Central Limit Theorem

Assume X_i are independent and identically distributed (iid) random variables (r.v.) with definite moments

$$\begin{cases} \mathbb{E}[X_i] = \mu & \forall i \\ \mathbb{V}[X_i] = \sigma^2 & \forall i \end{cases}$$

Let's define

$$z'_n = \sum_{i=1}^{n} X_i \Rightarrow \begin{cases} \mathbb{E}[z'_n] = n\mu \\ \mathbb{V}[z'_n] = n\sigma^2 \end{cases}$$

Standardized Variable z_n

$$z_n = \frac{z'_n - \mathbb{E}[z'_n]}{\sigma_{z'_n}} = \frac{\sum^n X_i - n\mu}{\sqrt{n\sigma^2}}$$

with $\begin{cases} \mathbb{E}[z_n] = 0 \\ \mathbb{V}[z_n] = 1 \end{cases}$

Standardized X_i :

$$Y_i = \frac{X_i - \mathbb{E}[X_i]}{\sigma_{X_i}} = \frac{X_i - \mu}{\sigma}$$

and $$\boxed{Z_n = \sum_{i=1}^{n} \frac{Y_i}{\sqrt{n}}}$$

- pdf for Z_2 $(n=2)$

$$f_{Z_2}(z) = \int f_{Y_1/\sqrt{2}}(u) f_{Y_2/\sqrt{2}}(z-u)\, du$$

- and associated Characteristic function

$$\phi_{Z_2}(t) = TF^{-1}[f_{Z_2}] = \mathbb{E}[e^{itZ_2}]$$

$$= \phi_{Y_1/\sqrt{2}}\, \phi_{Y_2/\sqrt{2}}$$

$$= \left(\phi_{Y/\sqrt{2}}\right)^2 \quad (Y_i \text{ are iid})$$

now: $\phi_{y/\sqrt{2}}(t) = E\left[e^{ity/\sqrt{2}}\right]$

$= \phi_y\left(t/\sqrt{2}\right)$

then: $$\boxed{\phi_{z_2}(t) = \left(\phi_y\left(\frac{t}{\sqrt{2}}\right)\right]^2}$$

Generalizing to n variables

$$\boxed{\phi_{z_n}(t) = \left[\phi_y\left(\frac{t}{\sqrt{n}}\right)\right]^n}$$

We also have

$$\phi_y\left(\frac{t}{\sqrt{n}}\right) = \sum_{k=0}^{\infty}\left(\frac{it}{\sqrt{n}}\right)^k \tilde{\nu}_k$$

with $\tilde{\nu}_k = \mathbb{E}[y^k]$ $\equiv$ (standardized) k^{th} order moment

[The Characteristic function is kind of $M \times (i)^k$ Generating Function]

$\tilde{\nu}_0 = 1$ y_i moments must exist!

$\tilde{\nu}_1 = 0 = \mathbb{E}[Y]$

$\tilde{\nu}_2 = 1 = \mathbb{V}[y]$

$$\hookrightarrow \phi_y\left(\frac{t}{\sqrt{n}}\right) = 1 - \frac{t^2}{n} + O\left(\frac{t^3}{n^{3/2}}\right) \xrightarrow[n\to\infty]{} 1 - \frac{t^2}{n}$$

$$\hookrightarrow \phi_{z_n}(t) = \left[\phi_y\left(\frac{t}{\sqrt{n}}\right)\right]^n \xrightarrow[n\to\infty]{} \left(1 - \frac{t^2}{n}\right)^n$$

Now: $$\left(1-\frac{t^2}{n}\right)^n = \sum_{k=0}^{\infty} \binom{n}{k} \left(\frac{-t^2}{n}\right)^k$$

$$= \sum_{k=0}^{\infty} \frac{n!}{(n-k)!\,k!} \frac{(-t^2)^k}{n^k} = \sum_{k=0}^{\infty} \frac{\overbrace{n(n-1)\cdots(n-k-1)}^{k \text{ terms}}}{n^k} \frac{(-t^2)^k}{k!}$$

$$\xrightarrow[n\to\infty]{} \sum_{k=0}^{\infty} \frac{\cancel{n^k}}{\cancel{n^k}} \frac{(-t^2)^k}{k!} = e^{-t^2}$$

then:

$$\phi_{z_n} \sim e^{-t^2} \quad (\text{Gaussian like})$$

and $$f_{z_2} = T.F[\phi_{z_n}] \simeq e^{-z^2}$$

(Gaussian is eigenvector of TF !)

Conclusion : Normal law is the limit

for $z = \sum_{i=1}^{n} x_i$ (iid)

$$f_z \xrightarrow[n \to \infty]{} e^{-\left(\frac{z-\mu_z}{\sigma_z}\right)^2} \quad \forall \phi_x$$

if $\mathbb{E}[X]$ and $\mathbb{V}[X]$ exist

This doesn't apply to the Cauchy distribution for instance