Some useful equalities

* Sample Mean $M = \frac{1}{n} \sum_{k=1}^{n} X_k$

$$\mathbb{E}[M] = \frac{1}{n} \sum_{k=1}^{n} \mathbb{E}[X_k] \quad \text{(linearity)}$$

X_i are i.i.d $\quad \mathbb{E}[X_k] = \mu \quad \forall k$

$$\mathbb{E}[M] = \frac{1}{n} n\mu = \mu = \mathbb{E}[X_k]$$

$$\boxed{\mathbb{E}[M] = \mathbb{E}[X_k] = \mu}$$

note weighted mean : $M_\omega = \frac{\sum \omega_k X_k}{\sum \omega_k}$

normalized weights : $\tilde{\omega}_k = \frac{\omega_k}{\sum \omega_k}$

$$M_\omega = \sum_{k=1}^{n} \tilde{\omega}_k X_k$$

$$\mathbb{E}[M_\omega] = \sum_{k=1}^{n} \tilde{\omega}_k \, \mathbb{E}[X_k] \quad \text{(linearity)}$$

$$= \mu\left(\sum_{k=1}^{n} \tilde{\omega}_k\right) = \mu = \mathbb{E}[X_k]$$

$$\forall\, \tilde{\omega}_k \quad \left(M = M_\omega \text{ for } \tilde{\omega} = \frac{1}{n} \right)$$

$$\mathbb{V}[M] = \mathbb{V}\left[\frac{1}{n} \sum_k X_k\right]$$

$$= \frac{1}{n^2} \mathbb{V}\left[\sum_k X_k\right]$$

$$\overset{iid}{=} \frac{1}{n^2} \sum_k \mathbb{V}[X_k]$$

$$\boxed{\mathbb{V}[M] = \frac{\mathbb{V}[X_k]}{n}}$$

$$\sigma_M = \frac{\sigma}{\sqrt{n}} \quad !$$

$$\mathbb{V}[M_\omega] = \mathbb{V}\left[\sum_k \tilde{\omega}_k X_k\right]$$

$$= \sum_k \tilde{\omega}_k^2 \, \mathbb{V}[X_k]$$

$$\boxed{\mathbb{V}[M_\omega] = \alpha_n \mathbb{V}[X_k]}$$

$$\alpha_n = \frac{\sum^n \omega_k^2}{\left(\sum^n \omega_k\right)^2}$$

if $\omega_k = 1/n$ $\qquad \alpha_n = 1/n$

+ Sample Variance (unbiased)

$$S_{n-1}^2 = \frac{1}{n-1} \sum_{k=1}^n (X_k - M)^2$$

$$\mathbb{E}[S_{n-1}^2] = \frac{1}{n-1} \sum_{k=1}^n \mathbb{E}[(X_k - M)^2] \text{ (linearity)}$$

$$= \frac{1}{n-1} \sum_{k=1}^n \mathbb{E}\left[\left((X_k - \mu) - (M - \mu)\right)^2\right]$$

$\mu = \mathbb{E}[X_k] \ \forall k$ (iid)

$$\mathbb{E}[S_{n-1}^2] = \frac{1}{n-1}\Big[\sum_{k=1}^{n} \mathbb{E}[(X_k-\mu)^2] \quad = \mathbb{V}[X_k]$$

$$+ \sum_{k=1}^{n} \mathbb{E}[(M-\mu)^2] \quad = \mathbb{V}[M]$$

$$- \sum_{k=1}^{n} \mathbb{E}[(x_k-\mu)(M-\mu)] \times 2\Big]$$

$$= \frac{1}{n-1}\Big(\sum_k \sigma^2 + \sum_k \frac{\sigma^2}{n}$$

$$-2 \sum_k \mathbb{E}[(X_k-\mu)(M-\mu)]\Big)$$

now $\quad M-\mu = \frac{1}{n}\sum_{k=1}^{n} X_k - \frac{\mu n}{n}$

$$= \frac{1}{n}\sum_k (X_k-\mu)$$

$$\sum_k (X_k-\mu) = n(M-\mu)$$

$$\sum_k (X_k-\mu)(M-\mu) = n(M-\mu)^2$$

and $\mathbb{E}\left[\sum_k (X_k-\mu)(\bar{X}-\mu)\right] = n\,\mathbb{E}[(\bar{X}-\mu)^2]$

$= n\,\mathbb{V}[\bar{X}] = n\,\frac{\sigma^2}{n} = \sigma^2$

then $\mathbb{E}[S^2_{n-1}] = \frac{1}{n-1}\left(n\sigma^2 + \sigma^2 - 2\sigma^2\right)$

$= \sigma^2 = \mathbb{V}[X]$

$$\boxed{\mathbb{E}[S^2_{n-1}] = \mathbb{V}[X] = \sigma^2}$$

$\mathbb{E}[S^2_n] = \frac{n-1}{n}\sigma^2$

is biased !

but $\mathbb{E}[S^2_n] \xrightarrow[n\to\infty]{} \sigma^2$

- $\mathbb{V}[XY] = \mathbb{E}[X^2Y^2] - \mathbb{E}[XY]^2$

$$\stackrel{iid}{=} \mathbb{E}[X^2]\mathbb{E}[Y^2] - \mathbb{E}[X]^2\mathbb{E}[Y]^2$$

$$= (\mathbb{V}[X] + \mathbb{E}^2[X])(\mathbb{V}[Y] + \mathbb{E}^2[Y])$$

$$- \mathbb{E}^2[X]\,\mathbb{E}^2[Y]$$

$$= \mathbb{V}[X]\mathbb{V}[Y]$$

$$+ \mathbb{E}^2[X]\,\mathbb{V}[Y] + \mathbb{E}^2[Y]\,\mathbb{V}[X]$$