1 Regularity and Completeness of Measures

Definition 1.1. *Given a measure space $(X, \mathcal{A}, \mu)$, we say that the measure μ (or the measure space itself) is complete if for any $B \subseteq A$ such that $\mu(A) = 0$, then $B \in \mathcal{A}$ and $\mu(B) = 0$.*

Example 1.1. *The measure space (X, M_{μ^*}, μ^*) mentioned in* **??**. *is complete.*

Given a set X and a σ-algebra $\mathcal{A}$, then we could complete the space through a completion process.

Definition 1.2. *Let $(X, \mathcal{A})$ be a measurable space with μ a measure on the pair, the completion of μ is the collection of subsets A of the form:*

$$E \subseteq A \subseteq F$$

for $E, F \in \mathcal{A}$ and $\mu(F \backslash E) = 0$. Denote this collection as $\mathcal{A}_\mu$.

Remark 1. Note that any subset $A \subseteq E$ where $\mu(E) = 0$ immediately implies that $E \in \mathcal{A}_\mu$.

We must check that this construction does not depend on the choice of E, F as above and thus the completion measure will be well-defined. We see that, by definition, $\mu(E) = \mu(F)$. Also, we see that $\mu(F) = \sup\{\mu(B) | B \in \mathcal{A}, B \subseteq A\}$. By symmetry, $\mu(E) = \inf\{\mu(B) | B \in \mathcal{A}, A \subseteq B\}$. Thus, we see that the choice of sets above does not affect the completion construction and we can attempt to extend the measure $\bar{\mu} : \mathcal{A}_\mu \to [0, \infty]$ such that $\bar{\mu}(A) = \mu(F) = \mu(E)$. It now suffices to prove that this indeed is a measure on $\mathcal{A}_\mu$.

Theorem 1.1. *Let $(X, \mathcal{A})$ be a measurable set and let μ be a measure on the space. Using Definition 1.2, the extended measure $\bar{\mu} : \mathcal{A}_{mu} \to [0, \infty]$ is indeed a complete measure on $\mathcal{A}_\mu$ which is a σ-algebra containing $\mathcal{A}$. Furthermore, $\bar{\mu} \restriction_{\mathcal{A}} = \mu$.*

Proof. We first proof that $\mathcal{A}_\mu$ is indeed a σ-algebra. Since $\mathcal{A} \subseteq \mathcal{A}_\mu$, $X \in \mathcal{A}_\mu$. Given two $A, B \in \mathcal{A}_\mu$, there exist $E_A, E_B \in \mathcal{A}$ and $F_A, F_B \in \mathcal{A}$ following the definition given in 1.2 respectively. Since $A \backslash B \subseteq (F_A \backslash E_A) \backslash (F_B \backslash E_B) \in \mathcal{A}$ and $\mu((F_A \backslash E_A) \backslash (F_B \backslash E_B)) = 0$. Next, suppose we had a countable family $\{A_i\} \subseteq \mathcal{A}_\mu$. Let $\{F_i\}, \{E_i\}$ be the sets mentioned in 1.2 respective by index. Then it follows that:

$$\bigcup_{i=1}^{\infty} E_i \subseteq \bigcup_{i=1}^{\infty} A_i \subseteq \bigcup_{i=1}^{\infty} F_i$$

and

$$\bigcup_{i=1}^{\infty} F_i \backslash \bigcup_{i=1}^{\infty} E_i \subseteq \bigcup_{i=1}^{\infty} (F_i \backslash E_i)$$

Thus,

$$\mu(\cup_{i=1}^{\infty} F_i \backslash \cup_{i=1}^{\infty} E_i) \leq \sum \mu(F_i \backslash E_i) = 0$$

and it follows that $\mathcal{A}_\mu$ is a σ-algebra and $\bar{\mu}$ is measure by the properties endowed by μ.

1. $\bar{\mu}(\emptyset) = 0$
2. Given a countable disjoint family $\{A_i\} \subset \mathcal{A}_\mu$, $\bar{\mu}(\cup_i A_i) = \mu(\cup_i E_i) = \sum_i \mu(E_i)$ since $E_i \subseteq A_i$ by our notation in Definition 1.2.

Completeness immediately follows from the construction of $\mathcal{A}_\mu$. □

Similar to our definition of an outer measure on a measurable space $(X, \mathcal{A}, \mu)$, we can define an inner measure as follows:

Definition 1.3. *Let $(X, \mathcal{A}, \mu)$ be a measure space. Define the inner measure $\mu_* : 2^X \to [0, \infty]$ as follows:*

$$\mu_*(A) = \sup\{\sum_I \mu(B_i) | \{B_i\}_I \subseteq \mathcal{A}, \bigcup_{i \in I}^{\infty} B_i \subseteq A, I \subseteq \mathbb{N}\}$$

In light of the set $\mathcal{A}_\mu$, we can find conditions in which the inner and outer measures coincide:

Theorem 1.2. *Let $(X, \mathcal{A}, \mu)$ be a measure space and let $A \subset X$ such that $\mu^*(A) < \infty$. Then $A \in \mathcal{A}_\mu$ if and only if $\mu^*(A) = \mu_*(A)$.*

Proof. If $A \in \mathcal{A}_\mu$, by our remark above and 1.1 , it immediately follows that $\mu^*(A) = \mu_*(A)$. Conversely, if $\mu^*(A) = \mu_*(A)$ there exists a countable families $\{E_i\}_{I_1}, \{F_i\}_{I_2}$ such that

$$\sum_{I_1} \mu(E_i) < \mu_*(A) + \epsilon$$

$$\sum_{I_2} \mu(F_i) < \mu^*(A) + \epsilon$$

Thus, $\mu(\cup_{I_2} F_i \backslash \cup_{I_1} E_i) < 2\epsilon$ for any $\epsilon \geq 0$. Hence, taking countable unions of such families for every $\epsilon < \frac{1}{n}$, there exists $E \subseteq A \subseteq F$ such that $\mu(F \backslash E) = 0$. □

Definition 1.4. *An outer measure μ on X is called regular if for any $A \subset X$, there exists a μ-measurable set $B \subset X$ such that $\mu(A) = \mu(B)$.*

Theorem 1.3. *Given an finite regular outer measure μ on X, for all $A \subset X$, A is μ-measurable if and only if $\mu(A) = \mu(A) + \mu(X \backslash A)$.*