Group members

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Topic

We will be formalising Niven's proof of the irrationality of π [Niven, 1947]. Should this task prove simpler than expected, we will be able to extend the scope of the project to alternative proofs of the irrationality of π [Roegel, 2020] or to smaller results that lead to the proof of the transcendence of π [Popescu, 2023].

Approach

We will follow the structure of Niven's proof, focusing on the following 'sub-problems':

1. Defining the polynomials $f(x)$ and $F(x)$ [Niven, 1947] and showing $n!f(x)$ has integral coefficients;
2. Proving facts about derivatives relating to $F(x)$ [Niven, 1947];
3. Defining the integral
$$\int_0^\pi f(x)\sin(x)dx$$
and showing this is equal to $F(\pi) + F(0)$;
4. Showing $F(\pi) + F(0)$ is an integer;
5. Proving the inequality
$0 < f(x)\sin(x) < \frac{\pi^n a^n}{n!}$
where a comes from the assumption that $\pi = \frac{a}{b}$; and
6. Concluding by contradiction that π is irrational.

Many of these sub-problems require the previous steps to be completed, so the project will be largely collaborative at each stage. However, step 5 and parts of step 3 can be done independently of the other results.

Existing Mathlib Results

After some brief preliminary research, we aim to use the following existing Mathlib results (modules and lemmas) at their respective stages of work as follows:

1. To define polynomials and show that they have integer coefficients, we expect to use results from **data.polynomial.basic**, **data.polynomial.integral_coeffs**, and **ring_theory.int.basic**;
2. To work with derivatives, including trigonometric functions, we expect to use results from **analysis.calculus.deriv**, **analysis.special_functions.trigonometric.deriv**, and **analysis.calculus.polynomial**;
3. We expect to make use of **measure_theory.integral.interval_integral** and **analysis.special_functions.trigonometric.integrals** to define the integral. We anticipate **fundamental_theorem_of_calculus** and **analysis.calculus.integration_by_parts** will be useful to show it is equal to $F(\pi) + F(0)$.

   In particular, the following lemmas may be useful to us:

- **interval_integral.integral_deriv_eq_sub**
- **interval_integral.integral_sin**
- **interval_integral.integral_mul**;

4. To show $F(\pi) + F(0)$ is an integer, we expect to use **data.polynomial.eval** and **data.int.basic**;
5. To prove the inequality on $f(x)\sin(x)$, the following lemmas may be useful to us:
   - **abs_sin_le_one**
   - **sin_pos_of_mem_Ioo $(0, \pi)$**, which states that $\sin(x) > 0$ for $x \in (0, \pi)$
   - **mul_pos**, **pow_pos**, **div_pos**;
6. To draw together the sub-problems into a contradiction, we anticipate using the following lemmas from **data.real.irrational.basic**:
   - **irrational_iff_ne_rational**, which states that a number is irrational if and only if it is not equal to any rational number
   - **exists_rat_ne_of_irrational**, which states that if a real number is irrational, then it is distinct from every rational number.

Division of Workload

As mentioned above, due to the cumulative nature of sub-problems in this project, we expect to undertake parts 1–4 in a fairly collaborative manner. This may involve working independently or in pairs at different points in time to build on existing work done. This approach can be adapted to fit around everyone's schedules.

Below is a rough division of which segments of the project each team member will complete, as well as responsibilities for the general upkeep of the project, including tasks completed at the date of writing:

- Tristan Hartwell - to research Lean syntax and conventions to support with more technical aspects. Tristan will primarily contribute to parts 3, 4, and 6.
- Kitty Powell - to act as informal project manager - created GitHub repository, found and dissected Niven's proof [Niven, 1947], drafted project outline. Kitty intends to contribute most to parts 1, 3, and 5.
- Lizzie Roxburgh - to keep an up-to-date list of progress, to research the Mathlib results listed above in greater detail - examined the inequality in part 5. Lizzie intends to contribute most significantly to parts 1, 2, and 6.
- Ben Wadsworth - to oversee updates and additions to GitHub repository - added a README file and a blank lean project to the repository. Ben will mainly undertake parts 2, 4, and 5.

References

Ivan Niven. A simple proof that π is irrational. *Bulletin of the American Mathematical Society*, 53(6):509, 1947.

Sever Angel Popescu. A simple and self-contained proof for the lindemann-weierstrass theorem, 2023. URL `https://arxiv.org/abs/2306.14352`.

Denis Roegel. Lambert's proof of the irrationality of Pi: Context and translation. Research report, LORIA, 2020. URL `https://hal.science/hal-02984214`.