Sheaves and their cohomology

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Recall: geometric tangent space

$$T_p^{\text{geo}}M = \{(I,c) \mid 0 \in I \overset{\circ}{\subseteq} K, c : I \to K, c(0) = p\}$$

Observation Let M be a premanifold, $p \in M$, $\gamma := [(I,c)] \in T_p^{\text{geo}}M$.
$\mathbb{K}$-linear map $D_\gamma : \mathcal{O}_{M,p} \to \mathbb{K}$, $f \mapsto (f \circ c)'(0)$ $([(U,f)])$ for $I \overset{\circ}{\supseteq} c^{-1}(U) \to \mathbb{K}$ with the property that $\forall f, g \in \mathcal{O}_{M,p} : D_\gamma(f \cdot g) = ((f \circ c)(g \circ c))'(0)$ which holds the product rule of derivative.

Def Let R be a ring, A be an R-algebra (commutative with unit), M be a model.
An R-derivation on A with values in M is an R-linear map, $\partial : A \to M$ such that $\forall f, g \in A : \partial(fg) = f\partial(g) + g\partial(f)$.

Notation. $\mathrm{Der}_R(A, M) = \{R\text{-derivations } A \to M\}$. This is an A-module.

Example D_γ from before $D_\gamma : \mathcal{O}_{M,p} \to \mathbb{K}$. Here $\mathbb{K}$ is an $\mathcal{O}_{M,p}$-module under the evaluation homomorphism $\mathrm{ev}_p : \mathcal{O}_{M,p} \to \mathbb{K}, f \mapsto f(p)$ $(\mathcal{O}_(M,p) \times \mathbb{K} \to \mathbb{K}, f, a \mapsto f \cdot a := f(p) \cdot a)$

By this way, we obtain an injective k-linear (!) map
$T_p^{\text{geo}}M \to \mathrm{Der}(\mathcal{O}_{M,p}, \mathbb{K})$ such that $\gamma = [(I,c)] \mapsto D_\gamma : f \mapsto (f \circ c)'(0)$.

Exercise. Verify this one (well-definedness and linearity)

Interlude: Locally ringed spaces

Let R be a ring (as usual commutative with unit)

Def. An R-ringed space is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of R-algebras $\mathcal{O}_X$ on X. A morphism of R-ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair $\varphi = (\varphi, \varphi^\flat)$ consisting of a continuous map $\varphi : X \to Y$ and a morphism of sheaves of R-algebras $\varphi^\flat : \mathcal{O}_Y \to \varphi_*\mathcal{O}_X$ (structure morphism.)

$U \overset{\circ}{\subseteq} Y$:

$$\begin{tikzcd}
\varphi_U^\flat : \mathcal{O}_Y(U) \arrow[r] \arrow[d] & \mathcal{O}_X(\varphi^{-1}U) \arrow[d] \\
\varphi_{U'}^\flat \mathcal{O}_Y(U') \arrow[r] & \mathcal{O}_X(\varphi^{-1}U')
\end{tikzcd}$$

Remark.

- In an earlier discussion of concrete k-spaces (where k is a field), the strucutre morphism of $\varphi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is given by $\varphi_U^\flat : \mathcal{O}_Y(U) \to \mathcal{O}_X(\varphi^{-1}U)$ such that $f \mapsto f \circ \varphi|_{\varphi^{-1}U}$.

- By adjointness of $(\varphi^{-1}, \varphi_*)$, we obtain a morphism $\varphi^\sharp : \varphi^{-1}\mathcal{O}_Y \to \mathcal{O}_X$, hence also $\varphi^\sharp_a : \mathcal{O}_{Y,\varphi(a)} \to \mathcal{O}_{X,a}$ for all $a \in X$. In case of concrete k-spaces, $\varphi^\sharp_a$ is again given by the composition with φ.
- $R = \mathbb{Z}$ (ringed spaces)

Def Let $\varphi = (\varphi, \varphi^\flat) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ and $\psi = (\psi, \psi^\flat) : (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$ be morphisms of R-ringed spaces. The composition $\psi \circ \varphi : (X, \mathcal{O}_X) \to (Z, \mathcal{O}_Z)$ is given by the compositions $X \xrightarrow{\varphi} Y \xrightarrow{\varphi} Z$ and $(\psi \circ \varphi)^\flat : \mathcal{O}_Z \xrightarrow{\psi^\flat} \psi_*\mathcal{O}_Y \xrightarrow{\psi_*(\varphi)} \psi_*(\varphi_*\mathcal{O}_X) = (\psi \circ \varphi)_*\mathcal{O}_X$

Def

1. A *local ring* is a ring A with exactly one maximal ideal, or equivalently, $\mathfrak{m} := A \backslash A^*$ is an ideal of A. In this case, $\kappa := A/\mathfrak{m}$ is called the residue field of A.
2. Let A and B be localring with maximal ideals $\mathfrak{m}_A$ and $\mathfrak{m}_B$ respectively. A *local ring homomorphism* $(A \to B)$ is a ring homomorphism $\varphi : A \to B$ such that $\varphi(\mathfrak{m}_A) \subseteq \mathfrak{m}_B$, or equivalently (!) $\varphi^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$.

Examples.

- Every field is a local ring.
- Let $(M, \mathcal{O}_M)$ be a premanifold, $p \in M$. (Claim that $\mathcal{O}_{M,p}$ is local.) Consider the evaluation homomorphism

  $$\mathrm{ev}_p : \mathcal{O}_{M,p} \to \mathbb{K}, f \mapsto f(p)$$

  that $\Updownarrow_{M,p} := \ker(\mathrm{ev}_p) = \{f \in \mathcal{O}_{M,p} \mid f(p) = 0\}$ is a maximal ideal of $\mathcal{O}_{M,p}$! (Apply the *first isomorphism theorem*, $\mathcal{O}_{M,p}/\mathfrak{m}_{M,p} \cong \mathbb{K}$ since ev_p is a surjective.)
  Furthermore, $\forall f \in \mathcal{O}_{M,p} \backslash \mathfrak{m}_{M,p}, \exists V \overset{\circ}{\subseteq} M, p \in V : f(x) \neq 0$ for all $x \in V$. ($[(v, 1/f)]$ is a multiplicative inverse of f in $\mathcal{O}_{M,p}$!)
  This means that $\mathcal{O}_{M,p} \backslash \mathfrak{m}_{M,p} = \mathcal{O}^\times_{M,p}$. Hence $\mathcal{O}_{M,p}$ is a local ring.

- Let $F : M \to N$ be a morphism of premanifolds, $a \in M$: $F^\sharp_a : \mathcal{O}_{N,F(a)} \to \mathcal{O}_{M,a}$ is given by composition with F. We see that $F^\sharp_a$ is a local ring homomorphism.

Def

- A *locally R-ringed space* is an R-ringed space $(X, \mathcal{O}_X)$ such that $\forall a \in X : \mathcal{O}_{X,a}$ is a local ring.
- A *morphism of locally R-ringed spaces* $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of R-ringed spaces $(\varphi, \varphi^\flat)$ such that $\forall a \in X : \varphi^\sharp_a : \mathcal{O}_{Y,\varphi(a)} \to \mathcal{O}_{X,a}$ is a local ring homomorphism.

Example morphisms of premanifolds.
Remark. The category of locally R-ringed spaces is a subset of the one of R-ringed spaces but not a full one!

Exercise Let k be a field. Show that a concrete k-space $(X, \mathcal{O}_X)$ is a locally k-ringed space if $\forall U \overset{\circ}{\subseteq} X, \forall f \in \mathcal{O}_X(U)$: f nowhere also on U: $\frac{1}{f} \in \mathcal{O}_X(U)$.

Proposition. Let k be a field, $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ be locally ringed concrete k-spaces, $\varphi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of k-ringed spaces.

Then $\forall U \overset{\circ}{\subseteq} Y, \forall f \in \mathcal{O}_Y(U) : \varphi_U^\flat(f) = f \circ \varphi|_{\varphi^{-1}U}$

In particular, every such a morphism of locally k-ringed spaces.

Proof. Let $U \overset{\circ}{\subseteq} Y$, $f \in \mathcal{O}_Y(U)$. (To show: $\forall a \in \varphi^{-1}(U), \varphi_U^\flat(f)(a) \overset{!}{=} f(\varphi(a))$).
Let $a \in \varphi^{-1}U$.
The commutative diagram $\to$

$$\begin{tikzcd}
f \arrow[rrr, mapsto] \arrow[dddd] & & & \varphi_a^\flat(f) \arrow[dddd] \\
& \mathcal{O}_Y(U) \arrow[r, "\varphi_U^\flat"] \arrow[d] & \mathcal{O}_X(\varphi^{-1}U) \arrow[d] & \\
& \mathcal{O}_{Y,\varphi(a)} \arrow[r, "\varphi_U^\sharp"] \arrow[d, "\mathrm{ev}_{\varphi(a)}"] & \mathcal{O}_{X,a} \arrow[d, "\mathrm{ev}_{\varphi(a)}"] & \\
& k \arrow[r, "?"] & k & \\
f(\varphi(a)) \arrow[rrr, mapsto] & & & \varphi_a^\flat(f)(a)
\end{tikzcd}$$

Claim. $\mathrm{ev}_a \circ \varphi_a^\sharp = \mathrm{ev}_\varphi(a) : \mathcal{O}_{Y,\varphi(a)} \to k$.
In fact, $\mathrm{ev}_a \circ \varphi_a^\sharp$ is k-linear and surjective, then $\ker(\mathrm{ev}_a \circ \varphi_a^\sharp)$ is a maximal ideal of $\mathcal{O}_{Y,\varphi_a}$.
Hence, $\ker(\mathrm{ev}_a \circ \varphi_a^\sharp) = \mathfrak{M}_{Y,\varphi(a)} (= \{f \in \mathcal{O}_{Y,\varphi(a)} \mid f(\varphi(a)) = 0\}$.
By then, $\mathrm{ev}_a \circ \varphi_a^\sharp$ and $\mathrm{ev}_{\varphi(a)}$ factor through a unique k-linear isomorphisms $\mathcal{O}_{Y,\varphi(a)}/\mathfrak{M}_{Y,\varphi(a)} \overset{\sim}{\to} k$.
Therefore, $\mathrm{ev}_a = \varphi_a^\sharp = \mathrm{ev}_a$. $\square$

Remark. For an example of a morphism of ringed spaces between locally ringed spaces which is not a morphism of locally ringed spaces, see (Hartshorne, Ch II. Example 2.3.2)

Back to the tangent spaces (Recall: $T_p^{\text{geo}} M \hookrightarrow \mathrm{Der}_{\mathbb{K}}(\mathcal{O}, \mathbb{K})$)

Def Let M be a C^∞/C^ω/complex-premanifold, $p \in M$, the algebraic tangent space of M at p

$$T_p^{\text{alg}} M = \mathrm{Der}_{\mathbb{K}}(\mathcal{O}_{M,p}, \mathbb{K})$$

Let $F : M \to N$ be a morphism of $(\dots)$-premanifolds, $a \in M$, then the tangent map is defined by
$T_p^{\text{alg}}(F) : T_p^{alg}(M) \to T_p^{alg}(N)$, that $\partial \mapsto \partial \circ F_a^\sharp$ (Check that it is a $\mathbb{K}$-derivation.)

Remark. Comparing T_p^{geo} and T_p^{alg}

$$
\begin{tikzcd}
\gamma \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & F_*\gamma \arrow[ddd, mapsto] \\
& T_p^{\mathrm{geo}}M \arrow[r, "F_*"] \arrow[d] & T_p^{\mathrm{geo}}N \arrow[d] & \\
& T_p^{\mathrm{alg}}M \arrow[r, "F_a^\sharp"] & T_p^{\mathrm{alg}}N & \\
D_\gamma \arrow[rrr, mapsto] & & & D_\gamma \circ F_a^\sharp \stackrel{?}{=} D_{F_*\gamma}
\end{tikzcd}
$$

Still to show that $T_p^{\mathrm{geo}}M \cong T_p^{\mathrm{alg}}M$ (so that in future, we can just write T_pM)

Lemma. Let k be a field, A be a k-algebra which is a local ring with maximal ideal $\mathfrak{m}$ and residue field $A/\mathfrak{m} \cong k$. There is an isomorphism of k-vector spaces.

$\mathrm{Der}_k(A,k) \xrightarrow{\sim} \mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k) = (\mathfrak{m}/\mathfrak{m}^2)^\vee$ that $\partial \mapsto \partial|_{\mathfrak{m}}$.

k can be interpreted as A-model via $A/\mathfrak{m}$. ($\mathfrak{m}^2$ = ideal of A generated by $f \circ g : f, g \in \mathfrak{m}$.)

Proof

- Well-definedness: why is $\partial(\mathfrak{m}^2) = 0$? Let $f, g \in \mathfrak{m}$. Then $\partial(fg) = [f]\partial(g) + [g]\partial(f) = 0$ ($f, g \in \mathfrak{m}$.)
  Hence $\partial|_{\mathfrak{m}}$ factors through $\mathfrak{m}/\mathfrak{m}^2$.

- k-linear: easy
- inverse map: $(\mathfrak{m}/\mathfrak{m}^2)^\vee \to \mathrm{Der}_k(A,k)$ that $\varphi \mapsto (D_\varphi : A \to k, f \mapsto \varphi(f - \bar{f}))$ $(A \to A/\mathfrak{m} \cong k \hookrightarrow A, \lambda \mapsto \lambda \cdot 1)$ Check that D_φ is really a derivation, $\partial \mapsto \partial|_{\mathfrak{m}}$ and $\varphi \mapsto D_\varphi$ are inverse to each other.

Def Let A be a local ring with maximal ideal $\mathfrak{m}$ and residue field k. We say that A satisfies *the tangent space condition* if

1. $\exists$ inclusion homomorphism $\iota : k \hookrightarrow A$ such that $(k \xhookrightarrow{\iota} A \twoheadrightarrow A/\mathfrak{m} \cong k)$ is id_k.
2. $\mathfrak{m}$ is finitely generated.

Example

- $k[[x_1, x_2, \dots, x_n]] = \{\text{power series over } k \text{ in } x_1, \dots, x_n\}$. $\mathfrak{m} = (x_1, x_2, \dots, x_n)$.
- Claim. $a \in \mathbb{R}^n$: $\mathcal{C}^\infty_{\mathbb{R}^n,a}, \mathcal{C}^\omega_{\mathbb{R}^n,a}$ satisfy the tangent space condition, also with $a \in \mathcal{C}^n$, so is $\mathcal{O}^h_{\mathbb{C}^n,a}$ (exercise: verify this. Arapura, 2.5.18)

In fact, we can show that

- $\mathfrak{m}_{\mathcal{C}^\infty_{\mathbb{R}^n},a}, \mathfrak{m}_{\mathcal{C}^\omega_{\mathbb{R}^n},a}$ are both generated by $\{x_1 - a_1, x_2 - a_2, \dots, x_n - a_n\}$.
- $\mathfrak{m}_{\mathcal{O}^h_{\mathbb{C}^n},a}$ is generated by $\{z_1 - a_1, z_2 - a_2, \dots, z_n - a_n\}$.

(Hint: Arapura, Exercise 2.5.18)

Based on this observation, we can show that

$$\dim T_p^{\mathrm{alg}} M = \dim M = \dim T_p^{\mathrm{geo}} M$$

The injective k-linear map $T_p^{\mathrm{geo}} M \hookrightarrow T_p^{\mathrm{alg}} M = \mathrm{Der}_{\mathbb{K}}(\mathcal{O}_{M,p}, \mathbb{K})$ is in fact an isomorphism!

Remark. To see what goes wrong with $C^\alpha, \alpha < \infty$ see (Wedhorn, Problem 5.3)

Def Let $(A, \mathfrak{m}, k)$ be a local ring which satisfies the tangent space condition.
The *Zariski cotangent space* of A is $T_A^\vee := \mathfrak{m}/\mathfrak{m}^2$ (This is a k-vector space.)
The Zariski tangent space is $T_A := (\mathfrak{m}/\mathfrak{m}^2)^\vee$ which $(T_A^\vee)^\vee \cong T_A$.

Hence by the previous discussion, we see that

$$T_{\mathcal{O}_{M,p}} \cong T_p^{\mathrm{alg}} M \cong T_p^{\mathrm{geo}} M.$$

(Quiz, 8 Sep.)