Sheaves and their cohomology

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Recall (in short)

$$M \otimes_R N = \left\{ \sum_{i=1}^{k} m_i \otimes n_i : k \in \mathbb{N}, m_i \in M, n_i \in N \right\}$$

subject to the following rules: ...

Basic properties

1. $R \otimes_R M \cong M \cong M \otimes_R R$
2. $(M \otimes_R N) \otimes_R L \cong M \otimes_R (N \otimes_R L)$
3. $M \otimes_R N \cong N \otimes_R M$
4. $(\bigoplus_{i \in I} M_i) \otimes_R N \cong \bigoplus_{i \in I}(M_i \otimes_R L)$

Remark.
(2) can be generalized to finitely many modules, i.e,

$$M_1 \otimes_R M_2 \otimes_R \cdots \otimes_R M_n \cong (M_1 \otimes_R (M_2 \otimes_R (\cdots))) \otimes_R M_n \cong M_1 \otimes_R (M_2 \otimes_R ((\cdots) \otimes_R M_n))$$

This also satisfies the universal property with R-multilinear maps $M_1 \times \cdots \times M_n \to N$ instead of R-bilinear maps.

(4) the tensor product of free module is free.

$$M = R^{(I)} = \bigoplus_{i \in I} Re_i, N = R^{(J)} = \bigoplus_{j \in J} Rf_j$$

$M \otimes_R N \cong \bigoplus_{(i,j) \in I \times J} R(e_i \otimes f_j)$

Functoriality
Let M_j, N_j be R-modules, $\varphi_j : M_j \to N_j$ for $j = 1, 2$. There exists uniquely R-linear map such that

$$\varphi_1 \otimes \varphi_2 : M_1 \otimes_R M_2 \to N_1 \otimes_R N_2$$

that is $(\varphi_1 \otimes_R \varphi_2)(m_1 \otimes m_2) = \varphi_1(m_1) \otimes \varphi_2(m_2)$ for all $m_1 \in M_1, m_2 \in M_2$.

Key step: apply the universal property to the R-bilinear map
$M_1 \times M_2 \to N_1 \otimes N_2$, $(m_1, m_2) \mapsto \varphi_1(m_1) \otimes_R \varphi(m_2)$.

Extension of scalars Let $\rho : R \to S$ be a ring homomorphism, M be an R-module.
$\rho^* M = M \otimes_R S$ is an S-module via $t(m \otimes_R s) := m \otimes ts$ that $m \in M, s, t \in S$.
(The multiplication with $t \in S$ is $\mathrm{id}_M \otimes_R (s \mapsto ts)$)

S is an R-module via $r \cdot s := \rho(r)s$, $r \in R$, $s \in S$.

Examples

- Let $M = \bigoplus_{i \in I} Re_i$ be a free R-module. Then, $M \otimes_R S \cong \oplus_{i\in I} Se_i$.
- Let V be an $\mathbb{R}$-vector space, $\rho : \mathbb{R} \hookrightarrow \mathbb{C}$. Then $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$ "complexification of V." $V_{\mathbb{C}} = \{v + iw \,|\, v, w \in V\}$ where $(\alpha + i\beta)(v + iw) = (\alpha v - \beta w) + i(\alpha w + \beta v)$.

Remark. By this way, we obtain a functor $\rho^* : (R\text{-Mod}) \to (S\text{-Mod})$

Exercise Let $\rho : R \to S$ be a ring homomorphism, M, N be R-modules. Show that $(\rho^* M) \otimes_S (\rho^* N) \cong \rho^*(M \otimes_R N)$.

Tensor Algebra & co.

Let M be an R-module.

Def

- Let $r \in \mathbb{N}_0$. The r-th tensor power of M is

$$T_R^r(M) := T^r(M) = M^{\otimes r} = \underbrace{M \otimes_R \cdots \otimes_R M}_{r \text{ times}}$$

  As $T^0(M) = R$ and $T^1(M) = M$.

- The tensor algebra of M is $T_R(M) := T(M) = \bigoplus_{r \geq 0} T^r(M)$ with the multiplication law obtained from $T^r(M) \times T^s(M) \to (T^r(M) \otimes_R T^s(M)) = T^{r+s}(M))$ such that $(m_1 \otimes \cdots \otimes m_r, n_1 \otimes \cdots n_s) \mapsto (m_1 \otimes \cdots \otimes m_r \otimes n_1 \otimes \cdots \otimes n_s)$. This is an associative R-algebra (not necessary commutative.)

The exterior algebra of M is

$$\Lambda_R(M) := \Lambda(M) := T(M) \Big/ (m \otimes m | m \in M)$$

where $(m \otimes m | m \in M)$ is two-sided ideal of $T(M)$. Then for this case $[m \otimes n] = -[n \otimes m]$.

This implies that $[m_1 \otimes \cdots \otimes m_n] = 0$ if $\exists i \neq j : m_i = m_j$.

The multiplication on $\Lambda(M)$ is typically denoted by $\wedge$ (wedge product.)

Therefore, the equivalence class of $m_1 \otimes m_2 \otimes \cdots \otimes m_n$ is $m_1 \wedge m_2 \wedge \cdots \wedge m_n$.

For $r \in \mathbb{N}_0$, we denoted by $\Lambda^r(M)$ the submodule of $\Lambda(M)$ generated by $m_1 \wedge \cdots \wedge m_r$ "r-th exterior product." $\Lambda(M) = \bigoplus \Lambda^r(M)$.

- The symmetric algebra of M is

$$\mathrm{Sym}_R(M) := \mathrm{Sym}(M) := T(M) \Big/ (m \otimes n - n \otimes m | m, n \in M)$$

  Similarly to $\Lambda(M)$, we have

$$\mathrm{Sym}(M) = \bigoplus_{r \in \mathbb{N}_0} \mathrm{Sym}^r(M)$$

where $\mathrm{Sym}^r(M)$ is generated by $[m_1 \otimes m_2 \otimes \cdot \otimes m_r]$ for $M_i \in M$.

Example Let $M := R^n$ (free R-module with basis $\{e_1, \dots, e_n\}$)

- $T^r(M)$ is free with basis $\{e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_r} \mid i_1, \dots, i_r \in \{1, 2, \dots, n\}\}$
- $\Lambda^r(M)$ is free with basis $\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_r} \mid i_1, \dots, i_r \in \{1, 2, \dots, n\}\}$ The rank of $\Lambda^r(M)$ is $\binom{n}{r}$.
- $\mathrm{Sym}(M) \cong R[x_1, \dots, x_n]$ as R-algebra.

Remark $\Lambda(M) = T(M) \Big/ (m \otimes m | m \in M)$

- $\Lambda^r(M)$ satisfies the following universal property

$$\begin{tikzcd} M \times \cdots \times M \arrow[rr] \arrow[dr] & & \Lambda^r M \arrow[dl, dotted, "\exists!"] \\ & N & \end{tikzcd}$$

$\forall R$-module $N, \varphi : M \times \cdots \times M \to N$ alternating r-multilinear map.
$\exists! \psi : \Lambda^r(M) \to N$ that is R-linear: $\psi(m_1 \wedge \cdots \wedge m_r) = \varphi(m_1, m_2, \dots, m_r)$ for all $m_i \in M$.

- $\Lambda(M)$ is "graded commutative." $\forall \omega \in \Lambda^r(M), \eta \in \Lambda^s(M) : \eta \wedge \omega = (-1)^{rs} \omega \wedge \eta_j$.
  (Note: $(-1)^{rs} = \mathrm{sgn}\left(\begin{cases} i + r & 1 \le i \le s \\ i - s & s < i \le r \end{cases}\right)$)

Exercise Let M be an R-module, $r \in \mathbb{N}$.

1. Let $\alpha_1, \dots, \alpha_r \in M^\vee = \mathrm{Hom}_R(M, R)$. Show that there is an R-linear map
$$\alpha_1 \wedge \cdots \wedge \alpha_n : \Lambda^r(M) \to R, m_1 \wedge \cdots \wedge m_r \mapsto \det(\alpha_i(m_j))_{i \times j}$$
2. Derive from (1) the R-linear map
$$\Phi : \Lambda^r(M^\vee) \to \Lambda^r(M)^\vee$$
Show that Φ is an isomorphism if M is free and finitely operated.

Back to sheaves of modules:
Let $(X, \mathcal{O}_X)$ be a ringed space.

Exercise Show that if $\mathcal{F}$ is a locally free $\mathcal{O}_X$-module of rank n, then so is $\mathcal{F}^\vee := \mathfrak{Hom}_{\mathcal{O}_X}(F, \mathcal{O}_X)$. (locally free of rank n: has open cover $\{U_i\} : \mathcal{F}_{U_i} \cong (\mathcal{O}_X)|_{U_i}$)

Def Let $\mathcal{F}, \mathcal{G}$ be $\mathcal{O}_X$-modules. The tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is a sheaf on X associated to the presheaf
$$(U \overset{\circ}{\subseteq} X) \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$$

This is an $\mathcal{O}_X$-module.

Some interesting properties. Let $\mathcal{F}, \mathcal{G}$ be $\mathcal{O}_X$-modules.

- $\forall x \in X : (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_a \cong \mathcal{F}_a \otimes_{\mathcal{O}_{X,a}} \mathcal{G}_a)$ ($\mathcal{F}_a$ is an $\mathcal{O}_{X,a}$-module via $[(U,f)][(V,s)] := [(U \cap V, (f|_{U\cap V})(s|_{U\cap V}))])$
- If $\mathcal{F}, \mathcal{G}$ are locally free of rank m, n respectively, then $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is locally free of rank mn.

Proof. Exercise. (Wedhorn, Problem 8.7)

Exercise (Wedhorn) Problems 8.7-8.8

Prop Let $\mathcal{L}$ be locally free $\mathcal{O}_X$-module of rank 1. Then, $\mathcal{L}^\vee = \mathfrak{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$ is locally free of rank 1 (Exercise: !) and $\mathcal{L} \otimes \mathcal{L}^\vee \cong \mathcal{O}_X$.

Proof. For $U \overset{\circ}{\subseteq} X$, consider $\mathcal{L}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{L}^\vee(U) \to \mathcal{O}_X$ arising from the $\mathcal{O}_X(U)$-bilinear map.

$$\mathcal{L}(U) \times \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{L}|_U, \mathcal{O}_X|_U) \to \mathcal{O}_X(U), (s, \varphi) \mapsto \varphi_U(s)$$

Sheafifying this yields $\mathcal{L} \otimes \mathcal{L}^\vee \to \mathcal{O}_X$. This is an isomorphism since it is an isomorphism on stalks (Check!!) $\square$

Remark Therefore, the isomorphism classes of locally free $\mathcal{O}_X$-modules of rank 1 together with the tensor product form an abelian group. This is called the Picard group and denoted by $\mathrm{Pic}(X)$.

Also a remark. We can also define the tensor products of finitely many $\mathcal{O}_X$-modules in a similar manner.

Let M be a C^∞-manifold.
(r, s)-tensors are sections of $(\mathcal{T}_M^1)^{\otimes r} \otimes (\mathcal{E}_M^1)^{\otimes s}$.

$$F^{i_1,\ldots,i_r}_{j_1,\ldots,j_n} \partial_{i_1} \otimes \cdots \partial_{i_r} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s}.$$

Def Let $\mathcal{F}$ be an $\mathcal{O}_X$-module, $r \in \mathbb{N}_0$.
The r-th exterior power of $\mathcal{F}$ is the sheaf associated to $U \mapsto \Lambda^r_{\mathcal{O}_X(U)}(\mathcal{F}(U))$. We call them $\Lambda^r \mathcal{F}$.
The r-symmetric power of $\mathcal{F}$ is the sheaf associated to $U \mapsto \mathrm{Sym}^r_{\mathcal{O}_X(U)}(\mathcal{F}(U))$. We call them $\mathrm{Sym}^r(\mathcal{F})$.

Example. Let M be a C^∞-manifold. Then, $\mathcal{E}^p_M := \Lambda^p(\mathcal{E}^1_M)$ (p-th differential forms: $dx^{i_1} \wedge \cdots \wedge dx^{i_p}$.)

Remark. If $\mathcal{F}$ is a locally free $\mathcal{O}_X$-module of rank n, then $\Lambda^r \mathcal{F}$ is locally free of rank $\binom{n}{r}$.
If $r = n$, $\det(\mathcal{F}) := \Lambda^n \mathcal{F}$ is locally free of rank 1 (the determinant of $\mathcal{F}$)

($T : V \to V$ of dimension n, $\dim V = n$: $\Lambda^n T : \Lambda^n V \to \Lambda^n V$ is given by multiplication with $\det T$)

Furthermore, there is an isomorphism (!)

$$\Lambda^r(\mathcal{F}^\vee) \xrightarrow{\sim} \Lambda^r(\mathcal{F})^\vee$$

Now let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces.

Def

- Let $\mathcal{F}$ be an $\mathcal{O}_X$-module, then $i_*\mathcal{F}$ is an $\mathcal{O}_Y$-module (a pushforward.) For $V \overset{\circ}{\subseteq} Y$: $i_*(V) = \mathcal{F}(f^{-1}(V))$ is an $\mathcal{O}_X(f^{-1}(V))$-module hence also an $\mathcal{O}_Y(V)$-module via $f^\flat_V : \mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}(V))$.
- Let $\mathcal{G}$ be an $\mathcal{O}_Y$-module. $f^{-1}\mathcal{G}$:= a sheafification of $\left(U \mapsto \varinjlim_{f(U) \subseteq V \overset{\circ}{\subseteq} Y} \mathcal{G}(V)\right)$ is an $f^{-1}\mathcal{O}_Y$-module. $f^\sharp : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. Then $f^*\mathcal{G} := f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$. This is an $\mathcal{O}_X$-module. (a pullback)

Remark for $x \in X$:

- $(f^*(\mathcal{G}))_X \cong \mathcal{G}_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x}$
- If $\mathcal{G}$ is locally free of rank n, so is $f^*\mathcal{G}$.

Exercise. Show that

$$f_* : (\mathcal{O}_X\text{-Mod}) \to (\mathcal{O}_Y\text{-Mod})$$

is a right adjoint of $f^* : (\mathcal{O}_Y\text{-Mod}) \to (\mathcal{O}_X\text{-Mod})$, i.e., $\mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F})$ for all ...