Sheaves and their cohomology

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Last session: T_pM for the geometric and algebraic sense are the same for C^∞-/C^ω-/$\mathbb{C}$-manifold.

$$T_p^{\mathrm{geo}}M \cong \mathrm{Der}_{\mathbb{K}}(\mathcal{O}_{M,p}, \mathbb{K}) \cong (\mathfrak{m}_{M,p}/\mathfrak{m}_{M,p}^2)^\vee$$

Let $(A, \mathfrak{m}, k)$ be a local ring which satisfies the tangent space conditions.

For $f \in A$, define

$$df := (f - \bar{f}) \bmod \mathfrak{m}^2 \in \mathfrak{m}/\mathfrak{m}^2$$

Note that $\bar{f}$ is the image of f under $A \twoheadrightarrow A/\mathfrak{m} \cong k \hookrightarrow A$.

Exercise Show that in this case, $d : A \to \mathfrak{m}/\mathfrak{m}^2, f \mapsto df$, is a k-derivation.

Example. $A = \mathcal{C}^\infty_{\mathbb{R}^n,0}$ that 0 is the identity of $\mathbb{R}^n$. $\mathfrak{m} = (x_1, \ldots, x_n)$ is the maximal ideal of A.

Let $f \in A$ then there is a Taylor expansion.

$$f(x_1, \ldots, x_n) = f(0) + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(0) \cdot x_i + \underbrace{r(x_1, \ldots, x_n)}_{\in \mathfrak{m}^2}$$

$df = f - \bar{f} = \sum_{i=1}^n \dfrac{\partial f}{\partial x_i}(0) \cdot x_i$ is in $\mathfrak{m}/\mathfrak{m}^2$.

Remark. Let M be a C^∞-/C^ω/$\mathbb{C}$-manifold, $p \in M$, $f \in \mathcal{O}_{M,p} \mapsto df \in T_pM^\vee$. What is $df(v)$ for $v \in T_pM$?

Use the algebraic definition $v \widehat{=} \partial_v : \mathcal{O}_{M,p} \to \mathbb{K}$. Under the isomorphism, $\mathrm{Der}_{\mathbb{K}}(\mathcal{O}_{M,p}, \mathbb{K}) \xrightarrow{\sim} (\mathfrak{m}_{M,p}/\mathfrak{m}_{M,p}^2)^\vee$, ∂_v is sent to $\partial_v|_{\mathfrak{m}_p}$. Hence $df(v) = \partial_v(f - \bar{f}) = \partial_v(f)$.

Exercise.

- Let M be a premanifold, $P(\epsilon)$ be a locally $\mathbb{K}$-ringed space consisting of one single point such that $\mathcal{O}_{P(\epsilon)} = P(\epsilon) = \mathbb{K}[\epsilon]/(\epsilon^2)$. Show that there is a natural bijection.

$$\mathrm{Hom}_{\mathbb{K}}(P(\epsilon), M) \cong \{(p, v) \mid p \in M, v \in T_pM\}$$

  (for a more general statement, see Wedhorn, Problem 5.5.)

- Arapura, Exercise 2.5.22

Def Let M be a C^∞-manifold. A C^∞ 1-form on $U \overset{\circ}{\subseteq} M$ is a finite linear combination $\sum_{i=1}^n g_i df_i,\ f, g \in \mathcal{C}^\infty(U)$ as a function $U \to \bigsqcup_{p\in M} T_pM^\vee$.

Then $\{$1-forms on $U\}$ is a $C^\infty(U)$ module.
The sheaf associated to $U \mapsto \{$1-forms on $U\}$ will be denoted by $\mathcal{E}_M^1$.
$\mathcal{E}_M^1(U)$ is again a $C^\infty(U)$ module.

Remark. let M be a C^∞-manifold, (U, x) be a chart of M where $x : U \xrightarrow{\sim} B \overset{\circ}{\subseteq} \mathbb{R}^n$. If say that $x = (x_1, \ldots, x_n), x_i$ is then smooth.

Discussion before:

$$\forall p \in U \forall f \in C^\infty(U), df|_p = \sum_{i=1}^n \frac{\partial(f \circ x^{-1})}{\partial x_i}(x(p))\, dx^i|_p$$

Based on this observation, we can show that

$$\mathcal{E}_n^1(U) = \bigoplus_{i=1}^n C^\infty(U)\, dx_i$$

How?: For each $j = \{1, \ldots, n\}$, define $\partial_j|_p \in T_pM$ by $\partial_j|_p(f) := \frac{\partial(f \circ x^{-1})}{\partial x_j}(x(p))$ when $dx^i|_p \partial_j|_p = \delta_{ij}$ (! Need to verify)

Def Let M be a C^∞-premanifold. A vector field on $V \overset{\circ}{\subseteq} M$ is a collection of vectors $(v_p \in T_pM)_{p\in V}$ such that $\forall U \overset{\circ}{\subseteq} V \forall f \in C^\infty(U)$ the function $p \in U \mapsto \langle v_p, df|_p\rangle$ is a C^∞ (i.e. lies in $C^\infty(U)$, $\langle \cdot, \cdot \rangle$ denotes the standard pairing $T_pM \times T_pM^\vee \to \mathbb{R}$).

$$\mathcal{T}_M(V) := \{C^\infty\text{-vector fields on } V\}$$

Remark.

- Therefore, each $D \in \mathcal{T}_M(V)$ defines a derivation $C_M^\infty(V) \to C_M^\infty(V), f \mapsto \langle D, df\rangle$.
- We can show that $\mathcal{T}_M$ is a sheaf on M.
- Let (U, x) be a chart on M, $x = (x^1, \cdots, x^n)$. Define $\partial_i := \dfrac{\partial}{\partial x^i} := (\partial_i|_p)_{p\in U}$ ($\partial_i|_p$ as before.) For each $D \in \mathcal{T}_M(U)$, there are $f_1, \ldots, f_n : U \to \mathbb{R}$ such that $D = \sum_{i=1}^n f_i \partial_i$.
  Consider $x^j \in C^\infty(U)$.
  $\langle D, dx^j\rangle = \sum_{i=1}^n f_i \langle \partial_i, dx^j\rangle = f_j$
  Hence $f_j \in C^\infty(U)$.

$$\mathcal{T}_M(U) = \bigoplus_{i=1}^n C^\infty(U) \partial_i$$

Sheaves of Modules

We want to generalize $\mathcal{E}_M^1, \mathcal{T}_M^1$.
Unless otherwise specified, let $(X, \mathcal{O}_X)$ be a ringed space.

Def A *sheaf of $\mathcal{O}_X$-modules* or simply an $\mathcal{O}_X$-module is a sheaf $\mathcal{F}$ of abelian groups on X together with a morphism of sheaves (of sets!)

$$\mathcal{O}_X \times \mathcal{F} \to \mathcal{F}$$

such that $\forall U \mathring{\subseteq} X$: the map $\mathcal{O}_X(U) \times \mathcal{F}(U) \to \mathcal{F}(U)$ yields an $\mathcal{O}_X(U)$-module structure on $\mathcal{F}(U)$.

(In particular, for $U \mathring{\subseteq} V \mathring{\subseteq} X, f \in \mathcal{O}_X(V), s \in \mathcal{F}(V)$, we have $(f \cdot s)|_V = (f|_V)(s|_V)$)

More definitions

- Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. An *$\mathcal{O}_X$-submodule of $\mathcal{F}$* in an $\mathcal{O}_X$-module $\mathcal{G}$ such that $\forall U \mathring{\subseteq} X : \mathcal{G}(U)$ is an $\mathcal{O}_X(U)$-submodule of $\mathcal{F}(U)$ and the restriction morphisms for $\mathcal{G}$ are obtained from those for $\mathcal{F}$.
- An *ideal sheaf* of $\mathcal{O}_X$ is an $\mathcal{O}_X$-submodule of $\mathcal{O}_X$.

Examples

- Let M be a C^∞-premanifold. $\mathcal{E}^1_M, \mathcal{T}^1_M$ are $\mathcal{C}^\infty_M$-modules.
- Let M be a C^∞-/$\mathbb{C}$-manifold, $N \subseteq M$ be a closed submanifold. Define $\mathcal{J}_{N/M}$ by

$$\forall U \mathring{\subseteq} \mathcal{J}_{N/M} := \{f \in \mathcal{O}_M(U) \mid f|_N = 0\} \trianglelefteq \mathcal{O}_M(U)$$

  or $\mathcal{J}_{N/M}$ is an ideal sheaf of $\mathcal{O}_M$.

Def A morphism of $\mathcal{O}_X$-modules $\mathcal{F} \to \mathcal{G}$ is a morphism φ of sheaves such that $\forall U \mathring{\subseteq} X$: $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is an $\mathcal{O}_X(U)$-module homomorphism.

Notation ($\mathcal{O}_X$-Mod) is the category of $\mathcal{O}_X$-modules.

Exercise. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of $\mathcal{O}_X$-modules

1. Show that $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ has a natural structure as $\mathcal{O}_X$-module.
2. Show that $\mathfrak{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) : U \mapsto \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ is a sheaf (!) of $\mathcal{O}_X$-modules.

Def We say that an $\mathcal{O}_X$-module $\mathcal{F}$ is *locally free of rank n* if $\exists$ open covering $X = \bigcup_{i \in I} U_i$ such that $\forall i \in I, \mathcal{F}|_{U_i} \cong (\mathcal{O}_X|_{U_j})^n$

Example. Let M be an n-dimensional C^∞-premanifold. $\mathcal{E}^1_M, \mathcal{T}_M$ are locally free of rank n.

Furthermore, $\mathcal{E}^1_M \cong \mathfrak{Hom}_{\mathcal{O}_M}(\mathcal{T}_M, \mathcal{O}_M)$.

Question: further construction of $\mathcal{O}_X$-modules?

Prop/Def Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of $\mathcal{O}_X$-modules.

1. $\ker(\varphi) : U \mapsto \ker(\varphi_U)$ is an $\mathcal{O}_X$-submodule of $\mathcal{F}$, called the *kernel* of φ.

2. The sheaf $\mathrm{im}(\varphi)$ associated to $\mathrm{im}(\varphi)^{\mathcal{P}} : U \mapsto \mathrm{im}(\varphi_U)$ is an $\mathcal{O}_X$-submodule of $\mathcal{G}$, called the *range* of φ.

($\mathcal{O}_X \times \mathrm{im}(\varphi)^{\mathcal{P}} \to \mathrm{im}(\varphi)^{\mathcal{P}}$, sheafification yields an $\mathcal{O}_X$-module structure on $\mathrm{im}(\varphi)$)

Def Let $\mathcal{F}$ be a $\mathcal{O}_X$-module, $\mathcal{G} \subseteq \mathcal{F}$ be an $\mathcal{O}_X$-submodule. The quotient $\mathcal{F}/\mathcal{G}$ is the sheaf associated to $U \mapsto \mathcal{F}(U)/\mathcal{G}(U)$.

Example. Let $N \subseteq M$ be a closed submanifold, $\mathcal{J}_{N/M} \trianglelefteq \mathcal{O}_M$ as before. $\overset{!}{\Longrightarrow}$ $\mathcal{O}_M/\mathcal{J}_{N/M} \cong i_*\mathcal{O}_N$, where $i : N \hookrightarrow M$ denotes the inclusion morphism.

Sketch proof: $i^\flat : \mathcal{O}_M \to i_*\mathcal{O}_N$
$\mathcal{O}_M(U)/\mathcal{J}_{N/M}(U) \to i_*\mathcal{O}_N(U)$ for each $U \overset{\circ}{\subseteq} M$. Then sheafify this one: $\mathcal{O}_M/\mathcal{J}_{N/M} \to i_*\mathcal{O}_N$

Excursion Tensor products

Let R be a commutative ring.

Prop/Def. Let M, N be R-modules. There exists an R-module $M \otimes_R N$ with an R-bilinear map.

$$\tau : M \times N \to M \otimes_R N$$

which satisfies the following universal property:

$\forall R$-module T $\forall R$-bilinear map $\varphi : M \times N \to T$, $\exists! R$-linear map $\psi : M \otimes_R N \to T$ such that $\psi \circ \tau = \varphi$

$$\begin{tikzcd}
M \times N \arrow[rr, "\tau"] \arrow[dr, "\varphi"'] & & M \otimes_R N \arrow[dl, dotted, "\psi"] \\
& T &
\end{tikzcd}$$

$M \otimes_R N$ is called the tensor product of M and N over R. It is unique up to unique isomorphism.

Construction.

$$M \otimes_R N = \left(\bigoplus_{(m,n) \in M \times N} R(m \otimes n) \right) \Bigg/ \left\langle \begin{matrix} m+m' \oplus n = m \oplus n + m' \oplus n, \\ (rm) \oplus n = r(m \oplus n), \\ m \oplus (n+n') = m \oplus n + m \oplus n', \\ m \oplus rn = r(m \oplus n) \end{matrix} \middle| m, m' \in M, n, n' \in N, r \in R \right\rangle$$

Remark. If M is generated by $\{m_i | i \in I\}$ and N by $\{n_j | j \in J\}$, then $M \otimes_R N$ is generated by $\{m_i \times n_j | i \in I, j \in J\}$

Example.

- $R^m \otimes R^n \cong R^{mn}$
- $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} = \{0\}$, since $([a]) \otimes r = (n[a]) \otimes \frac{r}{n}$ for all $[a] \in \mathbb{Z}/n\mathbb{Z}$, $r \in \mathbb{Q}$.