

Sheaves and their Cohomology

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Last session: introduced (pre-)sheaves and stalks (ad hoc def.)

Exercise Arapura, 3.1.15

A formal definition of stalks is based on the inductive limit (direct limit/filtered colimit)

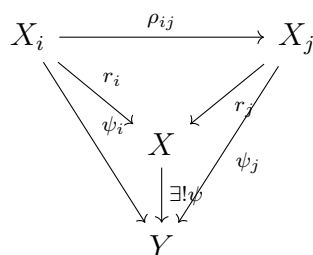
Let \mathcal{C} be a category and I be a directed set, i.e., a poset (I, \leq) such that $\forall i, j \in I \exists k \in I : i \leq k, j \leq k$. (e.g. $(\mathbb{N}, \text{divisibility})$)

Def An inductive system (a direct system) over I in \mathcal{C} is a pair $((X_i)_{i \in I}, (f_{ij})_{i \leq j})$ consisting of a family of objects $X_i \in \text{Ob}(\mathcal{C})$ and a family of morphisms $f_{ij} : X_i \rightarrow X_j$ for $i \leq j$ such that

- $\forall i \in I : f_{ii} = \text{id}_{X_i}$
- $\forall i \leq j \leq k : f_{ik} = f_{jk} \circ f_{ij}$

An inductive limit (a direct limit) of an inductive system $((X_i)_{i \in I}, (f_{ij})_{i \leq j})$ is an object X together with morphisms $r_i : X_i \rightarrow X$ for $i \in I$ such that $\forall i \leq j : r_i = r_j \circ f_{ij}$ which satisfies the following universal property:

For all $Y \in \text{Ob}(\mathcal{C})$ and morphisms $(\psi_i : X_i \rightarrow Y)_{i \in I}$ such that $\forall i \leq j : \psi_i = \psi_j \circ f_{ij}$, there exists a unique $\psi : X \rightarrow Y$ such that $\forall i : \psi \circ r_i = \psi_i$.



Remark: the inductive limit is unique up to unique isomorphism

Notation: $\varinjlim_{i \in I} X_i$

Examples: In (Sets)/(Grps)/(Rings)/(R-Mods)/...

$$\varinjlim_{i \in I} X_i = \left(\bigsqcup_{i \in I} X_i \right) / \left((x_i \in X_i) \sim (x_j \in X_j) \iff \exists k : i, j \leq k \text{ and } f_{ik}(x_i) = f_{jk}(x_j) \right)$$

If \mathcal{F} is a presheaf on a topological space X and $a \in X_i$ then

$$\mathcal{F}_a = \varinjlim_{a \in U \subseteq X} \mathcal{F}(U)$$

$f_{ij} - \rho_{VU} : \mathcal{F}(V) \rightarrow \mathcal{F}(U) \ (V \supseteq U); \mathcal{F}(U) \rightarrow \mathcal{F}_a, s \mapsto [(U, s)].$

Proposition Let X be a topological space, $a \in X$. Every morphism of presheaves $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ gives rise to a map (morphism) $\varphi_a : \mathcal{F}_a \rightarrow \mathcal{G}_a$ where $[U, s] \mapsto [(a, \varphi_a(s))]$.

In particular, $\mathcal{F} \mapsto \mathcal{F}_a$ is a functor $\text{PSh}(X) \rightarrow (\text{Sets})$, or also $\text{PAb}(X) \rightarrow (\text{Ab})$ etc.

Proof. Apply the universal property of $\varinjlim \mathcal{F}(U)$ to $(\mathcal{F}(U) \xrightarrow{\varphi_U} \mathcal{G}(U) \xrightarrow{\varphi_a} \mathcal{G}_a)$

$$\begin{array}{ccc}
 \mathcal{F}(V) & \longrightarrow & \mathcal{G}(V) \\
 \downarrow \rho_{VU} & \searrow & \downarrow \\
 \mathcal{F}(U) & \longrightarrow & \mathcal{G}(U)
 \end{array}
 \begin{array}{c}
 \nearrow \\
 \nearrow \\
 \nearrow
 \end{array}
 \mathcal{G}_a$$

This is possible since $\forall U \stackrel{\circ}{\subseteq} V \stackrel{\circ}{\subseteq} X, a \in U : (r_U^{\mathcal{F}} \circ \varphi_a) \circ \rho_{VU} = r_V^{\mathcal{G}} \circ \varphi_V$

Proposition Let X be a topological space, \mathcal{F}, \mathcal{G} are sheaves in X , $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ a morphism

1. $\forall U \stackrel{\circ}{\subseteq} X : \mathcal{F}(U) \rightarrow \prod_{a \in U} \mathcal{F}_a$ which $s \mapsto (s_a)_{a \in U}$ is injective $s_n := [(U, s)] \in \mathcal{F}_a$.
2. $(\forall U \stackrel{\circ}{\subseteq} X : \varphi_U : \mathcal{F}(U) \rightarrow \mathcal{G}(U))$ injective/bijective $\iff (\forall a \in X : \varphi_a : \mathcal{F}_a \rightarrow \mathcal{G}_a$ injective/bijective)
3. if $\psi : \mathcal{F} \rightarrow \mathcal{G}$ is another morphism then $\varphi = \psi \iff \forall a \in X : \varphi_a = \psi_a$.

Proof.

1. Let $s, t \in \mathcal{F}(U)$ be such that $\forall a \in U : s_a = t_a \implies \forall a \in U \exists V_a \stackrel{\circ}{\subseteq} U, a \in V_a : s|_{V_a} = t|_{V_a}$. since $\bigcup_{a \in U} V_a = U$, we get $s = t$ by the sheaf axiom.
2. Exercise
3. (\implies) Trivial, (\impliedby) To show: $\forall U \stackrel{\circ}{\subseteq} X \forall s \in \mathcal{F}(U) : \varphi_U(s) = \psi_U(s) \stackrel{(i)}{\iff} \forall U \stackrel{\circ}{\subseteq} X \forall s \in \mathcal{F}(U) \forall a \in U : \varphi_U(s)_a = \psi_U(s)_a$ but $\varphi_U(s)_a = \varphi_a(s_a)$ and $\psi_U(s)_a = \psi_a(s_a)$. Hence the claim follows. \square

Question: what about “surjective”?

Example. On $X = \mathbb{C}$, consider $D = \frac{d}{dz} : \mathcal{O}_{\mathbb{C}} = \mathcal{O}_{\mathbb{C}}$. This is given by $f \mapsto f'$ for all $U \stackrel{\circ}{\subseteq} \mathbb{C}$.

Then $U \stackrel{\circ}{\subseteq} V \stackrel{\circ}{\subseteq} \mathbb{C}$:

$$\begin{array}{ccc}
 \mathcal{O}_{\mathbb{C}}(V) & \xrightarrow{D} & \mathcal{O}_{\mathbb{C}}(V) \\
 \downarrow \text{res} & & \downarrow \text{res} \\
 \mathcal{O}_{\mathbb{C}}(U) & \xrightarrow{D} & \mathcal{O}_{\mathbb{C}}(V)
 \end{array}$$

Known: $\mathcal{O}_{\mathbb{C},a} = \left\{ \sum_{n \geq 0} c_n (z-a)^n, c_n \in \mathbb{C}, \text{ positive radius of convergence} \right\}$
 $\left(\frac{d}{dz} |_a : \mathcal{O}_{\mathbb{C},a} \rightarrow \mathcal{O}_{\mathbb{C},a} \right)$ is surjective.

But $\frac{1}{z} \in \mathcal{O}_{\mathbb{C}}(\mathbb{C} \setminus \{0\})$ has no preimage in $\mathcal{O}_{\mathbb{C}}(\mathbb{C} \setminus \{0\})$.

Problem: the presheaf defined by $U \mapsto \text{im}(D_U : \mathcal{O}_{\mathbb{C}}(U) \rightarrow \mathcal{O}_{\mathbb{C}}(U))$ is not a sheaf!

Proposition/Definition let X be a topological space, \mathcal{F} be a presheaf on X . There exists a sheaf \mathcal{F}^\dagger and a morphism $r : \mathcal{F} \rightarrow \mathcal{F}^\dagger$ with the following universal property:

For every sheaf \mathcal{G} on X and morphism $\varphi : \mathcal{F} \rightarrow \mathcal{G}$, there exists a morphism $\varphi^\dagger : \mathcal{F}^\dagger \rightarrow \mathcal{G}$ such that $\varphi^\dagger \circ r = \varphi$.

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\varphi} & \mathcal{G} \\ & \searrow r & \nearrow \varphi^\dagger \\ & \mathcal{F}^\dagger & \end{array}$$

(\mathcal{F}^\dagger, r) is unique up to unique isomorphism. It is called the sheafification or the associated sheaf of \mathcal{F} . Furthermore, the following properties hold:

1. $\forall a \in X : r_a : \mathcal{F}_a \rightarrow (\mathcal{F}^\dagger)_a$ is an isomorphism.
2. (Functoriality) for every morphism of presheaves $\mathcal{F} \xrightarrow{\varphi} \mathcal{G}$, there is a unique morphism $\varphi^\dagger : \mathcal{F}^\dagger \rightarrow \mathcal{G}^\dagger$ making the following diagram commutative.

$$\begin{array}{ccccc} \mathcal{F} & \xrightarrow{\varphi} & \mathcal{G} & \xrightarrow{\psi} & \mathcal{H} \\ \downarrow \varphi_{\mathcal{F}} & & \downarrow \varphi_{\mathcal{G}} & & \downarrow \psi_{\mathcal{H}} \\ \mathcal{F}^\dagger & \xrightarrow{\varphi^\dagger} & \mathcal{G}^\dagger & \xrightarrow{\psi^\dagger} & \mathcal{H}^\dagger \end{array}$$

Remark.

1. If \mathcal{F} is already a sheaf, then $r : \mathcal{F} \rightarrow \mathcal{F}^\dagger$ is an isomorphism.
2. Sheafification functor $\text{PSh}(X) \rightarrow \text{Sh}(X)$ (Sh to Ab)

Universal property: for every presheaf \mathcal{F} and a sheaf \mathcal{G} on X , there is a natural bijection (natural in sense of natural transformation)

$$\text{Hom}_{\text{Sh}(X)}(\mathcal{F}^\dagger, \mathcal{G}) \cong \text{Hom}_{\text{PSh}(X)}(\mathcal{F}, \mathcal{G})$$

The sheafification is left-adjoint to the inclusion functor.

*Proof.*¹

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||      |  |  |  |  <---- Stalks
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|||\    |  || || || || |
||||\--- section ----->

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¹feedback: the difference between sheaves and presheaves in sense of stalks.

$$\begin{array}{ccc}
f_*\mathcal{F}(V) & \xrightarrow{=} & \mathcal{F}(f^{-1}(V)) \\
\downarrow & & \downarrow \\
f_*\mathcal{F}(U) & \xrightarrow{=} & \mathcal{F}(f^{-1}(U))
\end{array}$$

Exercise. Show that if \mathcal{F} is a sheaf, then so is $f_*\mathcal{F}$.