Vv255 Lecture 16

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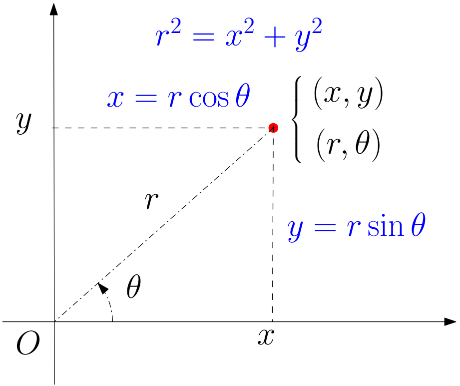
July 2, 2018

- Recall the polar coordinates in relation to Cartesian coordinates.



- For the same curve in the same space $\mathbb{R}^2$, we might have 2 representations

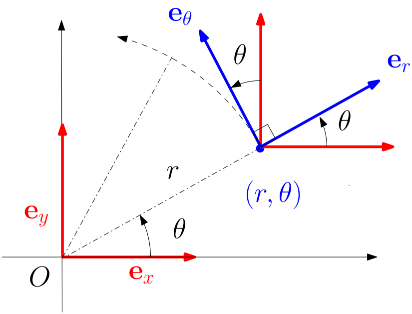
$$y = y(x) \qquad \text{e.g.} \quad y = \frac{3}{2}x - 1 \iff r = \frac{2}{3\cos\theta - 2\sin\theta}$$
$$r = r(\theta)$$

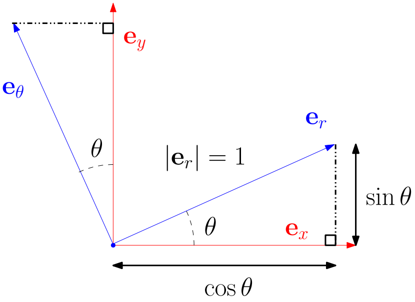
- In terms of the standard basis $\mathbf{e}_x$ and $\mathbf{e}_y$, we have

$$\mathbf{r} = \begin{bmatrix} x \\ y \end{bmatrix}_{\mathcal{C}} = x\mathbf{e}_x + y\mathbf{e}_y, \qquad \text{where} \quad \mathcal{C} = \{\mathbf{e}_x, \mathbf{e}_y\}$$

- We want to have an orthonormal basis $\mathcal{P} = \{\mathbf{e}_r, \mathbf{e}_\theta\}$,

$$\begin{bmatrix} r \\ \theta \end{bmatrix}_{\mathcal{P}} = r\mathbf{e}_r + \theta\mathbf{e}_\theta, \qquad \text{where} \quad \begin{array}{ll} \mathbf{e}_r & \text{is the direction of increasing } r. \\ \mathbf{e}_\theta & \text{is the direction of increasing } \theta. \end{array}$$





- Therefore, we expect the unit vector in the direction of increasing r to be

$$\mathbf{e}_r = \frac{(\cos\theta)\,\mathbf{e}_x + (\sin\theta)\,\mathbf{e}_y}{\sqrt{(\cos\theta)^2 + (\sin\theta)^2}} = (\cos\theta)\,\mathbf{e}_x + (\sin\theta)\,\mathbf{e}_y$$

- Similarly, the unit vector in the direction of increasing θ,

$$\mathbf{e}_\theta = \frac{(-\sin\theta)\,\mathbf{e}_x + (\cos\theta)\,\mathbf{e}_y}{\sqrt{(-\sin\theta)^2 + (\cos\theta)^2}} = (-\sin\theta)\,\mathbf{e}_x + (\cos\theta)\,\mathbf{e}_y$$

- We can verify the orthogonality of $\mathbf{e}_r$ and $\mathbf{e}_\theta$ by computing

$$\mathbf{e}_r \cdot \mathbf{e}_\theta = 0$$

- Note the Cartesian position vector $\mathbf{r}$ in terms of $\mathbf{e}_r$ and $\mathbf{e}_\theta$ is

$$\mathbf{r} = \begin{bmatrix} x \\ y \end{bmatrix}_{\mathcal{C}} = r\mathbf{e}_r \neq r\mathbf{e}_r + \theta\mathbf{e}_\theta = \begin{bmatrix} r \\ \theta \end{bmatrix}_{\mathcal{P}}$$

- Note both $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are functions of θ.

$$\mathbf{e}_r(\theta) \qquad \text{and} \qquad \mathbf{e}_\theta(\theta)$$

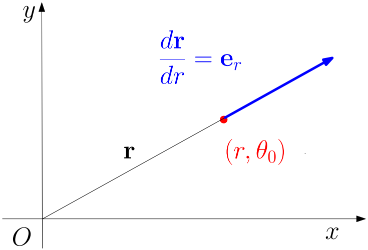
- Let $\theta = \theta_0$, then the Cartesian position vector is a vector-valued function,

$$\mathbf{r}(r) = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r\cos\theta_0 \\ r\sin\theta_0 \end{bmatrix} = r\begin{bmatrix} \cos\theta_0 \\ \sin\theta_0 \end{bmatrix}$$

- The rate of change of $\mathbf{r}$ with respect to r is

$$\frac{d\mathbf{r}}{dr} = \begin{bmatrix} \cos\theta_0 \\ \sin\theta_0 \end{bmatrix} = \mathbf{e}_r\Big|_{\theta=\theta_0}$$

which gives the tangential direction of the curve defined by $\mathbf{r}$ at r as usual.



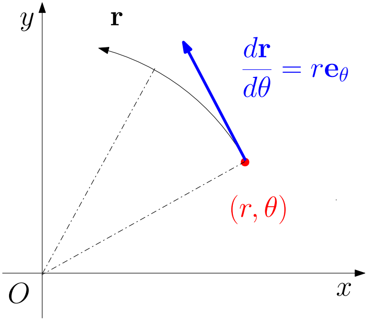
- If $r = r_0$, then the Cartesian position vector is a vector-valued function of θ,

$$\mathbf{r}(r) = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r_0 \cos\theta \\ r_0 \sin\theta \end{bmatrix}$$

- The rate of change of $\mathbf{r}$ with respect to θ is

$$\frac{d\mathbf{r}}{d\theta} = r_0 \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} = r_0 \mathbf{e}_\theta = r\mathbf{e}_\theta \Big|_{r=r_0}$$

which gives the tangential direction of the curve defined by $\mathbf{r}$ at θ as usual.



- We have two derivatives of

$$\mathbf{r}(r,\theta)$$

1. The rate of change of $\mathbf{r}$ with respect to r while holding θ fixed.

$$\frac{d\mathbf{r}}{dr}$$

2. The rate of change of $\mathbf{r}$ with respect to θ while holding r fixed.

$$\frac{d\mathbf{r}}{d\theta}$$

Q: Have you seen derivatives that are similar to those?

$$\frac{\partial \mathbf{r}}{\partial r} \qquad \text{and} \qquad \frac{\partial \mathbf{r}}{\partial \theta}$$

- The partial derivatives give the change of the function, here a vector-valued function, with respect to one independent variable, while holding other independent variables constant.

- Now consider the scalar-valued function of two variables,

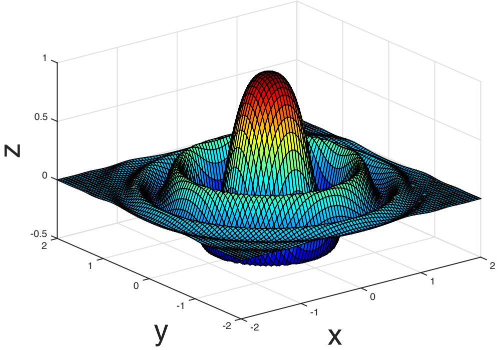
$$z = f(x, y)$$

- Recall a partial derivative of z is a directional derivative in the direction of

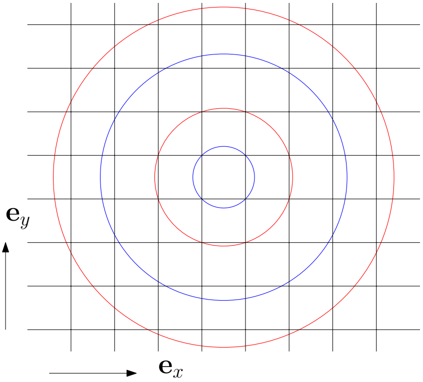
$$\mathbf{e}_x \qquad \text{or} \qquad \mathbf{e}_y$$

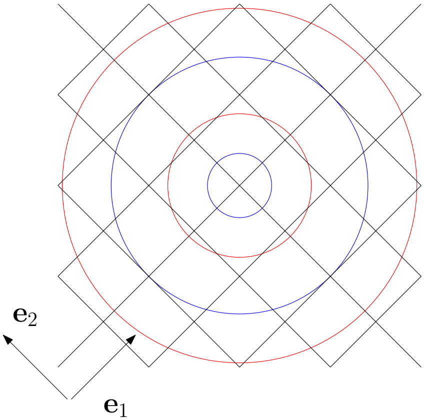
Q: Are these always the best direction to consider the rate of z?

$$f(x, y) = e^{-(x^2+y^2)} \cos\left(4(x^2 + y^2)\right)$$



- Clearly $\mathbf{e}_x$ and $\mathbf{e}_y$ are no better than any linear orthogonal directions.

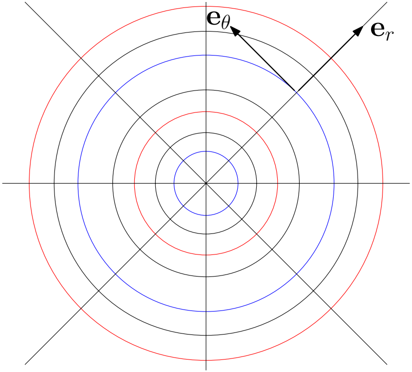




Q: Why using polar coordinates is a better choice here?

- Since it shows the radial symmetry of the function

$$
\begin{aligned}
f(x,y) &= e^{-(x^2+y^2)}\cos\Big(4\left(x^2+y^2\right)\Big) \\
&= e^{-r^2}\cos\left(4r^2\right)
\end{aligned}
$$

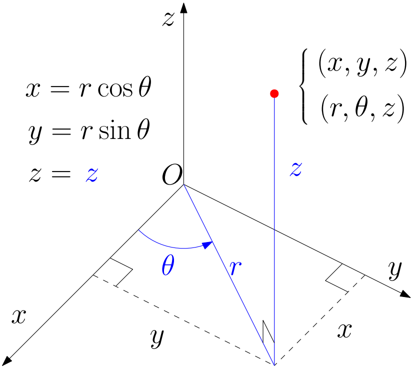


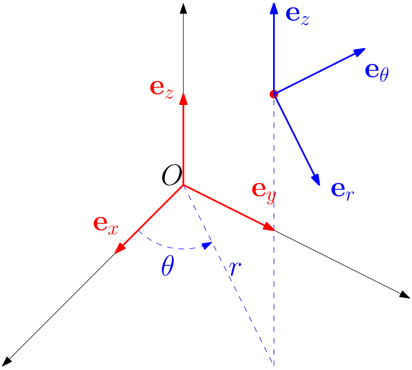
- It is often the symmetry of a given physical problem that points to the most convenient choice of basis or coordinates.
- If we add the usual z coordinate to the plane polar coordinates, then we will have a cylindrical coordinate system.

Definition

Cylindrical coordinates represent a point P in $\mathbb{R}^3$ by (r, θ, z) in which

1. r and θ are the polar coordinates for the projection of P onto the xy-plane
2. z is the Cartesian vertical coordinate.





- Similar to the plane polar basis, the cylindrical polar basis can be found by differentiating the Cartesian position vector

$$\mathbf{r}(r,\theta,z) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ z \end{bmatrix}$$

- As may be directly verified, the following basis is orthonormal everywhere,

$$\mathbf{e}_r = \frac{\frac{\partial \mathbf{r}}{\partial r}}{\left|\frac{\partial \mathbf{r}}{\partial r}\right|} = \frac{\partial \mathbf{r}}{\partial r} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}, \qquad \text{where } \mathbf{e}_r \text{ gives the direction of increasing } r.$$

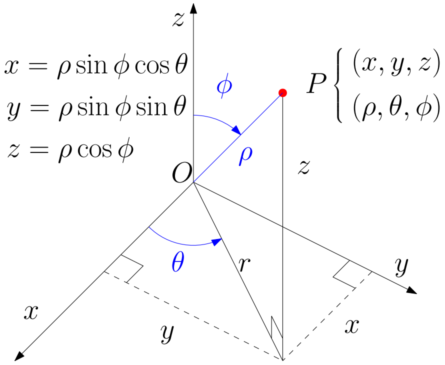
$$\mathbf{e}_\theta = \frac{\frac{\partial \mathbf{r}}{\partial \theta}}{\left|\frac{\partial \mathbf{r}}{\partial \theta}\right|} = \frac{1}{r}\frac{\partial \mathbf{r}}{\partial \theta} = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix}, \qquad \text{where } \mathbf{e}_\theta \text{ gives the direction of increasing } \theta.$$

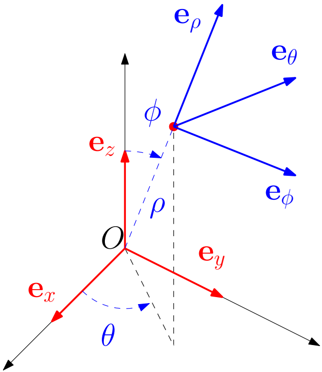
$$\mathbf{e}_z = \frac{\frac{\partial \mathbf{r}}{\partial z}}{\left|\frac{\partial \mathbf{r}}{\partial z}\right|} = \frac{\partial \mathbf{r}}{\partial z} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \text{where } \mathbf{e}_z \text{ gives the direction of increasing } z.$$

Definition

Spherical coordinates represent a point P in $\mathbb{R}^3$ by (ρ, θ, ϕ) in which

1. ρ is the distance between the point P to the origin O.
2. θ is the angular coordinate for the projection of P on the xy-plane.
3. ϕ is the angle $\mathbf{OP}$ makes with the positive z-axis.





- Again the spherical polar basis can be found by differentiating

$$\mathbf{r}(\rho,\theta,\phi)=\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}\rho\sin\phi\cos\theta\\\rho\sin\phi\sin\theta\\\rho\cos\phi\end{bmatrix}$$

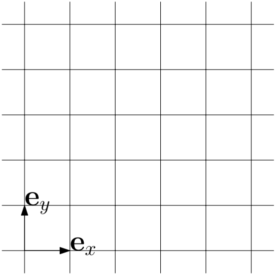
- Again it can be directly verified, the following basis is orthonormal in $\mathbb{R}^3$,

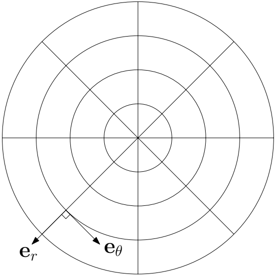
$$\mathbf{e}_\rho=\frac{\frac{\partial\mathbf{r}}{\partial\rho}}{\left|\frac{\partial\mathbf{r}}{\partial\rho}\right|}=\frac{\partial\mathbf{r}}{\partial\rho}=\begin{bmatrix}\sin\phi\cos\theta\\\sin\phi\sin\theta\\\cos\phi\end{bmatrix},$$ where $\mathbf{e}_\rho$ is the direction of increasing ρ.

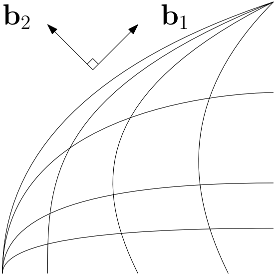
$$\mathbf{e}_\theta=\frac{\frac{\partial\mathbf{r}}{\partial\theta}}{\left|\frac{\partial\mathbf{r}}{\partial\theta}\right|}=\frac{1}{\rho\sin\phi}\frac{\partial\mathbf{r}}{\partial\theta}=\begin{bmatrix}-\sin\theta\\\cos\theta\\0\end{bmatrix},$$ where $\mathbf{e}_\theta$ is the direction of increasing θ.

$$\mathbf{e}_\phi=\frac{\frac{\partial\mathbf{r}}{\partial\phi}}{\left|\frac{\partial\mathbf{r}}{\partial\phi}\right|}=\frac{1}{\rho}\frac{\partial\mathbf{r}}{\partial\phi}=\begin{bmatrix}\cos\phi\cos\theta\\\cos\phi\sin\theta\\-\sin\phi\end{bmatrix},$$ where $\mathbf{e}_\phi$ is the direction of increasing ϕ.

- A big difference between Cartesian, and the plane polar, cylindrical polar and spherical polar is that the coordinate lines may be curved in the later ones.







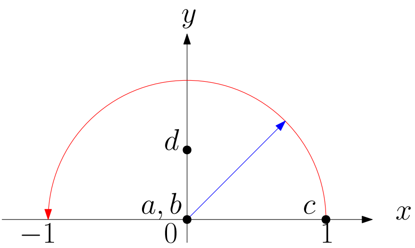
- In general, changing coordinates is a transformation from a space to another,

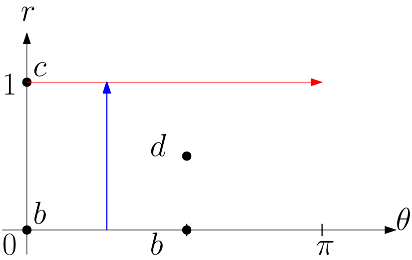
$$T\colon \mathbb{R}^2 \to \mathbb{R}^2 \qquad \text{and} \qquad T\colon \mathbb{R}^3 \to \mathbb{R}^3$$

- Specifically, it is an invertible transformation between points in the 2 spaces,

e.g. Polar $\to$ Cartesian and Cartesian $\to$ Polar

- Consider how points and vectors are transformed as we change the coordinates between Cartesian and the polar coordinates.





- Changing coordinates is similar to changing variables, we are interested in

1. changing a function of the old to be in terms of the new variables, e.g.

$$f(x, y) = F(u, v)$$

2. finding the derivatives with respect to the new variables, e.g.

$$f_u \qquad \text{and} \qquad f_v$$

- Suppose the transformation equations are given as "old in terms of new",

$$x = x(u, v) \qquad \text{and} \qquad y = y(u, v)$$

- If we actually know the function $z = f(x, y)$ explicitly, then it is easy to find

$$z = f(x, y) = f(x(u, v), y(u, v)) = F(u, v)$$

and the partial derivatives can be found directly. Even without $z = F(u, v)$,

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$
$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

the chain rule provides a way of finding the partial derivatives given we know

$$\frac{\partial z}{\partial x} \qquad \text{and} \qquad \frac{\partial z}{\partial y}$$

- Now let the transformation equations be given as “new in terms of old”,

$$u = u(x, y) \qquad \text{and} \qquad v = v(x, y)$$

- We might be able to solve x and y in terms of u and v, then as before

$$z = f(x, y) = f(x(u, v), y(u, v)) = F(u, v)$$

Q: What happens if we cannot solve for

$$x = x(u, v) \qquad \text{and} \qquad y = y(u, v)$$

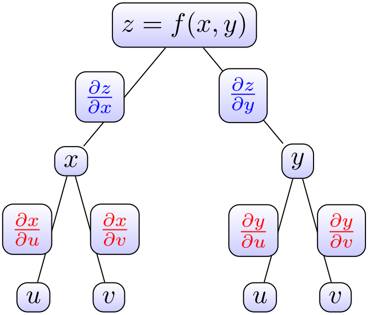
Q: Can we find $F(u, v)$ explicitly? How about F_u and F_v?

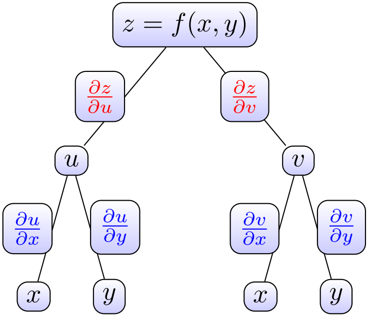
Q: Can we use the chain rule here to find the partial derivatives?

- Given $z = f(x, y)$, $u = u(x, y)$ and $v = v(x, y)$, we can find

$$\frac{\partial z}{\partial x}, \quad \frac{\partial z}{\partial y}, \quad \frac{\partial u}{\partial x}, \quad \frac{\partial u}{\partial y}, \quad \frac{\partial v}{\partial x} \quad \text{and} \quad \frac{\partial v}{\partial y}$$

- Here we have two versions of the chain rule, only one of those two is useful,





- No solvable

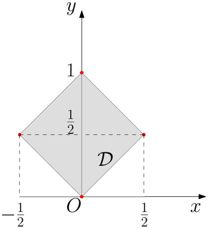
$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$
$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

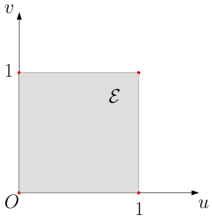
- Solvable

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$
$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

Exercise

Suppose that $T(x, y)$ is defined over the region $\mathcal{D}$, where $\mathcal{D}$ is indicated below





Now if we are interested in the rate of change of the function f along the edges, then it proves much easier to consider the transformation

$$u = y + x \qquad \text{and} \qquad v = y - x$$

Suppose that some physical quantity is defined to be

$$W = \frac{\partial^2 T}{\partial x^2} - \frac{\partial^2 T}{\partial y^2} + \frac{\partial T}{\partial x} + \frac{\partial T}{\partial y}$$

How can W be evaluated in terms of u and v?

- Therefore, for arbitrary

$$z = f(x,y), \qquad u = u(x,y) \qquad \text{and} \qquad v = v(x,y)$$

we can obtain the partial derivatives with respect to the new variables u and v by solving the linear equations, and in general, we have

$$\frac{\partial z}{\partial u} = \frac{\dfrac{\partial z}{\partial x}\dfrac{\partial v}{\partial y} - \dfrac{\partial z}{\partial y}\dfrac{\partial v}{\partial x}}{\dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial y} - \dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial x}} \qquad \text{and} \qquad \frac{\partial z}{\partial v} = \frac{\dfrac{\partial z}{\partial y}\dfrac{\partial u}{\partial x} - \dfrac{\partial z}{\partial x}\dfrac{\partial u}{\partial y}}{\dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial y} - \dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial x}}$$

- Note both derivatives share the same denominator, and it must be non-zero for the rate of change to be defined. Recall the determinant of a 2×2 matrix

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

$$\Longrightarrow \det(\mathbf{J}) = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}, \qquad \text{where} \quad \mathbf{J} = \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{bmatrix}.$$

Definition

The Jacobian of transformation $u = u(x, y)$ and $v = v(x, y)$ is the determinant

$$\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial(u, v)}{\partial(x, y)} = J(x, y)$$

- The matrix from which the Jacobian is defined is called the Jacobian matrix

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{bmatrix}$$

- For a transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$, the Jacobian is defined in a similar way.
- The Jacobian is of special interest, because it contains the information about the transformation between one set of coordinates (x, y) and another (u, v).

- Of course, we can have Cartesian coordinates in terms of other coordinates

$$x = x(u, v) \qquad \text{and} \qquad y = y(u, v)$$

- And the Jacobian matrix and the Jacobian for the transformation are

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{bmatrix} \qquad \text{and} \qquad J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \det(\mathbf{J})$$

- In this case, the Jacobian matrix can be understood as

$$\mathbf{J} = \begin{bmatrix} \nabla x^{\mathrm{T}} \\ \nabla y^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \mathbf{r}}{\partial u} & \dfrac{\partial \mathbf{r}}{\partial v} \end{bmatrix}$$

Q: What does it mean in terms of the partial derivatives of $z = f(u, v)$?

Q: Will any transformation work?

Q: What kinds of transformations will not provide a useful coordinate system?

- Suppose we were considering a new set of coordinates, (u, v), given by

$$u = x^2 + y + 1 \qquad \text{and} \qquad v = x^4 + 2x^2y + y^2 + x^2 - y$$

Q: Why this transformation is not going to provide useful coordinates system ?

$$v = (u-1)^2 - (u-1) = u^2 - 3u + 2$$

- There is a functional dependence between u and v, so it is not invertible.

Theorem

If $u(x, y)$ and $v(x, y)$ are functionally dependent, then

$$\frac{\partial(u, v)}{\partial(x, y)} = 0$$

Proof

- If $u(x, y)$ and $v(x, y)$ are functionally dependent, there is an equation

$$F(u, v) = 0$$

- Apply implicit differentiation,

$$\begin{aligned} F_u u_x + F_v v_x &= 0 \\ F_u u_y + F_v v_y &= 0 \end{aligned}$$

- For consistency, we must have

$$u_x = \alpha u_y \qquad \text{and} \qquad v_x = \alpha v_y \qquad \text{where } \alpha \text{ is a constant.}$$

- That is,

$$u_x v_y - v_x u_y = \frac{\partial(u, v)}{\partial(x, y)} = 0 \quad \square$$