Notes on Quantum Information Processing

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1 Quantum Mechanics

2 Qubits

A **qubit** $|\Psi\rangle$ is the linear combination of basis elements $|0\rangle$ and $|1\rangle$ interpreted as a superposition of $0, 1$:

$$|\Psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle\,, \quad \alpha, \beta \in \mathbb{C}$$

Taking n tensor products of $|0\rangle$,$|1\rangle$ yields entangled states of n-qubits:

$$\begin{gathered}
|0\rangle \otimes |0\rangle ... \otimes |0\rangle = |0....0\rangle \\
|0\rangle \otimes |0\rangle ... \otimes |1\rangle = |0....1\rangle \\
... \\
|1\rangle \otimes |1\rangle ... \otimes |1\rangle = |1....1\rangle
\end{gathered}$$

Let $\mathbb{C}^2$ be the 2 dimensional $\mathbb{C}$-vector space representing the space of superpostions of a single qubit. Then the n-qubit $\mathcal{H}_n$ can be represented as:

$$\mathcal{H}_n = (\mathbb{C}^2)^{\otimes n}$$

In other words, if $|\Psi\rangle \in \mathcal{H}_n$, then

$$|\Psi\rangle = \sum_{i=0}^{2^n-1} c_i\,|i\rangle \quad c_i \in \mathbb{C}$$

by definition of the n-tensor product of the 2-dimensional complex vector space.

3 Quantum Gates

Quantum gates are esstentially unitary operators on Hilbert spaces. These unitary operators act on single qubits by rotating them along the Bloch sphere.

3.1 Hadamard Gates

There are many gates that do not have classical analouges. One example is the Hadamard gate $H : \mathbb{C}^2 \to \mathbb{C}^2$

$$
\begin{aligned}
|0\rangle &\mapsto \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\
|1\rangle &\mapsto \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)
\end{aligned}
$$

which is represented as the matrix:

$$
H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}
$$

We can see from the definition that H takes a qubit and sends it to a superposition between $|0\rangle, |1\rangle$. H is unitary as $H^2 = I$:

$$
H^2 = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^2 = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = I
$$

Thus, H also takes superpositions of $|0\rangle, |1\rangle$ to single-state qubits.

We can directly calculate the effect the Hadamard gate has on n-qubits as such:

$$
H^{\otimes n} |j_{n-1}...j_0\rangle = \frac{1}{\sqrt{2^n}} \prod_{l=0}^{n-1} (|0\rangle + e^{i\pi j_l} |1\rangle)
$$

where $j_i = \{0, 1\}$ for $0 \leq i \leq n-1$.

Consequently, an *equal* superposition where the probabilities for measuring a given n-qubit state are uniform:

$$H^{\otimes n} |0...0\rangle = \frac{1}{\sqrt{2^n}} \sum_{i=0}^{2^n-1} |i\rangle$$

This superposition is instrumental in many quantum algorithms such as Grover's algorithm to extract useful information from oracles.

3.2 CNOT Gates

Define the CNOT (Controlled-Not Gate) as the operator which takes a 2-qubit system sends them as follows:

$$\begin{aligned} |0\rangle |0\rangle &\mapsto |0\rangle |0\rangle \\ |0\rangle |1\rangle &\mapsto |0\rangle |1\rangle \\ |1\rangle |0\rangle &\mapsto |1\rangle |1\rangle \\ |1\rangle |1\rangle &\mapsto |1\rangle |0\rangle \end{aligned}$$

The qubit on the bottom (target qubit) flips in respect to the value of the top qubit (control qubit). The corresponding matrix representation of $CNOT : \mathbb{C}^2 \to \mathbb{C}^2$ is:

$$CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

We can extend this idea to create a set of *generalized* CNOT gates with the following matrices:

4 Quantum Algorithms