The Hopkins-Levitzki Theorm: A Primer on Artinian Rings

Edward Kim

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Abstract

of investigate

In this paper, we shall develop some theory ~~behind the~~ Artianian Rings and ~~invest~~ some intriguing properties that they exhibit. The final goal of this paper is to introduce the Hopkins-Levitzki theorm for modules (or the Akizuki-Hopkins-Levitzki theorem). The theorm gives great insight into the relationship between the ascending chain condition of modules and descending chain condition of modules.

Introduction:

(A small historical note of some sort might be approporiate here, but not needed)

The Hopkins-Levitzki Theorm states the following:

Like when the result was first proved, any motivations of the authors, etc. It isn't important if you don't have time though.

For any right Artinian ring R and any right R-module M the following are equivalent:

1. M is Artinian.
2. M is Noetherian.
3. M has a composition series.
4. M is finitely generated.

(which are assoicated Artinian modules)

(which are associated to Noetherian rings) maybe explain a bit more

for modules over an Artinian ring.

This theorm connects the property of ascending chains and descending chains in that it states that one implies the other. This paper will attempt to develop the barebones theory needed to give a simple proof of the theorm. We shall first begin with some basic module theory and slowly introduce the concept of chain conditions, Artianian rings, and the semisimple property of rings and modules. Some technical proofs are simplified or omitted, but they can be found in any communtative algebra book such as COHN.

\cite{Cohn} or something

You should have a section called References (i.e. a bibliography) at the bottom.

Section 1: Some Module Theory

In this section, we shall develop the minimum module theory necessary for our journey of exploring Artinian Rings.

0.1 Definitions:

Given a ring R, we define a **left R-module** by the following: let M be an abelian group $(M, +)$ and let $\cdot : R \times M \to M$ such that following properties hold for all $r, s \in R$ and $a, b \in M$:

be

$$r \cdot (a + b) = r \cdot a + r \cdot b \tag{1}$$

$$(r + s) \cdot a = r \cdot a + s \cdot a \tag{2}$$

$$(r \cdot s) \cdot a = r \cdot (s \cdot a) \tag{3}$$

something like

$$1_R \cdot a = a \tag{4}$$

A **right R-module** is defined similarly with the map $\cdot : M \times R \to M$, ~~The corercesponding axioms also carry over~~. and taking the corersponding axioms above with the multiplication operation being applied on the right rather than the left.

Using these definitons, we can define a **submodule** as a subgroup of M closed under the operations of R.

Let us note that the ring R is itself a left and right R-module. If we replace the abelian group M above with R, we see that we get the original $(\cdot)$ ~~axioms~~ of a ring.

multiplication operation

Thus, we can understand the ideals of a ring by encapsulating them in terms of modules! Let R be a left R-module. If I is a submodule such that $RI \subseteq I$, the I is called a **left ideal**. Similarly, if we assume that R is a right R-module. Then we can define a **right ideal** as a submodule of R such that $IR \subseteq I$. A **two-sided ideal** is defined as an ideal that exhibits both properties. Throughout this paper, we shall be ~~handling~~ certain properties of Artianian rings by analyzing them by their modules.

studying? (maybe a different word)

We also denote a module M as **finitely generated** if there exists a finite subset which generates M.

2. Chain Conditions

the

to be

We define a chain of submodules C_i in module M ~~as a~~ sequences ~~of the form~~.

- reword?

having either the form

$$C_0 \subseteq C_1 \subseteq \ldots \subseteq C_n, \tag{5}$$

or the form

$$C_0 \supseteq C_1 \supseteq \ldots \supseteq C_n. \tag{6}$$

These chains are named as ascending chains and descending chains, respectively. If all ascending chains in a module can be shown to be finite, we shall call this module **Noetherian**. Similarly, if all descending chains are shown to be finite, the module is deemed **Artinian**. However, shouldn't we be talking about Artianian rings rather than modules? Later, we shall see that if a ring R is Artianian, then any finitely-generated R-module is also Artianian.

We shall now define some terms necessary to our understanding of special types of chains that may appear. We say that a module is **simple** if it is non-zero but has no non-zero proper submodules. We can think of simple modules as modules that cannot be "stripped" down any more. In other words, there is no smaller module contained in it. Now let $C_0 \subseteq C_1 \subseteq \ldots \subseteq C_n$ be a finite chain in the module M. We say ~~that a finite chain is~~ a **composition series** if and only if $M = \bigcup_{0 \leq i \leq n} C_i$ and C_i/C_{i-1}, for every i is a simple module. We can visualize a composition series as the "longest" chain in a sense. In other words, we cannot add more links to this chain since

swap these maybe

the

there are no more submodules in between these chains to “pull out”. The number of these factors, C_i, in the composition chain is called length. That being said, if we can add a link to our chain by “pulling out” a submodule, we obtain a **refinement**.

maybe make this more precise? This is good, deffinitely leave this in, but maybe for refinement you could also add another sentence with a more formal definition as well, just to give an idea of how the pulling out works precisely....particularly since you will be using it in proofs....

Two natural question to arise from the previous definitions:

1. Can every chain in a module M be extended to be a compostion series?
2. Will any two composition series in M have the same length?

Good! The motivating questions are nice :)

We will develop the necessary theorms to answer these questions in the following section.

3. Composition Series

In this section, I will sketch the proof rather than giving the entire proof for some of the theorms for the sake of brevity. The importance of the theorms stems mainly from their results which will be extensively used to understand Artinian rings.

the

First, we must create some tools to tackle the two questions that were presented in previous section. Let us turn our attention to the second one. It turns out that the **Schreier refinement theorm** answers this question for us:

numbered theorems? i.e. Theorem 1 (Schreier refinement Theorem):

Theorm: In a module M, any two submodules have isomorphic refinements.

Proof Sketch:

The idea is to prove that given any two finite chains in the module, we can find an isomorphism between their refinements. In other words, given two chains of the form:

, which will be specified by a permutation

$$M = R_0 \supseteq R_1 \supseteq ... \supseteq R_f = 0 \tag{7}$$

$$M = S_0 \supseteq S_1 \supseteq ... \supseteq S_g = 0 \tag{8}$$

we would then show that we

what objects does the permutation \sigma map between?

~~We~~ we can find some permutation mapping σ such that $C_{i-1}/C_i \cong D_{\sigma(i-1)}/D_{\sigma(i)}$. To prove this requires some theorms such as the modular law and the Butterfly Lemma (Zassenhaus Lemma). A full proof can be found in COHN ■

References (Numbered Thorems and \ref or \cite as needed)

Note

~~We see~~ that this theorm esstentially states that every composition series is isomorphic and, thus, must have the same length. Also, any chain of submodules that has no repetitions can be refined into a composition series since the theorm ~~generalizes for any two chains~~ of submodules. Using our new tool, we come to a significant piece of the Hopkins-Levitzki Theorm.

may be generalied to

Proposition 1: For a module M the following are equivalent:

1. M has a composition series.
2. All chains in M with no repetitions (i.e. no trivial factors) can be refined into a composition series.
3. M is Noetherian and Artinian.

Proof:

Perhaps another sentence to explain this? (A more explicit definition of refinement earlier may also help make this more clear)

$\underline{1 \to 2}$: This is a direct result of the Schreier refinement theorm.

$\underline{2 \to 3}$: Since every chain can be refined into a composition series, every ascending chain must terminate since a composition series is finite by definition. So the module must be Noetherian. ~~To~~ The same argument can be applied to any descending chain ~~and~~ and thus the module must also be Artianian.

$\underline{3 \to 1}$: If M is both Noetherian and Artianian then every ascending chain and descending chain must terminate and thus be finite. To ensure that a chain is a composition series we must be sure that there exists no non-simple submodules in between the links. So to begin constructing the composition series, take a maximal submodule of M called D_1. Then take the submodule of D_1 and name it D_2. Continue along this manner to get a chain of the form $M = D_0 \supseteq D_1 \supseteq ... \supseteq D_f$ where every D_{i-1}/D_i must be simple by construction. Since M is Artianian, this construction must terminate at some point. The result is a composition series. ■ Good!

4. Artianian Rings

which will

Now we get the meat of the paper! Here we develop some of the theory behind Artianian rings ~~so that it may~~ aid in our understanding of the Hopkins-Levitzki Theorm. First, let us get acquainted with some basic definitions.

Definition: We say that a ring R is **right Artianian** when we consider it as a right R-module over itself. A **left Artianian** ring is similarly defined as a left R-module over itself.
Before we prove the connection between Artianian rings and Artinian modules, we must prove a small theorm mentioned in the last section:

\; does a space in math mode

Theorm: (Modular Law) Let M be a module. If F,G,H where $F \subseteq H$, we have the following equality:

\begin{equation}
....
\label{eq:modLaw}
\end{equation}

To label for later reference.

$$F + (G \cap H) = (F + G) \cap H \tag{9}$$

Proof: To prove the forward inclusion, we see that $F \subseteq (F + G)$ and $F \subseteq H$. ~~Same for~~ Similarily we have that $(G \cap H) \subseteq (F + G)$ and $(G \cap H) \subseteq H$. So $F \subseteq (F + G) \cap H$ and $(G \cap H) \subseteq (F + G) \cap H$. ~~So~~ From this we see that $F + (G \cap H) \subseteq (F + G) \cap H$. For the reverse inclusion, suppose we have an $r \in (F + G) \cap H$. Then ~~So~~ r can be expressed as a sum: $r = f + g$ where $f \in F$ and $g \in G$ since $r \in (F + G)$. However, by the hypothesis, we see that $f \in H$ and $r \in H$. ~~So~~ $g = r - f \in H$, ~~So~~ giving $g \in (G \cap H)$. ~~And~~ By ~~that~~ the same token, $r = f + g \in F + (G \cap H)$. ~~So~~ This gives $F + (G \cap H) \supseteq (F + G) \cap H$. Hence, Putting this together we have that the equality is proven. ■

We have an easy corollary that is derived from this theorm:

Corollary: If $F + G = H + G$, $F \cap G = H \cap G$, then $F = H$.

To prove the corollary, simply replace the terms in (9) \eqref{eq:modLaw} ~~and reduce the expression proven.~~ with those in Corallary number (\ref{...}) and simplify the resulting expression. Now to the proposition.

Let us describe and prove some lemmas to aid us in our proof.

Lemma: Let R be a right Artianian Ring and A be a right ideal, then R/A is an Artianian

R-module.

chain

Proof: From the Third Isomorphism Theorm, we know that submodules of R/A correspond to the right ideals of R that contain A. Since R is an Artianian ring, any descending chain of right ideals should terminate. By our correspondence, this implies that any descending chain of submodules in R/A must termiante as well. So R/A is an Artianian R-module. ■

\begin{Lemma}

Lemma: A cyclic right R-module M is Artianian. \label{LemmaCyclic}

\end{Lemma}

Proof: Since M is cyclic, it must be generated by a single element m. Consider the ~~isomorphic~~ isomorphism $\phi : R \to M$ defined by $\phi(r) = mr$. The kernel of this map is precisely the **right annihilator** of M denoted by $r.\text{Ann}_R(M) = \{r \in R : mr = 0\}$. By the First Isomorphism Theorm, there exists a canonical isomorphism $\psi : R/(r.\text{Ann}_R(M)) \to M$. By the previous lemma, M must be Artianian. ■

Proposition 2: Let R be an Artianian Ring. Then all finitely generated R-modules are Artianian.

Proof: We shall proceed by induction on the cardinality of the generating set of M. Consider the base case where M is generated only by one element. # This implies that M is cyclic thus, by the previous lemma, M is Artinian and the base case is done. Assume that the cardinality is i for some $i > 1$. Take a subset G of the generating set of cardinality $i - 1$. Let us call the lone element we have left out u. G generates a submodule of M called M'. The quotient M/M' is cyclic since it is generated by a coset $u + M'$ and thus is Artianian. Consider a descending chain in M:

\ref{LemmaCyclic}

$$M = C_0 \supseteq C_1 ... \supseteq C_i \supseteq ... \tag{10}$$

or something

Let us prove that this descending chain terminates. Since M' is a submodule the chain determined by $C_i' = C_i \cap M'$ is a chain of submodules in M'. By the induction hypothesis, M' is Artianian and so the chain C' terminates. Now also consider the chain $C_i^* = C_i + M'$. This corresponds to the coset elements in M/M' which is also Artianian by above. So C_i^* also terminates. Suppose that the chains terminate at some $r \geq i$. So $C_r \cap M' = C_i \cap M'$ and $C_r + M' = C_i + M'$. So by the corollary of the Modular Law, we see that $C_r = C_i$. So the chain above terminates and M is Artianian. \#

Now we have a connection between finitely generated R-modules and R itself, ~~which~~ when R is Artianian. This gives us another connection to aid us in proving the Hopkins-Levitzki Theorm.

5. Semisimple Rings and Modules

Let us turn our attention to a special class of rings whose modules have nice structures and representations. We shall call a module **semisimple** if it can be expressed as a direct sum of simple right submodules. A ring R is called semisimple if it is a semisimple module over itself. From the definition of semisimpleness, we can deduce some properties of semisimple rings:

Proposition 3: A semisimple ring R is equivalent to a direct sum of a finite number of minimal right ideals and R is Noetherian and Artianian.

You should perhaps comment that, considering R as an R-module, then, since a simple module A has (0) as its only proper submodule then A must be a minimial right ideal of R, since by definition a minimial ideal contains no ideal other than the zero ideal... or something like this

This is of course obvious from the definitions, but it doesn't hurt to mention it.

Also it is that the direct sum of these ideals equals R, saying that the A_i are ideals which express R is somewhat unclear i.e. this could be taken to mean A_i=R, which is not true by in large...

Proof: Define $I_r = \bigoplus_i^r A_i$ where A_i are minimal right ideals that express R. We then have a ascending chain of ideals: $I_1 \subset I_2 \subset I_3 \subset ....$ By our definition, the union of these ideals should span R. Since R is semisimple, there must exist a right ideal I_r that contains $1 \in R$ for some r. (since the direct sum must gernate R) Then $I_1 \subset I_2 \subset I_3 \subset ... \subset I_r$ spans R and thus R is is generated by a finite number of minimal right ideals. Also, notice that the chain mentioned is a composition series since the ideals are minimal. Thus, by Propostion 1, R is Noetherian and Artinian. ■

I think making the minimal ideal <=> simple module relation more explicit makes this more immediate

This is correct, but a bit more detail would be good.

personally I may say "might" instead of can, and then comment that in fact we do see that, but this is more just a personal preference in writing, i.e. sometimes books call things intuitive and it seems like rather a stretch to say it is entirely intuitive... etc.

From this proposition, we can intuitively guess that modules over these rings will also be semisimple relative to simple modules. Therefore, we can now use what we know about semisimple rings to generalize to the modules over these rings!

Also it is not entirely clear to me what you mean by "semisimple relative to simple modules" I think you are trying to say from the Prop. above we may guess that the Prop. below should hold, which is accutate I think, and a nice way to lead into the next part...but the I might reword it a bit...

Proposition 4: A finitely generated module M is semisimple iff it is equivalent to a direct sum of simple submodules.

a

some collection (rather than handful)

reword

Proof: We shall construct such a direct sum of simple modules F inductively. Since M is finitely generated, we can find a set of simple modules C_i such that the finite sum of handful of these C_i's span M. Let us start with a starting direct sum of $F_1 = C_1$. If C_1 contains this generating set then we are done. Otherwise, assume that we have a direct sum of simple submodules $F_i = \bigoplus_i C_i$ such that F_i does not contain the entire generating set. Suppose we wanted to add another submodule C_{i+1} to this direct sum. $F_i \cap C_{i+1} \neq C_{i+1}$ since F_i cannot contain C_{i+1} by definition (Otherwise $F_i \cap C_{i+1} = C_{i+1}$ which will not help progress the induction). However, since all of the C_i's are simple, $F_i \cap C_{i+1} = 0$. Thus, we can construct the new direct sum $F_{i+1} = \bigoplus_{i+1} C_i$. If F_{i+1} spans M, then terminate the process. Otherwise, continue until we get a direct sum that spans M. Such a process must terminate since M is finitely generated. ■

i.e. to Start the direct sum with F_1=C_1 ...

this?

be more explicit maybe give the set a name or something..

Proposition 5: Any finitely generated module over a semisimple ring is semisimple.

Proof: Observe that R^n for some $n \geq 1$ is an R-module. Since M is finitely generated, we can find a surjective homomorphism $\phi : R^n \to M$ where n is the cardinality of the generating set of M (This is essentially a coordinate map). However, since R is semisimple, R^n must also be semisimple (as a module). We shall use the following theorm:

Move this theorem and proof above... having a theorem and proof inside a proof is a bit confusing...

Theorm: Any homomorphism between two simple R-modules is either an isomorphism or the zero homomorphism.

Proof: Assume that the homomorphism $f : M \to N$ is the not the zero map, then image(f) cannot be zero. However since N is simple, image$(f) = N$. Thus, kernel$(f) \neq M$ since f has a non-trivial image. But since M is also simple, it must be said that kernel$(f) = 0$. This implies that f is an isomorphism. ■

Using this, we see that under the homomorphism ϕ, the simple modules of R^n must map to simple modules in M or zero. Thus, we see that M is also made of a direct sum of simple modules. By Proposition 4, M is semisimple. ■

A natural question to ask is: "What are semisimple rings good for?" We see that by using semisimple rings, we can easily construct composition series for modules. In fact, such a method of construction is what we used to prove Propostion 3.

6. Hopkins-Levitzki Theorm: Radicals of Artianian Rings

Jacobson radical right?

For the last section we will need several more definitions...? Or just replace the define with something else perhaps...? define ...definitions sounds repeditive somehow

Let us define one last batch of definitions. The **radical** of a ring R is the smallest ideal such that R/A is a semisimple ring. We also define a module to be **nilpotent** if there exists a $r \in \mathbb{N}$ such that $N^r = 0$. It turns out that the radical of R is nilpotent through a surprising, deep lemma named after Japanese Mathematician, Tadashi Nakayama (1912-1964).

Theorm: (Nakayama's Lemma) If R is an Artianian ring, then the radical rad(R) is the sum of all nilpotent right ideals and is nilpotent itself.

Proof: The proof is a bit too technical and lengthy for this paper, but a detailed proof can be found in COHN.

Finally, we now have all the necessary pieces to assemble our proof of the Hopkin-Levitzki Theorm! Let us restate the theorm once again.

Theorm: (Hopkins-Levitzki Theorm)

For any right Artinian ring R and any right R-module M the following are equivalent:

1. M is Artinian.
2. M is Noetherian.
3. M has a composition series.
4. M is finitely generated.

Proof:
1 → 3: Let us denote N = rad(R). By definition, the quotient ring $S = R/N$ is semisimple and, by Nakayama's Lemma, N is nilpotent. Thus, $N^r = 0$ for some $r \in \mathbb{N}$. Consider the following descending chain:

$$M \supseteq MN \supseteq MN^2 ... \supseteq MN^r = 0 \tag{11}$$

easily

Let us denote the factors $F_i = MN^{i-1}/MN^i$. Notice that $N \subseteq r.\text{Ann}_R(F_i)$. We can easy see this by the following argument: consider any element $a \in MN^{i-1}$. For $\forall n \in N, a \cdot n \in MN^i$. However, this implies that $a \cdot n + MN^i = MN^i$ which is the zero coset. Thus, N annihilates F_i. Since N annihilates F_i, we can regard F_i as an S module. Also, S is semisimple so F_i is semisimple as a S module by Proposition 5. Since M is Artianian, we see that F_i is also Artianian as a an R module. Since S is just the quotient R/N, F_i is also Artianian as a S-module. However, S is semisimple, so we can easily construct a composition series for F_i. After getting all of the composition series for each F_i, glue each composition series in the order ~~of the~~ F_i's and we get a composition series for M.

we

in which the F_i's appeared. In this way we get a comp. series for M.