Problems in Category Theory (Maclane)

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1 Chapter 1

1.1 Section 4

Question 1.1. *Question: Let S be a fixed set and X^S be the set of all functions $f : S \to X$. Show that the map $X \mapsto X^S$ is the object functor* **Set** $\to$ **Set**, *and that the evaluation function $e_x : X^S \times S \to X$ defined by $e_x(h, s) = h(s)$ is a natural transformation.*

Proof. Let $X \mapsto X^S$ be the object functor of the functor $\mathcal{F}$ equipped with a morphism functor defined as follows: Given a morphism between two sets $f : X \to Y$, $\mathcal{F}(f) : X^S \to Y^S$ such that $\mathcal{F}(g_x) = f \circ g_x$ for $g_x \in X^S$. To see that e_x is a natural transformation: consider the functor $\mathcal{F} \times id :< X^S, S >\to< Y^S, S >$. Taking the evaluation mapping $e_x : X^S, S \to X$ yields one element set from X_θ: $\{h_X(s)\}$. Similarly for $e_Y :< Y^S, S >\to Y$, the evaluation function yields $Y_\theta = \{h_Y(s)\}$. Define the map between $f_\theta : X_\theta \to Y_\theta$ by $f_\theta(h_X(s)) = h_Y(s)$ By our definition above, the functor $\mathcal{F}$ respects the behavior of such a morphism $f : X \to Y$. Thus, we can define a functor $\mathcal{H}$ as above, yielding our natural transformation. $\square$

Question 1.2. *If H is a fixed group, show that $G \to H \times G$ defines a functor $H \times () :$* **Grp** $\to$ **Grp** *and that each morphism $f : H \to K$ defines a natural transformation such that $H \times () \to G \times ()$.*

Proof. We can define a functor $\mathcal{F}$ first by the natural object mapping from $G \mapsto H \times G$ and the morphism mapping for $g : G \to R$: $\mathcal{F}(f) : H \times G \to H \times R$. The natural transformation easily $H \times () \to G \times ()$ follows as such:

Given two groups G_1, G_2 let $f_1 : H \times G_1 \to H \times G_2$ and let $f_2 : G \times G_1 \to G \times G_2$. The natural transformation would be the morphism that takes $h : H \to G$ as this makes $h \circ f_1 = f_2 \circ h$. $\square$

Question 1.3. *For functors $S, T : C \to P$ where C is a category and P is a preorder, show that there is a natural transformation $S \to T$ iff $S \leq T$ for every object in C.*

Proof. If there is a natural transformation from $S \to T$, then the transformation must respect the morphims (which are the orderings) of the preorder P. Thus, according to P, $Sc \leq Tc$. Conversely, if such an ordering existed for every object $c \in C$, then we can define the natural transformation $S \to T$ as follows: Let $f_s : Sc \to Sc'$ and $f_c : Tc \to Tc'$, then define $\theta : S \to T$ as $\theta(Tc) = Sc$ and $\theta(Sc') = Tc'$. This map is well-defined as there can be only one morphism between two objects as defined by the preorder. □

1.2 Section 5

Question 1.4. *Find a category with an arrow which is both epi and monic, but not invertible.*

Proof. Consider the category of topological spaces and the arrow $\mathbb{Q} \xrightarrow{i} \mathbb{R}$. If Y is a topological space such that $Y \rightrightarrows \mathbb{Q} \to \mathbb{R}$ and both composite arrows are equal to each other, it follows that the parallel arrows $Y \rightrightarrows \mathbb{Q}$ are equal to each other, showing that the inclusion arrow is monic. To show that it is an epimorphism, consider $\mathbb{Q} \xrightarrow{i} \mathbb{R} \rightrightarrows X$ for some topological space X such that the composite arrows are equal. By the density of $\mathbb{Q}$ in $\mathbb{R}$, the two parallel arrows will converge by taking the rationals to converge to every point in $\mathbb{R}$ as a Cauchy sequence. Thus, the two parallel continuous functions must be equal. However, we immediately see that the inclusion function is not invertible. □

Question 1.5. *Prove that the composite of monics is monic and likewise for epis*

Proof. Let ϕ, θ be monics in a category $\mathcal{C}$. Suppose that for maps ξ_1, ξ_2, $\theta\phi\xi_1 = \theta\phi\xi_2$. Then by associativity:

$$\theta(\phi\xi_1) = \theta(\phi\xi_2) \implies \theta\xi_1 = \theta\xi_2 \implies \xi_1 = \xi_2$$

A similar procedure will determine the same property for epis. □

Question 1.6. *An arrow $f : a \to b$ in a category $\mathbb{C}$ is regular when there exists an arrow $g : b \to a$ such that $fgf = f$. Show that f is regular if it has a left or right inverse, and prove that every arrow in* **Set** *with $a \neq \emptyset$ is regular.*

Proof. Suppose that the arrow $f : a \to b$ as a right inverse $r : b \to a$ such that $fr = id_b$. Taking r be our regular arrow yields $(fr)f = f$. Similarly, for a left inverse $l : b \to a$ $f(lf) = f$. Now suppose that $\phi : X \to Y$ is a morphism in **Set**. Simply taking the coarsest inverse ϕ^{-1} will suffice. If the function is not injective such that there exists $x, y \in X$ such that $\phi(x) = \phi(y) = c \in T$. Then take $\phi^{-1}(c) = x$ or $\phi^{-1}(c) = y$. □

Question 1.7. *Consider the category with objects* $< X, e, t >$*, where* X *is a set,* $e \in X$*, and* $t : X \to X$ *and with morphisms* $f :< X, e, t > \to < X', e', t' >$ *the functions* $f : X \to X'$ *with* $fe = e'$ *and* $ft = t'f$*. Prove that this category has an initial object in which* X *is the set of natural numbers,* $e = 0$ *and* t *is the successor function.*

Proof. Given object $< X, e, t > \in \mathcal{C}$,we will define an unique morphism f_i from $< \mathbb{N}, 0, s >$ where $s : \mathbb{N} \to \mathbb{N}$ is the successor function. Define $f_i : \mathbb{N} \to X$ by first mapping $f_i(0) = e$ and $f_i(s^r 0) = t^r e$. It suffices to show that $tf_i = f_i s$ By our construction above, this diagram must commute as given any $n \in \mathbb{N}$, $n = s^n(0)$ so $tf_i(n) = t(s^n(0)) = t \cdot t^n e = t^{n+1}e = f_i(s^{n+1}(0)) = f_i(s(n))$. As f_i defined above is injective, the morphism is unique for every object in the category $\mathcal{C}$. □

2 Chapter 2

2.1 Section 3

Question 2.1. *Show that the product of categories includes the following known special cases: The product of monoids (categories with one object), of groups, of sets (discrete categories).*

Proof. Recall that we can construct a product category as follows: Consider two categories $\mathcal{C}, \mathcal{D}$. Define the product category $\mathcal{C} \times \mathcal{D}$ with objects as ordered pairs (c, d) where $c \in \mathcal{C}$ and $d \in \mathcal{D}$ and arrows $(c, d) \to (c', d')$ iff there exists arrows $c \to c'$ and $d \to d'$. We can naturally define projection functors $P_C : \mathcal{C} \times \mathcal{D} \to \mathcal{C}$ and $P_D : \mathcal{C} \times \mathcal{D} \to \mathcal{D}$. □

. The product category must have the following universal property: given two functors from a category $\mathcal{H}$ defined as $S : \mathcal{H} \to \mathcal{C}$ and $T : \mathcal{H} \to \mathcal{D}$, there exists a unique functor $F : \mathcal{H} \to \mathcal{C} \times \mathcal{D}$ such that the following diagram commutes:

$$\begin{tikzcd}
 & \mathcal{H} \arrow[dl, "S"'] \arrow[d, "F"] \arrow[dr, "T"] & \\
\mathcal{C} & \mathcal{C} \times \mathcal{D} \arrow[l, "P_C"] \arrow[r, "P_D"'] & \mathcal{D}
\end{tikzcd}$$

Similarly,we can define functors between product categories as $U \times V : \mathcal{C} \times \mathcal{D} \to \mathcal{C}' \times \mathcal{D}'$ for functors $U : \mathcal{C} \to \mathcal{D}$ and $V : \mathcal{C}' \to \mathcal{D}'$

The product of categories will necessarily include the product of monoids as two functors S, T : **Mon** $\to$ **Mon** map monoids to monoids such that for any two monoids $\mathcal{M}_1, \mathcal{M}_2 \in$ **Mon** and monoid homomorphisms(functors between monoids) $S : H \to \mathcal{M}_1$ and $T : H \to \mathcal{M}_2$ The following diagram commutes with a unique functor: $F : H \to \mathcal{M}_1 \times \mathcal{M}_2$

$$\begin{tikzcd}
 & H \arrow[dl, "S"'] \arrow[d, "F"] \arrow[dr, "T"] & \\
\mathcal{M}_1 & \mathcal{M}_1 \times \mathcal{M}_2 \arrow[l, "P_{M_1}"] \arrow[r, "P_{M_2}"'] & \mathcal{M}_2
\end{tikzcd}$$

Question 2.2. *Show that the product of two preorders is a preorder.*

Proof. Recall that a preorder is a category $\mathcal{C}$ such that for any two objects p, p' there exists at most one arrow $p \to p'$ We can define the product preorder category **Preord** $\times$ **Preord** with ordered tuples of objects (e, d) for objects $e, d \in C, D$ and we can define arrows between $(e, d) \to (e', d')$ iff there exists $e \to e'$ and $d \to d'$. As there can only be one arrow between objects in their respective catgories, the arrows in the product category are uniquely determined. Hence, the product category must also be a preorder. $\square$

2.2 Section 4

Question 2.3. *If* **Fin** *is the category of all finite sets and* G *is a finite group, describe* $\mathbf{Fin}^G$