Manifold (Contd.)

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Lecturer: Nithi Rungtanapirom
Transcriber: Kittapat Ratanaphupha

Def Let X be an n-dimensional C^α-manifold. A closed m-dimensional submanifold of X is a closed subset $Y \subseteq X$ such that for each $x \in Y$, there is a neighborhood $V \overset{\circ}{\subseteq} X$ such that $x \in V$ and an C^α-isomorphism $h : V \to B \overset{\circ}{\subseteq} \mathbb{R}^n$ such that $h(V \cap Y) = B \cap L$ for some m-dimensional subspace $L \subseteq \mathbb{R}^m$ (complex submanifold is similarly defined as the above.)

Lemma/Def Let $(X, \mathcal{C}_X^\alpha)$ be an n-dimensional C^α-manifold, $Y \subseteq X$ closed m-dimensional submanifold. Define $\mathcal{C}_Y^\alpha$ as follows:

For each $U \overset{\circ}{\subseteq} Y$, set

$$\mathcal{C}_Y^\alpha(U) = \{f : U \to \mathbb{R} \mid \forall x \in U \exists V_x \overset{\circ}{\subseteq} X, x \in V_x \text{ and } \exists \tilde{f} \in \mathcal{C}_X^\alpha(C_x) : \tilde{f}|_{V_x \cap U} = f|_{V_x \cap U}\}$$

Then $(Y, \mathcal{C}_Y^\alpha)$ is an m-dimensional C^α-manifold.

Proof. Suffices to show:
Each $x \in Y$ has a neighborhood $U \overset{\circ}{\subseteq} Y$ that is isomorphic to $(\tilde{B}, \mathcal{C}_{\mathbb{R}^m}^\alpha|_{\tilde{B}})$ for some $\tilde{B} \overset{\circ}{\subseteq} \mathbb{R}^m$.

Choose $V \overset{\circ}{\subseteq} X$ as in the definition of submanifold.
Without loss of generality, $L = \mathbb{R}^m \times \{0\} \subseteq \mathbb{R}^n$.
Define $\tilde{B} := B \cap L$ (identify L with $\mathbb{R}^m$) and $U = V \cap Y \overset{\circ}{\subseteq} Y$.

Claim. $h : V \xrightarrow{\sim} B$ induces an isomorphism $(U, \mathcal{C}_Y^\alpha|_U) \cong (\tilde{B}, \mathcal{C}_{\mathbb{R}^m}^\alpha|_{\tilde{B}})$.

In fact, every C^α-function $f(x_1, \dots, x_m)$ on $\tilde{B}$ can be extended trivially to B.
Hence, every function from $\mathcal{C}_Y^\alpha|_U$ yields a function from $\mathcal{C}_{\mathbb{R}^m}^\alpha|_{\tilde{B}}$ and vice versa. □

Consequence Let $f_1, \dots, f_r \in \mathcal{C}^\alpha(\mathbb{R}^n)$ and $X := \{a \in \mathbb{R}^n \mid f_1(a) = f_2(a) = \cdots = f_r(a) = 0\}$.
Assume that $\forall a \in X, \operatorname{rank}(J_{f_1,\dots,f_r}(a)) = n - m$ (Full-rank)
Then, X is an m-dimensional $\mathcal{C}^\alpha$-manifold (as closed submanifold of $\mathbb{R}^n$, the proposition can be proved by using the *Implicit function theorem.*)

Also, the same result holds for complex submanifold $\mathbb{C}^n$.

Examples. $S^n, O(n), U(n)$

Further examples

1. $\mathbb{T}^1 := \mathbb{R}/\mathbb{Z}$ (Real 1-torus as a quotient group) such that $U \subseteq \mathbb{T}^1 \iff \pi^{-1}(U) \subseteq \mathbb{R}$ is open. This topological space is Hausdorff and compact. (Check: $\mathbb{T}^1$ is homeomorphic with S^1 under the map $t \mapsto (\cos(2\pi t), \sin(2\pi t))$.)

Two possible ways to define $\mathcal{C}_{\mathbb{T}^1}^\alpha$:

- Use atlas: $U_1 := \{x+\mathbb{Z} : x \in (0,1)\}$ and $U_2 := \{x+\mathbb{Z} : x \in (-\frac{1}{2}, \frac{1}{2})\}$. This gives an homeomorphism $\varphi_1 : (0,1) \xrightarrow{\sim} U_1$ and $\varphi_2 : (-\frac{1}{2}, \frac{1}{2}) \xrightarrow{\sim} U_2$ with compatible change of charts.

$$(\varphi_2|)^{-1} \circ (\varphi_1|) : \left(0, \frac{1}{2}\right) \cap \left(\frac{1}{2}, 1\right) \xrightarrow{\sim} U_1 \cap U_2 \xrightarrow{\sim} \left(-\frac{1}{2}, 0\right) \cap \left(0, \frac{1}{2}\right)$$

With $x \mapsto \begin{cases} x & x < 1/2 \\ x-1 & x > 1/2 \end{cases}$

Define $\mathcal{C}^\alpha_{\mathbb{T}^1}(V) = \{f : V \to \mathbb{R} \,|\, f \circ (\varphi_1|) \in \mathcal{C}^\alpha_{\varphi_1^{-1}(V \cap U_1)}, f \circ (\varphi_2|) \in \mathcal{C}^\alpha_{\varphi_2^{-1}(V \cap U_2)}, \}$

- Define directly: for $V \overset{\circ}{\subseteq} \mathbb{T}^{-1}$, define

$$\mathcal{C}^\alpha_{\mathbb{T}^1}(V) = \{f : V \to \mathbb{R} \,|\, f \circ (\pi|) \in C^\alpha : \pi^{-1}(V) \to \mathbb{R}\}$$

Check: $\mathcal{C}^\alpha_{\mathbb{T}^1}$ defined by both ways are the same.
Remark. the method 2 defined by both ways are the same.

Exercise X is a manifold (C^α or complex), $\Gamma \leq \mathrm{Aut}(X) = \{X \xrightarrow{\sim} X$ is isomorphism.$\}$ Assume that the action of Γ on X has no fixed point ($\forall \sigma \in \Gamma \setminus \{\mathrm{id}\} \forall x \in X : \sigma(x) \neq x$) and is properly discontinuous ($\forall \sigma \in \Gamma \setminus \{\mathrm{id}\} \forall x \in X \exists V_x : x \in V_x$ and $V_x \cap \sigma(V_x) = \varnothing\}$)

Let $X := p\backslash X$ ($X/(x \sim \sigma(x)$ for $x \in X, \sigma \in \Gamma)$) be endowed with the quotient topology under the canonical projection $\pi : X \to Y$. Define $\mathcal{C}^\alpha_Y$ or $\mathcal{O}_Y$ by "f is $\mathcal{C}^\infty \iff f \circ \pi$ is $\ldots$"

Show that $(Y, \mathcal{C}^\alpha_Y)$ respect to is $(Y, \mathcal{O}_Y)$ is a manifold with $\dim Y = \dim X$.

Examples: $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ and $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$.
$\tau \in \mathbb{C}/\mathbb{R}$ that $E_\tau := \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ is a complex manifold of dimension 1.

2. The projective space. Let $k \in \{\mathbb{R}, \mathbb{C}\}$.

$$\mathbb{P}^n(k) = (k^{n+1} \setminus \{0\})/(x \sim \lambda x : x \in k^{n+1} \setminus \{0\}, \lambda \in k^*)$$

Notation: $[a_1 : a_2 : \cdots : a_n]$ = equivalence class of $(a_1, a_2, \ldots, a_n)$.
Another interpretation $P^n(k) = \{1 -$ dimensional subspaces of$k^{n+1}\}$
Endow $\mathbb{P}^n(k)$ with the quotient topology (check: which is Hausdorff and compact.)
Sheaf of functions: $f : U \to k$ ($U \overset{\circ}{\subseteq} \mathbb{P}^n(k)$) is C^α/holomorphic $\iff f \circ (\pi|)$ is C^α/holomorphic.

Claim. $\mathbb{P}^n(k)$ is a manifold.
Proof. Cover $\mathbb{P}^n(k)$ by $U_j := \{[a_0, a_1, \ldots, a_n] \,|\, a_j \neq 0\}$ for $j = 1, 2, \ldots, n$.
Check that $(U_j, \mathcal{C}^\alpha_{\mathbb{P}^n}|_{U_j}) \to (R^n, \mathcal{C}^\alpha_{\mathbb{R}^n}|_{U_j})$ (also with $\mathcal{O}_n$) is an isomorphism.
(By removing coordinates)

Exercises 2.2.14-2.2.20 (Arapura)

Sheaves

Excursion: Categories and Functors.
Def. A category $\mathcal{C}$ consists of the following data:

- A class $\mathrm{Ob}(\mathcal{C})$ of objects,
- For any objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, a set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of morphisms (or arrows)
- For any any $X \in \mathrm{Ob}(\mathcal{C})$, an identity morphism $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$,
- For any $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, a composition law $\mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$ that $(f, g) \mapsto g \circ f$. satisfies

- identity idempotency
- associativity of functions

Examples (Cat, Obs, Morph)

1. Sets, sets, mappings
2. Grps, groups, group homomorphism
3. Top, topological spaces, continuous maps
4. $C^\alpha - \mathrm{Mfd}$, C^α-manifolds, morphisms of concrete $\mathbb{R}$-space
5. $\mathbb{C} - \mathrm{Mfd}$, complex manifolds, morphisms of concrete $\mathbb{C}$-space (holomorphic maps)
6. $(P, \leq)$ (Poset), P, and $\mathrm{Hom}_{\mathcal{C}}(x, y) = \begin{cases} \{f_{x,y}\} & x \leq y \\ \varnothing & \text{otherwise} \end{cases}$

Def An *isomorphism* in $\mathcal{C}$ is a morphism $f : X \to Y$ such that $\exists g : Y \to X : g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.

Notation $X \cong Y$ if $\exists$ isomorphism $X \to Y$.
Observe that $\cong$ is an equivalence relation on $\mathrm{Ob}(\mathcal{C})$.

- g as the above is called the inverse of f and denoted by f^{-1}.
- An automorphism of X is an isomorphism on $X \to X$. $(\mathrm{Aut}_{\mathcal{C}}(X) := \{\text{automorphisms of } X\})$.

Examples (Isom).

- Grps is a bijective homomorphism.
- Top is a homeomorphism.
- $C^\alpha - \mathrm{Mfd}$ is a diffeomorphism.
- $\mathbb{C} - \mathrm{Mfd}$ is a biholomorphism.