The univalence axiom,
a brief & incomplete tour

School and Workshop on Univalent Mathematics

tslil clingman, 2019

Univalence & function extensionality

Univalence, the structure-identity principle, and you

Open questions the formulation of univalence

Equivalent reformulations

Notation

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Univalence & function extensionality

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We will prove that from a term ua we may deduce funext.

Proof idea

Let us fix $A : \mathsf{U_i}$, $B : A \to \mathsf{U_j}$, and $f : \prod(a : A), B$.

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Assuming only ua*, if* $q : C \to D$ *is an equivalence then so too is* $q \circ (-) : (E \to C) \to (E \to D)$.

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- By the *structure identity principle*, ua means that many algebraic things are univalent too.

Univalence, the structure-identity principle, and you

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OR

Change foundations

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…where does the S.I.P. fail?

Open questions the formulation of univalence

Putting the *fun* in function extensionality

Definition (Naïve function extensionality)

$$\mathsf{nfunext} : \prod_{(A : \mathsf{U_i})} \prod_{(B : \mathsf{U_j})} \prod_{(f, g : \prod (a : A), B)} f \sim g \to f = g$$

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Can we do better?

The setup

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It is known that nua+nuaβ $\leftrightarrow$ ua

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Equivalent reformulations

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Question

In the presence of funext, do (1.)-(3.) above imply (4.) and (5.), for possibly 'modified' unit and flip.