# Sheaves and their cohomology

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Lecturer: Nithi Rungtanapirom
Transcriber: Kittapat Ratanaphupha

Let $f : X \to Y$ be a continuous map.
Last session: $\mathcal{F}$ (pre-)sheaf on $X$: direct image $f_*\mathcal{F}$ on $Y$.

*Remark*

- This gives rise to a functor $f_* : \mathrm{PSh}(X) \to \mathrm{PSh}(Y)$ (also with $\mathrm{Sh}, \mathrm{PAb}, \mathrm{Ab}$.)
- If $g : Y \to Z$ is another continuous map, then $(g \circ f)_* = g_* \circ f_*$

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Q: What about the inverse direction?
The problem is: $U \mathring{\subseteq} X \not\Rightarrow f(U) \mathring{\subseteq} Y$ in general!

**Def** Let $f : X \to Y$ be a continuous map, $\mathcal{G}$ be a presheaf on $Y$. The *presheaf inverse image* $f^{\mathcal{P}}\mathcal{G}$ on $X$ is defined by

$$f^{\mathcal{P}}\mathcal{G}(U) := \varinjlim_{f(U) \subseteq V \mathring{\subseteq} Y} \mathcal{G}(V) \quad \text{for each } U \mathring{\subseteq} X.$$

The restriction morphisms $f^{\mathcal{P}}\mathcal{G}(U) \to f^{\mathcal{P}}\mathcal{G}(U')$.

For $U' \mathring{\subseteq} U \mathring{\subseteq} X$ are "given canonically."

The *sheaf inverse image* $f^{-1}\mathcal{G}$ is the sheaf associated to $f^{\mathcal{P}}\mathcal{G}$.

*Remark*:

- The easiest (?) case: $f : X \to Y$ is an open continuous map, then $f^{-1}\mathcal{G} = \mathcal{G}(f(U))$ for all $U \mathring{\subseteq} X$.
- This construction yields a functor

  $$f^{\mathcal{P}} : \mathrm{PSh}(Y) \to \mathrm{PSh}(X)$$

  (also in case of PAb.; $\mathcal{G} \to \mathcal{H}$ on $Y$: $\mathcal{G}(C) \to \mathcal{H}(V)$ for $V \mathring{\subseteq} Y$. $f^{\mathcal{P}}\mathcal{G} \to f^{\mathcal{P}}\mathcal{H}$.) Hence, also, $f^{-1} : \mathrm{PSh}(Y) \to \mathrm{PSh}(X)$.
- If $g : Y \to Z$ is another continuous map, then $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ (as functors)
- $a \in Y$ and $X = \{a\}$, $i : \{a\} \hookrightarrow Y$ an inclusion map. $\mathcal{F}$ is a presheaf on $Y$ defined by $i^{-1}\mathcal{F} = \mathcal{F}_a$.
- Hence if $\mathcal{G}$ is a pre-sheaf on $Y$ and $a \in X$, ($f : X \to Y$ a continuous map), then $(f^{-1}\mathcal{G})_a = \mathcal{G}_{f(a)}$

(Stalk is appropriate to inverse one, section: direct one)

*Example.* let $E$ be a set: sheaves $E_X$ and $E_Y$ of locally connected functions. (Constant sheaves.)

Claim: $f : X \to Y$ continuous map, then $f^{-1}E_Y = E_X$.

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**Prop**. Let $f : X \to Y$ be a continuous map, $\mathcal{F}$ be a presheaf on $X$, $\mathcal{G}$ be a (pre-)sheaf on $Y$.
There is a natural (transformation) bijection.

$$\mathrm{Hom}_{\mathrm{Sh}(X)}(f^{-1}\mathcal{G}, \mathcal{F}) \underset{\substack{\varphi \mapsto \varphi^\flat \\ \psi^\sharp \mapsto \psi}}{\cong} \mathrm{Hom}_{\mathrm{PSh}(Y)}(\mathcal{G}, f_*\mathcal{F}).$$

($f^{-1}$ is a left adjoint of $f_*$.)

*Proof.* (sketch)

- $\varphi : f^{-1}\mathcal{G} \to \mathcal{F}$: define $\varphi^\flat : \mathcal{G} \to f_*\mathcal{F}$ by $(\varphi^\flat)_V : \mathcal{G}(V) \to \varinjlim_{f(f^{-1}V) \subseteq W \mathring{\subseteq} Y} \mathcal{G}(W) \underset{\text{maps from the sheafification}}{\rightarrow} f^{-1}\mathcal{G}(f^{-1}V) \underset{\varphi_V}{\rightarrow} \mathcal{F}(f^{-1}V)$.
- $\psi : \mathcal{G} \to f_*\mathcal{F}$: define a morphism $\psi^{\mathcal{P}} : f^{\mathcal{P}}\mathcal{G} \to \mathcal{F}$ given by

$$(\psi^{\mathcal{P}})_U : f^{\mathcal{P}}\mathcal{G}(U) = \varinjlim_{f(U) \subseteq V \mathring{\subseteq} Y} \mathcal{G}(V) \xrightarrow{\varinjlim \psi_V}$$

$$\varinjlim_{f(U) \subseteq V \mathring{\subseteq} Y (U \subseteq f^{-1}V)} \to \mathcal{F}(U) \text{ (satisfies the universal property of } \varinjlim_{v} \rho_{f^{-1}V,U})$$

  for $U \mathring{\subseteq} X$. Then apply the universal property of the sheafification to obtain a morphims $\psi^\sharp : \mathcal{F}^{-1}\mathcal{G} \to \mathcal{F}$

A routine work shows that both constructions are inverse to each other! $\square$

**Exercises** (Wedhorn, Problems 3.6, 3.9)

Let $X$ be a topological space, $i : A \hookrightarrow X$ be an inclusion of a closed subspace, $\mathcal{F}$ be a sheaf of abelian groups on $A$.
Show that the morphism of sheaves $i^{-1}(i_*\mathcal{F}) \to \mathcal{F}$ corresponding to $\mathrm{id}_{i_*\mathcal{F}} : i_*\mathcal{F} \to i_*\mathcal{F}$ is an isomorphism!
## Tangent spaces[1]

3 equivalent definitions for $C^\infty$-/$C^\omega$-/complex premanifolds

1. "geometric definition"
2. as a space of derivation on a local ring
3. as the dual space of the "algebraic cotangent space."

Convention:

- Premanifold = $C^\alpha$-($\alpha \in \hat{\mathbb{N}}$) or complex premanifold
- $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Let $M$ be a premanifold and $p \in M$.

Heuristic: a tangent vector at $p$ should be $c'(0)$ for some curve $c : I \to M$ such that $0 \in I \mathring{\subseteq} \mathbb{K}, c(0) = p$.

[1]How to build a vector space on them

If $f \in \mathcal{O}_{M,p}$, the "chain rule" should imply that $(f \circ c)'(0) = Df|_p(c'(0))$.[2]

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**Def** Let $M$ be an $m$-dimensional premanifold and $p \in M$. The (geometric) tangent space of $M$ at $p$ is

$$T_p^{\mathrm{geo}} M = \{(I, c) \,|\, 0 \in I \mathring{\subseteq} K, c : I \to M \text{ morphism with } c(0) = p\} / \sim$$

where $(I_1, c_1) \sim (I_2, c_2) \iff \forall f \in O_{M,p} : (f \circ c_1)'(0) = (f \circ c_2)'(0)$.

The $k$-vector space structure $T_p^{\mathrm{geo}}(M)$ is defined as follows:
Let $(U, x)$ be a chart at $p$ ($p \in U \mathring{\subseteq} M, U \xrightarrow{\sim} B \subseteq \mathbb{K}^m$.)
Bijection (Check!) $T_p^{\mathrm{geo}} M \to \mathbb{K}^m$ ($[(I, c)] \to (x \circ c|)'(0)$; is a map $C^\alpha(U) \to \mathbb{K}^m$)

*Exercise.* Verify that in fact, the inverse map is $\mathbb{K}^m \to T_p^{\mathrm{geo}} M$, $v \mapsto (I, c_v)$ $(c_v(t) := x^{-1}(x(p) - tv))$

The $\mathbb{K}$-vector space structure on $T_p^{geo} M$ is inherited from the one on $\mathbb{K}^m$ under $T_p(x)$.
(This is independent of the choice of chart $(U, x)$!)

*Remark.* therefore, $\dim T_p^{\mathrm{geo}} M = m$.

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**Def** Let $F : M \to N$ be a morphism of premanifolds, $p \in M$. The *derivative of $F$ at $p$* or the *tangent map of $F$ at $p$* is $dF|_p = T_p(F) : T_p^{\mathrm{geo}}(M) \to T_p^{\mathrm{geo}}(N)$, that maps $[(I, c)] \mapsto [(I, F \circ c)]$.

**Why is it well-defined and linear?**
Let $(U, x)$ and $(V, y)$ be charts at $p \in M$ and $F(p) \in N$ respectively, $m = \dim M$ and $n = \dim N$.

$$\begin{array}{ccc} T_p^{\mathrm{geo}} M & \xrightarrow{T_p(F)} & T_{f(p)}^{\mathrm{geo}} N \\ \big\downarrow {\scriptstyle \cong T_p(x)} & & \big\downarrow {\scriptstyle \cong T_{f(p)}(y)} \\ \mathbb{K}^m & \xrightarrow{D(y \circ F \circ x^{-1})} & \mathbb{K}^n \end{array}$$

Chain rule in $\mathbb{K}$

$[(I, c)] \in T_p^{\mathrm{geo}} M \overset{T_p(x)}{\mapsto} (x \circ c|)'(0) \in \mathbb{K}^m$

$\overset{D(y \circ F \circ x^{-1})}{\mapsto} D(y \circ F \circ x^{-1})((x \circ c|)'(0)) = ((y \circ F \circ x^{-1}) \circ (x \circ c|))'(0) = (y \circ (F \circ c)|)'(0) = T_{F(p)}(y)([(I, F \circ c)])$

The above diagram is commutative. Since $T_p(x), D(y \circ F \circ x^{-1})$ and $T_{F(p)}(y)^{-1}$ are well-defined and linear, the same holds for $T_p(F)$.

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**Proposition**. $F : M \to N$, $G : N \to Q$ morphisms of premanifolds, $p \in M \implies T_p(G \circ F) = T_{F(p)}(G) \circ T_p(F)$.

[2] $= (f|_p)_*(c'(0))$

In particular, there is a functor.
$(C^\alpha\text{-PMfd}^*) \to (\mathbb{R}\text{-vect})$ $(\mathbb{C}\text{-PMfd}^*) \to (\mathbb{C}\text{-vect})$

from $(M, p) \mapsto T_p^{\text{geo}}(M)$

PMfd$^*$: pair $(M, p)$ with $M$ premanifold, $p \in M$, morphism: $(M, p) \xrightarrow{F} (N, q)$ = morphism $F : M \to N$ sending $p \mapsto q$.

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**Exercise** let $\phi : S^n \to \mathbb{R}$ be a smooth function. Show that there are at least two points $p, q \in S^n$ such that both $T_p(\phi)$ and $T_q(\phi)$ are zero maps.