

The univalence axiom, a brief & incomplete tour

School and Workshop on Univalent Mathematics
tslil clingman, 2019

Univalence & function extensionality

Univalence, the structure-identity principle, and you

Open questions the formulation of univalence

Equivalent reformulations



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Univalence & function extensionality

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An unfair and incorrect characterisation

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That didn’t work exactly. Perhaps there is a proof like this?

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We will prove that from a term **ua** we may deduce **funext**.

Proof idea

Let us fix $A : \mathcal{U}_i$, $B : A \rightarrow \mathcal{U}_j$, and $f : \prod (a : A), B$.

$$\sum_{(g : \prod (a : A), B)} f \sim g$$

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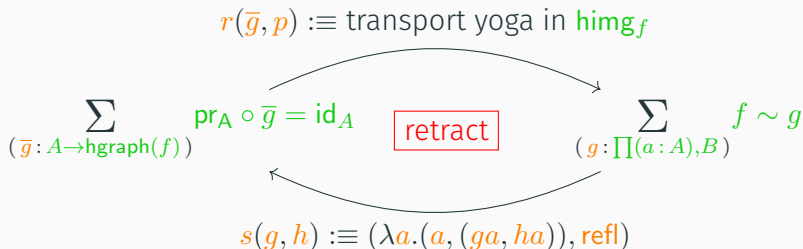
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Theorem (Voevodsky)

Assuming only **ua**, if $q: C \rightarrow D$ is an equivalence then so too is $q \circ (-): (E \rightarrow C) \rightarrow (E \rightarrow D)$.

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~ 50 of which is a single, uninteresting, technical fact about equivalences \rightsquigarrow good exercise?

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- Consequently, for any proposition $P : \mathbf{U} \rightarrow \mathbf{U}$ and witness $A \simeq B$, one cannot show $P(A)$ but not $P(B)$.
- By the *structure identity principle*, \mathbf{ua} means that many algebraic things are univalent too.

Univalence, the structure-identity
principle, and you

All forms of equality are equal, but some are more equal than others

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OR

Change foundations

Univalence for the working mathematician

For a flavour of the *structure-identity* principle, let's pick our favourite toy algebraic structure,

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If \mathcal{C} is a univalent category and T is a monad on \mathcal{C} then $\text{Alg}(T)$ is a univalent category.

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“isomorphism of algebraic structures is equivalent to equality”
...where does the S.I.P. fail?

Open questions the formulation of
univalence

Putting the *fun* in function extensionality

Definition (Naïve function extensionality)

$$\text{nfunext} : \prod_{(A : \mathcal{U}_i)} \prod_{(B : \mathcal{U}_j)} \prod_{(f, g : \prod_{(a : A)} B)} f \sim g \rightarrow f = g$$

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Can we do better?

The setup

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From `nfunext` we may derive `funext`.

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It is known that $\mathbf{nua} + \mathbf{nua}\beta \leftrightarrow \mathbf{ua}$

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Equivalent reformulations

The Orton-Pitts reformulation

Based on our previous discussion we'll assume ambient **funext**

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Definition (Orton-Pitts naïve univalence)

1. **unit** : $A = \sum (a : A), 1$

These imply **nua**. Given $(f, e) : A \simeq B$,

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Question

In the presence of **funext**, do (1.)-(3.) above imply (4.) and (5.), for possibly ‘modified’ **unit** and **flip**.