Irwin-Hall distubution

Let's X_k be independant random variables uniformly distubuted in $[0,1]$ (iid)

$$S_n = \sum_{k=0}^{n} X_k$$

↳ $\mathbb{E}[X] = 1/2$

$\mathbb{V}[X] = 1/12$ *

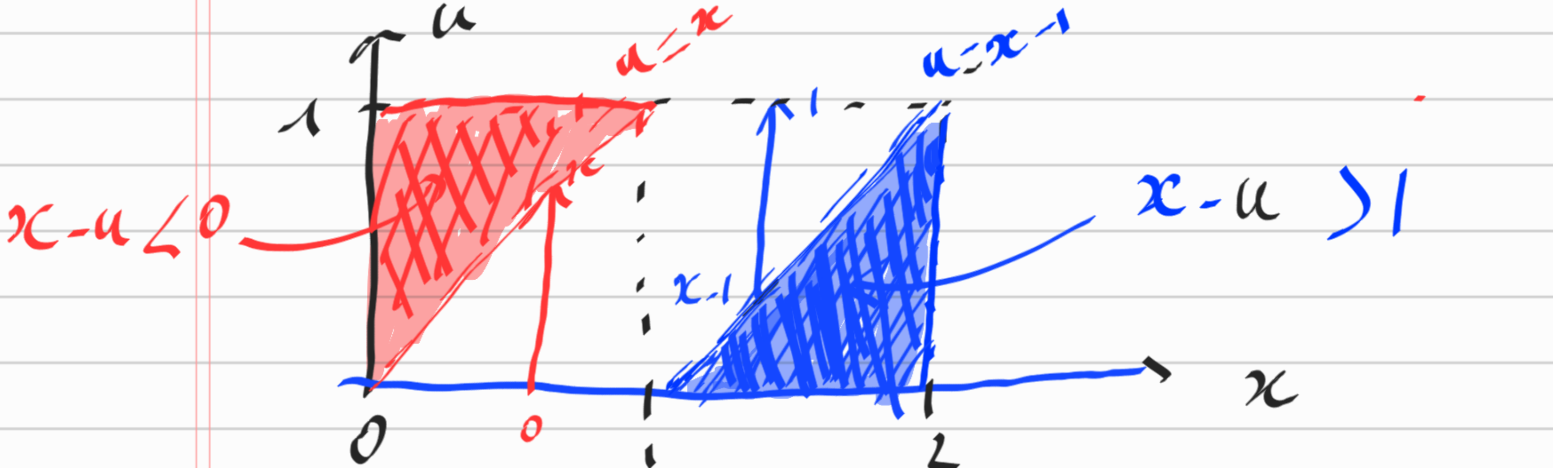
• $n=2$ $\quad S_2 = X_1 + X_2 \quad \in [0,2]$

p.d.f $\quad f_{S_1}(x) = 1 \qquad x \in [0,1)$

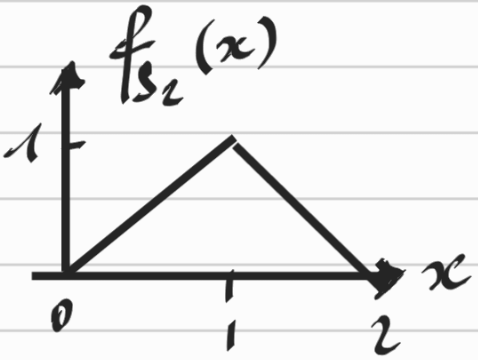
$$f_{S_2}(x) = \int_0^1 \underbrace{f_{S_1}(u)}\,\underbrace{f_{S_1}(x-u)}\,du \qquad x \in [0,2]$$

$\neq 0$ for $[0<u<1] \cap [0<x-u<1]$

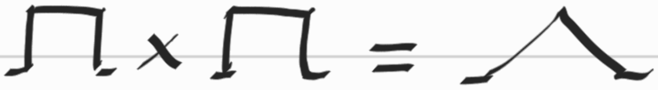
↳ $[0<u<x<1] \cup [1<x<u+1<2]$



$$\begin{cases} f_{S_2}(x) = \int_0^x du = x & x \in [0,1) \\ f_{S_2}(x) = \int_{x-1}^1 du = 2-x & x \in [0,2] \end{cases}$$



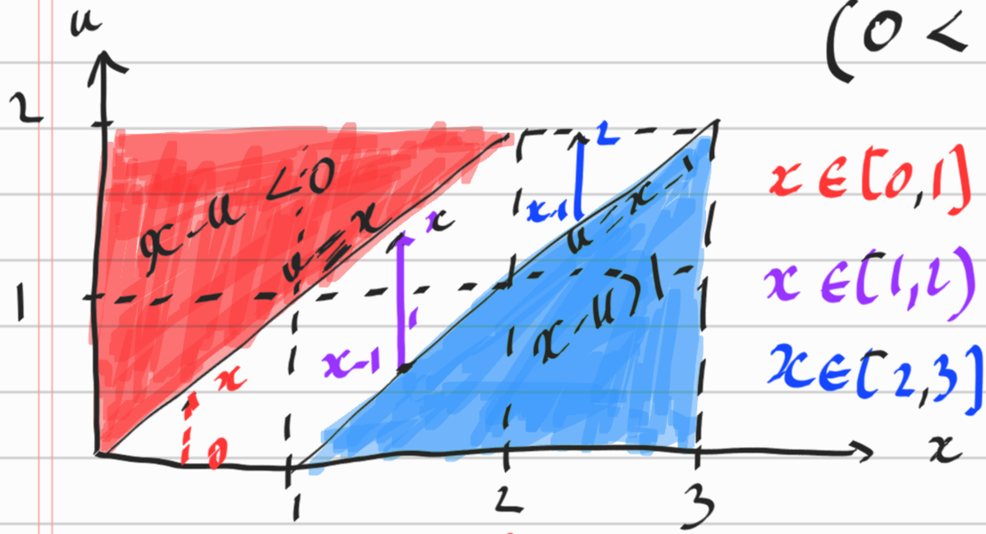
convolution product:



• $n = 3$ $S_3 = X_1 + X_2 + X_3$

pdf $f_{S_3}(x) = \int_0^1 \underbrace{f_{S_2}(u) f_{S_1}(x-u)}_{\neq 0 \text{ if } (0<u<2) \cap (0<x-u<1)} du$

$x \in [0,3]$



$x \in [0,1] \Rightarrow u \in [0,x]$

$x \in [1,2) \Rightarrow u \in [x-1, x]$

$x \in [2,3] \Rightarrow u \in [x-1, 2]$

a) $f_{S_3}(x) = \int_0^x f_{S_2}(u)\, du = \int_0^x u\, du = \left[u^2/2\right]_0^x$

b) $f_{S_3}(x) = \int_{x-1}^x f_{S_2}(u)\, du = \int_{x-1}^1 u\, du + \int_1^x (2-u)\, du$

c) $f_{S_3}(x) = \int_{x-1}^2 (2-u)\, du$

a) $x \in [0,1]$: $f_{S_3}(x) = x^2/2$

b) $x \in [1,2]$: $f_{S_3}(x) = \left[u^2/2\right]_{x-1}^{1} + \left[2u - u^2/2\right]_{1}^{x}$

$$= \frac{1}{2} - \frac{(x-1)^2}{2} + 2x - \frac{x^2}{2} - 2 + \frac{1}{2}$$

$$= -\frac{x^2}{2} + 2x - 1 - \frac{(x-1)^2}{2}$$

$$= \frac{x^2}{2} - (x^2 - 2x + 1) - \frac{(x-1)^2}{2}$$

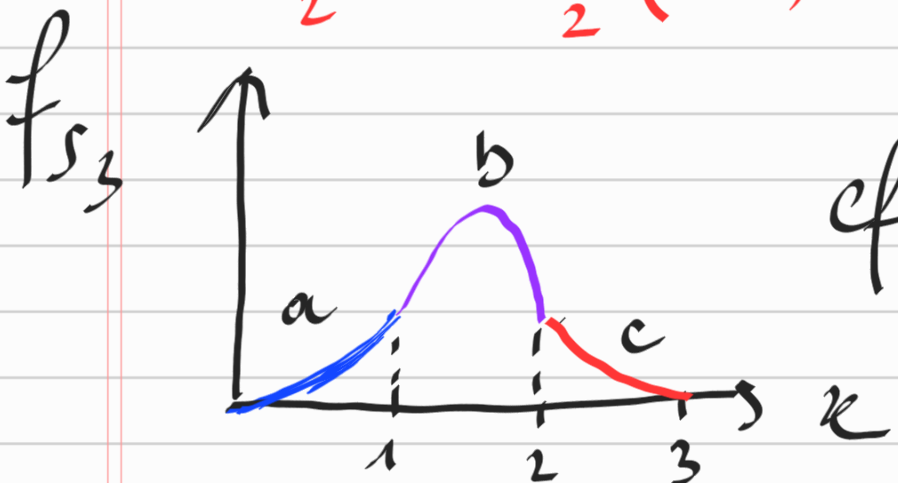
$$= \frac{x^2}{2} - \frac{3}{2}(x-1)^2$$

c) $x \in [2,3]$ $f_{S_3} = \int_{x-1}^{2} (2-u)\,du = \left[2u - \frac{u^2}{2}\right]_{x-1}^{2}$

$$= 4 - 2 - 2(x-1) + \frac{(x-1)^2}{2}$$

$$= 4 - 2x + \frac{(x-1)^2}{2}$$

$$= \frac{1}{2}x^2 - \frac{3}{2}(x-1)^2 + \frac{3}{2}(x-2)^2$$



cf "Spline"

Généralisation for S_n

Irwin-Hall distribution

$$f_{S_n}(x) = \frac{1}{(n-1)!} \sum_{k=0}^{\lfloor x \rfloor} (-1)^k \binom{n}{k} (x-k)^{n-1}$$

$$= \frac{1}{2(n-1)!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (x-k)^{n-1} \operatorname{sgn}(n-k)$$

$$x \in [0, n]$$

$$\mathbb{E}[S_n] = \mu = n/2 \overset{i.i.d}{=} n\,\mathbb{E}[X]$$

$$\mathbb{V}[S_n] = \sigma^2 = n/12 = n\,\mathbb{V}[X]$$

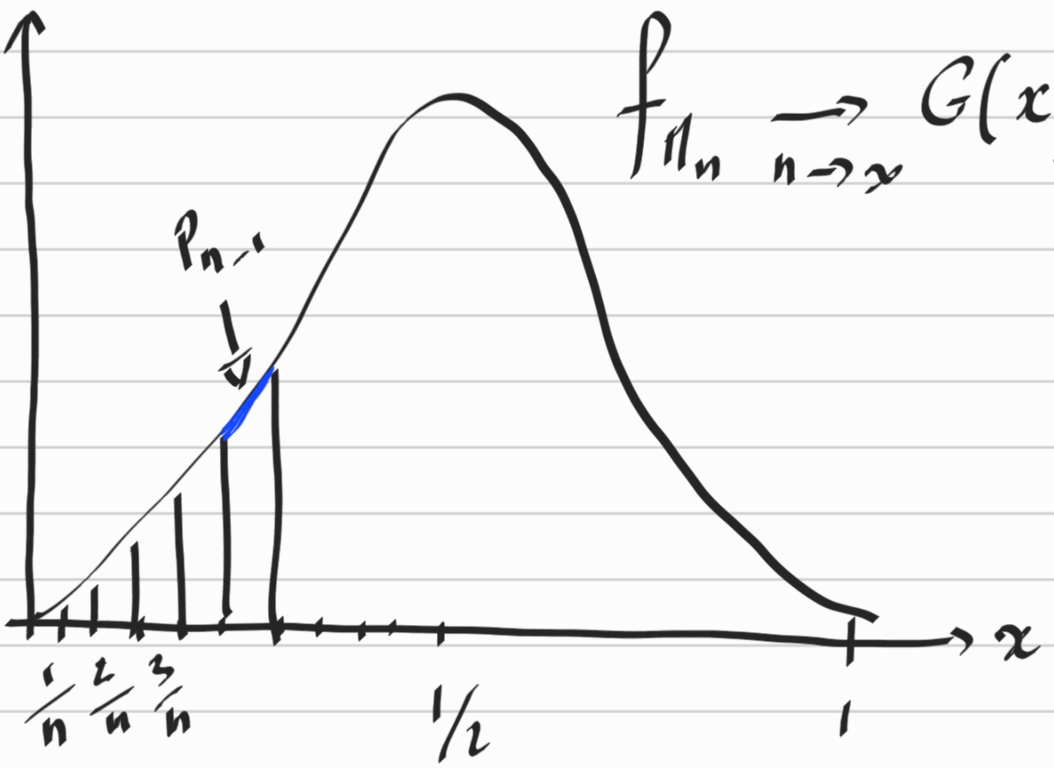
Variant : Bates distribution

$$M_n = \frac{1}{n} \sum_{k=1}^{n} X_n = \frac{1}{n} S_n$$

$$f_{M_n}(x) = \frac{n^n}{(n-1)!} \sum_{k=0}^{\lfloor nx \rfloor} (-1)^k \binom{n}{k} \left(x - \frac{k}{n}\right)^{n-1}$$

$$x \in [0, 1) \qquad \mathbb{E}[M_n] = 1/2 = \mathbb{E}[X]$$

$$\mathbb{V}[M_n] = \frac{1}{12n} = \frac{\mathbb{V}[X]}{n}$$



$$f_{M_n} \xrightarrow[n \to \infty]{} G\left(x; \frac{1}{2}, \frac{1}{\sqrt{12}}\right)$$

$$\underbrace{\mathbb{E}[X^2] - \mathbb{E}[X]^2}$$

* demo

$$\mathbb{V}[X] = \int_0^1 x^2\, dx - \left(\int_0^1 x\, dx\right)^2$$

$$= \left[\frac{x^3}{3}\right]_0^1 - \left(\left[\frac{x^2}{2}\right]_0^1\right)^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$