A Survey of Strong Stability Preserving High Order Time Discretizations

Chi-Wang Shu[*]

1 Introduction

Numerical solution for ordinary differential equations (ODEs) is an established research area. There are many well established methods, such as Runge-Kutta methods and multi-step methods, for such purposes. There are also many excellent books on this subject, for example [1], [13] and [17]. Special purpose ODE solvers, such as those for stiff ODEs, are also well studied. See, e.g., [6].

However, the class of methods surveyed in this article, the so-called strong stability preserving (SSP) methods, is somewhat special. These methods were designed specifically for solving the ODEs coming from a semi-discrete, spatial discretization of time dependent partial differential equations (PDEs), especially hyperbolic PDEs. Typically such ODEs are very large (the size of the system depends on the spatial discretization mesh size). More importantly, there are certain stability properties of the original PDE, such as total variation stability or maximum norm stability, which could be maintained by certain special spatial discretizations coupled with simple first order Euler forward time discretization, that would be desirable to maintain also for the high order time discretizations. SSP methods are designed to achieve such a goal.

We can thus highlight the main property of SSP time discretizations: if we *assume* that the first order, forward Euler time discretization of a method of lines semi-discrete scheme is stable under a certain norm, then a SSP high order time

[*]Division of Applied Mathematics, Brown University, Providence, RI 02912, USA. E-mail: shu@cfm.brown.edu. Research supported by ARO grant DAAD19-00-1-0405, NSF grants DMS-9804985 and ECS-9906606, NASA Langley grant NCC1-01035 and AFOSR grant F49620-99-1-0077.

discretization maintains this stability under a suitable restriction on the time step.

SSP time discretization methods were first developed by Shu in [20] and by Shu and Osher in [21], and were termed TVD (Total Variation Diminishing) time discretizations. The terminology was adopted because the method of lines ODE and its Euler forward version satisfy the total variation diminishing property when applied to scalar one dimensional nonlinear hyperbolic conservation laws. [21] contains a class of second to fifth order Runge-Kutta time discretizations which are proven SSP. [20] contains a class of first order SSP Runge-Kutta time discretizations which have very high CFL numbers, as well as a class of high order multi-step SSP methods. Later, Gottlieb and Shu [9] performed a systematic study of Runge-Kutta SSP methods, showing the optimal two stage, second order and three stage, third order SSP Runge-Kutta methods as well as low storage three stage, third order SSP Runge-Kutta methods, and proving the non-existence of four stage, fourth order SSP Runge-Kutta methods with non-negative coefficients. In [10], Gottlieb, Shu and Tadmor reviewed and further developed SSP Runge-Kutta and multi-step methods. The new results in [10] include the optimal explicit SSP linear Runge-Kutta methods, their application to the strong stability of coercive approximations, a systematic study of explicit SSP multi-step methods, and the study of the strong stability preserving property of implicit Runge-Kutta and multi-step methods. More recently, Spiteri and Ruuth [23] found a new class of high (up to fourth) order SSP Runge-Kutta methods by allowing the number of stages to be larger than the order of accuracy. The same authors also proved that luck runs out in this approach starting from fifth order: there is no SSP fifth order Runge-Kutta method with non-negative coefficients [18]. Gottlieb and Gottlieb in [8] obtained optimal linear SSP Runge-Kutta methods when the number of stages is larger than the order of accuracy. They have also made an interesting observation to use such methods for certain special variable coefficient ODEs, such as those coming from spatial discretizations for linear, constant coefficient PDEs such as the Maxwell's equations with time dependent boundary conditions.

One might ask the question whether it is worthwhile and necessary to use SSP time discretizations. Numerical examples shown in [9] indicate that oscillations and non-linear instability could occur when a linearly stable but non SSP Runge-Kutta method is applied to a TVD semi-discrete scheme, whose forward Euler first order time discretization can be proven stable. Thus it is at least safer to use SSP time discretizations whenever possible, especially when solving hyperbolic PDEs with shocks. In terms of computational cost, we remark that most SSP methods are of the same form and have the same cost as traditional ODE solvers. It is true that the time step Δt might need to be smaller when SSP is *proven* than when linear stability is proven, however in many situations Δt could be taken larger in practical calculations without encountering instability.

SSP time discretizations have been widely used in numerical solutions of time dependent PDEs, especially hyperbolic PDEs. ENO and WENO finite difference and finite volume schemes in [21], [22], [12] and [11], and Runge-Kutta discontinuous Galerkin finite element methods in [4] and [5], are such examples. Other examples of applications include the weighted L^2 SSP higher order discretizations of spectral methods [7], and the L^∞-SSP higher-order discretization for discontinuous Galerkin method in [3]. In fact, the (semi) norm can be replaced by any convex function, as the arguments of SSP are based on convex decompositions of high-order methods in terms of the first-order Euler method. An example of this is the cell entropy stability property of high order schemes studied in [16] and [15].

In this article we survey the current status of the development of SSP high order time discretizations. We provide a brief introduction with a simple example for the general framework of the method in section 2, and present the SSP Runge-Kutta methods in section 3 and the SSP multi-step methods in section 4. We give some concluding remarks in section 5.

2 General framework of SSP methods

We are interested in solving the following method of lines ODE

$$\frac{d}{dt}u(t) = L(u(t), t) \tag{1}$$

resulting from a spatial discretization to a time dependent partial differential equation. Here $u = u(t)$ is a (usually very long) vector and $L(u,t)$ depends on u either linearly or non-linearly. In many applications $L(u,t) = L(u)$ which does not explicitly depend on t. We would also consider an adjoint problem

$$\frac{d}{dt}u(t) = \tilde{L}(u(t), t), \tag{2}$$

which is closely related to (1). In fact, both L and $\tilde{L}$ approximate the same spatial derivatives in the original PDE. The difference is in their associated stability property when the ODEs (1) and (2) are discretized in time. Throughout this paper we *assume* that the first order Euler forward time discretization to (1):

$$u^{n+1} = u^n + \Delta t L(u^n, t^n), \tag{3}$$

where u^n is an approximation to $u(t^n)$, as well as the first order Euler backward time discretization to (2):

$$u^{n+1} = u^n - \Delta t \tilde{L}(u^n, t^n), \tag{4}$$

are stable under a certain (semi) norm

$$||u^{n+1}|| \leq ||u^n|| \tag{5}$$

with a suitable time step restriction

$$\Delta t \leq \Delta t_0. \tag{6}$$

Let us give a very simple example to illustrate this assumption. Assume we are solving the non-linear scalar one dimensional conservation law

$$v_t = f(v)_x$$

where $f'(v) \geq 0$, with periodic boundary conditions and a uniform spatial mesh $x_1 < x_2 < \cdots < x_N$. The following first order upwind scheme

$$v_j^{n+1} = v_j^n + \Delta t \left(\frac{f(v_{j+1}^n) - f(v_j^n)}{\Delta x} \right),$$

where v_j^n is an approximation to $v(x_j, t^n)$, as well as the adjoint scheme

$$v_j^{n+1} = v_j^n - \Delta t \left(\frac{f(v_j^n) - f(v_{j-1}^n)}{\Delta x} \right),$$

are both total variation diminishing (TVD). That is, if we define

$$u = (v_1, v_2, \cdots, v_N)^T, \tag{7}$$

$$L(u) = \left(\frac{f(v_2^n) - f(v_1^n)}{\Delta x}, \frac{f(v_3^n) - f(v_2^n)}{\Delta x}, \cdots, \frac{f(v_1^n) - f(v_N^n)}{\Delta x} \right)^T,$$

$$\tilde{L}(u) = \left(\frac{f(v_1^n) - f(v_N^n)}{\Delta x}, \frac{f(v_2^n) - f(v_1^n)}{\Delta x}, \cdots, \frac{f(v_N^n) - f(v_{N-1}^n)}{\Delta x} \right)^T,$$

and the total variation semi-norm by

$$||u|| = \sum_{j=1}^{N} |v_{j+1} - v_j|$$

for any u defined in (7) and satisfying the periodicity condition $v_{N+1} = v_1$, then we have the stability property (5) under the time step restriction (6) with

$$\Delta t_0 = \max_v |f'(v)| \Delta x,$$

for both (3) and (4). Higher order TVD spatial discretizations can also be designed to satisfy these properties. Examples would include the Runge-Kutta discontinuous Galerkin methods in [4].

With this assumption, we would like to find SSP time discretization methods to (1), sometimes with the help of (2), that are higher order accurate in time, yet still maintain the same stability condition (5), perhaps with a different restriction on the time step Δt than that in (6):

$$\Delta t \leq c \Delta t_0, \tag{8}$$

where c is called the *CFL coefficient* of the SSP method. The objective is to find such methods with simple format, low computational cost and less restriction on the time step Δt (larger CFL coefficient c). We will discuss SSP Runge-Kutta methods in the next section and SSP multi-step methods in section 4.

We remark that the strong stability assumption for the forward Euler step in (5) can be relaxed to the more general stability assumption

$$||u^{n+1}|| \leq (1 + O(\Delta t)) ||u^n||.$$

This general stability property will also be preserved by the high order SSP time discretizations. The total variation bounded (TVB) methods discussed in [19] and [4] belong to this category. However, if the forward Euler operator is not stable, the framework of SSP cannot be used to determine whether a high order time discretization is stable.

3 SSP Runge-Kutta methods

In [21], a general m stage Runge-Kutta method for (1) is written in the form:

$$\begin{aligned} u^{(0)} &= u^n, \\ u^{(i)} &= \sum_{k=0}^{i-1} \left(\alpha_{i,k} u^{(k)} + \Delta t \beta_{i,k} L(u^{(k)}, t^n + d_k \Delta t) \right), \qquad i = 1, ..., m \\ u^{n+1} &= u^{(m)} \end{aligned} \tag{9}$$

where d_k are related to $\alpha_{i,k}$ and $\beta_{i,k}$ by

$$d_0 = 0, \qquad d_i = \sum_{k=0}^{i-1} (\alpha_{i,k} d_k + \beta_{i,k}), \qquad i = 1, ..., m-1.$$

Thus we do not need to discuss the choice of d_k separately. Notice that in most ODE literature, e.g. [1], a Runge-Kutta method is written in the form of a Butcher array. Every Runge-Kutta method in the form of (9) can be easily converted in a unique way into a Butcher array, e.g., [21]. A Runge-Kutta method written in a Butcher array can also be rewritten into the form (9), however this conversion is in general *not* unique. This non-uniqueness in the representation (9) is explored in the literature to seek the largest provable time steps (8) for SSP.

We always need and require that $\alpha_{i,k} \geq 0$ in (9). If this is violated no SSP methods are possible.

If all the $\beta_{i,k}$'s in (9) are also nonnegative, $\beta_{i,k} \geq 0$, we have the following simple lemma which is the backbone of SSP Runge-Kutta methods:

Lemma 1. [21] If the forward Euler method (3) is stable in the sense of (5) under the time step restriction (6), then the Runge-Kutta method (9) with $\alpha_{i,k} \geq 0$ and $\beta_{i,k} \geq 0$ is SSP, i.e. its solution also satisfies the same stability (5), under the time step restriction (8) with the CFL coefficient

$$c = \min_{i,k} \frac{\alpha_{i,k}}{\beta_{i,k}}. \tag{10}$$

Proof: Since the forward Euler method (3) is stable in the sense of (5) under the time step restriction (6), we have

$$||u^{(k)} + \frac{\beta_{i,k}}{\alpha_{i,k}} \Delta t L(u^{(k)}, t^n + d_k \Delta t)|| \leq ||u^{(k)}||$$

if the time step restriction (8) is satisfied with c given by (10). Also notice that, by consistency,

$$\sum_{k=0}^{i-1} \alpha_{i,k} = 1.$$

We now use induction to prove

$$||u^{(k)}|| \leq ||u^n|| \tag{11}$$

for $k = 0, 1, ..., m$. Clearly (11) is valid for $k = 0$. Assuming that it is valid for all $k \leq i-1$, we then obtain

$$\begin{aligned}
||u^{(i)}|| &\leq \sum_{k=0}^{i-1} \alpha_{i,k} ||u^{(k)} + \frac{\beta_{i,k}}{\alpha_{i,k}} \Delta t L(u^{(k)}, t^n + d_k \Delta t)|| \\
&\leq \sum_{k=0}^{i-1} \alpha_{i,k} ||u^{(k)}|| \\
&\leq \sum_{k=0}^{i-1} \alpha_{i,k} ||u^n|| = ||u^n||.
\end{aligned}$$

The proof is thus completed.

■

Notice that the proof demonstrates that all the intermediate stages $u^{(i)}$ are also stable under the same norm. This is important for practical calculations, as instability in the middle stages could lead to unphysical artifacts such as negative density and pressure in gas dynamics calculations, thus preventing one from even finishing the Runge-Kutta step even though the complete step may be stable.

The most popular and successful SSP methods are those covered by Lemma 1. We will give some commonly used examples later.

If some of the $\beta_{i,k}$'s must be negative, we need the help of the adjoint operator $\tilde{L}$ in (2). Remember that the Euler backward time discretization to (2), given by (4), is assumed to be stable in the sense of (5). With a similar proof to that of Lemma 1, we then have the following lemma for this general case:

Lemma 2. [21] If the forward Euler method (3) and the backward Euler method (4) are both stable in the sense of (5) under the time step restriction (6), then the Runge-Kutta method (9) with $\alpha_{i,k} \geq 0$, and with $\beta_{i,k} L$ replaced by $\beta_{i,k} \tilde{L}$ whenever $\beta_{i,k}$ is negative, is SSP, i.e. its solution also satisfies the same stability (5), under the time step restriction (8) with the CFL coefficient

$$c = \min_{i,k} \frac{\alpha_{i,k}}{|\beta_{i,k}|}. \tag{12}$$

■

SSP methods covered by Lemma 2 with negative $\beta_{i,k}$'s are less popular in applications, partly because people do not want to deal with the unfriendly $\tilde{L}$ (although its construction for PDEs is in general very easy and similar to that for L, see for the simple example given in the previous section), and partly because of the extra computational and storage costs associated with the additional operator $\tilde{L}$.

We now give examples of some of the most useful SSP Runge-Kutta methods.

3.1 Linear SSP Runge-Kutta methods

The situation is dramatically simplified if $L(u,t) = L\,u$ is linear with constant coefficients.

The optimal m stage, m-th order SSP Runge-Kutta method for such linear constant coefficient case is given by [10]:

$$\begin{aligned} u^{(i)} &= u^{(i-1)} + \Delta t L u^{(i-1)}, \qquad i = 1, ..., m-1 \\ u^{(m)} &= \sum_{k=0}^{m-2} \alpha_{m,k} u^{(k)} + \alpha_{m,m-1} \left(u^{(m-1)} + \Delta t L u^{(m-1)} \right), \end{aligned} \tag{13}$$

where $\alpha_{1,0} = 1$ and

$$\begin{aligned} \alpha_{m,k} &= \frac{1}{k} \alpha_{m-1,k-1}, \qquad k = 1, ..., m-2 \\ \alpha_{m,m-1} &= \frac{1}{m!}, \qquad \alpha_{m,0} = 1 - \sum_{k=1}^{m-1} \alpha_{m,k}. \end{aligned} \tag{14}$$

This class of Runge-Kutta methods is SSP with the CFL coefficient $c = 1$ in (8).

Notice that, for this linear case, there exists only one m stage, m-th order Runge-Kutta method. So if we ignore the possible differences in the middle stages $u^{(i)}$ for $1 \le i \le m-1$, we are in effect claiming that this unique m stage, m-th order Runge-Kutta method is SSP.

Such methods are very useful when one solves a constant coefficient PDE such as the Maxwell's equation using a linear method, such as the discontinuous Galerkin method without using nonlinear limiters. Another application is to the linear and coercive approximations for parabolic type problems, see [10] and [14].

If one relaxes the condition on the number of stages so that it can be bigger than the order of accuracy, then the available Runge-Kutta method is no longer unique. For example, it is possible to get a $(m+1)$ stage, m-th order SSP Runge-Kutta method with CFL coefficient $c = 2$ in (8). See [8] for details.

Finally, we quote a very interesting result of [8], which claims that one can use this class of SSP Runge-Kutta methods on certain time dependent linear problems of the form

$$\frac{d}{dt} u(t) = L u(t) + f(t) \tag{15}$$

which could arise in, e.g. a discretization of a linear, constant coefficient PDE such as the Maxwell's equation with a time dependent boundary condition. The trick is to approximate $f(t)$ by a polynomial in t, for example using the Legendre or Chebyshev series approximations, or by any expansion

$$f(t) = \sum_{k=0}^{K} a_k \phi_k(t)$$

where

$$\frac{d}{dt}\phi(t) = B\phi(t)$$

for some constant matrix B. Here

$$\phi(t) = (\phi_0(t), \cdots, \phi_K(t))^T.$$

Trigonometric expansions and exponential expansions also belong to this class.

Clearly, if we denote

$$w = (u, \phi)^T, \qquad A = diag(a_0, a_1, ..., a_K)$$

then equation (15) becomes

$$\frac{d}{dt}w(t) = Cw(t)$$

where

$$C = \begin{pmatrix} L & A \\ 0 & B \end{pmatrix}$$

is a constant matrix. Hence SSP methods given in this subsection would apply.

3.2 Second order nonlinear SSP Runge-Kutta methods

The optimal second order, two stage nonlinear SSP Runge-Kutta method is given by [21], [9]:

$$\begin{aligned} u^{(1)} &= u^n + \Delta t L(u^n, t^n) \\ u^{n+1} &= \frac{1}{2}u^n + \frac{1}{2}u^{(1)} + \frac{1}{2}\Delta t L(u^{(1)}, t^n + \Delta t) \end{aligned} \tag{16}$$

with a CFL coefficient $c = 1$ in (8). This is just the classical Heun method or modified Euler method.

This method is clearly no more expensive or complicated than any other second order Runge-Kutta methods, with its added assurance of SSP with a healthy CFL coefficient $c = 1$. Thus it should be used whenever second order accuracy in time is desired and non-linear stability is a concern.

If one relaxes the condition on the number of stages so that it can be bigger than the order of accuracy, then SSP methods can be found to have larger CFL coefficients. For example, it is possible to get a three stage, second order SSP Runge-Kutta method with CFL coefficient $c = 2$ in (8), or a four stage, second order SSP Runge-Kutta method with CFL coefficient $c = 3$ in (8). See [23] for details.

3.3 Third order nonlinear SSP Runge-Kutta methods

The optimal third order, three stage nonlinear SSP Runge-Kutta method is given by [21], [9]:

$$\begin{aligned} u^{(1)} &= u^n + \Delta t L(u^n, t^n) \\ u^{(2)} &= \frac{3}{4}u^n + \frac{1}{4}u^{(1)} + \frac{1}{4}\Delta t L(u^{(1)}, t^n + \Delta t) \\ u^{n+1} &= \frac{1}{3}u^n + \frac{2}{3}u^{(2)} + \frac{2}{3}\Delta t L(u^{(2)}, t^n + \frac{1}{2}\Delta t), \end{aligned} \tag{17}$$

with a CFL coefficient $c = 1$ in (8).

This is by far the most widely used SSP method in the literature. It is clearly no more expensive or complicated than any other third order Runge-Kutta methods, with its added assurance of SSP with a healthy CFL coefficient $c = 1$. Thus it should be used whenever third order accuracy in time is desired and non-linear stability is a concern.

If one relaxes the condition on the number of stages so that it can be bigger than three, the order of accuracy, then SSP methods can be found to have larger CFL coefficients. For example, we quote the following four stage, third order SSP Runge-Kutta method from [23] which has a CFL coefficient $c = 2$ in (8):

$$\begin{aligned} u^{(1)} &= u^n + \frac{1}{2}\Delta t L(u^n, t^n) \\ u^{(2)} &= u^{(1)} + \frac{1}{2}\Delta t L(u^{(1)}, t^n + \frac{1}{2}\Delta t) \\ u^{(3)} &= \frac{2}{3}u^n + \frac{1}{3}u^{(2)} + \frac{1}{6}\Delta t L(u^{(2)}, t^n + \Delta t) \\ u^{n+1} &= u^{(3)} + \frac{1}{2}\Delta t L(u^{(3)}, t^n + \frac{1}{2}\Delta t). \end{aligned} \tag{18}$$

3.4 Third order nonlinear SSP Runge-Kutta methods with low storage

The general low-storage Runge-Kutta schemes can be written in the form [24], [2]:

$$\begin{aligned} u^{(0)} &= u^n, \qquad du^{(0)} = 0, \\ du^{(i)} &= A_i du^{(i-1)} + \Delta t L(u^{(i-1)}, t^n + d_{i-1}\Delta t), \qquad i = 1, \ldots, m, \\ u^{(i)} &= u^{(i-1)} + B_i du^{(i)}, \qquad i = 1, \ldots, m, \\ u^{n+1} &= u^{(m)}. \end{aligned} \tag{19}$$

Only u and du must be stored, resulting in two storage units for each variable.

Following Carpenter and Kennedy [2], the best SSP third order method found by numerical search in [9] is given by the system, with the free parameter b chosen as $b = 0.924574$:

$$z_1 = \sqrt{36b^4 + 36b^3 - 135b^2 + 84b - 12}$$

$$
\begin{aligned}
z_2 &= 2b^2 + b - 2 \\
z_3 &= 12b^4 - 18b^3 + 18b^2 - 11b + 2 \\
z_4 &= 36b^4 - 36b^3 + 13b^2 - 8b + 4 \\
z_5 &= 69b^3 - 62b^2 + 28b - 8 \\
z_6 &= 34b^4 - 46b^3 + 34b^2 - 13b + 2 \\
d_0 &= 0 \\
A_1 &= 0 \\
B_1 &= b \\
d_1 &= B_1 \\
A_2 &= \frac{-z_1(6b - 4b + 1) + 3z_3}{(2b+1)z_1 - 3(b+2)(2b-1)^2} \\
B_2 &= \frac{12b(b-1)(3z_2 - z_1) - (3z_2 - z_1)^2}{144b(3b-2)(b-1)^2} \\
d_2 &= B_1 + B_2 + B_2 A_2 \\
A_3 &= \frac{-z_1 z_4 + 108(2b-1)b^5 - 3(2b-1)z_5}{24z_1 b(b-1)^4 + 72bz_6 + 72b^6(2b-13)} \\
B_3 &= \frac{-24(3b-2)(b-1)^2}{(3z_2 - z_1)^2 - 12b(b-1)(3z_2 - z_1)}
\end{aligned}
$$

with a CFL coefficient $c = 0.32$ in (8). This is of course less optimal than (17) in terms of the CFL coefficient, but the low storage form is useful for large scale calculations.

This method can be coded up using only two arrays, one for u and the other for du. Thus this method is a favorite when storage is the paramount consideration, such as for large scale three dimensional calculations using small computers.

3.5 Fourth order nonlinear SSP Runge-Kutta methods

It is proven in [9] that all four stage, fourth order SSP Runge-Kutta scheme (9) with a nonzero CFL coefficient c in (8) must have at least one negative $\beta_{i,k}$. Such schemes are very ugly looking [21], [9] and they have never gained any popularity in the applications. So we will not list them here. Interested readers can find them in [21] and [9].

If one relaxes the condition on the number of stages so that it can be bigger than the order of accuracy, then SSP methods can be found to have non-negative $\alpha_{i,k}$ and $\beta_{i,k}$ and positive CFL coefficient c in (8), see [23]. For example, the following five stage, fourth order Runge-Kutta method [23] is SSP with a CFL coefficient $c = 1.508$ in (8):

$$
\begin{aligned}
u^{(1)} &= u^n + 0.39175222700392\,\Delta t L(u^n, t^n) \\
u^{(2)} &= 0.44437049406734\,u^n + 0.55562950593266\,u^{(1)} \\
&\quad +0.36841059262959\,\Delta t L(u^{(1)}, t^n + 0.39175222700392\,\Delta t) \\
u^{(3)} &= 0.62010185138540\,u^n + 0.37989814861460\,u^{(2)}
\end{aligned}
$$

$$
\begin{aligned}
&\quad +0.25189177424738\,\Delta t L(u^{(2)}, t^n + 0.58607968896780\,\Delta t) \\
u^{(4)} &= 0.17807995410773\,u^n + 0.82192004589227\,u^{(3)} \qquad (20)\\
&\quad +0.54497475021237\,\Delta t L(u^{(3)}, t^n + 0.47454236302687\,\Delta t) \\
u^{n+1} &= 0.00683325884039\,u^n + 0.51723167208978\,u^{(2)} + 0.12759831133288\,u^{(3)} \\
&\quad +0.08460416338212\,\Delta t L(u^{(3)}, t^n + 0.47454236302687\,\Delta t) \\
&\quad +0.34833675773694\,u^{(4)} \\
&\quad +0.22600748319395\,\Delta t L(u^{(4)}, t^n + 0.93501063100924\,\Delta t).
\end{aligned}
$$

4 SSP multi-step methods

In [20], a general m step method for (1) is written in the form:

$$
u^{n+1} = \sum_{i=1}^{m} \left(\alpha_i u^{n+1-i} + \Delta t \beta_i L(u^{n+1-i}, t^{n+1-i}) \right). \qquad (21)
$$

Again, we require that $\alpha_i \geq 0$ in (21). If this is violated no SSP methods are possible.

Similar to Lemma 1 and Lemma 2, we have the following simple lemmas for the multi-step methods in (21) to be SSP.

Lemma 3. [20] If the forward Euler method (3) is stable in the sense of (5) under the time step restriction (6), then the multi-step method (21) with $\alpha_i \geq 0$ and $\beta_i \geq 0$ is SSP, i.e. its solution also satisfies the same stability (5), under the time step restriction (8) with the CFL coefficient

$$
c = \min_i \frac{\alpha_i}{\beta_i}. \qquad (22)
$$

■

Lemma 4. [20] If the forward Euler method (3) and the backward Euler method (4) are both stable in the sense of (5) under the time step restriction (6), then the multi-step method (21) with $\alpha_i \geq 0$, and with $\beta_i L$ replaced by $\beta_i \tilde{L}$ whenever β_i is negative, is SSP, i.e. its solution also satisfies the same stability (5), under the time step restriction (8) with the CFL coefficient

$$
c = \min_i \frac{\alpha_i}{|\beta_i|}. \qquad (23)
$$

■

The proofs of these lemmas are similar to that for Lemmas 1 and 2, namely using the fact $\sum \alpha_i = 1$ from consistency, and the observation that u^{n+1} can be written as a convex combination of forward Euler steps with suitably scaled Δt's.

As before, we would like to have multi-step SSP methods with non-negative coefficients covered by Lemma 3, if at all possible, to avoid the complication and extra computational and storage costs associate with $\tilde{L}$. It is proven in [10] that we need at least $m+1$ steps for an m-th order SSP method with non-negative coefficients.

The SSP multi-step methods seem to be less popular in applications than the SSP Runge-Kutta methods, perhaps because of the complication of the start-up of the first few time steps, the difficulty of adjusting time steps in the middle of the calculation, and the relatively larger storage requirements, all these being generic problems with multi-step methods.

On the other hand, it is usually easier to satisfy the order of accuracy conditions for multi-step methods than for Runge-Kutta methods, especially for orders of accuracy higher than four.

We now give examples of some of the SSP multi-step methods.

4.1 Second order SSP multi-step methods

The optimal second order, three step SSP method with non-negative coefficients is given by [20], [10]:

$$u^{n+1} = \frac{3}{4}u^n + \frac{3}{2}\Delta t L(u^n, t^n) + \frac{1}{4}u^{n-1} \tag{24}$$

with a CFL coefficient $c = \frac{1}{2}$ in (8).

We remark that this SSP multi-step method has the same efficiency as the optimal two stage, second order Runge-Kutta method (16). This is because there is only one L evaluation per time step here, compared with two L evaluations in the two stage Runge-Kutta method. Of course, the storage requirement here is larger.

Again, if one increases the number of steps, then SSP methods can be found to have larger CFL coefficients. For example, it is possible to find a four step, second order SSP method with non-negative coefficients and a CFL coefficient $c = \frac{2}{3}$, see [20] and [10]. Notice that unlike in the case of Runge-Kutta methods, here the computational cost is not increased much with an increase of the number of steps, because the most expensive evaluation of L is performed only once. However, storage would be increased with this increase of the number of steps.

4.2 Third order SSP multi-step methods

The optimal third order, four step SSP method with non-negative coefficients is given by [20], [10]:

$$u^{n+1} = \frac{16}{27}u^n + \frac{16}{9}\Delta t L(u^n, t^n) + \frac{11}{27}u^{n-3} + \frac{4}{9}\Delta t L(u^{n-1}, t^{n-1}) \tag{25}$$

with a CFL coefficient $c = \frac{1}{3}$ in (8).

We remark that this SSP multi-step method has the same efficiency as the optimal three stage, third order Runge-Kutta method (17). This is because there is only one L evaluation per time step here, compared with three L evaluations in the three stage Runge-Kutta method. Of course, the storage requirement here is larger.

Again, if one increases the number of steps, then SSP methods can be found to have larger CFL coefficients. For example, it is possible to find a five step, third order SSP method with non-negative coefficients and a CFL coefficient $c = \frac{1}{2}$, see [20] and [10]. Because there is no significant increase in the computational cost when the number of steps is increased, when storage is not a consideration, it is advantageous to use a SSP multi-step methods with more steps and higher CFL coefficients.

5 Concluding remarks

We have given a brief survey of the strong stability preserving, or SSP, high order time discretizations for method of lines ODEs resulting from a spatial discretization of PDEs, especially the hyperbolic type PDEs. The most popular SSP methods in applications have been listed.

Bibliography

[1] J.C. Butcher, *The numerical analysis of ordinary differential equations: Runge-Kutta and general linear methods*, John Wiley, New York, 1987.

[2] M. Carpenter and C. Kennedy, *Fourth-order 2N-storage Runge-Kutta schemes*, NASA TM 109112, NASA Langley Research Center, June 1994.

[3] B. Cockburn, S. Hou and C.-W. Shu, *TVB Runge-Kutta local projection discontinuous Galerkin finite element method for conservation laws IV: The multidimensional case*, Math. Comp., 54, 1990, pp. 545–581.

[4] B. Cockburn and C.-W. Shu, *TVB Runge-Kutta local projection discontinuous Galerkin finite element method for conservation laws II: General framework*, Math. Comp., 52, 1989, pp. 411–435.

[5] B. Cockburn and C.-W. Shu, *Runge-Kutta Discontinuous Galerkin methods for convection-dominated problems*, J. Sci. Comput., 16, 2001, pp. 173–261.

[6] K. Dekkerand and J.G. Verwer, *Stability of Runge-Kutta methods for stiff nonlinear differential equations*, Elsevier Science Publishers B.V., Amsterdam, 1984.

[7] D. Gottlieb and E. Tadmor, *The CFL condition for spectral approximations to hyperbolic initial-boundary value problems*, Math. Comp., 56, 1991, pp. 565–588.

[8] S. Gottlieb and L.-A. Gottlieb, *Strong stability preserving properties of Runge-Kutta time discretization methods for linear operators*, J. Sci. Comput., submitted.

[9] S. Gottlieb and C.-W. Shu, *Total variation diminishing Runge-Kutta schemes*, Math. Comp., 67, 1998, pp. 73–85.

[10] S. Gottlieb, C.-W. Shu and E. Tadmor, *Strong stability preserving high order time discretization methods*, SIAM Rev., 43 (2001), pp. 89–112.

[11] C. Hu and C.-W. Shu, *Weighted essentially non-oscillatory schemes on triangular meshes*, J. Comput. Phys., 150 (1999), pp. 97–127.

[12] G. Jiang and C.-W. Shu, *Efficient implementation of weighted ENO schemes*, J. Comput. Phys., 126 (1996), pp. 202–228.

[13] J. D. Lambert, *Computational Methods in Ordinary Differential Equations*, John Wiley, New York, 1973.

[14] D. Levy and E. Tadmor, *From semi-discrete to fully discrete: stability of Runge-Kutta schemes by the energy method*, SIAM Rev., 40, 1998, pp. 40–73.

[15] H. Nessyahu and E. Tadmor, *Non-oscillatory central differencing for hyperbolic conservation laws*, J. Comput. Phys., 87, 1990, pp. 408–463.

[16] S. Osher and E. Tadmor, *On the convergence of difference approximations to scalar conservation laws*, Math. Comp., 50, 1988, pp. 19–51.

[17] A. Ralston, *A First Course in Numerical Analysis*, McGraw-Hill, New York, 1965.

[18] S. Ruuth and R. Spiteri, *Two barriers on strong-stability-preserving time discretization methods*, J. Sci. Comput., to appear.

[19] C.-W. Shu, *TVB uniformly high-order schemes for conservation laws*, Math. Comp., 49, 1987, pp. 105–121.

[20] C.-W. Shu, *Total-variation-diminishing time discretizations*, SIAM J. Sci. Stat. Comput., 9, 1988, pp. 1073–1084.

[21] C.-W. Shu and S. Osher, *Efficient implementation of essentially non-oscillatory shock-capturing schemes*, J. Comput. Phys., 77, 1988, pp. 439–471.

[22] C.-W. Shu and S. Osher, *Efficient implementation of essentially non- oscillatory shock capturing schemes II*, J. Comput. Phys., 83, 1989, pp. 32–78.

[23] R. Spiteri and S. Ruuth, *A new class of optimal high-order strong-stability-preserving time discretization methods*, SIAM J. Numer. Anal., submitted.

[24] J.H. Williamson, *Low-storage Runge-Kutta schemes*, J. Comput. Phys., 35, 1980, pp. 48–56.