

Sheaves and their cohomology

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Overview

1. Manifold
2. Sheaves
3. Derived functors \rightarrow Cohomology

General philosophy: a geometric object should consist of the following data:¹

1. a topological space: X
2. a collection of “distinguished” (real-/complex-/ k -; Körper) valued functions on open subsets of X (e.g., continuous, polynomial, holomorphic, smooth functions)

In what follows, let X be a topological space.

Def. Let T be a nonempty set. A presheaf of T -valued functions is a collection \mathcal{P} of subsets $\mathcal{P}(U) \subset \text{Map}(U, T)$ for $U \subseteq X$ such that $\forall V \subseteq U \subseteq X, \forall f \in \mathcal{P}(U) : f|_V \in \mathcal{P}(V)$.

Easiest example: $\text{Maps}(-, T)$

Less trivial examples:

- Constant presheaf: T^p where $T^p(U) = \{f : U \rightarrow T \text{ constant}\}$
- T is a topological space $\rightarrow \mathcal{C}_{X,T}$ is given by

$$\mathcal{C}_{X,T}(U) = \{f : U \rightarrow T \text{ cont.}\}$$

- $X = \mathbb{R}^n \rightarrow \mathcal{C}_{\mathbb{R}^n}^\infty$ is given by (replaced by smooth functions.)
- $X = \mathbb{C}^n \rightarrow \mathcal{O}_{\mathbb{C}^n}$ is given by (replaced by holomorphic functions.)

Def A presheaf of T -valued functions is called a *sheaf* (of T -valued functions) if the following holds for all open subsets $U \subseteq X$ and open covering $\{U_i\}_{i \in I}$ of U :

For any function $f : U_i \rightarrow T$, if $U_i \in \mathcal{P}(U)$ for all $i \in I$, then $f \in \mathcal{P}(U)$.

Example: smooth function

Non-example: T^p unless every open subset of X is connected. $\mathcal{L}^p(X)^2$ presheaf of p -integrable functions

¹Self note: Topological manifold: manifold with second-countable and Hausdorff

²In this case, $\mathcal{L}^p(X)$ is said to be a set of integrable functions, but in analysis, $L^p(X)$ is used more frequently that is defined by $L^p = \mathcal{L}^p / \mathcal{N}$ where $\mathcal{N} = \{f \in \mathcal{L}^p : f = 0 \text{ a.e.}\}$

Def \mathcal{P} presheaf of (T -valued) functions. The *sheafification* of \mathcal{P} is given by

$$\mathcal{P}^s(U) = \{f : U \rightarrow T \mid \forall x \in U, \exists U_x \overset{\circ}{\subseteq} U, x \in U_x, f|_{U_x} \in \mathcal{P}(U_x)\}$$

Examples:

- $(T^p)^s = T_X$ sheaf of locally constant functions.
 - $(\mathcal{L}^p(\mathbb{R}^n))^s$ is a locally p -integrable functions.
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Now let k be a field.

Note: $\text{Maps}(X, k)$ is a commutative k -algebra (a ring that has a compatible k -vector space structure.)

Def A *concrete k -space* is a pair (X, \mathcal{R}) consisting of a topological space X and a sheaf \mathcal{R} of k -valued functions on X such that $\forall U \overset{\circ}{\subseteq} X, \mathcal{R}(U)$ is a k -subalgebra of $\text{Maps}(U, k)$.

Examples.

- $(\mathbb{R}^n, \mathcal{C}_{\mathbb{R}}^\infty)$, $(\mathbb{R}^n, \mathcal{C}_{\mathbb{R}^n, \mathbb{R}})$ is a concrete \mathbb{R} -space
 - $(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}}^n)$ is a concrete \mathbb{C} -space
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Def Let (X, \mathcal{R}) and (Y, \mathcal{S}) be concrete k -spaces. A morphism of concrete k -spaces $(X, \mathcal{R}) \rightarrow (Y, \mathcal{S})$ is a continuous map $F : X \rightarrow Y$ such that

$$\forall U \overset{\circ}{\subseteq} Y \forall f \in \mathcal{S}(U) : F^*f := f \circ (F|_{F^{-1}U}) \in \mathcal{R}(F^{-1}U)$$

(F^*f is called the pullback of f along F .)

Example. $id_X : (X, \mathcal{R}) \rightarrow (X, \mathcal{R})$

Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a continuous map, say $F = (F_1, F_2, \dots, F_m)$ for $F_i : \mathbb{R}^n \rightarrow \mathbb{R}$

Claim: F induces a morphism $(\mathbb{R}^n, \mathcal{C}_{\mathbb{R}^n}^\infty) \rightarrow (\mathbb{R}^m, \mathcal{C}_{\mathbb{R}^m}^\infty)$ iff F_1, F_2, \dots, F_m are C^∞ .

Proof. (\Leftarrow) Trivial.

(\Rightarrow) $\forall U \overset{\circ}{\subseteq} \mathbb{R}^n \forall f \in \mathcal{C}_{\mathbb{R}^m}^\infty(U) : F^*f := f \circ (F|_{F^{-1}U}) \in \mathcal{C}_{\mathbb{R}^n}^\infty(F^{-1}U)$

Consider the function $\text{pr}_j : \mathbb{R}^m \rightarrow \mathbb{R}, (x_1, x_2, \dots, x_m) \mapsto x_j, j = 1, \dots, m$

Then, $\text{pr}_j \circ F = F_j$.

Since, pr_j is C^∞ , the same holds for F_j . □

Remark: similarly, a continuous map $F = (F_1, \dots, F_m) : \mathbb{C}^n \rightarrow \mathbb{C}^m$ induced a morphism of \mathbb{C} -spaces iff F_1, \dots, F_m are holomorphic.

Exercise Show that if $F : (X_1, \mathcal{R}_1) \rightarrow (X_2, \mathcal{R}_2)$ and $G : (X_2, \mathcal{R}_2) \rightarrow (X_3, \mathcal{R}_3)$ are morphisms of concrete k -spaces, then so $G \circ F$.

Prop/Def An isomorphism of concrete k -spaces $(X, \mathcal{R}) \xrightarrow{\cong} (Y, \mathcal{S})$ is a homeomorphism $F : X \rightarrow Y$ which satisfies one (and hence both) of the following equivalent properties.

1. Both F and F^{-1} are morphisms of concrete k -spaces.
2. $\forall U \subseteq Y, \forall f \in \text{Maps}(U, k) : f \in \mathcal{S}(U) \iff F^*f \in \mathcal{R}(F^{-1}U).$

Notation / Convention

$C_{\mathbb{R}^n}^\alpha$ is a set of n -th differentiable functions (where $\alpha \in \mathbb{N}$)

$C_{\mathbb{R}^n}^\infty$ smooth

$C_{\mathbb{R}^n}^\omega$ analytic

$\hat{\mathbb{N}}_0 = \mathbb{N}_0 \cup \{\omega, \infty\}$ with $\alpha < \infty < \omega$ for all $\alpha \in \mathbb{N}$. $\mathcal{O}_{\mathbb{C}^n}$ holomorphic

Def³ Let $\alpha \in \hat{\mathbb{N}}_0$. A (real) C^α -premanifold is a concrete \mathbb{R} -space $(X, \mathcal{C}_X^\alpha)$ which admits an open covering $X = \bigcup_{i \in I} U_i$ such that for each $i \in I$, there is $B_i \subseteq \mathbb{R}^n$ such that

$$(U_i, \mathcal{C}_X^\alpha|_{U_i}) \cong (B_i, \mathcal{C}_X^\alpha|_{B_i})$$

Terminology

- Atlas = collection of coordinate charts
- $\alpha = 0$, C^0 -manifold is a topological manifold
- $\alpha = \infty$, C^∞ -manifold is a smooth manifold
- $\alpha = \omega$, C^ω -manifold is an analytic manifold

Def Complex pre-manifold is the same as above, but with $(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n})$ instead of $(\mathbb{R}^n, \mathcal{C}_{\mathbb{R}^n}^\infty)$

Example of a premanifold which is not a manifold.

Def A real C^α /complex manifold is a real C^α /complex premanifold such that the topology of its underlying space is given by a metric (\iff the underlying topological space is Hausdorff and paracompact.)

Terminology

- C^α -diffeomorphism is an isomorphism of C^α -manifolds.
- Biholomorphism is an isomorphism of complex manifolds.
- Riemann surface is a one-dimensional complex manifold.

³Self-note: Original definition: a transition map; smooth compatible, The homeomorphism is a coordinate chart.

An n -dimensional complex manifold.

Let (X, \mathcal{O}_X) be a complex manifold of dimension n .

How to define a sheaf \mathcal{C}_X^∞ on X such that $(X, \mathcal{C}_X^\infty)$ become a smooth manifold of dimension $2n$.

Idea: X has a complex atlas $(B_j, \mathcal{O}_{\mathbb{C}^n}|_{B_j})_{j \in I}$ with isomorphisms $g_j : B_j \xrightarrow{\cong} U_j \subseteq X$
Define \mathcal{C}_X^∞ by

$$f \in \mathcal{C}_X^\infty \iff \forall j : f \circ g_j \text{ is } C^\infty$$

(on open subset of B_j in \mathbb{R}^{2n})

Check: $(X, \mathcal{C}_X^\infty)$ is a smooth manifold of dimension $2n$.