Straight line fit (2 points) p122

①.

$$\chi^2(a,b) = \sum_{i=1}^{2}\left(\frac{y_i - (ax_i + b)}{\sigma}\right)^2$$

$$\frac{\partial \chi^2}{\partial a} = \sum_{i=1}^{2} \frac{-2x_i(y_i - ax_i - b)}{\sigma^2} = 0$$

$$\Leftrightarrow -2x_1(y_1 - ax_1 - b) - 2x_2(y_2 - ax_2 - b) = 0 \quad (1)$$

$$\frac{\partial \chi^2}{\partial b} = \sum_{i=1}^{2} \frac{-2(y_i - ax_i - b)}{\sigma^2} = 0$$

$$\Leftrightarrow -2(y_1 - ax_1 - b) - 2(y_2 - ax_2 - b) = 0 \quad (2)$$

$$(1) \equiv 2a(x_1^2 + x_2^2) + 2b(x_1 + x_2) - 2(x_1y_1 + x_2y_2) = 0$$

$$(2) \equiv \cancel{2}a(x_1 + x_2) + \cancel{4}^{2}b - \cancel{2}(y_1 + y_2) = 0$$

$$\Leftrightarrow 2b = y_1 + y_2 - a(x_1 + x_2)$$

$$(1) \equiv 2a(x_1^2 + x_2^2) + \underbrace{\left[y_1 + y_2 - a(x_1 + x_2)\right]}_{=2b}(x_1 + x_2) - 2(x_1y_1 + x_2y_2) = 0$$

$$= 2a(x_1^2 + x_2^2) - a(x_1 + x_2)^2 + (x_1 + x_2)(y_1 + y_2) - 2(x_1y_1 + x_2y_2) = 0$$

$$= a\Big[\underbrace{\cancel{2}x_1^2 + \cancel{2}x_2^2 - \cancel{x_1^2} - \cancel{x_2^2} - 2x_1x_2}_{(x_2 - x_1)^2}\Big] + \Big[\underbrace{\cancel{x_1y_1} + \cancel{x_2y_2} + x_1y_2 + x_2y_1 - \cancel{2}x_1y_1 - \cancel{2}x_2y_2}\Big]$$

$$= (x_1 - x_2)y_2 + (x_2 - x_1)y_1$$

$$= (x_1 - x_2)(y_2 - y_1)$$

$$= a(x_2 - x_1)^2 - (x_2 - x_1)(y_2 - y_1) = 0 \Rightarrow \boxed{\hat{a} = \frac{y_2 - y_1}{x_2 - x_1}}$$

et $2\hat{b} = y_1 + y_2 - a(x_1 + x_2)$

$$= y_1 + y_2 - \frac{(y_2 - y_1)(x_1 + x_2)}{(x_2 - x_1)}$$

$$= \frac{(y_1 + y_2)(x_2 - x_1) - (y_2 - y_1)(x_1 + x_2)}{x_2 - x_1}$$

$$= \frac{y_1 x_2 + \cancel{y_2 x_2} - \cancel{y_1 x_1} - y_2 x_1 - y_2 x_1 - \cancel{y_2 x_2} + \cancel{y_1 x_1} + y_1 x_2}{x_2 - x_1}$$

$$2\hat{b} = \frac{2 y_1 x_2 - 2 y_2 x_1}{x_2 - x_1} \Rightarrow \boxed{\hat{b} = \frac{y_1 x_2 - y_2 x_1}{x_2 - x_1}}$$

- $\hat{b} = f_b(y_1, y_2)$

$$\sigma_{\hat{b}}^2 = \left(\frac{\partial f_b}{\partial y_1}\right)^2 \sigma_{y_1}^2 + \left(\frac{\partial f_b}{\partial y_2}\right)^2 \sigma_{y_2}^2 \qquad \sigma_{y_1} = \sigma_{y_2} = \sigma$$

$$= \left(\frac{x_2}{x_2 - x_1}\sigma\right)^2 + \left(\frac{-x_1}{x_2 - x_1}\sigma\right)^2 = \boxed{\frac{x_1^2 + x_2^2}{(x_2 - x_1)^2}\sigma^2 = \sigma_{\hat{b}}^2}$$

- $\hat{a} = f_a(y_1, y_2)$

$$\sigma_{\hat{a}}^2 = \left(\frac{\partial f_a}{\partial y_1}\right)^2 \sigma_{y_1}^2 + \left(\frac{\partial f_a}{\partial y_2}\right)^2 \sigma_{y_2}^2$$

$$= \left(\frac{-1}{x_2 - x_1}\sigma\right)^2 + \left(\frac{+1}{x_2 - x_1}\sigma\right)^2 = \boxed{\frac{2\sigma^2}{(x_2 - x_1)^2} = \sigma_{\hat{a}}^2}$$

- on a $y_2 = \hat{a} x_2 + \hat{b} = g(\hat{a}, \hat{b})$

$$\sigma_{y_2}^2 = \sigma^2 = \left(\frac{\partial g}{\partial a}\right)^2 \sigma_{\hat{a}}^2 + \left(\frac{\partial g}{\partial b}\right)^2 \sigma_{\hat{b}}^2 + 2 \frac{\partial g}{\partial a}\frac{\partial g}{\partial b}\sigma_{ab}$$

soit $\sigma^2 = x_2^2\sigma_a^2 + \sigma_b^2 + 2x_2\sigma_{ab}$

$$= x_2^2\frac{2\sigma^2}{(x_2-x_1)^2} + \frac{x_1^2+x_2^2}{(x_2-x_1)^2}\sigma^2 + 2x_2\sigma_{ab}$$

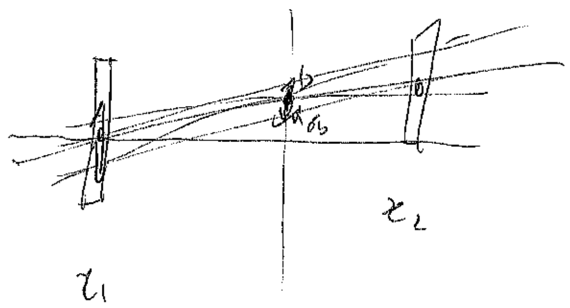
s.t $2x_2\sigma_{ab} = -\dfrac{2x_2^2 + x_1^2 + x_2^2 - (x_2-x_1)^2}{(x_2-x_1)^2}\sigma^2$

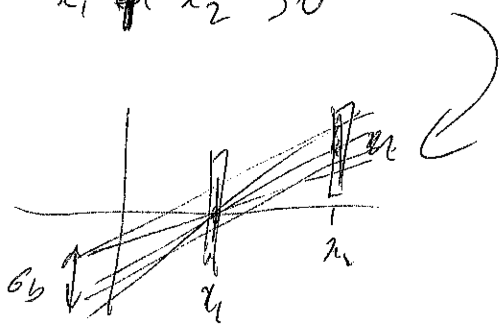
$$= -\frac{2x_2^2 + \cancel{x_1^2} + \cancel{x_2^2} - \cancel{x_1^2} - \cancel{x_2^2} + 2x_1x_2}{(x_2-x_1)^2}\sigma^2$$

$$\boxed{\sigma_{ab} = -\frac{x_2+x_1}{(x_2-x_1)^2}\sigma^2}$$

Uncorrelated when $x_2 = -x_1$

Anti-correlated $\sigma_{ab} < 0$ when $x_1 + x_2 > 0$





- Estimator are efficient (and unbiased)

$$Var[\hat{\theta}] = -E\left[\frac{\partial^2 \ln \mathcal{L}}{\partial \theta^2}\right]^{-1} \qquad \text{(RCF bound)}$$

$$\chi^2 = -2 \ln \mathcal{L}$$

- $\dfrac{\partial^2 \chi^2}{\partial \hat{a}^2} = +\dfrac{2(x_1^2+x_2^2)}{\sigma^2} = -2\dfrac{\partial^2 \ln \mathcal{L}}{\partial \hat{a}^2} \qquad -\dfrac{\partial^2 \ln \mathcal{L}}{\partial a^2} = +\dfrac{(x_1^2+x^2)}{\sigma^2}$

- $\dfrac{\partial^2 \chi^2}{\partial b^2} = -2\dfrac{\partial^2 \ln \mathcal{L}}{\partial b^2} = +\dfrac{4}{\sigma^2} \qquad -\dfrac{\partial^2 \ln \mathcal{L}}{\partial b^2} = +2/\sigma^2$

- $\dfrac{\partial^2 \chi^2}{\partial a \partial b} = \dfrac{2(x_1+x_2)}{\sigma^2} = -2\dfrac{\partial^2 \ln \mathcal{L}}{\partial a \partial b} \qquad -\dfrac{\partial^2 \ln \mathcal{L}}{\partial a \partial b} = +\dfrac{x_1+x_2}{\sigma^2}$

Warning multidimensional case $\ln \mathcal{L}(\vec{a}, \vec{b})$
RCF bound for efficient estimators

$$\underset{\substack{\uparrow \\ \text{matrix}}}{Cov_{ab}} = -E[\underbrace{\partial^2_{ab} \ln \mathcal{L}}_{\text{matrix of 2nd derivatives (Hesse Matrix)}}]^{-1}$$

matrix of 2nd derivatives (Hesse Matrix) $= \begin{pmatrix} +\dfrac{x_1^2+x_2^2}{\sigma^2} & +\dfrac{x_1+x_2}{\sigma^2} \\ +\dfrac{x_1+x_2}{\sigma^2} & +2/\sigma^2 \end{pmatrix}$

$$Cov_{ab} = \begin{pmatrix} \dfrac{x_1^2+x_2^2}{\sigma^2} & \dfrac{x_1+x_2}{\sigma^2} \\ \dfrac{x_1+x_2}{\sigma^2} & \dfrac{2}{\sigma^2} \end{pmatrix}^{-1} = \Delta^{-1}$$

$$\det(\Delta) = \frac{2(x_1^2+x_2^2)-(x_1+x_2)^2}{\sigma^4} = \frac{(x_2-x_1)^2}{\sigma^4}$$

$$Cov = \Delta^{-1} = \frac{1}{\det(\Delta)}\left(Com(\Delta)^T\right) = \frac{\sigma^4}{(x_2-x_1)^2}\begin{pmatrix} \dfrac{2}{\sigma^2} & -\dfrac{x_1+x_2}{\sigma^2} \\ -\dfrac{x_1+x_2}{\sigma^2} & \dfrac{x_1^2+x_2^2}{\sigma^2} \end{pmatrix} = \begin{pmatrix} \dfrac{2\sigma^2}{(x_2-x_1)^2} & -\dfrac{x_1+x_2}{(x_2-x_1)^2}\sigma^2 \\ -\dfrac{x_1+x_2}{(x_2-x_1)^2}\sigma^2 & \dfrac{x_1^2+x_2^2}{(x_2-x_1)^2}\sigma^2 \end{pmatrix}$$

(5)

$$Cov_{ab} = \begin{pmatrix} \sigma_a^2 & \sigma_{ab} \\ \sigma_{ab} & \sigma_b^2 \end{pmatrix} = \begin{pmatrix} 2 & -(x_1+x_2) \\ -(x_1+x_2) & x_1^2+x_2^2 \end{pmatrix} \times \frac{\sigma^2}{(x_2-x_1)^2}$$

i.e.:

$$\sigma_a^2 = \frac{2\sigma^2}{(x_2-x_1)^2}\,\sigma^2$$

$$\sigma_b^2 = \frac{x_1^2+x_2^2}{(x_2-x_1)^2}\,\sigma^2$$

$$\sigma_{ab} = -\frac{x_1+x_2}{(x_2-x_1)^2}\,\sigma^2$$

ok! identical to direct estimation (i.e. LS estimators are efficient)

$n = 2$

$$\hat{a} = \frac{n \sum_i x_i y_i - \sum_i \sum_j x_i y_j}{n \sum x_i^2 - (\sum x_i)^2}$$

$$= \frac{\cancel{2}(x_1 y_1 + x_2 y_2) - (\cancel{x_1 y_1} + x_1 y_2 + x_2 y_1 + \cancel{x_2 y_2})}{2(x_1^2 + x_2^2) - (x_1 + x_2)^2}$$

$$= \frac{x_1 y_1 + x_2 y_2 - x_1 y_2 - x_2 y_1}{\cancel{2}(x_1^2 + x_2^2) - (\cancel{x_1^2} + \cancel{x_2^2} + 2x_1 x_2)}$$

$$= \frac{x_1(y_1 - y_2) + x_2(y_2 - y_1)}{(x_1 - x_2)^2} = \frac{\cancel{(x_1 - x_2)}(y_1 - y_2)}{(x_1 - x_2)^{\cancel{2}}}$$

$$\boxed{\hat{a} = \frac{y_1 - y_2}{x_1 - x_2}}$$