

# CS1 Week 5

Transfer functions



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# Attention

- ETH no longer provides web hosting.
- All my exercise class materials have been moved to GitHub Pages.
- You can now find them under the following link:

<https://kissanv.github.io>

Or you can scan the QR-Code:



# Recap Last week

# Quiz 1

Into which parts do we split the solution of an ODE to obtain the time response?

**A.** Dynamic & Steady-state response

**B.** Natural & Forced response

**C.** Frequency & Time response

**C.** We don't split the solution

# Initial and Forced Response Solution

## 1. Initial Condition

$$x_{IC}(0) = x_0, u_{IC}(t) = 0, t \geq 0$$

Solve

$$\dot{x}(t) = Ax$$

Solution:

$$x_{IC}(t) = e^{At}x_0$$

$$y_{IC}(t) = Ce^{At}x_0$$

## 2. Forced Response

$$u_F(t) = u(t), x_F(0) = 0$$

Solve

$$\dot{x}(t) = Ax + Bu$$

Solution:

$$x_F(t) = \int_0^t e^{A(t-\tau)} B u(\tau) d\tau$$

$$y_F(t) = C \int_0^t e^{A(t-\tau)} B u(\tau) d\tau + Du(t)$$

$$y = y_{IC} + y_F = \underbrace{Ce^{At}x_0}_{\text{initial response}} + \underbrace{C \int_0^t e^{A(t-\tau)} B u(\tau) d\tau}_{\text{forced response}} + \underbrace{Du(t)}_{\text{feedthrough}}$$

**Time Response of an LTI system**

# Quiz 2

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}B u(\tau)d\tau$$

$$\int \tau e^{-\tau} d\tau = -\tau e^{-\tau} - e^{-\tau}$$

**Question 6** Choose the correct answer. (1 Point)

Consider a system with the following dynamics,

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) .$$

If  $u(t) = e^{-t}$ ,  $t \geq 0$ , and  $x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ , find  $x(t)$  for  $t \geq 0$ .

☐ A  $x(t) = \begin{bmatrix} -1 + t + e^{-t} \\ 1 - e^{-t} \end{bmatrix}$

☐ C  $x(t) = \begin{bmatrix} -1 + e^{-t} \\ 1 + t - e^{-t} \end{bmatrix}$

☐ B  $x(t) = \begin{bmatrix} 1 + t + e^{-t} \\ -1 + e^{-t} \end{bmatrix}$

☐ D  $x(t) = \begin{bmatrix} -1 - e^{-t} \\ -1 + t + e^{-t} \end{bmatrix}$

# Quiz 3

How is the stability of a system determined?

**A.** Imaginary part of eigenvalues of  $A$

**B.** Real part of eigenvalues of  $A$

**C.** Multiplicity of eigenvalues of  $A$

**D.** Real Part of eigenvalues of  $B$

# I.C Response: Real Eigenvalues

Let us now take a closer look at systems where  $A$  is diagonal. More specifically we will look at the initial condition response, i.e.  $u(t) = 0$

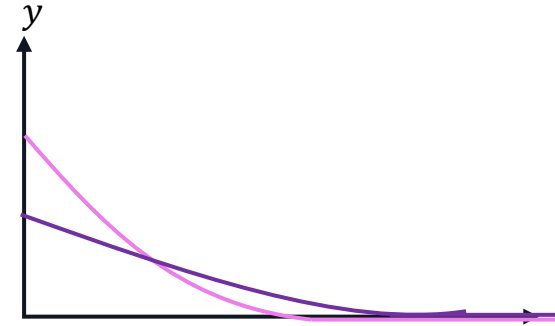
→ For a diagonal, real matrix:  $A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \lambda_i \in \mathbb{R}$

$$y(t) = c e^{At} x_0$$

where we can write out all terms and simplify for  $A$  being diagonal.

$$y(t) = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} \exp(\lambda_1 t) & 0 \\ 0 & \exp(\lambda_2 t) \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

$$y(t) = c_1 e^{\lambda_1 t} x_1(0) + c_2 e^{\lambda_2 t} x_2(0)$$



So for diagonal, real matrices the initial condition response is the linear combination of two exponentials.



# Quiz 4

What does Asymptotic stability mean?

- A.** All eigenvalues have strictly negative real part
- B.** All eigenvalues are purely imaginary
- C.** All solutions converge to 0 as  $t \rightarrow \infty$
- D.** every bounded inputs produces bounded output

# Quiz 5

What is BIBO stability?

- A.** Binary Inputs, Boundless Outputs
- B.** Bounded Inputs, Bounded Outputs
- C.** Bounded Inputs, Boundless Outputs
- D.** Big Inputs, Bigger Outputs

# Stability Conditions

A linearized, diagonalized system with  $A, B, C, D$  matrices is called

- **Lyapunov stable** if  $Re(\lambda_i) \leq 0 \forall i$
- **Asymptotically** stable if  $Re(\lambda_i) < 0 \forall i$
- **Unstable** if  $\exists Re(\lambda_i) > 0 \forall i$

A linearized system with non-diagonalizable  $A$  matrix is called

- **Lyapunov stable** if  $Re(\lambda_i) \leq 0 \forall i$  and there are no repeated eigenvalues with  $Re(\lambda_i) = 0$

For minimal LTI systems

- **Asymptotic stability = BIBO stability**

Bounded Input Bounded Output (BIBO) Stability: for every bounded input, the output will remain bounded

# Course Schedule

Subject		Week
Modeling	Introduction, Control Architectures, Motivation	1
	Modeling, Model examples	2
	System properties, Linearization	3
Analysis	Analysis: Time response, Stability	4
	Transfer functions 1: Definition and properties	5
	Transfer functions 2: Poles and Zeros	6
	Proportional feedback control, Root Locus	7
	Time-Domain specifications, PID control, Computer implementation	8
	Frequency response, Bode plots	9
	The Nyquist condition, Time delays	10
	Frequency-domain Specifications, Dynamic Compensation, Loop Shaping	11
Synthesis	Time delays, Successive loop closure, Nonlinearities	12
	Describing functions	13
	Intro to Uncertainty and Robustness	14

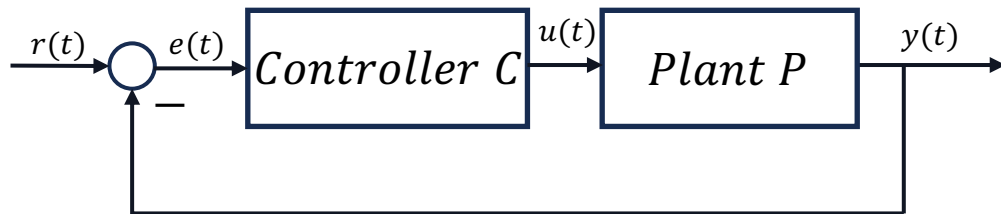
# Today

1. Intro and Definition of Transfer Functions
2. Forms of TF
3. Laplace Transform

# 1. Intro and Definition

of Transfer Functions

# Motivation



system  
modeling



Ferrari 812

Linearization

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t)\end{aligned}$$

Analysis

With information  
about our system

Synthesis

Controller C

Time response:

$$y = y_{IC} + y_F = Ce^{At}x_0 + C \int_0^t e^{A(t-\tau)} B u(\tau) d\tau + Du(t)$$

Last week

This Week

Last week: We analyzed the "natural dynamics of the system" by looking at the I.C. response.  
This week: We will analyze the forced response of the system. How do we analyze our system if there is an input  $u(t)$ ?

# Forced Response

The forced response is given by the convolution integral:

$$y_F = C \int_0^t e^{A(t-\tau)} B u(\tau) d\tau + D u(t)$$

- This is harder to interpret, and the integral is difficult to compute.

## *Thought Process*

- Since we are working with linear systems, we can decompose any input into smaller inputs  $\rightarrow u$  can be written as  $u = u_1 + \dots + u_n$
- We can then apply  $u_1, \dots, u_n$  separately to the system and sum all outputs  
 $y = y_1 + \dots + y_n$
- We know that any input  $u(t)$  can be expressed as an infinite sum of complex exponentials  $e^{st} \rightarrow$  See Fourier Series in Analysis III
- Idea: Compute the forced response to some general  $u(t) = e^{st}$ . Later we can easily compute the output to any input, since it will be a linear combination of  $e^{st}$  terms.



# Forced Response: Derivation (Optional)

$$y(t) = Ce^{At}x_0 + \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau + Du(t)$$

with  $u(t) = e^{st}$ :

$$y(t) = Ce^{At}x_0 + C \int_0^t e^{A(t-\tau)} Be^{s\tau} d\tau + De^{st}$$

rearrange:

$$y(t) = Ce^{At}x_0 + Ce^{At} \int_0^t e^{(sI-A)\tau} B d\tau + De^{st}$$

if  $(sI - A)$  is invertible:

$$y(t) = Ce^{At}x_0 + Ce^{At} \left[ (sI - A)^{-1} e^{(sI-A)\tau} B \right]_0^t + De^{st}$$

rearrange:

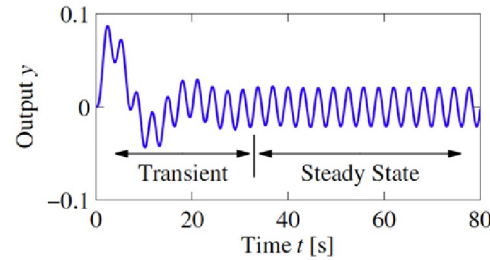
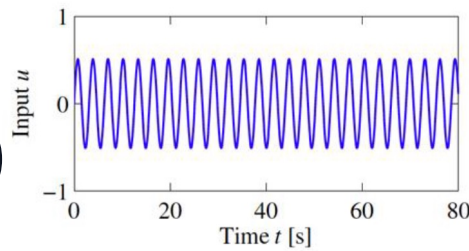
$$y(t) = Ce^{At}x_0 + Ce^{At} \left( (sI - A)^{-1} (e^{(sI-A)t} - I) B \right) + De^{st}$$

and finally:

$$y(t) = \underbrace{Ce^{At}(x_0 - (sI - A)^{-1}B)}_{\text{transient response}} + \underbrace{(C(sI - A)^{-1}B + D)e^{st}}_{\text{steady-state response } y_{ss}}$$

$\rightarrow 0$  if asy stable

# Transfer function $G(s)$



Rearranging:

$$y(t) = \underbrace{C e^{At} (x_0 - (sI - A)^{-1} B)}_{\text{transient response}} + \underbrace{(C(sI - A)^{-1} B + D) e^{st}}_{\text{steady-state response } y_{ss}}$$

$$y_{ss} = (C(sI - A)^{-1} B + D) e^{st} = G(s) e^{st}$$

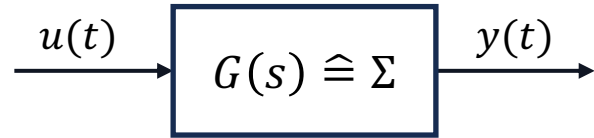
Recall more LinAlg for 2x2 Matrix  $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$M^{-1} = \frac{\text{adj}(M)}{\det(M)} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$G(s) = C \frac{\text{adj}(sI - A)}{\det(sI - A)} B + D$$

# Transfer Function $G(s)$

$$G(s) = C \frac{\text{adj}(sI - A)}{\det(sI - A)} B + D$$



- You can think of transfer function  $G(s)$  as the  $\Sigma$  in block diagrams
- The denominator of  $G(s)$  is the characteristic polynomial of the matrix  $A$
- In general, the transfer function is a rational function of the form:

$$G(s) = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \dots + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_0} + D$$

- **Poles** are the roots of the *Denominator* of  $G(s)$
- **Zeros** are the roots of the *Nominator* of  $G(s)$

# Old Exam Problem

$$G(s) = C \frac{\text{adj}(sI - A)}{\det(sI - A)} B + D$$

## Q16 (1 Points)

Given:

$$A = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}, B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, C = [0 \quad 1], D = -1.$$

Which of the following transfer functions  $G(s)$  is equivalent to the given state space system. Mark the correct answer.

Control Systems 1 Exam Fall 2022

**A.**  $G(s) = -\frac{s^2-3s+2}{s^2-4s+1}$

**C.**  $G(s) = \frac{s^2-3s+2}{s^2-4s+1}$

**B.**  $G(s) = -\frac{(s+2)(s-1)}{s^2-4s+1}$

**D.**  $G(s) = -\frac{s^2+3s+2}{s^2-4s+1}$

# Another One



$$G(s) = C \frac{\text{adj}(sI - A)}{\det(sI - A)} B + D$$

What is the transfer function of the following system (assuming  $x_0 = 0$ )

$$\dot{x}(t) = \begin{bmatrix} -4 & -1 \\ -1 & -4 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 3 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 3 & 1 \end{bmatrix} x(t)$$

**A.**  $G(s) = \frac{6s+15}{(s+4)(s+3)}$

**C.**  $G(s) = \frac{s+4}{s(s+1)}$

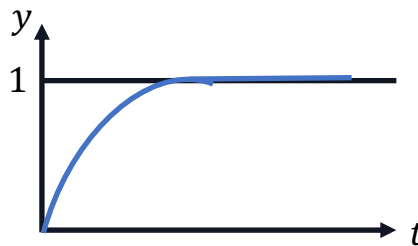
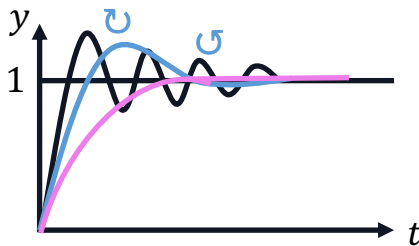
**B.**  $G(s) = \frac{6s+14}{s^2+8s+15}$

**D.**  $G(s) = \frac{s+4}{s^2+8s+15}$

# Order of a system

From a given output plot, how can we determine the order of a system?

→ We can't but there are some points to make



- In this plot we see oscillations in the output even though there weren't any in the input, this comes from complex EWs
  - always in pairs (complex conjugate)
  - min 2<sup>nd</sup> order

order  $\hat{=}$  number of EWs

oscillations → not 1<sup>st</sup> order

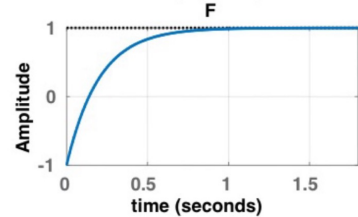
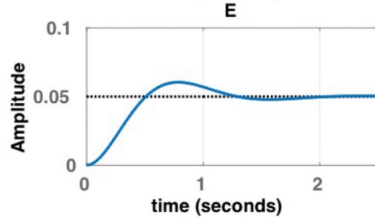
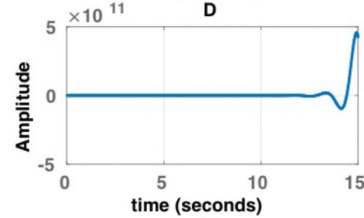
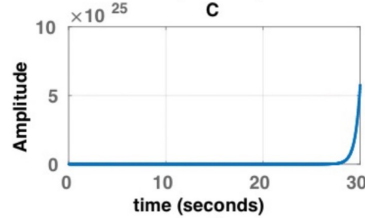
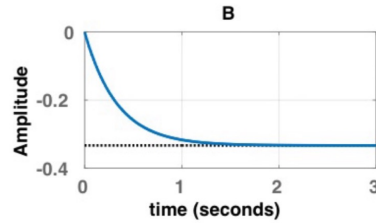
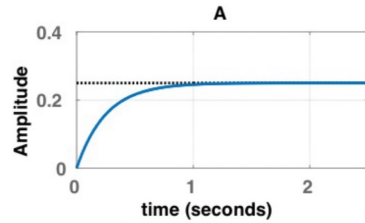
change in curvature → not 1<sup>st</sup> order

# Quiz

Question 11 Mark all correct statements. (2 Points)

In the Figure above various time responses are shown. However, only some of them correspond to a second order system; which ones are those?

- Question isn't clear: mark all systems that have to be at least 2<sup>nd</sup> order.



# Forms of TF



# Specific Inputs and TFs

**Sinusoidal Input:**  $u(t) = \cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$

output:  $y(t) = M \cos(\omega t + \phi)$

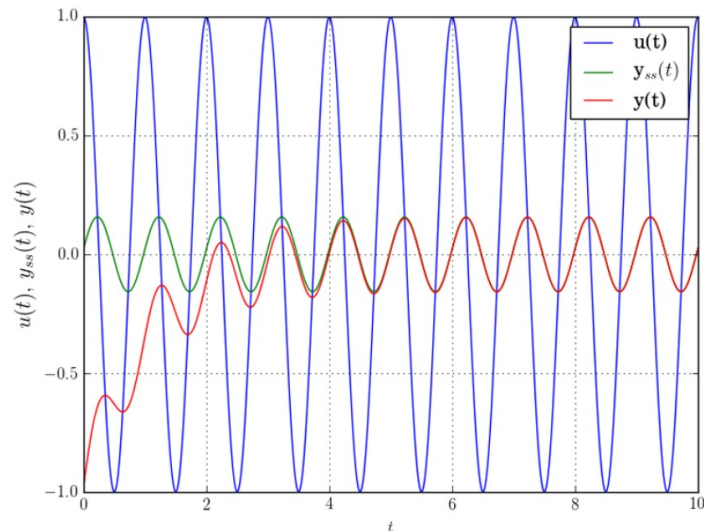
with  $M = |G(j\omega)|$ ,  $\phi = \angle G(j\omega)$

**Integrator:**  $u(t) \longrightarrow \boxed{\int} \longrightarrow y(t)$

For an input  $u(t) = e^{st}$ , the output will be  $y(t) = \frac{1}{s} e^{st}$

**Differentiator:**  $u(t) \longrightarrow \boxed{\frac{d}{dt}} \longrightarrow y(t) = \frac{du(t)}{dt}$

For an input  $u(t) = e^{st}$ , the output will be  $y(t) = s e^{st}$



$$G(s) = \frac{1}{s}$$

$$G(s) = s$$

} transfer functions

# Controllable Canonical Form

What if we now want to go from Transfer Function to a state-space model?

- Recall that there are many different state-space models for the same system!  $\rightarrow$  state-space model is **not** unique
- Generally, we are interested in the *minimal realization* of a system
- For a general TF  $G(s) = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \dots + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_0} + D$ , one minimal realization is given by the **Controllable Canonical Form**

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & & & \ddots & & 1 \\ -a_0 & -a_1 & \dots & & & -a_{n-1} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} b_0 & b_1 & \dots & b_{n-1} \end{bmatrix}, \quad D = [d]$$

# Diagonal realization

Another way to go from transfer function to state-space model is the **diagonal realization**

- If the transfer function is written as a partial fraction expansion of the form

$G(s) = \frac{p_1}{s-\lambda_1} + \frac{p_2}{s-\lambda_2} + \dots + \frac{p_n}{s-\lambda_n} + d$ , one minimal realization is given by:

$$\begin{aligned} A &= \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}, & B &= \begin{bmatrix} \sqrt{p_1} \\ \vdots \\ \sqrt{p_n} \end{bmatrix} \\ C &= \begin{bmatrix} \sqrt{p_1} & \dots & \sqrt{p_n} \end{bmatrix}, & D &= d. \end{aligned}$$

# Quiz (Old Exam HS22)

$$\left[ \begin{array}{ccccc|c} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} & 1 \\ \hline b_0 & b_1 & b_2 & \dots & \dots & d \end{array} \right]$$

Given:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -2 & 0 & -4 & -1 & -6 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} \frac{1}{2} & -2 & 0 & -\frac{1}{3} & 0 \end{bmatrix}, D = 0.$$

Which of the following transfer functions  $G(s)$  is equivalent to the given state space system. Mark the correct answer.

**A.**  $G(s) = \frac{\frac{1}{2}s^3 - 2s + \frac{1}{3}}{s^5 + 2s^4 + 4s^3 + 1.5s^2 + 6}$

**C.**  $G(s) = \frac{-\frac{1}{3}s^3 - 2s + \frac{1}{2}}{s \cdot (s^4 + 6s^3 + s^2 + 4s + 2)}$

**B.**  $G(s) = \frac{-\frac{1}{3}s^3 - 2s + \frac{1}{2}}{s^5 + 6s^4 + s^3 + 4s^2 + 2}$

**D.**  $G(s) = \frac{s \cdot (\frac{1}{2}s^3 - 2s + \frac{1}{3})}{s^5 + 2s^4 + 4s^3 + 1.5s^2 + 6}$

# Laplace Transform

# Laplace Transform

- Time response:  $y(t) = Ce^{At}x_0 + C \int_0^t e^{A(t-\tau)} B u(\tau) d\tau + Du(t)$

The convolution integral is difficult to compute

- Idea: Use Laplace-Space to describe I/O relation easier:

$$\begin{array}{ccc} y(t) = \Sigma u(t) & \text{(time domain)} & \\ Y(s) = G(s)U(s) & \text{(s-domain)} & \mathcal{L}\{\cdot\} \\ \text{Output in s-domain} \swarrow & \underbrace{\hspace{1cm}}_{\text{Transfer Function}} \swarrow & \nwarrow \text{Input in s-domain} \end{array}$$

- We can lastly take the inverse Laplace Transform of  $Y(s)$ , to get the time response in the time domain
- Video recommendation for visual interpretation of Laplace Transform:  
<https://www.youtube.com/watch?v=n2y7n6jw5d0>

# Derivation of $G(s)$

- Use  $\mathcal{L}\left\{\frac{d}{dt}f(t)\right\} = s \cdot F(s) - f(0)$

$$\begin{cases} \dot{y} \\ = Cx + Du\dot{x} \\ = Ax + Bu \end{cases} \xrightarrow[\text{transform the whole system}]{\mathcal{L}\{\cdot\}} \begin{cases} sX(s) = AX(s) + BU(s) \\ Y(s) = CX(s) + DU(s) \end{cases}$$

*Transfer Function*

$$\frac{Y}{U} = G(s) = C(sI - A)^{-1}B + D$$

solve for X

$$\begin{cases} sX(s) = AX(s) + BU(s) \\ Y(s) = CX(s) + DU(s) \end{cases} \xrightarrow{\text{plug in } Y}$$

→ instead of solving an ODE we have a function we can directly compute

# Laplace Table

For more complicated functions, we can instead of solving the integral, use the Laplace table

$f(t)$	$\mathcal{L}(f(t))$	$f(t)$	$\mathcal{L}(f(t))$
$f(t)$	$\mathcal{L}(f(t)) = F(s)$	$f * g(t)$	$F(s) \cdot G(s)$
1	$\frac{1}{s}$	$t^n \quad (n = 0, 1, 2, \dots)$	$\frac{n!}{s^{n+1}}$
$e^{at}f(t)$	$F(s - a)$	$\sin(kt)$	$\frac{k}{s^2 + k^2}$
$u(t - a)$	$\frac{e^{-as}}{s}$	$\sin^2(kt)$	$\frac{2k^2}{s(s^2 + 4k^2)}$
$f(t - a)u(t - a)$	$e^{-as}F(s)$	$\cos(kt)$	$\frac{s}{s^2 + k^2}$
$\delta(t)$	1	$\cos^2(kt)$	$\frac{s^2 + 2k^2}{s(s^2 + 4k^2)}$
$\delta(t - t_0)$	$e^{-st_0}$	$e^{at}$	$\frac{1}{s - a}$
$\frac{d^n}{dt^n} \delta(t)$	$s^n$	$\ln(at)$	$-\frac{1}{s} \left( \ln\left(\frac{s}{a}\right) + \gamma \right)$
$t^n f(t)$	$(-1)^n \frac{d^n F(s)}{ds^n}$	$\sinh(kt)$	$\frac{k}{s^2 - k^2}$
$f'(t)$	$sF(s) - f(0)$	$\cosh(kt)$	$\frac{s}{s^2 - k^2}$
$f^{(n)}(t) = \frac{d^n f(t)}{dt^n}$	$s^n F(s) - s^{n-1}f(0) - \dots - f^{(n-1)}(0)$		

2nd order system  
→ oscillations



# Summary

We looked at the following points today:

- How to compute the forced response without directly solving integrals
  - Using the input  $u(t) = e^{st}$ , since any signal can be expressed as a linear combination of such exponentials
  - Deriving the transfer function  $G(s)$ , which maps input to output in block diagrams
- Introducing the Controllable Canonical Form to go from transfer function to state-space representation
- Applying the Laplace Transform to compute transfer functions and output responses efficiently

# Tips for Problem Sets

Easy	Medium	Hard
1, 2, 4	3	5

- Nr. 1: Look at the slides if you need help with the derivation
- Nr. 2: Use the formula for the transfer function, you can skip 2b) if you don't want to solve it
- Nr. 3: Look at the slides for definitions of poles & for specific inputs
- Nr. 4: Apply the controllable canonical form
- Nr. 5: Make use of the Laplace Transform and the helpful tables. I would recommend working with the solutions and especially not wasting too much time on 5d)

# Questions?

# Feedback?

Too fast? Too slow? Less theory, more exercises?  
I would appreciate your feedback. Please let me know.

<https://docs.google.com/forms/d/e/1FAIpQLSdHI0kjWo63aNzDkAV0cnmQadCAj5L0D7v7aSh0BK7BBdEgpA/viewform?usp=header>