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Lemma 3.6.12

Assuming $MA(\aleph_1)$. Let X be a ccc space and $\{U_\alpha : \alpha < \omega_1\}$ be a family of nonempty open subsets of X. Then there is an uncountable $A \subseteq \omega_1$ such that $\{U_\alpha : \alpha \in A\}$ has the fip.

Proof. For each $\alpha \in \omega_1$, define $V_\alpha = \bigcup_{\xi \in \alpha} U_\xi$. Note that $\langle V_\alpha : \alpha \in \omega_1 \rangle$ is $\subseteq$-decreasing.
Claim there is some $\alpha \in \omega_1$ such that

$$\overline{V_\beta} = \overline{V_\alpha} \quad \text{for all } \beta \geq \alpha.$$

(Using ccc; we will only consider V_β for $\beta \geq \alpha$.)

Suppose not. There for each $\alpha \in \omega_1$, there is some $\beta > \alpha$ such that $\overline{V_\beta} \subsetneq \overline{V_\alpha}$. Then, there is a strictly increasing sequence $\langle \alpha_\xi : \xi \in \omega_1 \rangle$ such that $\overline{V_{\alpha_{\xi+1}}} \subsetneq \overline{V_{\alpha_\xi}}$ for each ξ.

So, for each ξ, $\overline{V_{\alpha_\xi}} \nsubseteq \overline{V_{\alpha_{\xi+1}}}$ which implies $V_{\alpha_\xi} \nsubseteq \overline{V_{\alpha_{\xi+1}}}$. Then, for each ξ, $V_{\alpha_\xi} \setminus \overline{V_{\alpha_{\xi+1}}}$ is open and non-empty.

Hence $\{V_{\alpha_\xi} \setminus \overline{V_{\alpha_{\xi+1}}} : \xi \in \omega_1\}$ is an uncountable family of (pointwise) disjoint open non-empty sets.

This is impossible where X has ccc.

By the claim, we may assume, WLOG, that $\overline{V_\beta}$'s are equal for all $\beta \in \omega_1$ (i.e., α in the claim is 0).

$V_0 \supseteq V_1 \supseteq V_2 \ldots$

Define a poset $(\mathcal{P}, \leq)$ by

$$\mathbb{P} = \{p \subseteq V_0 : p \text{ is open and non-empty}\}.$$

$$= \{p \in \mathbb{O}_X : p \subseteq V_0\}$$

and $p \leq q$ iff $p \subseteq q$. (Note: $\mathbb{P}$ is a subposet of $\mathbb{O}_X$.)

This poset is ccc since X (and V_0) has ccc (subposet that has ccc also has ccc.)

The set has FIP!

For each $\beta \in \omega_1$, define

$$D_\beta = \{p \in \mathbb{P} : \exists \xi > \beta \, (p \subseteq U_\xi)\}$$

To see that D_β is dense, let $p \in \mathbb{P}$. (Let $V_\beta = \bigcup_{\xi > \beta} U_\xi$.) Since $\overline{V_0} = \overline{V_\beta}$ and $p \subseteq V_0$,

$$p \cap V_\beta \neq \varnothing.$$

(If $p \cap V_\beta = \varnothing$ then $p \cap \overline{V_\beta} = \varnothing$.) so $p \cap U_\xi \neq \varnothing$ for some $\xi > \beta$. Choose $q = p \cap U_\xi$. Then, $q \leq p$ and $q \in D_\beta$.

Now, we consider $\mathcal{D} = \{D_\beta : \beta \in \omega_1\}$ is a collection of dense subsets of $\mathbb{P}$, and $|\mathcal{D}| \leq \omega_1$.

By $MA(\aleph_1)$, there is a filter $G \subseteq \mathbb{P}$ such that $G \cap D_\beta \neq \varnothing$ for all $\beta \in \omega_1$.

Define,

$$A = \{\xi \in \omega_1 : \exists p \in G(p \subseteq U_\xi)\}.$$

A is unbounded in ω_1: Let $\beta \in \omega_1$. Since $G \cap D_\beta \neq \varnothing$, we can pick $p \in G \cap D_\beta$. As $p \in D_\beta$, $p \subseteq U_\xi$ for some $\xi > \beta$. So, $\xi \in A$.

$\{U_\xi : \xi \in A\}$ has the fip: Let $U_{\alpha_1}, \ldots, U_{\alpha_n}$ with $\alpha_1, \ldots, \alpha_n \in A$. Then, for each $i \in \{1, \ldots, n\}$, $U_{\alpha_i} \supseteq p_i$ for some $p_i \in G$.

Since G is a filter, there is some $q \in G$ s.t. $q \leq p_i$ for all $i \in \{1, \ldots, n\}$. So $\varnothing \neq q \subseteq U_{\alpha_i}$ for all i. Hence, $\bigcap_{i=1}^{\infty} U_{\alpha_i} \neq \varnothing$.

Lemma 3.6.13

Assuming $MA(\aleph_1)$. Any product of two ccc spaces is also ccc.

Proof. Suppose X, Y are ccc spaces. Let $\mathcal{D} = \{U_\alpha \times C_\alpha : \alpha \in \omega_1\}$ be a collection of basic open subsets of $X \times Y$.

Idea: using Lemma 3.6.12 but the index may be not the same.

Consider $\{U_\alpha : \alpha \in \omega\}$ since X is ccc by Lemma 3.6.12, there is an uncountable $A \subseteq \omega_1$ such that

$$\{U_\alpha : \alpha \in A\}$$

has the fip. Consider $\{V_\alpha : \alpha \in A\}$. Since Y is ccc (and A is uncountable.) there exist distinct $\alpha, \beta \in A$ s.t. $V_\alpha \cap V_\beta \neq \varnothing$. By $(*)$, $U_\alpha \cap U_\beta \neq \varnothing$.

X is a topological space. For any collection $\{U_\alpha : \alpha \in \omega_1\}$ of non-empty open sets. There exists $\{A \subseteq \omega_1\}$ with $|A| = \omega_1$ and $\{U_\alpha : \alpha \in A\}$ has the fip.

(The property will call $\aleph_1$ is precaliber of X has $\aleph_1$ as a precaliber to the ccc by $MA(\aleph_1)$)

Lemma 3.6.7

There is a non-cc forcing poset $\mathbb{P}$ such that $MA(\aleph_1)$ fails. *Proof* Consider $\mathbb{P} = \mathrm{Fn}(I, J)$. For each $i \in I$ and $j \in J$, define

$$D_i = \{p \in P : i \in \mathrm{dom}(p)\}$$

and

$$R_j = \{p \in P : j\}$$

Since p is fintie $p \cup \{(i, j)\}$ and $p \cap \{(i', j')\}$. Similar as in the proof of Lemma 3.6.6, we can show taht D_i and R_j are dense in $\mathbb{P}$.

Consider $\mathcal{D} = \{D_i : i \in I\} \cup \{R_j : j \in J\}$. We have $|\mathcal{D}| \leq |I| + |J| = \aleph_1$.

By $MA(\aleph_1)$, we obtain a filter $G \subseteq \mathbb{P}$ s.t.

$$G \cap D_i \neq \varnothing \neq G \cap R_j$$

for any $i \in I$ and $j \in J$.

Similar as in the proof of Lemma 3.6.6. again. - f is a function (G is a filter) - $\mathrm{dom}(f) = I$ $(G \cap D_i \neq \varnothing.)$ - $\mathrm{ran}(f) = J$ $(G \cap R_i \neq \varnothing.)$ So $f : I \to J$ is onto, which is impossible.

Poset: Atom

Definition - $r \in \mathbb{P}$ is an atom if there is $p, q \leq r$ s.t. $p \perp q$.
- $\mathbb{P}$ is atomless if there are no atoms in $\mathbb{P}$; i.e. $\forall p \in \mathbb{P}\,(q \leq p \wedge r \leq p \wedge q \perp r)$
(ccc and atom are independent to each other.)

Lemma 3.6.10

1. If $\mathbb{P}$ has an atom, then $MA_{\mathbb{P}}(\kappa)$ holds for all κ

Proof. Define

$$G = \{p \in \mathbb{P} : p \not\perp r\}.$$

Claim 1: $G \cap D \neq \varnothing$ for any dense $D \subseteq \mathbb{P}$.
Suppose $D \subseteq \mathbb{P}$ is dense.
Then, there is some $s \leq r$ s.t. $s \in D$. So, $s \in G \cap D$.
$(\forall p \in \mathbb{P}\, \exists q \in D\, (q \leq p))$

Claim 2: G is a filter.
Closed upwardness is clear.
Next, let $p, q \in G$.
We have s extends p, r and t extends q, r,
since r is an atom, we have u extending s, t.
So u extends p and q, and $u \in G$ $(\because u \leq r.)$

2. If $\mathbb{P}$ is atomless, then $MA_{\mathbb{P}}(\kappa)$ is false for $\kappa = 2^{|\mathbb{P}|}$

Proof. Suppose that $\mathbb{P}$ is atomless.
We show that for any filter $G \subseteq P$, there is some dense set $D \subseteq \mathbb{P}$ such that $G \cap D = \varnothing$.

Let G be a filter in $\mathbb{P}$.
Define $D = \mathbb{P} \setminus G$.
D is dense.
Let $p \in \mathbb{P}$.
If $p \notin G$, $p \in D$.
Suppose $p \in G$, p is not atom.
So, there exists $p, q \leq r$ s.t. $p \perp q$.

Since G is a filter, r or s lies outside G (i.e., lies inside D)