Problems in Atiyah's Commutative Algebra

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1 Extensions and Contractions

Let $A \to B$ be a ring homomorphism. For an ideal $\mathfrak{a}$, $f(\mathfrak{a})$ is not an ideal in general. Consider the embedding $\phi : \mathbb{Z} \to \mathbb{Q}$, and take $f((m))$ for any $m \in \mathbb{Z}$. In general, for any $q \in \mathbb{Q}$, $qf(m) \notin (f((m)))$. To complete the image, we take the extension $\mathfrak{a}^e$ to be the ideal generated by $f(\mathfrak{a})$ in B. We define the contraction of an ideal $\mathfrak{b}^c$ in B to be $f^{-1}(B)$ which is always an ideal in A.

1.1 Properties of Extensions and Contractions

The following operations are closed under extensions and contractions. Let $\mathfrak{r}(\mathfrak{a})$ denote the radical of ideal $\mathfrak{a}$.

1. $(\mathfrak{a}_1 + \mathfrak{a}_2)^e = Bf(\mathfrak{a}_1 + \mathfrak{a}_2) = Bf(\mathfrak{a}_1) + Bf(\mathfrak{a}_2) = \mathfrak{a}_1^e + \mathfrak{a}_2^e$
2. $(\mathfrak{a}_1 \cap \mathfrak{a}_2)^e \subseteq \mathfrak{a}_1^e \cap \mathfrak{a}_2^e$. Easily follows since if $x \in (\mathfrak{a}_1 \cap \mathfrak{a}_2)^e$, then $x \in \mathfrak{a}_1^e$ and $x \in \mathfrak{a}_2)^e$. Howeverm, the converse is generally not true. $Let \phi : \mathbb{Z} \to \mathbb{Q}$ be the natural embedding and let $\mathfrak{a} = (p_1)$ and $\mathfrak{b} = (p_2)$ for two primes p_1, p_2. Then $\phi(\mathfrak{a}) \cap \phi(\mathfrak{b})$ (How about the example with the map $\mathbb{Z} \to \mathbb{Z}[i]$ with $\mathfrak{a} = (2)$ and $\mathfrak{b} = (5)$?)
3. $(\mathfrak{a}_1\mathfrak{a}_2)^e = Bf(\mathfrak{a}\mathfrak{b}) = Bf(\mathfrak{a})f(\mathfrak{b}) = Bf(\mathfrak{a})Bf(\mathfrak{b}) = \mathfrak{a}^e\mathfrak{b}^e$
4. $(\mathfrak{a} : \mathfrak{b})^e \subseteq (\mathfrak{a}^e : \mathfrak{b}^e)$ (Let $y \in (\mathfrak{a} : \mathfrak{b})$, then $Bf(y)f(\mathfrak{b}) \subseteq f(\mathfrak{a})$)
5. $\mathfrak{r}(\mathfrak{a})^e \subseteq \mathfrak{r}(\mathfrak{a}^e)$. If $y \in r(\mathfrak{a})^e$, then $y = bx$ where $x \in \mathfrak{r}(\mathfrak{a})$. Thus, $y \in Bf(\mathfrak{r}(\mathfrak{a}))$ and $y \in \mathfrak{r}(\mathfrak{a})^e$

2 Exercises

2.1 Problem 1

Question: If x is a nilpotent element in commutative ring A, show that $1+x$ is a unit. Deduce that the sum of a nilpotent element and a unit is a unit.

Proof. Since $x \in \mathfrak{R}$ where $\mathfrak{R}$ is the nilradical, there exists $n > 0$ such that $x^n = 0$. Multiplying by x^{n-1} yields $x^{n-1}(1+x) = x^{n-1} + 0 = x^{n-1}$. Thus, $1+x$ must be a unit. We can generalize this result as follows: Let u be a unit in the ring A and x as defined above. Performing a similar operation as above, we receive

$$x^{n-1}(u+x) = ux^{n-1}$$

. As u is a unit, let $y \in R$ be chosen such that $yu = 1$. Multiplying y on both sides gives us:

$$x^{n-1}(1+yx) = x^{n-1}$$

. As $yx \in \mathfrak{R}$, we see that the addition of a unit with a nilpotent element gives us a unit in R. $\square$

Problem 1 relates to the following results from Problem 10:

2.2 Problem 10

Question: Let A be a ring and $\mathfrak{R}$ be its nilradical. Show that the following are equivalent:

1. Show that A has exactly one prime ideal.
2. Every element of A is either a unit or nilpotent.
3. A / $\mathfrak{R}$ is a field.

Proof. For the implication $2 \implies 3.$ under the assumption of 2, the non-trivial cosets of $A/\mathfrak{R}$ will all contain units. Thus, for any $\bar{u} \in A/\mathfrak{R}$, the multiplication of coset of $\bar{u^{-1}}$ will yield $\bar{1}$. Thus, $A/\mathfrak{R}$ is a field.

For $3 \implies 1$, If $A/\mathfrak{R}$, then $\mathfrak{R}$ must be maximal. Thus, $\mathfrak{R}$ is prime. However as $\mathfrak{R}$ is the intersection of all prime ideals, it follows that it must be the only prime ideal. For $1 \implies 2$, by proposition 1.9, we know that any $x \in A - \mathfrak{R}$ must be a unit as if there is only one prime ideal, it must exactly be a maximal ideal. Thus, A is only composed of units and nilpotent elements. $\square$

Remark: This ring seems to be partitioned into two classes: one with elements that are inextricably linked to the muliplicative identity 1 and the other consisting of elements eventually return to 0.

2.3 Problem 2

Let A be a commutative ring and $A[x]$ its corresponding polynomial ring with an indeterminate x with coefficients in A.

1.

2.4 Problem 6

Let A be a commutative ring such that every ideal not contained in the nilradical contains a non-zero idempotent (element e such that $e^2 = e \neq 0$. Prove that the nilradical and the Jacobson radical are equal.

Proof. Let $\mathfrak{R}$ denote the nilradical and let $\mathfrak{J}$ denote the Jacobson radical. Since every maximal ideal is prime, it follows that $\mathfrak{R} \subseteq \mathfrak{J}$. Suppose for the sake of contradicion that the inclusion is proper, then by our assumption there exists an idempotent $e \in \mathfrak{J}$. Then, we know that $1 - ey$ is a unit for all $y \in A$. Then the following equality holds: there exists $s \in R$ such that

$$(1 - ey)s = 1 \implies e(1 - ey)s = e \implies e(1 - y)s = e$$

Hence, $(1 - y)s = 1$ and $(1 - y)$ is a unit for all $y \in A$. So by the same proposition, $1 \in \mathfrak{J}$, a contradiction. $\square$

2.5 Problem 11

Let A be a commutative ring. A is deemed as Boolean if $x^2 = x$ for all $x \in A$. In a Boolean ring A, show that

1. $2x = 0$
2. every prime ideal $\mathfrak{p}$ is maximal, and $A/\mathfrak{p}$ is a field with two elements.
3. every finitely generated ideal in A is principal.

Proof.

1. $(2x)^2 = 2x \implies 4x^2 = 2x$. Since $x^2 = x$, the equality would only hold if $2x = 0$.
2. If $\mathfrak{p}$ is a prime ideal in A, then $A/\mathfrak{p}$ is an integral domain. Since for $2x = 0$ and $x^2 = x$ for every $x \in A$, there can only be two equivalence classes, namely the equivalence classes of $x + \mathfrak{p} = 1 + \mathfrak{p}$ where $x \neq 0$. and $2(x + \mathfrak{p}) = 0 + \mathfrak{p}$. This gives us the field of two elements with the inverses being the classes themselves. This directly gives us that $A/\mathfrak{p}$ is a field and hence every prime ideal is maximal.
3. Let $\mathfrak{a}$ be a finitely generated ideal in A generated by the set $\{a_1, a_2, ..., a_n\}$. We note the following properties expressed by the Boolean conditions above show us that the element
$$m = \sum_{i=1}^{n} a_i$$
Note that we can express any of the a_i by multiplying m with $a_1...a_{i-1}a_{i+1}...a_n$

□

2.6 Problem 14

In a ring A, let Σ be the set of ideals where all of its elements are zero divisors. Show that Σ has a maximal element and that this maximal element is a prime ideal. Hence, the set of zero-divisors is a union of prime ideals.

Proof. Order Σ by set inclusion. For any chain of set inclusions $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \ldots \subseteq \mathfrak{a}_\mathfrak{n}$, the union

$$\mathfrak{a}_m = \bigcup \mathfrak{a}_\mathfrak{i}$$

will be the maximal element in this finite chain as every element will be a zero-divisor in R. By Zorn's Lemma, we conclude that Σ has a maximal element denoted as Σ_m. Suppose that $x, y \in R$ are elements such that $xy \in \Sigma_m$. Then $xyb = 0$ for some $b \neq 0$. However, $x(yb) = 0$ and $x, yb \neq 0$, making x a zero-divisor. So $x \in \Sigma_m$. By symmetry, we can also arrive that $y \in \Sigma_m$. If both $x, y \in \Sigma_m$, this would contradict the maximality of Σ_m. □

Compare this to the definition using the annhilator of $x \in R$. If D is the set of the zero-divisors in R

$$D = \bigcup_{x \in R} r(\mathrm{Ann}(x))$$

3 The Prime Spectrum of a Ring

3.1 Problem 15

Let A be a ring and let X be the set of all prime ideals of A. For each $E \subseteq A$, denote $V(E) \subseteq X$ be the set of prime ideals containing E. Prove that

1. If $\mathfrak{a}$ is the ideal generated E, then $V(E) = V(\mathfrak{a}) = V(r(\mathfrak{a}))$

   Proof. If a prime ideal contains subset E, then it must contain the smallest ideal generated by E. Recall that the ideal generated by E can be defined as

   $$\bigcap_{E \subseteq I \text{ ideal}} I$$

   Hence $V(\mathfrak{a}) \subseteq V(E)$. Since we are considering sets of prime ideals, $V(r(\mathfrak{a})) \subseteq V(\mathfrak{a})$ as $x^n \in \mathfrak{a} \subseteq \mathfrak{p}$ for some $\mathfrak{p}$ prime and $n > 0$ implies that $x \in \mathfrak{p}$. Since $E \subset \mathfrak{a}$, it is clear that $V(E) \subseteq V(r(\mathfrak{a}))$ which yields the equality. □

2. $V(0) = X$, $V(1) = \emptyset$

Proof. $V(0) = X$ is clear from the definition of an ideal. $V(1) = \emptyset$ no prime ideal can contain the entire ring R by definition. $\square$

3. If $(E_i)_{i \in I}$ is any family of subsets (say countable for now), then

$$V(\bigcup_{i \in I} E_i) = \bigcap_{i \in I} V(E_i)$$

Proof. The inclusion

$$\bigcap_{i \in I} V(E_i) \subseteq V(\bigcup_{i \in I} E_i)$$

follows from the observation that for any E_i,

$$V(E_i) \subseteq V(\bigcup_{i \in I} E_i)$$

for any prime ideal that contains the collective union of $\{E_i\}$ must contain each individual set. On the other hand, if a prime ideal $\mathfrak{p}$ contains the collective union $\{E_i\}$ then

$$\mathfrak{p} \in V(E_i)$$

for all $i \in I$. Thus,

$$\mathfrak{p} \in \bigcap_{i \in I} V(E_i)$$

and we have the converse inclusion. $\square$

4. Let $\mathfrak{a}, \mathfrak{b}$ be any ideals in R.

$V(\mathfrak{a} \cap \mathfrak{b}) \subseteq V(\mathfrak{a}\mathfrak{b})$ since $\mathfrak{a} \cap \mathfrak{b} \subseteq \mathfrak{a}\mathfrak{b}$. The inclusion $V(\mathfrak{a}\mathfrak{b}) \subseteq V(\mathfrak{a}) \cup V(\mathfrak{b})$ is clear as $V(\mathfrak{a}\mathfrak{b}) \subseteq V(\mathfrak{a})$.

Finally, $V(\mathfrak{a}) \cup V(\mathfrak{b}) \subseteq V(\mathfrak{a} \cap \mathfrak{b})$ as $V(\mathfrak{a}) \subseteq V(\mathfrak{a} \cap \mathfrak{b})$ a la Problem 3.

We see that this defines a topology on the set of prime ideals as the sets $V(E)$ are defined as closed sets in the topology as part 4 shows us that they are closed under finite unions. This topology is called the Zariski topology and the topological space X is the spectrum of the ring A denoted as $\mathrm{Spec}(A)$

Let us consider Spec($\mathbb{Z}$). The prime ideals are exactly the principal ideals (p) for p prime. Consider any finite set $E \subset \mathbb{Z}$. We see from the axioms of the Zariski topology on Spec($\mathbb{Z}$) that

$$V(E) = V(\bigcup_{n \in E} \{n\}) = \bigcap_{n \in E} V(n)$$

Let $n = p_1^{r_1} ... p_r^{r_n}$ be the prime decomposition of n then $(p_i) \in V(n)$ for $1 \leq i \leq r$. Thus, the intersection will contain all the prime divisors all the elements in E.

3.2 Problem 17

Question: For each $f \in A$, let X_f denote the complement of $V(f)$ in $X = \text{Spec}(f)$. The sets X_f are open. Show that they form a basis of open sets for the Zariski topology.

Proof. Consider any open set $\mathcal{O}$ of the Zariski topology on Spec(A). Taking its complement $\bar{\mathcal{O}}$ will a closed set which can be represented as $V(E)$ where $E = \cap_{P \in \mathcal{O}} P$. Now $E \subseteq A$, it can be decomposed into singletons $E = \cup_{s \in E}\{s\}$ and thus, $V(E) = V(\cup_{s \in E}\{s\}) = \cap_{s \in E} V(\{s\})$. Now taking it's complement yields us a union of open sets $\cup_{s \in E} \bar{V}(\{s\}) = \cup_{s \in E} X_s$. Hence, we see that every open sets can be expressed as a countable union of sets X_s as desired. $\square$

For $f, g \in A$

1. $X_f \cap X_g = \overline{V(f) \cup V(g)} = \overline{V(fg)} = X_{fg}$
2. $X_f = \emptyset \implies V(f) = X \implies f \in \mathfrak{R}$. The converse holds true as well.
3. Suppose $X_f = X \implies V(f) = \emptyset \implies f$ is a unit. As if f was a non-unit, f would be contained in some maximal ideal as hence in a prime ideal. The converse follows by reversing the argument.
4. Suppose $X_f = X_g \implies V(f) = V(g) \implies V((f)) = V((g)) \implies V(r((f))) = V(r((g)))$

3.3 Problem 18: Investigating the Closure of Sets

Question: For psychological reasons, it sometimes convienient to denote a prime ideal of A by a letter such as x or y when thinking of it as a point of $X = \mathrm{Spec}(A)$. When thinking of x as a prime ideal of A, we denote it by $\mathfrak{p}_x$. Show that

1. the set $\{x\}$ is closed if and only if $\mathfrak{p}_x$ is maximal.

   Proof. Suppose that $\{x\}$ is a closed set. Then $\{x\} = V(E) = V(\mathfrak{a})$ where $\mathfrak{a}$ is the ideal generated by subset $E \subset A$. It must follows that $\mathfrak{a}$ is a maximal ideal for if this was not true then there exists an element $r \in A$ such that $r + (a)$ would generate a proper ideal containing (a) which is in turn contained in some maximal ideal y of A. However, this would contradict the assumption that $\{x\} = V(\mathfrak{a})$. The converse follows by reversing the arguments. $\square$

2. $\overline{\{x\}} = V(\mathfrak{p}_x)$. This follows by examining the definition of closure in the Zariski topology. Suppose for the sake of contradiction, there exists a prime ideal such that $p \notin V(\mathfrak{p}_x)$, then there exists an open set $\mathcal{O}$, namely $\overline{V(\mathfrak{p})}$, such that $p \in \mathcal{O}$, but $\mathfrak{p}_x \notin \mathcal{O}$. Hence $p \notin \overline{\{x\}}$.

3. $y \in \overline{\{x\}}$ if and only if $\mathfrak{p}_x \subseteq \mathfrak{p}_y$.

   Proof. If $y \in \overline{\{x\}}$, then by part two, $y \in V(\mathfrak{p}_x)$, hence $\mathfrak{p}_x \subseteq \mathfrak{p}_y$. Conversely, if $\mathfrak{p}_x \subseteq \mathfrak{p}_y$. then $y \in V(\mathfrak{p}_x)$, and thus $y \in \overline{\{x\}}$ as part two dictates an equality. $\square$

4. X is a T_0 space.

   Proof. Let x, y be two distinct points in X. Let $a \in A$ be a element of the ring such that a only contained in either x or y. Consider the closed set $V((a))$. Without the loss of generality, suppose that $a \in x$, but $a \notin y$. then $x \notin \overline{V((a))}$, but $y \in \overline{V((a))}$, showing the T_0 property. $\square$

3.4 Problem 19: Irreducible Zariski Topology

Question: A topological space X is said to be irreducible if $X \neq \emptyset$ and if every pair of non-empty open sets in X intersect, or equivalently if every non-empty open set is dense in X. Show that $\text{Spec}(A)$ is irreducible if and only if the nilradical of A is a prime ideal.

Proof. We show the converse first. If the nilradical $\mathfrak{R}$ is a prime ideal, then for every $x \in X$, $\mathfrak{R} \subseteq x$. Hence, any of the basic open sets must contain $\mathfrak{R}$, giving us non-empty intersection for all open sets in the topology. Now suppose that $\text{Spec}(A)$ is irreducible. Let $x, y \in A$ be two common elements of the complement of three disjoint closed sets $V(E_1), V(E_2), V(E_3)$ in X such that $x, y \notin \mathfrak{R}$ If $xy \in \mathfrak{R}$, then $xy \subset V(E_1), V(E_2), V(E_3)$. However, this directly contradicts our definition of the open sets of the topology. $\square$