Sheaves and their cohomology

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Logistics

Score arrangements:

- Assignments 30%
- Quizzes 30%
- Oral Examination 40%

Recommended textbooks:

- Arapura, *Algebraic geometry over the complex numbers*
- Wedhorn, *Manifolds, Sheaves and Cohomology*

Overview

1. Manifold
2. Sheaves
3. Derived functors → Cohomology

General philosophy: a geometric object should consist of the following data:[1]

1. a topological space: X
2. a collection of "distinguished" (real-/complex-/k-; Körper) valued functions on open subsets of X (e.g., continuous, polynomial, holomorphic, smooth functions)

In what follows, let X be a topological space.

Def. Let T be a nonempty set. A presheaf of T-valued functions is a collection $\mathcal{P}$ of subsets $\mathcal{P}(U) \subset \operatorname{Map}(U, T)$ for $U \overset{\circ}{\subseteq} X$ such that $\forall V \overset{\circ}{\subseteq} U \overset{\circ}{\subseteq} X, \forall f \in \mathcal{P}(U) : f|_V \in \mathcal{P}(V)$.

Easiest example: $\operatorname{Maps}(-, T)$
Less trivial examples:

- Constant presheaf: T^p where $T^p(U) = \{f : U \to T \text{ constant}\}$
- T is a topological space $\to \mathcal{C}_{X,T}$ is given by

$$\mathcal{C}_{X,T}(U) = \{f : U \to T \text{ cont.}\}$$

[1] Self note: Topological manifold: manifold with second-countable and Hausdorff

- $X = \mathbb{R}^n \to \mathcal{C}^\infty_{\mathbb{R}^n}$ is given by (replaced by smooth functions.)
- $X = \mathbb{C}^n \to \mathcal{O}_{\mathbb{C}^n}$ is given by (replaced by holomorphic functions.)

Def A presheaf of T-valued functions is called a *sheaf* (of T-valued functions) if the following holds for all open subsets $U \overset{\circ}{\subseteq} X$ and open covering $\{U_i\}_{i\in I}$ of U:
For any function $f : U_i \to T$, if $U_i \in \mathcal{P}(U)$ for all $i \in I$, then $f \in \mathcal{P}(U)$.

Example: smooth function
Non-example: T^p unless every open subset of X is connected. $\mathcal{L}^p(X)$[2] presheaf of p-integrable functions

Def $\mathcal{P}$ presheaf of (T-valued) functions. The *sheafification of* $\mathcal{P}$ is given by

$$\mathcal{P}^s(U) = \{f : U \to T \mid \forall x \in U, \exists U_x \overset{\circ}{\subset} U, x \in U_x, f|_{U_x} \in \mathcal{P}(U_x)\}$$

Examples:

- $(T^p)^s = T_X$ sheaf of locally constant functions.
- $(\mathcal{L}^p(\mathbb{R}^n))^s$ is a locally p-integrable functions.

Now let k be a field.
Note: $\mathrm{Maps}(X, k)$ is a commutative k-algebra (a ring that has a compatible k-vector space structure.)

Def *A concrete k-space* is a pair $(X, \mathcal{R})$ consisting of a topological space X and a sheaf $\mathcal{R}$ of k-valued functions on X such that $\forall U \overset{\circ}{\subseteq} X, \mathcal{R}(U)$ is a k-subalgebra of $\mathrm{Maps}(U, k)$.

Examples.

- $(\mathbb{R}^n, \mathcal{C}^\infty_{\mathbb{R}})$, $(\mathbb{R}^n, \mathcal{C}_{\mathbb{R}^n,\mathbb{R}})$ is a concrete $\mathbb{R}$-space
- $(\mathbb{C}^n, \mathcal{O}^n_{\mathcal{C}})$ is a concrete $\mathbb{C}$-space

Def Let $(X, \mathcal{R})$ and $(Y, \mathcal{S})$ be concrete k-spaces. A morphism of concrete k-spaces $(X, \mathcal{R}) \to (Y, \mathcal{S})$ is a continuous map $F : X \to Y$ such that

$$\forall U \overset{\circ}{\subseteq} Y \forall f \in \mathcal{S}(U) : F^* f := f \circ (F|_{F^{-1}U}) \in \mathcal{R}(F^{-1}U)$$

($F^* f$ is called the pullback of f along F.)

Example. $id_X : (X, \mathcal{R}) \to (X, \mathcal{R})$

Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a continuous map, say $F = (F_1, F_2, \dots, F_m)$ for $F_i : \mathbb{R}^n \to \mathbb{R}$

Claim: F induces a morphism $(\mathbb{R}^n, \mathcal{C}^\infty_{\mathbb{R}^n}) \to (\mathbb{R}^m, \mathbb{C}^\infty_{\mathcal{R}^m})$ iff $F_1, F_2, \dots, F_m$ are C^∞.
Proof. ($\Longleftarrow$) Trivial.
($\Longrightarrow$) $\forall U \overset{\circ}{\subseteq} \mathbb{R}^n \forall f \in \mathcal{C}^\infty_{\mathbb{R}^m}(U) : F^* f := f \circ (F|_{F^{-1}U}) \in \mathcal{C}^\infty_{\mathbb{R}^m}(F^{-1}U)$
Consider the function $\mathrm{pr}_j : \mathbb{R}^m \to \mathbb{R}, (x_1, x_2, \dots, x_m) \mapsto x_j, j = 1, \dots, m$
Then, $\mathrm{pr}_j \circ F = F_j$.
Since, pr_j is C^∞, the same holds for F_j.

[2] In this case, $\mathcal{L}^p(X)$ is said to be a set of integrable functions, but in analysis, $L^p(X)$ is used more frequently that is defined by $L^p = \mathcal{L}^p/\mathcal{N}$ where $\mathcal{N} = \{f \in \mathcal{L}^p : f = 0 \text{ a.e.}\}$

Remark: similarly, a continuous map $F = (F_1, \dots, F_m) : \mathbb{C}^n \to \mathbb{C}^m$ induced a morphism of $\mathbb{C}$-spaces iff $F_1, \dots, F_m$ are holomorphic.

Exercise Show that if $F : (X_1, \mathcal{R}_1) \to (X_2, \mathcal{R}_2)$ and $G : (X_2, \mathcal{R}_2) \to (X_3, \mathcal{R}_3)$ are morphisms of concrete k-spaces, them so $G \circ F$.

Prop/Def An isomorphism of concrete k-spaces $(X, \mathcal{R}) \overset{\cong}{\to} (Y, \mathcal{S})$ is a homeomorphism $F : X \to Y$ which satisfies one (and hence both) of the following equivalent properties.

1. Both F and F^{-1} are morphisms of concrete k-spaces.
2. $\forall U \mathring{\subseteq} Y, \forall f \in \mathrm{Maps}(U, k) : f \in \mathcal{S}(U) \iff F^* f \in \mathcal{R}(F^{-1}U)$.

Notation / Convention
$C^{\alpha}_{\mathbb{R}^n}$ is a set of n-th differentiable functions (where $\alpha \in \mathbb{N}$)
$C^{\infty}_{\mathbb{R}^n}$ smooth
$C^{\omega}_{\mathbb{R}^n}$ analytic
$\hat{\mathbb{N}}_0 = \mathbb{N}_0 \cup \{\omega, \infty\}$ with $\alpha < \infty < \omega$ for all $\alpha \in \mathbb{N}$. $\mathcal{O}_{\mathbb{C}^n}$ holomorphic

Def[3] Let $\alpha \in \hat{\mathbb{N}}_0$. *A (real) C^{α}-premanifold* is a concrete $\mathbb{R}$-space $(X, \mathcal{C}^{\alpha}_X)$ which admits an open covering $X = \bigcup_{i \in I} U_i$ such that for each $i \in I$, there is $B_i \mathring{\subseteq} \mathbb{R}^n$ such that

$$(U_i, \mathcal{C}^{\alpha}_X|_{U_i}) \cong (B_i, \mathcal{C}^{\alpha}_X|_{B_i})$$

Terminology

- Atlas = collection of coordinate charts
- $\alpha = 0$, C^0-manifold is a topological manifold
- $\alpha = \infty$, C^{∞}-manifold is a smooth manifold
- $\alpha = \omega$, C^{ω}-manifold is an analytic manifold

Def Complex pre-manifold is the same as above, but with $(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n})$ instead of $(\mathbb{R}^n, \mathcal{C}^{\infty}_{\mathbb{R}^n})$

Example of a premanifold which is not a manifold.

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Def A real C^{α}/complex manifold is a real C^{α}/complex premanifold such that the topology of its underlying space is given by a metric ($\iff$ the underlying topological space is Hausdorff and paracompact.)

Terminology

- C^{α}-diffeomorphism is an isomorphism of C^{α}-manifolds.
- Biholomorphism is an isomorphism of complex manifolds.
- Riemann surface is a one-dimensional complex manifold.

An n-dimensional complex manifold.
Let $(X, \mathcal{O}_X)$ be a complex manifold of dimension n.

[3]Self-note: Original definition: a transition map; smooth compatible, The homeomorphism is a coordinate chart.

How to define a sheaf C^∞_X on X such that (X, C^∞_X) become a smooth manifold of dimension $2n$.

Idea: X has a complex atlas $(B_j, \mathcal{O}_{\mathbb{C}^n}|_{B_j})_{j\in I}$ with isomorphsims $g_j : B_j \xrightarrow{\cong} U_j \mathring{\subseteq} X$
Define $\mathcal{C}^\infty_X$ by

$$f \in \mathcal{C}^\infty_X \iff \forall j : f \circ g_j \text{ is } C^\infty$$

(on open subset of B_j in $\mathbb{R}^{2n}$)

Check: $(X, \mathcal{C}^\infty_X)$ is a smooth manifold of dimension $2n$.