Winning The "Graph Wars"

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December 13, 2022

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1 Introduction

1.1 Rationale

Not longer than two days ago I was challenged to a mathematical standoff by a good friend of mine, the weapons would be polynomial functions and their graphs. Winning this competition was a matter of honor, so I used all of my skill and wits to prepare. The fight would be held in "Graphwar", a small competitive 1v1 game published on the 23rd of February 2022.

The goal of the game is to design a function that can be plotted from one point to another without intersecting any circles in shortest time. You are given 60 seconds to input as many functions as you want, then the turn is passed onto your opponent. This is wrapped into a war-like setting where your function is a projectile, directed by your soldiers against your enemy's.

It is also usually played without assistance of any external programs like plotters or calculators, that's for good reason. Most importantly, game is considered solved: for any initial state of the field there can be deduced a function to guarantee a "hit".

1.2 Aim

My ultimate goal set out to be simply designing an algorithm, which when given a list of all circle obstacles, their radii, and the edge points would express the path to go from to another without colliding with any of the obstacles.

The plane that the function is plotted on spans 40 units vertically and 100 units horizontally. The units and the obstacles can only be placed at integer coordinates.

You can sketch any function using the list of allowed operations:

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2 Investigation

The task consists of the small independents subtasks: finding the points and fitting the function. The found points are passed into the fitting algorithm, so it in general it is a 2-step pipeline. Some alternative approaches use more stages, as mentioned in the "Limitations & Reflection" section.

2.1 Finding The Points

This problem can be boiled down to pathfinding. However, usual pathfinding techniques operate on discrete spaces like graphs and grids which our plane is not, we need to convert it to one. Cells of the grid can be in two states: occupied or empty. The occupied cells are the ones covered by an obstacle. Since obstacle circles only occur at integer coordinates and the smallest radius of a circle is 1, we can safely assume that the smallest size of a grid cell we might need is 1 respectively. After that we can apply the **Theta***. It is preferable to Dijkstra's, A* and others since it is designed to fit any-angle pathfinding. It can find near-optimal paths with runtimes comparable to A*. Even though our points are aligned on a grid, the "projectile" is allowed to move in all directions, not just the cardinal ones. Only additional piece of information it requires is a line of sight predicate P, which in turn requires a function to count the amount of objects that block the sight, let's name that function L and try to define it. Let $I(p_1, p_2)$ be some function, equal to the amount of intersections of a line between points p_1 and p_2 with all the circles of a set C containing $\langle x, y, r \rangle$. So $I_n(p_1, p_2)$ is the amount of intersections of a line from p_1 to p_2 with some circle C_n. Using that we can define $I(p_1, p_2) = \sum_{n=1}^{|C|} I_n(p_1, p_2)$.

For simplicity let's assume the circle is at $\langle 0, 0\rangle$ with some radius r. But first let's find the point $p_0 = (x_0, y_0)$ on the line closest to the circle. We can use an alternative line equation $ax + by + c = 0$. It also defines a vector $\vec{v'} = (a, b)$ perpendicular to it. To ensure that $\vec{v'}$ is perpendicular to the line we can proove it. Imagine two points $u_1 = (0, 0)$ and $u_2 = (1, -\frac{a}{b})$, they both line on the previously described line and form a vector $\vec{u} = \vec{u_1 u_2} = (1, -\frac{a}{b})$. $\pi/2 = \arccos(\vec{u} \cdot \vec{v'})$, therefore the vectors are perpendicular. Since $\vec{u}$ is parallel to the line any vectors perpendicular to it will be also perpendicular to the line.

Coordinates of p_0 should be proportional to the vector $\vec{v'}$. Now we may just normalize the vector to the length.

We also need the distance from the origin to the line. It can be expressed as a height of a right-angles triangle with sides of x-intercept $(-\frac{C}{A}, 0)$ and y-intercept $(0, -\frac{C}{B})$.

First we normalize by dividing the vector by it's length.

$$\vec{v}_{normalized} = \left(\frac{a}{\sqrt{a^2 + b^2}}, \frac{b}{\sqrt{a^2 + b^2}}\right) \tag{1}$$

Then we scale the vector by the distance between the line and origin.

$$\vec{v} = \vec{v}_{normalized} \times \frac{c}{\sqrt{a^2 + b^2}} \tag{2}$$

By simplifying the denominators we get

$$\vec{v} = \left(a\frac{c}{a^2 + b^2}, b\frac{c}{a^2 + b^2}\right) \tag{3}$$

This vector is inverted, so by multiplying everything by -1 we get the actual coordinates of the point. Since the vector and the point lie in the same coordinate system and are relative to $(0, 0)$, we can claim their equivalence, therefore

$$p_0 = -\vec{v} = (-a\frac{c}{a^2 + b^2}, -b\frac{c}{a^2 + b^2}) = (x_0, y_0) \tag{4}$$

So p_0 is the closest point to the circle. Now, based on this we can calculate the amount of intersections: if p_0 is inside the circle there are 2 intersections, if it lies on the radius there is 1 intersection and there are none otherwise.

$$\begin{cases} I_n(p_1, p_2) = 0, |\vec{v}| > C_{n_r} \\ I_n(p_1, p_2) = 1, |\vec{v}| = C_{n_r} \\ I_n(p_1, p_2) = 2, |\vec{v}| < C_{n_r} \end{cases} \tag{5}$$

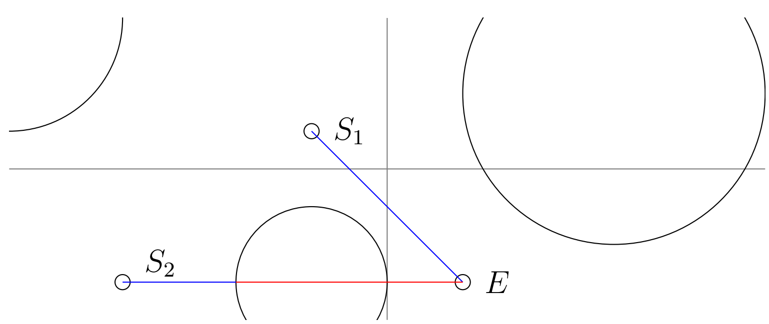
When evaluating the function I_n for simplicity we just subtract the coordinates (C_{n_x}, C_{n_y}) from all the other coordinates. This shifts the entire plane so that the circle is at the origin, the precise usecase the solution above covers. Using that we can define our line of sight function.

$$L(A, B) = \sum_{n=1}^{|C|} I_n(p_1, p_2)$$

This function is 0 when no intersections are found and > 0 when there is at least one intersection, we can also convert it into a predicate $P(p_1, p_2)$ like this.

$$\begin{cases} P(p_1, p_2) = 1, L(p_1, p_2) = 0 \\ P(p_1, p_2) = 0, L(p_1, p_2) > 0 \end{cases} \tag{6}$$

Assuming the algorithm is not ill-formed and given correct input information it should formulate a near-optimal path, some list A of cardinality n, where each point is represted as a pair of x and y coordinates $\langle x, y \rangle$. It is required that all points of A satisfy $\forall a_i, a_{i+1} \in A : a_{i_x} < a_{i+1_x}$. If this is not true at some point the path turns back on itself. This game situation is unresolvable (more of that is mentioned in the "Limitations & Reflection" section). The example below should make everything described above more graspable.



In the figure above the blue lines represent the line of sight. The line is blue all the way if the value of P for the two points the line connects is 1, it becomes red when the first intersection occurs. There exists a clear line of sight between points S_1 and E, but not S_2 and E.

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2.2 Fitting The Function

The fitting algorithm will operate on the list A, output by the pathfinding algorithm and assume the same ordering.

A_n is guaranteed to be the rightmost point, which will make the calculations a lot more intuitive. We may also ignore all the x values in ranges $(-\infty; a_{1_x})$ and $(a_{n_x}; \infty)$, since the function will not be evaluated there at any point.

There is an infinite number of ways to make a function that goes through a set of points, the simplest one is construcing the Lagrange Interpolation Polynomial which does fit this usecase perfectly. We may follow a specific example and try to generalise it later.

For example $A = \{\langle 0, 1\rangle, \langle 6, 9\rangle, \langle 17, 4\rangle, \langle 19, 22\rangle\}$. All the coordinates are purposfully picked to be in the first quadrant, since in this way they are much easier to visualize.

First we may try to make a function $\phi_i(x)$ such that it equals to 0 at every point, but x_i. For simplicity we may make it be equal to 1, this will allow for better composability in the future. For example the process for the second point might look something like this.

First we use a helper function $\hat{\phi}_2(x)$ make it be zero at all the points, but x_i.

$$\hat{\phi}_2(x) = (x-0)(x-17)(x-19) \tag{7}$$

Now the value at x_2 is non-zero, we can just devide it by $\hat{\phi}_2(x_2)$ so it is 1 at x_2.

$$\hat{\phi}_2(x_2) = \hat{\phi}_2(6) = (6-0)(6-17)(6-19) \tag{8}$$

$$\phi_2(x) = \frac{(x-0)(x-17)(x-19)}{\hat{\phi}_2(x_2)} \tag{9}$$

$$\phi_2(x) = \frac{(x-0)(x-17)(x-19)}{(6-0)(6-17)(6-19)} \tag{10}$$

x	x_1	x_2	x_3	x_4
$\phi_2(x)$	0	1	0	0

Formally, in this context ϕ_i is called a "Lagrange polynomial basis". To make ϕ_i completely representative of the point at x_i we need to scale it's vertical component by y_i. Let's define a new helper function $\psi_i(x)$.

$$\psi_2(x) = y_2\phi_2(x) \tag{11}$$

$$\psi_2(x) = \frac{(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)} * y_2 \tag{12}$$

Based on this we can express $\psi_i(x)$'s values for any x_i.

	$\psi_1(x)$	$\psi_2(x)$	$\psi_3(x)$	$\psi_4(x)$
x_1	y_1	0	0	0
x_2	0	y_2	0	0
x_3	0	0	y_3	0
x_4	0	0	0	y_4

Looking at the table of all the values of ψ_i we can deducde that their linear combination will satisfy our condition of $\psi_i(x_i) = y_i$ and $\psi_i(x_j) = 0, j \neq i$, yielding an $|A|$th order polynomial. So for the example set A the fit function is $f(x) = \psi_1(x) + \psi_2(x) + \psi_3(x) + \psi_4(x)$.

If we generalise this function to any set A it would look like the following.

$$f(x) = \sum_{i=1}^{|A|} \psi_i(x) \tag{13}$$

We can also expand our helper functions ϕ and ψ, so we get it to our final form.

$$\psi_i(x) = y_i\phi_i(x)$$

$$\phi_i = \prod_{j=1, j\neq i}^{|A|} \frac{x-x_j}{x_i-x_j} \tag{14}$$

The final form is the following, a proper Lagrange Interpolation Polynomial, a member of P_n. The

$$f(x) = \sum_{i=1}^{|A|} \left(y_i * \prod_{j=1, j \neq i}^{|A|} \frac{x - x_j}{x_i - x_j} \right) \tag{15}$$

This is the final form of the function f, that given a set of points fits a function through them. When applying it in practise one should not forget to account for the previously applied transformations, such as the offset of the soldier's origin. It's (naïve) computational complexity is $O(n^2)$, which is subpar, possible improvements are mentioned in the "Limitations & Reflection" section.

3 Limitations & Reflection

I used a geometric approach to finding intersections of lines with circles. The algebraic solution has a higher computational error due to imprecisions of floating-point arithmetic. This doesn't impact the current usecase, however can be important when dealing with more fine point placement.

The origin of each soldiers' shot is unique. The x coordinate is 0, but the y is same as that of the soldier. The investigation conciously ignores this restriction since it would make the solution overwhelming and difficult to read. Accounting for that requires subtracting the soldier's position from the each of the path points.

The described method of fitting covers a lot of scenarios, but it doesn't yield correct results when the next intended path point a_{i+1} on the path which goes after some a_i. In this case the pathfinding algorithm requires a path to turn back, which is impossible to do. This issue is most visible when reindexing: the order of points will be changed. Beyond that, this arises only in situations when the $f(x)$ **can't be a well-defined function**, since that requires it to have to two different values at same x. Luckily, those scenarios never occured in a real game. Perhaps the game accounts for that and intentionally avoids it.

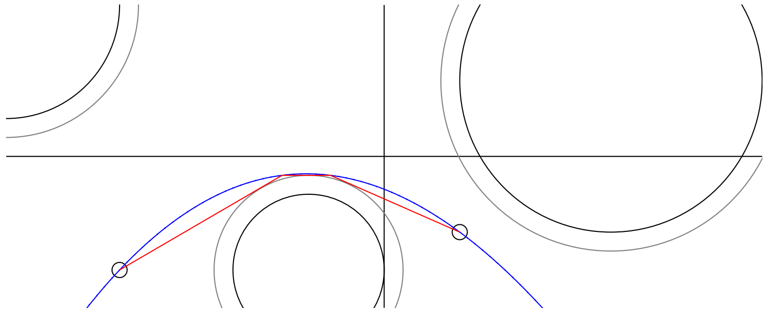
My method produces good results on the given dataset of densely packed points, however, the more spread apart the points are the greater the algorithm overshoots in spaces between two points. That can be a problem since it may actually accidentally go through one of the spheres. This effect can be mitigated by sampling more intermediate points, which in turn increases computational complexity. Since the game field is limited to span 100 units horizontally and 40 units vertically this is not a big problem. One of the alternative solutions would be using splines to form Bézier Curves, Hermite Curves, Catmul-Rom or even B-Splines, but those are **barely touched on in the book** if at all. Using splines would be optimal, beyond that a spline of degree ∞ for a set of points *is* the Lagrange Interpolation. I would really love to see them included in the curriculum.

All the solutions later developed by other people interested in the game employ similar approaches with some optimisations on the algorithm side. Some also make use of CV (Computer Vision) to automatically dispatch the data to the calculator which later forms a polynomial. Several applications use an additional step of the pipeline to optimise the resulting polynomial by approximating it further using other techniques.

In some edge-case scenarios errors in floating point arithmetic might cause an unintentional collision. Since built paths might be touching the radius of the circle (due to the nature of the algorithm) it might be useful to artifically increase the radius through some function $h(r) = r + t$, to ensure that the path is built at least t units away from the cirles.

4 Conclusion

In conclusion I can for sure regard this research as successful. All the described techniques were applied and achieved great results. I won the standoff and got to explain my methodology to the defeated opponent, which was delightful. Beyond that, several possible improvements were already mentioned above. I would be interested in redoing that research in some time to see how my understanding of the topic has changed with the knowledge I will develop.



A representation of a game field with function f generated using my algorithm. The radii provded to the algorithm were increased by 2.5, they are drawn in gray. The path generated by Theta* is drawn in red. The path of the generated function is in blue. In this specific case the simplified polynomial is like this (the entire process of expansion is ommited for the sake of brevity, it would take an unreasonable amount of time to compute manually and an unreasonable amount of paper to print).

$$f(x) = 4.833... \times 10^{-5}x^3 - 1.181... \times 10^{-2}x^2 - (0.389...)x - 15 - 4.335...$$

The grayed out term is the offset of the soldier from origin and is not part of the Lagrange Interpolation, however it plays an important role in the game and is not to be forgotten about.

TODO

5 Bibliography

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