Sheaves and their cohomology

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Last session: introducing categories (objects/morphisms/iso-, automorphisms)

Categories (Contd.)

Def Let $\mathcal{C}$ be a category

- An *initial object* is an object $P \in \mathrm{Ob}(\mathcal{C})$ such that $\forall X \in \mathrm{Ob}(C) : \exists! f : P \to X$.
- A *terminal object* is an object $Q \in \mathrm{Ob}(\mathcal{C})$ such that $\forall x \in \mathrm{Ob}(\mathcal{C}) : \exists! f : X \to Q$.
- A *zero object* is an object which is both initial and terminal.

Example

- In (Sets), an initial object is $\varnothing$, and a terminal object is $\{a\}$ (a singleton.)
- In (Grps), $\{e\}$ is a zero object.
- In (Rings), an initial object is $\mathbb{Z}$ and a terminal object is $\{0\}$.

Remark If exists, an initial/a terminal object is unique up to unique isomorphism.
Proof. If P and P' are initial objects, then $\exists! f : P \to P'$ and $\exists! g : P' \to P$ means $g \circ f : P \to P$ that it must be id_P, also $f \circ g = \mathrm{id}_{P'}$. Hence, f, g are isomorphisms i.e. $P \cong P'$. (Similar to the terminal object.) □

Def Let $\mathcal{C}$ be a category. A subcategory of $\mathcal{C}$ is a category $\mathcal{C}'$ such that $\mathrm{Ob}(\mathcal{C}') \subseteq \mathrm{Ob}(\mathcal{C})$ and $\forall x, y \in \mathrm{Ob}(C') : \mathrm{Hom}_{\mathcal{C}'}(X, Y) \subseteq \mathrm{Hom}_{\mathcal{C}}(X, Y)$ and the composition law of morphisms and the identity morphisms of objects in $\mathcal{C}'$ are the same as in $\mathcal{C}$.

A full subcategory of $\mathcal{C}$ is a subcategory $\mathcal{C}'$ such that $\forall x, y \in \mathrm{Ob}(C') : \mathrm{Hom}_{C'}(X, Y) = \mathrm{Hom}_{C'}(X, Y)$.

Examples:

- (Rings) is a subcategory of (Rngs)
- (Ab) is a full subcategory of (Grps)

Def Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A (covariant) functor $F : \mathcal{C} \to \mathcal{D}$ consists of the following data:

- An assignment $\mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$
- for each $X, Y \in \mathrm{Ob}(\mathcal{C})$, a map $\mathrm{Hom}_C(X, Y) \to \mathrm{Hom}_{C'}(X, Y)$, is a map.

such that the following conditions are satisfied:

1. $\forall x \in \mathrm{Ob}(\mathcal{C}) : F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$
2. $forall x, y, z \in \mathrm{Ob}(\mathcal{C}) : \forall f : X \to Y, g : Y \to Z, F(g \circ f) = F(g) \circ F(f)$.

Examples

- GL_n : (Rings) $\to$ (Grps)
- Forgetful functors, e.g., (Rings) $\to$ Ab, $(R, +, \cdot) \mapsto (R, +)$, (Grps)/(Top)/($C^\alpha$-Mfds) $\to$ (Sets)
- Inclusion functors, e.g., (Ab) $\to$ (Grp), (fields) $\to$ (CommRings) $\to$ (Rings)

Def Contravariant functors $F : \mathcal{C} \to \mathcal{D}$ is a functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ ($\mathcal{C}^{\mathrm{op}}$ is $\mathcal{C}$ with reversing arrows) where $\mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{C})$ and $\forall X, Y \in \mathrm{Ob}(\mathcal{C}) : \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$.

Examples.

- $\mathrm{Hom}_{\mathcal{C}}(-, X) : \mathcal{C}^{\mathrm{op}} \to$ (Sets)
- $\mathcal{C}^\alpha : (C^\alpha - \mathrm{Mfd}) \to (\mathbb{R} - vect), (X, \mathcal{C}^\alpha_X) \mapsto \mathcal{C}^\alpha_X(X)$

Def $\mathcal{C}, \mathcal{D}$ categories. $F, G : \mathcal{C} \to \mathcal{D}$ functors. *A natural transformation* (morphism of functors) $\phi : F \to G$ ($\phi : F \implies G$) is collection $(\phi_X : F(X) \to G(X))_{X \in \mathrm{Ob}(\mathcal{C})}$ of morphisms in $\mathcal{D}$ such that $\forall f : X \to Y$ in $\mathcal{C}$: the diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{\phi_X} & G(X) \\ \downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow{\phi_Y} & G(Y) \end{array}$$

is commutative.

Natural isomorphism: $\forall X \in \mathrm{Ob}(\mathcal{C}) : \phi_X$ is isomorphism ("$F \cong G$")

Notation
$\mathrm{Nat}(F, G) := \{$natural transformations $F \implies G\}$

Examples

- F is a field: $\mathrm{id}, (-)^{**}, :$ (F-vect) $\to$ (F-vect) is a natural transformation given by $V \to V^{**}$, $v \mapsto (\phi \mapsto \phi(v))$ ($V^{**} := \mathrm{Hom}_F(\mathrm{Hom}_F(V, F), F)$

$$\begin{array}{ccc} V & \longrightarrow & V^{**} \\ \downarrow & & \downarrow \\ W & \longrightarrow & W^{**} \end{array}$$

  If we replace (F-vect) by (f.d. F-vect), we obtain a natural isomorphism.

- det as a natural transformation. Consider the functors $GL_n, (\cdot)^\times$: (CRings) $\to$ (Grps) that det is a natural transformation $\det : GL_n \to (\cdot)^\times$.

Remark Similarly, natural transformations for contravariant functors are defined in the same way.

Back to Sheaves

Let X be a topological space.

Def A *presheaf* $\mathcal{F}$ (of sets) on X consist of the following data:

1. for each $U \overset{\circ}{\subseteq} X$, a set $\mathcal{F}(U)$.
2. for each $U \overset{\circ}{\subseteq} V \overset{\circ}{\subseteq} X$, a map $\rho_{VU} : \mathcal{F}(V) \to \mathcal{F}(U)$ (Restriction map)

such that the following conditions hold:

1. $\forall U \overset{\circ}{\subseteq} X : \rho_{UU} = \mathrm{id}_{\mathcal{F}(U)}$
2. $\forall U \overset{\circ}{\subseteq} V \overset{\circ}{\subseteq} W \overset{\circ}{\subseteq} X : \rho_{WU} = \rho_{VU} \circ \rho_{WV}$

Notation

- $U \overset{\circ}{\subseteq} V \overset{\circ}{\subseteq} X$, $s \in \mathcal{F}(V)$, the restriction $s|_U = \rho_{VU}(s)$.
- An element of $\mathcal{F}(U)$ is called a section of $\mathcal{F}$ over U.

Def A morphism of presheaves $\varphi : \mathcal{F} \to \mathcal{G}$ on X is a collection of maps $(\varphi_a : \mathcal{F}(U) \to \mathcal{G}(U))_{U \overset{\circ}{\subseteq} X}$ such that $\forall U \overset{\circ}{\subseteq} V \overset{\circ}{\subseteq} X$: $\varphi_U \circ \rho^{\mathcal{F}}_{VU} = \rho^{\mathcal{G}}_{VU} \circ \varphi_V$.

The composition of morphisms of presheves $\varphi : \mathcal{F} \to \mathcal{G}$ and $\psi : \mathcal{G} \to \mathcal{H}$ is given by

$$(\psi \circ \varphi)_U := \psi_U \circ \varphi_U : \mathcal{F}(U) \to \mathcal{H}(U)$$

for each $U \overset{\circ}{\subseteq} X$.

Remark Alternative defintiion of presheves.
Let (Ouv_X) (Ouvert) be the category of open subsets of X with includsion maps as morphisms.

- Presheaf is a contravariant function $\mathcal{F} : (Ouv_X) \to$ (sets).
- Morphism of presheaf is a natural transformation.

By this way, we can define presheaves of groups/rings/(in general: with values in a category $\mathcal{C}$.) In this case, ρ_{VU} is a morphism in that category (restriction morphism.)

Def A *sheaf* on X is a presheaf $\mathcal{F}$ on X which satisfies the following condition for all $U \overset{\circ}{\subseteq} X$ and open covering $\{U_i\}_{i \in I}$ of U:

(Sh) For every family $\{s_i \in \mathcal{F}(U_i)\}_{i \in I}$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all $i, j \in I$. There exists a unique $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all $i \in I$.

A morphism of sheaves is a morphism of presheaves.

Notation

- $\mathrm{Sh}(X)$ is a category of sheaves on X and this is a full subset of $\mathrm{PSh}(X)$.
- $\mathrm{PAb}(X), \mathrm{Ab}(X)$: category of (pre-)sheaves of abelian groups on X.

Example

$\mathcal{F}$ given by $\mathcal{F}(U) := \begin{cases} \mathbb{Z}, & U = X. \\ \{0\}, & \text{otherwise.} \end{cases}$ is a presheaf of abelian groups which is not a sheaf

(*Pf.* X is not a union of two proper open subsets.)

Remark. T a set: every presheaf of T-valued functions on X in the sense of the section 1 is a presheaf here. It is a sheaf of function in sense of the section 1 iff it satisfies (Sh).

$f : U \to T : f \in \mathcal{F}(U) \iff (\forall i \in I : f|_{U_i} \subseteq \mathcal{F}(U_i))$

Examples

- $(X, \mathcal{R})$ is a concrete k-space. $\mathcal{R}$ is a sheaf of rings (k-algebras) on X. $f \in \mathcal{R}(x)$ if $\mu_f : \mathcal{R} \to \mathcal{R}$ that $g \in \mathcal{R}(U) \mapsto g \cdot (f|_U) \in \mathcal{R}(U)$ is a morphism of sheaves of k-vector space.
- $X = \mathbb{C}^n$: the partial derivative yields a morphism of sheaves of $\mathbb{C}$-vector spaces.

$$\frac{\partial}{\partial z_j} : \mathcal{O}_{\mathbb{C}^n} \to \mathcal{O}_{\mathbb{C}^n}$$

Def $\mathcal{F}$ presheaf on X, $a \in X$. The *stalk of* $\mathcal{F}$ at a is

$$\mathcal{F}_a := \{(U, s) \mid a \in U \mathring{\subseteq} X, s \in \mathcal{F}(U)\} / \sim$$

where $(U, s) \sim (U', s') \iff \exists V \mathring{\subseteq} U \cap U', a \in V : s|_U = s|_V$ (Check: this is an equivalence relation.)

An element of $\mathcal{F}_a$ is called of a *germ* of $\mathcal{F}$ at a.

Remark The stalk of presheaves of group/rings/... can be defined similarly with the corresponding algebraic structure.

For example, if $\mathcal{F}$ is a presheaf of rings, then $\mathcal{F}_a$ is a ring under $+, \cdot$ given by

$$[(U, s)] \dot{+} [(v, t)] = [(U \cap V, s|_{U \cap V} \dot{+} t|_{U \cap V})].$$

(Check: well-definedness and ring axioms)

Example $X = \mathbb{C}$ and $a \in \mathbb{C}$. What is $\mathcal{O}_{\mathbb{C},a}$? ($\mathcal{O}_{\mathbb{C}}$ is a sheaf of holomorphic functions.)

Identity theorem. two holomorphic functions f and g agree on a neighborhood of a iff they have the same power series expansion around a.

$$\mathcal{O}_{\mathbb{C},a} = \left\{ \sum_{n=0}^{\infty} c_n (z - a)^n \text{ power series with positive radius of convergence.} \right\}$$

Exercise. Arapura 3.1.15.