Lang Chapter 3 (Modules)

Problem 14

Consider the following commutative diagram:

$$
\begin{tikzcd}
 & M' \arrow[r, "\phi_1"] \arrow[d, "f"] & M \arrow[r, "\phi_2"] \arrow[d, "g"] & M'' \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & N' \arrow[r, "\psi_1"] & N \arrow[r, "\psi_2"] & N'' &
\end{tikzcd}
$$

1. Let us prove that if f, h are monomorphisms, then g is also a monomorphism. It suffices to show that if $g(x) = 0_N$, then $x = 0_M$. Since h is a monomorphism, $\ker h = \{0_{M''}\}$. Thus, by the exactness of the top row and commutativity of the rightmost square, $x \in \text{img } \phi_1 \cap \ker g$. Since $x \in \text{img } \phi_1$, there exists $a \in M'$ such that $\phi(a) = x$. By commutativity of the leftmost square, $\psi_2(f(a)) = g(\phi_1(a)) = 0_N$. However, f is a monomorphism and so is ψ_1 by the exactness of the bottom row. Hence, $a = 0_{M'}$ and $x = \phi_1(a) = 0_M$.

2. Now suppose that f, h are surjective. Let us show that g is also surjective. Given $c \in N$, let us consider the case where $\psi_2(c) = 0$. By the exactness of the bottom row, there exists $b \in N'$ such that $\psi_1(b) = c$. By our assumption of the surjectivity of f, there must exist $a \in M'$ such that $f(a) = b$. By commutativity of the leftmost square, $g(\phi_1(a)) = \psi_1(f(a))) = c$. Thus, $\phi_1(a)$ is the element in M we desire, making g surjective. Now we turn to the case where $\psi_2(x) \neq 0$. By surjectivity of h, there exists $r \in M''$ such that $h(r) = \psi_2(c)$. By the exactness of the top row, there must exist $q \in M$ such that $\phi_2(q) = r$. Let $s = g(q)$. By commutativity of the rightmost square, $\psi_2(s) - \psi_2(c) = \psi_2(s - c) = 0$. Thus, by exactness of the bottom row, there exists $t \in n'$ such that $\psi_1(t) = s - c$. By surjectivity of f and the commutativity of the leftmost square, we have an element $z \in M'$ such that $g(\phi_1(z)) = f - c$. Thus,

   $$g(\phi_1(z)) - f = g(\phi_1(z) - q) = c$$

   . Taking our desired element to be $\phi_1(z) - q$ shows that q is surjective.

Problem 18

Consider an nversely directed system of commutative rings $\{A_i\}$ and an inveresly directed $n+1$ system of A-modules $\{M_i\}$ such that the following diagram commutes: