Diagonalization

Edward Kim

August 4, 2018

Diagonalization is a time-tested tool dating back to Cantor's famous proof on the cardinality of $\mathbb{R}$. The of the undecidability of halting problem uses diagonalization to show that no TM can decide whether an arbitrary TM will halt on a given input x. This technique will be used to show some basic results on the natural hierarchy of deterministic and non-deterministic TMs based on the amount of resources one allocates to it.

1 The Time Hierarchy Theorems

Definition 1.1. *Given a function $f : \mathbb{N} \to \mathbb{N}$, we say that f is time-constructible if the map $x \mapsto f(|x|)$ can be computed in $O(f(|x|))$ time*

Theorem 1.1. *Let f, g be time-constructible functions such that $f(n) \log f(n) = o(g(n))$, then*

$$DTIME(f(n)) \subset DTIME(g(n))$$

Proof. We assume that simulating a machine by universal TM, $\mathcal{U}$ is at most logarithmic. Consider the language L determined by the machine D which exhibits the following behavior: on input x, run the TM $M_x(x)$ for $g(|x|)$ steps. If the machine halts at any point with output y, output its negation $\neg y$. Otherwise, output 0.

By construction, $L \in DTIME(g(n))$. We shall prove that L cannot be decided by any machine in $DTIME(f(n))$. For the sake of contradiction, suppose that there exists machine G such that G decides L and for any input x, there exists a constant c such that G runs for at most $c|x|$. By our assumption above, we can simulate G on $\mathcal{U}$ with $c|x| \log |x|$ steps. Since $f(n) \log f(n) = o(g(n))$, there exists an string y such that $g(|y|) > f(|y|) \log |y|$ such that $M_y = G$ (this follows from the padding lemma). Hence, $D(y) = \neg b = G(y)$. This gives us our contradiction. $\square$

Theorem 1.2. *If f, g are time-constructible functions such that $f(n+1) = o(g(n))$, then*

$$NTIME(f(n)) \subset NTIME(g(n))$$