COHOMOLOGY OF CONNECTED COMPACT LIE GROUPS

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CONTENTS

1. DE RHAM COHOMOLOGY

We rapidly review some definitions and concepts from de Rham cohomology before introducing the main topics. Recall that for a n-dimensional smooth manifold M, we define $\Omega^p(M) = \Gamma(M, \bigwedge^p(T^*M))$ to be the p^{th} *differential forms* on M i.e smooth sections of the p-alternating tensor bundle. The direct sum of these vector spaces:

$$\Omega^*(M) = \bigoplus_{p=0}^{n} \Omega^p(M)$$

along with the wedge product $\wedge : \Omega^n(M) \times \Omega^m(M) \to \Omega^{n+m}(M)$ turns $\Omega^*(M)$ into an associative, anticommutative graded $C^\infty(M)$-algebra. Furthermore, there exists an *exterior derivative*: $d : \Omega^n(M) \to \Omega^{n+1}(M)$ defined as follows on basis forms:

$$d(f dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n) = df \wedge dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$$

We will frequently use the global formula for the exterior derivative on $\Omega^p(M)$:

$$d\omega(X_1, \cdots, X_{p+1}) = \sum_{i=1}^{p+1} (-1)^i X_i \omega(X_1, \cdots, \widehat{X_i}, \cdots, X_{p+1}) \tag{1.1}$$

$$+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \cdots, \widehat{X_i}, \cdots, \widehat{X_j}, \cdots, X_{p+1}) \tag{1.2}$$

where $X_1, \cdots, X_{p+1}$ are smooth vector fields on M. Now it turns out that $d \circ d = 0$, defining the following cochain complex:

Definition 1.1. *The* p^{th} de Rham cohomology group *is defined to be the quotient groups*

$$H^p(M) = Z^p(M)/B^p(d : \Omega^{p-1}(M) \to \Omega^p) \tag{1.3}$$

where

$$Z^p(M) = \textit{Ker}\left(d : \Omega^p(M) \to \Omega^{p+1}\right)$$
$$B^p(M) = \textit{Im}\left(d : \Omega^p(M) \to \Omega^{p+1}\right)$$

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For $\omega \in \Omega^p(M)$, recall that ω is called *closed* if $d\omega = 0$ and is called *exact* if there exists $\eta \in \Omega^{p-1}(M)$ such that $\omega = d\eta$. Thus, $Z^p(M)$ denotes the closed p-forms on M while $B^p(M)$ denotes the exact p-forms on M. The following proposition shows that smooth maps induce homomorphisms between the cohomology groups via their pullbacks:

Proposition 1.1.1. *Let $F : M \to N$ be a smooth map. The pullback $F^* : \Omega^p(N) \to \Omega^p(M)$ takes forms from $Z^p(N)$ into $Z^p(M)$ and $B^p(N)$ into $B^p(M)$, inducing a homomorphism between cohomology groups $H^p(N) \to H^p(M)$.*

Proof. If $\omega \in \Omega^p(N)$ is closed, then $dF^*\omega = F^*d\omega = 0$ since the exterior derivative commutes with pullbacks . Similarly, if $\omega = d\eta$ for some $\eta \in \Omega^{p-1}(N)$, then $F^*\omega = F^*d\eta = dF^*\eta$. This shows the proposition with the induced cohomology map being defined as:

$$F^*[\omega] = [F^*\omega], \quad \omega \in \Omega^p(N)$$

□

This further shows that diffeomorphisms between manifolds induce isomorphisms between their respective cohomology groups. For more information about background topics, see [3].

2. Left-invariant Forms

We base a good bit of our presentation on the articles by Fok and Zhang ([2],[5]). The discussions on invariant forms are based on the treatment in [1]. The original paper detailing these arguments is [4].

We begin by deeming G to be a connected compact Lie group. First, we start with the definition of a left-invariant form

Definition 2.1. *Let $\omega \in \Omega^p(G)$. Then ω is said to be* left-invariant *if*

$$L_g^*\omega = \omega, \quad g \in G \tag{2.1}$$

We denote $\Omega_L^p(G)$ to be the subspace of left-invariant p-forms on G.

Lemma 2.2. *Let $\omega \in \Omega_L^p(G)$. Then $d\omega \in \Omega_L^{n+1}$*

Proof. For all $g \in G$:

$$L_g^*d\omega = dL^*\omega = d\omega$$

□

This shows that $\Omega_L^*(G)$ form a cochain complex under d. Define $H_L^p(G)$ be cohomology of $\Omega_L^p(G)$. Furthermore, note that since pullbacks distribute over the wedge product:

$$L_g^*(\omega_1 \wedge \omega_2) = L_g^*\omega_1 \wedge L_g^*\omega_2 = \omega_1 \wedge \omega_2$$

This shows that when $\omega_1 \in \Omega_L^n(G)$ and $\omega_2 \in \Omega_L^m(G)$, then $\omega_1 \wedge \omega_2 \in \Omega_L^{n+m}(G)$. In fact, the wedge product becomes a well-defined map: $\cup : H_L^n(G) \times H_L^m(G) \to H_L^{n+m}(G)$ given by

$$([\omega_1], [\omega_2]) \mapsto [\omega_1 \wedge \omega_2]$$

This endows a ring structure on $H_L^*(G)$, and an identical argument shows that $H^*(G)$ is also a ring. Now as subspaces, there exists an inclusion map $i : \Omega_L^p(G) \hookrightarrow \Omega^p(G)$ which induces a ring homomorphism between $i' : H_L^*(G) \to H^*(G)$.

Theorem 2.3. *i' is a ring isomorphism*

Proof. To show surjectivity, given a form $\omega \in \Omega^p(G)$, there is an invariant form $\omega' \in \Omega_L^p(G)$ defined as

$$\omega' = \int_G L_g^*\omega \, dg \tag{2.2}$$

Recall that a similar trick is used to create an invariant inner product from an existing inner product on some representation space. We summarize the proof of [1]. The main goal is to show that ω', ω lie in the same cohomology class.

Recall the de Rham theorem (see [3], Theorem 18.14) stating that $H^p(G)$ is isomorphic to $S^p(M, \mathbb{R})$ where $S^p(M, \mathbb{R})$ refers to the singular cohomology groups of G. This isomorphism is defined by the de Rahm homomorphism $d_\ell : H^p(M) \to S^p(M, \mathbb{R})$:

$$d_\ell[\omega][c] = \sum_i r_i \int_{Z_i} \omega$$

where $c = \sum_i r_i Z_i$ is a linear combination of p-simplexes on G. If we can show that

$$\int_Z \omega - \omega' = 0$$

for all p-simplexes Z, the de Rham isomorphism will force $[\omega - \omega'] = 0$, yielding surjectivity. Since G is connected, exp is surjective which means that we find a continuous curve connecting the identity e to any $g \in G$. The curve along with its corresponding left-invariant vector field shows that it is possible to continuously deform the identity map Id_G to the left-translation map G, proving the two maps are homotopic. By homotopy-invariance of singular cohomology ([3], Prop 18.5), the induced cohomology map $(L_g)_* : S^p(G, \mathbb{R}) \to S^p(G, \mathbb{R})$ must simply be identity. To state this explicitly, $[gZ] = [Z]$ for all $g \in G$. This just leaves a calculation:

$$\begin{aligned}
\int_Z \omega - \omega' &= \int_Z \omega - \int_Z \int_G L_g^* \omega dg \\
&= \int_Z \omega - \int_G \int_Z L_g^* \omega dg \\
&= \int_Z \omega - \int_G \int_{gZ} \omega dg \\
&= \int_Z \omega - \int_G \int_Z \omega dg \\
&= \int_Z \omega - \int_Z \omega \\
&= 0
\end{aligned}$$

The second equality follows from Fubini's theorem. The third equality follows from the definition of the $L_g^* \omega$. The second-to-last equality is just $\int_G dg = 1$.

To show injectivity: let $\omega \in \Omega_L^p(G)$ such that $\omega = d\eta$ for some $\eta \in \Omega^{p-1}(G)$. Then

$$\omega = \omega' = \int_G L_g^* \omega \; dg = \int_G L_g^* d\eta \; dg = d \int_G L_g^* \eta \; dg = d\eta'$$

The first equality is by left-invariance of ω. Then ω must be an exact in respect to cochain complex $\Omega_L^*(G)$, showing injectivity. □

Theorem 2.3 states that the classes of the de Rham complex of G, $H^p(G)$ can be represented through its left-invariant counterparts $H_L^p(G)$. The additional structure placed on the left-invariant forms on G give rise to an isomorphism between alternating forms on the Lie algebra of G, $\mathfrak{g}$ and the left-invariant forms

Lemma 2.4. *Define*

$$\psi : \Omega_L^*(G) \to \bigwedge^* \mathfrak{g}^* \tag{2.3}$$

$$\omega \mapsto \{\omega\} := \omega|_{\wedge^n T_e G} \tag{2.4}$$

or explicitly: $\{\omega\}(X_1, \cdots X_n) = \omega_e((X_1)_e, \cdots, (X_n)_e)$. *The map* ψ *will be a ring isomorphism.*

Proof. Showing that ψ is a ring homomorphism just involves reasoning about the wedge product. To show surjectivity, take some $\mu \in \bigwedge^n \mathfrak{g}^*$ and define the p-form

$$\omega_g((X_1)_g, \cdots, (X_n)_g) = \mu((L_{g^{-1}})_*(X_1)_g, \cdots, (L_{g^{-1}})_*(X_n)_g)$$

where $(L_{g^{-1}})_*(X_i)$ is the pushforward vector field of X_i in respect to the diffeomorphism $L_{g^{-1}}$. To check that $\omega \in \Omega_L^n(G)$:

$$\begin{aligned}
(L_g^*\omega)_h((X_1)_h, \cdots, (X_n)_h) &= \omega_{gh}((L_g)_*(X_1)_h, \cdots, (L_g)_*(X_n)_h) \\
&= \mu((L_{h^{-1}g^{-1}}L_g)_*(X_1)_h, \cdots (L_{h^{-1}g^{-1}}L_g)_*(X_n)_h) \\
&= \mu((L_{h^{-1}})_*(X_1)_h), \cdots, (L_{h^{-1}})_*(X_n)_h) \\
&= \omega_h((X_1)_h, \cdots, (X_n)_h)
\end{aligned}$$

so $L_g^*\omega = \omega$ as desired. As for injectivity, suppose we had $\alpha \in \Omega_L^n(G)$ such that $\{\alpha\} = 0$. By a similar line of thinking as above:

$$\begin{aligned}
\alpha_h((X_1)_h, \cdots, (X_n)_h) &= (L_h^*\alpha)_e((L_{h^{-1}})_*(X_1)_h, \cdots, (L_{h^{-1}})_*(X_n)_h) \\
&= \alpha_e((L_{h^{-1}})_*(X_1)_h, \cdots, (L_{h^{-1}})_*(X_n)_h) \\
&= 0
\end{aligned}$$

□

There is actually a stronger property based on this ring isomorphism ψ. We will now show that there exists a homomorphism $\delta : \bigwedge^n \mathfrak{g}^* \to \bigwedge^{n+1} \mathfrak{g}^*$ which commutes with the ψ, turning $\bigwedge^* \mathfrak{g}^*$ into a cochain complex.

Lemma 2.5. *There exist a linear map* δ *which makes the below diagram commute:*

$$\begin{tikzcd}
\Omega_L^n \arrow[r, "d"] \arrow[d, "\psi"] & \Omega_L^{n+1} \arrow[d, "\psi"] \\
\bigwedge^n \mathfrak{g}^* \arrow[r, "\delta"] & \bigwedge^{n+1} \mathfrak{g}^*
\end{tikzcd}$$

with the explicit form

$$\delta\omega(\xi_1, \cdots, \xi_{n+1}) = \sum_{i<j} \omega([\xi_i, \xi_j], \xi_1, \cdots, \widehat{\xi_i}, \cdots, \widehat{\xi_j}, \cdots, \xi_{n+1}) \tag{2.5}$$

Proof. Let us first take the $\xi_1, \cdots, \xi_{n+1}$ which are tangent vectors living in T_eG and extend them to left-invariant vector fields on G via a pushforward, $(X_i)_g = (L_g)_*(\xi_i)$. Expand $\psi \circ d$ as follows:

$$\begin{aligned}
(\psi d\omega)(\xi_1, \cdots, \xi_{n+1}) &= d\omega(X_1, \cdots, X_{n+1}) \\
&= \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \cdots, X_i, \cdots, \widehat{X_i}, \cdots, \widehat{X_j}, \cdots, X_{n+1}) \\
&= \sum_{i<j} (-1)^{i+j} \{\omega\}([(X_i)_e, (X_j)_e], (X_1)_e, \cdots, (X_i)_e, \cdots, \widehat{X_i}, \cdots, \widehat{X_j}, \cdots, (X_{n+1})_e) \\
&= (\delta\psi\omega)(\xi_1, \cdots, \xi_{n+1})
\end{aligned}$$

The first equality follows since the restriction is $(X_i)_e = \xi$ by construction. The second equality arises from the left-invariance of ω and the X_i. Recall the global formula in Equation 1.1 and the terms:

$$\begin{aligned} X_i\omega(X_1, \cdots, \widehat{X_i}, \cdots, X_{n+1}) &= \mathcal{L}_{X_i}(\omega(X_1, \cdots, \widehat{X_i}, \cdots, X_{n+1}) \\ &= (\mathcal{L}_{X_i})(\omega(X_1, \cdots, \widehat{X_i}, \cdots, X_{n+1}) + \sum_{j=0, j\neq i} \omega(X_1, \cdots [X_i, X_j], \cdots X_{n+1}) \\ &= \sum_{j=0, j\neq i} \omega(X_1, \cdots, \widehat{X_i}, \cdots, [X_i, X_j], \cdots X_{n+1}) \end{aligned}$$

since $\mathcal{L}_{X_i}\omega = 0$ and $\mathcal{L}_{X_i}X_i = [X_i, X_i] = 0$. Taking the total sum reveals that:

$$\sum_{i=1}^{n+1} \sum_{j=1, j\neq i} \omega(X_1, \cdots, \widehat{X_i}, \cdots, [X_i, X_j], \cdots X_{n+1}) = 0 \tag{2.6}$$

The third equality is once again just from the restrictions of the X_i and the definition of ψ. $\square$

The lemma shows that the $\bigwedge^* \mathfrak{g}^*$ under the δ map is a cochain complex with ψ inducing a cochain complex isomorphism.

Theorem 2.6. $H^*_L(G) \cong H^*(\bigwedge^* \mathfrak{g}^*, \delta)$

$H^*(\mathfrak{g}) = H^*(\bigwedge^* \mathfrak{g}^*, \delta)$ is called the *Lie algebra cohomology* of $\mathfrak{g}$. We have shown that for connected compact groups, its Lie algabra cohomology isomorphic to its left-invariant cohomology groups.

3. Bi-invariant Forms

We can extend these results to the so-called *Bi-invariant classes* of G

Definition 3.1. *Let $\omega \in \Omega^n(G)$. ω is bi-invariant if $\omega \in \Omega^n_B(G) = \Omega^n_L(G) \cap \Omega^n_R(G)$. That is:*

$$\begin{aligned} L_g^*\omega = \omega \\ R_g^*\omega = \omega \end{aligned}$$

for all $g \in G$

The use of the average to discover bi-invariant cohomology representatives in $[\omega]$ also apply:

$$\omega' = \int_G \int_G L_g^* R_g^* \omega \; dgdg \tag{3.1}$$

A similar argument to Theorem 2.3 yields that the inclusion of subspaces $\Omega^n_B(G) \hookrightarrow \Omega^n(G)$ is also a ring isomorphism

Theorem 3.2. $H^*_B(G) \cong H^*(G)$ *as rings*

Define $(\bigwedge^* \mathfrak{g}^*)^G$ to be the subspace of all alternating tensors invariant under Ad_g action with:

$$\begin{aligned} \rho : & \Omega^*_B(G) \to \left(\bigwedge^* \mathfrak{g}^*\right)^G \\ & \omega \mapsto \{\omega\} \end{aligned}$$

where $\{\omega\}$ refers to the restriction of ω to $\mathfrak{g}$. As before, ρ is a ring isomorphism.

Proposition 3.2.1. *ρ is a ring isomorphism*

Proof. Showing ρ is a ring homomorphism follows by reasoning on the restriction of wedge products of biinvariant forms. As pullbacks distribute over the wedge product, products of bi-invariant forms restrict to Ad_g-invariant alternating forms on $\mathfrak{g}$. To show surjectivity, we define an identical form to the one shown in Lemma 2.4, namely for $\mu \in \left(\bigwedge^n \mathfrak{g}^*\right)^G$,

$$\omega_h((X_1)_h, \cdots, (X_n)_h) = \mu(L_{h^{-1}}(X_1)_h, \cdots, L_{h^{-1}}(X_n)_h)$$

Lemma 2.4 gave us left-invariance. To show right-invariance:

$$\begin{aligned}
(R_g^*\omega)_h((X_1)_h, \cdots, (X_n)_h) &= \omega_{hg}((R_g)_*(X_1)_h, \cdots, (R_g)_*(X_n)_h) \\
&= \mu((L_{g^{-1}h^{-1}}R_g)_*(X_1)_h, \cdots, (L_{g^{-1}h^{-1}}R_g)_*(X_n)_h) \\
&= \mu((L_{g^{-1}}R_gL_{h^{-1}})_*(X_1)_h, \cdots, (L_{g^{-1}}R_gL_{h^{-1}})_*(X_n)_h) \\
&= \mu(Ad_g(L_{h^{-1}})_*(X_1)_h, \cdots, Ad_g(L_{h^{-1}})_*(X_n)_h) \\
&= \mu((L_{h^{-1}})_*(X_1)_h, \cdots, (L_{h^{-1}})_*(X_n)_h) \\
&= \omega_h((X_1)_h, \cdots, (X_n)_h)
\end{aligned}$$

The second-to-last equality is by Ad_g-invariance of μ, and the last equality is by left-invariance of ω. $\square$

Lemma 3.3. *There exist a linear map δ which makes the below diagram commute:*

$$\begin{tikzcd}
\Omega_B^n \arrow[r, "d"] \arrow[d, "\rho"] & \Omega_B^{n+1} \arrow[d, "\rho"] \\
\left(\bigwedge^n \mathfrak{g}^*\right)^G \arrow[r, "\delta"] & \left(\bigwedge^{n+1} \mathfrak{g}^*\right)^G
\end{tikzcd}$$

with the same explicit form.

Theorem 3.4. $H_B^*(G) \cong H^*(\left(\bigwedge^* \mathfrak{g}^*\right)^G, \delta)$

References

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