CS1 Week 4
Empty

Time response, Stability

Recap Last week

Quiz 1

Which properties do linear systems need to fulfil?

A. Linearity

B. Superposition

C. Homogeneity

D. Additivity

Quiz 2

Classify the following system

$$x_1(t) = \sin(t) - 5$$
$$x_2(t) = \cos(t)$$
$$y(t) = u(t)\left(x_1^2(t) + t \cdot x_2^2(t)\right)$$

A. Linear

B. Static

C. Time invariant

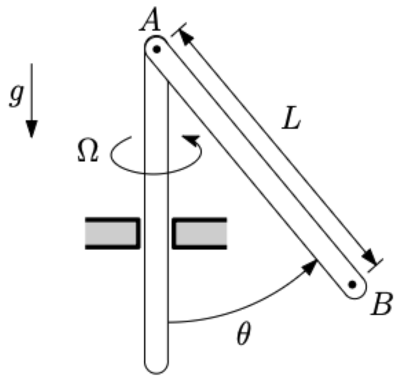
D. Causal

Quiz 3 (Spring 2019 Q4,5,6,7)

Box 1: Questions 4, 5, 6, 7

You are given the mechanical system depicted below with the following equation of motion:

$$\frac{1}{3}\ddot{\theta}(t) - \frac{1}{3}\Omega^2 \sin(\theta(t))\cos(\theta(t)) + \frac{g}{2L}\sin(\theta(t)) = 0.$$



Question 4 *Choose the correct answer. (1 Point)*

Which of the following is the correct state representation of the above system?

$\boxed{\text{A}}$ $\dot{x}(t) = \begin{bmatrix} x_2(t) \\ \frac{1}{3}\Omega^2 \sin x_1(t) \cos x_1(t) + \frac{3g}{2L}\sin x_1(t) \end{bmatrix}$

$\boxed{\text{B}}$ $\dot{x}(t) = \begin{bmatrix} x_2(t) \\ \frac{1}{3}\Omega^2 \sin x_1(t) \cos x_1(t) - \frac{g}{2L}\sin x_1(t) \end{bmatrix}$

$\boxed{\text{C}}$ $\dot{x}(t) = \begin{bmatrix} x_2(t) \\ \Omega^2 \sin x_1(t) \cos x_1(t) - \frac{3g}{2L}\sin x_1(t) \end{bmatrix}$

$\boxed{\text{D}}$ $\dot{x}(t) = \begin{bmatrix} x_1(t) \\ \Omega^2 \sin x_1(t) \cos x_1(t) - \frac{3g}{2L}\sin x_1(t) \end{bmatrix}$

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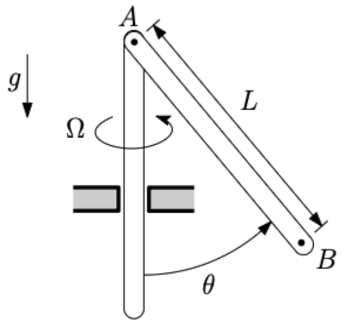

$$\dot{x}(t) = \begin{bmatrix} x_2(t) \\ \Omega^2 \sin x_1(t) \cos x_1(t) - \frac{3g}{2L}\sin x_1(t) \end{bmatrix}$$

Question 5 *Mark all correct statements. (2 Points)*
Which of the following points x_e are equilibrium points of the system?

A $x_e = \begin{bmatrix} \arccos\left(\frac{3g}{2\Omega^2 L}\right) + 3\pi \\ 0 \end{bmatrix}$.

B $x_e = \begin{bmatrix} c \\ 0 \end{bmatrix}, \quad \forall c \in \mathbb{R}$.

C $x_e = \begin{bmatrix} \arccos\left(\frac{3\Omega^2 g}{2L}\right) \\ 1 \end{bmatrix}$.

D $x_e = \begin{bmatrix} \arccos\left(\frac{3g}{2\Omega^2 L}\right) + 2\pi \\ 0 \end{bmatrix}$.

E $x_e = \begin{bmatrix} 0 \\ 3\pi \end{bmatrix}$.

F $x_e = \begin{bmatrix} 3\pi \\ 0 \end{bmatrix}$.

G $x_e = \begin{bmatrix} 0 \\ \frac{\pi}{2} \end{bmatrix}$.

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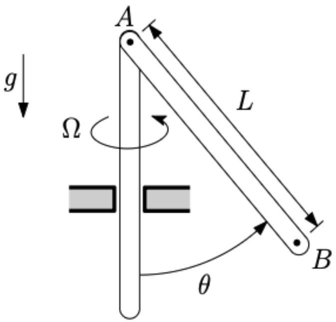

$$\dot{x}(t) = \begin{bmatrix} x_2(t) \\ \Omega^2 \sin x_1(t) \cos x_1(t) - \frac{3g}{2L} \sin x_1(t) \end{bmatrix}$$

Question 6 *Choose the correct answer. (1 Point)*

Consider the equilibrium point $x_e = \begin{bmatrix} -5\pi \\ 0 \end{bmatrix}$. Linearize the system. Which matrix A describes the linearized dynamics?

- [A] $A = \begin{bmatrix} 0 & 1 \\ \Omega^2 + \frac{3g}{2L} & 0 \end{bmatrix}$.
- [B] $A = \begin{bmatrix} 0 & 1 \\ -\Omega^2 + \frac{3g}{2L} & 0 \end{bmatrix}$.
- [C] Since the given equilibrium is not stable, a matrix A does not exist.
- [D] $A = \begin{bmatrix} 1 & 0 \\ \Omega^2 - \frac{3g}{2L} & 0 \end{bmatrix}$.
- [E] $A = \begin{bmatrix} 0 & 1 \\ \Omega^2 - \frac{3g}{2L} & 0 \end{bmatrix}$.

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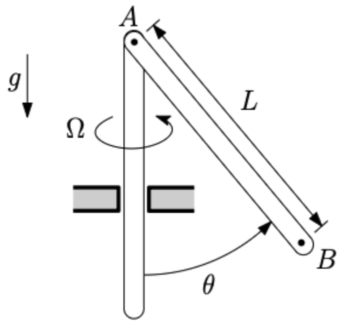


$$\dot{x}(t) = \begin{bmatrix} x_2(t) \\ \Omega^2 \sin x_1(t) \cos x_1(t) - \frac{3g}{2L} \sin x_1(t) \end{bmatrix}$$

Question 7 *Mark all correct statements. (2 Points)*

Which of the following statements about the system are true?

- [A] The system is dynamic.
- [B] The system is time-varying.
- [C] The dimension of the system is 2.
- [D] The dimension of the system can be 1.

Course Schedule

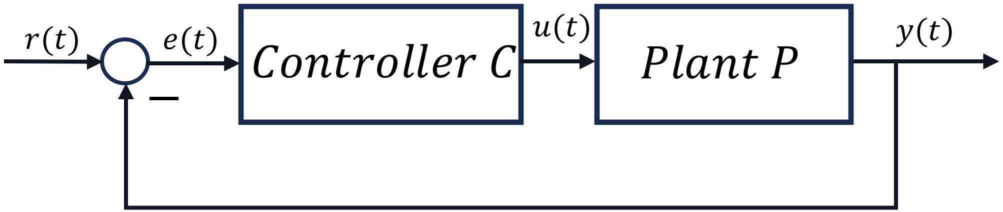
	Subject	Week
Modeling	Introduction, Control Architectures, Motivation	1
	Modeling, Model examples	2
	System properties, Linearization	3
Analysis	Analysis: Time response, Stability	4
	Transfer functions 1: Definition and properties	5
	Transfer functions 2: Poles and Zeros	6
	Proportional feedback control, Root Locus	7
	Time-Domain specifications, PID control, Computer implementation	8
	Frequency response, Bode plots	9
	The Nyquist condition, Time delays	10
Synthesis	Frequency-domain Specifications, Dynamic Compensation, Loop Shaping	11
	Time delays, Successive loop closure, Nonlinearities	12
	Describing functions	13
	Intro to Uncertainty and Robustness	14

Today

1. Time response
2. Stability

1. Time response

Motivation

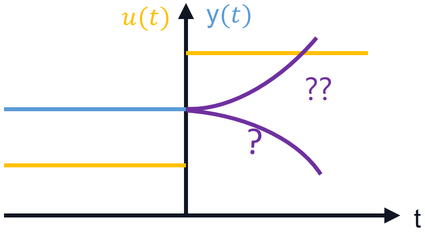


system modeling → **Plant P** → **Linearization** →

state-space representation

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

"what will $y(t)$ look like?"



$\dot{x}(t) = \cdots$

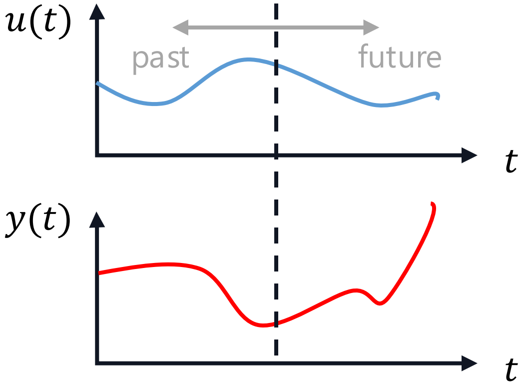
solve ODE so we get an expression for $x(t)$ and therefore an solution for $y(t)$

Analysis

Controller C ← **Synthesis**

- Recall modeling from week 2
- Recall linearization from week 3
- Today: Analysis
- End goal is a robust system

Time response



$u = ua + ub$ → $\dot{x}(t) = Ax(t) + Bu(t)$, $y(t) = Cx(t) + Du(t)$ → $y = ya + yb$

- Since we deal with linearized systems, input & output can be split into two parts

→ To know the behavior of a system for all time t, we split input into

$u \ = \ upast \ + \ ufut_{ure}$, this is called the *time response* of a system

- Additionally, since we deal with real systems, causality holds (i.e. current output only depends on past and current inputs)

→ effects of u_{past} can be summarized by $x(t^*)$ (state at any given time t^*)

- Usually time invariance holds → reference time doesn't matter, we pick $t^* = 0$

→ this leads to Initial Condition response

Initial and Forced Response

- As said, since the system is linear, we take advantage of linearity and consider two separate cases:

Initial-Conditions response:
No external inputs

$$y_{\mathrm{IC}} \begin{cases} u_{IC}(t) = 0, t \geq 0, x_{IC}(0) = x_0 \end{cases}$$

Forced response:
due to external inputs or disturbances

$$y_F \begin{cases} u_F(t) = u(t), t \geq 0, x_F(0) = 0 \end{cases}$$

After solving each case separately, we just add y_{IC} and y_F to get the complete output. This separation allows us to analyze the effects of non-zero initial conditions and non-zero inputs separately.

Initial and Forced Response Solution

1. Initial Condition

$$x_{IC}(0) = x_0, u_{IC}(t) = 0, t \geq 0$$

Solve $\dot{x}(t) = Ax$

Solution: $x_{IC}(t) = e^{At}x_0$

$$y_{IC}(t) = Ce^{At}x_0$$

2. Forced Response

$$u_F(t) = u(t),\ x_F(0) = 0$$

Solve $\dot{x}(t) = Ax + Bu$

Solution: $x_F(t) = \int_0^t e^{A(t-\tau)}B\,u(\tau)\,d\tau$

$$y_F(t) = C\int_0^t e^{A(t-\tau)}B\,u(\tau)\,d\tau\ +\ Du(\mathrm{t})$$

today

$$\boxed{y = y_{IC} + y_F = \underbrace{\boxed{Ce^{At}x_0}}_{\text{initial response}} + \underbrace{C\int_0^t e^{A(t-\tau)}B\,u(\tau)d\tau}_{\text{forced response}} + \underbrace{Du}_{\text{feedthrough}}}$$

Time Response of an LTI system

Matrix Exponential

If we take a closer look, we see that some terms contain the matrix exponential e^{At}.

But how do we compute it? Throwback to Linear Algebra II…

The matrix exponential can be defined through a Taylor-series:

$$e^{At} = \sum_{n=0}^{\infty} \frac{1}{n!}(At)^n = I + At + \frac{1}{2}(At)^2 + \cdots + \frac{1}{n}(At)^n$$

For some matrices we can avoid infinitely many calculations and simplify calculations:

⟶ Diagonal: $\exp\left(\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} t\right) = \begin{bmatrix} \exp(\lambda_1 \mathrm{t}) & 0 \\ 0 & \exp(\lambda_2 \mathrm{t}) \end{bmatrix}$

⟶ Jordan Form: $\exp\left(\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} t\right) = \begin{bmatrix} \exp(\lambda \mathrm{t}) & t\exp(\lambda \mathrm{t}) \\ 0 & \exp(\lambda \mathrm{t}) \end{bmatrix}$

Where λ_i are the eigenvalues of the respective matrix

Coordinate Transformation (LinAlg Recap)

To facilitate calculations, we can therefore do a coordinate transformation, $x = T\tilde{x}$ such that is e^{At} easier to compute.

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + D\,u(t) \end{cases} \xrightarrow{x = T\tilde{x}} \begin{cases} T\dot{\tilde{x}}(t) = AT\tilde{x}(t) + Bu(t) \\ y(t) = CT\tilde{x}(t) + Du(t) \end{cases}$$

$$\begin{cases} \dot{\tilde{x}}(t) = (T^{-1}AT)\tilde{x}(t) + (T^{-1}B)u(t) \\ y(t) = CT\tilde{x}(t) + Du(t) \end{cases}$$

$$\begin{cases} \dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{B}u(t) \\ y(t) = \tilde{C}\tilde{x}(t) + \tilde{D}u(t) \end{cases}$$

For a matrix $A \in R^{n\times n}$ with n eigenvalues $\lambda_1, \dots, \lambda_n$ and n *<u>linearly independent</u> eigenvectors* $v_1, \dots, vn$ one can do a coordinate transformation such that $\tilde{A} = T^{-1}AT = diag(\lambda_1, \dots, \lambda_n)$ where $\tilde{A}$ is a diagonal matrix with the eigenvalues $\lambda_1, \dots, \lambda_n$ on the diagonal and T a transformation matrix containing the eigenvectors as $v_1, \dots, vn$ columns.

Note: the time response remains *<u>unchanged</u>*. Through the transformation, we simple use a different realization of the system, i.e. a different state vector.

I.C Response: Real Eigenvalues

Let us know take a closer look at systems where A is diagonal. More specific we will look at the initial condition response, i.e. $u(t) = 0$

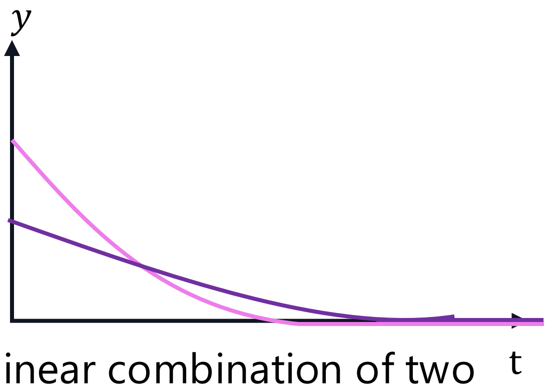
$\longrightarrow$ For a diagonal, real matrix: $A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \lambda_i \in R$

$$y(t) = Ce^{At}x_0$$

where we can write out all terms and simplify for A being diagonal.

$$y(t) = [c_1 \; c_2] \begin{bmatrix} \exp(\lambda_1 t) & 0 \\ 0 & \exp(\lambda_2 t) \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

$$y(t) = c_1 e^{\lambda_1 t} x_1(0) + c_2 e^{\lambda_2 t} x_2(0)$$



So for diagonal, real matrices the initial condition response is the linear combination of two exponentials.

I.C Response: Complex Eigenvalues

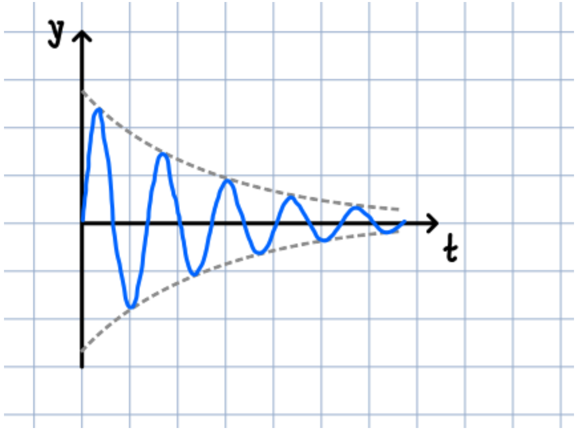
⟶ For a diagonal, complex matrix: $A = \begin{bmatrix} \sigma + j\omega & 0 \\ 0 & \sigma - j\omega \end{bmatrix}$

$$y(t) = CeAtx_0$$

where we can write out all terms and simplify for A being diagonal.

$$
\begin{aligned}
y &= c_1 e^{\sigma t} e^{j\omega t} x_1(0) + c_2 e^{\sigma t} e^{-j\omega t} x_2(0) \\
&= e^{\sigma t}\left(c_1 e^{j\omega t} x_1(0) + c_2 e^{-j\omega t} x_2(0)\right) \\
&= e^{\sigma t}(\alpha_1 \sin(\omega t) + \alpha_2 \cos(\omega t)) \\
&= \alpha e^{\sigma t} \sin(\omega t + \phi)
\end{aligned}
$$

→ if signal decays or not depends on $Re(\lambda)$!
→ complex poles generate new frequencies
→ oscillations in output without any in input



I.C Response: Repeated Eigenvalues

- If repeated eigenvalues appear (where algebraic and geometric multiplicity do not match), we cannot diagonalize the matrix
- We can still bring into Jordan form
- For a 2nd order system, the initial condition response would be:

$$y(t) = Ce^{At}x_0 = C \exp\left(\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} t\right) x_0 = [c_1\ c_2] \begin{bmatrix} \exp(\lambda t) & t\exp(\lambda t) \\ 0 & \exp(\lambda t) \end{bmatrix} x_0 = c_1\ e^{\lambda t} x_{0,1} + c_2 t e^{\lambda t} x_{0,1}$$

- In general: the initial condition response is a linear combination of terms of the form $\exp(\lambda t)$ and $t^m \exp(\lambda t)$
- Often, repeated eigenvalues occur at $\lambda = 0$:

$$y(t) = c_1 x_{1,0} + c_1 t x_{1,0}$$

Time Response Overview

- Any matrix A can be transformed into a diagonal or Jordan matrix

The response of a system will always be a linear combination of terms in the following form:

- Real eigenvalues: $e^{\lambda_\mathrm{i}\mathrm{t}}$
- Complex conjugate eigenvalues: $e^{\sigma t}\sin(\omega \mathrm{t} + \phi)$
- Repeated real eigenvalues: $te^{\lambda_\mathrm{i}\mathrm{t}}$
- The input and its derivatives: $u(t),\ \dot{u}(t), \ldots$

Important takeaway:

The stability of a system can be determined by the real part of the eigenvalues of A

Quiz

$$\delta(x-a) := \begin{cases} \infty & x = a \\ 0 & x \neq a \end{cases} \quad a \in [0, \infty) \quad \delta \notin \mathcal{H}$$

$$\int_0^\infty \delta(x-a)\,dx = 1 \qquad \int_0^\infty g(x)\delta(x-a)\,dx = g(a)$$

What is the time response of a **first order system** with $u(t) = \delta(t), d = 0, x_0 \neq 0$?

$$\boxed{y = y_{IC} + y_F = Ce^{At}x_0 + C\int_0^t e^{A(t-\tau)}B\,u(\tau)d\tau + Du}$$

A. $y(t) = Ce^{at}x_0 - \frac{cb}{a}(1 - e^{-at})$

B. $y(t) = Ce^{at}(x_0 + b)$

C. You can't apply impulse as input

D. quantum state detected

2. Stability

Stability

We observed that the time response is linked to *exponential terms*.

$$y(t) = c_1 e^{\lambda_1 t} x_1(0) + c_2 e^{\lambda_2 t} x_2(0)$$
$$y(t) = \alpha e^{\sigma t} \sin(\omega t + \phi)$$

The growth of these terms is dictated by the real part of the eigenvalues of A. We can see that if the eigenvalues λ have a positive real part, the output will grow exponentially over time, i.e. become unstable. ($y \longrightarrow \infty$)

But what does stability really mean? There are a few ways to classify stability…

Stability Conditions

A linearized, diagonalized system with A, B, C, D matrices is called

- **Lyapunov stable** if $Re(\lambda_i) \leq 0 \ \forall i$
- **Asymptotically** stable if $Re(\lambda_i) < 0 \ \forall i$
- **Unstable** if $\exists \ Re(\lambda_i) > 0 \ \forall i$

A linearized system with non-diagonizable A matrix is called

- **Lyapunov stable** if $Re(\lambda_i) \leq 0 \ \forall i$ and there are no repeated eigenvalues with $Re(\lambda_i) = 0$

For *minimal LTI systems*:

- ***Asymptotic stability = BIBO stability***

Bounded Input Bounded Output **(BIBO) Stability**: for every bounded input, the output will remain bounded

Old Exam Question (Summer 2018)

Question 7 *Choose the correct answer. (1 Point)*

Consider the two systems with output signal $y(t)$ and input signal $u(t)$, described below

1. $y(t) = \sin(t)u(t)$
2. $y(t) = \int_0^t \sin(\tau)u(\tau)d\tau$

Which system is BIBO stable?

A. None

A. System 2

B. Both

D. System 1

Questions?

Feedback?

Too fast? Too slow? Less theory, more exercises?

I would appreciate your feedback. Please let me know.

https://docs.google.com/forms/d/e/1FAIpQLSdHl0kjWo63aNzDkAV0cnmQadCAj5L0D7v7aSh0BK7BBdEgpA/viewform?usp=header