recall that our goal is to predict and understand strategic behavior
- Set of best response strategies: too large, barely says anything (e.g. Matching Pennies)
- IESDS: better, but still may contain multiple strategies (for instance, if there are 3 players and each has 2 strategies left after IESDS, then there are still 8 possible outcomes)
- IEWDS: further reduces the strategy set, but may delete reasonable strategies

III. Nash equilibrium

[Can we have precision, existence and justification at the same time?]

Nash equilibrium

Definition

$s^* = (s_1^*, ..., s_n^*) \in S$ is a Nash equilibrium if $s_i^* \in BR_i(s_{-i}^*)$ $\forall i \in N$.

A Nash equilibrium is a **strategy profile** in which each player is best responding to her opponents (mutual best responses).

Nash equilibrium

Simplest way of finding NE: underline method

		2	
		A	B
1	A	0,0	$\boxed{\underline{4}, \underline{1}}$
	B	$\boxed{\underline{1}, \underline{4}}$	3,3

The Nash equilibria are (A, B) and (B, A) (*not* $(4, 1)$ and $(1, 4)$!)

Nash equilibrium

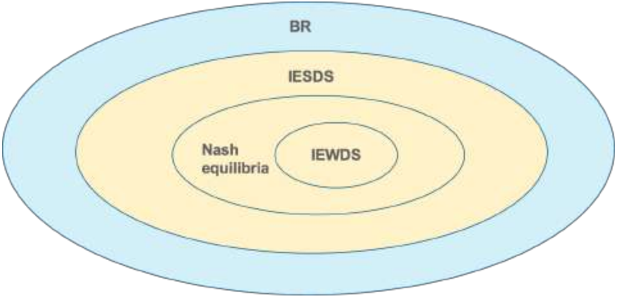

Figure: Relation between the solution concepts

Nash equilibrium

Some properties of NE

- If a player has a strictly dominant strategy, that strategy must occur in every NE
- If a player has a strictly dominated strategy, that strategy never occurs in any NE
- If a player has a weakly dominant strategy, that strategy must occur in some NE
- If a player has a weakly dominated strategy, that strategy may occur in some NE

Nash equilibrium

Interpretation of NE

- Self-enforcing behavior: believe NE $\rightarrow$ play NE.
- Steady state of learning: behavior converges $\rightarrow$ must converge to NE.

Problems with NE:

- Coordination failure: > 1 NE
- Equilibrium selection: > 1 NE
- Non-Nash behavior (especially in short run)

Existence

Not all games have a NE

- Matching pennies
- Bertrand competition where $P = MC$ is not allowed
- Beauty contest where choosing 0 is not allowed

Existence

What are some unique features of the above game?

- Either the set of strategies is open, i.e. boundary not included...
- ...or there is a “jump” in payoffs

We aim to summarize general criteria for NE’s existence.

Existence

Theorem

Suppose that A_i is a nonempty compact convex subset of $\mathbb{R}^k$ and u_i is continuous and quasi-concave in A_i given any $a_{-i} \in A_{-i}$ $\forall i \in N$ for strategic game $(N, (A_i), (u_i))$. Then there exists a NE.

- Review the concepts: compact, convex, quasi-concave
- Heuristic and graphical proof

IV. Auctions (complete information)

[How to identify NE when the underline method is not feasible?]

Bidding for $100

Possible auction forms

- First price sealed bid auction
- Second price sealed bid auction
- Dutch auction (clock descending)
- English auction (outcry)
- All pay auction

SPA

Second-price sealed bid auction

- Two bidders, A and B
- Each writes a number simultaneously and put it in a sealed envelope
- Higher bidder wins the $100 and pays the lower bidder's bid
- For example, if A bids $86 and B bids $70, then A wins and pays $70, earning $30
- Lower bidder pays nothing and earns nothing
- In the case of a tie, winner is selected randomly with equal probabilities, and pays the equal bid

SPA

Finding NE

- Cannot draw matrix
- Even if we can (e.g. when bids are integers only), matrix is too large to handle
- Try thinking in "blocks"
    - Step 1: categorize strategies and see if any category can *never* be in a NE
    - Step 2: inspect the strategy profiles surviving step 1 and see if each can be a NE

SPA

Elimination of impossibilities

- Both bid < 100? $\rightarrow$ The one with positive probability to lose has incentive to deviate (why?)
- Both bid > 100? $\rightarrow$ The one with positive probability to win has incentive to deviate (why?)
- Remaining strategy profiles: one bids ≤ 100 and the other bids ≥ 100

SPA

Checking for best response

- Suppose A bids ≥ 100 while B bids ≤ 100
- For A: increasing bid cannot strictly improve payoff; decreasing bid always weakly decreases payoff
- For B: increasing bid only weakly decreases payoff; decreasing bid does not change payoff (0)
- Thus every such strategy profile is a NE

SPA

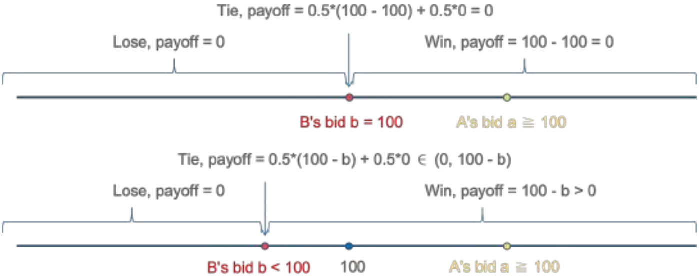

Figure: An illustration from A's perspective

SPA

Discussion: SPA is equivalent to English auction in terms of allocation rule and bidders' strategy formulation. Why?

FPA

First-price sealed bid auction

- Two bidders, A and B
- Each writes a number simultaneously and put it in a sealed envelope
- Higher bidder wins the $100 and pays the lower bidder's bid
- For example, if A bids $86 and B bids $70, then A wins and pays $86, earning $14
- Lower bidder pays nothing and earns nothing
- In the case of a tie, winner is selected randomly with equal probabilities, and pays the equal bid

FPA

Assume that bids can only be integers. Again we try thinking in "blocks":

- A bids < 98? $\rightarrow$ B bids $+1$ $\rightarrow$ A has incentive to deviate
- A bids > 100? $\rightarrow$ B bids less and lose $\rightarrow$ A has incentive to deviate
- Remaining strategy profiles: Each player bids either 98, 99 or 100

FPA

Identify NE

- The bidders bid different numbers? $\rightarrow$ At least one bidder will deviate (why?)
- Remaining strategy profiles: both bid 98, or both bid 99, or both bid 100
- Verify that each is a NE

FPA

With continuous bids

- No one bids < 100 in any NE
- No one bids > 100 in any NE
- Both bid 100 $\rightarrow$ verify that it is the unique NE
- Analogous to Bertrand competition (with identical costs)

FPA

Discussion: FPA is equivalent to Dutch auction in terms of allocation rule and bidders' strategy formulation. Why?

All-pay auction

Now suppose that each bidder needs to pay their bid

- Two bidders, A and B
- Each writes a number simultaneously and put it in a sealed envelope
- Higher bidder wins the $100 and pays her own bid
- Lower bidder pays her own bid and earns nothing
- For example, if A bids $86 and B bids $70, then A wins and pays $86, earning $14; B loses and pays $70, earning -$70
- In the case of a tie, winner is selected randomly with equal probabilities, and both bidders pay the equal bid

All-pay auction

Can there be certain winner or loser?

- In any NE, if there is a certain loser, their best response is unique: bid 0
- Hence, for the other bidder (the certain winner), their best response is to bid 1 (if bids can only be integers) or slightly above 0 (if bids can be any real number)
- However, now the loser will best respond by bidding slightly higher than the winner!
- Therefore, there is no NE with certain winner and loser

All-pay auction

How about equal bids?

- With equal bids, each bidder's expected benefit is 50
- Hence, no bidder will bid > 50 since it's all-pay
- However, there is always incentive to outbid the competitor!
- Therefore this cannot be a NE as well

Conclusion: there is no (pure-strategy) NE in an all-bid auction

V. Partnership

[How to identify NE when payoffs are differentiable?]

Partnership

Suppose two firms are considering a joint operation

- Each firm chooses its effort, x (for firm 1) and y (for firm 2), independently
- x and y must lie within $[0, 4]$
- The joint company's profit is $4(x + y + 0.2xy)$
- Each firm shares half of the profit
- Effort is costly: it costs x^2 (for firm 1) or y^2 (for firm 2)

Partnership

To find NE

- Clearly, no matrix can be drawn for this game
- Meanwhile, it is not clear at first what strategies will never occur in NE
- However, as payoffs are differentiable, we may find the best-response *functions* of players
- NE is then characterized by intersection of these functions

Partnership

Standard procedure

- Firm 1's problem

$$\max_x 2(x + y + 0.2xy) - x^2$$

- First-order condition

$$2 + 2 \times 0.2y - 2x = 0$$

- Check second-order condition: FOC indeed implies maximum
- $BR_1(y) = 1 + 0.2y$; likewise, $BR_2(x) = 1 + 0.2x$
- Unique NE: $x = y = 1.25$

Partnership

The game is actually also *dominance solvable*, i.e. IESDS leaves only one strategy profile

FIGURE 8.2
Partnership game
best-response functions.

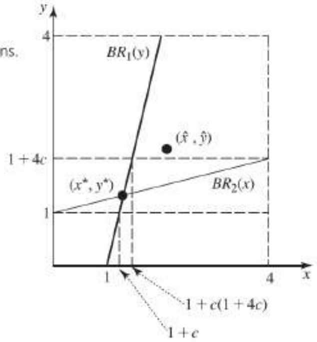

Figure: Dominance solvability

VI. Discussion

[What is NE missing?]

Equilibrium selection

What would you do in the following game?

	A	B
A	1000,1000	0,0
B	0,0	0.1,0.1

Information

Incomplete information in auctions

- Usually bidders' valuation is not common knowledge
- In a 2-bidder auction, what happens if each bidder knows their own valuation, and that their opponent's valuation is uniformly distributed on $[0, 1]$?
- For a second price auction, it doesn't matter – bidding own valuation is a weakly dominant strategy anyway
- For a first price auction, will a bidder still bid their own valuation?

Lecture 4: Mixed-Strategy NE and Efficiency

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August 31, 2025

I. Mixed-strategy NE

[How do we make predictions in a finite game in general?]

Rationale

Games with no pure-strategy NE

- Matching pennies
- Rock-paper-scissors
- Almost every athletic sport

Games where pure-strategy NE does not make sense

- Battle of sexes
- Almost every coordination game with no prior communication

Rationale

Mixed-strategy NE as a way out

- Recall the existence theorem: compact and concave strategy set + quasi-concave and continuous payoff = NE existence
- Set of mixed strategies: automatically compact and concave
- Expected payoff: automatically linear and continuous
- Interpretation as randomization

Definition

Definition

A (possibly mixed) profile $s = (s_1, \ldots, s_n)$ is a Nash equilibrium if $s_i \in BR_i(s_{-i})$ for all i.

- Same definition extended to mixed strategies
- Each player chooses a distribution over actions to maximize *expected* payoff
- Direct "underline and compare" no longer works because the strategy sets are infinite

Definition

Understanding mixture in practice

- Not literally "rolling dice"–can reflect fluctuating tastes, moods, or environments
- Approximates long-run variation: rotating tasks, alternating patterns, repeated attempts to best-respond to others

Methodology

Matching pennies

		2	
		H	T
1	H	1,-1	-1,1
	T	-1,1	1,-1

A mixed strategy NE is a strategy profile

$$(p \times H + (1-p) \times T, q \times H + (1-q) \times T)$$

that satisfy mutual best responses.

Methodology

Standard way: BR functions

- Player 1 solves

$$\max_p pq + (1-p)(1-q) - p(1-q) - q(1-p) = 1 + (4q-2)p - 2q$$

- Likewise, player 2 solves

$$\max_p -pq - (1-p)(1-q) + p(1-q) + q(1-p) = -1 + (2-4p)q + 2p$$

- Hence the BR functions are

$$BR_1(q) = \begin{cases} 1 & \text{if } q > 1/2, \\ [0,1] & \text{if } q = 1/2, \\ 0 & \text{if } q < 1/2, \end{cases} \qquad BR_2(p) = \begin{cases} 0 & \text{if } p > 1/2, \\ [0,1] & \text{if } p = 1/2, \\ 1 & \text{if } p < 1/2. \end{cases}$$

Methodology

Deriving NE

- Claim: no equilibrium can have $p \neq 1/2$ or $q \neq 1/2$
- Otherwise a pure best response cycle would occur
- Hence the unique NE is

$$p^* = \tfrac{1}{2}, \qquad q^* = \tfrac{1}{2}.$$

Methodology

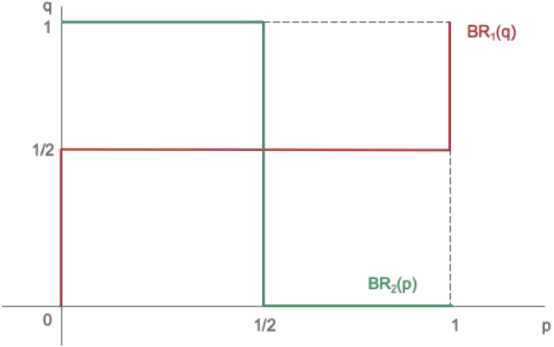


Figure: Illustration of BR functions

Methodology

Simple way: indifference principle

- There is no NE where either player uses a pure strategy
- Hence every NE must be in "totally mixed strategies"
- In such an equilibrium, what does it mean to use a mixed strategy as BR?
- Every player must be *indifferent* between pure strategies

Methodology

Applying the principle

- Make player 1 indifferent: q must satisfy $\mathbb{E}[u_1(H)] = \mathbb{E}[u_1(T)]$:

$$q - (1 - q) = (1 - q) - q \Rightarrow q = 1/2$$

- Make player 2 indifferent: p must satisfy $\mathbb{E}[u_2(H)] = \mathbb{E}[u_2(T)]$:

$$-p + (1 - p) = -(1 - p) + p \Rightarrow p = 1/2$$

Thus $(p^*, q^*) = (1/2, 1/2)$.

Methodology

Remarks

- To solve player 1's mixing probabilities, use *player 2's* indifference condition (and vice versa).
- A totally mixed NE need not exist (e.g., Prisoners' Dilemma).
- In any finite game, at least one NE (pure or mixed) exists.

Example

Battle of sexes

		2	
		A	B
1	A	a, b	$0, 0$
	B	$0, 0$	b, a

- Two pure-strategy NE: (A, A) and (B, B)
- No possibility of a NE with one player using a pure strategy and the other mixing
- Hence we are left with one task: finding possible "totally mixed" NE

Example

Let $p = \Pr_1(A)$ and $q = \Pr_2(A)$. Indifference yields:

- Player 1: $aq = b(1-q) \Rightarrow q^* = \frac{b}{a+b}$
- Player 2: $bp = a(1-p) \Rightarrow p^* = \frac{a}{a+b}$
- Unique totally mixed NE:

$$\left(p^* A + (1-p^*)B, \;\; q^* A + (1-q^*)B\right)$$

with $p^* = \frac{a}{a+b}$, $q^* = \frac{b}{a+b}$

Comparative statics: As a rises (or b falls), player 1 plays A more often and player 2 plays A less often.

Example

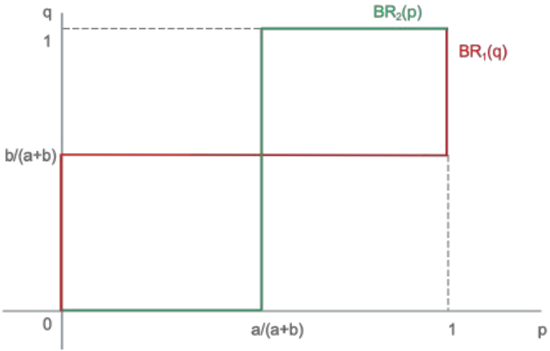


Figure: Illustration of BR functions

General approach (finite games)

1. Find all pure-strategy NE.
2. Check for NE where some players are pure and others mix.
3. If needed, search for totally mixed NE using indifference.

General approach (finite games)

Exercise

		2: C	D	E
1	A	1,2	1,2	3,0
	B	0,1	0,0	4,2

II. Case study: Volunteer's Dilemma

[How can mixed-stratege NE be applied in actual economic prediction?]

Kitty Genovese and the bystander effect

On March 13, 1964, a lady named Kitty Genovese was murdered in Queens, NYC. Reports suggested many witnesses but few interventions. The situation maps to a "bystander effect" in sociology and psychology: each bystander could have helped but few actually did.

Volunteer's dilemma

Model the situation as a game

- A number of people observe an incident
- The incident requires at least one volunteer to fix it
- Everyone would like to see the incident fixed
- As long as there is at least one volunteer, adding the number of volunteers won't help
- Volunteering is costly: time-consuming or dangerous

Volunteer's dilemma

		Others: At least one volunteers	Others: None volunteers
Player 1	Volunteer	0	0
	Not volunteer	1	-10

- Pure-strategy NE: Exactly one player volunteers (more than one cannot be a best response; zero is not stable)
- However, a mixed-strategy NE would make more sense among strangers (why?)

Volunteer's dilemma

Symmetric mixed-strategy NE

- Let each player volunteer with probability p.
- If player 1 volunteers, payoff $= 0$
- If she does not, payoff $= 1$ with prob. $1 - (1 - p)^{n-1}$ (someone else volunteers) and $= -10$ with prob. $(1 - p)^{n-1}$
- Indifference:

$$(1-(1-p)^{n-1})-10(1-p)^{n-1} = 0 \Rightarrow (1-p)^{n-1} = \frac{1}{11} \Rightarrow p^* = 1-11^{-\frac{1}{n-1}}.$$

Volunteer's dilemma

Implications

- Probability the incident is fixed:

$$1-(1-p^*)^n = 1-11^{-\frac{n}{n-1}},$$

  which *decreases* in n.
- As group size n increases, each individual volunteers less often so that the chance no one else volunteers stays at $1/11$ (by indifference).
- Social optimum is "exactly one volunteer"—which does not arise under symmetric mixing without coordination.

Volunteer's dilemma

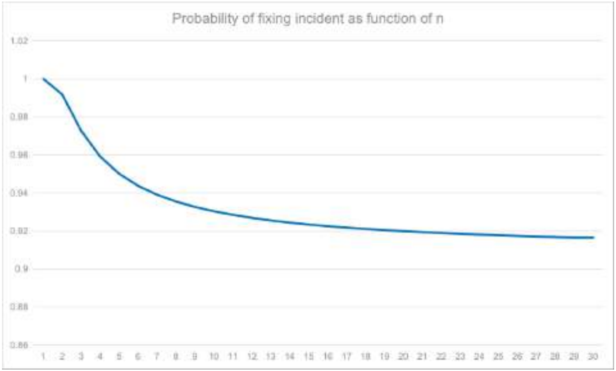


Figure: Probability of fixing incident as function of n

III. Efficiency

[What does NE say about achieving social optimum?]

Individual rationality vs. Social optimum

From cases like Volunteer's dilemma, we observe that

- Equilibrium justifies players' behavior when they act *solely for themselves*
- Equilibrium does not always produce *the greater good*

How to identify the greater good?

- Pareto efficiency: no one can be improved without hurting someone else
- Strict (utilitarian) efficiency: maximize the sum of payoffs

Motivating scenario: Insurance markets

Two consumer types: high risk (H) and low risk (L). Two insurers compete (Bertrand) and offer a contract for each type.

- Competitive zero profit: each contract breaks even
- Adverse selection: H prefers the L contract (lower premium) $\Rightarrow$ only L-contract is demanded, losses ensue, market unravels

Motivating scenario: Insurance markets

Typically, government intervention is required to restore efficiency

- Policy pitfalls: a uniform mandatory plan priced at average risk attracts only H; forcing L to buy may create further issues.
- Remedies: subsidies, and (costly) verification/screening
- The optimal policy is a product of theoretical arguments and empirical verification

From the insurance story...

A bigger picture

- Everyone wants *efficiency*, but what we get is *equilibrium*.
- Bending equilibrium toward efficiency needs costs and/or clever design.
- Analogies: parenting middle-schoolers; political propaganda; team management styles.

Example 1: Car wash stations

Figure: Evolution of car wash stations

Example 1: Car wash stations

Setup

- Several car wash stations in the same location (players)
- The price for a car wash, p, is fixed at least in the short run, due to price rigidity
- Every car wash station has the option of upgrading its service or sticking to the current service (strategies)
- Upgrading its service will cost c for a station (payoff)
- Given the same price, the consumers will go to the station with better service when service quality is different and just pick a station randomly if service quality is the same (payoff)

Example 1: Car wash stations

Assume n stations and k customers. If m stations upgrade:

- Non-upgraders earn 0
- Each upgrader earns $p \cdot \frac{k}{m} - c$

Deviations:

- An upgrader who downgrades loses $p \cdot \frac{k}{m} - c$
- A non-upgrader who upgrades earns $p \cdot \frac{k}{m+1} - c$

Example 1: Car wash stations

A NE is characterized by m and must satisfy (for $0 < m < n$):

$$\text{Upgrader nonnegativity:} \qquad p\frac{k}{m} - c \geq 0,$$

$$\text{Non-upgrader deviation not profitable:} \qquad p\frac{k}{m+1} - c \leq 0.$$

Boundary cases:

- $m = n$: require $p\frac{k}{n} - c \geq 0$.
- $m = 0$: require $pk - c \leq p\frac{k}{n}$.

Example 1: Car wash stations

Comparative statics

- m increases with p and k: more stations incentivized to upgrade when upgrade is more profitable
- m decreases with c: less stations incentivized to upgrade when upgrade is more costly
- Incentive to upgrade is zero at $n = 1$, peaks at $n = 2$, and falls with larger n

Example 1: Car wash stations

Efficiency

- *Stations only:* a prisoner's dilemma flavor–upgrading can be (weakly) dominant at high p or large k, yet joint non-upgrade saves costs with the same total revenue
- *Including customers:* assume baseline benefit b; upgraded benefit $B > b$. If $(B - b)k > c$, the strictly efficient outcome is *exactly one* upgrader (to avoid redundant costs). Accounting for externalities such as waiting-time externalities complicates the optimum
- Mixed-strategy NE: homework

Example 2: Tragedy of the commons

Figure: Highway congestion

Example 2: Tragedy of the commons

Highway use as a game

- Continuum of players with mass a
- Each chooses *Highway* or *Local*
- Benefit from highway: $b - cx$ where x is highway mass
- Local benefit: constant $d > 0$
- Assume $b - ca < d < b$

Example 2: Tragedy of the commons

Solving for NE

- x_{NE} solves $b - cx_{\text{NE}} = d \Rightarrow x_{\text{NE}} = \frac{b-d}{c}$ (unique)
- If the inequality does not hold, either some players going local will switch to highway, or the converse
- At NE, highway and local benefits equalize; highway becomes *useless* on average due to self-imposed congestion externality

Example 2: Tragedy of the commons

Social optimum: strict efficiency

- Strictly efficient x: maximize $(b - cx)x + d(a - x) \Rightarrow$ FOC $b - 2cx = d$ $\Rightarrow x^{\text{EFF}} = \frac{b-d}{2c} < x^{\text{NE}}$
- Policy tools: quantity limits, congestion pricing, or subsidies for local routes
- Equity vs efficiency trade-offs in large cities

Example 2: Tragedy of the commons

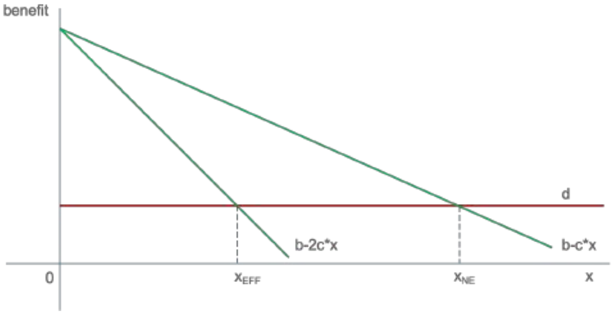


Figure: Illustration of NE vs. efficient allocation. What does it look like?

Example 3: Public good provision

Figure: Typical public goods

Example 3: Public good provision

Voluntary contribution to public good

- n players, budget a each
- Player i contributes $x_i \in [0, a]$
- Project value $= b \sum_j x_j$, equally shared, so payoff

$$u_i = a - x_i + \frac{b}{n} \sum_j x_j.$$

Example 3: Public good provision

Marginal private return: $-1 + \frac{b}{n}$ (independent of others).

Nash equilibrium

- If $\frac{b}{n} > 1$, contributing all is strictly dominant
- If $\frac{b}{n} < 1$, contributing 0 is strictly dominant

Example 3: Public good provision

Social welfare: $\sum_i u_i = an + (b-1)\sum_i x_i$.

Efficiency

- If $b > 1$, everyone contributing all is optimal.
- If $b < 1$, zero contribution is optimal.

Inefficiency: When $1 < b < n$, NE under-provides the public good $\Rightarrow$ micro-foundation for taxation.

IV. Discussion

[Why is achieving efficiency so difficult?]

Free-market advocacy

Why markets succeed

- Markets allocate capital efficiently via prices
- Interventions (wage floors, price caps, unions) help one side but may harm others/overall efficiency
- Long-run, global efficiency often favors freer markets

Free-market advocacy

Why markets fail

- Short-run frictions (e.g. unemployment)
- Externalities (e.g. environmental issues)
- Information problems (e.g. adverse selection)
- Alternatives (e.g., planning) face knowledge and incentive limits and require concentrated power with its own risks

Is efficiency always preferred?

Random serial dictatorship

- Three people split \$1,000,000 via rock-paper-scissors; winner takes all
- Every outcome is Pareto efficient
- Would you prefer this or paying \$10 for a notarized equal split?

Efficiency vs. risk and fairness

Do we agree on Pareto improvements?

Who Wants to Be a Millionaire

- Suppose two players team up for the game
- Double-or-nothing next question vs secure split
- Will the players always agree on whether to go for the next question?

Efficiency vs. heterogeneous perception

Are efficient outcomes equally good?

Matching

- Two men m_1, m_2 and two women w_1, w_2
- Both m_1 and m_2 like w_1 better; however, w_1 likes m_1 better while w_2 likes m_2 better
- Two efficient matchings: "$m_1 - w_1$, $m_2 - w_2$" and "$m_1 - w_2$, $m_2 - w_1$"
- However, in the second one, m_1 likes w_1 better than his current match while w_1 likes m_1 better than her current match

Efficiency vs. stability

Lecture 5: Sequential Games and Subgame Perfection

School of Management and Economics, CUHK (Shenzhen)

I. Sequential games

[How do sequential games make a difference, and (why) do we need new solutions?]

Centipede game

Consider the following game between two players

- There is \$100 on the table
- Round 1: player 1 decides whether to take the money; if she does, game ends
- Otherwise, the \$100 increases to \$200 and game proceeds to round 2: player 2 decides whether to take the money; if she does, game ends
- ...
- When the money increases to \$1000, whether or not the player decides to take it, the game ends after her decision

Centipede game

A game theorist's thoughts

- Whether 1 would take the money in round 1 depends on whether 2 would take the money in round 2
- Whether 2 would take the money in round 2 depends on whether 1 would take the money in round 3
- ...
- Finally, it depends on whether 2 would take the money in round 10 (when the money grows to $1000)

Centipede game

Two game theorists' decisions

- Clearly, in round 10 player 2 would definitely take the money
- Then in round 9, player 1 would take the money
- Then in round 8, player 2 would take the money
- ...
- Finally, in round 1, player 1 would take the money
- This is a reasonable solution, given common knowledge of rationality in every possible round

Centipede game

Is this a NE?

- Player 1 plays 11take the money in round 1/3/5/7/9" and player 2 plays "take the money in round 2/4/6/8/10"
- Given that player 2 will take the money in round 2, taking the money in round 1 is a BR for player 1
- Given that player 1 will take the money in round 1, there's nothing player 2 can do to change her payoff, so it is also a BR for her
- Conclusion: NE indeed

Centipede game

Is this the only NE?

- Consider the following strategy profile: player 1 plays “take the money in round 1, but do not take the money in round 3/5/7/9” and player 2 plays “take the money in round 2, but do not take the money in round 4/6/8/10”
- Given that player 2 will take the money in round 2, every strategy with taking the money in round 1 is a BR for player 1
- Given that player 1 will take the money in round 1, there’s nothing player 2 can do to change her payoff, so every strategy is a BR for her
- Conclusion: No. There are many other NE
- Hence, our new solution must have some special properties that NE does not

Deeper epistemological issue

With static games, we

- Assume common knowledge of rationality (for once)
- I am rational; I know that you are rational; I know that you know that I am rational; and so on

However, if we do the same in sequential games...

- No rationality requirement if the game proceeds to the next stage
- Hence could result in many NE with irrational off-path behavior

Remark: need stricter requirement of rationality, which is reflected in a new notion of equilibrium

Deeper epistemological issue

Practice of "sequential rational thinking": suppose there are 23 coins on the table. Starting with you, you and your opponent may take 1–3 coins at a time in turn. The one taking the last coin wins. What is your sure-win strategy?

Deeper epistemological issue

Realistic scenarios requiring sequential thinking

Figure: Bigger issues that need sequentially optimal plans

Subgame perfection

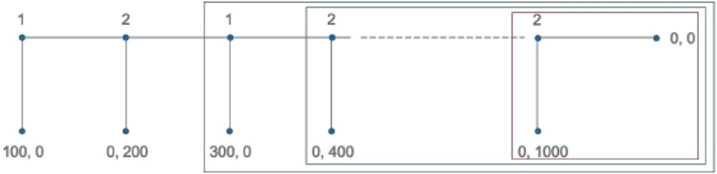


Figure: Smaller games in the centipede game

Subgame perfection

The idea of subgames

- Each “subgame” is a small game tree
- A subgame is a smaller game that starts from a decision node, and includes all nodes that go after it, as well as the associated payoffs
- If a decision node is included in an information set containing > 1 nodes, it cannot be regarded as the initial node of a subgame

Subgame perfection

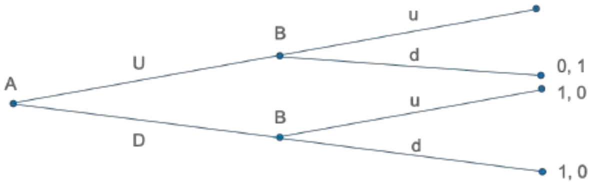


Figure: Counting subgames

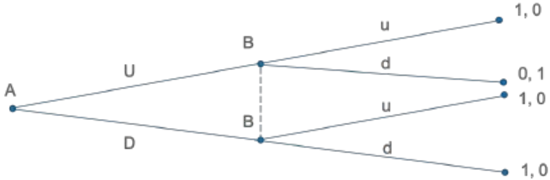


Figure: Never cut through information sets

Subgame perfection

Our previous analysis

- Start from round 10 and determined player 2's best response $\rightarrow$ NE in round 10 (the smallest subgame)
- Given the NE in round 10, determine player 1's best response in round 9 $\rightarrow$ this must be a NE from round 9 (the second smallest subgame)
- ...
- Finally, given the NE in rounds 2 — 10, we determined player 1's best response in round 1 $\rightarrow$ this must be a NE from round 1 (the largest subgame, or the original game)
- Hence, our solution represents a NE in *every subgame*

Subgame perfection

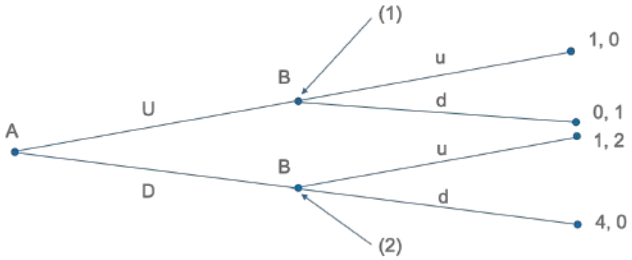
Generalizing our idea

- *Definition*: a strategy profile that constitutes a NE in every subgame is called a Subgame Perfect Nash Equilibrium (SPNE)
- The way to find SPNE is called *backward induction*: (1) start from the smallest subgame(s) and find the NE; (2) Given the NE of previous step, find NE in the second smallest subgame(s) ... (n) Given the NE of all the "proper" subgames, find NE of the original game
- Typically, the number of SPNE is less than the number of NE (is it always unique?)

II. Simple examples

[What are some basic rules to follow when identifying SPNE?]

Example 1



Example 1

Finding SPNE

- Three subgames: the original game, the one starting from (1), and the one starting from (2)
- B's optimal choice: d in subgame (1) and u in subgame (2)
- Given the above, A's optimal choice is D
- SPNE: (D, "$d|U, u|D$")

Example 1

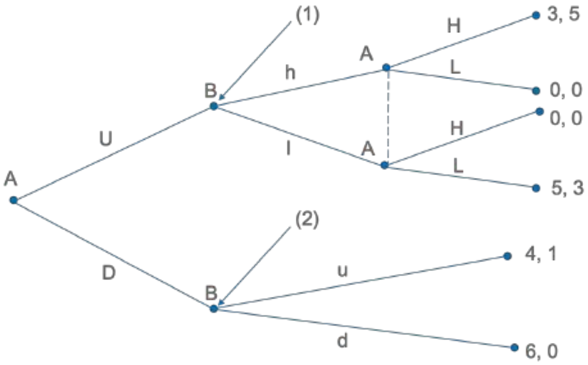
Finding NE

		B			
		u|U, u|D	u|U, d|D	d|U, u|D	d|U, d|D
A	U	$\underline{1}$, 0	1, 0	0, $\underline{1}$	0, $\underline{1}$
	D	$\underline{1}$, $\underline{2}$	$\underline{4}$, 0	$\underline{1}$, $\underline{2}$	$\underline{4}$, 0

Figure: NE in normal form

How many mixed-strategy NE are there?

Example 2



Example 2

Finding SPNE

- Three subgames: the original game, the one starting from (1), and the one starting from (2)
- NE in subgame (2) is easy – B will choose u
- 3 possible NE in subgame (1): (a) B chooses h, A chooses H; (b) B chooses l, A chooses L; (c) B chooses $(5/8) \times h + (3/8) \times l$, A chooses $(3/8) \times H + (5/8) \times L$
- Assign one NE to subgame (1) and then proceed to finding NE for original game

Example 2

Finding SPNE (cont'd)

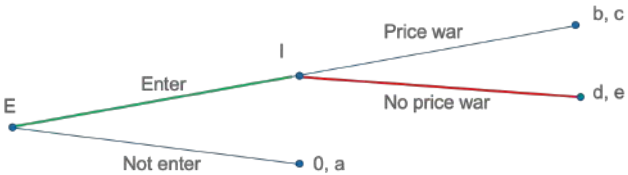
- Assign (a) to (1) $\rightarrow$ If A chooses U, she gets 3; if she chooses D she gets 4 $\rightarrow$ she will choose D $\rightarrow$ SPNE: A chooses D at first, and chooses H in (1); B chooses h in (1), and chooses u in (2)
- Assign (b) to (1) $\rightarrow$ If A chooses U, she gets 5; if she chooses D she gets 4 $\rightarrow$ she will choose U $\rightarrow$ SPNE: A chooses U at first, and chooses L in (1); B chooses l in (1), and chooses u in (2)
- Assign (c) to (1) $\rightarrow$ If A chooses U, she gets 15/8; if she chooses D she gets 4 $\rightarrow$ she will choose D $\rightarrow$ SPNE: A chooses D at first, and chooses $(3/8) \times H + (5/8) \times L$ in (1); B chooses $(5/8) \times h + (3/8) \times l$ in (1), and chooses u in (2)
- Three SPNE in total

Example 3

Chain store paradox

- Suppose that a small retail store (the entrant) is thinking of entering the downtown of Shenzhen
- There is already a large chain store (the incumbent) which has been dominating the retail market
- Knowing the existence of the entrant, the incumbent threatens to start a price war if the entrant enters
- The price war will destroy the entrant, but will hurt the incumbent as well
- Question: is this threat credible?

Example 3



- Assume $b < 0$, $c < e < a$, $d > 0$
- SPNE: entrant enters and incombent accommodates (no price war)
- The threat is non-credible

Example 3

		Incumbent	
		Price war|Enter	No price war|Enter
Entrant	Enter	b, c	d, e
	Not enter	0, a	0, a

- Two NE: (not enter, price war—enter) and (enter, no price war—enter)
- In NE, we require common knowledge of rationality – but only in the original game
- In SPNE, we require common knowledge of *sequential* rationality – there is common knowledge of rationality in every subgame

III. Bargaining and credible threat

[What are some important applications of subgame perfection?]

The road so far

We have covered

- Basic elements of a game
- Categorization and solution concepts
- Basic principles for solving for equilibria

Next: more specific methodologies in applications

Bargaining

A typical scenario in business

- Divide profits with partners
- Tacit collusion with competitors
- Bargaining allocates an economic surplus via proposal/acceptance rounds

Bargaining

Ultimatum game (one round)

- Two players, 1 and 2
- Player 1 proposes $(a, 1-a)$; Player 2 accepts or rejects
- If accept: payoffs $(a, 1-a)$
- If reject: payoffs $(0, 0)$

Bargaining

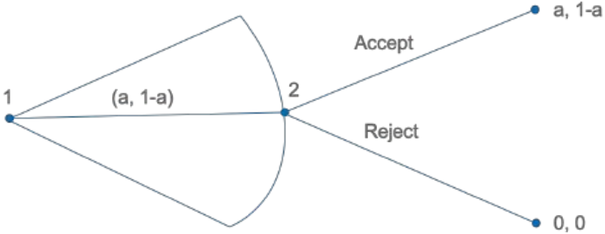


Figure: Illustration of strategy continuum – a "fan" diagram

Bargaining

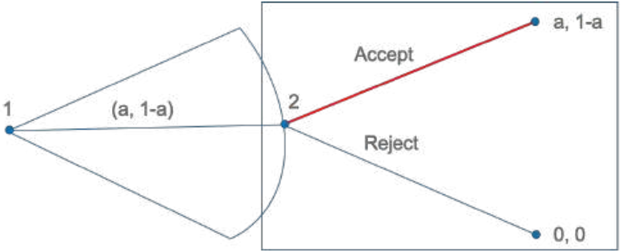


Figure: Backward induction – player 2's best response

Player 2 accepts every proposal with $a < 1$ and is indifferent between acceptance and rejection when $a = 1$.

Bargaining

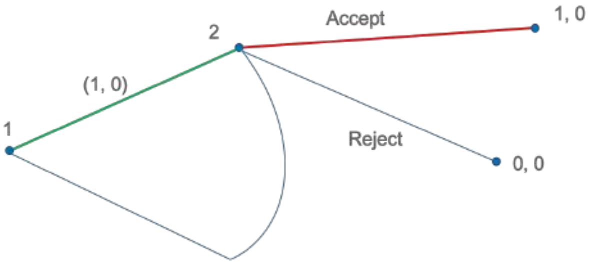


Figure: Backward induction – player 1's optimal proposal

Unique SPNE: 1 proposes (1, 0) and 2 will accept every proposal. (Why unique given that 2 has indifference somewhere?)

Bargaining

Two-round bargaining (no discount)

- Round 1: 1 proposes $(a, 1-a)$; 2 accepts/rejects. If reject,
- Round 2: 2 proposes $(1-b, b)$; 1 accepts/rejects; else $(0, 0)$.

Backward induction

- In round 2, 1 always accepts, so 2 proposes $(0, 1)$.
- Knowing this, in round 1, 2 may only accept $(0, 1)$.
- 1 is indifferent over proposals (gets 0 anyway).

SPNE: 1 makes arbitrary proposal; 2 accepts iff $(0, 1)$; in round 2, 2 proposes $(0, 1)$; 1 always accepts.

Bargaining

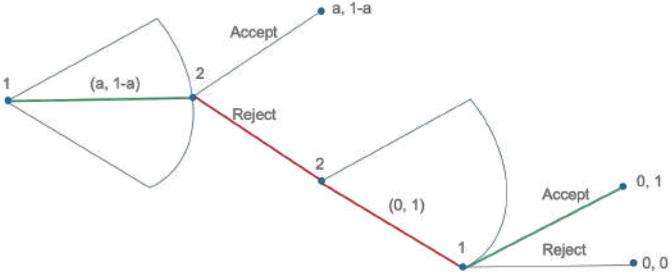


Figure: Equilibrium path in two-round bargaining

Bargaining

n-round bargaining (no discount)

- Last-round proposer keeps the whole dollar
- Any earlier offer that gives the last-round proposer < 1 will either not be proposed, or be rejected
- Many SPNE for large n, but all imply: last proposer gets 1, the other gets 0.

Bargaining

Two-round bargaining with discount $\delta \in (0, 1)$

- Game: Same sequence as before, but round-2 payoffs are discounted by δ in round 1, i,e. if $(1 - b, b)$ is proposed and accepted in round 2, the players value it at payoffs $(\delta(1 - b), \delta b)$ in round 1
- Backward induction: In round 2, 2 proposes $(0, 1)$; 1 accepts. In round 1, 2's outside option is δ (from round 2), so 2 accepts any offer giving her at least δ
- SPNE: 1 proposes $(1 - \delta, \delta)$; 2 accepts offers $\geq \delta$ and rejects others; round 2 as above.

Bargaining

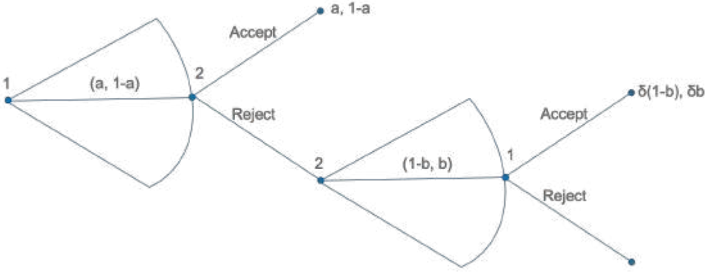


Figure: Two-round bargaining with discount

Bargaining

Discount and bargaining power

- With discounting, the last-round proposer has only *partial* bargaining power.
- If $\delta = 0$, last-round proposer has no power; agreement occurs early in practice.
- As n grows, how may equilibrium shares depend on impatience?

Bargaining

Exercise (three players, discount δ)

- Round 1: player 1 proposes; 2 decides; regardless,
- Round 2: 3 decides on 1's plan; if both accept, implement; else
- Round 3: 3 proposes; 2 decides; if reject, all get 0

Find SPNE shares as functions of δ.

Credible threat

		2	
		Confess	Deny
1	Confess	$-c, -c$	$a, -b$
	Deny	$-b, a$	$0, 0$

- A prisoner's dilemma ($a, b, c > 0,\ b > c$) played twice
- Each player's payoff is the sum from the two rounds

Credible threat

Naive plan: "Deny in both rounds; if someone deviates in round 1, retaliate."

- Backward induction: round 2 has unique NE: (Confess, Confess)
- Hence no incentive to deny in round 1
- SPNE: (Confess, Confess) in round 1; no matter what happened in round 1, (Confess, Confess) in round 2.

Credible threat

Adding a harsh option: Testify

		Player 2		
		Confess	Deny	Testify
Player 1	Confess	$\underline{-c}$, $\underline{-c}$	$\underline{a}$, -b	$\underline{-d}$, -d
	Deny	-b, $\underline{a}$	0, 0	$\underline{-d}$, -d
	Testify	-d, $\underline{-d}$	-d, $\underline{-d}$	$\underline{-d}$, $\underline{-d}$

Figure: PD with variation

Credible threat

Candidate SPNE

- Round 1 both *Deny*
- Round 2: if both denied in round 1 $\Rightarrow$ play (Confess, Confess); otherwise play (Testify, Testify)
- The “good” NE as carrot and the “bad” one as stick

Credible threat

Backward induction

- Round 2: each plays a NE contingent on history
- Round 1: deviator gains a now but gets $-d$ next round; compliant player gets $-c$ next round
- The candidate is a SPNE if and only if $-c \geq a - d \Leftrightarrow d \geq a + c$

Credible threat

Discounting the punishment

- Suppose that round-2 payoffs are discounted by $\delta \in (0, 1)$
- The condition tightens to

$$-d\delta \leq a - d\delta - \delta c \quad \Rightarrow \quad d \geq \frac{a}{\delta} + c.$$

- Impatience makes cooperation harder; as $\delta \to 0$, cooperation becomes impossible

Exercise

Two-stage game. Round 1, player 1 chooses A or B.
If A, play the following subgame G_A:

		Player 2	
		C	D
Player 1	C	1, 1	3, 0.5
	D	0.5, 3	2, 2

If B, play the following subgame G_B:

		Player 2	
		C	D
Player 1	C	1.5, 1.5	0, 0
	D	0, 0	1.5, 1.5

Characterize all SPNE.

IV. Discussion

[Are there further strategic concerns for humans in a dynamic environment?]

Checklist for finding SPNE

- Finite pure-strategy sets: draw the tree and label best responses.
- Infinite strategies (e.g., proposals): tree is illustrative; describe BR formally.
- Simultaneous subgames: solve NE of each subgame; then apply backward induction.
- Multiple NE in a subgame $\Rightarrow$ multiple SPNE overall.
- Take expectations whenever there is randomness.

Trial and error

Suppose player 1 now has two sequential proposals to make to player 2

- First, how to divide a dollar, $(a, 1 - a)$
- Second, how to divide 1 million dollars, $(b, 1000000 - b)$

How would player 1 use the first proposal to make the most out of the second?

Rationality and humanity

The dilemma in *Batman: the Dark Knight*

- Two ships with bomb planted
- Passengers of each ship can decide whether to detonate the other ship for their own survival
- If no bomb is detonated by midnight, the Clown will blow up both ships

Besides altruism, can you think of any other reason why people would not choose the (seemingly) rational action in such a scenario?

Lecture 6: Business Competition

School of Management and Economics, CUHK (Shenzhen)

I. Classical models of quantity competition

[What is the nature of competition, and what are the most typical ways of modeling it?]

Nature of competition

Competition is a way of allocating scarce resources

- For object/wealth: auction, contest, lottery
- For property/right to use: patent, exam
- Our focus: competing for customers/market share

Live examples

Figure: Real competition scenarios in business

Business competition landscape

Basic categorization

- Market structures: oligopoly, monopolistic competition, perfect competition.
- Modes: quantity (Cournot/Stackelberg), effort (Tullock), price (Bertrand).
- Initial positions: identical vs heterogeneous costs.

Cournot model

Two firms choose quantities q_1, q_2 simultaneously

- Inverse demand $P = a - Q$ with $Q = q_1 + q_2$
- Identical marginal cost $c > 0$ (no fixed cost)
- Profit of firm i: $\pi_i = (a - Q)q_i - cq_i$

Cournot model

FOC approach

- Firm 1's problem: $\max_{q_1} \ (a-(q_1+q_2))q_1 - cq_1 \Rightarrow a - q_2 - 2q_1 - c = 0$
- Best response: $BR_1(q_2) = \frac{a-c-q_2}{2}$
- Symmetrically, $BR_2(q_1) = \frac{a-c-q_1}{2}$
- Unique NE:

$$q_1^* = q_2^* = \frac{a-c}{3}$$

Cournot model

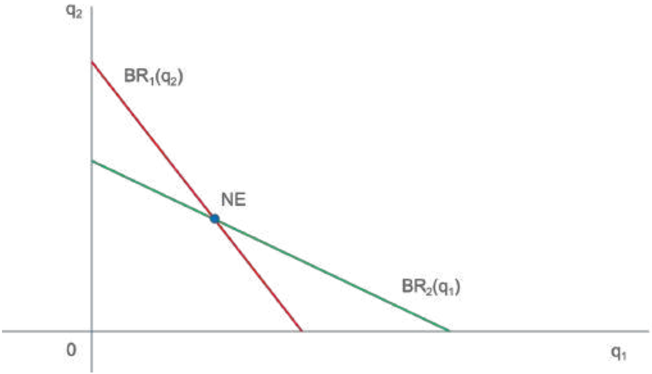


Figure: Intersection of BR

Cournot model

Shortcut by symmetry

- Firms are symmetric $\rightarrow$ BR function is identical
- Set $q_1 = q_2 = q$ in a single BR:

$$q = \frac{a - c - q}{2} \Rightarrow q = \frac{a - c}{3}.$$

- Can we set the same q earlier in the problem (before getting to BR)?

Cournot model

Variation 1: n identical firms

- Each firm i chooses q_i
- $Q = \sum_i q_i$, $P = a - Q$, $\mathrm{MC} = c$
- From $BR_i(q_{-i}) = \frac{a-c-\sum_{j\neq i} q_j}{2}$ and symmetry $q_i = q$,

$$q = \frac{a-c-(n-1)q}{2} \Rightarrow q = \frac{a-c}{n+1}$$

Cournot model

Implications on market structure

- Overall quantity: $Q = \frac{n(a-c)}{n+1}$
- Price and profits:

$$P = \frac{a+nc}{n+1}, \qquad \pi_i = (P-c)q = \frac{(a-c)^2}{(n+1)^2}.$$

- As $n \to \infty$, $P \to c$ and $\pi_i \to 0$ (perfect-competition limit)

Cournot model

Variation 2: different marginal costs ($c_1 \neq c_2$)

- BRs:

$$q_1 = \frac{a - c_1 - q_2}{2}, \qquad q_2 = \frac{a - c_2 - q_1}{2}.$$

- Solving yields unique NE

$$q_1^* = \frac{a + c_2 - 2c_1}{3}, \quad q_2^* = \frac{a + c_1 - 2c_2}{3}$$

Cournot model

Implications on market position

- Overall quantity and price

$$Q^* = \frac{2a - c_1 - c_2}{3}, \quad P^* = \frac{a + c_1 + c_2}{3}.$$

- If $c_2 > c_1$, then $q_1^* > q_2^*$ and $\pi_1 > \pi_2$

Cournot model

Differentiated products

- Demands: $P_1 = a - q_1 + bq_2$, $P_2 = a - q_2 + bq_1$
- Common MC $c > 0$
- $b > 0$: complements; $b < 0$: substitutes

Cournot model

Differentiated products (cont'd)

- Firm 1: $BR_1(q_2) = \frac{a+c^- - bq_2}{2}$ with FOC $a + bq_2 - 2q_1 - c = 0$; similarly for firm 2
- By symmetry,

$$q_1^* = q_2^* = \frac{a-c}{2-b} \quad (b < 2).$$

Cournot model

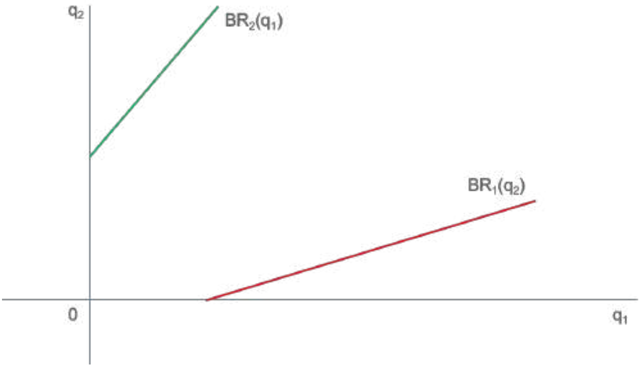


Figure: Non-existence of equilibrium when $b > 2$ (strong complement)

Stackelberg model

Sequential quantity competition

- Leader (1) chooses q_1
- Follower (2) observes q_1 then chooses q_2
- Similar market demand $p = a - Q$ and constant MC $c > 0$
- SPNE: leader chooses *quantity* while follower chooses *function of quantity*

Stackelberg model

Solution

- Follower BR: $q_2^*(q_1) = \frac{a-c-q_1}{2}$
- Leader solves

$$\max_{q_1} (a - q_1 - q_2(q_1))q_1 - cq_1 = \max_{q_1} \frac{(a - c - q_1)q_1}{2}.$$

FOC $\Rightarrow q_1^* = \frac{a-c}{2}$

- SPNE: $q_1^* = \frac{a-c}{2}$, $q_2^*(q_1) = \frac{a-c-q_1}{2}$

Stackelberg model

Comparison to Cournot ($q_1 = q_2 = \frac{a-c}{3}$)

- Higher total production, lower price
- Leader produces more than under Cournot, follower less
- Why?

Stackelberg model

First-mover advantage

- Leader knows it can "force" follower to shrink production by producing more
- $q^*(q_1)$ has a slope of $-\frac{1}{2}$ $\Rightarrow$ per unit of additional production by leader leads to half a unit of reduced production by follower
- Overall effect: higher total production

II. Contest and related models of competition

[How do some alternative models compare to quantity competition?]

Tullock contest

Effort-based competition

- n players
- Choose efforts $x_i \geq 0$ simultaneously
- Prize value $V > 0$ to a single winner (e.g. monopoly profit from owning whole market)

Tullock contest

Costly effort and payoffs

- Player i's winning probability:

$$p_i = \frac{x_i}{\sum_j x_j} \text{ (for } \sum_j x_j > 0)$$

- Cost of effort is cx_i, $c > 0$
- Expected payoff:

$$\pi_i = V \frac{x_i}{\sum_j x_j} - cx_i$$

Tullock contest

Solve the model: in-class exercise.

Hotelling, R&D, and networks

- *Hotelling location:* strategic differentiation along space/attributes.
- *R&D:* quality improvements shift demand; multi-dimensional budgets.
- *Network/attention:* contagion and viral marketing as contests for attention.

III. Price competition

[How do we analyze competition from the other typical angle?]

From quantity to price

Conclusions from quantity competition

- Producing more is relatively better
- With two firms market looks oligopolistic; as $n \to \infty$, price $\to$ MC
- First-mover advantage (Stackelberg)

Question: Do similar forces hold when firms choose prices instead of quantities?

Bertrand competition

Setting

- Two firms set prices p_1, p_2 simultaneously
- Demand $Q = a - P$ with market price $P = \min\{p_1, p_2\}$
- If $p_1 = p_2$, firms split demand
- MC $c > 0$ is identical

Bertrand competition

The game mirrors a first-price sealed-bid auction

	Bertrand competition	First-price auction
Type of game	Simultaneous	Simultaneous
Strategy	A price in real numbers	A bid in real numbers
Rule	Lower price wins	Higher bid wins
Trade-off	Lower price yields lower profit	Higher bid yields lower payoff

Figure: Bertrand vs FPA

Bertrand competition

Solution by best responses

- If rival sets $p > c$: undercut slightly to win the whole market
- If rival sets $p < c$: price above it to lose less (selling below cost hurts more)
- Only mutual best response: $p_1 = p_2 = c$
- Conclusion: unique NE is *competitive pricing* $p = c$ with zero profits

Bertrand competition

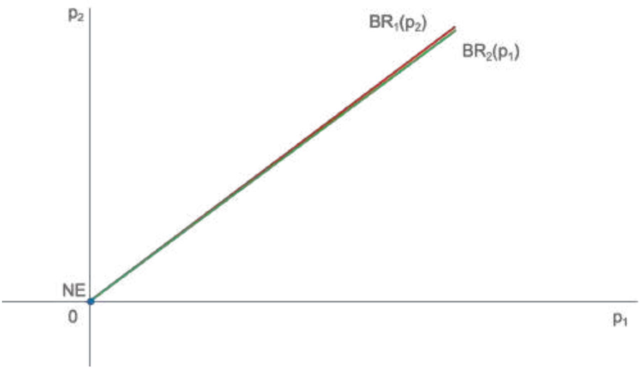


Figure: Best responses in Bertrand competition

Bertrand competition

Solution by exclusion

- In any NE: (i) no firm prices $> c$ (can be undercut; original price not optimal); (ii) no firm prices $< c$ (profitable to raise)
- The only candidate that survives is $p_1 = p_2 = c$
- Check incentives to confirm

Bertrand competition

Implications for market structure

- Adding one competitor (duopoly) already pushes price to c (perfect-competition outcome)
- In contrast with Cournot where duopoly still earns positive profits
- With one firm, monopoly pricing obtains in both models

Bertrand competition

Many firms

- If $n > 2$, equilibrium market price is still c
- A NE requires at least two firms pricing at c while the rest (weakly) price above c
- Otherwise a lone c-pricer would want to deviate (would it necessarily raise the price?)

Bertrand competition

Different costs

- Let $c_1 > c_2 > 0$
- equilibrium price $p \in [c_2, \min\{p^*, c_1\}]$ where p^* is firm 2's monopoly price
    - $p < p^*$ vs. $p = p^*$
- The high-cost firm earns zero profit in equilibrium; the low-cost firm captures the market by (slightly) undercutting

Bertrand competition

Differentiated products

- Demands $q_1 = a - p_1 + bp_2$, $q_2 = a - p_2 + bp_1$
- Substitutes if $b > 0$, complements if $b < 0$ (opposite sign convention from Cournot)
- Write profits and solve FOCs for BR functions
- Intersection gives (interior) NE

Examples

Figure: Examples supporting our model

Examples

Figure: Examples contradicting our model

Sequential price setting

Suppose leader chooses p_1

- Follower's BR: undercut if $p_1 > c$, raise price if $p_1 < c$, and choose $\geq c$ if $p_1 = c$
- SPNE: Leader chooses $p_1 \geq c$ and follower best responds as above
- Leader never makes positive profit, but follower may $\rightarrow$ second-mover advantage unlike Stackelberg

Coase conjecture

Durable-good monopoly over time

- A monopolist sells a durable good over multiple dates to long-lived buyers
- It can only set prices (not quantities)
- Consumers can wait
- This behaves like a firm competing in prices with its *future self*

Coase conjecture

Two-buyer illustration

- Two valuations $x > y \geq MC$, one unit demand each
- With no discounting, any attempt to charge $> y$ today unravels because the firm will profit by cutting price tomorrow
- Strategic buyers anticipate this and wait
- Outcome price collapses to y when trade happens

Coase conjecture

Continuum of buyers

- With many valuations and no discounting, any price $> MC$ leaves some buyers out, creating incentives to cut price later
- Anticipation of future cuts drives the only sustainable selling price to MC

IV. Midterm review

Describing a game

Normal-form essentials

- Count players and pure strategies carefully
- Matrices work only with ≤ 3 players and finite strategies
- Remember canonical games: PD, BoS, Matching Pennies, etc.
- In sequential games, count *strategies*, not actions.

Describing a game

Extensive form essentials

- Order of moves
- Decision vs terminal nodes
- Information sets (dashed lines)
- Count subgames
- Use “fan” diagrams when strategy spaces are continuous

A 3-player example

1 and 2 play Matching Pennies; outcome is observed by 2 and 3; then 2 and 3 play BoS if match, PD otherwise.

Compute numbers of information sets, subgames, and each player's pure strategies (matrices would be huge: two matrices with 16×32 blocks each).

Solution tools

Simultaneous games

- Delete strictly dominated strategies
- Derive BRs
- Use indifference for mixed NE (odd or infinite number of NEs are common)

Solution tools

Sequential games

- Start from smallest subgames
- Use backward induction or matrix for simultaneous subgames
- SPNE $\subseteq$ NE
- Write strategy profiles as complete plans

Efficiency vs equilibrium

Measure for equilibrium to be "good"

- Pareto efficiency vs strict (utilitarian) efficiency
- NE may match (BoS) or mismatch (PD, public good) efficiency
- To find strictly efficient profiles, treat society as a single player maximizing sum of payoffs

Typical applications

Try exercise questions about

- Auctions
- Tragedy of the commons
- Bargaining
- Wwice-repeated PD
- Quantity/effort competition
- Price competition

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