📚 Table of Contents


```
begin
    using PlutoUI    # visualization purpose
    TableOfContents(title="📚 Table of Contents", indent=true, depth=3, aside=true)
end
```

Lab 01: Linear regression

- Student name:
- ID:

How to do your homework

- You will work directly on this notebook; the word **TODO** indicates the parts you need to do.
- You can discuss the ideas as well as refer to the documents, but *the code and work must be yours*.

How to submit your homework

- Before submitting, save this file as `<ID>.jl`. For example, if your ID is 123456, then your file will be `123456.jl`. And export to PDF with name `123456.pdf` then submit zipped source code and pdf into `123456.zip` onto Moodle.

> Note
>
> **Note that you will get 0 point for the wrong submit**.

Content of the assignment:

- Linear regression
- Polinomial linear regression

1. The hypothesis set

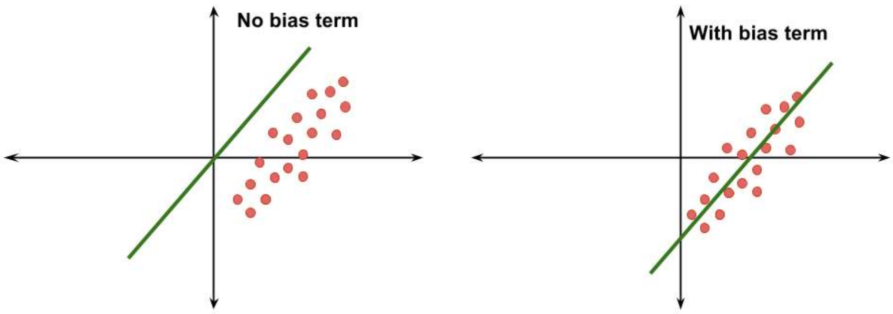
- Linear regression is a linear model, e.g. a model that assumes a linear relationship between the input variables (x) and the single output variable (y). More specifically, that y can be calculated from a linear combination of the input variables (x).
- Generally, a linear model will make predictions by calculating a weighted sum of the input features (independent variables).

$$\hat{y} = w_0 + \sum_{i=1}^{d} w_i x_i$$

  - $\hat{y}$ is the predicted value.
  - d is the number of features.
  - x_i is the i^{th} feature value.
  - w_j is the j^{th} model parameter (including the bias term w_0 and the feature weights $w_1, w_2, \ldots w_d$)
- You can rewrite the first equation in the matrix form as following:

$$\hat{y} = h_{\mathbf{w}}(\mathbf{x}) = \mathbf{w}^T \cdot \mathbf{x}$$

  - $\mathbf{w}$ is the model **parameter vector** (including the bias term w_0 and the feature weights $w_1, w_2, \ldots w_n$).
  - $\mathbf{w}^T$ is a transpose of $\mathbf{w}$ (a row vector insteade of column vector).
  - $\mathbf{x}$ is the instance's **feature vector**, *containing* x_0 to x_d, with x_0 *always equal to* 1.
  - $\mathbf{w}^T \cdot \mathbf{x}$ is the dot product of $\mathbf{w}^T$ and $\mathbf{x}$.
  - $h_{\mathbf{w}}$ is the hypothesis function, using the parameters $\mathbf{w}$.



2. Performance measure and the learning goal

- Before we start to train the model, we need to determine how good the model fits the training data. There are a couple of ways to determine the level of quality, but we are going to use the most popular one and that is the **MSE** (Mean Square Error). We need to find the value for $\mathbf{w}$ that will minimize the MSE:

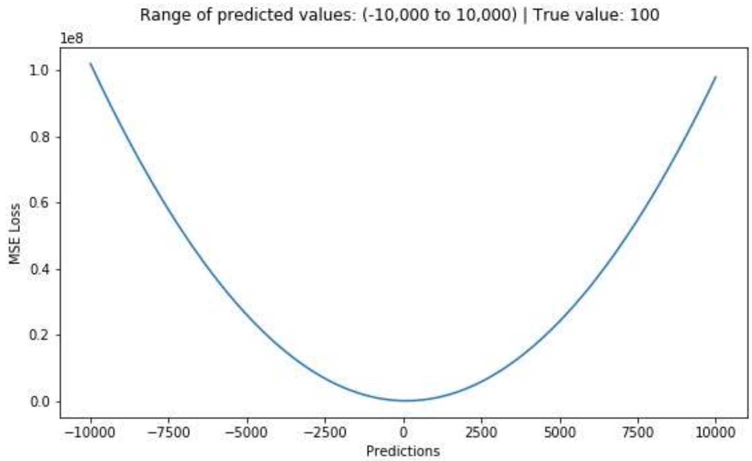
$$\mathbf{w} = \arg\min MSE_{\mathcal{D}_{train}}$$

- MSE on the train set $\mathcal{D}_{train}$ denoted as $(\mathbf{X}, \mathbf{y})$ including n samples $\{(\mathbf{x}_1, y_1), \ldots (\mathbf{x}_n, y_n)\}$

$$MSE(X, h_{\mathbf{w}}) = \frac{1}{m}\sum_{i=1}^{m}\left(\mathbf{w}^T \cdot \mathbf{x}_i - y_i\right)^2$$

$$MSE(X, h_{\mathbf{w}}) = \frac{1}{m}\|\mathbf{X}\mathbf{w} - \mathbf{y}\|^2$$

- Example below is a plot of an MSE function where the true target value is 100, and the predicted values range between -10,000 to 10,000. The MSE loss (Y-axis) reaches its minimum value at prediction (X-axis) = 100. The range is 0 to ∞.



- To find the value of $\mathbf{w}$ that minimizes the MSE cost function the most common way (*we have known since high school*) is to solve the derivative (gradient) equation.

$$\hat{\mathbf{w}} = \left(\mathbf{X}^T\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{y}$$

  - $\hat{\mathbf{w}}$ is the value of $\mathbf{w}$ that minimizes the cost function
- In order to reduce the complexity of this approach, I will provide you another version of computing the optimal $\mathbf{w}$:

$$\mathbf{X}\mathbf{w} = \mathbf{y}$$

$$\Leftrightarrow \mathbf{X}^T\mathbf{X}\mathbf{w} = \mathbf{X}^T\mathbf{y}$$

$$\Leftrightarrow \mathbf{w} = \left(\mathbf{X}^T\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{y}$$

3. Implementation

Import library

```
PlotlyBackend()
```

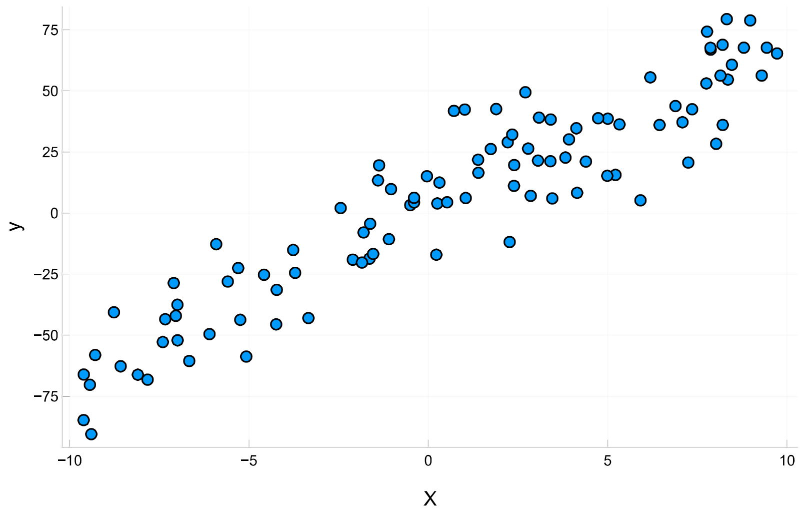
```
begin
    using Plots    , Distributions
    plotly()
end
```

Create data

([0.250974, 2.26921, 3.41137, 2.70392, -4.24132, 4.12887, 8.02454, 7.87131, 8.20504, mo

```
begin
    function create_data(f, num_data=100)
        # f is the function that is used to generate data
        X = Base.rand(Distributions.Uniform(-10,10), num_data)
        y = f.(X);
        return X, y
    end

    a = Base.rand(Distributions.Uniform(-10,10), 1)[1]
    f(x) = a*x + Base.rand(Distributions.Normal(1,15), 1)[1]
    X, y = create_data(f)
end
```

Visualize data



TODO:

- Your observations about data:
- All the dots are scattered but together they form a pattern, which resembles a straight line. We can say that they form a linear pattern

Training function

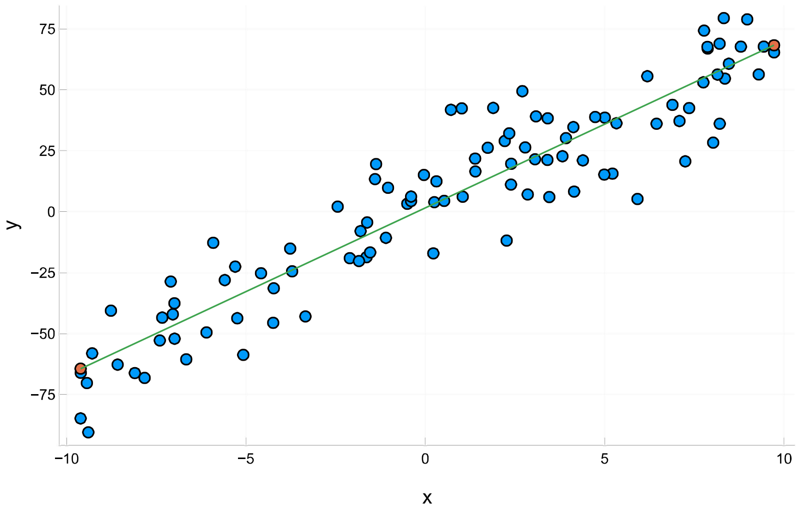
train_linear_regression (generic function with 1 method)

```
function train_linear_regression(X, y)
    """
    Trains Linear Regression on the dataset (X, y).

    Parameters
    ----------
    X : Matrix, shape (n, d + 1)
        The matrix of input vectors (each row corresponds to an input vector);
        the first column of this matrix is all ones (corresponding to x_0).
    y : Matrix, shape (n, 1)
        The vector of outputs.

    Returns
    -------
    w : Matrix, shape (d + 1, 1)
        The vector of parameters of Linear Regression after training.
    """
    # TODO: compute your weight
    w = inv(X' * X) * X' * y
    return w
end
```

```
begin
    # Construct one_added_X
    # TODO: First column of one_added_X is all ones (corresponding to x_0).
    function addOne(X)
        # Create a column of ones
        one = ones(Float64, size(X, 1))

        # Concatenate the column of ones to the beginning of X
        one_added = hcat(one, X)
        #one_added = reshape(one_added,(100,2))
        return one_added
    end
    one_added_X = addOne(X)
    println(one_added_X)
    println("size of one_added_X = ", Base.size(one_added_X))
    println("size of y = ", Base.size(y))
end
```

```
[1.0 0.25097445179201117; 1.0 2.2692066851047734; … ; 1.0 -5.59092719460
0046; 1.0 7.355659932629969]
size of one_added_X = (100, 2)
size of y = (100,)
```

Train our model and visualize




```
begin
    w = train_linear_regression(one_added_X, y)
    # Visualize result
    predicted_ys = one_added_X*w
    scatter(X, y, xlabel="x", ylabel="y", legend=false)
    x_min = minimum(X)
    x_max = maximum(X)
    xs = [x_min x_max]'
    # Construct ones_added_xs
    # TODO: First column of ones_added_xs is all ones (corresponding to x_0).
    ones_added_xs = addOne(xs)
    predicted_ys = ones_added_xs*w
    print(predicted_ys)
    scatter!(xs, predicted_ys, legend=false)
    plot!(xs, predicted_ys, legend=false)

end
```

```
[-64.37117539751817, 68.3067580809821]
```

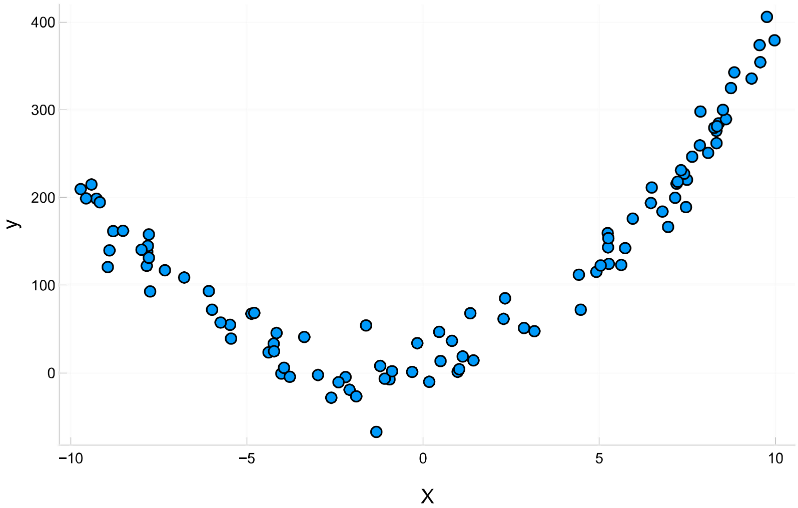
4. Polinomial regression

- Observe the following dataset. Can we use linear regression model to fit this data?

  TODO: While the dots still create a pattern, it resembles a curve, not a line, so linear regression will not be suitable for fitting this data.

```
([-5.47787, -0.952473, -8.79672, -4.23891, 9.5709, 2.32436, 6.79368, 0.493792, 5.27171,
```

```
begin
    a_ = Base.rand(Distributions.Uniform(-5,5))
    b_ = Base.rand(Distributions.Uniform(-10,10))
    # c_ = Base.rand(Distributions.Uniform(-10,10))
    g(x) = a_*x^2 + b_*x + Base.rand(Distributions.Normal(1,20),1)[1]
    X_, y_ = create_data(g)
end
```



Abstract the problem

- For this kind of datasets, you have to **change the hypothesis set**. For example, in this case, we assume that our data is in the parabol form. Therefore, the hypothesis set will be $\hat{y} = ax^2 + bx + c$. Another assumption: $\hat{y} = ax^3 + bx^2 + cx + d$.
- In general, we have the polinomial regression form:

$$\hat{y} = w_0 + \sum_{i=1}^{d} w_i x_i^i$$

- Re-write the equation above:

$$\hat{y} = h_{\mathbf{w}}(\mathbf{x}) = \mathbf{w}^T \cdot \mathbf{x}, \text{ where } \mathbf{x} = \begin{bmatrix} x_0 = 1 \\ x_1^1 \\ \vdots \\ x_d^d \end{bmatrix} \in \mathbb{R}^{d+1}$$

- To solve this problem, we have to find $\mathbf{w}$ such that:

$$\min_{\mathbf{w}} MSE(X, h_{\mathbf{w}}) = \min_{\mathbf{w}} \frac{1}{m} \|\mathbf{X}\mathbf{w} - \mathbf{y}\|^2$$

- Recall the solution of MSE in section 2:

$$\hat{\mathbf{w}} = \left(\mathbf{X}^T\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{y}$$

- Now, it's time for coding!

Solve the problem in code

Step 1: Create polynomial feature

$$\mathbf{X} = \{\mathbf{x}_i = (x_0 = 1, x_1^1, x_2^2, \ldots, x_d^d)\}_{i=1}^{n}$$

```julia
# TODO: Create X

begin
    function poly_features(X, K)
        # X: inputs of size N x 1
        # K: degree of the polynomial
        X_poly = ones(Float64, length(X), K + 1)
        # Iterate over the degree of the polynomial
        for i in 1:K
            # Compute the i-th degree polynomial feature
            X_poly[:, i + 1] = X[:, 1] .^ i
        end

        # Return the polynomial features
        return X_poly
    end

    # assume that our data is in form of a parabol
    X = poly_features(X_, 2)

    println(size(X))
end
```

```
(100, 3)
```

Step 2: Train our model and find **w**

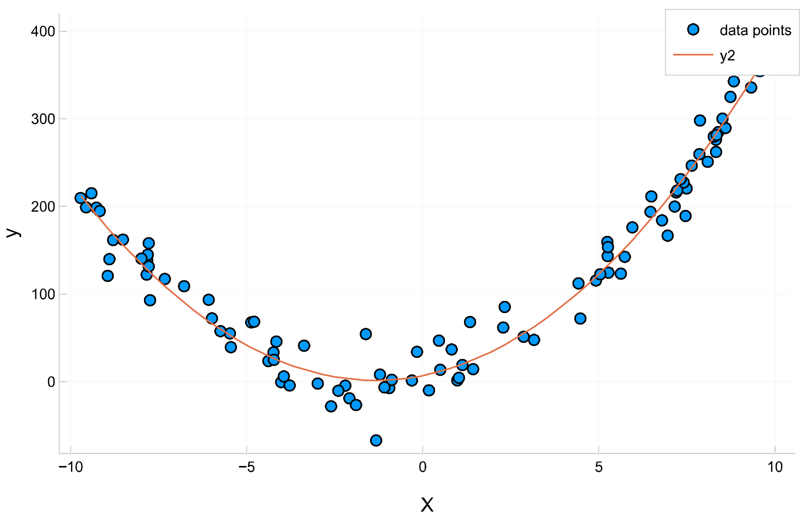
```
train_polynomial_regression (generic function with 1 method)
```

```julia
# TODO: train our model
function train_polynomial_regression(X, y)
    W =   inv(X' * X) * X' * y
    return W
end
```

```
[6.75779, 8.09666, 3.00812]
```

```julia
begin
    w = train_polynomial_regression(X, y_)
end
```

Step 3: Visualize our result




```julia
# visualize

begin
    Xₜₑₛₜ = poly_features(
        Base.collect(Base.range(Base.minimum(X_),Base.maximum(X_),length=50)),2)
    ŷ = Xₜₑₛₜ * w

    scatter(X_, y_, label="data points", xlabel="X", ylabel="y")
    plot!(Base.collect(Base.range(Base.minimum(X_),Base.maximum(X_),length=50)), ŷ)
end
```