Design and Analysis of Data Structures and Algorithms :: Amortized Analysis

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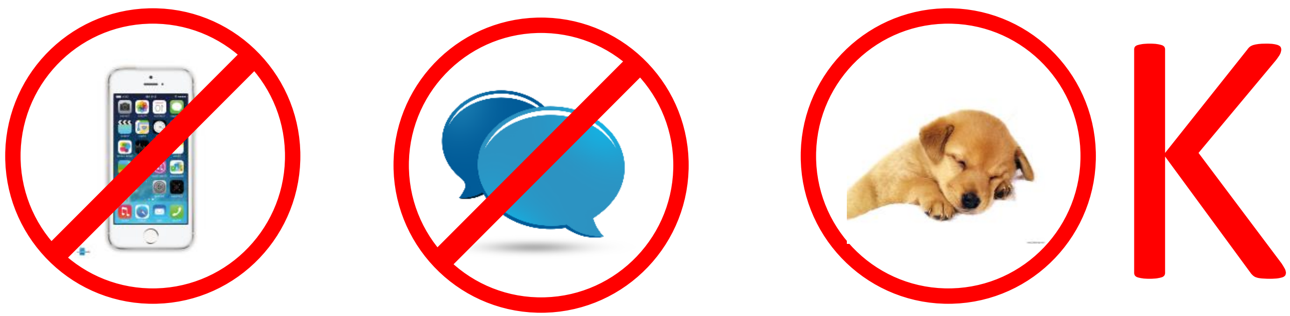
กฎการฟังบรรยาย



Algorithm and Complexity

PART 5:

ALGORITHM ANALYSIS: AMORTIZED ANALYSIS

Outline

- Definition
- Aggregate Method
- Accounting Method
- Potential Method
- Examples : Binary Counter and Dynamic Table

Cost of A Sequence of Operations

- Let operation A requires $\Theta(n)$ cost in worst-case
- Calling A m times costs $\Theta(\ m\ n\ )$?
- Not necessary : it may cost $O(\ m\ n\ )$
- Sometimes worst cases do not happen consecutively in a sequence of calls
- Actual worst-case cost may be $o(\ m\ n\ )$

Amortized Analysis

- The worst-case cost for any sequence of m operations
- Average performance of each operation in worst case (no probability is involved).

$$\text{amortized time} = \frac{\text{worst time for a sequence of } m \text{ ops}}{m}$$

Example

- Given a list of n elements
- Sort this list m times using InsertionSort
- First time : worst-case $\Theta(n^2)$
- 2^{nd} - m^{th} times : worst-case $\Theta(n)$
- Total worst-case time : $\Theta(n^2 + mn)$
- Amortized time : $\Theta(n^2 / m + n)$

Example

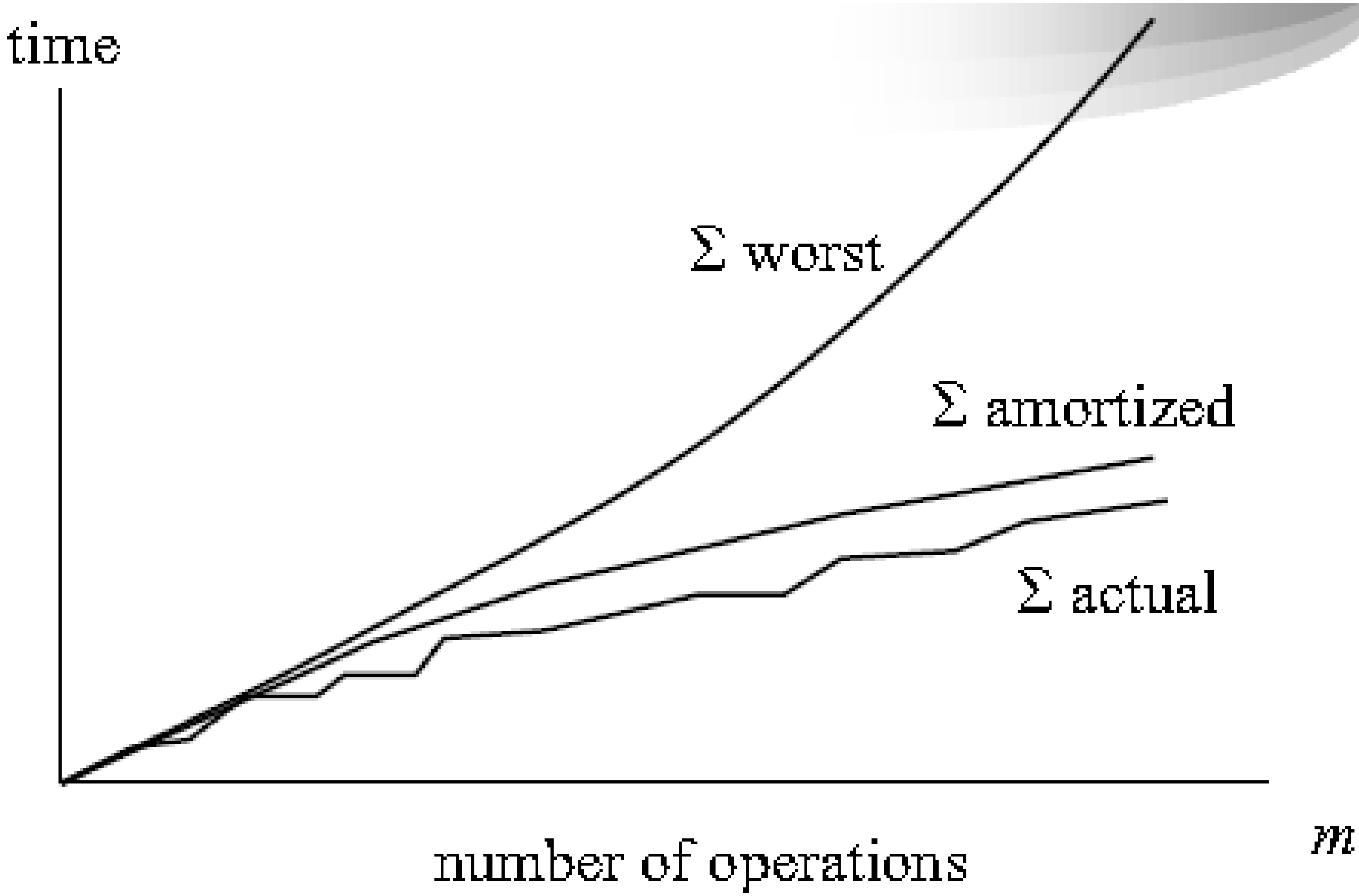
- Given a list of n elements
- Sort this list m times using <u>SelectionSort</u>
- First time : worst-case $\Theta(n^2)$
- 2^{nd} - m^{th} times : worst-case $\Theta(n^2)$
- Total worst-case time : $\Theta(mn^2)$
- Amortized time : $\Theta(n^2)$

Disjoint Sets

- Union : Union by rank : $\mathrm{O}(1)$
- Find : Path compression : $\mathrm{O}(\log n)$
- A sequence of m Union and Find operations perform on n elements : $\mathrm{O}(m \log^* n)$
- Amortized cost : $\mathrm{O}(\log^* n)$

$$\log* 2^{2^{2^{2}}} = 4$$

Amortized Analysis



Amortized Analysis Techniques

- Aggregate Method
- Accounting Method
- Potential Method

Aggregate Method

- Compute the worst-case time $T(m)$ in total of a sequence of m operations
- Amortized cost = $T(m) / m$
- Amortized cost of each operation is the same even when there are several types of operations in the sequence.

Accounting Method

- Assign (***guess***) amortized cost for each operation
- If an operation's actual cost is less than its amortized cost, the difference is assigned to specific objects in the data structure as <u>credit</u>
- Credit can be used later for operations whose actual cost exceeds their amortized cost
- Correct if the credit is nonnegative at all times

Potential Method

- Assign (***guess***) a potential function for the entire data structure
- Potential energy increases if amortized cost exceeds actual cost
- Potential energy decreases if amortized cost is less than actual cost
- Amortized cost = Actual cost + Δ(potential)
- Correct if potential is never less than its initial

Binary Counter

- a k-bit binary counter A[0..k-1]
- A[k-1] : MSB, A[0] : LSB
- count upward from 0 m times

```
0 0 0 0
0 0 0 1
0 0 1 0
0 0 1 1
0 1 0 0
```

Increment Binary Counter

```
Increment( A[0..k-1] )
{
      i = 0
      while ( i < k and A[i] = 1 )
            A[i] = 0
            i = i+1
      if ( i < k )
            A[i] = 1
}
```

Actual cost, Worst cost, Amortized cost = ?

Increment : Actual Cost

- Actual cost is linear in the number of bits flipped

0	0	0	0	Actual cost
0	0	0	1	1
0	0	1	0	2
0	0	1	1	1
0	1	0	0	3
0	1	0	1	1
0	1	1	0	2

Increment : Worst Cost

- A[0..k-1] : k bits
- Worst when all bits are flipped $\Theta(\ k\ )$

```
1 1 1 1
0 0 0 0
```

- m Increments on an initially zero k-bit counter takes time $O(\ mk\ )$

Increment : Aggregate Method

- Tighten the worst-case cost of m Increments
- $A[0]$ flips every time : m times
- $A[1]$ flips every other time : $m/2$ times
- $A[2]$ flips every fourth time : $m/4$ times $(m/2^2)$…
- $A[i\ ]$ flips every 2^ith time : $m/2^i$ times
- The total number of flips in m Increments is

$$\boxed{\sum_{i=0}^{\lfloor \lg m \rfloor} \frac{m}{2^i} < m \sum_{i=0}^{\infty} \frac{1}{2^i} = 2m = O(m)}$$

- Amortize cost = $O(m)/m = O(1)$

Increment : Accounting Method

- Assign 2-baht amortized cost for an Increment
- 1 baht for flipping a 0-bit to 1-bit
- 1 baht kept at the 1-bit for later flipping back to 0

Increment : Accounting Method

Σ (amortized cost)

0

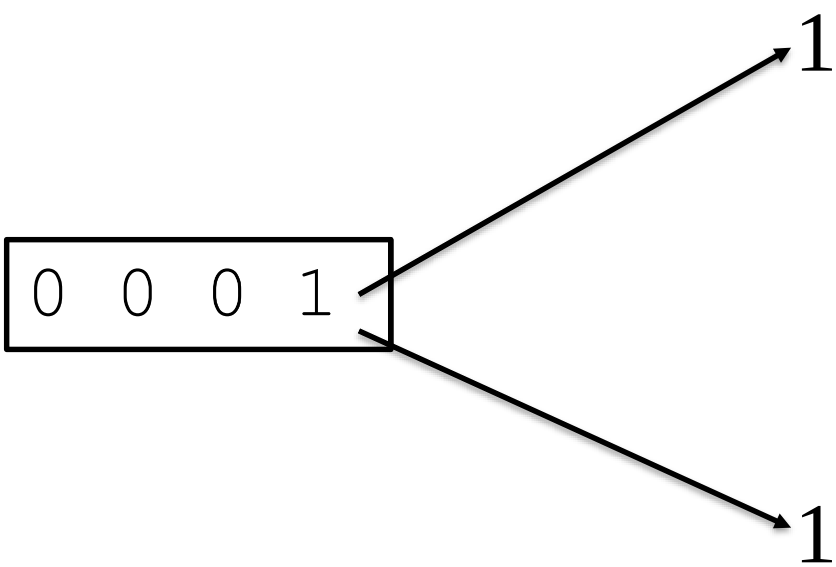
$\boxed{0 \quad 0 \quad 0 \quad 0}$

0

Σ (actual cost)

Increment : Accounting Method

Σ (amortized cost)



Σ (actual cost)

1st Increment

Increment : Accounting Method

Σ (amortized cost)

2

$\boxed{0 \quad 0 \quad 0 \quad 1^{1}}$

1

Σ (actual cost)

1st Increment

Increment : Accounting Method

Σ (amortized cost)

2+0

0	0	0	0

1+1

Σ (actual cost)

2^{nd} Increment

Increment : Accounting Method

Σ (amortized cost)

2+1

| 0 0 1 0 |

1+2

Σ (actual cost)

2nd Increment

Increment : Accounting Method

Σ (amortized cost)

2+2

$\boxed{0 \quad 0 \quad 1^{1} \quad 0}$

1+2

Σ (actual cost)

2^{nd} Increment

Increment : Accounting Method

Σ (amortized cost)

2+2+1

0 0 1^1 1

1+2+1

Σ (actual cost)

3rd Increment

Increment : Accounting Method

Σ (amortized cost)

2+2+2

$\boxed{0 \quad 0 \quad 1^{1} \; 1^{1}}$

1+2+1

Σ (actual cost)

3rd Increment

Increment : Accounting Method

Σ (amortized cost)

2+2+2+0

| 0 | 0 | 1[1] | 0 |

1+2+1+1

Σ (actual cost)

4th Increment

Increment : Accounting Method

Σ (amortized cost)

2+2+2+0

0 0 0 0

1+2+1+2

Σ (actual cost)

4th Increment

Increment : Accounting Method

Σ (amortized cost)

2+2+2+1

| 0 1 0 0 |

1+2+1+3

Σ (actual cost)

4^{th} Increment

Increment : Accounting Method

Σ (amortized cost)

2+2+2+2

$\boxed{0 \quad 1^{1} \quad 0 \quad 0}$

1+2+1+3

Σ (actual cost)

4^{th} Increment

Increment : Accounting Method

Σ (amortized cost)

2+2+2+2

0	1^1	0	0

1+2+1+3

Σ (actual cost)

4^{th} Increment

Increment : Accounting Method

- Each `Increment` costs at most two bahts since at most one bit is set to “1”
- No negative credit happens
- Σ(amortized cost) $\geq$ Σ(actual cost) at all times
- Increment is $O(1)$ amortized time

Increment : Potential Method

- Let Φ_i be potential function of the data structure after the i^{th} operation
- Let c_i be the actual cost of the i^{th} operation
- Let a_i be the amortized cost of the i^{th} operation

Potential Function

- Potential energy increases if $a_i > c_i$
- Potential energy decreases if $a_i < c_i$

$$\Phi_i = \Phi_{i-1} + (a_i - c_i)$$

$$a_i = c_i + \Phi_i - \Phi_{i-1}$$

$$\begin{aligned} \Sigma a_i &= \Sigma c_i + \Sigma\Phi_i - \Sigma\Phi_{i-1} \\ &= \Sigma c_i + \Phi_m - \Phi_0 \end{aligned}$$

$$\Sigma a_i \geq \Sigma c_i \;\text{ if }\; \Sigma\Phi_m \geq \Sigma\Phi_0$$

Increment : Potential Method

- Let b_i be the number of 1's in the counter after the i^{th} `Increment`
- Suppose that the i^{th} operation resets r_i bits

$$c_i \leq r_i + 1$$

$$b_i \leq b_{i-1} - r_i + 1$$

0	1	0	1	1
0	1	1	0	0

$$b_{11} = 3$$

$$b_{12} = b_{11} - r_{12} + 1$$

$$= 3 - 2 + 1 = 2$$

$$c_{12} = 3$$

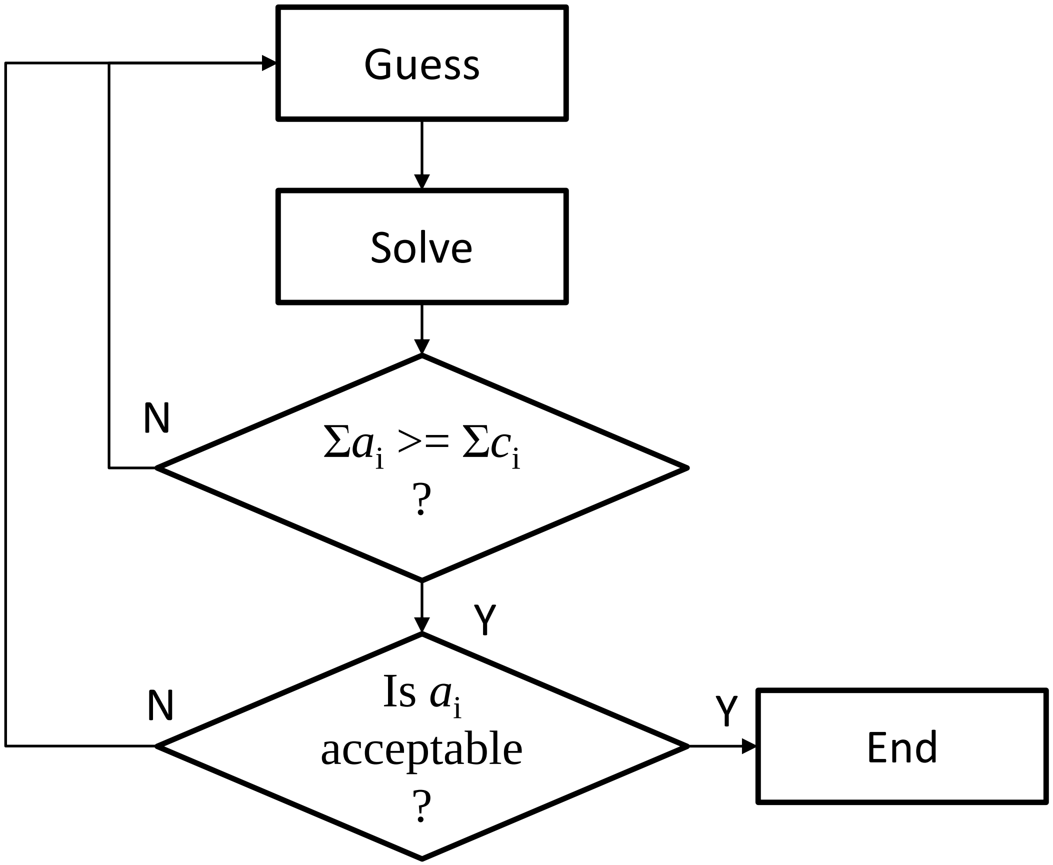
Increment : Potential Method

Let $\Phi_i = b_i, \Phi_0 = 0, \Phi_i \geq \Phi_0$

$$\begin{aligned} \Phi_i - \Phi_{i-1} &= b_i - b_{i-1} \\ &\leq (b_{i-1} - r_i + 1) - b_{i-1} \\ &= 1 - r_i \end{aligned}$$

$$\begin{aligned} a_i &= c_i + \Phi_i - \Phi_{i-1} \\ &\leq (r_i + 1) + 1 - r_i \\ &= 2 = O(1) \end{aligned}$$

Analysis Loop



Dynamic Table

- Table that can expand or contract as needed

```
Table_Insert( T, x )
        if ( T.size = 0 )
                T = CreateTable( 1 )
        if ( T.num = T.size )
                S = CreateTable( 2*T.size )
                insert all items of T into S          <- O(n)
                freeTable( T )
                T = S
        insert x into Table[T]                        <- O(n)
```

Dynamic Table : Sequence of Inserts

- A sequence of n `Table_Insert` on initially empty table
- Double the table size when inserting into the full table
- Worst case of `Table_Insert` is $O(n)$
- Worst case of the sequence is $O(n^2)$
- Not tight : table expansion occurs infrequently

Worst Case of n Insertions

- Aggregate Method
- Observe that $c_i = i$ if i-1 is an exact power of 2, otherwise $c_i = 1$
- Amortized cost is $3 = \mathrm{O}(1)$

$$\sum_{i=1}^{n} c_i \le n + \sum_{j=0}^{\lfloor \lg n \rfloor} 2^j$$

$$\le n + 2^{\lfloor \lg n \rfloor + 1} - 1$$

$$< n + 2n$$

$$= 3n$$

Dynamic Table : Accounting Method

- Why is amortized cost of a `Table_Insert` 3 ?
- 1 is to move the item into the table (actual cost)
- 1 is kept for future movement during expansion
- 1 is kept for another older item in the table for future movement during expansion

Dynamic Table : Accounting Method

Σ (amortized cost)

0

0

Σ (actual cost)

Dynamic Table : Accounting Method

Σ (amortized cost)

new item

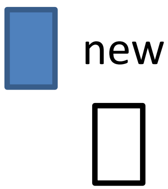
0

0

Σ (actual cost)

Dynamic Table : Accounting Method

Σ (amortized cost)



new item

0

0

Σ (actual cost)

Dynamic Table : Accounting Method

Σ (amortized cost)

1

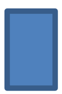
1

Σ (actual cost)

Dynamic Table : Accounting Method

Σ (amortized cost)

2

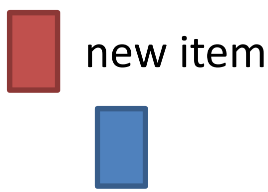


1

Σ (actual cost)

Dynamic Table : Accounting Method

Σ (amortized cost)

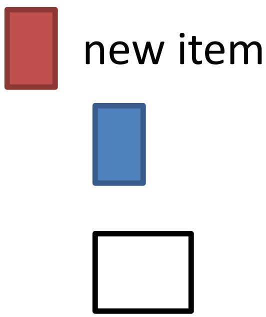


3

1

Σ (actual cost)

Dynamic Table : Accounting Method



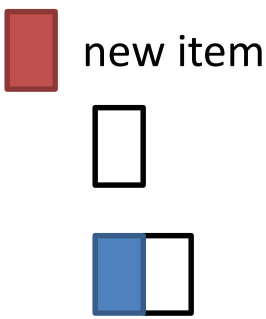
Σ (amortized cost)

3

1

Σ (actual cost)

Dynamic Table : Accounting Method



Σ (amortized cost)

3

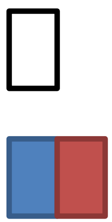
1

Σ (actual cost)

Dynamic Table : Accounting Method

Σ (amortized cost)

3+0



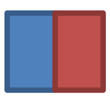
1+1

Σ (actual cost)

Dynamic Table : Accounting Method

Σ (amortized cost)

3+1



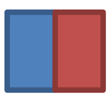
1+2

Σ (actual cost)

Dynamic Table : Accounting Method

Σ (amortized cost)

3+2



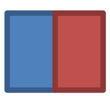
1+2

Σ (actual cost)

Dynamic Table : Accounting Method

Σ (amortized cost)

3+3

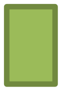


1+2

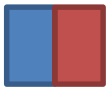
Σ (actual cost)

Dynamic Table : Accounting Method

Σ (amortized cost)

 new item

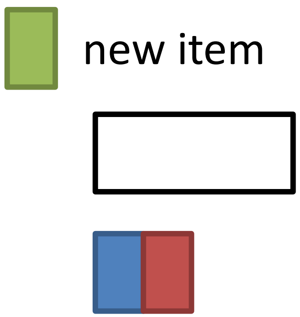
3+3



1+2

Σ (actual cost)

Dynamic Table : Accounting Method



Σ (amortized cost)

3+3

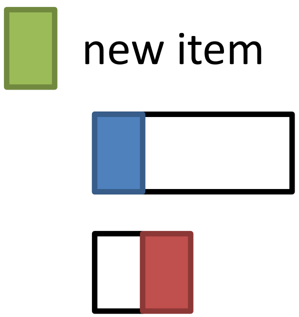
1+2

Σ (actual cost)

Dynamic Table : Accounting Method

Σ (amortized cost)

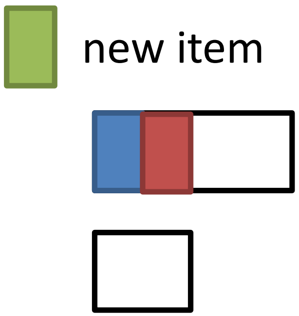
$3+3+0$



$1+2+1$

Σ (actual cost)

Dynamic Table : Accounting Method



Σ (amortized cost)

3+3+0

1+2+2

Σ (actual cost)

Dynamic Table : Accounting Method

Σ (amortized cost)

3+3+1

1+2+3

Σ (actual cost)

Dynamic Table : Accounting Method

$$\Sigma \text{ (amortized cost)}$$

$$3+3+2$$

$$1+2+3$$

$$\Sigma \text{ (actual cost)}$$

Dynamic Table : Accounting Method

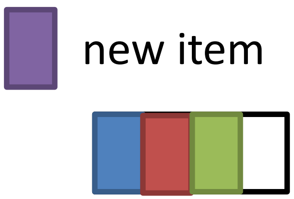
Σ (amortized cost)

3+3+3

1+2+3

Σ (actual cost)

Dynamic Table : Accounting Method



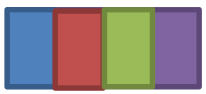
Σ (amortized cost)

3+3+3

1+2+3

Σ (actual cost)

Dynamic Table : Accounting Method



Σ (amortized cost)

3+3+3+1

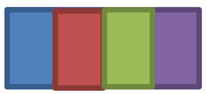
1+2+3+1

Σ (actual cost)

Dynamic Table : Accounting Method

Σ (amortized cost)

3+3+3+2



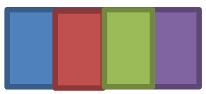
1+2+3+1

Σ (actual cost)

Dynamic Table : Accounting Method

Σ (amortized cost)

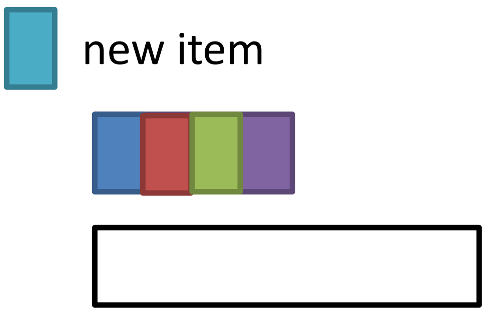
3+3+3+3



1+2+3+1

Σ (actual cost)

Dynamic Table : Accounting Method



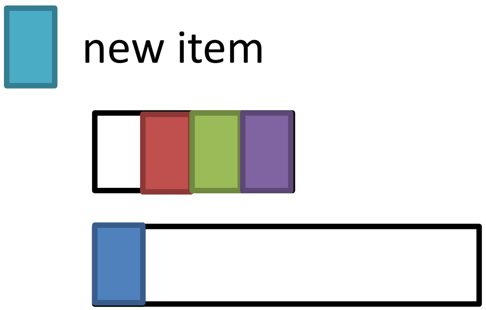
Σ (amortized cost)

$3+3+3+3$

$1+2+3+1$

Σ (actual cost)

Dynamic Table : Accounting Method



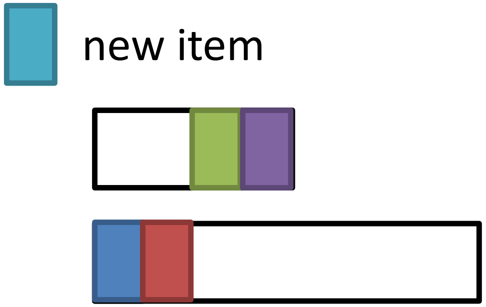
Σ (amortized cost)

3+3+3+3+0

1+2+3+1+1

Σ (actual cost)

Dynamic Table : Accounting Method



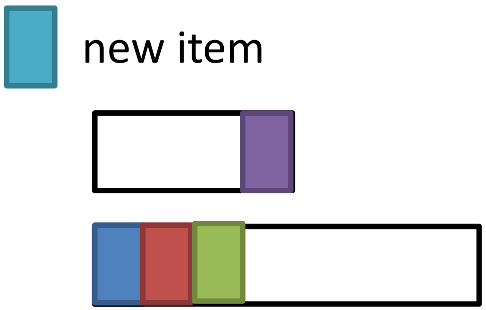
Σ (amortized cost)

$3+3+3+3+0$

$1+2+3+1+2$

Σ (actual cost)

Dynamic Table : Accounting Method



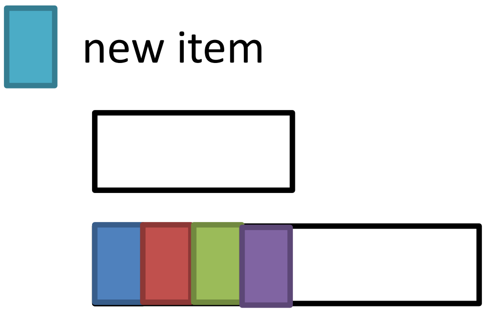
Σ (amortized cost)

3+3+3+3+0

1+2+3+1+3

Σ (actual cost)

Dynamic Table : Accounting Method



Σ (amortized cost)

3+3+3+3+0

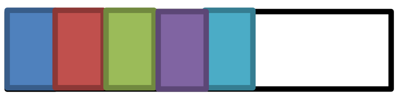
1+2+3+1+4

Σ (actual cost)

Dynamic Table : Accounting Method

Σ (amortized cost)

3+3+3+3+1



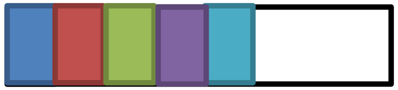
1+2+3+1+5

Σ (actual cost)

Dynamic Table : Accounting Method

Σ (amortized cost)

3+3+3+3+2



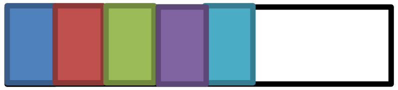
1+2+3+1+5

Σ (actual cost)

Dynamic Table : Accounting Method

Σ (amortized cost)

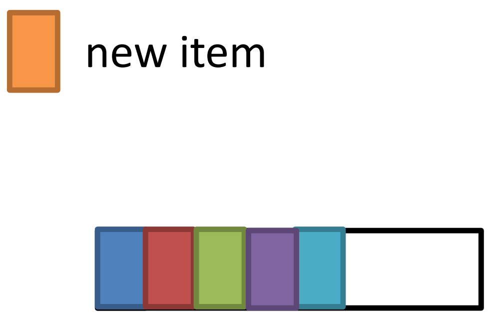
3+3+3+3+3



1+2+3+1+5

Σ (actual cost)

Dynamic Table : Accounting Method



Σ (amortized cost)

3+3+3+3+3

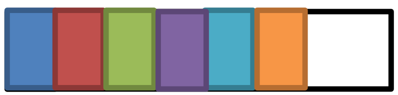
1+2+3+1+5

Σ (actual cost)

Dynamic Table : Accounting Method

Σ (amortized cost)

3+3+3+3+3+1



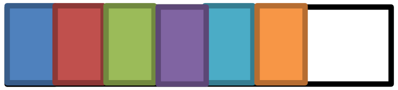
1+2+3+1+5+1

Σ (actual cost)

Dynamic Table : Accounting Method

Σ (amortized cost)

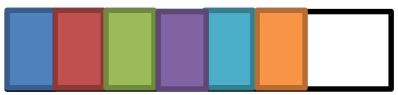
3+3+3+3+3+2



1+2+3+1+5+1

Σ (actual cost)

Dynamic Table : Accounting Method



Σ (amortized cost)

$3+3+3+3+3+3$

$>=$

$1+2+3+1+5+1$

Σ (actual cost)

$\boxed{\text{amortized cost} \leq 3 = O(1)}$

Dynamic Table : Potential Method

- Let n_i be the number of items in the table
- Let s_i be size of the table
- We want the potential to be zero initially and after every expansion
- We want the potential to build to the table size when the table is full
- Potential is used for moving items during expansion

Dynamic Table :Potential Function

$\Phi_i = 2n_i - s_i \qquad \Phi_0 = 0 \text{ when } i = 0 \textit{ or } n_i = s_i/2$

If the i^{th} insertion does not trigger an expansion

$$
\begin{aligned}
a_i &= c_i + \Phi_i - \Phi_{i-1} \\
&= 1 + (2n_i - s_i) - (2n_{i-1} - s_{i-1}) \\
&= 1 + (2n_i - s_i) - (2(n_i - 1) - s_i) \\
&= 3
\end{aligned}
$$

Dynamic Table :Potential Function

$\Phi_i = 2n_i - s_i \qquad \Phi_0 = 0 \text{ when } i = 0 \; or \; n_i = s_i/2$

If the i^{th} insertion does trigger an expansion ($n_{i-1} = s_{i-1}$)

$$
\begin{aligned}
a_i &= c_i + \Phi_i - \Phi_{i-1} \\
&= n_i + (2n_i - s_i) - (2n_{i-1} - s_{i-1}) \\
&= n_i + (2n_i - 2s_{i-1}) - (2s_{i-1} - s_i) \\
&= 3n_i - 3s_{i-1} \\
&= 3n_i - 3(n_i - 1) \\
&= 3
\end{aligned}
$$

Dynamic Table : Contraction

- Idea : halve the size of the table when deletion causes the table to become less than 1/4 full.

Homework

Conclusion

- Amortized cost is the worst cost of sequence of operations
- Better estimate than m * (worst cost per operation)
- Used in the analysis of advanced data structures
- Accounting : guess a_i → no negative credit
- Potential function : guess $\Phi_i \longrightarrow \Phi_i \geq \Phi_i \longrightarrow a_i$