Problems in Atiyah's Commutative Algebra

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1 Extensions and Contractions

Let $A \to B$ be a ring homomorphism. For an ideal $\mathfrak{a}$, $f(\mathfrak{a})$ is not an ideal in general. Consider the embedding $\phi : \mathbb{Z} \to \mathbb{Q}$, and take $f((m))$ for any $m \in \mathbb{Z}$. In general, for any $q \in \mathbb{Q}$, $qf(m) \notin (f((m)))$. To complete the image, we take the extension $\mathfrak{a}^e$ to be the ideal generated by $f(\mathfrak{a})$ in B. We define the contraction of an ideal $\mathfrak{b}^c$ in B to be $f^{-1}(B)$ which is always an ideal in A.

1.1 Properties of Extensions and Contractions

The following operations are closed under extensions and contractions. Let $\mathfrak{r}(\mathfrak{a})$ denote the radical of ideal $\mathfrak{a}$.

1. $(\mathfrak{a}_1 + \mathfrak{a}_2)^e = Bf(\mathfrak{a}_1 + \mathfrak{a}_2) = Bf(\mathfrak{a}_1) + Bf(\mathfrak{a}_2) = \mathfrak{a}_1^e + \mathfrak{a}_2^e$
2. $(\mathfrak{a}_1 \cap \mathfrak{a}_2)^e \subseteq \mathfrak{a}_1^e \cap \mathfrak{a}_2^e$. Easily follows since if $x \in (\mathfrak{a}_1 \cap \mathfrak{a}_2)^e$, then $x \in \mathfrak{a}_1^e$ and $x \in \mathfrak{a}_2)^e$. Howeverm, the converse is generally not true. $Let \phi : \mathbb{Z} \to \mathbb{Q}$ be the natural embedding and let $\mathfrak{a} = (p_1)$ and $\mathfrak{b} = (p_2)$ for two primes p_1, p_2. Then $\phi(\mathfrak{a}) \cap \phi(\mathfrak{b})$ (How about the example with the map $\mathbb{Z} \to \mathbb{Z}[i]$ with $\mathfrak{a} = (2)$ and $\mathfrak{b} = (5)$?)
3. $(\mathfrak{a}_1\mathfrak{a}_2)^e = Bf(\mathfrak{a}\mathfrak{b}) = Bf(\mathfrak{a})f(\mathfrak{b}) = Bf(\mathfrak{a})Bf(\mathfrak{b}) = \mathfrak{a}^e\mathfrak{b}^e$
4. $(\mathfrak{a} : \mathfrak{b})^e \subseteq (\mathfrak{a}^e : \mathfrak{b}^e)$ (Let $y \in (\mathfrak{a} : \mathfrak{b})$, then $Bf(y)f(\mathfrak{b}) \subseteq f(\mathfrak{a})$)
5. $\mathfrak{r}(\mathfrak{a})^e \subseteq \mathfrak{r}(\mathfrak{a}^{\mathfrak{e}})$. If $y \in r(\mathfrak{a})^e$, then $y = bx$ where $x \in \mathfrak{r}(\mathfrak{a})$. Thus, $y \in Bf(\mathfrak{r}(\mathfrak{a}))$ and $y \in \mathfrak{r}(\mathfrak{a})^e$

2 Exercises

2.1 Problem 1

Question: If x is a nilpotent element in commutative ring A, show that $1 + x$ is a unit. Deduce that the sum of a nilpotent element and a unit is a unit.

Proof. Since $x \in \mathfrak{R}$ where $\mathfrak{R}$ is the nilradical, there exists $n > 0$ such that $x^n = 0$. Multiplying by x^{n-1} yields $x^{n-1}(1 + x) = x^{n-1} + 0 = x^{n-1}$. Thus, $1 + x$ must be a unit. We can generalize this result as follows: Let u be a unit in the ring A and x as defined above. Performing a similar operation as above, we receive

$$x^{n-1}(u + x) = ux^{n-1}$$

. As u is a unit, let $y \in R$ be chosen such that $yu = 1$. Multiplying y on both sides gives us:

$$x^{n-1}(1 + yx) = x^{n-1}$$

. As $yx \in \mathfrak{R}$, we see that the addition of a unit with a nilpotent element gives us a unit in R. □

Problem 1 relates to the following results from Problem 10:

2.2 Problem 10

Question: Let A be a ring and $\mathfrak{R}$ be its nilradical. Show that the following are equivalent:

1. Show that A has exactly one prime ideal.
2. Every element of A is either a unit or nilpotent.
3. A / $\mathfrak{R}$ is a field.

Proof. For the implication $2 \implies 3$. under the assumption of 2, the non-trivial cosets of $A/\mathfrak{R}$ will all contain units. Thus, for any $\bar{u} \in A/\mathfrak{R}$, the multiplication of coset of $\bar{u^{-1}}$ will yield $\bar{1}$. Thus, $A/\mathfrak{R}$ is a field.

For $3 \implies 1$, If $A/\mathfrak{R}$, then $\mathfrak{R}$ must be maximal. Thus, $\mathfrak{R}$ is prime. However as $\mathfrak{R}$ is the intersection of all prime ideals, it follows that it must be the only prime ideal. For $1 \implies 2$, by proposition 1.9, we know that any $x \in A - \mathfrak{R}$ must be a unit as if there is only one prime ideal, it must exactly be a maximal ideal. Thus, A is only composed of units and nilpotent elements. $\square$

Remark: This ring seems to be partitioned into two classes: one with elements that are inextricably linked to the muliplicative identity 1 and the other consisting of elements eventually return to 0.

2.3 Problem 2

Let A be a commutative ring and $A[x]$ its corresponding polynomial ring with an indeterminate x with coefficients in A.

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