CS1 Week 5

Transfer functions

Attention

- ETH no longer provides web hosting.
- All my exercise class materials have been moved to GitHub Pages.
- You can now find them under the following link:

https://kissanv.github.io

Or you can scan the QR-Code:

Recap Last week

Quiz 1

Into which parts do we split the solution of an ODE to obtain the time response?

A. Dynamic & Steady-state response

B. Natural & Forced response

C. Frequency & Time response

C. We don't split the solution

Initial and Forced Response Solution

1. Initial Condition

$$x_{IC}(0) = x_0, u_{IC}(t) = 0, t \geq 0$$

Solve $\dot{x}(t) = Ax$

Solution: $x_{IC}(t) = e^{At}x_0$

$$y_{IC}(t) = Ce^{At}x_0$$

2. Forced Response

$$u_F(t) = u(t),\ x_F(0) = 0$$

Solve $\dot{x}(t) = Ax + Bu$

Solution: $x_F(t) = \int_0^t e^{A(t-\tau)}B\,u(\tau)\,d\tau$

$$y_F(t) = C\int_0^t e^{A(t-\tau)}B\,u(\tau)\,d\tau + Du(\mathrm{t})$$

$$\boxed{y = y_{IC} + y_F = \underbrace{Ce^{At}x_0}_{\text{initial response}} + \underbrace{C\int_0^t e^{A(t-\tau)}B\,u(\tau)d\tau}_{\text{forced response}} + \underbrace{Du}_{\text{feedthrough}}}$$

Time Response of an LTI system

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)} B\, u(\tau) d\tau$$

$$\int \tau e^{-\tau}\, d\tau = -\tau e^{-\tau} - e^{-\tau}$$

Quiz 2

Question 6 *Choose the correct answer. (1 Point)*

Consider a system with the following dynamics,

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \ .$$

If $u(t) = e^{-t}$, $t \geq 0$, and $x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, find $x(t)$ for $t \geq 0$.

$\boxed{\text{A}}$ $x(t) = \begin{bmatrix} -1 + t + e^{-t} \\ 1 - e^{-t} \end{bmatrix}$

$\boxed{\text{B}}$ $x(t) = \begin{bmatrix} 1 + t + e^{-t} \\ -1 + e^{-t} \end{bmatrix}$

$\boxed{\text{C}}$ $x(t) = \begin{bmatrix} -1 + e^{-t} \\ 1 + t - e^{-t} \end{bmatrix}$

$\boxed{\text{D}}$ $x(t) = \begin{bmatrix} -1 - e^{-t} \\ -1 + t + e^{-t} \end{bmatrix}$

Quiz 3

How is the stability of a system determined?

A. Imaginary part of eigenvalues of A

B. Real part of eigenvalues of A

C. Multiplicity of eigenvalues of A

D. Real Part of eigenvalues of B

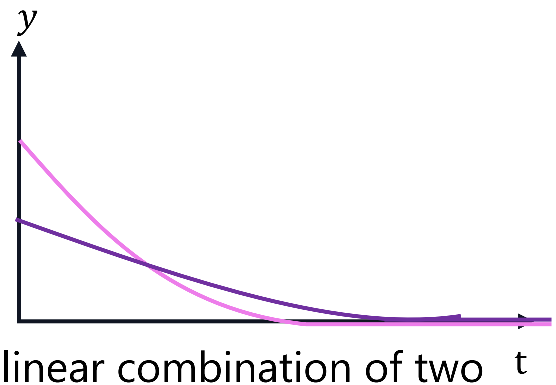
I.C Response: Real Eigenvalues

Let us know take a closer look at systems where A is diagonal. More specific we will look at the initial condition response, i.e. $u(t) = 0$

$\longrightarrow$ For a diagonal, real matrix: $A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \lambda_i \in R$

$$y(t) = Ce^{At}x_0$$

where we can write out all terms and simplify for A being diagonal.



$$y(t) = [c_1\ c_2] \begin{bmatrix} \exp(\lambda_1 t) & 0 \\ 0 & \exp(\lambda_2 t) \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

$$y(t) = c_1 e^{\lambda_1 t} x_1(0) + c_2 e^{\lambda_2 t} x_2(0)$$

So for diagonal, real matrices the initial condition response is the linear combination of two exponentials.

Quiz 4

What does Asymptotic stability mean?

A. All eigenvalues have strictly negative real part

B. All eigenvalues are purely imaginary

C. All solutions converge to 0 as $t \rightarrow \infty$

D. every bounded inputs produces bounded output

Quiz 5

What is BIBO stability?

A. Binary Inputs, Boundless Outputs

B. Bounded Inputs, Bounded Outputs

C. Bounded Inputs, Boundless Outputs

D. Big Inputs, Bigger Outputs

Stability Conditions

A linearized, diagonalized system with A, B, C, D matrices is called

- **Lyapunov stable** if $Re(\lambda_i) \leq 0 \; \forall i$
- **Asymptotically** stable if $Re(\lambda_i) < 0 \; \forall i$
- **Unstable** if $\exists \; Re(\lambda_i) > 0 \; \forall i$

A linearized system with non-diagonizable A matrix is called

- **Lyapunov stable** if $Re(\lambda_i) \leq 0 \; \forall i$ and there are no repeated eigenvalues with $Re(\lambda_i) = 0$

For minimal LTI systems

- ***Asymptotic stability = BIBO stability***

Bounded Input Bounded Output (BIBO) Stability: for every bounded inout, the output will remain bounded

Course Schedule

	Subject	Week
Modeling	Introduction, Control Architectures, Motivation	1
	Modeling, Model examples	2
	System properties, Linearization	3
Analysis	Analysis: Time response, Stability	4
	Transfer functions 1: Definition and properties	5
	Transfer functions 2: Poles and Zeros	6
	Proportional feedback control, Root Locus	7
	Time-Domain specifications, PID control, Computer implementation	8
	Frequency response, Bode plots	9
	The Nyquist condition, Time delays	10
Synthesis	Frequency-domain Specifications, Dynamic Compensation, Loop Shaping	11
	Time delays, Successive loop closure, Nonlinearities	12
	Describing functions	13
	Intro to Uncertainty and Robustness	14

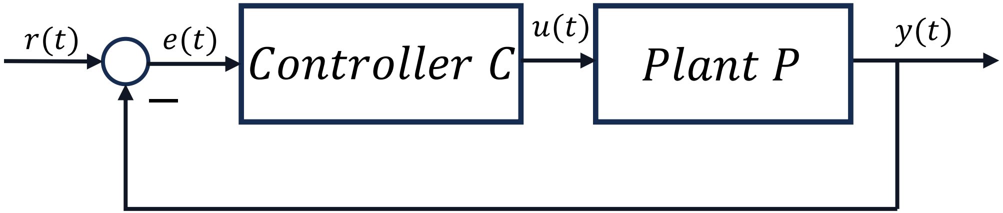
Today

1. Intro and Definition of Transfer Functions
2. Forms of TF
3. Laplace Transform

1. Intro and Definition

of Transfer Functions

Motivation



system modeling → **Plant P** → **Linearization** → state-space representation

Ferrari 812

state-space representation

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

Analysis

With information about our system

Synthesis → **Controller C**

Time response:

$$y = y_{IC} + y_F = \underbrace{Ce^{At}x_0}_{\text{Last week}} + \underbrace{C\int_0^t e^{A(t-\tau)}B\,u(\tau)d\tau + Du(t)}_{\text{This Week}}$$

Last week: We analyzed the "natural dynamics of the system" by looking at the I.C. response.
This week: We will analyze the forced response of the system. How do we analyze our system if there is an input $u(t)$?

Forced Response

The forced response is given by the convolution integral:

$$\boxed{y_F = C\int_0^t e^{A(t-\tau)}B\,u(\tau)d\tau + Du(t)}$$

- This is harder to interpret, and the integral is difficult to compute.

Thought Process

- Since we are working with linear systems, we can decompose any input into smaller inputs → u can be written as $u = u_1 + \cdots + u_n$
- We can then apply $u_1, \ldots, u_n$ separately to the system and sum all outputs $y = y_1 + \cdots + y_n$
- We know that any input $u(t)$ can be expressed as an infinite sum of complex exponentials e^{st} → See Fourier Series in Analysis III
- Idea: Compute the forced response to some general $u(t) = e^{st}$. Later we can easily compute the output to any input, since it will be a linear combination of e^{st} terms.

Forced Response: Derivation (Optional)

$$y(t) = Ce^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

with $u(t) = e^{st}$:

$$y(t) = Ce^{At}x_0 + C\int_0^t e^{A(t-\tau)}Be^{s\tau}d\tau + De^{st}$$

rearrange:

$$y(t) = Ce^{At}x_0 + Ce^{At}\int_0^t e^{(sI-A)\tau}B\ d\tau + De^{st}$$

if $(sI - A)$ is invertible:

$$y(t) = Ce^{At}x_0 + Ce^{At}\left[(sI-A)^{-1}e^{(sI-A)\tau}B\right]_0^t + De^{st}$$

rearrange:

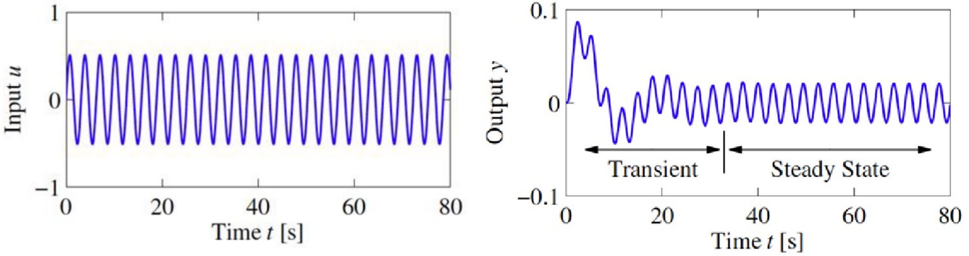
$$y(t) = Ce^{At}x_0 + Ce^{At}\left((sI-A)^{-1}\left(e^{(sI-A)t} - I\right)B\right) + De^{st}$$

and finally:

$$y(t) = \underbrace{Ce^{At}\left(x_0 - (sI-A)^{-1}B\right)}_{\text{transient response}} + \underbrace{\boxed{\left(C(sI-A)^{-1}B + D\right)e^{st}}}_{\text{steady-state response } y_{ss}}$$

→ 0 if asy stable

Transfer function $G(s)$



Rearranging:

$$y(t) = \underbrace{Ce^{At}(x_0 - (sI - A)^{-1}B)}_{\text{transient response}} + \underbrace{(C(sI - A)^{-1}B + D)e^{st}}_{\text{steady-state response } y_{ss}}$$

$$y_{ss} = (C(sI - A)^{-1}B + D)e^{st} = G(s)\, e^{st}$$

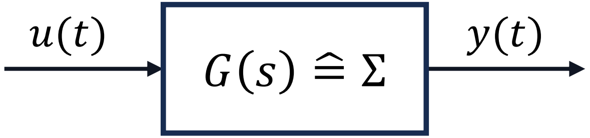
Recall more LinAlg for 2x2 Matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$M^{-1} = \frac{adj(M)}{det(M)} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\boxed{G(s) = C\frac{adj(sI - A)}{det(sI - A)}B + D}$$

Transfer Function $G(s)$

$$\boxed{G(s) = C\frac{adj(sI - A)}{det(sI - A)}B + D}$$



- You can think of transfer function $G(s)$ as the Σ in block diagrams
- The denominator of $G(s)$ is the characteristic polynomial of the matrix A
- In general, the transfer function is a rational function of the form:

$$G(s) = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0} + D$$

- **Poles** are the roots of the *Denominator* of $G(s)$
- **Zeros** are the roots of the *Nominator* of $G(s)$

Old Exam Problem

$$G(s) = C\frac{adj(sI - A)}{\det(sI - A)}B + D$$

Q16 **(1 Points)**

Given:

$$A = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}, B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 \end{bmatrix}, D = -1.$$

Which of the following transfer functions $G(s)$ is equivalent to the given state space system. Mark the correct answer.

Control Systems 1 Exam Fall 2022

A. $G(s) = -\frac{s^2-3s+2}{s^2-4s+1}$

C. $G(s) = \frac{s^2-3s+2}{s^2-4s+1}$

B. $G(s) = -\frac{(s+2)(s-1)}{s^2-4s+1}$

D. $G(s) = -\frac{s^2+3s+2}{s^2-4s+1}$

Another One

$$G(s) = C\frac{adj(sI-A)}{\det(sI-A)}B + D$$

What is the transfer function of the following system (assuming $x_0 = 0$)

$$\dot{x}(t) = \begin{bmatrix} -4 & -1 \\ -1 & -4 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 3 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 3 & 1 \end{bmatrix} x(t)$$

A. $G(s) = \frac{6s+15}{(s+4)(s+3)}$

C. $G(s) = \frac{s+4}{s(s+1)}$

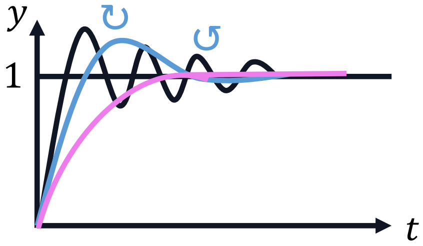
B. $G(s) = \frac{6s+14}{s^2+8s+15}$

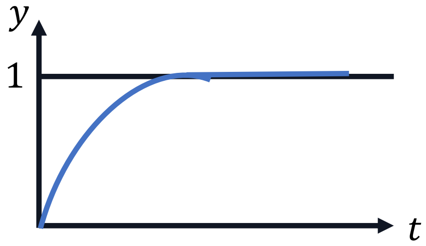
D. $G(s) = \frac{s+4}{s^2+8s+15}$

Order of a system

From a given output plot, how can we determine the order of a system?

→We can't but there are some points to make





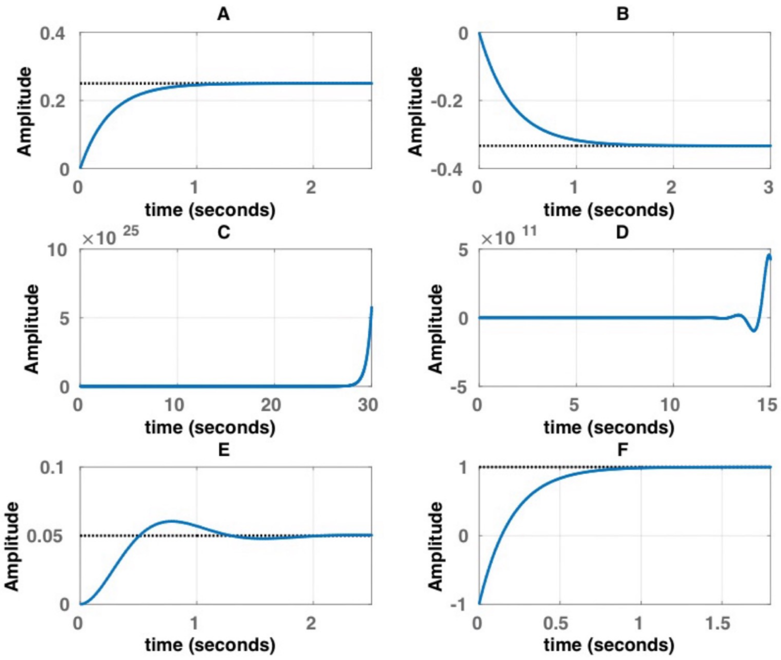
- In this plot we see oscillations in the output even though there weren't any in the input, this comes from complex EWs
  - always in pairs (complex conjugate)
  - min 2nd order

order ≙ number of EWs
oscillations → not 1st order

change in curvature → not 1st order

Quiz

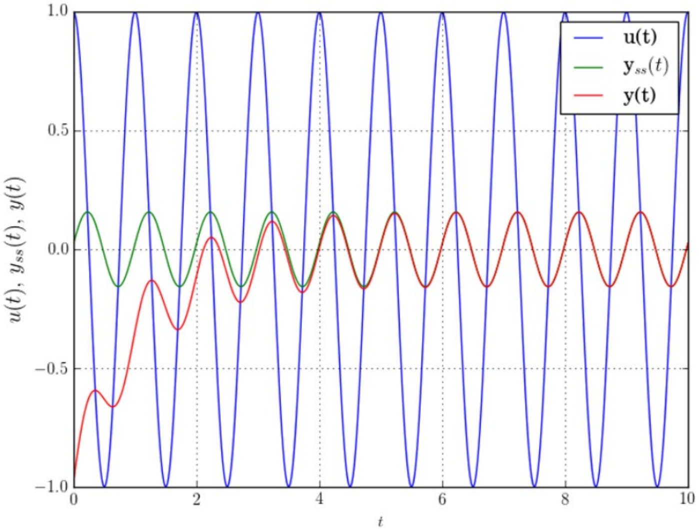
Question 11 *Mark all correct statements. (2 Points)* In the Figure above variuos time responses are shown. However, only some of them correspond to a second order system; which ones are those?



- Question isn't clear: mark all systems that have to be at least 2nd order.

Forms of TF

Specific Inputs and TFs



Sinusoidal Input: $u(t) = \cos(\omega t) = \dfrac{e^{j\omega t} + e^{-j\omega t}}{2}$

output: $y(t) = M\cos(\omega t + \phi)$

with $M = |G(j\omega)|, \quad \phi = \angle G(j\omega)$

Integrator: $u(t) \longrightarrow \boxed{\int} \longrightarrow y(t)$

For an input $u(t) = e^{st}$, the output will be $y(t) = \frac{1}{s}e^{st}$

Differentiator: $u(t) \longrightarrow \boxed{\frac{d}{dt}} \longrightarrow y(t) = \frac{du(t)}{dt}$

For an input $u(t) = e^{st}$, the output will be $y(t) = se^{st}$

$$\boxed{G(s) = \frac{1}{s}}$$

$$\boxed{G(s) = s}$$

transfer functions

minimal realization: (A, B, C, D) matrices have minimal dimension (i.e. equal to the degree of the denominator of $G(s)$)

Controllable Canonical Form

What if we now want to go from Transfer Function to a state-space model?

- Recall that there are many different state-space models for the same system! → state-space model is **not** unique
- Generally, we are interested in the *minimal realization* of a system
- For a general TF $G(s) = \frac{b_{n-1}s^{n-1}+b_{n-2}s^{n-2}+\cdots+b_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_0} + d$, one minimal realization is given by the ***Controllable Canonical Form***

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & & & \ddots & & 1 \\ -a_0 & -a_1 & \dots & & & -a_{n-1} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} b_0 & b_1 & \dots & b_{n-1} \end{bmatrix}$$

$$D = [d]$$

minimal realization: (A, B, C, D) matrices have minimal dimension (i.e. equal to the degree of the denominator of $G(s)$)

Diagonal realization

Another way to go from transfer function to state-space model is the **diagonal realization**

- If the transfer function is written as a partial fraction expansion of the form

$G(s) = \frac{p_1}{s-\lambda_1} + \frac{p_2}{s-\lambda_2} + \cdots + \frac{p_n}{s-\lambda_n} + d$, one minimal realization is given by:

$$A = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}, \qquad B = \begin{bmatrix} \sqrt{p_1} \\ \vdots \\ \sqrt{p_n} \end{bmatrix}$$

$$C = \begin{bmatrix} \sqrt{p_1} & \cdots & \sqrt{p_n} \end{bmatrix}, \qquad D = d.$$

Quiz (Old Exam HS22)

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cccccc|c} 0 & 1 & 0 & \cdots & 0 & & 0 \\ 0 & 0 & 1 & \cdots & 0 & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & & 0 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} & & 1 \\ \hline b_0 & b_1 & b_2 & \cdots & \cdots & & d \end{array}\right]$$

Given:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -2 & 0 & -4 & -1 & -6 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} \frac{1}{2} & -2 & 0 & -\frac{1}{3} & 0 \end{bmatrix}, D = 0.$$

$$\boxed{G(s) = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0} + d}$$

Which of the following transfer functions $G(s)$ is equivalent to the given state space system. Mark the correct answer.

A. $G(s) = \frac{\frac{1}{2}s^3 - 2s + \frac{1}{3}}{s^5 + 2s^4 + 4s^3 + 1.5s^2 + 6}$

C. $G(s) = \frac{-\frac{1}{3}s^3 - 2s + \frac{1}{2}}{s \cdot (s^4 + 6s^3 + s^2 + 4s + 2)}$

B. $G(s) = \frac{-\frac{1}{3}s^3 - 2s + \frac{1}{2}}{s^5 + 6s^4 + s^3 + 4s^2 + 2}$

D. $G(s) = \frac{s \cdot \left(\frac{1}{2}s^3 - 2s + \frac{1}{3}\right)}{s^5 + 2s^4 + 4s^3 + 1.5s^2 + 6}$

Laplace Transform

Laplace Transform

- Time response: $y(t) = Ce^{At}x_0 + C\int_0^t e^{A(t-\tau)}B\,u(\tau)d\tau + Du(t)$

The convolution integral is difficult to compute

- Idea: Use Laplace-Space to describe I/O relation easier:

$$\begin{aligned} y(t) &= \Sigma u(t) \quad \text{(time domain)} \\ \underbrace{Y(s)}_{\text{Output in s-domain}} &= \underbrace{G(s)}_{\text{Transfer Function}}\underbrace{U(s)}_{\text{Input in s-domain}} \quad \text{(s-domain)} \end{aligned}$$

$\mathcal{L}\{\cdot\}$

- We can lastly take the inverse Laplace Transform of $Y(s)$, to get the time response in the time domain
- Video recommendation for visual interpretation of Laplace Transform: https://www.youtube.com/watch?v=n2y7n6jw5d0

Derivation of $G(s)$

- Use $\mathcal{L}\left\{\frac{d}{dt}f(t)\right\} = s \cdot F(s) - f(0)$

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \xrightarrow{\mathcal{L}\{\cdot\}} \begin{cases} sX(s) = AX(s) + BU(s) \\ Y(s) = CX(s) + DU(s) \end{cases}$$

transform the whole system

solve for X

$$\begin{cases} sX(s) = AX(s) + BU(s) \\ Y(s) = CX(s) + DU(s) \end{cases}$$

plug in Y

Transfer Function

$$\boxed{\frac{Y}{U} = G(s) = C(sI - A)^{-1}B + D}$$

→ instead of solving an ODE we have a function we can directly compute

Laplace Table

For more complicated functions, we can instead of solving the integral, use the Laplace table

$f(t)$	$\mathcal{L}(f(t))$
$f(t)$	$\mathscr{L}(f(t)) = F(s)$
1	$\frac{1}{s}$
$e^{at}f(t)$	$F(s-a)$
$u(t-a)$	$\frac{e^{-as}}{s}$
$f(t-a)u(t-a)$	$e^{-as}F(s)$
$\delta(t)$	1
$\delta(t-t_0)$	e^{-st_0}
$\frac{d^n}{dt^n}\delta(t)$	s^n
$t^n f(t)$	$(-1)^n \frac{d^n F(s)}{ds^n}$
$f'(t)$	$sF(s) - f(0)$
$f^n(t) = \frac{d^n f(t)}{d^n t}$	$s^n F(s) - s^{n-1}f(0) - \ldots - f^{n-1}(0)$

$f(t)$	$\mathcal{L}(f(t))$
$f * g(t)$	$F(s) \cdot G(s)$
$t^n \quad (n = 0, 1, 2, \ldots)$	$\frac{n!}{s^{n+1}}$
$\boxed{sin(kt)}$	$\boxed{\frac{k}{s^2+k^2}}$
$sin^2(kt)$	$\frac{2k^2}{s(s^2+4k^2)}$
$\boxed{cos(kt)}$	$\boxed{\frac{s}{s^2+k^2}}$
$cos^2(kt)$	$\frac{s^2+2k^2}{s(s^2+4k^2)}$
e^{at}	$\frac{1}{s-a}$
$ln(at)$	$-\frac{1}{s}\left(ln\left(\frac{s}{a}\right) + \gamma\right)$
$sinh(kt)$	$\frac{k}{s^2-k^2}$
$cosh(kt)$	$\frac{s}{s^2-k^2}$

2nd order system
→ oscillations

Summary

We looked at the following points today:

- How to compute the forced response without directly solving integrals
  - Using the input $u(t) = e^{st}$, since any signal can be expressed as a linear combination of such exponentials
  - Deriving the transfer function $G(s)$, which maps input to output in block diagrams
- Introducing the Controllable Canonical Form to go from transfer function to state-space representation
- Applying the Laplace Transform to compute transfer functions and output responses efficiently

Tips for Problem Set 04

Easy	Medium	Hard
$1, 2, 4$	3	5

- Nr. 1: Look at the slides if you need help with the derivation
- Nr. 2: Use the formula for the transfer function, you can skip 2b) if you don't want to solve it
- Nr. 3: Look at the slides for definitions of poles & for specific inputs
- Nr. 4: Apply the controllable canonical form
- Nr. 5: Make use of the Laplace Transform and the helpful tables. I would recommend working with the solutions and especially not wasting too much time on 5d)

Questions?

Feedback?

Too fast? Too slow? Less theory, more exercises?

I would appreciate your feedback. Please let me know.

https://docs.google.com/forms/d/e/1FAIpQLSdHl0kjWo63aNzDkAV0cnmQadCAj5L0D7v7aSh0BK7BBdEgpA/viewform?usp=header