Problems in Lang's Algebra

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Lang Chapter 1 (Groups)

Problem 6

Question: Prove that the group of inner automorphisms of a group G are normal in $Aut(G)$.

Proof. Let $\mathcal{L} \in Aut(G)$ and $Inn(G)$ be the subgroup of inner automorphisms of G. It follows from our definitions that for any $y \in G$,

$$(\mathcal{L} \circ Inn(G) \circ \mathcal{L}^{-}1)(y) = (\mathcal{L} \circ Inn(G))(\mathcal{L}^{-}1(y)) = \mathcal{L}(x\mathcal{L}^{-1}(y)x^{-1}) = \mathcal{L}(x)y\mathcal{L}^{-1}(x) \in Inn(G)$$

Since $x \in G$ was arbitrary, we see that $\mathcal{L} \circ Inn(G) \circ \mathcal{L}^{-}1 \subseteq Inn(G)$. Hence, $Inn(G) \trianglelefteq Aut(G)$. $\square$

Problem 7

Question: Let G be a group such that $Aut(G)$ is cyclic. Prove that G is abelian.

Proof. By Proposition 4.2 (Lang 42), the group of inner of automorphisms $Inn(G)$ is also cyclic. Let $G \to Inn(G)$ be the surjective homomorphism $x \mapsto c_x$. The kernel of this map is the center of $Z(G)$ of the group. Hence, $G/Z(G) \cong Inn(G)$. It suffices to prove that if $G/Z(G)$ is abelian, then G must be abelian. We can simply take this in two cases. If we have $a, b \in G$, let's take the case where they both occupy the same coset. Thus, $ab^{-1} \in Z(G)$. We can use this for the following equivalences:

$$ab = a(b^{-1}b)b = (ab^{-1})bb = b(ab^{-1})b = ba$$

Hence, any two elements in the same coset must commute. As the factor group is cyclic, let x be a generator for $G/Z(G)$. Then given that $a, b \in G$ live in different cosets, a, b must occupy a coset with some power of x. Since the powers of x trivially commute and a, b commute with the generators, it follows that a, b must commute and the proposition follows. □

Problem 9

1. Question: Let G be a group and H a subgroup of finite index. Show that there exists a normal subgroup N of G contained in H and also of finite index.

   Proof. Let $(G : H) = n$. We can construct a homomorphism $\phi : G \to S_n$ by conjugating the coset representatives labeled up to n for all $g \in G$. It follows that $Ker(\phi) \subseteq H$. Let $N = Ker(\phi)$. Then by our isomorphism theorems, G/N is isomorphic to it's image which is finite and the proposition follows. □

2. Question: Let G be a group and let H_1, H_2 be subgroups of finite index. Prove that $H_1 \cap H_2$ has finite index.

   Proof. Consider the cosets of G/H_1. By assumption, the number of cosets $(G : H_1)$ is finite. Pick any coset C and consider how the elements partition into the cosets of $(G/(H_1 \cap H_2)$ by cases. Let $a, b \in C$ be any two elements of our coset. If $ab^{-1} \in H_2$, then it aligns with a coset of H_2. Otherwise, the two elements split into separate cosets in $G/(H_1 \cap H_2)$, namely a coset where $ab^{-1} \in H_2$ as well. However, the number of cosets it could move into is bounded by $(G : H_2)$. Thus, the number of unique cosets that could be generated by this method is bounded by $(G : H_1)(G : H_2)$ which is finite. Hence, $H_1 \cap H_2$ is of finite index. □

0.1 Problem 12 (Semidirect Product)

1. $x \mapsto \gamma_x$ induces a homomorphism $f : H \to Aut(N)$ since

$$f(yz) = x(yz)x^{-1} = (xyx^{-1})(xzx^{-1}) = f(x)f(y)$$

2. We first note that the kernel of the map $\phi : H \times N \to HN$ is trivial. Let $f(e, n_1) = e$, then $en = e$ and $n = e$. A symmetric argument can be done on the left. Since $H \cap N = \{e\}$, it follows that the kernel is trivial. Surjection follows immediately from the definition of the map. Hence, ϕ is a bijective map. To show that this map is an isomorphism of groups, it suffices to prove that ϕ is a homomorphism if and only if f is the trivial map. Starting with the converse, if f is the trivial map, then $hnh^{-1} = n$ for any $n \in N, h \in H$. Hence,

$$\phi(h_1h_2, n_1n_2) = (h_1h_2)(n_1n_2) = (h_1h_2)(n_1(h_2^{-1}h_2)n_2) =$$
$$h_1(h_2n_1h_2^{-1})h_2n_2 = (h_1n_1)(h_2n_2) - \phi(h_1, n_1)\phi(h_2, n_2)$$

Now if ϕ is a homomorphism, then by our observation above,

$$(h_1h_2)(n_1n_2) = (h_1n_1)(h_2n_2)$$

Taking the proper left and right inverses yield the following:

$$h_2n_1h_2^{-1} = n_1$$

By taking h_2, n_1 to be any element in H, N respectively, we conclude that f is indeed trivial.

3. Let N, H be subgroups. Define $\psi : N \to Aut(H)$ to be our above homomorphism. We will construct the semidirect product as follows: Let $x \in N, h \in H$ and construct the set made of elements of the form (x, h). Define the composition law as follows:

$$(x_1, h_1)(x_2, h_2) = (x_1\psi(h_1)x_2, h_1h_2)$$

We will first show that this indeed follows the group laws. Associativity easily follows from the underlying property of groups N, H. The identity exists from $e_N, e_H, \psi(e_N) = id_H$. The inverse also exists since

$$(x, h)(\psi^{-1}(h)x^{-1}, h^{-1}) = (e_N, e_H)$$

Hence, the set is a group. We will now show that this yields a semidirect product on N and H by associating N with $(n, 1)$ and H with $(1, h)$.

We immediately note that $N \cap H = \{e\}$. Thus, it suffices to prove that NH is isomorphic to our group through a map $\theta : NH \to G, nh \mapsto (n, h)$. First, we show that θ is indeed a homomorphism.

Note: We can extend this definition for H be the normalizer of any subgroup N. Then the map f would be the homomorphism from $Norm(N) \to Aut(N)$.

Note: We note why the map f above is so significant. What's stopping the map $H \times N \to HN$ is precisely the conjugation issue ($hnh^{-1} = n$) for all $n \in N, h \in H$.

Problem 17

Question: Let X, Y be finite sets and let $C \subseteq X \times Y$. For $x \in X$, let $\phi(x) = |\{y \in Y | (x, y) \in C\}|$. Verify that $|C| = \sum_{x \in X} \phi(x)$

Proof. Let $|X| = n_1$ and $|Y| = n_2$. □

Problem 18

Proof. We use Problem 17 by taking our set to be $S \times T$ and our subset C to be the entirety of $S \times T$. The result follows from the finite sum shown above. □

Problem 19

1. *Proof.* This follows by weighting the cardinality of orbit G_s for each element. □
2. *Proof.* By Lagrange's Theorem, we know that $|G : G_s| = |G|/|G_s|$ where G_s is the

stabilizer of $s \in S$. By the part above:

$$\sum_{t \in Gs} \frac{1}{|Gt|} = \frac{1}{|G|} \sum_{t \in Gs} |G_t| = 1$$

Hence, we can add this sum over all coset representatives S_r:

$$\frac{1}{|G|} \sum_{s_r \in \mathcal{O}} \sum_{t \in Gs_r} |G_t| = |\{\mathcal{O}_i\}|$$

where $\{\mathcal{O}_i\}$ is the set of orbits of G in S. Finally, we note that the following sums are equivalent:

$$\sum_{s \in Gs} |Gs| = \sum_{g \in G} f(x)$$

where $f(x)$ is the number of fixed points $x \in G$ exhibits. Substituting the sum yields our desired result:

$$\frac{1}{|G|} \sum_{g \in G} f(x) = |\{\mathcal{O}_i\}|$$

$\square$