Problems in Category Theory (Maclane)

Edward Kim

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1 Chapter 1

1.1 Section 4

Question 1.1. Question: Let S be a fixed set and X^S be the set of all functions $f: S \to X$. Show that the map $X \mapsto X^S$ is the object functor $\mathbf{Set} \to \mathbf{Set}$, and that the evaluation function $e_x: X^S \times S \to X$ defined by $e_x(h,s) = h(s)$ is a natural transformation.

Proof. Let $X \mapsto X^S$ be the object functor of the functor \mathcal{F} equipped with a morphism functor defined as follows: Given a morphism between two sets $f: X \to Y$, $\mathcal{F}(f): X^S \to Y^S$ such that $\mathcal{F}(g_x) = f \circ g_x$ for $g_x \in X^S$. To see that e_x is a natural transformation: consider the functor $\mathcal{F} \times id : \langle X^S, S \rangle \to \langle Y^S, S \rangle$. Taking the evaluation mapping $e_x: X^S, S \to X$ yields one element set from X_{θ} : $\{h_X(s)\}$. Similarly for $e_Y: \langle Y^S, S \rangle \to Y$, the evaluation function yields $Y_{\theta} = \{h_Y(s)\}$. Define the map between $f_{\theta}: X_{\theta} \to Y_{\theta}$ by $f_{\theta}(h_X(s)) = h_Y(s)$ By our definition above, the functor \mathcal{F} respects the behavior of such a morphism $f: X \to Y$. Thus, we can define a functor \mathcal{H} as above, yielding our natural transformation. \square

Question 1.2. If H is a fixed group, show that $G \to H \times G$ defines a functor $H \times ()$: $\mathbf{Grp} \to \mathbf{Grp}$ and that each morphism $f: H \to K$ defines a natural transformation such that $H \times () \to G \times ()$.

Proof. We can define a functor \mathcal{F} first by the natural object mapping from $G \mapsto H \times G$ and the morphism mapping for $g: G \to R$: $\mathcal{F}(f): H \times G \to H \times R$. The natural transformation easily $H \times () \to G \times ()$ follows as such:

Given two groups G_1, G_2 let $f_1: H \times G_1 \to H \times G_2$ and let $f_2: G \times G_1 \to G \times G_2$. The natural transformation would be the morphism that takes $h: H \to G$ as this makes $h \circ f_1 = f_2 \circ h$.

Question 1.3. For functors $S,T:C\to P$ where C is a category and P is a preorder, show that there is a natural transformation $S\to T$ iff $S\le T$ for every object in C.

Proof. If there is a natural transformation from $S \to T$, then the transformation must respect the morphims (which are the orderings) of the preorder P. Thus, according to P, $Sc \leq Tc$. Conversely, if such an ordering existed for every object $c \in C$, then we can define

the natural transformation $S \to T$ as follows: Let $f_s : Sc \to Sc'$ and $f_c : Tc \to Tc'$, then define $\theta : S \to T$ as $\theta(Tc) = Sc$ and $\theta(Sc') = Tc'$. This map is well-defined as there can be only one morphism between two objects as defined by the preorder.

1.2 Section 5

Question 1.4. Find a category with an arrow which is both epi and monic, but not invertible.

Proof. Consider the category of topological spaces and the arrow $\mathbb{Q} \xrightarrow{i} \mathbb{R}$. If Y is a topological space such that $Y \rightrightarrows \mathbb{Q} \to \mathbb{R}$ and both composite arrows are equal to each other, it follows that the parallel arrows $Y \rightrightarrows \mathbb{Q}$ are equal to each other, showing that the inclusion arrow is monic. To show that it is an epimorphism, consider $\mathbb{Q} \xrightarrow{i} \mathbb{R} \rightrightarrows X$ for some topological space X such that the composite arrows are equal. By the density of \mathbb{Q} in \mathbb{R} , the two parallel arrows will converge by taking the rationals to converge to every point in \mathbb{R} as a Cauchy sequence. Thus, the two parallel continuous functions must be equal. However, we immediately see that the inclusion function is not invertible.

Question 1.5. Prove that the composite of monics is monic and likewise for epis

Proof. Let ϕ , θ be monics in a category \mathcal{C} . Suppose that for maps $\xi_1, \xi_2, \theta \phi \xi_1 = \theta \phi \xi_2$. Then by associativity:

$$\theta(\phi\xi_1) = \theta(\phi\xi_2) \implies \theta\xi_1 = \theta\xi_2 \implies \xi_1 = \xi_2$$

A similar procedure will determine the same property for epis.

Question 1.6. An arrow $f: a \to b$ in a category \mathbb{C} is regular when there exists an arrow $g: b \to a$ such that fgf = f. Show that f is regular if it has a left or right inverse, and prove that every arrow in **Set** with $a \neq \emptyset$ is regular.

Proof. Suppose that the arrow $f: a \to b$ as a right inverse $r: b \to a$ such that $fr = id_b$. Taking r be our regular arrow yields (fr)f = f. Similarly, for a left inverse $l: b \to a$ f(lf) = f. Now suppose that $\phi: X \to Y$ is a morphism in **Set**. Simply taking the coarsest inverse ϕ^{-1} will suffice. If the function is not injective such that there exists $x, y \in X$ such that $\phi(x) = \phi(y) = c \in T$. Then take $\phi^{-1}(c) = x$ or $\phi^{-1}(c) = y$.

Question 1.7. Consider the category with objects $\langle X, e, t \rangle$, where X is a set, $e \in X$, and $t: X \to X$ and with morphisms $f: \langle X, e, t \rangle \to \langle X', e', t' \rangle$ the functions $f: X \to X'$ with fe = e' and ft = t'f. Prove that this category has an initial object in which X is the set of natural numbers, e = 0 and t is the successor function.

Proof. Given object $\langle X, e, t \rangle \in \mathcal{C}$, we will define an unique morphism f_i from $\langle \mathbb{N}, 0, s \rangle$ where $s : \mathbb{N} \to \mathbb{N}$ is the successor function. Define $f_i : \mathbb{N} \to X$ by first mapping $f_i(0) = e$ and $f_i(s^r0) = t^re$. It suffices to show that $tf_i = f_is$ By our construction above, this diagram must commute as given any $n \in \mathbb{N}$, $n = s^n(0)$ so $tf_i(n) = t(s^n(0)) = t \cdot t^n e = t^{n+1}e = f_i(s^{n+1}(0)) = f_i(s(n))$. As f_i defined above is injective, the morphism is unique for every object in the category \mathcal{C} .

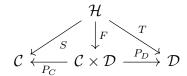
2 Chapter 2

2.1 Section 3

Question 2.1. Show that the product of categories includes the following known special cases: The product of monoids (categories with one object), of groups, of sets (discrete categories).

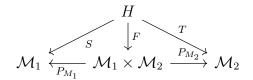
Proof. Recall that we can construct a product category as follows: Consider two categories \mathcal{C}, \mathcal{D} . Define the product category $\mathcal{C} \times \mathcal{D}$ with objects as ordered pairs (c, d) where $c \in \mathcal{C}$ and $d \in \mathcal{D}$ and arrows $(c, d) \to (c', d')$ iff there exists arrows $c \to c'$ and $d \to d'$. We can naturally define projection functors $P_C : \mathcal{C} \times \mathcal{D} \to \mathcal{C}$ and $P_D : \mathcal{C} \times \mathcal{D} \to \mathcal{D}$.

. The product category must have the following universal property: given two functors from a category \mathcal{H} defined as $S: \mathcal{H} \to \mathcal{C}$ and $T: \mathcal{H} \to \mathcal{D}$, there exists a unique functor $F: \mathcal{H} \to \mathcal{C} \times \mathcal{D}$ such that the following diagram commutes:



Similarly, we can define functors between product categories as $U \times V : \mathcal{C} \times \mathcal{D} \to \mathcal{C}' \times \mathcal{D}'$ for functors $U : \mathcal{C} \to \mathcal{D}$ and $V : \mathcal{C}' \to \mathcal{D}'$

The product of categories will necessarily include the product of monoids as two functors $S, T: \mathbf{Mon} \to \mathbf{Mon}$ map monoids to monoids such that for any two monoids $\mathcal{M}_1, \mathcal{M}_2 \in \mathbf{Mon}$ and monoid homomorphisms(functors between monoids) $S: H \to \mathcal{M}_1$ and $T: H \to \mathcal{M}_2$ The following diagram commutes with a unique functor: $F: H \to \mathcal{M}_1 \times \mathcal{M}_2$



Question 2.2. Show that the product of two preorders is a preorder.

Proof. Recall that a preorder is a category \mathcal{C} such that for any two objects p, p' there exists at most one arrow $p \to p'$ We can define the product preorder category **Preord** \times **Preord** with ordered tuples of objects (e,d) for objects $e,d \in \mathcal{C}, D$ and we can define arrows between $(e,d) \to (e',d')$ iff there exists $e \to e'$ and $d \to d'$. As there can only be one arrow between objects in their respective categories, the arrows in the product category are uniquely determined. Hence, the product category must also be a preorder.

2.2 Section 4

Question 2.3. If Fin is the category of all finite sets and G is a finite group, describe Fin^G