Poisson bracket for one electron hybrid model

immediate

1 Electron hybrid model

Here we just set $\epsilon_0 = \mu_0 = m_e = q_e = 1$. The electron hybrid model is as follows,

$$
\begin{aligned}
&\frac{\partial \mathbf{j}_c}{\partial t} = \Omega_{pe}^2 \mathbf{E} + \mathbf{j}_c \times \mathbf{B}_0(\mathbf{x}), && (1)\\
&\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + (\mathbf{E} + \mathbf{v} \times (\mathbf{B} + \mathbf{B}_0)) \cdot \nabla_{\mathbf{v}} f = 0, && (2)\\
&\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}, && (3)\\
&\frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{B} - \mathbf{j}_c - \int \mathbf{v} f \mathrm{d}\mathbf{v}, && (4)
\end{aligned}
$$

where $\Omega_{pe}^2 = n_{c0}(\mathbf{x})$. The energy (Hamiltonian) is

$$
\mathcal{H} = \frac{1}{2}\int |\mathbf{E}|^2 \mathrm{d}\mathbf{x} + \frac{1}{2}\int |\mathbf{B}|^2 \mathrm{d}\mathbf{x} + \frac{1}{2}\int \frac{1}{\Omega_{pe}^2}\mathbf{j}_c^2 \mathrm{d}\mathbf{x} + \frac{1}{2}\int |\mathbf{v}|^2 f \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v}. \tag{5}
$$

2 Poisson structure

We define one bracket as follows,

$$
\begin{aligned}
\{F, G\}[\mathbf{j}_c, f, \mathbf{E}, \mathbf{B}] = &\int f[\frac{\delta F}{\delta f}, \frac{\delta G}{\delta f}]_{xv} \mathrm{d}\mathbf{v}\mathrm{d}\mathbf{x}\\
&+ \int f \left(\nabla_{\mathbf{v}} \frac{\delta F}{\delta f} \cdot \frac{\delta G}{\delta \mathbf{E}} - \nabla_{\mathbf{v}} \frac{\delta G}{\delta f} \cdot \frac{\delta F}{\delta \mathbf{E}} \right) \mathrm{d}\mathbf{v}\mathrm{d}\mathbf{x}\\
&+ \int f(\mathbf{B} + \mathbf{B}_0) \left(\nabla_{\mathbf{v}} \frac{\delta F}{\delta f} \times \nabla_{\mathbf{v}} \frac{\delta G}{\delta f} \right) \mathrm{d}\mathbf{v}\mathrm{d}\mathbf{x}\\
&+ \int \left(\nabla \times \frac{\delta F}{\delta \mathbf{E}} \cdot \frac{\delta G}{\delta \mathbf{B}} - \nabla \times \frac{\delta G}{\delta \mathbf{E}} \cdot \frac{\delta F}{\delta \mathbf{B}} \right) \mathrm{d}\mathbf{x}\\
&+ \int \Omega_{pe}^2 \left(\frac{\delta F}{\delta \mathbf{j}_c} \cdot \frac{\delta G}{\delta \mathbf{E}} - \frac{\delta G}{\delta \mathbf{j}_c} \cdot \frac{\delta F}{\delta \mathbf{E}} \right) \mathrm{d}\mathbf{x}\\
&+ \int \Omega_{pe}^2 \mathbf{B}_0 \cdot \left(\frac{\delta F}{\delta \mathbf{j}_c} \times \frac{\delta G}{\delta \mathbf{j}_c} \right) \mathrm{d}\mathbf{x}.
\end{aligned} \tag{6}
$$

The two terms in red are new compared with the Poisson bracket of Vlasov–Maxwell equations. We can verify the system can be obtained by the above bracket and Hamiltonian.

Theorem 2.1. *The above bracket is a Poisson bracket.*

Proof. We here prove the Jacobi identity is satisfied. The above bracket can be reformulated as follows,

$$
\begin{aligned}
\{F, G\} = &= \int f[\frac{\delta F}{\delta f}, \frac{\delta G}{\delta f}]_{xv} \mathrm{d}\mathbf{v}\mathrm{d}\mathbf{x} \\
&+ \int f \left(\nabla_{\mathbf{v}} \frac{\delta F}{\delta f} \cdot \frac{\delta G}{\delta \mathbf{E}} - \nabla_{\mathbf{v}} \frac{\delta G}{\delta f} \cdot \frac{\delta F}{\delta \mathbf{E}} \right) \mathrm{d}\mathbf{v}\mathrm{d}\mathbf{x} \\
&+ \int f \left(\mathbf{B} + \mathbf{B}_0\right) \left(\nabla_{\mathbf{v}} \frac{\delta F}{\delta f} \times \nabla_{\mathbf{v}} \frac{\delta G}{\delta f} \right) \mathrm{d}\mathbf{v}\mathrm{d}\mathbf{x} \\
&+ \int \left(\nabla \times \frac{\delta F}{\delta \mathbf{E}} \cdot \frac{\delta G}{\delta \mathbf{B}} - \nabla \times \frac{\delta G}{\delta \mathbf{E}} \cdot \frac{\delta F}{\delta \mathbf{B}} \right) \mathrm{d}\mathbf{x} \\
&+ \int \Omega_{pe}^2 \left(\frac{\delta F}{\delta \mathbf{j}_c} \cdot \frac{\delta G}{\delta \mathbf{E}} - \frac{\delta G}{\delta \mathbf{j}_c} \cdot \frac{\delta F}{\delta \mathbf{E}} \right) \mathrm{d}\mathbf{x} \\
&+ \int \Omega_{pe}^2 \mathbf{B}_0 \cdot \left(\frac{\delta F}{\delta \mathbf{j}_c} \times \frac{\delta G}{\delta \mathbf{j}_c} \right) \mathrm{d}\mathbf{x}, \\
&=: \{F, G\}_{\mathrm{VM}} + \{F, G\}_{jE}.
\end{aligned} \tag{7}
$$

Then the Jacobi identity can be written as follows,

$$
\begin{aligned}
&\{\{F, G\}, H\} + \{\{G, H\}, F\} + \{\{H, F\}, G\} \\
&= \{\{F, G\}_{\mathrm{VM}}, H\}_{\mathrm{VM}} + \{\{F, G\}_{\mathrm{VM}}, H\}_{jE}, \\
&+ \{\{F, G\}_{jE}, H\}_{\mathrm{VM}} + \{\{F, G\}_{jE}, H\}_{jE} + \mathrm{cyc}.
\end{aligned} \tag{8}
$$

From the bracket theorem in [1], we only need to consider the functional derivatives of $\{F, G\}$ modulo the second derivative terms, which, with abuse of notations, gives

$$
\begin{aligned}
&\frac{\delta\{F, G\}_{\mathrm{VM}}}{\delta \mathbf{j}_c} = \mathbf{0}, \quad \frac{\delta\{F, G\}_{\mathrm{VM}}}{\delta \mathbf{E}} = \mathbf{0}, \\
&\frac{\delta\{F, G\}_{jE}}{\delta f} = 0, \quad \frac{\delta\{F, G\}_{jE}}{\delta \mathbf{E}} = \mathbf{0}, \quad \frac{\delta\{F, G\}_{jE}}{\delta \mathbf{B}} = \mathbf{0}.
\end{aligned} \tag{9}
$$

So the second term and third term in (8) are all zeros. As $\frac{\delta\{F,G\}_{jE}}{\delta \mathbf{E}} = \mathbf{0}$, and $\frac{\delta\{F,G\}_{jE}}{\delta \mathbf{j}_c} = \mathbf{0}$ (in the sense of modulo the second variational derivatives), we know that the fourth term in (8) is zero.

Next only the first term in (8) is left. The starting point is the classical Poisson bracket of Vlasov–Maxwell equations,

$$
\begin{aligned}
\{\{\bar{F},\bar{G}\}\}(f,\mathbf{E},\bar{\mathbf{B}}) = & = \int f[\frac{\delta \bar{F}}{\delta f},\frac{\delta \bar{G}}{\delta f}]_{xv}\mathrm{d}\mathbf{v}\mathrm{d}\mathbf{x} \\
& + \int f\left(\nabla_{\mathbf{v}}\frac{\delta \bar{F}}{\delta f}\cdot\frac{\delta \bar{G}}{\delta \mathbf{E}} - \nabla_{\mathbf{v}}\frac{\delta \bar{G}}{\delta f}\cdot\frac{\delta \bar{F}}{\delta \mathbf{E}}\right)\mathrm{d}\mathbf{v}\mathrm{d}\mathbf{x} \\
& + \int f\bar{\mathbf{B}}\left(\nabla_{\mathbf{v}}\frac{\delta \bar{F}}{\delta f}\times\nabla_{\mathbf{v}}\frac{\delta \bar{G}}{\delta f}\right)\mathrm{d}\mathbf{v}\mathrm{d}\mathbf{x} \\
& + \int\left(\nabla\times\frac{\delta \bar{F}}{\delta \mathbf{E}}\cdot\frac{\delta \bar{G}}{\delta \bar{\mathbf{B}}} - \nabla\times\frac{\delta \bar{G}}{\delta \mathbf{E}}\cdot\frac{\delta \bar{F}}{\delta \bar{\mathbf{B}}}\right)\mathrm{d}\mathbf{x}
\end{aligned} \tag{10}
$$

We give a coordinate transformation, $\bar{\mathbf{B}} = \mathbf{B} - \mathbf{B}_0$. Given any functional $F(f,\mathbf{E},\mathbf{B})$, we define a functional $\bar{F}(f,\mathbf{E},\bar{\mathbf{B}})$ as

$$
\bar{F}(f,\mathbf{E},\bar{\mathbf{B}}) = F(f,\mathbf{E},\bar{\mathbf{B}} - \bar{\mathbf{B}}_0) = F(f,\mathbf{E},\mathbf{B}),
$$

we have the following relations of variational derivatives,

$$
\frac{\delta \bar{F}}{\delta f} = \frac{\delta F}{\delta f}, \quad \frac{\delta \bar{F}}{\delta \mathbf{E}} = \frac{\delta F}{\delta \mathbf{E}}, \quad \frac{\delta \bar{F}}{\delta \bar{\mathbf{B}}} = \frac{\delta F}{\delta \mathbf{B}}. \tag{11}
$$

Substituting the above variational derivatives into (10), we have

$$
\begin{aligned}
\{\{\bar{F},\bar{G}\}\}(f,\mathbf{E},\bar{\mathbf{B}}) = & = \int f[\frac{\delta F}{\delta f},\frac{\delta G}{\delta f}]_{xv}\mathrm{d}\mathbf{v}\mathrm{d}\mathbf{x} \\
& + \int f\left(\nabla_{\mathbf{v}}\frac{\delta F}{\delta f}\cdot\frac{\delta G}{\delta \mathbf{E}} - \nabla_{\mathbf{v}}\frac{\delta G}{\delta f}\cdot\frac{\delta F}{\delta \mathbf{E}}\right)\mathrm{d}\mathbf{v}\mathrm{d}\mathbf{x} \\
& + \int f\left(\mathbf{B}+\mathbf{B}_0\right)\left(\nabla_{\mathbf{v}}\frac{\delta F}{\delta f}\times\nabla_{\mathbf{v}}\frac{\delta G}{\delta f}\right)\mathrm{d}\mathbf{v}\mathrm{d}\mathbf{x} \\
& + \int\left(\nabla\times\frac{\delta F}{\delta \mathbf{E}}\cdot\frac{\delta G}{\delta \mathbf{B}} - \nabla\times\frac{\delta G}{\delta \mathbf{E}}\cdot\frac{\delta F}{\delta \mathbf{B}}\right)\mathrm{d}\mathbf{x} \\
& =: \{F,G\}(f,\mathbf{E},\mathbf{B}).
\end{aligned} \tag{12}
$$

Then we know that the 'first term + cyc = 0' in (8), i.e., $\{\{F,G\}_{\mathrm{VM}},H\}_{\mathrm{VM}} + \mathrm{cyc} = 0$, which is a result of the Jacobi identity of Eq. (10). □

3 Hamiltonian splitting

From the previous section, we derive a time splitting to numerically solve (1)-(4). Indeed, reformulating the hybrid model as

$$
\partial_t U = \{U,\mathcal{H}\},
$$

where $\mathcal{H}$ denotes the Hamiltonian (5), U a functional of the unknown $(f,\mathbf{E},\mathbf{B},\mathbf{j}_c)$, and $\{\cdot,\cdot\}$ the bracket defined in (6). we construct the splitting from the decomposition of the Hamiltonian

$$
\dot{U} = \{U,\mathcal{H}_{\mathbf{E}}\} + \{U,\mathcal{H}_{\mathbf{B}}\} + \{U,\mathcal{H}_{\mathbf{j}_c}\} + \{U,\mathcal{H}_f\}.
$$

In the following we write down the resulting equations to be solved within this splitting.

First, let us remark the the unknown has to expressed in the following

$$U(t,x,v) = \int\int U(t,y,w)\delta(\mathbf{x}-\mathbf{y})\delta(\mathbf{v}-\mathbf{w})\mathrm{d}\mathbf{w}\mathrm{d}\mathbf{y},$$

so that their Fréchet derivatives becomes (with respect to f here)

$$\frac{\delta U}{\delta f} = \delta(\mathbf{x}-\mathbf{y})\delta(\mathbf{v}-\mathbf{w}).$$

3.1 Equations for $\mathcal{H}_{\mathbf{E}}$

We have to compute $\{U, \mathcal{H}_{\mathbf{E}}\}$, for $U = f, \mathbf{E}, \mathbf{B}, \mathbf{j}_C$. We have the following relations

$$\frac{\delta \mathcal{H}_{\mathbf{E}}}{\delta(f,\mathbf{B},\mathbf{j}_C)} = 0, \quad \text{and } \frac{\delta \mathcal{H}_{\mathbf{E}}}{\delta \mathbf{E}} = \mathbf{E},$$

so that

$$\begin{aligned}
\partial_t f(t,\mathbf{x},\mathbf{v}) &= \{f, \mathcal{H}_{\mathbf{E}}\}(t,\mathbf{x},\mathbf{v}) \\
&= \int f(t,\mathbf{y},\mathbf{w})\left(\nabla_v \delta(\mathbf{x}-\mathbf{y})\delta(\mathbf{v}-\mathbf{w})\cdot\mathbf{E}\right)\mathbf{dydw} \\
&= -\mathbf{E}(t,\mathbf{x})\cdot\nabla_v f(t,\mathbf{x},\mathbf{v}).
\end{aligned}$$

Similar calculations enable to derive the equations for $\mathbf{E}, \mathbf{B}, \mathbf{j}_C$ so that we obtain

$$\partial_t \mathbf{E} = 0, \partial_t \mathbf{B} = -\nabla \times \mathbf{E}, \partial_t \mathbf{j}_C = \Omega_{pe}^2 \mathbf{E}.$$

This system can be solved exactly in time.

3.2 Equations for $\mathcal{H}_{\mathbf{B}}$

We obtain

$$\partial_t f = 0, \partial_t \mathbf{E} = \nabla \times \mathbf{B}, \partial_t \mathbf{B} = 0, \partial_t \mathbf{j}_C = 0.$$

This system can be solved exactly in time.

3.3 Equations for $\mathcal{H}_{\mathbf{j}_C}$

We obtain

$$\partial_t f = 0, \partial_t \mathbf{E} = -\mathbf{j}_C, \partial_t \mathbf{B} = 0, \partial_t \mathbf{j}_C = \mathbf{j}_C \times \mathbf{B}_0.$$

This system can be solved exactly in time.

3.4 Equations for $\mathcal{H}_f$

We obtain

$$\partial_t f = -\mathbf{v}\cdot\nabla f - (\mathbf{v}\times(\mathbf{B}+\mathbf{B}_0))\cdot\nabla_{\mathbf{v}} f, \partial_t \mathbf{E} = -\int \mathbf{v} f d\mathbf{v}, \partial_t \mathbf{B} = 0, \partial_t \mathbf{j}_C = 0.$$

This system can *not* be solved exactly in time. However, we can split again the term $\mathcal{H}_f := \int |\mathbf{v}|^2 f d\mathbf{v}$ as follows

$$\mathcal{H}_f := \int |\mathbf{v}|^2 f d\mathbf{v} = \int v_1^2 f d\mathbf{v} + \int v_2^2 f d\mathbf{v} + \int v_3^2 f d\mathbf{v} =: \mathcal{H}_{f_1} + \mathcal{H}_{f_2} + \mathcal{H}_{f_3}.$$

Then, we get for $\mathcal{H}_{f_i}$ (denoting $(\mathbf{v}\times(\mathbf{B}+\mathbf{B}_0) := \hat{B}\mathbf{v}$, with $\hat{B}$ a matrix)

$$\partial_t f = -v_i \partial_{x_i} f - \hat{B}_{j,i} v_i \partial_{v_j} f, \partial_t \mathbf{E} = -\int v_i f d\mathbf{v}.$$

It turns out that this system can be solved exactly in time (see [3] for more details).

4 Reduced models and Hamiltonian splitting

$1dx - 3dv$ model In [2], they consider $\mathbf{B}_0(\mathbf{x}) = B_0 e_z$ and $\Omega_{pe}(\mathbf{x}) = const$ together with the following unknown

$$f(t,z,\mathbf{v}), E_x(t,z), E_y(t,z), B_x(t,z), B_y(t,z), j_{c,x}(t,z), j_{c,y}(t,z),$$

with $\mathbf{v} = (v_x, v_y, v_z)$ which satisfy

$$\frac{\partial j_{c,x}}{\partial t} = \Omega_{pe}^2 E_x + j_{c,y} B_0, \tag{13}$$

$$\frac{\partial j_{c,y}}{\partial t} = \Omega_{pe}^2 E_y - j_{c,x} B_0, \tag{14}$$

$$\frac{\partial f}{\partial t} + v_z \partial_z f + (E_x + v_y B_0 - v_z B_y)\,\partial_{v_x} f \tag{15}$$

$$+ (E_y - v_x B_0 + v_z B_x)\,\partial_{v_y} f + (v_x B_y - v_y B_x)\,\partial_{v_z} f = 0, \tag{16}$$

$$\frac{\partial B_x}{\partial t} = \partial_z E_y, \tag{17}$$

$$\frac{\partial B_y}{\partial t} = -\partial_z E_x, \tag{18}$$

$$\frac{\partial E_x}{\partial t} = -\partial_z B_y - j_{c,x} - \int v_x f \mathrm{d}\mathbf{v}, \tag{19}$$

$$\frac{\partial E_y}{\partial t} = \partial_z B_x - j_{c,y} - \int v_y f \mathrm{d}\mathbf{v}. \tag{20}$$

The Poisson structure reduces to (6) except that the integration is not done on the whole 3 dimensional spatial domain but only on $z \in \mathbb{R}$. All the vectorial operators have to be expressed in detail and one can hope that the Jacobi identity is satisfied. In [2], it is said that $\int v_z f \mathrm{d}\mathbf{v} = 0$ to ensure that there is no creation of E_z in the Ampere equation. This has to be understood.

In this case, the Hamiltonian splitting writes as

- $\mathcal{H}_{\mathbf{E}}$: $\partial_t(E_x, E_y) = 0, \partial_t f + E_x \partial_{v_x} f + E_y \partial_{v_y} f = 0, \partial_t B_x = \partial_z E_y, \partial_t B_y = -\partial_z E_x, \partial_t j_{c,x} = \Omega_{pe}^2 E_x, \partial_t j_{c,y} = \Omega_{pe}^2 E_y$.

- $\mathcal{H}_{\mathbf{B}}$: $\partial_t(f, B_x, B_y, j_{c,x}, j_{c,y}) = 0, \partial_t E_x = -\partial_z B_y, \partial_t E_y = \partial_z B_x$.

- $\mathcal{H}_{\mathbf{j}_C}$: $\partial_t(f, B_x, B_y) = 0, \partial_t E_x = -j_{c,x}, \partial_t E_y = -j_{c,y}, \partial_t j_{c,x} = j_{c,y} B_0, \partial_t j_{c,y} = -j_{c,x} B_0$.

- $\mathcal{H}_f$: $\partial_t(B_x, B_y) = 0$, $\partial_t f + v_z \partial_z f + (v_y B_0 - v_z B_y)\partial_{v_x} f + (-v_x B_0 + v_z B_x)\partial_{v_y} f + (v_x B_y - v_y B_x)\partial_{v_z} f = 0$, $\partial_t E_x = -\int v_x f \mathrm{d}\mathbf{v}$, $\partial_t E_y = -\int v_y f \mathrm{d}\mathbf{v}$

As explained above, the last part $\mathcal{H}_f$ ca not be solved exactly in time and should be split again into $\mathcal{H}_f = \mathcal{H}_{f,1} + \mathcal{H}_{f,2} + \mathcal{H}_{f,3}$. Then, it leads

- $\mathcal{H}_{f,1}$: $\partial_t(B_x, B_y, E_y) = 0$, $\partial_t f + \left(-v_x B_0 \partial_{v_y} f + B_y v_x \partial_{v_z} f\right) = 0$, $\partial_t E_x = -\int v_x f \mathrm{d}\mathbf{v}$.

- $\mathcal{H}_{f,2}$: $\partial_t(B_x, B_y, E_x) = 0$, $\partial_t f + (B_0 v_y \partial_{v_x} f - B_x v_y \partial_{v_z} f) = 0$, $\partial_t E_y = -\int v_y f \mathrm{d}\mathbf{v}$.

- $\mathcal{H}_{f,3}$: $\partial_t(B_x, B_y, E_x, E_y) = 0$, $\partial_t f + v_z \partial_z f + \left(-B_y v_z \partial_{v_x} f + v_z B_x \partial_{v_y} f\right) = 0$.

The first two parts $\mathcal{H}_{f,1}, \mathcal{H}_{f,2}$ can be solved exactly (using a simple directional splitting). The last one $\mathcal{H}_{f,3}$ deserves more attention (we follow the strategy from [3]).

First, we introduce a new function $g(t, z, \mathbf{v}) := f(t, z + tv_z, \mathbf{v})$ which satisfies

$$\partial_t g - B_y(z + tv_z) v_z \partial_{v_x} g + v_z B_x(z + tv_z) \partial_{v_y} g = 0. \tag{21}$$

We observe in the sequel that the characteristics can be solved exactly

$$\dot{v}_x = -B_y(z + tv_z) v_z, \quad \dot{v}_y = B_x(z + tv_z) v_z.$$

We recall that z, v_z and $B_{x,y}$ are constant in time, and expanding the magnetic field $B_{x,y}$ into Fourier series in the z variable leads to the following computation for the equation on v_x

$$\begin{aligned} v_x(t) &= v_x(0) - v_z(0) \int_0^t \sum_k \hat{B}_{k,y}(0) e^{ik(z(0) + tv_z(0))} ds \\ &= v_x(0) - v_z(0) \sum_k \hat{B}_{k,y}(0) e^{ikz(0)} \int_0^t e^{iktv_z(0)} ds \\ &= v_x(0) - \sum_k \hat{B}_{k,y}(0) \frac{1}{ik} e^{ikz(0)} (e^{iktv_z(0)} - 1), \end{aligned}$$

whereas for the equation on v_y, we have

$$\begin{aligned} v_y(t) &= v_y(0) + v_z(0) \int_0^t \sum_k \hat{B}_{k,x}(0) e^{ik(z(0) + tv_z(0))} ds \\ &= v_y(0) + \sum_k \hat{B}_{k,x}(0) \frac{1}{ik} e^{ikz(0)} (e^{iktv_z(0)} - 1). \end{aligned}$$

Then, the f dynamics of $\mathcal{H}_{f,3}$ is computed by first solving (21) using a directional splitting (which is exact here)

$$g(t,z,\mathbf{v}) = g\left(0, z, v_x + v_z\sum_k \hat{B}_{k,x}(0)\frac{1}{ik}e^{ikz}(e^{iktv_z}-1), v_y - v_z\sum_k \hat{B}_{k,x}(0)\frac{1}{ik}e^{ikz}(e^{iktv_z}-1), v_z\right),$$

and then by performing the inverse change of unknown

$$f(t,z,\mathbf{v}) = g(t, z - tv_z, \mathbf{v}).$$

Then, the $\mathcal{H}_{f,3}$ can be solved exactly in time using 3 one-dimensional linear advections.

Another (non Hamiltonian) splitting would be

- $\partial_t(E_x, E_y) = 0, \partial_t f + E_x\partial_{v_x} f + E_y\partial_{v_y} f = 0, \partial_t B_x = \partial_z E_y, \partial_t B_y = -\partial_z E_x, \partial_t j_{c,x} = \Omega_{pe}^2 E_x, \partial_t j_{c,y} = \Omega_{pe}^2 E_y$.
- $\partial_t(B_x, B_y, j_{c,x}, j_{c,y}) = 0, \partial_t E_x = -\partial_z B_y, \partial_t E_y = \partial_z B_x, \partial_t f + (v_y B_0 - v_z B_y)\,\partial_{v_x} f + (-v_x B_0 + v_z B_x)\,\partial_{v_y} f + (v_x B_y - v_y B_x)\,\partial_{v_z} f = 0$.
- $\partial_t(f, B_x, B_y) = 0, \partial_t E_x = -j_{c,x}, \partial_t E_y = -j_{c,y}, \partial_t j_{c,x} = j_{c,y} B_0, \partial_t j_{c,y} = -j_{c,x} B_0$.
- $\partial_t(B_x, B_y) = 0$, $\partial_t f + v_z\partial_z f = 0$, $\partial_t E_x = -\int v_x f \mathrm{d}\mathbf{v}$, $\partial_t E_y = -\int v_y f \mathrm{d}\mathbf{v}$

In this splitting, there are 4 stages, instead of 6 in the Hamiltonian one. A question would be: do we need the Hamiltonian splitting ? (since errors in space may break the geometric structure...)

5 To check, remarks

- check the calculations !! The Poisson structure is ok in the full case, one has to check how it works in the reduced case $1d - 3v$.
- check that Poisson equation and $\nabla \cdot \mathbf{B} = 0$ are satisfied through this splitting.
- this splitting enables to use Eulerian methods in space and velocity whereas in [2], PIC (and FEM) is used. We can use Fourier in space and velocity for example, or semi-Lagrangian methods. We can benefit from the test case described in [2] to test the two splittings (Hamiltonian and not Hamiltonian). The drawback of the second splitting comes from the fact that the Poisson bracket used for this splitting does not enjoy the Jacobi identity. Then the question is: is it really important in practice to satisfy the Jacobi identity ?
- Exponential integrators are also very good candidates to approximate this model.
- In [2], there is test which shows that geometric schemes present better behavior than standard ones. In this working document, we have a Hamiltonian splitting, an almost Hamiltonian splitting and a standard scheme (exponential) which represent good materials to explore this.

References

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