Sheaves and their Cohomology

Date: 2023-08-18 8:00-10:00 AM
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Last session: introduced (pre-)sheaves and stalks (ad hoc def.)

Exercise Arapura, 3.1.15

A formal definition of stalks is based on the injective limit (direct limit/filtered colimit)

Let $\mathcal{C}$ be a category and I be a directed set, i.e., a poset $(I, \leq)$ such that $\forall i, j \in I \exists k \in I : i \leq k, j \leq k$. (e.g. ($\mathbb{N}$, divisibility))

Def An injective system (a direct system) over I in $\mathcal{C}$ is a pair $((X_i)_{i \in I}, (f_{ij})_{i \leq j})$ consisting a family of objects $X_i \in \mathrm{Ob}(\mathcal{C})$ and a family of morphisms $f_{ij} : X_i \to X_j$ for $i \leq j$ such that

- $\forall i \in I : f_{ii} = \mathrm{id}_X$
- $\forall i \leq j \leq k : f_{ik} = f_{jk} \circ f_{ij}$

An injective limit (a direct limit) of an injective system $((X_i)_{i \in I}, (f_{ij})_{i \leq j})$ is an object X together with morphisms $r_i : X_i \to X$ for $i \in I$ such that $\forall i \leq j : r_i = r_j \circ f_{ij}$ which satisfies the following universal property:

For all $Y \in \mathrm{Ob}(\mathcal{C})$ and morphisms $(\psi_i : X_i \to Y)_{i \in I}$ such that $\forall i \leq j : \psi_i = \psi_j \circ f_{ij}$, there exists a unique $\psi : X \to Y$ such that $\forall i : \psi \circ r_i = \psi_i$.

$$\begin{tikzcd}
X_i \arrow[rr, "\rho_{ij}"] \arrow[dr, "r_i"] \arrow[ddr, "\psi_i"'] & & X_j \arrow[dl, "r_j"'] \arrow[ddl, "\psi_j"] \\
& X \arrow[d, dashed, "\exists!\psi"] & \\
& Y &
\end{tikzcd}$$

Remark: the injective limit is unique up to unique isomorphism

Notation: $\varinjlim_{i \in I} X_i$

Examples: In (Sets)/(Grps)/(Rings)/(R-Mods)/...

$$\varinjlim_{i \in I} X_i = \left(\coprod_{i \in I} X_i \right) \Big/ ((x_i \in X_i) \sim (x_j \in X_j) \iff \exists k : i, j \leq k \text{ and } f_{ik}(x_i) = f_{jk}(x_j))$$

If $\mathcal{F}$ is a presheaf on a topological space X and $a \in X_i$, then

$$\mathcal{F}_a = \varinjlim_{a \in U \overset{\circ}{\subseteq} X} \mathcal{F}(U)$$

f_{ij} - $\rho_{VU} : \mathcal{F}(V) \to \mathcal{F}(U)$ $(V \supseteq U)$; $\mathcal{F}(U) \to \mathcal{F}_a$, $s \mapsto [(U, s)]$.

Proposition Let X be a topological space, $a \in X$. Every morphism of presheaves $\varphi : \mathcal{F} \to \mathcal{G}$ gives rise to a map (morphism) $\varphi_a : \mathcal{F}_a \to \mathcal{G}_a$ where $[(U, s)] \mapsto [(u, \varphi_u(s))]$.

In particular, $\mathcal{F} \mapsto \mathcal{F}_a$ is a functor $\mathrm{PSh}(X) \to (\mathrm{Sets})$, or also $\mathrm{PAb}(X) \to (\mathrm{Ab})$ etc.

Proof. Apply the universal property of $\varinjlim \mathcal{F}(U)$ to $(\mathcal{F}(U) \overset{\varphi_U}{\to} \mathcal{G}(U) \overset{\varphi_a}{\to} \mathcal{G}_a)$

$$\begin{tikzcd}
\mathcal{F}(V) \arrow[rr] \arrow[dd, "\rho_{VU}"] \arrow[dr] \arrow[drrr] & & \mathcal{G}(V) \arrow[dd] \arrow[dr] & \\
& \mathcal{F}_a \arrow[rr, dotted] & & \mathcal{G}_a \\
\mathcal{F}(U) \arrow[rr] \arrow[ur] \arrow[urrr] & & \mathcal{G}(U) \arrow[ur] &
\end{tikzcd}$$

This is possible since $\forall U \overset{\circ}{\subseteq} V \overset{\circ}{\subseteq} X, a \in U : (\iota_U^{\mathcal{F}} \circ \varphi_a) \circ \rho_{VU} = \iota_V^{\mathcal{G}} \circ \varphi_V$

Proposition Let X be a topological space, $\mathcal{F}, \mathcal{G}$ be sheaves on X and $\varphi : \mathcal{F} \to \mathcal{G}$ a morphism

1. $\forall U \overset{\circ}{\subseteq} X : \mathcal{F}(U) \to \prod_{a \in U} \mathcal{F}_a$ which $s \mapsto (s_a)_{a \in U}$ is injective $s_n := [(U, s)] \in \mathcal{F}_a$.
2. $(\forall U \overset{\circ}{\subseteq} X : \varphi_U : \mathcal{F}(U) \to \mathcal{G}(U))$ injective/bijective $\iff$ $(\forall a \in X : \varphi_a : \mathcal{F}_a \to \mathcal{G}_a$ injective/bijective$)$
3. if $\psi : \mathcal{F} \to \mathcal{G}$ is another morphism then $\varphi = \psi \iff \forall a \in X : \varphi_a = \psi_a$.

Proof.

1. Let $s, t \in \mathcal{F}(U)$ be such that $\forall a \in U : s_a = t_a \implies \forall a \in U \exists V_a \overset{\circ}{\subseteq} U, a \in V_a : s|_{V_a} = t|_{V_a}$. since $\bigcup_{a \in U} V_a = U$, we get $s = t$ by the sheaf axiom.
2. Exercise
3. $(\implies)$ Trivial, $(\impliedby)$ To show: $\forall U \overset{\circ}{\subseteq} X \forall s \in \mathcal{F}(U) : \varphi_U(s) = \psi_U(s) \overset{\text{(i)}}{\iff} \forall U \overset{\circ}{\subseteq} X \forall s \in \mathcal{F}(U) \forall a \in U : \varphi_U(s)_a = \psi_U(s)_a$ but $\varphi_U(s)_a = \varphi_a(s_a)$ and $\psi_U(s)_a = \psi_a(s_a)$. Hence the claim follows. $\square$

Question: what about "surjective"?

Example. On $X = \mathbb{C}$, consider $D = \frac{d}{dz} : \mathcal{O}_{\mathbb{C}} = \mathcal{O}_{\mathbb{C}}$. This is given by $f \mapsto f'$ for all $U \overset{\circ}{\subseteq} \mathbb{C}$.

Then $U \overset{\circ}{\subseteq} V \overset{\circ}{\subseteq} \mathbb{C}$:

$$\begin{tikzcd}
\mathcal{O}_{\mathbb{C}}(V) \arrow[r, "D"] \arrow[d, "\mathrm{res}"] & \mathcal{O}_{\mathbb{C}}(V) \arrow[d, "\mathrm{res}"] \\
\mathcal{O}_{\mathbb{C}}(U) \arrow[r, "D"] & \mathcal{O}_{\mathbb{C}}(V)
\end{tikzcd}$$

Known: $\mathcal{O}_{\mathbb{C},a} = \left\{\sum_{n\geq 0} c_n(z-a)^n, c_n \in \mathbb{C},\ \text{positive radius of convergence}\right\}$
$\left(\frac{d}{dz}|_a : \mathcal{O}_{\mathbb{C},a} \to \mathcal{O}_{\mathbb{C},a}\right)$ is surjective.

But $\frac{1}{z} \in \mathcal{O}_{\mathbb{C}}(\mathbb{C} \setminus \{0\})$ has no preimage in $\mathcal{O}_{\mathbb{C}}(\mathbb{C} \setminus \{0\})$.

Problem: the presheaf defined by $U \mapsto \mathrm{im}(D_u : \mathcal{O}_{\mathbb{C}}(U) \to \mathcal{O}_{\mathbb{C}}(U))$ is not a sheaf!

Proposition/Definition let X be a topological space, $\mathcal{F}$ be a presheaf on X. There exists a sheaf $\mathcal{F}^+$ and a morphism $\iota : \mathcal{F} \to \mathcal{F}^+$ with the following universal property:

For every sheaf $\mathcal{G}$ on X and morphism $\varphi : \mathcal{F} \to \mathcal{G}$, there exists a morphism $\varphi^+ : \mathcal{F}^+ \to \mathcal{G}$ such that $\varphi^+ \circ \iota = \varphi$.

$$\begin{tikzcd}
\mathcal{F} \arrow[rr, "\varphi"] \arrow[dr, "\iota"'] & & \mathcal{G} \\
& \mathcal{F}^+ \arrow[ur, dotted, "\varphi^+"'] &
\end{tikzcd}$$

(F^+, ι) is unique up to unique isomorphism. It is called the sheafification or the associated sheaf of $\mathcal{F}$. Furthermore, the following properties hold:

1. $\forall a \in X : \iota_a : \mathcal{F}_a \to (\mathcal{F}^+)_a$ is an isomorphism.
2. (Functoriality) for every morphism of presheaves $\mathcal{F} \xrightarrow{\varphi} \mathcal{G}$, there is a unique morphism $\varphi^+ : \mathcal{F}^+ \to \mathcal{G}^+$ making the following diagram commutative.

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\varphi"] \arrow[d, "\varphi_{\mathcal{F}}"] & \mathcal{G} \arrow[r, "\psi"] \arrow[d, "\varphi_{\mathcal{G}}"] & \mathcal{H} \arrow[d, "\psi_{\mathcal{H}}"] \\
\mathcal{F}^+ \arrow[r, "\varphi^+"] & \mathcal{G}^+ \arrow[r, "\psi^+"] & \mathcal{H}^+
\end{tikzcd}$$

Remark.

1. If $\mathcal{F}$ is already a sheaf, then $r : \mathcal{F} \to \mathcal{F}^+$ is an isomorphism.
2. Sheafification functor $\mathrm{PSh}(X) \to \mathrm{Sh}(X)$ (Sh to Ab)

Universal property: for every presheaf $\mathcal{F}$ and a sheaf $\mathcal{G}$ on X, there is a national bijection (natural in sense of natural transformation)

$$\mathrm{Hom}_{\mathrm{Sh}(X)}(\mathcal{F}^+, \mathcal{G}) \cong \mathrm{Hom}_{\mathrm{PSh}(X)}(\mathcal{F}, \mathcal{G})$$

The sheafification is left-adjoint to the inclusion functor.

Proof.[1]

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[1] feedback: the difference between sheaves and presheaves in sense of stalks.

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$U \overset{\circ}{\subseteq} X, \mathcal{F}^+(U) = \{(s_a) \in \prod_{a\in U} \mathcal{F}_a : \forall a \in U \exists V_a \overset{\circ}{\subseteq} U, a \in V_a \exists t \in \mathcal{F}(V_a), s_b = t_b$ for all $b \in V_a\}$

- $\mathcal{F}^+$ is a sheaf, $\iota : \mathcal{F} \to \mathcal{F}^+$ is obvious (!)
- $\forall a \in X : \iota_a : \mathcal{F}_a \to (\mathcal{F}^+)_a$ is bijective.

Proof. The inverse map $(\mathcal{F}^+)_a \to \mathcal{F}_a$ is given by $\mathcal{F}^+(U) \to \mathcal{F}_a$, $(s_b)_{b\in U} = s_a$for $U \overset{\circ}{\subseteq} X$ such that $a \in U$ + universal property of $\varinjlim_{a\in U \overset{\circ}{\subseteq} X} \mathcal{F}^+(X)$

- (ii) is obvious by this construction of $\mathcal{F}^+$.

Still to show: the universal property.

$\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism, $\mathcal{G}$ be a sheaf.

by (ii), there is a unique morphism of sheaves $\varphi^+ : \mathcal{F}^+ \to \mathcal{G}^+$ such that $\varphi^+ \circ \iota_{\mathcal{F}} = \iota_{\mathcal{G}} \circ \varphi$.

But $\mathcal{G}$ is a sheaf, it implies that $\mathcal{G}$ can be identified with $\mathcal{G}^+$ under $\iota_{\mathcal{G}}$, i.e., $\varphi^+ \circ \iota_{\mathcal{F}} = \varphi$.$\square$

Proposition Let X be a topological space, $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves of groups on X. The sheaf $\mathrm{im}(\varphi)$ on X associated to the presheaf $(U \overset{\circ}{\subseteq} X) \mapsto \mathrm{im}(\varphi_m) \subseteq \mathcal{G}(U)$ can be identified with a subsheaf of $\mathcal{G}$. The equality occurs iff $\forall a \in X : \mathrm{im}(\varphi_a) = \mathcal{G}_a$.

The proof of the proposition. Obvious: $\mathrm{im}(\varphi)^{\mathcal{F}}$ is a subpresheaf of $\mathcal{G}$.
The inclusion morphism: $\mathrm{im}(\varphi)^{\mathcal{F}} \hookrightarrow \mathcal{G}$.
Morphism: $\mathrm{im}(\varphi) \to \mathcal{G}$.
Here, $\forall U \overset{\circ}{\subseteq} X : \mathrm{im}(\varphi)(U) \to \mathcal{G}(U)$ is injective since it is injective on stalks.
$\mathrm{im}(\varphi)$ can be identified with a subsheaf of $\mathcal{G}$.

Furthermore, $\mathrm{im}(\varphi) = \mathcal{G}$
$\iff \forall U \overset{\circ}{\subseteq} X \forall t \in \mathcal{G}(U) \forall a \in U \exists V_a \overset{\circ}{\subseteq} U, a \in V_a \exists \mathcal{F} \in \mathrm{im}(\varphi_{V_a}) : t_b = \mathcal{F}_b$ for all $b \in V_a$ (since $\exists \mathcal{F} \ldots, \iff \exists s \in \mathcal{F}(V_a) : \varphi_{V_a}(s) = t|_{V_a}$).
$\iff \forall a \in X, \forall \hat{t} = [(U, t)] \in \mathcal{G}, \exists \hat{s} = [(V, s)] \in \mathcal{F}_a : \varphi_a(\hat{s}) = \hat{t}$ $\square$

Remark. It also follows that ($\forall a \in X : \varphi_a$ surjective) $\iff$

$$\forall U \overset{\circ}{\subseteq} X \forall t \in \mathcal{G}(U) \exists \text{ open covering } \{U_i\}_{i\in I} \text{ of U} \exists s_i \in \mathcal{F}(U_i) : t|_{U_i} = \varphi_{U_i}(s_i) \text{ for all } i \in I$$

Example. sheafification of a presheaf of functions (see the section 1).

Question: Given a continuous map $f : X \to Y$. Can we construct a sheaf on Y from one on X or vice versa?

Definition Let $f : X \to Y$ be a continuous map and $\mathcal{F}$ be a presheaf on X.
The direct image of $\mathcal{F}$ under f is a presheaf $f_*\mathcal{F}$ on Y given by

$$(f_*\mathcal{F})(U) := \mathcal{F}(f^{-1}(U))$$

for all $U \overset{\circ}{\subseteq} Y$. Here the restriction morphism for $f_*\mathcal{F}$ is obtained directly from $\mathcal{F}$. $U \overset{\circ}{\subseteq} V$:

$$\begin{tikzcd}
f_*\mathcal{F}(V) \arrow[r, "="] \arrow[d] & \mathcal{F}(f^{-1}(V)) \arrow[d] \\
f_*\mathcal{F}(U) \arrow[r, "="] & \mathcal{F}(f^{-1}(U))
\end{tikzcd}$$

Exercise. Show that fi $\mathcal{F}$ is a sheaf, then so is $f_*\mathcal{F}$.