ON THE FOURIER ANALYSIS OF BOOLEAN FUNCTIONS AND THE LINAI-MANSOUR-NISAN THEOREM

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ABSTRACT. We briefly summarize the basic tools of Fourier Analysis applied to the finite abelian group $\mathbb{Z}_2^n$. By viewing boolean functions $f : \{0,1\}^n \to \{0,1\}$ as square-integrable maps in the Hilbert space $L^2(\mathbb{Z}_2^n)$ under the uniform probability measure, we derive the Linai-Mansour-Nisan theorem concerning the tail of the Fourier spectrum of boolean functions in AC^0. Finally, we highlight some applications of the LMN theorem in the context of sensitivity and influence of functions in AC^0.

1. FOURIER ANALYSIS OF BOOLEAN FUNCTIONS

1.1. **Characters.** We introduce some stepping stones defining the tools used in the Fourier Analysis of boolean functions. Many of these tools can be defined using techniques from Representation Theory, but for the sake of briefity, we will not delve into those concepts here in detail. Instead, we will provide remarks and snippets on relevant topics as needed along with references to the standard literature at the end of the section.

We first define some general concepts for any finite abelian group G. A group homomorphism from $\chi : G \to \mathbb{C}^*$ is called a *character* of G where $\mathbb{C}^* = \mathbb{C}/\{0\}$ is the multiplicative group of the complex numbers. We call the homomorphism $\chi_0 : G \to \mathbb{C}^*,\ \chi_0 = 1$ the *trivial character* of G. By definition, for $a, b \in G$,

$$\chi(a+b) = \chi(a)\chi(b) \tag{1.1}$$

for all characters χ.

Let us now consider the simplest finite abelian group, $\mathbb{Z}_n$ for some $n \in \mathbb{N}$. It follows from basic properties of cyclic groups that the only characters of $\mathbb{Z}_n$ are:

$$\chi_j(x) = e^{2\pi i j x/n} \quad j \in [n],\ x \in \mathbb{Z}_n \tag{1.2}$$

We can decompose any finite abelian group G into a direct product of finite cyclic groups $G \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \cdots \times \mathbb{Z}_{n_k}$. If we decompose $x = \sum_{i\in[k]} x_i$ for $x \in G$, then the characters of G are defined as

$$\chi_a(x) = \prod_{i\in[k]} e^{2\pi i a_k x_k/n_k} \tag{1.3}$$

We see that every element $a \in G$ has an unique associated character χ_a. Furthermore

$$\chi_{a+b}(x) = \chi_a(x)\chi_b(x) \tag{1.4}$$

$$\chi_{a^{-1}}(x) = \frac{1}{\chi_a(x)} \tag{1.5}$$

This shows that the set whose elements are the characters of G endowed with pointwise multiplication is an abelian group. Denote this set as $\hat{G}$. From the map $a \mapsto \chi_a$, we have a group isomorphism between $G \cong \hat{G}$. The group $\hat{G}$ is known as the *character group* of G.

To define the Fourier transform, we must consider the Hilbert space $L_2(G)$ where G endowed with the uniform probability measurel i.e the discrete measure mapping each subset $H \subseteq G$ to $\frac{|H|}{|G|}$. This yields the standard inner product for maps $f, g \in L_2(G)$:

$$\langle f, g \rangle = \frac{1}{|G|} \sum_{x \in G} f(x)\overline{g(x)} = \mathbb{E}_x f(x)\overline{g(x)} \tag{1.6}$$

Note that this induces the norm $||f||_2 = \sqrt{\langle f, f \rangle} = \mathbb{E}_x |f(x)|^2$ on $L_2(G)$. It is simple to check that the characters of G indeed lie in $L_2(G)$. However, we can prove something stronger: the characters of G, χ_a form an *orthonormal basis* of $L_2(G)$.

Theorem 1.1. *The characters $\chi_a \in \hat{G}$ form an orthonormal basis for $L_2(G)$*

Proof. To show this, we must employ a lemma: $\square$

We restrict our attention to the finite abelian group $\mathbb{Z}_2^n$ as a natural group to define boolean functions $f : \{0,1\}^n \to \{0,1\}$. By the observations above, we conclude that our characters for $\mathbb{Z}_2^n$ are defined as:

$$\chi_a(x) = (-1)^{\sum_{i \in [n]} x_i a_i} = (-1)^{\sum_{i \in [n],\ a_i = 1} x_i} \tag{1.7}$$

where x_i is the i^{th} bit of $x \in \{0,1\}^n$.

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