Sheaves and their Cohomology

Date: 2023-08-18 8:00-10:00 AM Lecturer: Nithi Rungtanapirom Transcriber: Kittapat Ratanaphupha

Last session: introduced (pre-)sheaves and stalks (ad hoc def.)

Exercise Arapura, 3.1.15

A formal definition of stalks is based on the injective limit (direct limit/filtered colimit)

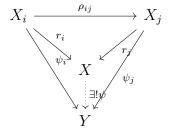
Let \mathcal{C} be a category and I be a directed set, i.e., a poset (I, \leq) such that $\forall i, j \in I \exists k \in I : i \leq k, j \leq k$. (e.g. $(\mathbb{N}, \text{divisibility}))$

Def An injective system (a direct system) over I in \mathcal{C} is a pair $((X_i)_{i \in I}, (f_{ij})_{i \leq j})$ consisting a family of objects $X_i \in \text{Ob}(\mathcal{C})$ and a family of morphisms $f_{ij}: X_i \to X_j$ for $i \leq j$ such that

- $\forall i \in I : f_{ii} = \mathrm{id}_X$
- $\forall i \leq j \leq k : f_{ik} = f_{jk} \circ f_{ij}$

An injective limit (a direct limit) of an injective system $((X_i)_{i \in I}, (f_{ij})_{i \leq j})$ is an object X together with morphisms $r_i: X_i \to X$ for $i \in I$ such that $\forall i \leq j: r_i = r_j \circ f_{ij}$ which satisfies the following universal property:

For all $Y \in \text{Ob}(\mathcal{C})$ and morphisms $(\psi_i : X_i \to Y)_{i \in I}$ such that $\forall i \leq j : \psi_i = \psi_j \circ f_{ij}$, there exists a unique $\psi : X \to Y$ such that $\forall i : \psi \circ r_i = \psi_i$.



Remark: the injective limit is unique up to unique isomorphism

Notation: $\lim_{i \in I} X_i$

Examples: In (Sets)/(Grps)/(Rings)/(R-Mods)/...

$$\lim_{i \in I} X_i = \left(\bigsqcup_{i \in I} X_i\right) \middle/ \left((x_i \in X_i) \sim (x_j \in X_j) \iff \exists k : i, j \le k \text{ and } f_{ik}(x_i) = f_{jk}(x_j)\right)$$

If \mathcal{F} is a presheaf on a topological space X and $a \in X_i$, then

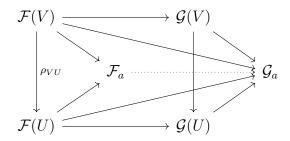
$$\mathcal{F}_a = \varinjlim_{a \in U \subseteq X} \mathcal{F}(U)$$

$$f_{ij} - \rho_{VU} : \mathcal{F}(V) \to \mathcal{F}(U) \ (V \supseteq U); \ \mathcal{F}(U) \to \mathcal{F}_a, \ s \mapsto [(U, s)].$$

Proposition Let X be a topological space, $a \in X$. Every morphism of presheaves $\varphi : \mathcal{F} \to \mathcal{G}$ gives rise to a map (morphism) $\varphi_a : \mathcal{F}_a \to \mathcal{G}_a$ where $[(U, s)] \mapsto [(u, \varphi_u(s))]$.

In particular, $\mathcal{F} \mapsto \mathcal{F}_a$ is a functor $PSh(X) \to (Sets)$, or also $PAb(X) \to (Ab)$ etc.

Proof. Apply the universal property of $\varinjlim \mathcal{F}(U)$ to $(\mathcal{F}(U) \xrightarrow{\varphi_U} \mathcal{G}(U) \xrightarrow{\varphi_a} \mathcal{G}_a)$



This is possible since $\forall U \subseteq V \subseteq X, a \in U : (\iota_U^{\mathcal{F}} \circ \varphi_a) \circ \rho_{VU} = \iota_V^{\mathcal{G}} \circ \varphi_V$

Proposition Let X be a topological space, \mathcal{F}, \mathcal{G} be sheaves on X and $\varphi : \mathcal{F} \to \mathcal{G}$ a morphism

- 1. $\forall U \subseteq X : \mathcal{F}(U) \to \prod_{a \in U} \mathcal{F}_a$ which $s \mapsto (s_a)_{a \in U}$ is injective $s_n := [(U, s)] \in \mathcal{F}_a$.
- 2. $(\forall U \subseteq X : \varphi_U : \mathcal{F}(U) \to \mathcal{G}(U))$ injective/bijective \iff $(\forall a \in X : \varphi_a : \mathcal{F}_a \to \mathcal{G}_a \text{ injective/bijective})$
- 3. if $\psi: \mathcal{F} \to \mathcal{G}$ is another morphism then $\varphi = \psi \iff \forall a \in X: \varphi_a = \psi_a$.

Proof.

- 1. Let $s, t \in \mathcal{F}(U)$ be such that $\forall a \in U : s_a = t_a \implies \forall a \in U \exists V_a \subseteq U, a \in V_a : s|_{V_a} = t|_{V_a}$. since $\bigcup_{a \in U} V_a = U$, we get s = t by the sheaf axiom.
- 2. Exercise
- 3. (\Longrightarrow) Trivial, (\Longleftrightarrow) To show: $\forall U \subseteq X \forall s \in \mathcal{F}(U) : \varphi_U(s) = \psi_U(s) \iff \forall U \subseteq X \forall s \in \mathcal{F}(U) \forall a \in U : \varphi_U(s)_a = \psi_U(s)_a \text{ but } \varphi_U(s)_a = \varphi_a(s_a) \text{ and } \psi_U(s)_a = \psi_a(s_a)$. Hence the claim follows.

Question: what about "surjective"?

Example. On $X = \mathbb{C}$, consider $D = \frac{d}{dz} : \mathcal{O}_{\mathbb{C}} = \mathcal{O}_{\mathbb{C}}$. This is given by $f \mapsto f'$ for all $U \stackrel{\circ}{\subseteq} \mathbb{C}$. Then $U \stackrel{\circ}{\subseteq} V \stackrel{\circ}{\subseteq} \mathbb{C}$:

$$\mathcal{O}_{\mathbb{C}}(V) \xrightarrow{D} \mathcal{O}_{\mathbb{C}}(V)
\downarrow^{\text{res}} \qquad \downarrow^{\text{res}}
\mathcal{O}_{\mathbb{C}}(U) \xrightarrow{D} \mathcal{O}_{\mathbb{C}}(V)$$

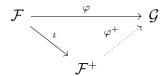
Known: $\mathcal{O}_{\mathbb{C},a} = \left\{ \sum_{n\geq 0} c_n (z-a)^n, c_n \in \mathbb{C}, \text{ positive radius of convergence} \right\}$ $\left(\frac{d}{dz} |_a : \mathcal{O}_{\mathbb{C},a} \to \mathcal{O}_{\mathbb{C},a} \right)$ is surjective.

But $\frac{1}{z} \in \mathcal{O}_{\mathbb{C}}(\mathbb{C} \smallsetminus \{0\})$ has no preimage in $\mathcal{O}_{\mathbb{C}}(\mathbb{C} \smallsetminus \{0\})$.

Problem: the presheaf defined by $U \mapsto \operatorname{im}(D_u : \mathcal{O}_{\mathbb{C}}(U) \to \mathcal{O}_{\mathbb{C}}(U))$ is not a sheaf!

Proposition/Definition let X be a topological space, \mathcal{F} be a presheaf on X. There exists a sheaf \mathcal{F}^+ and a morphism $\iota : \mathcal{F} \to \mathcal{F}^+$ with the following universal property:

For every sheaf \mathcal{G} on X and morphism $\varphi : \mathcal{F} \to \mathcal{G}$, there exists a morphism $\varphi^+ : \mathcal{F}^+ \to \mathcal{G}$ such that $\varphi^+ \circ \iota = \varphi$.



 (F^+, ι) is unique up to unique isomorphism. It is called the sheafification or the associated sheaf of \mathcal{F} . Furthermore, the following properties hold:

- 1. $\forall a \in X : \iota_a : \mathcal{F}_a \to (\mathcal{F}^+)_a$ is an isomorphism.
- 2. (Functoriality) for every morphism of presheaves $\mathcal{F} \stackrel{\varphi}{\to} \mathcal{G}$, there is a unique morphism $\varphi^+: \mathcal{F}^+ \to \mathcal{G}^+$ making the following diagram commutative.

$$\begin{array}{ccc} \mathcal{F} & \stackrel{\varphi}{\longrightarrow} & \mathcal{G} & \stackrel{\psi}{\longrightarrow} & \mathcal{H} \\ \downarrow^{\varphi_{\mathcal{F}}} & \downarrow^{\varphi_{\mathcal{G}}} & \downarrow^{\psi_{\mathcal{H}}} \\ \mathcal{F}^{+} & \stackrel{\varphi^{+}}{\longrightarrow} & \mathcal{G}^{+} & \stackrel{\psi^{+}}{\longrightarrow} & \mathcal{H}^{+} \end{array}$$

Remark.

- 1. If \mathcal{F} is already a sheaf, then $r: \mathcal{F} \to \mathcal{F}^+$ is an isomorphism.
- 2. Sheafification functor $PSh(X) \rightarrow Sh(X)$ (Sh to Ab)

Universal property: for every presheaf \mathcal{F} and a sheaf \mathcal{G} on X, there is a national bijection (natural in sense of natural transformation)

$$\operatorname{Hom}_{\operatorname{Sh}(X)}(\mathcal{F}^+,\mathcal{G}) \cong \operatorname{Hom}_{\operatorname{PSh}(X)}(\mathcal{F},\mathcal{G})$$

The sheafification is left-adjoint to the inclusion functor.

Proof.1

¹feedback: the difference between sheaves and presheaves in sense of stalks.

 $U \stackrel{\circ}{\subseteq} X, \mathcal{F}^+(U) = \{(s_a) \in \prod_{a \in U} \mathcal{F}_a : \forall a \in U \exists V_a \stackrel{\circ}{\subseteq} U, a \in V_a \exists t \in \mathcal{F}(V_a), s_b = t_b \text{ for all } b \in V_a\}$

- \mathcal{F}^+ is a sheaf, $\iota: \mathcal{F} \to \mathcal{F}^+$ is obvious (!)
- $\forall a \in X : \iota_a : \mathcal{F}_a \to (\mathcal{F}^+)_a$ is bijective.

Proof. The inverse map $(\mathcal{F}^+)_a \to \mathcal{F}_a$ is given by $\mathcal{F}^+(U) \to \mathcal{F}_a$, $(s_b)_{b \in U} = s_a$ for $U \subseteq X$ such that $a \in U$ + universal property of $\lim_{a \in U \subseteq X} \mathcal{F}^+(X)$

• (ii) is obvious by this construction of \mathcal{F}^+ .

Still to show: the universal property.

 $\varphi: \mathcal{F} \to \mathcal{G}$ be a morphism, \mathcal{G} be a sheaf.

by (ii), there is a unique morphism of sheaves $\varphi^+: \mathcal{F}^+ \to \mathcal{G}^+$ such that $\varphi^+ \circ \iota_{\mathcal{F}} = \iota_{\mathcal{G}} \circ \varphi$. But \mathcal{G} is a sheaf, it implies that \mathcal{G} can be identified with \mathcal{G}^+ under $\iota_{\mathcal{G}}$, i.e., $\varphi^+ \circ \iota_{\mathcal{F}} = \varphi$.

Proposition Let X be a topological space, $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves of groups on X. The sheaf $\operatorname{im}(\varphi)$ on X associated to the presheaf $(U \subseteq X) \mapsto \operatorname{im}(\varphi_m) \subseteq \mathcal{G}(U)$ can be identified with a subsheaf of \mathcal{G} . The equality occurs iff $\forall a \in X : \operatorname{im}(\varphi_a) = \mathcal{G}_a$.

The proof of the proposition. Obvious: $\operatorname{im}(\varphi)^{\mathcal{F}}$ is a subpresheaf of \mathcal{G} .

The inclusion morphism: $\operatorname{im}(\varphi)^{\mathcal{F}} \hookrightarrow \mathcal{G}$.

Morphism: $im(\varphi) \to \mathcal{G}$.

Here, $\forall U \subseteq X : \operatorname{im}(\varphi)(U) \to \mathcal{G}(U)$ is injective since it is injective on stalks.

 $im(\varphi)$ can be identified with a subsheaf of \mathcal{G} .

Furthermore, $im(\varphi) = \mathcal{G}$

 $\iff \forall U \subseteq X \forall t \in \mathcal{G}(U) \forall a \in U \exists V_a \subseteq U, a \in V_a \exists \mathcal{F} \in \operatorname{im}(\varphi_{V_a}) : t_b = \mathcal{F}_b \text{ for all } b \in V_a \text{ (since } \exists \mathcal{F} \dots, \iff \exists s \in \mathcal{F}(V_a) : \varphi_{V_a}(s) = t|_{V_a}).$ $\iff \forall a \in X, \forall \hat{t} = [(U, t)] \in \mathcal{G}, \exists \hat{s} = [(V, s)] \in \mathcal{F}_a : \varphi_a(\hat{s}) = \hat{t}$

Remark. It also follows that $(\forall a \in X : \varphi_a \text{ surjective}) \iff$

 $\forall U \subseteq X \forall t \in \mathcal{G}(U) \exists \text{ open covering } \{U_i\}_{i \in I} \text{ of } U \exists s_i \in \mathcal{F}(U_i) : t|_{U_i} = \varphi_{U_i}(s_i) \text{ for all } i \in I$

Example. sheafification of a presheaf of functions (see the section 1).

Question: Given a continuous map $f: X \to Y$. Can we construct a sheaf on Y from one on X or vice versa?

Definition Let $f: X \to Y$ be a continuous map and \mathcal{F} be a presheaf on X. The direct image of \mathcal{F} under f is a presheaf $f_*\mathcal{F}$ on Y given by

$$(f_*\mathcal{F})(U) := \mathcal{F}(f^{-1}(U))$$

for all $U \stackrel{\circ}{\subseteq} Y$. Here the restriction morphism for $f_* \mathcal{F}$ is obtained directly from \mathcal{F} . $U \stackrel{\circ}{\subseteq} V$:

$$f_*\mathcal{F}(V) \stackrel{=}{\longrightarrow} \mathcal{F}(f^{-1}(V))$$

$$\downarrow \qquad \qquad \downarrow$$

$$f_*\mathcal{F}(U) \stackrel{=}{\longrightarrow} \mathcal{F}(f^{-1}(U))$$

Exercise. Show that fi \mathcal{F} is a sheaf, then so is $f_*\mathcal{F}$.