A review of mathematical topics in collisional kinetic theory

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This text has appeared with minor modifications in the *Handbook of Mathematical Fluid Dynamics* (Vol. 1), edited by S. Friedlander and D. Serre, published by Elsevier Science (2002).

I have performed some last-minute addenda and corrections to take into account very recent advances; they are put as footnotes within the body of the text.

Some other corrections have been performed after publication and are incorporated within the text, without modification of the numbering of formulas, subsections or footnotes. These modifications are listed on the Errata sheet for this text, which can be downloaded from

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INTRODUCTION

The goal of this review paper is to provide the reader with a concise introduction to the mathematical theory of collision processes in (dilute) gases and plasmas, viewed as a branch of kinetic theory.

The study of collisional kinetic equations is only part of the huge field of nonequilibrium statistical physics. Among other things, it is famous for historical reasons, since it is in this setting that Boltzmann proved his celebrated theorem about entropy. As of this date, the mathematical theory of collisional kinetic equations cannot be considered to be in a mature state, but it has undergone spectacular progress in the last decades, and still more is to be expected.

I have made the following choices for presentation :

1) The emphasis is definitely on the mathematics rather than on the physics, the modelling or the numerical simulation. About these topics the survey by Carlo Cercignani will say much more. On the other hand, I shall always be concerned with the physical relevance of mathematical results.

2) Most of the presentation is limited to a small number of widely known, mathematically famous models which can be considered as archetypes — mainly, variants of the Boltzmann equation. This is not only for the sake of mathematics : also in modelling do these equations play a major role.

3) Two important interface fields are hardly discussed : one is the transition from particle systems to kinetic equations, and the other one is the transition from kinetic equations to hydrodynamics. For both problematics I shall only give basic considerations and adequate references.

4) Not all mathematical theories of kinetic equations (there are many of them !) are "equally" represented : for instance, fully nonlinear theories occupy much more space than perturbative approaches, and the Boltzmann equation without cut-off is discussed in about the same detail than the Boltzmann equation with cut-off (although the literature devoted to the latter case is considerably more extended). This partly reflects the respective vivacity of the various branches, but also, unavoidably, personal tastes and areas of competence. I apologize for this !

5) I have sought to give more importance to mathematical methods and ideas, than to results. This is why I have chosen a "transversal" presentation : for each problem, corresponding tools and ideas are first explained, then the various results obtained by their use are carefully described in their respective framework. As a typical example, and unlike most textbooks, this review does not treat spatially homogeneous and spatially inhomogeneous theories separately, but insists on tools which apply to both frameworks.

6) At first I have tried to give extensive lists of references, but soon realized that it was too ambitious...

The plan of the survey is as follows.

First, a presentation chapter discusses models for collisional kinetic theory and introduces the reader to the various mathematical problems which arise in their study. A central position is given to the Boltzmann equation and its variants.

Chapter 2 bears on the Cauchy problem for the Boltzmann equation and variants. The main questions here are propagation of regularity and singularities, regularization effects, decay and strict positivity of solutions. The influence of the Boltzmann collision kernel (satisfying Grad's angular cut-off or not) is discussed with care.

Chapter 3 considers the trend to equilibrium, insisting on constructive approaches. Boltzmann's H theorem and entropy dissipation methods have a central role here.

The shorter, but important chapter 4 studies in detail the case of so-called Maxwell collision kernels, and several links between the theory of the Boltzmann equation and information theory. The ideas in this chapter crucially lie behind some of the most notable results in chapter 3, even though, strictly speaking, these two chapters are to a large extent independent.

Finally, chapter 5 discusses selected open problems and promising new trends in the field.

Apart from the numerous references quoted in the text, the reader may find useful the short bibliographical notes which are included before the bibliography, to help orientate through the huge literature on the subject.

Let me add one final word about conventions: it is quite customary in kinetic theory (just as in the field of hyperbolic systems of conservation laws) to use the vocable "entropy" for Boltzmann's H functional; however the latter should rather be considered as the negative of an entropy, or as a "quantity of information". In the present review I have followed the custom of calling H an entropy, however I now regret this choice and recommend to call it an information (or just the H functional); accordingly the "entropy dissipation functional" should rather be called "entropy production functional" or "dissipation of information" (which is both closer to physical intuition and maybe more appealing).

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C.V.

CHAPTER 1

GENERAL PRESENTATION

The goal of this chapter is to introduce, and make a preliminary discussion of, the mathematical models and problems which will be studied in more detail thereafter. The first section addresses only physical issues, starting from scratch. We begin with an introduction to kinetic theory, then to basic models for collisions.

Then in section 2, we start describing the mathematical problems which arise in collisional kinetic theory, restricting the discussion to the ones that seem to us most fundamental. Particular emphasis is laid on the Boltzmann equation. Each paragraph contains at least one major problem which has not been solved satisfactorily.

Next, a specific section is devoted to the classification of collision kernels in the Boltzmann collision operator. The variety of collision kernels reflects the variety of possible interactions. Collision kernels have a lot of influence on qualitative properties of the Boltzmann equation, as we explain.

In the last two sections, we first present some basic general tools and considerations about the Boltzmann operator, then give an overview of existing mathematical theories for collisional kinetic theory.

Contents

1. Models for collisions in kinetic theory

1.1. Distribution function. The object of kinetic theory is the modelling of a gas (or plasma, or any system made up of a large number of particles) by a distribution function in the particle phase space. This phase space includes macroscopic variables, i.e. the position in physical space, but also microscopic variables, which describe the "state" of the particles. In the present survey, we shall restrict ourselves, most of the time, to systems made of a single species of particles (no mixtures), and which obey the laws of classical mechanics (non-relativistic, non-quantum). Thus the microscopic variables will be nothing but the velocity components. Extra microscopic variables should be added if one would want to take into account non-translational degrees of freedom of the particles : internal energy, spin variables, etc.

Assume that the gas is contained in a (bounded or unbounded) domain $X \subset \mathbb{R}^N$ ($N = 3$ in applications) and observed on a time interval $[0,T]$, or $[0,+\infty)$. Then, under the above simplifying assumptions, the corresponding kinetic model is a nonnegative function $f(t,x,v)$, defined on $[0,T] \times X \times \mathbb{R}^N$. Here the space $\mathbb{R}^N = \mathbb{R}^N_v$ is the space of possible velocities, and should be thought of as the tangent space to X. For any fixed time t, the quantity $f(t,x,v)\,dx\,dv$ stands for the density of particles in the volume element $dx\,dv$ centered at (x,v). Therefore, the minimal assumption that one can make on f is that for all $t \geq 0$,

$$f(t,\cdot,\cdot) \in L^1_{\rm loc}(X; L^1(\mathbb{R}^N_v));$$

or at least that $f(t,\cdot,\cdot)$ is a bounded measure on $K \times \mathbb{R}^N_v$, for any compact set $K \subset X$. This assumption means that a bounded domain in physical space contains only a finite amount of matter.

Underlying kinetic theory is the modelling assumption that the gas is made of so many particles that it can be treated as a continuum. In fact there are two slightly different ways to consider f : it can be seen as an approximation of the true density of the gas in phase space (on a scale which is much larger than the typical distance between particles), or it can reflect our lack of knowledge of the true positions of particles. Which interpretation is made has no consequence in practice[1].

The kinetic approach goes back as far as Bernoulli and Clausius; in fact it was introduced long before experimental proof of the existence of atoms. The first true bases for kinetic theory were laid down by Maxwell [**335, 337, 336**]. One of the main ideas in the model is that all measurable macroscopic quantities ("observables") can be expressed in terms of microscopic averages, in our case integrals of the form $\int f(t,x,v)\,\varphi(v)\,dv$. In particular (in adimensional form), at a given point x and a

[1]For instance, assume that the microscopic description of the gas is given by a cloud of n points $x_1,\dots,x_n$ in $\mathbb{R}^N_x$, with velocities $v_1,\dots,v_n$ in $\mathbb{R}^N_v$. A microscopic configuration is an element $(x_1,v_1,\dots,x_n,v_n)$ of $(\mathbb{R}^N_x \times \mathbb{R}^N_v)^n$. The "density" of the gas in this configuration is the empirical measure $(1/n)\sum_{i=1}^n \delta_{(x_i,v_i)}$; it is a probability measure on $\mathbb{R}^N_x \times \mathbb{R}^N_v$. In the first interpretation, $f(x,v)\,dx\,dv$ is an approximation of the empirical measure. In the second one, there is a symmetric probability density f^n on the space $(\mathbb{R}^N_x \times \mathbb{R}^N_v)^n$ of all microscopic configurations, and f is an approximation of the one-particle marginal

$$P_1 f^n(x_1,v_1) = \int f^n(x_1,v_1,\dots,x_n,v_n)\,dx_2\,dv_2\,\dots\,dx_n\,dv_n.$$

Thus the first interpretation is purely deterministic, while the second one is probabilistic. It is the second interpretation which was implicitly used by Boltzmann, and which is needed by Landford's validation theorem, see paragraph 2.1.

given time t, one can define the local density ρ, the local macroscopic velocity u, and the local temperature T, by

$$\rho = \int_{\mathbb{R}^N} f(t,x,v)\,dv, \qquad \rho u = \int_{\mathbb{R}^N} f(t,x,v) v\,dv, \tag{1}$$

$$\rho|u|^2 + N\rho T = \int_{\mathbb{R}^N} f(t,x,v)|v|^2\,dv.$$

For much more on the subject, we refer to the standard treatises of Chapman and Cowling [**154**], Landau and Lipschitz [**304**], Grad [**250**], Kogan [**289**], Uhlenbeck and Ford [**433**], Truesdell and Muncaster [**430**], Cercignani and co-authors [**141, 148, 149**].

1.2. Transport operator. Let us continue to stick to a classical description, and neglect for the moment the interaction between particles. Then, according to Newton's principle, each particle travels at constant velocity, along a straight line, and the density is constant along characteristic lines $dx/dt = v$, $dv/dt = 0$. Thus it is easy to compute f at time t in terms of f at time 0 :

$$f(t,x,v) = f(0, x - vt, v).$$

In other words, f is a weak solution to the equation of **free transport**,

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = 0. \tag{2}$$

The operator $v \cdot \nabla_x$ is the (classical) transport operator. Its mathematical properties are much subtler than it would seem at first sight; we shall discuss this later. Complemented with suitable boundary conditions, equation (2) is the right equation for describing a gas of noninteracting particles. Many variants are possible; for instance, v should be replaced by $v/\sqrt{1+|v|^2}$ in the relativistic case.

Of course, when there is a macroscopic force $F(x)$ acting on particles, then the equation has to be corrected accordingly, since the trajectories of particles are not straight lines any longer. The relevant equation would read

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + F(x) \cdot \nabla_v f = 0 \tag{3}$$

and is sometimes called the linear Vlasov equation.

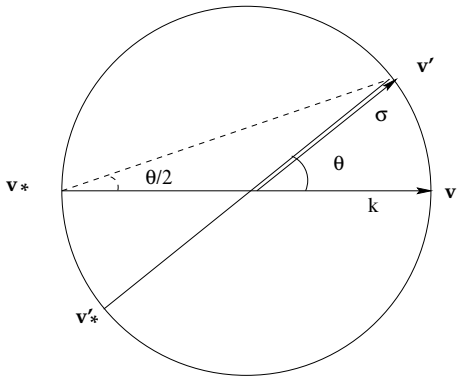
1.3. Boltzmann's collision operator. We now want to take into account interactions between particles. We shall make several postulates.

1) We assume that particles interact via **binary collisions** : this is a vague term describing the process in which two particles happen to come very close to each other, so that their respective trajectories are strongly deviated in a very short time. Underlying this hypothesis is an implicit assumption that the gas is **dilute enough** that the effect of interactions involving more than two particles can be neglected. Typically, if we deal with a three-dimensional gas of n hard spheres of radius r, this would mean

$$nr^3 \ll 1, \qquad nr^2 \simeq 1.$$

2) Moreover, we assume that these collisions are **localized** both in space and time, i.e. they are brief events which occur at a given position x and a given time t. This means that the typical duration of a collision is very small compared to the

FIGURE 1. A binary elastic collision



typical time scale of the description, and also quantities such as the impact parameter (see below) are negligible in front of the typical space scale (say, a space scale on which variations due to the transport operator are of order unity).

3) Next, we further assume these collisions to be **elastic** : momentum and kinetic energy are preserved in a collision process. Let v', v'_* stand for the velocities before collision, and v, v_* stand for the velocities after collision : thus

$$\begin{cases} v' + v'_* = v + v_* \\ \\ |v'|^2 + |v'_*|^2 = |v|^2 + |v_*|^2. \end{cases} \tag{4}$$

Since this is a system of $N+1$ scalar equations for $2N$ scalar unknowns, it is natural to expect that its solutions can be defined in terms of $N-1$ parameters. Here is a convenient representation of all these solutions, which we shall sometimes call the **σ-representation** :

$$\begin{cases} v' = \dfrac{v + v_*}{2} + \dfrac{|v - v_*|}{2}\sigma \\ \\ v'_* = \dfrac{v + v_*}{2} - \dfrac{|v - v_*|}{2}\sigma. \end{cases} \tag{5}$$

Here the parameter $\sigma \in S^{N-1}$ varies in the $N-1$ unit sphere. Fig. 1 pictures a collision in the velocity phase space. The **deviation angle** θ is the angle between pre- and post-collisional velocities.

Very often, particles will be assumed to interact via a given **interaction potential** $\phi(r)$ (r = distance between particles); then v' and v'_* should be computed as the result of a classical scattering problem, knowing v, v_* and the *impact parameter* between the two colliding particles. We recall that the impact parameter is what would be the distance of closest approach if the two particles did not interact.

4) We also assume collisions to be **microreversible**. This word can be understood in a purely deterministic way : microscopic dynamics are time-reversible; or in a probabilistic way : the probability that velocities (v', v'_*) are changed into (v, v_*) in a collision process, is the same as the probability that (v, v_*) are changed into (v', v'_*).

5) And finally, we make the Boltzmann **chaos** assumption : the velocities of two particles which are about to collide are uncorrelated. Roughly speaking, this means that if we randomly pick up two particles at position x, which have not collided yet, then the joint distribution of their velocities will be given by a tensor product (in velocity space) of f with itself. Note that *this assumption implies an asymmetry between past and future* : indeed, in general if the pre-collisional velocities are uncorrelated, then post-collisional velocities have to be correlated[2] !

Under these five assumptions, in 1872 Boltzmann (Cf. [**93**]) was able to derive a **quadratic collision operator** which accurately models the effect of interactions on the distribution function f :

$$\left.\frac{\partial f}{\partial t}\right|_{\rm coll}(t,x,v) = Q(f,f)\,(t,x,v) \tag{6}$$

$$= \int_{\mathbb{R}^N} dv_* \int_{S^{N-1}} d\sigma\, B(v-v_*,\sigma)\Big(f'f'_* - ff_*\Big). \tag{7}$$

Here we have used standard abbreviations : $f' = f(t,x,v')$, $f_* = f(t,x,v_*)$, $f'_* = f(t,x,v'_*)$. Moreover, the nonnegative function $B(z,\sigma)$, called the **Boltzmann collision kernel**, depends only on $|z|$ and on the scalar product $\langle \frac{z}{|z|},\sigma\rangle$. Heuristically, it can be seen as a probability measure on all the possible choices of the parameter $\sigma \in S^{N-1}$, as a function of the relative velocity $z = v - v_*$. But truly speaking, this interpretation is in general false, because B is not integrable... The Boltzmann collision kernel is related to the **cross-section** Σ by the identity

$$B(z,\sigma) = |z|\Sigma(z,\sigma).$$

By abuse of language, B is often called the cross-section.

Let us explain (7) a little bit. This operator can formally be split, in a self-evident way, into a **gain** and a **loss** term,

$$Q(f,f) = Q^+(f,f) - Q^-(f,f).$$

The loss term "counts" all collisions in which a given particle of velocity v will encounter another particle, of velocity v_*. As a result of such a collision, this particle will in general change its velocity, and this will make less particles with velocity v. On the other hand, each time particles collide with respective velocities v' and v'_*, then the v' particle may acquire v as new velocity after the collision, and this will make more particles with velocity v : this is the meaning of the gain term.

It is easy to trace back our modelling assumptions : 1) the quadratic nature of this operator is due to the fact that only binary collisions are taken into account; 2) the fact that the variables t, x appear only as parameters reflects the assumption that collisions are localized in space and time; 3) the assumption of elastic collisions results in the formulas giving v' and v'_*; 4) the microreversibility implies the particular structure of the collision kernel B; 5) finally, the appearance of the tensor products[3] $f'f'_*$ and ff_* is a consequence of the chaos assumption.

On the whole, the **Boltzmann equation** reads

$$\frac{\partial f}{\partial t} + v\cdot\nabla_x f = Q(f,f), \qquad t \geq 0,\ x\in\mathbb{R}^N,\ v\in\mathbb{R}^N; \tag{8}$$

[2]See for instance the discussion in section 2.4.

[3]in the sense that $ff_* = (f\otimes f)(v,v_*)$, $f'f'_* = (f\otimes f)(v',v'_*)$.

or, when a macroscopic force $F(x)$ is also present,

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + F(x) \cdot \nabla_v f = Q(f,f), \qquad t \geq 0,\ x \in \mathbb{R}^N,\ v \in \mathbb{R}^N; \tag{9}$$

The deepest physical and mathematical properties of the Boltzmann equation are linked to the subtle interaction between the linear transport operator and the non-linear collision operator.

We note that this equation was written in weak formulation by Maxwell as early as 1866 : as recalled in paragraph 2.3 below, Maxwell [**335, 337**] wrote down the equation satisfied by the observables $\int f(t,x,v)\varphi(v)\,dv$ (see in particular [**335**, eq. 3]). However Boltzmann did a considerable job on the interpretation and consequences of this equation, and also made them widely known in his famous treatise [**93**], which was to have a lot of influence on theoretical physics during several decades.

Let us make a few comments about the modelling assumptions. They may look rather crude, however they can in large part be completely justified, at least in certain particular cases. By far the deepest assumption is Boltzmann's chaos hypothesis ("Stosszahlansatz") which is intimately linked to the questions of irreversibility of macroscopic dynamics and the arrow of time. For a discussion of these subtle topics, the reader may consult the review paper by Lebowitz [**293**], or the enlightening treatise by Kac [**284**], as well as the textbooks [**149, 410**] for the most technical aspects. We shall say just a few words about the subject in paragraphs 2.1 and 2.4.

1.4. Collision kernels. Since the collision kernel (or equivalently the cross-section) only depends on $|v - v_*|$ and on $\langle \frac{v-v_*}{|v-v_*|}, \sigma \rangle$, i.e. the cosine of the deviation angle, throughout the whole text we shall abuse notations by writing

$$B(v - v_*, \sigma) = B\big(|v - v_*|, \cos\theta\big), \qquad \cos\theta = \left\langle \frac{v - v_*}{|v - v_*|}, \sigma \right\rangle. \tag{10}$$

Maxwell [**335**] has shown how the collision kernel should be computed in terms of the interaction potential ϕ. In short, here are his formulas (in three dimensions of space), as they can be found for instance in Cercignani [**141**], for a repulsive potential. For given impact parameter $p \geq 0$ and relative velocity $z \in \mathbb{R}^3$, let the deviation angle θ be

$$\theta(p,z) = \pi - 2p \int_{s_0}^{+\infty} \frac{ds/s^2}{\sqrt{1 - \frac{p^2}{s^2} - 4\frac{\phi(s)}{|z|^2}}} = \pi - 2 \int_0^{\frac{p}{s_0}} \frac{du}{\sqrt{1 - u^2 - \frac{4}{|z|^2}\phi\left(\frac{p}{u}\right)}},$$

where s_0 is the positive root of

$$1 - \frac{p^2}{s_0^2} - 4\frac{\phi(s_0)}{|z|^2} = 0.$$

Then the collision kernel B is implicitly defined by

$$B\left(|z|, \cos\theta\right) = \frac{p}{\sin\theta}\frac{dp}{d\theta}|z|. \tag{11}$$

It can be made explicit in two cases :

- hard spheres, i.e. particles which collide bounce on each other like billiard balls : in this case, in dimension 3, $B(|v-v_*|,\cos\theta)$ is just proportional to $|v-v_*|$ (the cross-section is constant);
- Coulomb interaction, $\phi(r) = 1/r$ (in adimensional variables and in three dimensions of space) : then B is given by the famous Rutherford formula,

$$B\big(|v-v_*|,\cos\theta\big) = \frac{1}{|v-v_*|^3\sin^4(\theta/2)}. \tag{12}$$

A dimensional factor of $(e^2/4\pi\epsilon_0 m)$ should multiply this kernel (e = charge of the particle, ϵ_0 = permittivity of the vacuum, m = mass of the particle). Unfortunately, Coulomb interactions cannot be modelled by a Boltzmann collision operator; we shall come back to this soon.

In the important[4] model case of inverse-power law potentials,

$$\phi(r) = \frac{1}{r^{s-1}}, \qquad s > 2,$$

then the collision kernel cannot be computed explicitly, but one can show that

$$B\big(|v-v_*|,\cos\theta\big) = b(\cos\theta)|v-v_*|^\gamma, \qquad \gamma = \frac{s-(2N-1)}{s-1}. \tag{13}$$

In particular, in three dimensions of space, $\gamma = (s-5)/(s-1)$. As for the function b, it is only implicitly defined, locally smooth, and has a *nonintegrable singularity* for $\theta \to 0$:

$$\sin^{N-2}\theta\, b(\cos\theta) \sim K\theta^{-1-\nu}, \qquad \nu = \frac{2}{s-1} \qquad (N=3). \tag{14}$$

Here we have put the factor $\sin^{N-2}\theta$ because it is (up to a constant depending only on the dimension) the Jacobian determinant of spherical coordinates on the sphere S^{N-1}.

The nonintegrable singularity in the "angular collision kernel" b is an effect of the huge amount of **grazing** collisions, i.e. collisions with a very large impact parameter, so that colliding particles are hardly deviated. This is not a consequence of the assumption of inverse-power forces; in fact a nonintegrable singularity appears as soon as the forces are of infinite range, no matter how fast they decay at infinity. To see this, note that, according to (11),

$$\int_0^\pi B\big(|z|,\cos\theta\big)\sin\theta\, d\theta = |z|\int_0^\pi p\frac{dp}{d\theta}\, d\theta = |z|\int_0^{p_{\max}} p\, dp = \frac{|z|\, p_{\max}^2}{2}. \tag{15}$$

By the way, it seems strange to allow infinite-range forces, while we assumed interactions to be localized. This problem has never been discussed very clearly, but in principle there is no contradiction in assuming the range of the interaction to be infinite at a microscopic scale, and negligible at a macroscopic scale. The fact that the linear Boltzmann equation can be rigorously derived from some particle dynamics with infinite range [**179**] also supports this point of view.

As one sees from formula (13), there is a particular case in which the collision kernel does not depend on the relative velocity, but only on the deviation angle :

[4] Inverse power laws are moderately realistic, but very important in physics and in modelling, because they are simple, often lead to semi-explicit results, and constitute a one-parameter family which can model very different phenomena. Van der Waals interactions typically correspond to $s = 7$, ion-neutral interactions to $s = 5$, Manev interactions [**88, 279**] to $s = 3$, Coulomb interactions to $s = 2$.

particles interacting via a inverse $(2N-1)$-power force ($1/r^5$ in three dimensions). Such particles are called **Maxwellian molecules**. They should be considered as a theoretical model, even if the interaction between a charged ion and a neutral particle in a plasma may be modelled by such a law (see for instance [**164**, t.1, p. 149]). However, Maxwell and Boltzmann used this model a lot[5], because they had noticed that it could lead to many explicit calculations which, so did they believe, were in agreement with physical observations. Also they believed that the choice of molecular interaction was not so important, and that Maxwellian molecules would behave pretty much the same as hard spheres[6].

Since the time of Maxwell and Boltzmann, the need for results or computations has led generations of mathematicians and physicists to work with more or less artificial variants of the collision kernels given by physics. Such a procedure can also be justified by the fact that for many interesting interactions, the collision kernel is not explicit at all : for instance, in the case of the Debye potential, $\phi(r) = e^{-r}/r$. Here are two categories of artificial collision kernels :

- when one tames the singularity for grazing collisions and replaces the collision kernel by a locally integrable one, one speaks of **cut-off collision kernel**;

- collision kernels of the form $|v - v_*|^\gamma$ $(\gamma > 0)$ are called variable hard spheres collision kernels.

It is a common belief among physicists that the properties of the Boltzmann equation are quite a bit sensitive to the dependence of B upon the relative velocity, but very little to its dependence upon the deviation angle. True as it may be for the behavior of macroscopic quantities, this creed is definitely wrong at the microscopic level, as we shall see.

In all the sequel, we shall consider general collision kernels $B(|v - v_*|, \cos\theta)$, in arbitrary dimension N, and make various assumptions on the form of B without always caring if it corresponds to a true interaction between particles (i.e., if there is a ϕ whose associated collision kernel is B). Our goal, in a lot of situations, will be to understand how the collision kernel affects the properties of the Boltzmann equation. However, we shall always keep in mind the collision kernels given by physics, in dimension three, to judge how satisfactory a mathematical result is.

1.5. Boundary conditions. Of course the Boltzmann equation has to be supplemented with boundary conditions which model the interaction between the particles and the frontiers of our domain $X \subset \mathbb{R}^N$ (wall, etc.)

The most natural boundary condition is the **specular reflection** :

$$f(x, R_x v) = f(x, v), \qquad R_x v = v - 2\big(v \cdot n(x)\big)n(x), \qquad x \in \partial X, \tag{16}$$

where $n(x)$ stands for the outward unit normal vector at x. In the context of optics, this condition would be called the Snell-Descartes law : particles bounce back on the wall with an postcollision angle equal to the precollision angle.

However, as soon as one is interested in realistic modelling for practical problems, equation (16) is too rough... In fact, a good boundary condition would have to take into account the fine details of the gas-surface interaction, and this is in general a

[5] See Boltzmann [**93**, chapter 3].

[6] Further recall that at the time, the "atomic hypothesis" was considered by many to be a superfluous complication.

very delicate problem[7]. There are a number of models, cooked up from modelling assumptions or phenomenological a priori constraints. As good source for these topics, the reader may consult the books by Cercignani [**141, 148**] and the references therein. In particular, the author explains the relevant conditions that a scattering kernel K has to satisfy for the boundary condition

$$f(x, v_{\text{out}}) = \int K(v_{in}, v_{out}) f(x, v_{in})\, dv_{in}$$

to be physically plausible. Here we only list a few common examples.

One is the **bounce-back condition**,

$$f(x, -v) = f(x, v) \qquad x \in \partial X. \tag{17}$$

This condition simply means that particles arriving with a certain velocity on the wall will bounce back with an opposite velocity. Of course it is not very realistic, however in some situations (see for instance [**148**, p. 41]) it leads to more relevant conclusions than the specular reflection, because it allows for some transfer of tangential momentum during collisions.

Another common boundary condition is the **Maxwellian diffusion**,

$$f(x, v) = \rho_-(x) M_w(v), \qquad v \cdot n(x) > 0 \tag{18}$$

where $\rho_-(x) = \int_{v\cdot n<0} f(x, v)\, |v \cdot n|\, dv$ and M_w is a particular gaussian distribution depending only on the wall,

$$M_w(v) = \frac{e^{-\frac{|v|^2}{2T_w}}}{(2\pi)^{\frac{N-1}{2}} T_w^{\frac{N+1}{2}}}.$$

In this model, particles are absorbed by the wall and then re-emitted according to the distribution M_w, corresponding to a thermodynamical equilibrium between particles and the wall.

Finally, one can combine the above models. Already Maxwell had understood that a convex combination of (16) and (18) would certainly be more realistic than just one of these two equations. No need to say, since the work of Maxwell, much more complicated models have appeared, for instance the Cercignani-Lampis (CL) model, see [**148**].

In mathematical discussions, we shall not consider the problem of boundary conditions except for the most simple case, which is specular reflection. In fact, most of the time we shall simply avoid this problem by assuming the position space to be the whole of $\mathbb{R}^N$, or the torus $\mathbb{T}^N$. Of course the torus is a mathematical simplification, but it is also used by physicists and by numerical analysts who want to avoid taking boundary conditions into account...

[7]Here we assume that the fine details of the surface of the wall are invisible at the scale of the spatial variable, so that the wall is modelled as a smooth surface, but we wish to take these details into account to predict velocities after collision with the wall. Another possibility is to assume that the roughness of the wall results in irregularities which can be seen at spatial resolution. Then it is natural in many occasions to assume ∂X to be very irregular. For some mathematical works about this alternative approach, see [**33, 272, 273**].

1.6. Variants of the Boltzmann equation. There are many variants. Let us only mention

- **relativistic models**, see [**50, 158, 199, 200, 233, 234, 235, 14**];
- **quantum models**, see [**196, 328, 209**] and the references therein. They will be discussed in section 3 of chapter 5. Also we should mention that models of quantum Boltzmann equation have recently gained a lot of interest in the study of semi-conductors, see in particular [**65, 66**] and the works by Poupaud and coworkers on related models [**386, 387, 388, 244, 360**], also [**354, 355**] (in two dimensions) and [**15**] (in three dimensions);
- **linear models**, in particular the linear Boltzmann equation,

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \int_{\mathbb{R}^N \times S^{N-1}} dv_* \, d\sigma \, \widetilde{B}(v - v_*, \sigma) \big[F(v'_*) f(v') - F(v_*) f(v) \big],$$

  where F is a given probability distribution. As a general rule, such linear equations model the influence of the environment, or background, on a test-particle (think of a particle in an environment of random scatterers, like in a random pinball game). The distribution F is the distribution of the background, and is usually assumed to be stationary, which means that the environment is in statistical equilibrium. Linear Boltzmann-like models are used in all areas of physics, most notably in quantum scattering [**206**] and in the study of transport phenomena associated with neutrons or photons [**148**, p. 165–172]. A general mathematical introduction to linear transport equations can be found in [**157**, chapter 21].
- **diffusive models**, like the Fokker-Planck equation, which is often used in its linear version,

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \nabla_v \cdot (\nabla_v f + f v), \tag{19}$$

  or in its nonlinear form,

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \rho^\alpha \nabla_v \cdot \big[T \nabla_v f + f(v - u) \big], \tag{20}$$

  where $\alpha \in [0, 1]$, and ρ, u, T are the local density, velocity and temperature defined by (1). When $\alpha = 1$, this model has the same quadratic homogeneity as the Boltzmann equation. Of course it is also possible to couple the equation only via T and not u, etc. A classical discussion on the use of the Fokker-Planck equation in physics can be found in the important review paper by Chandrasekhar [**153**].

  The Landau equation, which is described in detail in the next paragraph, is another diffusive variant of the Boltzmann equation;
- **energy-dissipating models**, describing inelastic collisions. These models are particularly important in the theory of granular materials : see section 2 of chapter 5;
- **model equations**, like the (simplistic) **BGK model**, see for instance [**141, 148**]. In this model one replaces the complicated Boltzmann operator by $M^f - f$, where M^f is the Maxwellian distribution[8] with the same local

[8] see paragraph 2.5 below.

density, velocity and temperature than f. Also variants are possible, for instance multiplying this operator by the local density ρ...

Another very popular model equation for mathematicians is the **Kac model** [**283**]. It is a one-dimensional caricature of the Boltzmann equation which retains some of its interesting features. The unknown is a time-dependent probability measure f on $\mathbb{R}$, and the equation reads

$$\frac{\partial f}{\partial t} = \frac{1}{2\pi}\left(\int_{\mathbb{R}} dv_* \int_0^{2\pi} d\theta\, f' f'_*\right) - f, \tag{21}$$

where (v', v'_*) is obtained from (v, v_*) by a rotation of angle θ in the plane $\mathbb{R}^2$. This model preserves mass and kinetic energy, but not momentum;

- **discrete-velocity models** : these are approximations of the Boltzmann equation where particles are only allowed a finite number of velocities. They are used in numerical analysis, but their mathematical study is (or once was) a popular topic. About them we shall say nothing; references can be found for instance in [**230, 141, 148**] and also in the survey papers [**384, 62**]. Among a large number of works, we only mention the original contributions by Bony [**94, 95**], Tartar [**416, 417**], and the consistency result by Bobylev et al [**367**], for the interesting number-theoretical issues that this paper has to deal with.

  We note [**148**, p. 265] that discrete-velocity models were once believed by physicists to provide miraculously efficient numerical codes for simulation of hydrodynamics. But these hopes have not been materialized...

  Also some completely unrealistic discrete-velocity equations have been studied as simplified mathematical models, without caring whether they would approximate or not the true Boltzmann equation. These models sometimes have no more than two or three velocities ! Some well-known examples are the Carleman equation, with two velocities, the Broadwell model, with four velocities in the plane, or the Cabannes equation with fourteen velocities. For instance the Carleman equation reads

$$\begin{cases} \dfrac{\partial f_1}{\partial t} + \dfrac{\partial f_1}{\partial x} = f_{-1}^2 - f_1^2, \\ \\ \dfrac{\partial f_{-1}}{\partial t} - \dfrac{\partial f_{-1}}{\partial x} = f_1^2 - f_{-1}^2. \end{cases} \tag{22}$$

  For a study of these models see [**230, 384, 141, 148, 60, 61, 108, 25**] and references included. Of course, these models are so oversimplified[9] that they cannot be considered seriously from the physical point of view, even if one may expect that they keep some relevant features of the Boltzmann equation. By the way, as noticed by Uchiyama [**432**], the Broadwell model *cannot* be derived from a fictitious system of deterministic four-velocity particles ("diamonds") in the plane, see [**149**, Appendix 4C]. Only at the price of some extra stochasticity assumption can the derivation be fixed [**117**].

All the abovementioned models should be considered with appropriate boundary conditions. These conditions can also be replaced by the effect of a **confinement**

[9] As a word of caution, we should add that even if they are so simplified, their mathematical analysis is not trivial at all, and many problems in the field still remain open.

potential $V(x)$: this means that there is a macroscopic force of the form $F(x) = -\nabla V(x)$ acting on the system.

Finally, it is important to note that Boltzmann's model is obtained as a result of the assumption of localized interaction; in particular, it does not take into account a possible interaction of long (macroscopic) range which would result in a macroscopic mean-field force, typically

$$F(x) = -\nabla\Phi(x), \qquad \Phi(x) = \rho *_x \phi,$$

where ϕ is the interaction potential and ρ the local density. The modelling of the interaction by such a coupled force is called a **Vlasov** description.

When should one prefer a Vlasov, or a Boltzmann description ? A dimensional analysis by Bobylev and Illner [**88**] shows that for inverse-power forces like $1/r^s$, and under natural scaling assumptions,

- for $s > 3$, the Boltzmann term should prevail on the mean-field term;
- for $s < 3$, the Boltzmann term should be negligible in front of the mean-field term.

The separating case, $s = 3$, is the so-called Manev interaction [**88, 279**].

There are subtle questions here, which are not yet fully understood, even at a formal level. Also the uniqueness of the relevant scaling is not clear. From a physicist's point of view, however, it is generally accepted that a good description is obtained by adding up the effects of a mean-field term and those of a Boltzmann collision operator, with suitable dimensional coefficients.

Another way to take into account interactions on a macroscopically significant scale is to use a description à la Povzner. In this model (see for instance [**389**]), particles interact through **delocalized collisions**, so that the corresponding Boltzmann operator is integrated with respect to the position y of the test particle, and reads

$$\int_{\mathbb{R}^N_x} dy \int_{\mathbb{R}^N_v} dv_* \, \widetilde{B}\big(v - v_*, x - y\big)\Big[f(y, v'_*)f(x, v') - f(y, v_*)f(x, v)\Big]. \tag{23}$$

Note that the kernel $\widetilde{B}$ now depends on $x - y$. On the other hand, there is no collision parameter σ any more, the outgoing velocities being uniquely determined by the positions x, y and the ingoing velocities v', v'_*. It was shown by Cercignani [**140**] that this type of equations could be retrieved as the limit of a large stochastic system of "soft spheres".

A related model is the Enskog equation for dense gases, which has never been clearly justified. It resembles eq. (23) but the multiplicity of the integral is $2N - 1$ instead of $2N$, because there is no integration over the distance $|x - y|$. Mathematical studies of the Enskog model have been performed by Arkeryd, Arkeryd and Cercignani [**24, 27, 28**] — in particular, reference [**24**] provides well-posedness and regularity under extremely general assumptions (large data, arbitrary dimension) by a contraction method. See paragraph 2.1 in chapter 5 for an inelastic variant which is popular in the study of granular material.

The study of these models is interesting not only in itself, but also because numerical schemes always have to perform some delocalization to simulate the effect of collisions. This explains why the results in [**140**] are very much related to some of

the mathematical justifications for some numerical schemes, as performed in [**453, 396**].

1.7. Collisions in plasma physics. The importance and complexity of interaction processes in plasma physics justifies that we devote a special paragraph to this topic. A plasma, generally speaking, is a gas of (partially or totally) ionized particles. However, this term encompasses a huge variety of physical situations : the density of a plasma can be extremely low or extremely high, the pressures can vary considerably, and the proportion of ionized particles can also vary over several orders of magnitude. Nonelastic collisions, recombination processes may be very important. We do not at all try to make a precise description here, and refer to classical textbooks such as Balescu [**46**], Delcroix and Bers [**164**], the very nice survey by Decoster in [**160**], or the numerous references that can be found therein. All of these sources put a lot of emphasis on the kinetic point of view, but [**160**] and [**164**] are also very much concerned with fluid descriptions. A point which should be made now, is that the classical collisional kinetic theory of gases a priori applies when the density is low (we shall make this a little bit more precise later on) and when nonelastic processes can be neglected.

Even taking into account only elastic interactions, there are a number of processes going on in a plasma : Maxwell-type interactions between ions and neutral particles, Van der Waals forces between neutral particles, etc. However, the most important feature, both from the mathematical and the physical point of view, is the presence of **Coulomb interactions between charged particles**. The basic model for the evolution of the density of such particles is the **Vlasov-Poisson equation**

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + F(x) \cdot \nabla_v f = 0, \tag{24}$$

$$F = -\nabla V, \quad V = \frac{e^2}{4\pi\epsilon_0 r} *_x \rho, \qquad \rho(t,x) = \int f(t,x,v)\, dv.$$

For simplicity, here we have written this equation for only one species of particles, and also we have not included the effect of a magnetic field, which leads to the Vlasov-Maxwell system (see for instance [**191**]). We have kept the physical parameters $e =$ charge of the particle and $\epsilon_0 =$ permittivity of vacuum for the sake of a short discussion about scales.

Even though this is not the topic of this review paper, let us say just a few words on the Vlasov-Poisson equation. Its importance in plasma physics (including astrophysics) cannot be overestimated, and thousands of papers have been devoted to its study. We only refer to the aforementioned textbooks, together with the famous treatise by Landau and Lipschitz [**304**]. From the mathematical point of view, the basic questions of existence, uniqueness and (partial) regularity of the solution to the Vlasov equation have been solved at the end of the eighties, see in particular Pfaffelmoser [**382**], DiPerna and Lions [**191**], Lions and Perthame [**318**], Schaeffer [**402**]. Reviews can be found in Glassey [**233**] and Bouchut [**96**]. Stability, and (what is more interesting !) instability of several classes of equilibrium distributions to the Vlasov-Poisson equation have recently been the object of a lot of studies by Guo and Strauss [**264, 265, 266, 267, 268**]. Several important questions, however, have not been settled, like the derivation of the Vlasov-Poisson equation from particle systems (see Spohn [**410**] and Neunzert [**356**] for related topics) and the explanation of the famous and rather mysterious Landau damping effect.

There is no doubt that the Vlasov-Poisson equation is the correct equation to describe a classical plasma on a short time scale. However, when one wants to consider long periods of time, it is necessary to take into account collisions between particles. For this it is natural to introduce a Boltzmann collision operator in the right-hand side of (24). However, the Boltzmann equation for Coulomb interactions does **not** make sense ! Indeed, the collision integral would be infinite even for very smooth (or analytic) distribution functions[10]. This is due to the very slow decay of the Coulomb potential, and the resulting very strong angular singularity of the collision kernel given by Rutherford's formula (12). A standard remedy to this problem is to assume that there is a **screening** (due to the presence of two species of particles, for instance), so that the effective interaction potential between charged particles is not the Coulomb interaction, but the so-called **Debye potential**

$$\phi(r) = \frac{e^{-r/\lambda_D}}{4\pi\epsilon_0 r}. \tag{25}$$

Here λ_D is the *Debye length*, i.e. a typical screening distance. In the classical theory of plasmas,

$$\lambda_D = \sqrt{\frac{\epsilon_0 kT}{\rho e^2}},$$

where k is Boltzmann's constant, T is the temperature of the plasma (rigorously speaking, it should depend on x...) and ρ its mean density (same remark). The resulting collision kernel is no longer explicit, but at least makes sense, because the very strong angular singularity in Rutherford's formula (12) is tamed.

The replacement of Coulomb by Debye potential can be justified by half-rigorous, half-heuristic arguments (see the references already mentioned). However, in most of the cases of interest, the Debye length is *very large* with respect to the characteristic length r_0 for collisions, called the *Landau length* :

$$r_0 = \frac{e^2}{4\pi\epsilon_0 kT}.$$

More precisely, in so-called classical plasmas (those for which the classical kinetic description applies), one has

$$r_0 \ll \rho^{-1/3} \ll \lambda_D.$$

This means first that the Landau distance is very small with respect to other scales (so that collisions can be considered as localized), and secondly that the plasma is so dilute that the typical distance between particles is very small with respect to the screening distance, which is usually considered as the relevant space scale.

By a formal procedure, Landau [**291**] showed that, as the ratio $\Lambda \equiv 2\lambda_D/r_0 \to \infty$, the Boltzmann collision operator for Debye potential behaves as

$$\frac{\log\Lambda}{2\pi\Lambda}\, Q_L(f,f),$$

where Q_L is the so-called **Landau collision operator** :

$$Q_L(f,f) = \nabla_v \cdot \left(\int_{\mathbb{R}^3} dv_*\, a(v-v_*)[f_*(\nabla f) - f(\nabla f)_*] \right). \tag{26}$$

[10]More precisely, the natural definition of the collision operator would lead to the following nonsense : whatever f, $Q(f,f)$ is an element of $\{-\infty, 0, +\infty\}$, see [**450**, Annex I, Appendix A].

Here $a(z)$ is a symmetric (degenerate) nonnegative matrix, proportional to the orthogonal projection onto $z^\perp$:

$$a_{ij}(z) = \frac{L}{|z|}\left[\delta_{ij} - \frac{z_i z_j}{|z|^2}\right], \tag{27}$$

and L is a dimensional constant.

The resulting equation is the **Landau equation**. In adimensional units, it could be written as

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + F(x) \cdot \nabla_v f = Q_L(f, f). \tag{28}$$

Many mathematical and physical studies also consider the simplified case when only collisions are present, and the effect of the mean-field term is not present. However, the collision term, normally, should be considered only as a long-time correction to the mean-field term.

We now present a variant of the Landau equation, which appears when one replaces the function $L/|z|$ in (27) by $|z|^2$, so that

$$a_{ij}(z) = |z|^2 \delta_{ij} - z_i z_j. \tag{29}$$

This approximation is called "Maxwellian". It is not realistic from the physical point of view, but has become popular because it leads to simpler mathematical properties and useful tests for numerical simulations.

Under assumption (29), a number of algebraic simplifications arise in the Landau operator (26); since they are independent on the dimension, we present them in arbitrary dimension $N \geq 2$. Without loss of generality, choose an orthonormal basis of $\mathbb{R}^N$ in such a way that

$$\int_{\mathbb{R}^N} f\, dv = 1, \qquad \int_{\mathbb{R}^N} f v\, dv = 0, \qquad \int_{\mathbb{R}^N} f|v|^2\, dv = N$$

(unit mass, zero mean velocity, unit temperature), and assume moreover that

$$\int_{\mathbb{R}^N} f v_i v_j\, dv = T_i \delta_{ij}$$

(the T_i's are the directional temperatures; of course $\sum_i T_i = N$). Then the Landau operator with matrix (29) can be rewritten as

$$\sum_i (N - T_i)\partial_{ii} f + (N-1)\nabla \cdot (fv) + \Delta_S f. \tag{30}$$

Here Δ_S stands for the Laplace-Beltrami operator,

$$\Delta_S f = \sum_{ij} (|v|^2 \delta_{ij} - v_i v_j)\partial_{ij} f - (N-1) v \cdot \nabla_v f,$$

i.e. a diffusion on centered spheres in velocity space. Thus the Landau operator looks like a nonlinear Fokker-Planck type operator, with some additional isotropisation effect due to the presence of the Laplace-Beltrami operator. The diffusion is enhanced in directions where the temperature is low, and slowed down in directions where the temperature is high : this is normal, because in the end the temperature along all directions should be the same.

Formulas like (30) show that in the isotropic case, and under assumption (29), the nonlinear Landau equation reduces (in a well-chosen orthonormal basis) to the linear Fokker-Planck equation ! By this remark [**447**] one can construct many explicit

solutions (which generalize the ones in [**296**]). These considerations explain why the Maxwellian variant of the Landau equation has become a popular test case in numerical analysis [**106**].

Let us now review other variants of the Landau equation. First of all, there are relativistic and quantum versions of it [**304**]. For a mathematically-oriented presentation, see for instance [**298**]. There are also other, more sophisticated models for collisions in plasmas : see for instance [**164**, section 13.6] for a synthetic presentation. The most famous of these models is the so-called **Balescu-Lenard collision operator**, whose complexity is just frightening for a mathematician. Its expression was established by Bogoljubov [**90**] via a so-called BBGKY-type hierarchy, and later put by Lenard under the form that we give below. On the other hand, Balescu [**46, 47**] derived it as part of his general perturbative theory of approximation of the Liouville equation for many particles. Just as the Landau operator, the Balescu-Lenard operator is in the form

$$\sum_{ij} \frac{\partial}{\partial v_i} \left\{ \int_{\mathbb{R}^3} dv_* \, a_{ij}(v, v_*) \left(f_* \frac{\partial f}{\partial v_j} - f \frac{\partial f_*}{\partial v_{*j}} \right) \right\}, \tag{31}$$

but now the matrix a_{ij} depends on f in a strongly nonlinear way :

$$a_{ij}(v, v_*) = \int_{k \in \mathbb{R}^3,\, |k| \leq K_{\max}} \delta[k \cdot (v - v_*)] \, \frac{k_i k_j}{|k|^4} \, \frac{1}{|\epsilon|^2} \, dk, \tag{32}$$

where ϵ is the "longitudinal permittivity" of the plasma,

$$\epsilon = 1 - \frac{8\pi}{|k|^2} \int \frac{1}{k \cdot (v - v_*) - i0} k \cdot (\nabla f)_* \, dv_*.$$

Here δ is the Dirac measure at the origin, and

$$\frac{1}{x - i0} = \lim_{\varepsilon \to 0+} \frac{1}{x - i\varepsilon} = \mathcal{P}\left(\frac{1}{x}\right) + i\pi\delta$$

is a complex-valued distribution on the real line ($\mathcal{P}$ stands for the Cauchy principal part). Moreover, $K_{\max}$ is a troncature parameter (whose value is not very clearly determined) which corresponds to values of the deviation angles beyond which collisions cannot be considered grazing. This is not a Debye cut !

Contrary to Landau, Balescu and Lenard derived the operator Q_L (26) as an approximation of (31). To see the link between these two operators, let us set $k = \mu\omega$, $\mu \geq 0$, $\omega \in S^2$; then one can rewrite (32) as

$$\int_{S^2} d\omega \, \delta[\omega \cdot (v - v_*)] \, \omega_i \omega_j \left(\int_0^{K_{max}} \frac{d\mu}{\mu |\epsilon|^2} \right). \tag{33}$$

But

$$\int_{S^2} d\omega \, \delta(\omega \cdot z) \omega_i \omega_j = \frac{\pi}{|z|} \left(\delta_{ij} - \frac{z_i z_j}{|z|^2} \right).$$

So one can replace the operator (31) by the Landau collision operator if one admits that the integral in $d\mu$ in (33) depends very little on v, v_*, ω. Under this assumption, it is natural to replace f by a Maxwellian[11], and one finds for this integral an expression of the form $1 + (\mu_D^2)/\mu^2$, where μ_D has the same homogeneity as the inverse of a Debye length. Then one can perform the integration; see Decoster [**160**] for much more details.

[11] This is the natural statistical equilibrium, see paragraph 2.5.

In spite of its supposed accuracy, the interest of the Balescu-Lenard model is not so clear. Due to its high complexity, its numerical simulation is quite tricky. And except in very particular situations, apparently one gains almost nothing, in terms of physical accuracy of the results, by using it as a replacement for the Landau operator. In fact, it seems that the most important feature of the Balescu-Lenard operator, to this date, is to give a theoretical basis to the use of the Landau operator !

There exist in plasma physics some even more complicated models, such as those which take into account magnetic fields (Rostoker operator, see for instance [**465**] and the references therein). We refrain from writing up the equations here, since this would require several pages, and they seem definitely out of reach of a mathematical treatment for the moment... We also mention the simpler **linear Fokker-Planck operator for Coulomb interaction**, derived by Chandrasekhar. This is a linear operator of Landau-type, which describes the evolution of a test-particle interacting with a "bath" of Coulomb particles in thermal equilibrium. A formula for it is given in Balescu [**46**, paragraph 37], in terms of special functions.

1.8. Physical validity of the Boltzmann equation. Experience has shown that the Boltzmann equation and its variants realistically describe phenomena which occur in dilute atmosphere, in particular aeronautics at high altitude, or interactions in dilute plasmas. In many situations, predictions based on the Navier-Stokes equation are not accurate for low densities; a famous historical example is provided by the so-called Knudsen minimum effect in a Poiseuille flow [**148**, p. 99] : if the difference of pressure between the entrance and the exit of a long, narrow channel is kept fixed, then the flow rate through a cross-section of this channel is not a monotonic function of the average pressure, but exhibits a minimum for a certain value of this parameter. This phenomenon, established experimentally by Knudsen, remained controversial till the 1960's.

Also the Boltzmann equation cannot be replaced by fluid equations when it comes to the study of *boundary layers* (Knudsen layer, Sone sublayer due to curvature...) and the gas-surface interaction.

Nowadays, with the impressive development of computer power, it is possible to perform very precise numerical simulations which seem to fully corroborate the predictions based on the Boltzmann equation — within the right range of physical parameters, of course.

All these questions are discussed, together with many numerical simulations and experiments, in the recent broad-audience survey book by Cercignani [**148**] (see also the review paper [**136**] by the same author in the present volume).

2. Mathematical problems in collisional kinetic theory

In this section, we try to define the most interesting mathematical problems which arise in the study of Boltzmann-like equations. At this point we should make it clear that the Boltzmann equation can be studied for the sake of its applications to dilute gases, but also as one of the most basic and famous models for nonequilibrium statistical mechanics.

2.1. Mathematical validity of the Boltzmann equation. In the last paragraph, we have mentioned that, in the right range of physical parameters, the physical validity of the Boltzmann equation now seems to be beyond any doubt. On the other hand, the mathematical validity of the Boltzmann equation poses a more

challenging problem. For the time being, it has been investigated only for the hard-sphere model.

Let us give a short description of the problem in the case of hard spheres. But before that, we add a word of caution about the meaning of "mathematical validity" : it is not a proof that the model is the right one in a certain range of physical parameters (whatever this may mean). It is only a rigorous derivation of the model, in a suitable asymptotic procedure, from another model, which is conceptually simpler but contains more information (typically : the positions of all the particles, as opposed to the density of particles). Of course, the Boltzmann equation, just like many models, can be derived either by mathematical validation, or by direct modelling assumptions, and the second approach is more arbitrary, less interesting from the mathematical point of view, but not less "respectable" if properly implemented ! This is why, for instance, Truesdell [**430**] refuses to consider the problem of mathematical validity of the Boltzmann equation.

The validation approach to be discussed now is due to Grad [**249**], and it is particularly striking because the starting point is nothing but the model given by Newton's laws of classical mechanics. It was not before 1972 that Cercignani [**139**] showed Grad's approach to be mathematically consistent, in the sense that it can be rigorously implemented if one is able to prove some "reasonable" estimates on the solutions[12].

Grad's approach : The starting point is the equation of motion, according to Newton's laws, for a system of n spherical particles of radius r in $\mathbb{R}^3$, bouncing elastically on each other with billiard reflection laws. The state of the system is described by the positions and velocities of the centers, $x_1, v_1, \dots, x_n, v_n$, and the phase space is the subset of $(\mathbb{R}^3_x \times \mathbb{R}^3_v)^n$ (or $(X \times \mathbb{R}^3)^n$) such that $|x_i - x_j| \geq r$ $(i \neq j)$. On this phase space there is a flow $(S_t)_{t \geq 0}$, well-defined up to a zero-probability set of initial configurations, which is neglected. We now consider symmetric probability densities $f^n(x_1, v_1, \dots, x_n, v_n)$ on the phase space (symmetry reflects the physical assumption of undiscernability of particles). Of course, the flow (S_t) on the phase space induces a flow on such probability densities, the solution of which is denoted by $(f^n_t)_{t \geq 0}$. Moreover, by integrating f^n_t over all variables but the k first position variables and the k first velocity variables, one defines the k-particle distribution function $P_k f^n(x_1, v_1, \dots, x_k, v_k)$ (think of P_k as a projection operator). In probabilistic terminology, $P_1 f^n$ is the first marginal of f^n.

Assume now that

(i) $n \to \infty$, $r \to 0$ (continuum limit) in such a way that $nr^2 \to 1$ [the gas is sufficiently dilute, but not too much, so that only binary interactions play a significant role, and a typical particle collides about once in a unit of time. This limit is called the **Boltzmann-Grad limit**];

(ii) $P_1 f^n_0 \longrightarrow f_0$, where f_0 is a given distribution function [this assumption means that the one-particle function at time 0 can be treated as "continuous" as the number of particles becomes large];

(iii) $P_2 f^n_0 \longrightarrow f_0 \otimes f_0$, and more generally, for fixed k,

$$P_k f^n_0 \longrightarrow f_0 \otimes \cdots \otimes f_0 \tag{34}$$

[12]This remark is important because many people doubted the possibility of a rigorous derivation, see the discussion in section 2.4.

[this is the chaos assumption at time 0];

the problem is then to prove that $P_1 f_t^n \longrightarrow f_t$, where $f_t = f_t(x,v)$ is the solution of the Boltzmann equation with hard-sphere kernel, and with initial datum f_0.

This problem seems exceptionally difficult. The main result in the field is the 1973 Lanford's theorem [**292**]. He proved the result for small time, and under some strong assumptions on the initial probability distributions $P_k f_0^n$: they should be continuous, satisfy appropriate gaussian-type bounds, and converge uniformly[13] towards their respective limits. Later his proof was rewritten by Illner and Pulvirenti, and extended to arbitrarily large time intervals, under a smallness assumption on the initial datum, which enabled to treat the Boltzmann equation as a kind of perturbation of the free transport equation: see [**275, 276**] and the nice reviews in [**149, 394, 410**]. For sure, one of the outstanding problems in the theory of the Boltzmann equation is to extend Lanford's result to a more general framework, without smallness assumption. Another considerable progress would be its extension to long-range interactions, which is not clear even from the formal point of view (see for instance Cercignani [**137**]).

The Boltzmann-Grad limit is also often called the **low-density limit**, and presented in the following manner [**149**, p. 60] : starting from the equations of Newtonian dynamics, blow-up the scales of space and time by a factor ε^{-1} (thus ε is the ratio of the microscopic scale by the macroscopic scale), and require the number of particles to be of the order of ε^{-2}, then let ε go to 0. In particular, the density will scale as $\varepsilon^{2/3}$ since the volume will scale as ε^{-3}, and this explains the terminology of "low density limit".

Remarks about the chaos assumption :

1. Heuristically, the relevance of the chaos assumption in Boltzmann's derivation can be justified as follows : among all probability distributions f^n which have a given marginal $f = P_1 f^n$, the most likely, in some sense which we do not make precise here[14], is the tensor product $f \otimes \cdots \otimes f$. And as $n \to \infty$, this distribution becomes by far the most likely. Thus, Boltzmann's chaos assumption may be justified by the fact that we choose the most likely microscopic probability distribution f^n which is compatible to our macroscopic knowledge (the one-particle distribution, or first marginal, f).

2. In fact, the chaos property is automatically satisfied, in weak sense, by all sequences of probability measures (f^n) on $(\mathbb{R}^3 \times \mathbb{R}^3)^n$ which are compatible with the density f. More precisely : let us say that a microscopic configuration $z =$

[13] on compact subsets of $(\mathbb{R}^3 \times \mathbb{R}^3)^k_{\neq}$, which is the set obtained from $(\mathbb{R}^3 \times \mathbb{R}^3)^k$ by deleting all configurations with $x_i = x_j$ for some distinct indices i, j.

[14] This is related to the fact that the tensor product $f^{\otimes n}$ has minimal entropy among all n-particles probability distribution functions f^n with given first marginal f, and to the fact that the negative of the entropy yields a measure of "likelihood", see paragraph 2.4 below. [In this review, the entropy of a distribution function f is defined by the formula $H(f) = \int f \log f$; note the sign convention.] For hard spheres, a subtlety arises from the fact that configurations in which two spheres interpenetrate are forbidden, so f^n cannot be a tensor product. Since however the total volume of the spheres, nr^3, goes to 0 in the limit, it is natural to assume that this does not matter. On the other hand, contact points in the n-particle phase space play a crucial role in the way the chaos property is propagated in time.

$(x_1, v_1, \dots, x_n, v_n) \in (\mathbb{R}^3_x \times \mathbb{R}^3_v)^n$ is admissible if its empirical density,

$$\omega_z = \frac{1}{n} \sum_{i=1}^{n} \delta_{(x_i, v_i)},$$

is a good approximation to the density $f(x,v)\,dx\,dv$, and let $(f^n)_{n\in\mathbb{N}}$ be a sequence of symmetric probability densities on $(\mathbb{R}^3_x \times \mathbb{R}^3_v)^n$ respectively, such that the associated measures μ^n give very high probability to configurations which are admissible. In a more precise writing, we require that for every bounded continuous function $\varphi(x,v)$ on $\mathbb{R}^3 \times \mathbb{R}^3$, and for all $\varepsilon > 0$,

$$\mu^n \left[z \in (\mathbb{R}^3_x \times \mathbb{R}^3_v)^n;\ \left| \int_{\mathbb{R}^3_x \times \mathbb{R}^3_v} \varphi(x,v) \big[\omega_z(dx\,dv) - f(t,x,v)\,dx\,dv \big] \right| > \varepsilon \right] \xrightarrow[n\to\infty]{} 0.$$

Then, (f^n) satisfies the chaos property, in weak sense [**149**, p. 91] : for any $k \geq 1$,

$$P_k f^n \xrightarrow[n\to\infty]{} f^{\otimes k}$$

in weak-$*$ measure sense. This statement expresses the fact that f^n is automatically close to the tensor product $f^{\otimes n}$ in the sense of weak convergence of the marginals. However, weak convergence is not sufficient to derive the Boltzmann equation, because of the problem of *localization of collisions.* Therefore, in Lanford's theorem one imposes a stronger (uniform) convergence of the marginals, or strong chaos property.

3. This is the place where probability enters the Boltzmann equation : via the initial datum, i.e. the probability density (f^n) ! According to [**149**, p. 93], the conclusion of Lanford's theorem can be reformulated as follows : for all time $t > 0$, if f^n_t is obtained from f^n_0 by transportation along the characteristics of the microscopic dynamics, and if μ^n_t is the probability measure on $(\mathbb{R}^3_x \times \mathbb{R}^3_v)^n$ whose density is given by f^n_t, then for all $\varepsilon > 0$, and for all bounded, continuous function $\varphi(x,v)$ on $\mathbb{R}^3_x \times \mathbb{R}^3_v$,

$$\mu^n_t \left[\left\{ z \in (\mathbb{R}^3_x \times \mathbb{R}^3_v)^n;\ \left| \int_{\mathbb{R}^3_x \times \mathbb{R}^3_v} \varphi(x,v) \big[\omega_z(dx\,dv) - f_t(x,v)\,dx\,dv \big] \right| > \varepsilon \right\} \right] \xrightarrow[n\to\infty]{} 0,$$

where f_t is the solution to the Boltzmann equation with initial datum f_0. In terms of μ^n :

$$\mu^n \left[\left\{ z \in (\mathbb{R}^3_x \times \mathbb{R}^3_v)^n;\ \left| \int_{\mathbb{R}^3_x \times \mathbb{R}^3_v} \varphi(x,v) \big[\omega_{S_t z}(dx\,dv) - f_t(x,v)\,dx\,dv \big] \right| > \varepsilon \right\} \right] \xrightarrow[n\to\infty]{} 0.$$

In words : for *most* initial configurations, the evolution of the density under the microscopic dynamics is well approximated by the solution to the Boltzmann equation. Of course, this does not rule out the existence of "unlikely" initial configurations for which the solution of the Boltzmann equation is a very bad approximation of the empirical measure.

4. If the chaos property is the crucial point behind the Boltzmann derivation, then one should expect that it *propagates* with time, and that

$$\forall t > 0, \qquad P_k f^n_t \to f_t \otimes \cdots \otimes f_t. \tag{35}$$

However, this propagation property only holds in a weak sense. Even if the convergence is strong (say, uniform convergence of all marginals) in (34), it *has* to be weaker in (35), say almost everywhere, see the discussion in Cercignani et al. [**149**].

The reason for this weakening is the appearance of microscopic correlations (under evolution by the microscopic, reversible dynamics). In particular, if the initial microscopic datum is "very likely", this does not imply at all that the microscopic datum at later times should be very likely ! On the contrary, it should present a lot of correlations...

5. In fact, one has to be extremely cautious when handling (35). To illustrate this, let us formally show that for $t > 0$ the approximation

$$P_2 f_t^n(x, v; y, w) \simeq f_t(x, v) f_t(y, w) \tag{36}$$

cannot be true in strong sense, uniformly in all variables[15], as $n \to \infty$ (the symbol $\simeq$ here means "approaches, in L^∞ norm, uniformly in all variables x, y, v, w, as $n \to \infty$"). Indeed, assume that (36) holds true uniformly in x, y, v, w, and choose $y = x + r\sigma$, $\langle v - w, \sigma \rangle > 0$, i.e. an ingoing collisional configuration in the two-particle phase space. Then, presumably

$$P_2 f_t^n(x, v; x + r\sigma, w) \simeq f_t(x, v) f_t(x + r\sigma, w) \simeq f_t(x, v) f_t(x, w) \tag{37}$$

as $n \to \infty$. But from the specular reflection condition, for any $t > 0$,

$$P_2 f_t^n(x, v; x + r\sigma, w) = P_2 f_t^n(x, v'; x + r\sigma, w'),$$

where v' and w' are post-collisional velocities,

$$v' = v - \langle v - w, \sigma \rangle \, \sigma, \qquad w' = w + \langle v - w, \sigma \rangle \, \sigma.$$

Applying (36) again, this would result in

$$P_2 f_t^n(x, v; x + r\sigma, w) = P_2 f_t^n(x, v'; x + r\sigma, w') \simeq f_t(x, v') f_t(x, w'),$$

which is not compatible with (37) (unless f_t solves eq. (53) below). This contradiction illustrates the fact that (36) cannot be propagated by the dynamics of hard spheres. It is actually property (37), sometimes called **one-sided chaos**, which is used in the derivation of the Boltzmann equation, and which should be propagated for positive times : it means that the velocities of particles which are just about to collide are not correlated. But it is a very difficult problem to handle eq. (37) properly, because it involves the restriction of f^n to a manifold of codimension 1, and may be violated even for initial data which satisfy the conditions of Lanford's theorem ! So an appropriate generalized sense should be given to (37). Lanford's argument cleverly avoids any discussion of (37), and only assumes (36) at time 0, the approximation being uniform outside collisional configurations. So he plainly avoids discussing one-sided chaos, and does not care what is propagated for positive times, apart from weak chaos[16]. To sum up : the physical derivation of the Boltzmann equation is based on the *propagation of one-sided chaos*, but no one knows how this property should be expressed mathematically — if meaningful at all.

An easier variant of the validation problem is the derivation of **linear** transport equations describing the behavior of a **Lorentz gas** : a test-particle in a random pinball game, with scatterers randomly distributed according to (say) a Poisson law. Under a suitable scaling, the law of this test-particle converges towards the solution of a linear Boltzmann equation, as was first formalized by Gallavotti [**226**], before several improvements appeared [**409, 91**]. See Pulvirenti [**394**] for a review and

[15] constrained by $|x - y| \geq r$.

[16] This is possible because he uses a perturbative proof, based on an iterative Duhamel formula, in which everything is expressed in terms of the initial datum...

introduction of the subject. The convergence actually holds true for almost all (in the sense of Poisson measure) fixed configuration of scatterers, but *fails* for certain specific configurations, for instance a periodic array, as shown in Bourgain, Golse and Wennberg [**102**]. We also note that Desvillettes and Pulvirenti [**179**] are able to rigorously justify the linear Boltzmann equation for some interactions with *infinite range*.

Kac's approach : To conclude this paragraph, we mention another line of approach towards the mathematical justification of the Boltzmann equation. It goes via the construction of some many-particle *stochastic* system, such that the first marginal of its law at a given time t should be an approximation of the solution to the Boltzmann equation if the initial datum is chaotic. This subject was initiated by Kac[17] [**283**], and developed by Sznitman [**412**] in connection with the problem of propagation of chaos. Recent progress on this have been achieved by Graham and Méléard [**256, 344**].

The main conceptual difference between both approaches lies in the moment where probability is introduced, and irreversibility[18] as well. In Lanford's approach, the starting point is a deterministic particle system; it is only the particular "chaotic" choice of the initial datum which leads to the macroscopic, irreversible Boltzmann equation in the limit. On the other hand, for Kac the microscopic particle system is already stochastic and irreversible from the beginning. Then the main effect of the limit is to turn a linear equation on a large n-particle phase space, into a nonlinear equation on a reduced, one-particle phase space.

Of course Kac's approach is less striking than Grad's, because the starting point contains more elaborate modelling assumptions, since stochasticity is already built in. Kac formulated his approach in a spatially homogeneous[19] setting, while this would be meaningless for Grad's approach. In fact, it is as if Kac wanted to treat the positions of the particles (which, together with ingoing velocities, determine the outgoing velocities) as hidden probabilistic variables. Then, all the subtleties linked to one-sided chaos can be forgotten, and it is sufficient to study just propagation of (weak) chaos.

Moreover, Kac's approach becomes important when it comes to make an interpretation of the Monte Carlo numerical schemes which are often used to compute approximate solutions of the Boltzmann equation. These schemes are indeed based on large stochastic particle systems. See Pulvirenti [**394, 453, 396**] for references about the study of these systems, in connection with the validation problem. We do not develop here on the problem of the rigorous justification of numerical schemes, but this topic is addressed in the companion review [**136**] by Cercignani.

2.2. The Cauchy problem. From the mathematical point of view, the very first problem arising in the study of the Boltzmann equation is the **Cauchy problem** : given a distribution function $f_0(x,v)$ on $\mathbb{R}^N_x \times \mathbb{R}^N_v$ (or $X \times \mathbb{R}^N_v$), satisfying appropriate and physically realistic assumptions, show that there exists a (unique)

[17] See paragraph 1.5 in chapter 5.

[18] See paragraph 2.4.

[19] See paragraph 5.2.

solution of

$$\begin{cases} \dfrac{\partial f}{\partial t} + v \cdot \nabla_x f = Q(f,f), \\ \\ f(0,\cdot,\cdot) = f_0. \end{cases} \tag{38}$$

Needless to say, the Boltzmann equation seems impossible to solve explicitly[20], except in some very particular situations : semi-explicit solutions by Bobylev [**79**], Bobylev and Cercignani [**81**]; self-similar solutions of infinite mass by Nikolskii, see [**289**, p. 286]; particular solutions in a problem of shear flow by Truesdell, see [**430**, chap. 14–15], some simple problems of modelling with a lot of symmetries [**148**]... Explicit solutions are discussed in the review paper [**207**]. These exact solutions are important in certain modelling problems, but they are exceptional. This justifies the study of a general Cauchy problem.

Of course, the question of the Cauchy problem should be considered as a preliminary for a more detailed study of qualitative properties of solutions of the Boltzmann equation. The main qualitative properties in which one is interested are : smoothness and singularities, conservation laws, strict positivity, existence of Lyapunov functionals, long-time behavior, limit regimes. We shall come back on all of this in the next chapters. As recalled in section 3, the properties of the solutions may depend heavily on the form of the collision kernel.

As of this date, the Cauchy problem has still not received satisfactory answers. As we shall describe in section 5, there are several "competing" theories which either concern (more or less) simplified cases, or are unable to answer the basic questions one may ask about the solutions. Yet this problem has spectacularly advanced since the end of the eighties.

Another fundamental problem in many areas of modelling by Boltzmann equation, as explained for instance in Cercignani [**148**], is the existence of **stationary solutions** : given a box X, prove that there exists a (unique ?) stationary solution of the Boltzmann equation in the box :

$$v \cdot \nabla_x f = Q(f,f), \qquad x \in X,\ v \in \mathbb{R}^N$$

together with well-chosen boundary conditions (ideally, dictated by physical assumptions).

The stationary problem has been the object of a lot of mathematical studies in the past few years; see for instance [**31, 25, 34, 36, 26, 37, 38**]. We shall not consider it here, except for a few remarks. This is first because the theory is less developed than the theory of the Cauchy problem, secondly because we wish to avoid the subtle discussion of boundary conditions for weak solutions.

2.3. Maxwell's weak formulation, and conservation laws. The change of variables $(v, v_*, \sigma) \longrightarrow (v', v'_*, k)$, with $k = (v - v_*)/|v - v_*|$, has unit Jacobian and is involutive. Since $\sigma = (v' - v'_*)/|v' - v'_*|$, one can abuse terminology by referring to this change of variables as $(v, v_*) \longrightarrow (v', v'_*)$. It will be called the pre-postcollisional change of variables. As a consequence of microreversibility, it leaves the collision kernel B invariant.

[20] Although no theorem of non-solvability has been proven !

The fact that this change of variable has unit Jacobian is not a general feature of Boltzmann-like equations, actually it is false for energy-dissipating models[21]... Also the change of variables $(v, v_*) \longrightarrow (v_*, v)$ is clearly involutive and has unit Jacobian. As a consequence, if φ is an arbitrary continuous function of the velocity v,

(39)
$$\int_{\mathbb{R}^N_v} Q(f,f)\,\varphi\,dv = \int_{\mathbb{R}^N\times\mathbb{R}^N} dv\,dv_* \int_{S^{N-1}} d\sigma\, B(v-v_*,\sigma)(f'f'_* - ff_*)\varphi$$

(40)
$$= \int_{\mathbb{R}^N\times\mathbb{R}^N} dv\,dv_* \int_{S^{N-1}} d\sigma\, B(v-v_*,\sigma) ff_*(\varphi' - \varphi)$$

(41)
$$= \frac{1}{2}\int_{\mathbb{R}^N\times\mathbb{R}^N} dv\,dv_* \int_{S^{N-1}} d\sigma\, B(v-v_*,\sigma) ff_*(\varphi' + \varphi'_* - \varphi - \varphi_*).$$

This gives a weak formulation for Boltzmann's collision operator. From the mathematical point of view, it is interesting because expressions like (40) or (41) may be well-defined in situations where $Q(f,f)$ is not. From the physical point of view, it expresses the change in the integral $\int f(t,x,v)\varphi(v)\,dv$ which is due to the action of collisions. Actually, this formulation is so natural for a physicist, that equation (40) was written by Maxwell[22] [**335**, eq. 3] before Boltzmann gave the explicit expression of $Q(f,f)$!

Let f be a solution of the Boltzmann equation (8), set in the whole space $\mathbb{R}^N_x$ to simplify. By the conservative properties of the transport operator, $v\cdot\nabla_x$,

(42)
$$\frac{d}{dt}\int f(t,x,v)\varphi(v)\,dx\,dv = \int Q(f,f)\varphi\,dx\,dv,$$

and the right-hand side is just the x-integral of any one of the expressions in formulas (39)–(41). As an immediate consequence, whenever φ satisfies the functional equation

(43)
$$\forall (v, v_*, \sigma) \in \mathbb{R}^N\times\mathbb{R}^N\times S^{N-1}, \quad \varphi(v') + \varphi(v'_*) = \varphi(v) + \varphi(v_*)$$

then, at least formally,

$$\frac{d}{dt}\int f(t,x,v)\varphi(v)\,dx\,dv = 0$$

along solutions of the Boltzmann equation. The words "at least formally" of course mean that the preceding equations must be rigorously justified with the help of some integrability estimates on the solutions to the Boltzmann equation.

It can be shown under very weak conditions[23] [**142, 29**], [**149**, p. 36–42] that solutions to (43), as expected, are only linear combinations of the *collision invariants* :

$$\varphi(v) = 1,\ v_i,\ \frac{|v|^2}{2}, \qquad 1\le i\le N.$$

[21] See section 2 in chapter 5.

[22] Actually it is not so easy to recognize the Boltzmann equation in Maxwell's notations !

[23] This problem was first treated by Gronwall [**259, 260**] and Carleman [**119**] under stronger conditions. Then people started to study it under weaker and weaker assumptions. Its interest lies not only in checking that there are no hidden conservation laws in the Boltzmann equation, but also in solving the important eq. (53) below, for which simpler methods are however available.

This leads to the (formal) **conservation laws** of the Boltzmann equation,

$$\frac{d}{dt}\int f(t,x,v)\begin{pmatrix}1\\ v_i\\ \dfrac{|v|^2}{2}\end{pmatrix}dx\,dv=0,\qquad 1\le i\le N$$

meaning that the total mass, the total momentum and the total energy of the gas are preserved.

These conservation laws should hold true when there are no boundaries. In presence of boundaries, conservation laws may be violated : momentum is not preserved by specular reflection, neither is energy if the gas is in interaction with a wall kept at a fixed temperature. See Cercignani [**141, 148**] for a discussion of general axioms of the classical modelling of gas-surface interaction, and resulting laws.

If one disregards this possible influence of boundaries, then the preservation of mass, momentum and energy under the action of the Boltzmann collision operator is clearly the least that one can expect from a model which takes into account only elastic collisions. Yet, to this date, *no mathematical theory has been able to justify these simple rules at a sufficient level of generality.* The problem is of course that too little is known about how well behaved are the solutions to the Boltzmann equation. Conservation of mass and momentum are no problem, but no one knows how to obtain an a priori estimate which would imply a little bit more integrability than just finite energy.

Another crucial topic for a fluid description is the validity of **local conservation laws**, i.e. continuity equations obtained by integrating the Boltzmann equation with respect to v only. With notations (1), these equations are

$$\text{(44)}\qquad \begin{cases}\dfrac{\partial\rho}{\partial t}+\nabla_x\cdot(\rho u)=0,\\[2ex] \dfrac{\partial}{\partial t}(\rho u)+\nabla_x\cdot\left(\displaystyle\int_{\mathbb{R}^N} f v\otimes v\,dv\right)=0,\\[2ex] \dfrac{\partial}{\partial t}(\rho|u|^2+N\rho T)+\nabla_x\cdot\left(\displaystyle\int_{\mathbb{R}^N} f|v|^2\,v\,dv\right)=0.\end{cases}$$

At this moment, only the first of these equations has been proven in full generality [**308**].

2.4. Boltzmann's H theorem and irreversibility. In this paragraph, we discuss some of the most famous aspects of the Boltzmann equation. This will justify a few digressions to make the topic as clear as possible.

Let us symmetrize the integral (39) once more, fully using all the symmetries of the collision operator. We obtain

$$\text{(45)}\qquad \int Q(f,f)\varphi\,dv=-\frac{1}{4}\int dv\,dv_*\,d\sigma\,B(v-v_*,\sigma)\Big(f'f'_*-ff_*\Big)\big(\varphi'+\varphi'_*-\varphi-\varphi_*\big).$$

We shall refer to this formula as **Boltzmann's weak formulation**.

Without caring about integrability issues, we plug $\varphi = \log f$ into this equation, and use the properties of the logarithm, to find

$$\int_{\mathbb{R}^N} Q(f,f) \log f \, dv = -D(f), \tag{46}$$

where D is the **entropy dissipation functional**,

$$D(f) = \frac{1}{4} \int_{\mathbb{R}^{2N} \times S^{N-1}} dv \, dv_* \, d\sigma \, B(v - v_*, \sigma) \Big(f' f'_* - f f_* \Big) \log \frac{f' f'_*}{f f_*} \geq 0. \tag{47}$$

That $D(f) \geq 0$ just comes from the fact that the function $(X, Y) \longmapsto (X - Y)(\log X - \log Y)$ is nonnegative.

Next, we introduce Boltzmann's H **functional**,

$$H(f) = \int_{\mathbb{R}^N_x \times \mathbb{R}^N_v} f \log f. \tag{48}$$

Of course, the transport operator $-v \cdot \nabla_x$ does not contribute in any change of the H functional in time[24]. As a consequence, if $f = f(t,x,v)$ is a solution of the Boltzmann equation, then $H(f)$ will evolve in time because of the effects of the collision operator :

$$\frac{d}{dt} H\big(f(t,\cdot,\cdot)\big) = - \int_{\mathbb{R}^N_x} D\big(f(t,x,\cdot)\big) \, dx \leq 0. \tag{49}$$

This is the famous **Boltzmann's H theorem** : the H-functional, or entropy, is nonincreasing with time. This theorem is "proven" and discussed at length in Boltzmann's treatise [**93**]. Before commenting on its physical implications, let us give a few analytical remarks :

1. For certain simplified models of the Boltzmann equation, McKean [**342**] has proven that the H-functional is, up to multiplicative and additive constants, the only "local" (i.e. of the form $\int A(f)$) Lyapunov functional.

2. There are some versions with boundary conditions; actually, it was emphasized by Cercignani that the H theorem still holds true for a modified H-functional (including the temperature of the wall, for instance, if the wall is kept at fixed temperature) as soon as a certain number of general axioms are satisfied. See [**141**] for precise statements.

3. The argument above, leading to formula (47), does not work for certain variants of the Boltzmann equation, like mixtures. Actually Boltzmann had also given a (false) proof in this case, and once the error was discovered, produced a totally obscure argument to fix it (see the historical references in [**148**]). As pointed out by Cercignani and Lampis [**150**], the most robust way to prove the H theorem is to use again Maxwell's weak formulation, and to note that

$$\frac{1}{2} \int B f f_* \log \frac{f' f'_*}{f f_*} \, dv \, dv_* \, d\sigma = \frac{1}{2} \int B f f_* \left(\log \frac{f' f'_*}{f f_*} - \frac{f' f'_*}{f f_*} + 1 \right) \leq 0, \tag{50}$$

because $\int Q(f,f) \, dv = 0$ and $\log X - X + 1 \leq 0$. This line of proof can be generalized to mixtures [**148**, paragraph 6.4] and other models.

[24] More generally, the transport operator does not contribute to any change of a functional of the form $\int A(f) \, dx \, dv$.

4. The Landau equation also satisfies a H theorem. The corresponding entropy dissipation is (formally)

$$D_L(f) = \frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3} ff_* \left|\sqrt{a(v-v_*)}\Big(\nabla(\log f) - [\nabla(\log f)]_*\Big)\right|^2 dv\, dv_*. \tag{51}$$

What is the physical sense of Boltzmann's H theorem ? First of all, we note that the H functional should coincide with the usual entropy of physicists up to a change of sign. Also, it is a dynamical entropy, in the sense that is defined for nonequilibrium systems. Thus Boltzmann's H theorem is a manifestation of the second law of thermodynamics (Clausius' law) which states that the physical entropy of an isolated system should not decrease in time. In particular, it demonstrates that the Boltzmann model has some **irreversibility** built in. This achievement (produce an analytical proof of the second law for some specific model of statistical mechanics) was one of the early goals of Boltzmann, and was later considered as one of his most important contributions to statistical physics.

But the H theorem immediately raised a number of objections, linked to the fact that the starting point of the derivation of the Boltzmann equation is just classical, reversible mechanics — so where does the irreversibility come from ? Of course, since the existence of atoms was controversial at the time, Boltzmann's construction seemed very suspect... About the controversy between Boltzmann and opponents, and the way to resolve apparent contradictions in Boltzmann's approach, one may consult [**202, 284, 149, 293**]. Here we shall explain in an informal way the main arguments of the discussion, warning the reader that the following considerations have not been put on a satisfactory mathematical basis.

Zermelo pointed out that Boltzmann's theorem seemed to contradict the famous Poincaré recurrence theorem[25]. Boltzmann replied that the scales of time on which Poincaré's theorem applied in the present setting were much larger than the age of the universe, and therefore irrelevant. This answer is justified by the fact that the Boltzmann equation should be a good approximation of the microscopic dynamics, only on a time scale which depends on the actual number of particles — see the discussion in the end of paragraph 2.5.

Then Loschmidt came out with the following paradox. Let be given a gas of particles, evolving from time 0 to time $t_0 > 0$. At time t_0, reverse all velocities and let the gas evolve freely otherwise. If Boltzmann were right, the same Boltzmann equation should describe the behavior of the gas on both time intervals $[0, t_0]$ and $[t_0, 2t_0]$. Since the reversal of velocities does not change the entropy of a distribution function, the entropy at time $2t_0$ should be strictly less than the entropy at time 0. But, by time-reversibility of classical mechanics, at time $2t_0$ the system should be back to its initial configuration, which would be a contradiction... To this argument Boltzmann is reported to have replied "Go on, reverse the velocities !"

The answer to Loschmidt's paradox is subtle and has to do with the probabilistic content of the Boltzmann equation. Starting with the classical monograph by P. and T. Ehrenfest [**202**], it was understood that reversible microdynamics and

[25]For almost all choice of the initial datum, a conservative system with a compact phase space will always come arbitrarily close to its initial configuration for large enough times. This theorem applies to a system of n particles obeying the laws of classical mechanics, interacting via elastic collisions, enclosed in a box.

irreversible macrodynamics are not contradictory, provided that the right amount of probability is used in the interpretation of the macroscopic model. This view is very well explained in the excellent book by Kac [**284**].

In the case of the Boltzmann derivation, everything seems deterministic : neither the microscopic model, nor the macroscopic equation are stochastic. But the probabilistic content is hidden in the choice of the initial datum. As we mentioned earlier[26], with very high probability the Boltzmann equation gives a good approximation to the evolution of the density of the gas. And here randomness is in the choice of the microscopic initial configuration, among all configurations which are compatible with the density f (see paragraph 2.1 for more precise formulations). But for exceptional configurations which are not chaotic, the derivation of the Boltzmann equation fails.

Now comes the tricky point in Loschmidt's argument. As we discussed in paragraph 2.1, the chaos property, in strong sense, is not preserved by the microscopic, reversible dynamics. What should be preserved is the one-sided chaos : ingoing collisional configurations are uncorrelated. More precisely, if the strong chaos assumption holds true at initial time, then chaos should be true for ingoing configurations at positive times, and for outgoing configurations at negative times.

So, reversing all velocities as suggested by Loschmidt is an innocent operation at the level of the limit one-particle distribution (which has forgotten about correlations), but by no means at the level of the microscopic dynamics (which keeps all correlations in mind). It will transform a configuration which is chaotic as far as ingoing velocities are concerned, into a configuration which is chaotic as far as outgoing velocities are concerned[27]... Then, the microscopic dynamics will preserve this property that outgoing velocities are not correlated, and, by repeating the steps of the derivation of the Boltzmann equation, we find that the correct equation, to describe the gas from time t_0 on, should be the *negative Boltzmann equation* (with a minus sign in front of the collision operator). Therefore in Loschmidt's argument, of course the entropy is unchanged at time t_0, but then it should start to increase, and be back at its original value at time $2t_0$.

At the moment, a fully satisfactorily mathematical discussion of Loschmidt's paradox is not possible, since we do not know what one-sided chaos should really mean, mathematically speaking. But one can check, as is done in [**149**, section 4.7], that strong chaos[28] is not propagated in time — so that it will be technically impossible to repeat Lanford's argument when taking as initial datum the microscopic configuration at time t_0, be it before or after reversal of velocities.

With this in mind, one also easily answers to the objection, why would the Boltzmann equation select a direction of time ? Actually, it does not[29], and this can be seen by the fact that if strong[30] chaos is assumed at initial time, then the correct equation should be the Boltzmann equation for positive times, and the negative Boltzmann equation for negative times. *We* have selected a direction of time by

[26] Recall the remarks about propagation of chaos in paragraph 2.1.

[27] In particular, this configuration is very unlikely as an initial distribution. Thus Loschmidt's paradox illustrates very well the fact that the Boltzmann derivation works for *most* initial data, but not for all !

[28] roughly speaking, in the sense of uniform convergence of the marginals towards the tensor product distributions, recall paragraph 2.1.

[29] By the way, Boltzmann himself believed that the direction of positive times should be *defined* as the direction in which the H-functional has a decreasing behavior...

[30] "Double-sided" should be the right condition here !

assuming the distribution function to be "very likely" at time 0 and studying the model for positive times. In fact, in Boltzmann's description, the entropy is maximum at time 0, and decreases for positive times, increases for negative times. As an amusing probabilistic reformulation : knowing the one-particle distribution function at some time t_0, with very high probability the entropy is a maximum at this time t_0 ! Related considerations can be found in Kac [**284**, p. 79] for simpler models, and may explain the cryptic statement by Boltzmann that "the H-functional is always, almost surely, a local maximum".

Most of the explanations above are already included in Boltzmann's treatise [**93**], in physicist's language; in particular Boltzmann was very well aware of the probabilistic content of his approach. But, since so many objections had been raised against Boltzmann's theory, many physicists doubted for a long time that a rigorous derivation of the Boltzmann equation, starting from the laws of classical mechanics, could be possible. This is one of the reasons why Lanford's theorem was so spectacular.

After this digression about irreversibility, let us now briefly comment on Boltzmann's H-functional itself. Up to the sign, it coincides with Shannon's entropy (or information) quantity, which was introduced in communication theory at the end of the forties[31]. In the theory of Shannon, the entropy measures the redundancy of a language, and the maximal compression rate which is applicable to a message without (almost) any loss of information : see [**156**] and the many references therein. In this survey, we shall make precise some links between information theory and the kinetic theory of gases, in particular via some variants of famous information-theoretical inequalities first proven by Stam [**165**].

From the physical point of view, the entropy measures the volume of *microstates* associated to a given macroscopic configuration[32]. This is suggested by the following computation, due to Boltzmann (see [**93**]). Let us consider n particles taking p possible different states; think of a state as a small "box" in the phase space $\mathbb{R}^N_x \times \mathbb{R}^N_v$. Assume that the only information to which we have access is the number n_i of particles in each state i. In other words, we are unable to distinguish particles with different states; or, in a probabilistic description, we only have access to the one-particle marginal. This macroscopic description is in contrast with the microscopic description, in which we can distinguish all the particles, and know the state of each of them. Given a macroscopic configuration $(n_1, \cdots, n_p)$, the number of compatible microscopic configurations is

$$\Omega = \frac{n!}{n_1! \cdots n_p!}.$$

Let us set $f_i = n_i/n$, and let all n_i's go to infinity. By Stirling formula (or other methods, see [**156**, p. 282]), one shows that, up to an additive constant which is

[31] Here is a quotation by Shannon, extracted from [**331**], which we learnt in [**16**]. "My greatest concern was how to call it. I thought of calling it 'information'. But the word was overly used, so I decided to call it 'uncertainty'. When I discussed it with John Von Neumann, he had a better idea. He told me :"You should call it entropy, for two reasons. In first place your uncertainty has been used in statistical mechanics under that name, so it already has a name. In second place, and more important, no one knows what entropy really is, so in a debate you will always have the advantage."

[32] Here, the one-particle probability distribution f is the macroscopic description of the system, while the many-particle probability distribution f^n is the microscopic state.

independent of the f_i's,

$$\frac{1}{n} \log \Omega \longrightarrow - \sum_{i=1}^{p} f_i \log f_i.$$

This result explains the link between the H-functional and the original definition of the entropy by Boltzmann, as the logarithm of the volume of microstates[33]. So we see that it is the exponential of the negative of the entropy, which plays the role of a "volume" in infinite dimension. Up to a normalization, this quantity is known in information theory as the **entropy power**[34] :

$$\mathcal{N}(f) = \exp(-2H(f)/N). \tag{52}$$

More remarks about the physical content of entropy, or rather entropies, are formulated in Grad [**251**].

In the discussion of Boltzmann's derivation and irreversibility, we have seen two distinct entropies : the macroscopic entropy $H(f)$, which is fixed by the experimenter at initial time, and then wants to decrease as time goes by; and the microscopic entropy, $H(f^n)$, which is more or less assumed to be minimal at initial time (among the class of microscopic distributions f^n which are compatible with f), and is then kept constant in time by the microscopic dynamics. There is no contradiction between the fact that $H(f_t)$ is decreasing and $H(f_t^n)$ is constant, because there is no link between both objects[35] if there are correlations at the level of f_t^n.

As a last comment, the decrease of the entropy is a fundamental property of the Boltzmann equation, but the H-functional is far from containing all the information about the Boltzmann equation. This is in contrast with so-called gradient flows, which are partial differential equations of the form $\partial f/\partial t = -\text{grad}\, E(f)$, for some "entropy" functional E and some gradient structure. For such an equation, in some sense the entropy functional encodes all the properties of the flow...

The main, deep reason for the fact that the Boltzmann equation cannot be seen as a gradient flow, is the fact that the collision operator depends only on the velocity space; but even if we restrict ourselves to solutions which do not depend on space, then the Boltzmann equation is not (to the best of our knowledge) a gradient flow. Typical gradient flows, in a sense which will be made more precise later, are the linear Fokker-Planck equation (19), or the McNamara-Young model for granular media (see [**70**] and section 2 in chapter 5). The lack of gradient flow structure contributes to the mathematical difficulty of the Boltzmann equation.

2.5. Long-time behavior. Assume that $B(v-v_*, \sigma) > 0$ for almost all (v, v_*, σ), which is always the case in applications of interest. Then equality in Boltzmann's

[33]This is the famous formula $S = k \ln \Omega$, which was written on Boltzmann's grave.

[34]The analogy between power entropy and volume can be pushed so far that, for instance, the Shannon-Stam entropy power inequality, $\mathcal{N}(f * g) \geq \mathcal{N}(f) + \mathcal{N}(g)$, can be seen as a consequence of the Brunn-Minkowski inequality on the volume of Minkowski sums of compact sets $|X + Y|^{1/d} \geq |X|^{1/d} + |Y|^{1/d}$. This (very) nontrivial remark was brought to our attention by F. Barthe.

[35]Except the inequality $H(f_t) \leq \liminf H(f_t^n)/n$... By the way [**149**, p. 99–100], the function $\phi(X) = X \log X$ is, up to multiplication by a constant or addition of an affine function, the only continuous function ϕ which satisfies the inequality $\int \phi(P_1 f^n) \leq \int \phi(f^n)/n$ for all f^n's, with equality for the tensor product distribution.

H theorem occurs if and only if for almost all x, v, v_*, σ,

$$f' f'_* = f f_*. \tag{53}$$

Under extremely weak assumptions on f [**149, 307, 377**], this functional equality forces f to be a **local Maxwellian**[36], i.e. a probability distribution function of the form

$$f(x,v) = \rho(x) \frac{e^{-|v-u(x)|^2/2T(x)}}{(2\pi T(x))^{N/2}}.$$

Thus it is natural to guess that the effect of collisions is to bring $f(t,\cdot)$ closer and closer to a local Maxwellian, as time goes by. This is compatible with **Gibbs' lemma :** among all distributions on $\mathbb{R}^N_v$ with given mass, momentum and energy, the minimum of the entropy is achieved by the corresponding Maxwellian distribution.

Of course, equation (53) implies $Q(f,f) = 0$. As a corollary, we see that solutions of the functional equation

$$Q(f,f) = 0, \tag{54}$$

where the unknown $f(v)$ is a distribution function on $\mathbb{R}^N_v$ with finite mass and energy, are precisely Maxwellian distributions.

As we have seen, local Maxwellian states are precisely those distribution functions for which the dissipation of entropy vanishes. But if the position space X is a bounded domain on $\mathbb{R}^N$ (with suitable boundary conditions, like specular reflection, or Maxwellian re-emission), then one can show that there are very few time-dependent local Maxwellian distributions which satisfy the Boltzmann equation. Except in particular cases (domains with symmetries, see [**254**]), such a solution has to take the form

$$f(t,x,v) = \rho \, \frac{e^{-|v-u|^2/2T}}{(2\pi T)^{N/2}}, \tag{55}$$

for some parameters ρ, u, T which depend neither on t nor on x. A state like (55) is called a **global Maxwellian**, or global equilibrium state. It is uniquely determined by its total mass, momentum and energy.

The problem of the **trend to equilibrium** consists in proving that the solution of the Cauchy problem (38) converges towards the corresponding global equilibrium as $t \to +\infty$, and to estimate, in terms of the initial datum, the speed of this convergence. On this subject, the (now outdated) paper by Desvillettes [**168**] accurately surveys existing methods up to the beginning of the nineties. Since that time, new trends have emerged, with the research for constructive estimates and the development of entropy dissipation techniques. To briefly summarize recent trends, we should say that the problem of trend to equilibrium has received rather satisfactory answers in situations where the Cauchy problem is known to have well-behaved solutions. We shall make a detailed review in chapter 3.

[36] The result follows from the characterization of solutions to (43), but can also be shown directly by Fourier transform as in [**377**]. However, no proof is more enlightening than the one due to Boltzmann (see paragraph 4.3 in chapter 3). His proof requires C^1 smoothness, but Lions [**307**, p. 423] gave a beautiful proof that L^1 solutions to (53) have to be smooth. And anyway, one can always easily reduce to the case of smooth densities by the remarks in section 4.7 of chapter 3.

On the other hand, if more complicated boundary conditions are considered, it may happen that the global equilibrium be no longer Maxwellian; and that the mere existence of a global equilibrium already be a very difficult problem[37].

Also, when the gas gets dispersed in the whole space, then things become complicated : in certain situations, and contrary to what was previously believed by many authors, solutions to the Boltzmann equation *never* get close to a local Maxwellian state [**383, 419, 327**]. The main physical idea behind this phenomenon is that the dispersive effects of the transport operator may prevent particles to undergo a sufficient number of collisions. In the whole space setting, the relevant problem is therefore not trend to equilibrium, but rather **dispersion** : find estimates on the speed at which the gas is dispersed at infinity. We shall not develop on this problem; to get information the reader may consult Perthame [**376**] (see also [**277**] for similar estimates in the context of the Vlasov-Poisson equation). Dispersion estimates play a fundamental role in the modern theory of the Schrödinger equations, and there is a strong analogy with the estimates appearing in this field [**135**].

We conclude this paragraph with an important remark about the meaning of $+\infty$ in the limit $t \to +\infty$. It is only in a suitable asymptotic regime that the Boltzmann equation is expected to give an accurate description of, say, a system of n particles. But for a given, large number n of particles, say 10^{23}, the quality of this description cannot be uniform in time. To get convinced of this fact, just think that Poincaré's recurrence theorem will apply to the n-particle system after a very, very long time. In fact, since the Boltzmann equation is established on physical scales such that each particle typically encounters a finite number of collisions in a unit of time, we may expect the Boltzmann approximation to break down on a time at most $O(n)$, i.e. a time on which a typical particle will have collided with a nonnegligible fraction of its fellow particles, so that finite-size effects should become important. This means that any theorem involving time scales larger than 10^{23} is very likely to be irrelevant[38] ... Such a conclusion would only be internal to Boltzmann's equation and would not yield any information about physical "reality" as predicted by the model. So what is interesting is not really to prove that the Boltzmann equation converges to equilibrium as $t \to +\infty$, but rather to show that it becomes very close to equilibrium when t is very large, yet not unrealistically large. Of course, from the mathematical point of view, this may be an extremely demanding goal, and the mere possibility of proving explicit rates should already be considered as a very important achievement, as well as identifying the physical factors (boundary conditions, interaction, etc.) which should slow down, or accelerate the convergence.

2.6. Hydrodynamic limits. The H theorem was underlying the problem of the trend to global equilibrium in the limit $t \to +\infty$. It also underlies the assumption of local thermodynamical equilibrium in the hydrodynamical limits.

Generally speaking, the problem of the hydrodynamical limit can be stated as follows : pass from a Boltzmann description of a dilute gas (on microscopic scales of space and time, i.e. of the order of the mean free path and of the mean time

[37] See the references in paragraph 2.2.

[38] Besides being of no practical value, since these time scales are much much larger than reasonable physical scales.

between collisions, respectively) to a hydrodynamic description, holding on macroscopic scales of space and time. And the scaling should be such that f "looks like" a local Maxwellian, even if local Maxwellians cannot be solutions of the Boltzmann equation...

To make this more concrete, assume that one contracts the measurements of lengths and time by a factor ε, the velocity scale being preserved (ε can be thought of as the Knudsen number, which, roughly speaking, would be proportional to the ratio between the mean free path and a typical macroscopic length). Then, the new distribution function[39] will be

$$f_\varepsilon(t,x,v) = f\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, v\right).$$

If f solves the Boltzmann equation, then f_ε solves the *rescaled* Boltzmann equation

$$\frac{\partial f_\varepsilon}{\partial t} + v \cdot \nabla_x f_\varepsilon = \frac{1}{\varepsilon} Q(f_\varepsilon, f_\varepsilon).$$

Hence the role of the macroscopic parameter ε is to considerably enhance the role of collisions. In view of the H theorem, one expects f_ε to resemble more and more a local Maxwellian when $\varepsilon \to 0$: this is the assumption of **local thermodynamical equilibrium**, whose mathematical justification is in general a delicate, still open problem.

Here is an equivalent, nonrigorous way of seeing the limit : the time scale of trend to local equilibrium should be of the order of the mean time between collisions, which should be much smaller than the macroscopic time.

For fixed ε, the macroscopic quantities (density, momentum, temperature) associated to f_ε via (1) satisfy the equations

$$\begin{cases} \dfrac{\partial \rho_\varepsilon}{\partial t} + \nabla_x \cdot (\rho_\varepsilon u_\varepsilon) = 0 \\ \dfrac{\partial (\rho_\varepsilon u_\varepsilon)}{\partial t} + \nabla_x \cdot \left(\displaystyle\int_{\mathbb{R}^N} f_\varepsilon v \otimes v \, dv \right) = 0 \\ \dfrac{\partial}{\partial t} \big(\rho_\varepsilon |u_\varepsilon|^2 + N \rho_\varepsilon T_\varepsilon \big) + \nabla_x \cdot \left(\displaystyle\int_{\mathbb{R}^N} f_\varepsilon |v|^2 \, v \, dv \right) = 0. \end{cases}$$

The assumption of local thermodynamical equilibrium enables one to *close* this system in the limit $\varepsilon \to 0$, and to formally obtain

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \nabla_x \cdot (\rho u) = 0 \\ \dfrac{\partial (\rho u)}{\partial t} + \nabla_x \cdot (\rho u \otimes u + \rho T I_N) = 0 \\ \dfrac{\partial}{\partial t} (\rho |u|^2 + N \rho T) + \nabla_x \cdot \left(\rho |u|^2 u + (N+2) \rho T u \right) = 0. \end{cases} \tag{56}$$

[39] Note that f_ε is not a probability density.

with I_N standing for the identity $N \times N$ matrix. System (56) is nothing but the system of the **compressible Euler equations**, when the pressure is given by the law of perfect gases, $p = \rho T$. See [**431**]

Other scalings are possible, and starting from the Boltzmann equation one can get many other equations in fluid mechanics [**56**]. In particular, by looking at perturbations of a global equilibrium, it is possible to recover Navier-Stokes type equations. This is one of several possible ways of interpreting the Navier-Stokes equation, see [**467**] for remarks about other interpretations. From a physicist's point of view, the interesting aspect of this limit is the appearance of the viscosity from molecular dynamics. From a mathematician's perspective, another interesting thing is that there are some well-developed mathematical theories for the Navier-Stokes equation, for instance the famous theory of weak solutions by Leray [**299, 300, 301**], see Lions [**313, 314**] for the most recent developments — so one can hope to prove theorems !

An interesting remark, due to Sone and coworkers [**408, 407**], shows that sometimes a hydrodynamic equation which looks natural is actually misleading because some kinetic effects should have an influence even at vanishing Knudsen number; this phenomenon was called "ghost effect".

There are formal procedures for "solving" the Boltzmann equation in terms of a series expansion in a small parameter ε (like our ε above), which are known as Hilbert and Chapman-Enskog expansions. These procedures have never received a satisfactory mathematical justification in general, but have become very popular tools for deriving hydrodynamical equations. This approach is described in reference textbooks such as [**154, 430, 141, 148, 48**], and particularly [**250**, section 22 and following]. But it also underlies dozens of papers on formal hydrodynamical limits, which we do not try to review. By the way, it should be pointed out that equations obtained by keeping "too many" (meaning 3 or 4) terms of the Hilbert or Chapman-Enskog series, like the so-called Burnett or super-Burnett equations, seem to be irrelevant (a discussion of this matter, an ad hoc recipe to fix this problem, and further references, can be found in Jin and Slemrod [**282**]). Also, these expansions are not expected to be convergent, but only "asymptotic". In fact, a solution of the Boltzmann equation which could be represented as the sum of such a series would be a very particular one[40] : it would be entirely determined by the fields of local density, mean velocity and pressure associated with it.

All these problems illustrate the fact that the Hilbert and Chapman-Enskog methods rely on very sloppy grounds. Rather violent attacks on their principles are to be found in [**430**]. In spite of this, these methods are still widely used. Without any doubt, their popularity lies in their systematic character, which enables one to formally derive the correct equations in a number of situations, without having to make any guess.

An alternative approach, which is conceptually simpler, and apparently more effective for theoretical purposes, is a moment-based procedure first proposed by Grad [**249**]. For Maxwell molecules, this procedure leads to hydrodynamic equations whose accuracy could be expected to be of high order; the method has been developed in particular by Truesdell and collaborators [**274, 430**] under the name of "Maxwellian iteration".

[40] "normal" in the terminology of Grad [**250**].

The problem of *rigorous* hydrodynamical limits has been studied at length in the literature, but most of these results have been obtained in a perturbative setting. Recently, some more satisfactory results (in the large) have been obtained after the development of a spectacular machinery (see [**441**] for a presentation). We shall discuss a few references on both lines of approach in section 5, as we shall go along the presentation of the mathematical theories for the Cauchy problem.

On the other hand, the Boltzmann equation, as a model, does not capture the full range of hydrodynamical equations that one would expect from dynamical systems of interacting particles. At the level of the Euler equation above, this can be seen by the fact that the pressure law is of the form $p = \rho T$. See [**363, 397, 467**] for more general equations and partial results on the problem of the direct derivation of hydrodynamical equations from particle systems. This however does not mean that it is not worth working at the level of the Boltzmann equation; first because the limit should be simpler to rigorously perform, than the limit for "raw" particle systems; secondly because the Boltzmann equation is one of the *very few* models of statistical physics which have been derived from mechanical first principles. This kind of preoccupations meets those expressed by Hilbert in the formulation of his sixth problem[41] about the axiomatization of physics : can one put the equations of fluid mechanics on a completely rigorous basis, starting from Newton's laws of microscopic motion ?

2.7. The Landau approximation. This problem occurs in the kinetic theory of dilute plasmas, as briefly described in paragraph 1.7. Starting from the Boltzmann equation for screened Coulomb potential (Debye potential), can one justify the replacement of this operator by the Landau operator (26) as the Debye length becomes large in comparison with the space length scale ? This problem can easily be generalized to an arbitrary dimension of space.

A few remarks are in order as regards the precise meaning of this problem. First of all, this is not a derivation of the Landau equation from particle systems (such a result would be an outstanding breakthrough in the field). Instead, one considers the Boltzmann equation for Debye potential as the starting point. Secondly, if one wants to stick to the classical theory of plasmas, then

- either one neglects the effects of the mean-field interaction; in this case the problem has essentially been solved recently, as we shall discuss in section 5 of chapter 2.

- or one takes into account this effect, and then the Landau equation should only appear as a long-time correction to the Vlasov-Poisson equation. To the best of our knowledge, no such result has ever been obtained even in simplified regimes.

2.8. Numerical simulations. The literature about numerical simulation for the Boltzmann equation is considerable. All methods used to this day consider separately the effects of transport and collision. This **splitting** has been theoretically justified by Desvillettes and Mischler [**178**], for instance, but a thorough discussion of the best way to implement it seems to be still lacking.

Dominant methods are based on Monte Carlo simulation, and introduce "particles" interacting by collisions. Of course, transport is no problem for a particles-based method : just follow the characteristics, i.e. the trajectories of particles in

[41] directly inspired by Boltzmann's treatise, among other things.

phase space. Then one has to implement the effect of collisions, and many variants are possible. Sometimes the dynamics of particles obey the Newton laws of collision only on the average, in such a way that their probability density still comply with the Boltzmann dynamics. We do not try to review the literature on Monte Carlo simulation, and refer to the very neat survey in Cercignani [**148**, chap. 7], or to the review paper [**136**] by the same author. Elements of the theoretical justification of Monte Carlo simulation, in connection with chaos issues, are reviewed in Cercignani et al. [**149**, chapter 10], Pulvirenti [**394**] and Graham and Méléard [**257**].

Let us however say a few words about deterministic methods which have emerged recently, thanks to the increase of computational capacity[42]. For a long time these methods were just unaffordable because of the high computational cost of the $(2N-1)$-fold integral in the Boltzmann collision oeprator, but they are now becoming more and more competitive.

Deterministic schemes based on conservation laws have been devised in the last years by Buet, Cordier, Degond, Lemou, Lucquin [**106, 297, 105**]. In these works, the simulated distribution function is constrained to satisfy conservation of mass, momentum and energy, as well as decreasing of entropy. This approach implies very clever procedures, in particular to handle the discretization of the spheres appearing in the Boltzmann representation (this problem is pretty much the same than the consistency problem for discrete velocity models). For variants such as the Landau equation, these difficulties are less important, but then one has to hunt for possible undesirable symmetries, which may introduce spurious conservation laws, etc.

Another deterministic approach is based on Fourier transform, and has been developed by Bobylev and Rjasanow, Pareschi, Perthame, Russo, Toscani [**82, 89, 370, 371, 373, 372**]). We shall say a little bit more on these schemes in paragraph 4.8, when introducing Fourier transform tools.

At the moment, both methods have been competing, especially in the framework of the Landau equation [**107, 215**]. It seems that spectral schemes are useful to give extremely accurate results, but cannot beat conservation-based schemes in terms of speed and efficiency. Certainly more is to be expected on the subject.

2.9. Miscellaneous. We gather here a few other basic questions about the Boltzmann equation, which, even though less important than the ones we have already presented, do have mathematical interest.

Phenomenological derivation of the Boltzmann equation. We have seen two ways to introduce the Boltzmann equation : either by direct modelling assumptions (dilute gas, chaos, etc.) or by rigorous theorems starting from particle systems. There is a third way towards it, which is by making some "natural" phenomenological assumptions on the form of the collision operator, and try to prove that these assumptions uniquely determine the form of the collision operator. A classical discussion by Bobylev [**84**] exemplifies this point of view. To roughly sum up the very recent work by Desvillettes and Salvarani [**180**], it is shown that (essentially) the only smooth quadratic forms Q acting on probability densities such that

1) evolution by $\partial_t f = Q(f,f)$ preserves nonnegativity,

2) Q has Galilean invariance,

[42]The same increase of computational capacity seems to make discrete-velocity models less and less attractive.

3) $Q(M,M)=0$ for any Maxwellian M,

have to be linear combinations of a Boltzmann and a Landau operator. Note the inversion of the point of view : in paragraph 2.4 we have checked that the form of Boltzmann's operator imposed a Maxwellian form to an equilibrium distribution, while here we see that the assumption of Maxwellian equilibrium states[43] contributes to determine the structure of the Boltzmann operator.

Image of Q. This problem is very simply stated : for any function $f(v)$ with enough integrability, we know that $\int Q(f,f)\varphi(v)\,dv=0$ whenever $\varphi(v)$ is a linear combination of $1, v_i$ $(1\le i\le N)$, $|v|^2$. Conversely, let h be a function satisfying $\int h\varphi = 0$ for the same functions φ, is it sufficient to imply the existence of a probability distribution f such that $Q(f,f)=h$? This problem apparently has never received even a partial answer for a restricted subclass of functions h.

Divergence form of Boltzmann's collision operator. Unlike several other operators, particularly Landau or Fokker-Planck, the Boltzmann operator is not written in divergence form, even though it is conservative. Physicists consider this not surprising, in as much as Boltzmann's operator models sudden (as opposed to continuous) changes of velocities.

However, generally speaking, any function Q whose integral vanishes can be written as the divergence of something. This writing is in general purely mathematical and improper to physical interpretation. But a nice feature of Boltzmann's operator, as we have shown in [**448**], is that an explicit and (relatively) simple expression exists. We obtained it by going back to the physical interpretation in terms of collisions, as in Landau [**291**]. Since there is no well-defined "flux", one is led to introduce fictitious trajectories, linked to the parametrization of the pre-collisional variables. Here is one possible expression for the flux, which is a priori not unique : $Q(f,f)=-\nabla_v\cdot J(f,f)$, where

$$J(f,f)=-\int_{(v-v_*,\omega)>0} dv_*\,d\omega\,B(v-v_*,\omega)\int_0^{(v-v_*,\omega)} dr\,f(v+r\omega)f(v_*+r\omega)\omega \tag{57}$$

$$=\int_{(v_*-v,v_\circ-v)<0} dv_\circ\,dv_*\,f(v_\circ)f(v_*)B\left(v_\circ-v_*,\frac{v-v_\circ}{|v-v_\circ|}\right)\frac{v-v_\circ}{|v-v_\circ|^N}. \tag{58}$$

The Boltzmann operator can even be written as a double divergence,

$$Q(f,f)=\sum_{ij}\frac{\partial^2}{\partial v_i\,\partial v_j}A_{ij}(f,f),\qquad \sum_i A_{ii}(f,f)=-\nabla_v\cdot J^E(f,f),$$

for some explicit functionals A_{ij} and J^E, $1\le i,j\le N$. Each of the "fluxes" J, A_j, J^E is related to a conservation law.

Besides casting some more light on the structure of Boltzmann's operator and its relations with divergence operators, these formulas may be used for devising conservative numerical schemes via the "velocity diffusion" method (implemented in works by Mas-Gallic and others).

Eternal solutions. Let us introduce this problem with a quotation from Truesdell and Muncaster [**430**, p.191] about irreversibility : "In a much more concrete way

[43] which can be taken as physically reasonable from outer considerations, like central limit theorem, etc.

than Boltzmann's H theorem, [the time-behavior of the moments of order 2 and 3 for the Boltzmann equation with Maxwellian[44] molecules] illustrate[s] the *irreversibility* of the behavior of the kinetic gas. This irreversibility is particularly striking if we attempt to trace the origin of a grossly homogeneous condition by considering past times instead of future ones. Indeed *the magnitude of each component of [the pressure tensor] and [the tensor of the moments of order 3] that is not 0 at* $t = 0$ *tends to* ∞ *as* $t \to -\infty$. Thus any present departure from kinetic equilibrium must be the outcome of still greater departure in the past."

Appealing as this image may be, it is our conviction that it is actually *impossible*[45] to let $t \to -\infty$. More precisely, heuristic arguments in [**450**, Annex II, Appendix] make the following conjecture plausible : *let* $f = f(t, v)$ *be a solution of the Boltzmann equation for* $v \in \mathbb{R}^N$, $t \in \mathbb{R}$, *with finite kinetic energy. Then* f *is a Maxwellian, stationary distribution.*

In the case of the spatially inhomogeneous Boltzmann equation, the conjecture should be reformulated by saying that f is a local Maxwellian (which would allow "travelling Maxwellians", of the form $M(x - vt, v)$, where $M(x, v)$ is a global Maxwellian in the x and v variables).

This problem, which is reminiscent of Liouville-type results for parabolic equations [**468**], may seem academic. But in [**450**, Annex II, Appendix] we were able to connect it to the important problem of the *uniformity of trend to equilibrium*, and to prove this conjecture for some simplified models; Cabannes [**109, 110**] also was able to prove it for some simplified discrete-velocity approximations of the Boltzmann equation. At this stage we mention that the *nonnegativity* of the solution is fundamental, since nontrivial, partially negative solutions may exist. By the way, the nonnegativity of the solutions to the Boltzmann equation is also very important from the mathematical point of view : if it is not imposed, then irrelevant blow-up of the distribution function may occur.

Very recently, Bobylev and Cercignani [**81**] made a considerable progress in this problem, proving the conjecture for so-called Maxwellian collision kernels, under the additional requirement that the solution has all of its moments finite for all time. Simultaneously, they discovered some very interesting self-similar eternal solutions of the Boltzmann equation with infinite kinetic energy, which may describe the asymptotic behavior of the Boltzmann equation in some physical regimes where the energy is very large.

3. Taxonomy

After having introduced the main mathematical problems in the area, we are prepared for a more precise discussion of Boltzmann-like models. This section may look daunting to the non-specialist, and we shall try to spare him. The point is that there are, truly speaking, as many different Boltzmann equations as there are collision kernels. Hence it is absolutely necessary, before discussing rigorous results, to sketch a "zoological" classification of these. For simplicity, and because this case is much better understood, we limit the presentation to the classical, elastic case. For background about relativistic, quantum, nonelastic collision kernels, see paragraph 1.6, chapter 5, or the survey papers [**209**] and [**146**].

[44] See section 3.

[45] which may be an even more striking manifestation of irreversibility ?

We shall conclude this section with a brief description of the main mathematical and physical effects that the choice of the collision kernel should have on the properties of the Boltzmann equation.

3.1. Kinetic and angular collision kernel. Recall that the collision kernel $B = B(v - v_*, \sigma) = B(|v - v_*|, \cos\theta)$ [by abuse of notations] only depends on $|v - v_*|$ and $\cos\theta$, the cosine of the deviation angle. Also recall that for a gas of three-dimensional hard spheres,

$$B\big(|v - v_*|, \cos\theta\big) = K|v - v_*|, \qquad K > 0$$

while for inverse s-power forces, B factors up like

$$B\big(|v - v_*|, \cos\theta\big) = \Phi(|v - v_*|)\, b(\cos\theta), \tag{59}$$

where $\Phi(|z|) = |z|^\gamma$, $\gamma = (s-5)/(s-1)$ in dimension $N = 3$, and $b(\cos\theta)\sin^{N-2}\theta \sim K\theta^{-(1+\nu)}$, $\nu = 2/(s-1)$ in dimension $N = 3$ also.

Exactly what range of values of s should be considered is by no means clear in the existing literature. Many authors [**111, 18, 170**] have restricted their discussion to $s > 3$. Klaus [**288**, p. 895] even explains this restriction by the impossibility of defining the Boltzmann linearized collision operator for $s \leq 3$. However, as we shall explain, at least a weak theory of the Boltzmann equation can be constructed for any exponent $s \in (2, +\infty)$. The limit value $s = 2$ corresponds to the Coulomb interaction, which strictly speaking does not fit into the framework of the Boltzmann equation, as we have discussed in paragraph 1.7.

Remark : What may possibly be true, and anyway requires clarification, is that the derivation of the Boltzmann equation from particle systems may fail for $s \leq 3$, because of the importance of the mean-field interaction. But even in this case, the Boltzmann description of collisions should be rehabilitated in the investigation of the long-time behavior.

Even though one is naturally led to deal with much more general collision kernels, products like (59) are the basic examples that one should keep in mind when discussing assumptions. By convention, we shall call Φ the **kinetic collision kernel**, and b the **angular collision kernel**. We shall discuss both quantities separately.

From the mathematical point of view, the control of Boltzmann's collision operator is all the more delicate that the collision kernel is "large" (in terms of singularities, or behavior as $|v - v_*| \to \infty$). On the contrary, when one is interested in such topics as trend to equilibrium, it is good to have a strictly positive kernel because this means more collisions; then the difficulties often come from the vanishings of the collision kernel. In short, one should keep in mind the heuristic rule that *the mathematical difficulties encountered in the study of the Cauchy problem often come from large values of the collision kernel, those encountered in the study of the trend to equilibrium often come from small values of the collision kernel.*

3.2. The kinetic collision kernel. It is a well-established custom to consider the cases $\Phi(|v - v_*|) = |v - v_*|^\gamma$, and to distinguish them according to

- $\gamma > 0$: hard potentials;
- $\gamma = 0$: Maxwellian potentials;
- $\gamma < 0$: soft potentials.

For inverse-power forces in dimension 3, hard potentials correspond to $s > 5$, soft potentials to $s < 5$.

We shall stick to this convention, but insist that it is quite misleading. First of all, "hard potentials" are not necessarily associated to an interaction potential ! It would be better to speak of "hard kinetic collision kernel". But even this would not be a neat classification, because it involves at the same time the behavior of the collision kernel for large and for small values of the relative velocity, which makes it often difficult to appreciate the assumptions really needed in a theorem. Sometimes a theorem which is stated for hard potentials, would in fact hold true for all kinetic collision kernels which are bounded below at infinity, etc. As typical examples, trouble for the study of the Cauchy problem may arise due to large relative velocities for hard potentials, or due to small relative velocities for soft potentials...

How positive may γ be ? For hard spheres, $\gamma = 1$, hence a satisfactory theory should be able to encompass this case. In many cases one is able to treat $\gamma < 2$ or $\gamma \leq 2$, or even less stringent assumptions.

Conversely, how negative may γ be ? Contrarily to what one could think, critical values of the exponent s do not, in general, correspond to critical values of γ. As a striking example, think of Coulomb potential ($s = N - 1$), which normally should correspond to a power law $\gamma = N/(N-2)$ in dimension N. Besides the fact that this is meaningless when $N = 2$, this exponent is less and less negative as the dimension increases; hence the associated Cauchy problem is more and more easy because of the weaker singularity.

The following particular values appear to be most critical : $\gamma = -2$, $\gamma = -N$. The appearance of the limit exponent -2 in the study of several mathematical properties [**437, 247, 248, 446**] has led us in [**446**] to suggest the distinction between moderately soft potentials ($-2 < \gamma < 0$) and very soft potentials ($\gamma < -2$). It is however not clear whether the border corresponds to a change of mathematical properties, or just an increase in difficulty.

Note that dimension 3 is the only one in which the Coulomb potential coincides with the limit exponent $-N$, which makes its study quite delicate[46] !

3.3. The angular collision kernel. We now turn to the angular collision kernel $b(\cos\theta) = b(k \cdot \sigma)$, $k = (v - v_*)/|v - v_*|$. First of all, without loss of generality one may restrict the deviation angle to the range $[0, \pi/2]$, replacing if necessary b by its "symmetrized" version, $[b(\cos\theta) + b(\cos(\pi - \theta))]1_{0 \leq \theta \leq \pi/2}$. From the mathematical point of view, this is because the product $f'f'_*$ appearing in the Boltzmann collision operator is invariant under the change of variables $\sigma \to -\sigma$; from the physical point of view this reflects the undiscernability of particles.

As mentioned above, for inverse-power law forces, the angular collision kernel presents a nonintegrable singularity as $\theta \to 0$, and is smooth otherwise. The fact that the collision kernel presents a nonintegrable singularity with respect to the angular variable is not a consequence of the choice of inverse-power forces, recall (15).

[46] All the more that $\gamma = -3$ also seems to have some special, bad properties independently of the dimension, see [**429**].

By analogy with the examples of inverse-power forces, one would like to treat the following situations :

$$b(\cos\theta)\sin^{N-2}\theta \sim K\theta^{-(1+\nu)} \qquad \text{as } \theta \to 0, \tag{60}$$

$$0 \leq \nu < 2.$$

Grad's angular cut-off [**250, 141**] simply consists in postulating that the collision kernel is integrable with respect to the angular variable. In our model case, this means

$$\int_{S^{N-1}} b(k\cdot\sigma)\,d\sigma = |S^{N-2}| \int_0^\pi b(\cos\theta)\sin^{N-2}\theta\,d\theta < \infty. \tag{61}$$

The utmost majority of mathematical works about the Boltzmann equation crucially rely on Grad's cut-off assumption, from the physical point of view this could be considered as a short-range assumption.

3.4. The cross-section for momentum transfer. Let $M(|v-v_*|)$ be defined by

$$\int_{S^{N-1}} d\sigma\, B(v-v_*,\sigma)[v-v'] = \frac{1}{2}(v-v_*)M(|v-v_*|). \tag{62}$$

Indeed, by symmetry, the left-hand side is parallel to $v-v_*$. The quantity M is called the cross-section for momentum transfer. In our model case, $M(|v-v_*|) = \mu\Phi(|v-v_*|)$, where

$$\mu = |S^{N-2}| \int_0^\pi b(\cos\theta)(1-\cos\theta)\,\sin^{N-2}\theta\,d\theta. \tag{63}$$

From the physical point of view, the cross-section for momentum transfer is one of the basic quantities in the theory of binary collisions (see for instance [**405, 164**]) and its computation via experimental measurements is a well-developed topic.

On the other hand, the mathematical importance of the cross-section for momentum transfer has not been explicitly pointed out until very recently. From the mathematical point of view, finiteness of M (for almost all $v-v_*$) is a necessary condition for the Boltzmann equation to make sense [**450**, Annex I, Appendix A], in the sense that if it is not satisfied, then $Q(f,f)$ should only take values in $\{-\infty, 0, +\infty\}$. On the other hand, together with Alexandre we have recently shown [**12**] that suitable assumptions on M are essentially what one needs to develop a coherent theory for the Boltzmann equation without Grad's angular cut-off assumption. Note that in our model case, finiteness of M requires that $b(\cos\theta)(1-\cos\theta)$, or equivalently $b(\cos\theta)\theta^2$, be integrable on S^{N-1}. This precisely corresponds to the range of admissible singularities $\nu \in [0,2)$.

When $\nu = 2$, the integral (63) diverges logarithmically for small values of θ : this is one reason for the failure of the Boltzmann model to describe Coulomb collisions. Very roughly, the Debye truncature yields a finite μ, which behaves like the logarithm of the Debye length; this is what physicists call the *Coulomb logarithm*. And due to the fact that the divergence is only logarithmic, they expect the cross-section for momentum transfer to depend very little on the precise value of the Debye length.

3.5. The asymptotics of grazing collisions. The mathematical links between the Boltzmann and the Landau collision operator can be made precise in many ways. As indicated for instance by Degond and Lucquin [**162**], for a fixed, smooth f, one can consider $Q_L(f,f)$ as the limit when $\varepsilon \to 0$ of a Boltzmann collision operator for Coulomb potential, with a truncated angular collision kernel,

$$b_\varepsilon(\cos\theta) = \frac{1}{\log \varepsilon^{-1}} b(\cos\theta) 1_{\theta \geq \varepsilon}.$$

The factor $\log \varepsilon^{-1}$ compensates for the logarithmic divergence of the Coulomb cross-section for momentum transfer. As for the parameter ε, it is proportional to the "plasma parameter", which is very small for classical plasmas, and actually goes to 0 as the Debye length goes to infinity.

Also in the case of non-Coulomb potentials can one define an asymptotic regime in which the Boltzmann equation turns into a Landau equation. Such a formal study was performed by Desvillettes [**169**] : he considered the limit of the Boltzmann operator when the collision kernel is of the form

$$\sin^{N-2}\theta\, b_\varepsilon(\cos\theta) = \frac{1}{\varepsilon^3} \sin^{N-2}\left(\frac{\theta}{\varepsilon}\right) b\left(\cos\left(\frac{\theta}{\varepsilon}\right)\right), \qquad \varepsilon \to 0.$$

These two asymptotic procedures seem very different both from the mathematical and the physical point of view. Also, the truncation in the Coulomb case does not correspond to the Debye cut : indeed, for the Debye interaction potential, the collision kernel does not factor up. However, all limits of the type Boltzmann $\to$ Landau can be put into a unified formalism, first sketched in our work [**446**], then extended and made more precise in Alexandre and Villani [**12**]. The idea is that all that matters is that 1) all collisions become grazing in the limit, 2) the cross-section for momentum transfer keep a finite value in the limit. For simplicity we state precise conditions only in our model case where the collision kernel factors up as $\Phi(|v-v_*|)b_\varepsilon(\cos\theta)$:

$$\text{(64)} \qquad \begin{cases} \forall\, \theta_0 > 0, \quad b_\varepsilon(\cos\theta) \underset{\varepsilon\to 0}{\longrightarrow} 0 \qquad \text{uniformly in } \theta \geq \theta_0, \\ \\ \mu_\varepsilon \equiv |S^{N-2}| \displaystyle\int_0^\pi b_\varepsilon(\cos\theta)\,(1-\cos\theta)\,\sin^{N-2}\theta\, d\theta \underset{\varepsilon\to 0}{\longrightarrow} \mu. \end{cases}$$

This limit will be referred to as the asymptotics of grazing collisions. It can be shown that in this limit, the Boltzmann operator turns into the general Landau operator

$$\text{(65)} \qquad Q_L(f,f) = \nabla_v \cdot \left(\int_{\mathbb{R}^N} dv_*\, a(v-v_*) \big[f_*(\nabla f) - f(\nabla f)_* \big] \right),$$

$$\text{(66)} \qquad a_{ij}(z) = \Psi(|z|) \left[\delta_{ij} - \frac{z_i z_j}{|z|^2} \right],$$

taking into account the identity

$$\text{(67)} \qquad \Psi(|z|) = \frac{\mu}{4(N-1)} |z|^2 \Phi(|z|).$$

According to (67), we shall use the terminology of hard, Maxwellian, or soft potentials for the Landau operator depending on whether $\Psi(|z|)$ in (66) is proportional to $|z|^{\gamma+2}$ with $\gamma > 0$, $\gamma = 0$, $\gamma < 0$ respectively. Let us insist that the most

relevant situation is the particular case introduced in (26)–(27) to describe Coulomb collisions. The general Landau operator (65) can be considered in several ways,

- either like an approximation of the Landau equation with Coulomb potential : from the (theoretical or numerical) study of the corresponding equations one estblishes results which might be extrapolated to the Coulomb case;

- either as an approximation of the effect of grazing collisions in the Boltzmann equation without cut-off : one could postulate that such an operator may be well approximated by $Q_1 + Q_2$, where Q_1 is a Boltzmann operator satisfying Grad's cut-off assumption and Q_2 is a Landau operator (see [**141**] for similar considerations).

- or like a mathematical auxiliary in the study of the Boltzmann equation. This point of view will prove useful in chapter 3.

To conclude this paragraph, we mention that limits quite similar to the asymptotics of grazing collisions appear in the study of kinetic equations modelling quantum effects such as the Bose-Einstein condensation (like the Kompaneets equation, see chapter 5), as can be seen in the recent work of Escobedo and Mischler [**209**].

3.6. What do we care about collision kernels ? We shall now try to explain in a non-rigorous manner the influence of the collision kernel (or equivalently, of the cross-section) on solutions to the Boltzmann equation. By the way, it is a common belief among physicists that the precise structure of the collision kernel (especially the angular collision kernel) has hardly any influence on the behavior of solutions. Fortunately for us mathematicians, this belief has proven to be wrong in several respects. We make it clear that the effects to be discussed only reflect the influence of the collision kernel, but may possibly come into conflict with boundary condition effects[47], for instance. We also point out that although some illustrations of these effects are known in many regimes, none of them has been shown to hold at a satisfactory degree of generality.

Distribution tails. First of all, let us be interested in the behavior of the distribution function for large velocities : how fast does it decay ? can large distribution tails occur ? The important feature here is the behavior of the *kinetic collision kernel* as the relative velocity $|v - v_*|$ goes to infinity. If the collision kernel becomes unbounded at infinity, then solutions should be well-localized, and automatically possess finite moments of very high order, even if the initial datum is badly localized[48]. On the other hand, if the collision kernel decreases at infinity, then a slow decay at infinity should be preserved as time goes by.

This effect is best illustrated by the dichotomy between hard and soft potentials. In certain situations, it has been proven that for hard potentials, no matter how slowly the initial datum decays, then at later times the solution has finite moments of all orders. For soft potentials on the other hand, a badly localized initial datum leads to a badly localized solution at later times. Precise statements and references will be given in chapter 2.

[47] In particular, it is not clear for us whether specular reflection in a non-convex domain would not entail appearance of singularities. The troubles caused by non-convex domains have been well-studied in linear transport theory, but not, to our knowledge, in the context of the Boltzmann equation.

[48] See the heuristic explanations in paragraph 2.2 of chapter 2.

As a consequence, it is also expected that the trend to equilibrium be faster for hard potentials than for soft potentials. We shall come back to this in chapter 3.

Regularization effects. We now turn to the smoothness issue. Two basic questions are in order : if the initial datum is smooth, does it imply that the solution remains smooth ? If the initial datum is nonsmooth, can however the solution become smooth ? This time the answer seems strongly dependent on the *angular collision kernel*. If the angular collision kernel is integrable (Grad's cut-off assumption), then one expects that smoothness *and lack of smoothness* are propagated in time. In other words, the solution at positive times should have precisely the same smoothness as the initial datum. The understanding of this property has made very significant progress in the past years.

On the other hand, when the collision kernel presents a *nonintegrable singularity*, then the solution should become infinitely smooth for positive times, as it does for solutions to the heat equation. This idea emerged only in the last few years, and its very first mathematical implementation was done by Desvillettes [**171**] for a one-dimensional caricature of the Boltzmann equation. Since then this area has been very active, and nowadays the regularization effect begins to be very well understood as well. Yet much more is to be expected in this direction.

Effects of kinetic singularity. If we summarize our classification of collision kernels, there are only three situations in which they can become very large : small deviation angles (as illustrated by the cut-off vs. non-cutoff assumption), large relative velocities (as illustrated by hard vs. soft potentials), and small relative velocities (as illustrated by hard vs. soft potential, but in the reverse way). If the influence of the former two is now fairly well understood, it is not so at all for the latter. It is known that a singularity of the collision kernel at small relative velocities is compatible with propagation of *some* smoothness, but no one knows if it preserves all the smoothness or if it entails blow-up effects in certain norms, or conversely regularization phenomena. In chapter 5, we shall say a little bit more on this issue, so far inexistent, and which we believe may lead to very interesting developments in the future.

4. Basic surgery tools for the Boltzmann operator

Here we describe some of the most basic, but most important tools which one often needs for a fine study of the Boltzmann operator,

$$Q(f,f) = \int_{\mathbb{R}^N} dv_* \int_{S^{N-1}} d\sigma\, B(v-v_*,\sigma)\Big(f'f'_* - ff_*\Big).$$

Later on, we shall describe more sophisticated ingredients which apply in specific situations.

4.1. Symmetrization of the collision kernel. In view of formulas (5), the quantity

$$f'f'_* - ff_*$$

is invariant under the change of variables $\sigma \to -\sigma$. Thus one can replace (from the very beginning, if necessary) B by its "symmetrized" version

$$\overline{B}(z,\sigma) = [B(z,\sigma) + B(z,-\sigma)]1_{z\cdot\sigma>0}.$$

In other words, one can always assume the deviation angle θ to be at most $\pi/2$ in absolute value. This is why all spherical integrals could be written with an angular variable going from 0 to $\pi/2$, instead of π. From the physical point of view, this constatation rests on the undiscernability of particles (and this principle does not hold for mixtures). From the mathematical point of view, this trick is very cheap, but quite convenient when one wants to get rid of *frontal* collisions (deviation angle close to π, which almost amounts to an exchange between the velocities).

4.2. Symmetric and asymmetric point of view. There are (at least) two entirely different ways to look at the Boltzmann operator $Q(f, f)$.

The first is the symmetric point of view : the important object is the "tensor product" $ff_* = f \otimes f$, and the Boltzmann operator is obtained by integrating $(f \otimes f)(v', v'_*) - (f \otimes f)(v, v_*)$ with respect to the variable v_* and the parameter σ. This point of view is often the most efficient in problems which have to do with the trend to equilibrium, because the H theorem rests on this symmetry.

On the other hand, one can consider $Q(f, f)$ as the action upon f of a linear operator which depends on f : $Q(f, f) = \mathcal{L}_f(f)$. This introduces an asymmetry between f_* (defining the operator) and f (the object on which the operator acts). This point of view turns out to be almost always the most effective in a priori estimates on the Boltzmann equation.

For many asymmetric estimates, it is important, be it for the clarity of proofs or for the methodology, to work with the bilinear (but not symmetric !) Boltzmann operator

$$Q(g, f) = \int_{\mathbb{R}^N} dv_* \int_{S^{N-1}} d\sigma\, B(v - v_*, \sigma)(g'_* f' - g_* f). \tag{68}$$

Note that we have reversed the natural order of the arguments to make it clear that $Q(g, f)$ should be understood as $\mathcal{L}_g(f)$...

4.3. Differentiation of the collision operator. The following simple identities were proven in Villani [**445**] (but certainly someone had noticed them before) :

$$\nabla Q^{\pm}(g, f) = Q^{\pm}(\nabla g, f) + Q^{\pm}(g, \nabla f). \tag{69}$$

These formulas enable one to differentiate the collision operator at arbitrary order via a Leibniz-type formula.

4.4. Joint convexity of the entropy dissipation. Remarkably, Boltzmann's entropy dissipation functional

$$D(f) = \frac{1}{4} \int dv\, dv_*\, d\sigma\, B(v - v_*, \sigma) \Big(f' f'_* - f f_* \Big) \log \frac{f' f'_*}{f f_*}$$

is a convex functional of the tensor product ff_*— but not a convex functional of f !

This property also holds for Landau's entropy dissipation, which can be rewritten as

$$D_L(f) = \frac{1}{2} \int dv\, dv_*\, \Psi(|v - v_*|) \frac{\big| \Pi(v - v_*)(\nabla - \nabla_*)(f f_*) \big|^2}{f f_*},$$

so that convexity of D_L results from convexity of the function $(x, y) \mapsto |x|^2/y$ in $\mathbb{R}^N \times \mathbb{R}_+$.

Such convexity properties may be very interesting in the study of some weak limit process, because weak convergence is preserved by tensor product. But beware ! ff_*

is a tensor product only with respect to the velocity variable, not with respect to the x variable.

4.5. Pre-postcollisional change of variables. A universal tool in the Boltzmann theory is the involutive change of variables with unit Jacobian[49]

$$(v, v_*, \sigma) \longrightarrow (v', v'_*, k), \tag{70}$$

where k is the unit vector along $v - v_*$,

$$k = \frac{v - v_*}{|v - v_*|}.$$

Since $\sigma = (v' - v'_*)/|v' - v'_*|$, the change of variables (70) formally amounts to the exchange of (v, v_*) and (v', v'_*). As a consequence, under suitable integrability conditions on the measurable function F,

$$\begin{aligned} &\int F(v, v_*, v', v'_*) B\big(|v - v_*|, k \cdot \sigma\big)\, dv\, dv_*\, d\sigma \\ &= \int F(v, v_*, v', v'_*) B\big(|v - v_*|, k \cdot \sigma\big)\, dv'\, dv'_*\, dk \\ &= \int F(v', v'_*, v, v_*) B\big(|v - v_*|, k \cdot \sigma\big)\, dv\, dv_*\, d\sigma. \end{aligned}$$

Here we have used $|v' - v'_*| = |v - v_*|$, $\sigma \cdot k = k \cdot \sigma$ to keep the arguments of B unchanged; also recall the abuse of notations $B(v - v_*, \sigma) = B(|v - v_*|, k \cdot \sigma)$.

Note that the change of variables $(v, v_*) \to (v', v'_*)$ for given σ is illegal !

4.6. Alternative representations. There are other possible parametrizations of pre- and postcollisional velocities. A very popular one is the **ω-representation**,

$$v' = v - \langle v - v_*, \omega \rangle\, \omega, \qquad v'_* = v_* + \langle v - v_*, \omega \rangle\, \omega, \qquad \omega \in S^{N-1}. \tag{71}$$

In this representation, the bilinear collision operator[50] reads

$$Q(g, f) = \frac{1}{2} \int dv_*\, d\omega\, \widetilde{B}(v - v_*, \omega) \Big(g'_* f' - g_* f \Big), \tag{72}$$

where

$$\frac{1}{2} \widetilde{B}(z, \omega) = \left| 2 \left\langle \frac{z}{|z|}, \omega \right\rangle \right|^{N-2} B(z, \sigma).$$

We have kept a factor $1/2$ in front of $\widetilde{B}$ to recall that each pair (v', v'_*) corresponds to two distinct values of ω.

One of the advantages of the ω-representation is that it is possible to change variables $(v, v_*) \longleftrightarrow (v', v'_*)$ for fixed ω and this is again an involutive transformation with unit Jacobian. Another advantage is that it is a linear change of variables. Yet, as soon as one is interested in fine questions where the symmetries of the Boltzmann operator play an important role, the σ-representation is usually more convenient.

A third representation is the one introduced by Carleman [**119**], particularly useful for the study of the gain operator Q^+ when the collision kernel satisfies Grad's

[49] A way to see that this change of variables has unit Jacobian is to use the ω-representation of paragraph 4.6. In this representation, very clearly the pre-postcollisional change of variables has unit Jacobian; and the Jacobian from the σ-representation to the ω-representation is the same for pre-collisional and for post-collisional velocities.

[50] All the representations formulas below for Q also work just the same for its gain and loss terms (Q^+ and Q^-) separately, with obvious changes.

angular cut-off. The principle of Carleman's representation is to choose as new variables v' and v'_*, the pre-collisional velocities. Of course, not all values of v' and v'_* are admissible. If v and v' are given, then the set of admissible velocities v'_* is the hyperplane $E_{vv'}$, orthogonal to $v - v'$ and going through v. Using the identity $v - v_* = 2v - v' - v'_*$, one gets

$$(73)\quad Q(g,f) = \int_{\mathbb{R}^N} dv' \int_{E_{vv'}} dv'_* \, \frac{1}{|v-v'|^{N-1}} \tilde{B}\left(2v - v' - v'_*, \frac{v'-v'_*}{|v'-v'_*|}\right) \big[g(v'_*)f(v') - g(v'+v'_*-v)f(v)\big].$$

To conclude this paragraph, we mention Tanaka's representation [**415**], which is equivalent to Maxwell's weak formulation :

$$(74)\qquad Q^+(g,f) = \int dv\, dv_*\, g_* f\Big(\Pi'_{v,v_*} - \Pi_{v,v_*}\Big),$$

where Π_{v,v_*} (resp. Π'_{v,v_*}) is a measure on the sphere S^{N-1}, $\Pi_{v,v_*} = B(v-v_*,\sigma)\, d\sigma\, \delta_v$ (resp. $\Pi'_{v,v_*} = B(v-v_*,\sigma)\, d\sigma\, \delta_{v'}$).

4.7. Monotonicity. Each time one has to handle an expression involving a nonnegative integrand and the collision kernel, it may be useful to consider it as a monotone function of the collision kernel. This point of view is particularly interesting for the entropy dissipation (47), which obviously is an increasing function of the collision kernel. Therefore, to bound (47) from below for a given collision kernel B, it is sufficient to bound it below for an auxiliary, simplified collision kernel B_0 such that $B \geq B_0$.

Most of the time, the "simplified" collision kernel will be a Maxwellian one. As we shall see in chapter 4, Maxwellian collision kernels have specific properties.

4.8. Bobylev's identities. We now turn to more intricate tools introduced by Bobylev. Even though the Boltzmann operator has a nice weak formulation (Maxwell's formula in paragraph 2.3), it is a priori quite painful to find out a representation in Fourier space. It turns out that such a representation is not so intricate, at least when the collision kernel is Maxwellian ! This fact was first brought to the attention of the mathematical community by Bobylev, who was to make Fourier transform an extremely powerful tool in the study of the Boltzmann operator with Maxwellian collision kernel (see the review in [**79**]).

Here is **Bobylev's identity :** let $b(\cos\theta)$ be a collision kernel depending only on the cosine of the deviation angle, and let

$$Q(f,f) = \int_{\mathbb{R}^N \times S^{N-1}} dv_*\, d\sigma\, b(\cos\theta)[f'f'_* - ff_*]$$

be the associated Boltzmann operator. Then its Fourier transform is

$$(75)\qquad \mathcal{F}\big[Q(g,f)\big](\xi) = \int_{S^{N-1}} \Big[\widehat{g}(\xi^-)\widehat{f}(\xi^+) - \widehat{g}(0)\widehat{f}(\xi)\Big] b\left(\frac{\xi}{|\xi|}\cdot\sigma\right) d\sigma,$$

where $\widehat{f}$ stands for the Fourier transform of f, ξ is the Fourier variable, and

$$(76)\qquad \xi^\pm = \frac{\xi \pm |\xi|\sigma}{2}.$$

Note that $|\xi^+|^2 + |\xi^-|^2 = 1$.

A remarkable feature about (75) is that the integral is now $(N-1)$-fold, instead of $(2N-1)$-fold. This formula is actually a particular case of a more general one which does not assume Maxwellian collision kernel [**10**, Appendix] :

$$(77)\quad \mathcal{F}[Q(g,f)](\xi) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N \times S^{N-1}} \widehat{B}\left(|\xi_*|, \frac{\xi}{|\xi|}\cdot\sigma\right) \Big[\widehat{g}(\xi^- + \xi_*)\widehat{f}(\xi^+ - \xi_*) - \widehat{g}(\xi_*)\widehat{f}(\xi - \xi_*)\Big] d\xi_*\, d\sigma,$$

where the Fourier transform $\widehat{B}$ of $B \equiv B(|z|, \cos\theta)$ is with respect to the variable z only. Of course, in the particular case $B(|z|,\cos\theta) = b(\cos\theta)$, we have $\widehat{B}(|\xi_*|,\cos\theta) = (2\pi)^{N/2}\delta[\xi_* = 0]\, b(\cos\theta)$, and this entails formula (75). Thus we see that the reduction of the multiplicity in the integral is directly linked to the assumption of Maxwellian collision kernel.

As a consequence of (75), results **Bobylev's lemma**[51] : if Q is a Boltzmann operator with Maxwellian collision kernel, then, whatever the Maxwellian probability distribution M,

$$Q\big(g * M, f * M\big) = Q(g,f) * M.$$

This is a very useful regularization lemma when dealing with Maxwellian collision kernels.

4.9. Application of Fourier transform to spectral schemes. Here we digress a little bit to briefly discuss numerical schemes based on Fourier transform, which are related to Bobylev's ideas. Here are the main ideas of these "spectral schemes" :

1) truncate the support of the distribution function f, then extend f into a periodic function on $\mathbb{R}^N$;

2) expand f in Fourier series, and compute the expression of the collision operator $Q(f,f)$ in terms of the Fourier coefficients of f.

Special attention must be given to the way the support is truncated ! As explained in [**372**] for instance, if the support of f is reduced to a compact set with diameter R, then it should be extended by periodicity with period $T \geq (2+\sqrt{2})R$, in order to avoid overlap problems in the computation of the collision integral. Assume for instance $T = \pi$, $R = \lambda\pi$, $\lambda = 2/(3+\sqrt{2})$.

After passing in Fourier representation,

$$\widehat{f}_k = \frac{1}{(2\pi)^N} \int_{[-\pi,\pi]^N} f(v) e^{-ik\cdot v}\, dv,$$

and truncating high Fourier modes, a very simple expression is obtained for the k-th mode $\widehat{Q}(k)$ of $Q(f,f)$:

$$\widehat{Q}(k) = \sum_{\ell+m=k;\ |\ell|,|m|\leq K} \widehat{f}_\ell\, \widehat{f}_m\, \widehat{\beta}(\ell,m), \qquad |k| \leq K$$

[51] This lemma was actually proven, for a constant collision kernel, by Morgenstern [**351**, section 10] in the fifties ! However, Bobylev was the author who devised a general proof, made this lemma widely known and linked it to other properties of Maxwellian collision kernels.

where

$$\widehat{\beta}(\ell,m)=\int_{|z|\le 2\lambda\pi} dz \int_{S^{N-1}} d\sigma B(z,\sigma)\big[e^{i\ell\cdot z^+ + im\cdot z^-}-e^{iz\cdot m}\big], \tag{78}$$

$$z^\pm=\frac{z\pm|z|\sigma}{2}.$$

Of course, this formula is very much reminiscent of (77). Spectral schemes have several advantages : once in Fourier space, the numerical simulation is immediate (just a simple first-order system of coupled ODE's). Moreover, all coefficients $\widehat{\beta}(\ell,m)$ can be computed once for all with extreme precision; this may demand some memory space, but is quite a gain in the speed of the computation. They are able to give extremely precise results. However, these schemes are rather rigid and do not allow for all the numerical tricks which can be used by other methods to reduce computational time. Moreover, since conservation laws are not built in the schemes, numerical simulations have to be conducted with a certain minimal precision in order to get realistic results.

From the mathematical point of view, this method presents very interesting features, think that it simulates the Boltzmann equation in weak formulation. In particular, the method works just the same with or without Grad's angular cutoff assumption. This can be used in theoretical works like [**374**] (analysis of the behavior of the method in the asymptotics of grazing collisions).

5. Mathematical theories for the Cauchy problem

In this section, we try to survey existing mathematical frameworks dealing with the Cauchy problem (and also more quantitative issues) for Boltzmann-like equations. These theories are all connected, but more or less tightly. Before this, in the first paragraph we describe the most apparent problems in trying to construct a general, good theory.

5.1. What minimal functional space ? In the full, general situation, known a priori estimates for the Boltzmann equation are only those which are associated to the basic physical laws :

- formal conservation of mass and energy,
- formal decrease of the entropy.

Of course, the latter means *two* estimates ! an estimate on the entropy, and another one on the dissipation of entropy.

When the position space is unbounded, say the whole of $\mathbb{R}^N$, then the properties of the transport operator also make it rather easy to get local (in time) estimates on $\int f(t,x,v)|x|^2\,dx\,dv$. For this it suffices to use the identity

$$\frac{d}{dt}\int f(t,x,v)|x-vt|^2\,dx\,dv=0.$$

On the whole, one gets for free the a priori estimates

$$\int_{\mathbb{R}^N\times\mathbb{R}^N} f(t,x,v)\big[1+|v|^2+|x|^2+\log f(t,x,v)\big]\,dx\,dv + \int_0^t\int_{\mathbb{R}^N_x} D(f(\tau,x,\cdot))\,d\tau\,dx \tag{79}$$

$$\leq \int_{\mathbb{R}^N \times \mathbb{R}^N} f(0,x,v)\big[1+2|x|^2+(2t^2+1)|v|^2+\log f(0,x,v)\big]\,dx\,dv.$$

Apart from the term in $D(f)$, estimate (79) does not depend on the collision kernel B. Disregarding this entropy dissipation estimate which is difficult to translate in terms of size and smoothness of the distribution function, all that we know for the moment about f is

$$f \in L^\infty\Big([0,T]; L^1_2(\mathbb{R}^N_x \times \mathbb{R}^N_v) \cap L\log L(\mathbb{R}^N_x \times \mathbb{R}^N_v)\Big).$$

Here

$$\|f\|_{L^1_2} = \int_{\mathbb{R}^N \times \mathbb{R}^N} f(x,v)(1+|x|^2+|v|^2)\,dx\,dv.$$

It is easily seen that the estimate in $L\log L$, combined with the moment estimate, entails an L^1 control of $f(\log f)_+$.

Remark : One may be interested in situations where the total mass is infinite (gas in the whole space). The estimates can then be adapted to this situation : see in particular Lions [**310**].

By the Dunford-Pettis compactness criterion, sequences of solutions to the Boltzmann equation with a uniform entropy estimate will be weakly compact, say in $L^p([0,T]; L^1(\mathbb{R}^N_x \times \mathbb{R}^N_v))$, and this ensures that their cluster points cannot be singular measures.

This looks like a mathematical convenience : after all, why not try to handle the Boltzmann equation in a genuine measure setting, which seems to best reflect physical intuition ? It turns out that the space of measures *is not stable*[52] — and thus irrelevant — for the study of the Boltzmann equation. This can be seen by the following remark. Consider as initial datum a linear combination of Dirac masses approximating a continuous profile (think of each of these Dirac masses as clusters of particles with a common velocity, each cluster being located at a different position). Even though the meaning of the Boltzmann equation for so singular data is not a priori clear, it is easy to figure out what weak solution we should expect : each cluster should keep moving with constant velocity, until two clusters happen to be in the same position and collide. Should this eventuality occur, particles in each cluster would be scattered in all directions, with respective probabilities given by the Boltzmann collision kernel. But the point is, by modifying very slightly the initial positions of the clusters, one can always make sure (in dimension $N \geq 2$) that *no* collision ever occurs ! Therefore, this example shows that one can construct a sequence of weak solutions of the Boltzmann equation, converging in weak measure sense towards a continuous solution of the *free transport* equation. Thus there is no stability in measure space... It would be tedious, but quite interesting, to extend this counterexample to absolutely continuous initial data, say linear combinations of very sharply peaked Maxwellians.

The aforegoing discussion is strongly linked to the fact that both x and v variables are present in the Boltzmann equation. If one is interested in solutions which do not depend on the space variable, then it is perfectly possible to construct a meaningful theory of measure solutions, as first noticed by Povzner [**389**].

[52] On the other hand, the space of functions with bounded entropy is stable, as shown by the works of DiPerna and Lions discussed below.

To come back to our original discussion, we have just seen that it is in general impossible to deal with arbitrary measures, and therefore the entropy estimate is welcome to prevent concentration. But is it enough to handle the Boltzmann operator ? Very very roughly, it seems that to control the Boltzmann operator, we would like to have a control of $\int dv_* f f_*$, which is just $\|f\|_{L^1_v} f$. The $L^1_{x,v}$ norm of this quantity is just the norm of f in $L^2_x(L^1_v)$. This is the kind of spaces in which we would like to have estimates on f. But such estimates are a major open problem ! It seems that the entropy dissipation estimate is not sufficient for that purpose, one of the troubles being caused by local Maxwellian states, which make the entropy dissipation vanish but can have arbitrarily large macroscopic densities.

Thus the only a priori estimates which seem to hold in full generality do not even allow us to give a meaningful sense to the equation we wish to study... this major obstruction is one of the reasons why the Cauchy problem for the Boltzmann equation is so tricky — another reason being the intricate nature of the Boltzmann operator. Due to this difficulty, most theories only deal with more or less simplified situations. On the other hand, for simpler models like the BGK equation, in which the collision operator is homogeneous of degree 1, the Cauchy problem is much easier [**375, 378, 398**].

By the way, the lack of a priori estimates for the Boltzmann equation is also very cumbersome when dealing with boundary conditions, especially in the case of Maxwellian diffusion. Indeed, the treatment of boundary conditions in a classical sense requires the trace of the solution to be well-defined on the boundary, which is not trivial [**271, 149**]. Weak formulations are available (see in particular Mischler [**347**]), but rather delicate.

We distinguish six main theories, inequally represented and inequally active — and by no means hermetically independent, especially in recent research. By order of historical appearance : spatially homogeneous theory, theory of Maxwellian molecules, perturbative theory, solutions in the small, renormalized solutions, one-dimensional problems.

5.2. The spatially homogeneous theory. In the spatially homogeneous theory one is interested in solutions $f(t,x,v)$ which do not depend on the x variable. This approach is rather common in physics, when it comes to problems centered on the collision operator : this looks reasonable, since the collision operator only acts on the velocity dependence. Moreover, the spatially homogeneous theory naturally arises in numerical analysis, since all numerical schemes achieve a splitting of the transport operator and the collision operator. Finally, it is expected that spatial homogeneity is a stable property, in the sense that a weakly inhomogeneous initial datum leads to a weakly inhomogeneous solution of the Boltzmann equation. Under some ad hoc smallness assumptions, this guess has been mathematically justified by Arkeryd et al. [**32**], who developed a theory for **weakly inhomogeneous solutions** of the full Boltzmann equation.

Thus, the **spatially homogeneous Boltzmann equation** reads

$$\frac{\partial f}{\partial t} = Q(f,f), \tag{80}$$

and the unknown $f = f(t, v)$ is defined on $\mathbb{R}_+ \times \mathbb{R}^N$. Note that in this situation, the only stationary states are Maxwellian distributions, by the discussion about eq. (54) in paragraph 2.5.

The spatially homogeneous theory was the very first to be developed, thanks to the pioneering works by Carleman [**118**] in the thirties. Carleman proved existence and uniqueness of a solution to the spatially homogeneous problem for a gas of hard spheres, and an initial datum f_0 which was assumed to be radially symmetric, continuous and decaying in $O(1/|v|^6)$ as $|v| \to +\infty$. He was also able to prove Boltzmann's H theorem, and convergence towards equilibrium in large time. Later he improved his results and introduced new techniques in his famous treatise [**119**]. Then in the sixties, Povzner [**389**] extended the mathematical framework of Carleman and relaxed the assumptions.

In the past twenty years, the theory of the Cauchy problem for spatially homogeneous Boltzmann equation for **hard potentials with angular cut-off** was completely rewritten and extensively developed, first by Arkeryd, then by DiBlasio, Elmroth, Gustafsson, Desvillettes, Wennberg, A. Pulvirenti, Mischler [**17, 20, 23, 186, 187, 204, 269, 270, 170, 458, 456, 457, 392, 393, 349**]. It is now in a very mature state, with statements of existence, uniqueness, propagation of smoothness, moments, positivity... Optimal conditions for existence and uniqueness have been identified by Mischler and Wennberg [**349, 463**]. Recent works by Toscani and the author [**428**] have led to almost satisfactory results about the H theorem and trend to equilibrium, even though some questions remain unsettled. Also, one can work in a genuine measure framework.

The study of singular kernels (be it soft potentials or potentials with nonintegrable angular singularities) is much more recent [**18, 172, 171, 173, 446, 449, 10**]. This area is still under construction, but currently very active.

In view of the last advances, it is quite likely that very soon, we shall have a fairly complete picture of the spatially homogeneous theory, with or without cut-off, at least when the kinetic collision kernel is not too singular for small relative velocities. A review on the state of the art for the spatially homogeneous theory is performed in Desvillettes [**174**]. A task which should be undertaken is to systematically extend all of these achievements to the framework of weakly inhomogeneous solutions [**32**].

5.3. Maxwellian molecules. After spatial homogeneity, a further simplification is the assumption that the collision kernel be Maxwellian, i.e. do not depend on the relative velocity but only on the (cosine of the) deviation angle. The corresponding theory is of course a particular case of the preceding one, but allows for a finer description and presents specific features. It was first developed by Wild and Morgenstern in the fifties [**464, 350, 351**]. Then Truesdell [**274**] showed that all spherical moments of the solutions could be "explicitly" computed. Simple explicit solutions, important for numerical simulations, were produced independently by Bobylev [**78**] and Krook and Wu.

Later, Bobylev set up and completed an ambitious program based on the Fourier transform (see the survey in [**79**]). Included were the classification of several families of semi-explicit solutions, and a fine study of trend to equilibrium. Key tools in this program were the identities of paragraph 4.8. As of now, the theory can be

considered as complete, with the exception of some non-standard problems like the identification of the image of Q or the classification of all eternal solutions[53]...

5.4. Perturbation theory. Another regime which has been extensively studied is the case when the distribution function is assumed to be very close to a global Maxwellian equilibrium, say the centered, unit-temperature Maxwellian M.

Under this assumption, it is natural to try to linearize the problem, in such a way that quadratic terms in Boltzmann's operator become negligible. In order to have a *self-adjoint* linearized Boltzmann operator, the relevant change of unknown is $f = M(1+h)$. Then, one can expand the Boltzmann operator $Q(f,f)$ by using its bilinearity, and the identity $Q(M,M) = 0$. This leads to the definition of the **linearized Boltzmann operator**,

$$Lh = M^{-1}\Big[Q(M,Mh) + Q(Mh,M)\Big].$$

L is a symmetric, nonpositive operator in $L^2(M)$ (endowed with the scalar product $(h_1,h_2)_M = \int h_1 h_2 M$), nonpositivity being nothing but the linearized version of the H theorem. Moreover, the remainder in the nonlinear Boltzmann operator, $Q(Mh,Mh)$, can be considered small if h is very close to 0 in an appropriate sense. Note that smallness of h in $L^2(M)$ really amounts to smallness of $f - M$ in $L^2(M^{-1})$, which, by the standards of all other existing theories, is an extremely strong assumption !

The linearized Boltzmann equations also have their own interest, of course, and the spectral properties of the linearized Boltzmann operator have been addressed carefully; see [**141**] for discussion and references. The different analysis required by the linearized Landau equation is performed in [**161, 298**]. A remarkable feature is that when the collision kernel is Maxwellian, then *the spectrum can be computed explicitly* (and eigenfunctions too : they are Hermite polynomials). This calculation was first performed in a classical paper by Wang Chang and Uhlenbeck [**454**], then simplified by Bobylev [**79**, p. 135] thanks to the use of Fourier transform.

Grad has set up the foundations for a systematic study of the linearized Boltzmann equation, see [**250, 252, 255**]. At the same time he initiated in this perturbative framework a "rigorous" study of hydrodynamical equations based on Chapman-Enskog or moments expansion [**249, 253**]. Later came the pioneering works of Ukai [**434, 435**] on the nonlinear perturbation [**434, 435**], followed by a huge literature, among which we quote [**333, 359, 111, 437, 403, 436, 286**]. This theory is now in an advanced stage, with existence and uniqueness theorems, and results of trend to equilibrium. The proofs often rely on the theory of linear operators, abstract theory of semigroups, abstract Cauchy-Kowalewski theorems.

As we just said, with the development of this branch of the Cauchy problem for the Boltzmann equation, came the first rigorous discussions on the transition to hydrodynamical equations, in a perturbative framework, after a precise spectral analysis of the linearized operator was performed. On this approach we quote for instance [**112, 113, 159, 203, 358, 438, 44, 334, 58, 43, 212, 213**]. Actually, in the above we have mixed references dealing with stationary and with evolutionary problems, for which the setting is rather similar... An up-to-date account of the

[53] These problems have been explained in paragraph 2.9.

present theory can be found in [**214**]. However, all known results in this direction deal with *smooth* solutions of the hydrodynamic equations.

Also related to this linearized setting is a large literature addressing more qualitative issues, like **half-space problems**, to be understood as a modelization for kinetic layers[54], or the description of simple shock waves. Among many works, here are a few such papers : [**42, 357, 115, 258, 52, 51, 332, 155, 151, 188, 214**] (the work [**37**] is an exception, in the sense that it deals with the Milne problem in the fully nonlinear case).

It is certainly mathematically and physically justified to work in a perturbative setting, as long as one keeps in mind that this only covers situations where the distribution function is *extremely close to equilibrium*. Thus this theory can in no way be considered as a general answer to the Cauchy problem.

Even taking this into account, some criticisms can be formulated, for instance the use of abstract spectral theory which often leads to nonexplicit results. Also, we note that the great majority of these works is only concerned with hard potentials with cut-off, and most especially hard spheres. Early (confusing) remarks on the Boltzmann operator without cut-off can be found in Pao [**369**] and Klaus [**288**], but this is all. Soft potentials with cut-off have been studied by Caflisch [**111**], Ukai and Asano [**437**].

Theoretical research in the area of linearized or perturbative Boltzmann equation is not so intensive as it used to be[55]. We shall not come back to this theory and for more details we address the reader to the aforementioned references.

5.5. Theories in the small. Another line of approach deals with **short-time** results. This may seem awkward for statistical equations which are mostly interesting in the long-time limit ! but may be interesting when it comes to validation issues. Conversely — but this is more delicate—, it is possible to trade the assumption of small time for an assumption of small initial datum expanding in the vacuum. This case could be described as **perturbation of the vacuum**, and exploits the good "dispersive" properties of the transport operator. The modern approach starts with the classical papers of Kaniel and Shinbrot for the small-time result [**285**], Illner and Shinbrot for the small-datum result [**278**].

Also it should be clear that Lanford's theorem[56] contains a proof of local in time existence under rather stringent assumptions on the initial datum. It was computed that Lanford's bounds allowed about 15% of the particles to collide at least once ! The extensions of Lanford's results by Illner and Pulvirenti [**275, 276**] were also adaptations of small-datum existence theorems.

Theories in the small were further developed by Toscani and Bellomo [**64, 425, 418, 419, 420, 368**], some early mistakes being corrected by Polewczak [**385**] who also proved smoothness in the x variable on this occasion. See the book of Bellomo, Palczewski and Toscani [**63**] for a featured survey of known techniques at the end of the eighties. Then the results were improved by Goudon [**246, 247**] (introducing

[54] A kinetic layer describes the transition between a domain of space where a hydrodynamic description is relevant, and another domain in which a more precise kinetic description is in order; for general background see [**148**].

[55] After completion of this review, two important works by Guo, about the Landau equation and the Boltzmann equation for soft potentials, in a close-to-equilibrium, periodic setting, popped up just to contradict this statement.

[56] See paragraph 2.1.

some new monotonicity ideas), and also Mischler and Perthame [**348**] in the context of solutions with infinite total kinetic energy.

One of the main ideas is that if the initial datum is bounded from above by a well-chosen Maxwellian (or squeezed between two Maxwellians), then this property remains true for all times, by some monotonicity argument. Therefore, solutions built by these methods usually satisfy gaussian-type bounds. To get an idea of the method, a very pedagogical reference is the short proof in Lions [**308**] which covers smooth, fast-decaying collision kernels.

Bellomo and Toscani have also studied cases where the decay of the solution is not gaussian, but only polynomial. It is in this framework that Toscani [**419**] was able to construct solutions of the Boltzmann equation in the whole space, which do not approach local equilibrium[57], as time becomes large.

From the mathematical point of view, these theories cannot really be considered in a mature stage, due to a certain rigidity. For instance, it is apparently an open problem to treat boundary conditions : only the whole space seems to be allowed. Also, since the proofs strongly rely on Grad's splitting between the gain and loss part, the treatment of non-cutoff potentials is open. Finally, the limitations of small time, or small initial datum, are quite restrictive, even though not so much as for the linearized theory (the bounds here have the noticeable advantage to be explicit). However, as mentioned above, theories in the small have led to two quite interesting, and physically controversial, results : the validation of the Boltzmann equation for hard-spheres via the Boltzmann-Grad limit, at least in certain cases; and a simple construction of solutions of the Boltzmann equation which do not approach local equilibrium as time becomes large. Recent works by Boudin and Desvillettes [**101**] in this framework also resulted in interesting proofs of propagation of regularity and singularities, which were implicitly conjectured by physicists [**148**].

We shall not develop further on this line of approach, and address the reader to the above references for more.

5.6. The theory of renormalized solutions. Introduced by DiPerna and Lions at the end of the eighties, this theory is at the moment the only framework where existence results for the full Boltzmann equation, without simplifying assumptions, can be proven [**190, 192, 194, 167, 307, 308, 309, 311, 310, 306, 316, 12, 13**]. Apart from a high technical level, this theory mainly relies on two ingredients :

- **the velocity-averaging lemmas** : under appropriate conditions, these lemmas, initiated by Golse, Perthame and Sentis [**243**] and further developed in [**242, 232, 195, 379, 99, 312, 315**], yield partial regularity (or rather regularization) results for *velocity-averages* $\int g(t,x,v)\varphi(v)\,dv$ of solutions of transport equations $\partial g/\partial t + v\cdot\nabla_x g = S$. The regularity is of course with respect to the time-space variables, and thus the physical meaning of these lemmas is that macroscopic observables enjoy some smoothness properties even when the distribution function itself does not.

- **the renormalization** : this trick allows one to give a distributional sense to the Boltzmann equation even though there does not seem to be enough a priori estimates for that. It consists in formally multiplying (8) by the

[57] See paragraph 2.5.

nonlinear function of the density, $\beta'(f)$, where β belongs to a well-chosen class of admissible nonlinearities. By chain-rule, the resulting equation reads

$$\frac{\partial \beta(f)}{\partial t} + v \cdot \nabla_x \beta(f) = \beta'(f) Q(f,f). \tag{81}$$

Assume now that $\beta'(f) \leq C/(1+f)$, for some $C > 0$. Then, since the Boltzmann operator $Q(f,f)$ is quadratic, one may expect $\beta'(f)Q(f,f)$ to be a sublinear operator of f... in which case the basic a priori estimates of mass, energy and entropy[58] would be enough to make sense of (81). Distribution functions satisfying (81) in distributional sense are called *renormalized solutions*. Strictly speaking, these solutions are neither weaker, nor stronger than distributional solutions. Typical choices for β are $\beta(f) = \delta^{-1} \log(1+\delta f)$ $(\delta > 0)$ or $\beta(f) = f/(1+\delta f)$.

Apart from the study of the Boltzmann equation, renormalization and velocity-averaging lemmas have become popular tools for the study of various kinetic equations [**191, 375, 318, 176, 229**], ordinary differential equations with nonsmooth (Sobolev-regular) coefficients [**193, 97**], or the reformulation of some hyperbolic systems of conservation laws as kinetic systems [**319, 320, 280, 381, 380**][59]. The idea of renormalization has even been exported to such areas of partial differential equations as nonlinear parabolic equations (see [**76, 77**] and the references therein). In fact, renormalization is a general tool which can be applied outside the field of renormalized solutions; in this respect see the remark at the level of formula (130).

As regards the Boltzmann equation, many fundamental questions are still unsolved : in particular uniqueness, propagation of smoothness, energy conservation, moment estimates, positivity, trend to equilibrium... Therefore, as of this date, this theory cannot be considered as a satisfactory answer to the Cauchy problem. However, it provides a remarkable answer to the *stability* problem.

The techniques are robust enough to adapt to boundary-value problems [**271, 144, 30, 347, 346**] (be careful that some of the proofs in [**271**] are wrong and have been corrected in [**30**]; the best results are those of Mischler [**347**]). As an important application of the theory of renormalized solutions, Levermore [**302**] proved the validity of the linearization approximation if the initial datum is very close to a global Maxwellian. Also the hydrodynamical transition towards some models of fluid mechanics can be justified without assumption of smoothness of the limit hydrodynamic equations : see in particular Bardos, Golse and Levermore [**57, 55, 54, 53**], Golse [**239**], Golse et al. [**241**], Golse and Levermore [**240**], Lions and Masmoudi [**317**], Golse and Saint-Raymond [**245**], Saint-Raymond [**401**]. The high point of this program is certainly the rigorous limit from the DiPerna-Lions renormalized solutions to Leray's weak solutions of the incompressible Navier-Stokes equation, which was performed very recently in [**245**]; see [**441**] for a review.

The original theory of DiPerna and Lions heavily relied on Grad's cut-off assumption, but recent progress have extended it to cover the full range of physically realistic collision kernels [**12**]. This extended theory has set a framework for the

[58] See paragraph 5.1.

[59] Recently, Bouchut has shown how to use velocity-averaging lemmas to study classical hypoellipticity in certain kinetic equations.

study of very general effects of propagation of "regularity", in the form of propagation of strong compactness [**308**], or "smoothing", in the form of appearance of strong compactness [**311, 316, 12, 13**].

Moreover, even if a uniqueness result is not available, it appears that renormalized solutions are strong enough to prove some results of weak-strong uniqueness [**308, 324**] : under certain assumptions on the collision kernel, if we know that there exists a strong solution to the Boltzmann equation, then there exists a unique renormalized solution, and it coincides with the strong solution. On the occasion of this study, Lions [**308**] pointed out the possibility to construct very weak solutions, called "dissipative solutions", which are of very limited physical value, but have been used in various areas as a powerful tool for treating some limit regimes, be it in fluid mechanics for such degenerate equations as the three-dimensional Euler equation [**313**], in hydrodynamical limits [**239, 241, 401**] or stochastic fully nonlinear partial differential equations [**321, 322**]. Thus ramifications of the DiPerna-Lions theory have been a source of inspiration for problems outside the field.

In view of these achievements and of the current vitality of the theory of renormalized solutions, we shall come back to it in more detail in the next chapters.

5.7. Monodimensional problems. It is of course impossible to speak of a monodimensional Boltzmann equation, since elastic collisions are meaningless in dimension 1. But in many problems of modelling [**148**], symmetry assumptions enable one to consider solutions depending on the position in space, x, through only one variable. From the mathematical point of view, such problems seem to present specific features, one of the reasons being that the dispersive power of the transport operator is very strong in dimension 1, so that dispersion estimates can be used to (almost) control the collision operator.

In the end of the eighties, Arkeryd [**22**] was able to apply a contraction method similar to the one in [**24**] in order to get existence results for the Boltzmann equation in one dimension of space, however he needed a physically unrealistic damping in the collision operator for small relative velocities in the space direction.

Then, building on original works by Beale [**61**] and especially Bony [**94, 95**] on discrete-velocity Boltzmann equations, Cercignani [**145, 147**] was able to extract some new estimates in this one-dimensional situation, and prove existence of "strong" solutions to the Boltzmann equation, under rather stringent assumptions on the collision kernel. Here "strong" means that $Q^{\pm}(f,f) \in L^1_{\rm loc}(\mathbb{R}_x \times \mathbb{R}^N_v)$. For some time this line of research was quite promising, but it now seems to be stalled...

CHAPTER 2

CAUCHY PROBLEM

The meaning of "Cauchy problem" in this chapter is to be understood in an extended sense : we shall not only be concerned in existence and uniqueness of solutions, but also in a priori estimates. Three main issues will be addressed : decay of the solutions at large velocities (and also at large positions, but large velocities are the main concern), smoothness, and strict positivity. As we explained above, the decay of the solutions mainly depends on the behavior of the kinetic collision kernel, while their smoothness heavily relies on the angular collision kernel. As for the strict positivity, the matter is not very clear yet.

We have adopted the following presentation : first, we recall a bit about velocity-averaging lemmas, which have become a universal tool in the study of transport equations, and we shall comment on their use in the particular context of the Boltzmann equation. In section 2, we address moment estimates, and discuss the influence of the kinetic collision kernel. Then in section 3, we first enter the core of the study of Boltzmann's operator, and we discuss issues of propagation of smoothness and propagation of singularities when the angular collision kernel is integrable (Grad's angular cut-off). Conversely, in section 4, we explain the structure of Boltzmann's operator when the angular collision kernel presents a nonintegrable singularity for grazing collisions, and associated theorems of regularization. Since the Landau equation is linked to the Boltzmann equation via the emphasis on grazing collisions, this will lead us to discuss the Landau approximation in section 5. We conclude in section 6 with lower bound estimates.

In many places the picture is incomplete, especially in the full, spatially inhomogeneous situation.

Our discussion is mainly based on a priori estimates. We have chosen not to discuss existence proofs, strictly speaking. Sometimes these proofs follow from the a priori estimates by rather standard PDE arguments (fixed point, monotonicity, compactness), sometimes they are very, very complicated. In any case they are unlikely to be of much interest to the non-specialist reader, and we shall skip them all. Complete proofs of the most famous results can be found in [**149**].

Also, in this review we insist that a priori estimates should be explicit, but we do not care whether solutions are built by a constructive or non-constructive method. This is because we are mainly concerned with qualitative statements to be made about the solutions, and their physical relevance. If we were more concerned about practical aspects like numerical simulation, then it would be important that existence results be obtained by constructive methods.

As a last remark, we note that we have excluded from the discussion all references which include nonstandard analysis [**19, 21, 25**] – just because we are not familiar with these techniques.

Contents

1. Use of velocity-averaging lemmas

1.1. Reminders. Velocity-averaging lemmas express the local smoothness in macroscopic variables (t,x) of averages of the distribution function with respect to the microscopic variable (the velocity). Here is a basic, important example : assume that f satisfies

$$\frac{\partial f}{\partial t} + v\cdot\nabla_x f = S, \qquad f\in L^2_{t,x,v}, \quad S\in L^2_{t,x}(H^{-s}_v). \tag{82}$$

Then, for any $\varphi\in C^\infty_0(\mathbb{R}^N_v)$,

$$\int_{\mathbb{R}^N} f(t,x,v)\varphi(v)\,dv \;\in\; H^{\frac{1}{2(1+s)}}_{t,x}.$$

Here H^α is the Sobolev space of order α, and when we write "$\in$", this really means "lies in a bounded subset of".

From the physical point of view, averaging lemmas express the fact that observables (typically, the local density) are smoother than the distribution function f itself. From the mathematical point of view, they are consequences of a "geometric" fact which we shall describe briefly. Consider the Fourier transform of f with respect to the variables t and x, write (τ,ξ) for the conjugate variables, then (82) becomes

$$(\tau + v\cdot\xi)\widehat{f} = \widehat{S},$$

so that

$$|\widehat{f}|^2 = \frac{|\widehat{S}|^2}{|\tau+v\cdot\xi|^2}.$$

Since the numerator vanishes for well-chosen values of v, this does not tell us much about the decay of $\widehat{f}$ as τ and ξ go to infinity. But when v varies in a compact set of $\mathbb{R}^N$, the set of values of v such that $\tau+v\cdot\xi$ is small will itself be very small; this is why *on the average* $|\widehat{f}|$ will decay at infinity faster than $|\widehat{S}|$.

Many variants are possible, see in particular [**242, 195**]. A pedagogical introduction about velocity-averaging lemmas is provided by Bouchut [**96**]. Let us make a few comments :

1. The L^2 a priori bound for f may be replaced by a L^p bound, $p>1$ (then the regularization holds in some $W^{k,q}$ Sobolev space), but not by an L^1 bound. Some replacements with L^1 estimates can be found e.g. in Saint-Raymond [**400**][1].

2. It is possible to cover cases in which the right-hand side also lies in a negative Sobolev space with respect to the x variable, provided that the exponent of differentiation be less than 1. Obviously, if the exponent is greater than 1, then the transport operator, which is first-order differential in x, cannot regularize... The case where the exponent is exactly 1 is critical, see Perthame and Souganidis [**379**].

3. The above theorem considers time and space variables $(t,x)\in\mathbb{R}\times\mathbb{R}^N$, but there are local variants, see in particular [**99**].

4. The transport operator $v\cdot\nabla_x$ may be replaced by $a(v)\cdot\nabla_x$ under various conditions on a.

5. Some vector-valued variants show that convolution products of the form $f*_v\varphi$ are smooth in all t,x,v variables.

[1] See also a recent note by Golse and Saint-Raymond.

A remarkable aspect of averaging lemmas is that they do not rely on the explicit solution to the linear transport equation (82) (at least nobody knows how to use the explicit solution for that purpose !). Instead, they are usually based on Fourier transform, or more generally harmonic analysis.

There are variants of averaging lemmas which do not lead to smoothness but to a gain of integrability, with estimates in $L^p_t(L^q_x(L^r_v))$, and sometimes apply in a larger range of exponents. Developed by Castella and Perthame [**135**] with a view of applications to the Vlasov-Poisson equation, these estimates are analogous to a famous family of inequalities due to Strichartz for the Schrödinger equation. Even though these estimates also give more information about the transport operator (which appears to be much more complex than it would seem !), it is still not very clear what to do with them. A discussion of the links between these Strichartz-like estimates and velocity-averaging lemmas can be found in Bouchut [**96**].

In the next two paragraphs, we briefly describe the interest of averaging lemmas in the context of the theory of renormalized solutions for the Boltzmann equation.

1.2. How to use velocity-averaging lemmas in the Boltzmann context ? No need to say, it would be very useful to get regularity results on averages of solutions of the Boltzmann equation. Since the Boltzmann collision operator looks a little bit like a convolution operator with respect to the v variable, we could hope to recover partial smoothness for it, etc...

However, if we try to rewrite the Boltzmann equation like (82), with $S = Q(f,f)$, we run into unsurmountable difficulties. First of all, we do not have the slightest a priori estimate on S ! Something like integrability would be sufficient, since measures can be looked as elements of negative Sobolev spaces, but even this is not known in general[2].

Next, we only have $f \in L\log L$, and this seems to be a limit case where averaging lemmas do not apply[3]... As pointed out to us by F. Bouchut, $L\log^{1+\varepsilon} L$ would be feasible, although extremely technical, but for $L\log L$ this seems to be linked with deep unsolved questions of harmonic analysis in Hardy spaces.

This is the place where the clever DiPerna-Lions renormalization trick will save the game. After rewriting the Boltzmann equation in renormalized formulation,

$$\frac{\partial \beta(f)}{\partial t} + v\cdot\nabla_x \beta(f) = \beta'(f)Q(f,f),$$

we see an opportunity to apply averaging lemmas to the function $\beta(f)$, which lies for instance in $L^1\cap L^\infty$ as soon as $\beta(f) \le Cf/(1+f)$. If we take it for granted that we shall find a meaningful definition of $\beta'(f)Q(f,f)$, i.e. a *renormalized formulation of the Boltzmann operator*, with bounds like $\beta'(f)Q(f,f) \in L^1([0,T]\times\mathbb{R}^N_x; H^{-\alpha}(\mathbb{R}^N_v))$, then we shall get a smoothness bound for $\int \beta(f)\varphi\,dv$, or $\beta(f) *_v \varphi$...

Of course, smoothness for averages of $\beta(f)$ does not mean smoothness for averages of f. But since β may be chosen to vary over a large range of admissible nonlinearities, by using approximation arguments combined with the a priori bounds on mass, energy, entropy, it is not difficult to show that averages like $\int f(v)\,\varphi(v)\,dv$, $f * \varphi$ ($\varphi \in L^\infty$) will lie in a uniformly strongly compact set of L^1.

[2]Except when the x-variable is one-dimensional, see paragraph 5.7 in chapter 1.

[3]Note however the following result by Golse and Saint-Raymond [**245**]. If $\partial_t f^n + v\cdot\nabla_x f^n = S_n$ with (f^n) weakly compact in L^1 and (S_n) bounded in L^1, then $\int \varphi(v) f^n\,dv$ is strongly compact in (t,x).

The interest of such approximation arguments is that they are robust and easy, and retain the softest information, which is gain of compactness. However, their nonexplicit nature is one of the drawbacks of the theory, and one can expect that important efforts will be devoted in the future to turn them into quantitative statements (as in [**101, 176**] for instance).

1.3. Stability/propagation/regularization. Some of the main principles in the theory of renormalized solutions are 1) as a starting point, try to work with just the basic known a priori estimates of mass, energy, entropy, entropy dissipation, 2) treat the Cauchy problem as a stability problem, and 3) replace smoothness by strong compactness.

Point 2) means the following : consider a sequence $f^n(t,x,v)_{n\in\mathbb{N}}$ of solutions, or approximate solutions, satisfying uniform a priori estimates of mass, energy, entropy, entropy dissipation. Without loss of generality, $f^n \longrightarrow f$ weakly in $L^p([0,T];L^1(\mathbb{R}^N_x\times\mathbb{R}^N_v))$, $1\le p<\infty$. Then one would like to prove that f also solves the Boltzmann equation. As a corollary, this will yield a result of existence and stability of solutions. Since a priori estimates are so poor, this is a very bad, unsatisfactory existence result. But for the same reason, this is an extremely good stability result.

Now for point 3), it consists in replacing the statement "f is a smooth function", which is meaningless in a framework where so little information is available, by the statement "f^n lies in a strongly compact set of L^1". For instance,

- "smoothness propagates in time" is replaced by "if $f^n(0,\cdot,\cdot)$ lies in a strongly compact set of $L^1(\mathbb{R}^N_x\times\mathbb{R}^N_v)$, then for all $t>0$, so does $f^n(t,\cdot,\cdot)$";
- "singularities propagate in time" is replaced by "if $f^n(0,\cdot,\cdot)$ does not lie in a strongly compact set of $L^1(\mathbb{R}^N_x\times\mathbb{R}^N_v)$, then for all $t>0$, neither does $f^n(t,\cdot,\cdot)$";
- "there is an immediate regularization of the solution" is replaced by "for (almost all) $t>0$, $f^n(t,\cdot,\cdot)$ lies in a strongly compact set of $L^1(\mathbb{R}^N_x\times\mathbb{R}^N_v)$".

Note that the second item in the list can be rephrased as "smoothness propagates backwards in time".

One of the nice features of the theory of renormalized solutions is that, with the help of averaging lemmas, these goals can be achieved by a good understanding of the structure of the Boltzmann operator alone. This approach has been developed by Lions, especially in [**307**] and in [**311**]. As a typical example, if we suspect some regularization effect due to collisions and wish to prove appearance of strong compactness, then, it essentially suffices to derive some smoothness estimate in the velocity variable, coming from an a priori estimate where the effect of collisions would be properly used[4], together with a meaningful renormalized formulation. Indeed, the velocity-smoothness estimate would imply that whenever φ is an approximation of a Dirac mass, then $f^n *_v \varphi$ should be close to f^n, in strong sense. On the other hand, from the use of averaging lemmas one would expect something like : $f^n * \varphi$ is "smooth" in t,x,v. Then the strong compactness would follow. This strategy was introduced by Lions [**311**].

Of course, the technical implementation of these fuzzy considerations turns out to be very intricate. All of the statements of the previous lines are only approximately satisfied : for instance we will not know that $f^n *_v \varphi$ is close to f^n, but

[4]Most typically, the entropy dissipation estimate.

rather that $\gamma(f^n) *_v \varphi$ is close to $\gamma(f^n)$, and we will not know whether this holds for almost all t, x, but only for those t, x at which the local mass, energy, entropy are not too high, etc. In all the sequel we shall conscientiously wipe out all of these difficulties and address the reader to the references above for details. On the other hand, we shall carefully describe the structure of the Boltzmann operator, its renormalized formulation, and how these properties relate to statements of propagation of smoothness or regularization.

2. Moment estimates

Moment estimates are the first and most basic estimate for the Boltzmann equation. Since one wants to control the energy (= second moment), it is natural to ask for bounds on moments higher than 2. In fact, if one wants to rigorously justify the identity

$$\int_{\mathbb{R}^N} Q(f,f)|v|^2\,dv = 0,$$

and if the kinetic collision kernel in the Boltzmann operator behaves like $|v-v_*|^\gamma$, then it is natural to ask for bounds on the moments of order $2+\gamma$.

Of course, once the question of local (in time) estimates is settled, one would like to have information on the long-time behavior of moments.

In the spatially homogeneous situation, moment estimates are very well understood, and constitute the first step in the theory. In the case of the full, spatially-inhomogeneous Boltzmann equation there is absolutely no clue of how to get such estimates. This would be a major breakthrough in the theory. As for perturbative theories, they are not really concerned with moment estimates : by construction, solutions have a very strong (typically, gaussian) decay at infinity.

As for the long-time behavior of moments, it is also well controlled in the spatially homogeneous case. In the full setting, even for much simpler, linear variants of the Boltzmann equation, the problem becomes much trickier, and satisfactory answers are only beginning to pop out now.

In all the sequel, we shall only discuss the **spatially homogeneous situation**. The starting point of most estimates [**389, 204, 170, 460, 349**] is the weak formulation

(83)
$$\int_{\mathbb{R}^N_v} Q(f,f)\varphi(v)\,dv = \frac{1}{2}\int_{\mathbb{R}^{2N}} dv\,dv_*\,f f_* \int_{S^{N-1}} B\big(|v-v_*|,\cos\theta\big)\big(\varphi'+\varphi'_*-\varphi-\varphi_*\big)\,d\sigma,$$

applied to $\varphi(v)=|v|^s$, $s>2$, or more generally to $\psi(|v|^2)$, where ψ is an increasing convex function.

2.1. Maxwellian collision kernels. The most simple situation is when the collision kernel is Maxwellian. As noticed by Truesdell [**274**], integral expressions like $\int_{S^{N-1}} b(\cos\theta)\varphi(v')\,d\sigma$ can be explicitly computed when φ is a homogeneous polynomial of the velocity variable. As a consequence, the integral in (83) can be expressed in terms of moments of f and angular integrals depending on b. This makes it possible to establish a closed system of differential equations for all "homogeneous" moments. So in principle, the exact values of all moments can be explicitly computed for any time. Then, Truesdell showed that all moments which are bounded at initial time converge exponentially fast to their equilibrium values. Moreover, if some moment is infinite at initial time, then it can never become finite.

2.2. Hard potentials. In the case of hard potentials, or more generally when the kinetic collision kernel grows unbounded at infinity, then the solution to the Boltzmann equation is expected to be well-localized at infinity, even if the decay at initial time is relatively slow. Heuristically, this can be understood as follows : if the collision kernel diverges for very large relative velocity, then very fast particles have a very high probability to collide with rather slow particles, which always constitute the majority of the gas. Thus, these fast particles will certainly be slowed down very quickly.

At the level of weak formulations like (83), this means that the "dominant" part will be negative (as soon as φ is a convex function of $|v|^2$). More precisely, if, say, $B(|v-v_*|,\cos\theta) = |v-v_*|^\gamma b(\cos\theta)$, $\gamma > 0$, then, for some constants $K > 0$, $C < +\infty$ depending only on s, N, γ and b,

$$\int_{\mathbb{R}^N} Q(f,f)|v|^s\,dv \le -K\left(\int_{\mathbb{R}^N} f\,dv\right)\left(\int_{\mathbb{R}^N} f|v|^{s+\gamma}\,dv\right) + C\left(\int_{\mathbb{R}^N} f|v|^\gamma\,dv\right)\left(\int_{\mathbb{R}^N} f|v|^s\,dv\right). \tag{84}$$

This inequality is just one example among several possible ones. It easily follows from the **Povzner inequalities** or their variants, introduced in [**389**] and made more precise by Elmroth [**204**], Wennberg [**460**], Bobylev [**85**], Lu [**328**]. Here is a typical Povzner inequality from [**328**] : for any $s > 2$, and $\gamma \le \min(s/2, 2)$, $|\theta| \le \pi/2$,

$$|v'|^s + |v'_*|^s - |v|^s - |v_*|^s \le -\kappa_s(\theta)|v|^s + C_s\Big(|v|^{s-\gamma}|v_*|^\gamma + |v|^\gamma|v_*|^{s-\gamma}\Big),$$

where $\kappa_s(\theta)$ is an explicit function of θ, strictly positive for $0 < \theta \le \pi/2$.

Let us now look at the applications to the solutions of the spatially homogeneous Boltzmann equation. As a consequence of (84) and the conservation of mass ($\int f = 1$), Elmroth [**204**] proved uniform boundedness of all moments which are finite at initial time. This kind of estimates has been simplified and lies at the basis of spatially homogeneous theory. Let us explain the argument without entering into details, or looking for best possible constants. Multiplying the inequality above by $|v - v_*|^\gamma$ ($0 < \gamma < 2$, say), and integrating against the *angular* collision kernel, one easily gets

$$\frac{1}{4}\int_{S^{N-1}} b(\cos\theta)|v-v_*|^\gamma\Big(|v'|^s + |v'_*|^s - |v|^s - |v_*|^s\Big)\,d\sigma \le -K|v|^{s+\gamma} + C\Big(|v|^s|v_*|^\gamma + |v_*|^s|v|^\gamma\Big) \tag{85}$$

(additional terms, like $|v|^{s-\gamma}|v_*|^{2\gamma}$, are easily absorbed into the last term in the right-hand side by Young's inequality). Then, let us integrate (85) against ff_* : after application of Fubini's identity, we find (84), which can be rewritten as

$$\frac{d}{dt}\int_{\mathbb{R}^N} f(t,v)|v|^s\,dv \le -K\left(\int_{\mathbb{R}^N} f\,dv\right)\left(\int_{\mathbb{R}^N} f|v|^{s+\gamma}\,dv\right) + C\left(\int_{\mathbb{R}^N} f|v|^s\,dv\right)\left(\int_{\mathbb{R}^N} f|v|^\gamma\,dv\right) \tag{86}$$

The last integral is bounded because of the energy bound and $\gamma < 2$. Since $\int f = 1$, one finds that the s-order moments,

$$M_s(t) = \int f(t,v)|v|^s \, dv,$$

satisfy some system of differential inequalities

$$\frac{d}{dt} M_s \leq -K_s M_{s+\gamma} + C_s M_s. \tag{87}$$

Now, by Hölder's inequality and $\int f = 1$ again,

$$M_{s+\gamma} \leq M_s^{1+\gamma/s}. \tag{88}$$

Since solutions of $dX/dt \leq -CX^{1+\alpha} + CX$, $\alpha > 0$, are uniformly bounded, Elmroth's theorem follows.

Desvillettes [**170**] pointed out that the conclusion is much stronger : if at least one moment of order $s > 2$ is finite at initial time, then *all* moments immediately become finite for positive times — and then of course, remain uniformly bounded as time goes to infinity. The result was further extended by Wennberg [**460**], Wennberg and Mischler [**349**], in particular the assumption of finite moment of order $s > 2$ at initial time can be dispended with. Moreover, these results hold for cut-off or non-cutoff angular collision kernels.

Bobylev [**85**] has given a particularly clear discussion of such moment estimates, with various explicit bounds of $M_s(t)$ in terms of $M_s(0)$ and s. A very interesting by-product of this study was the proof of *gaussian tail* estimates. By precise estimates of the growth of the bounds on $M_s(t)$, he was able to prove that if the initial datum satisfies $\int \exp(\alpha_0 |v|^2) \, f_0(v) \, dv < +\infty$ for some $\alpha_0 > 0$, then, at least when $\gamma = 1$ (hard spheres), there exists some $\alpha > 0$ such that

$$\sup_{t \geq 0} \int_{\mathbb{R}^N} \exp(\alpha |v|^2) \, f(t,v) \, dv < +\infty.$$

Anticipating a little bit on precise results for the Cauchy problem, we can say that moment estimates have been a key tool in the race for optimal uniqueness results in the context of *hard potentials with cut-off*. In fact, progress in this uniqueness problem can be measured by the number of finite moments required for the initial datum : Carleman needed 6, Arkeryd [**17**] only 4, Sznitman [**412**] was content with 3, Gustafsson [**270**] with $2 + \gamma$, Wennberg [**458**] needed only $2 + \varepsilon$ ($\varepsilon > 0$). Finally, Mischler and Wennberg [**349**] proved uniqueness under the sole assumption of finite energy. On this occasion they introduced "reversed" forms of Povzner inequalities, which show that the kinetic energy of weak solutions to the Boltzmann equation can only *increase* or stay constant; hence the uniqueness result holds in the class of weak solutions whose kinetic energy is nonincreasing.

More surprisingly, these moment estimates can also be used for proving *nonuniqueness* results ! in the class of weak solutions whose kinetic energy is not necessarily constant, of course. The idea, due to Wennberg [**463**], is quite simple : consider a sequence $(f_0^n)_{n \in \mathbb{N}}$ of initial data, made up of a Maxwellian (equilibrium) distribution, plus a small bump centered near larger and larger velocities as $n \to \infty$. The bump is chosen in such a way that its contribution to the total mass is negligible as $n \to \infty$, but not its contribution to the kinetic energy; so that the total kinetic

energy of f^n is, say, twice the energy E of the Maxwellian. For each n, one can solve the corresponding Boltzmann equation with hard potentials, and it has energy $2E$. One can check that, as $n \to \infty$, this sequence of solutions converges, up to extraction of a subsequence, to a weak solution of the Boltzmann equation, with Maxwellian initial datum. But, by means of some precise uniform moment bounds, one can prove that for positive times, the kinetic energy passes to the limit :

$$\forall t > 0, \qquad \lim_{n\to\infty} \int_{\mathbb{R}^N} f^n(t,v)|v|^2\,dv = \int_{\mathbb{R}^N} f(t,v)|v|^2\,dv.$$

Hence this weak solution f of the Boltzmann equation has energy E at time 0, and energy $2E$ for any time $t > 0$, in particular it is not the stationary solution...

2.3. Soft potentials. When the kinetic collision kernel decays as $|v-v_*| \to \infty$, or more generally when it is uniformly bounded, then local in time moment estimates are much easier to get. On the other hand, the result is much weaker, since only those moments which are initially bounded, can be bounded at later times.

When the collision kernel presents a singularity for zero relative velocity, say $|v - v_*|^\gamma$ with $\gamma < 0$, then additional technical difficulties may arise. When $\gamma \leq -1$, and even more when $\gamma < -2$, it is not a priori clear that power laws $|v|^s$ (or their mollified versions, $(1 + |v|^2)^{s/2}$) are admissible test-functions for the spatially homogeneous Boltzmann equation. This difficulty is overcome for instance by the method in Villani [**446**], or the remarks in [**10**].

In all cases, anyway, one proves local in time propagation of moments. But now, due to the decay of the collision kernel at infinity, it becomes considerably more difficult to find good *long time* estimates. Obtaining polynomial bounds is quite easy, but this is not always a satisfactory answer. The main result on this problem is due to Desvillettes [**170**] : he showed that when the kinetic collision kernel behaves at infinity like $|v - v_*|^\gamma$, $-1 < \gamma < 0$, then all those moments which are initially bounded, can be bounded by $C(1 + t)$, as the time t goes to infinity. By interpolation, if the initial datum has a very good decay at infinity (many finite moments), then the growth of "low"-order moments will be very slow. Thus, even if uniform boundedness is not proven, the "escape of moments at infinity" has to be slow.

The results of Desvillettes can be extended to the case where $\gamma > -2$, though there is no precise reference for that (at the time where Desvillettes proved his result, weak solutions were not known to exist for $\gamma \leq -1$). On the other hand, when $\gamma \leq -2$, then it is still possible to prove local in time propagation of moments of arbitrary order, but the bounds are in general polynomial and quite bad...

In the case of the Landau equation where things are less intricate, one can derive the following estimate [**429**] when the decay at infinity is like $|v-v_*|^\gamma$, $-3 \leq \gamma \leq -2$: then the moment of order s grows no faster than $O((1 + t)^\lambda)$, with $\lambda = (s - 2)/3$.

2.4. Summary. We now sum up all the preceding discussion in a single theorem. As we said above, for the time being it is only in the spatially homogeneous setting that relevant moment estimates have been obtained. The conditions of the following theorem are enough to guarantee existence of weak solutions, but not necessarily uniqueness.

THEOREM 1. *Let $B(|v - v_*|, \cos\theta) = |v - v_*|^\gamma b(\cos\theta)$ be a collision kernel, $-3 \leq \gamma \leq 1$, $\int_0^\pi b(\cos\theta)(1-\cos\theta)\,\sin^{N-2}\theta\,d\theta < +\infty$. Let $f_0 \in L^1_2(\mathbb{R}^N_v)$ be an initial datum*

with finite mass and energy, and let $f(t,v)$ be a weak solution of the Boltzmann equation, $f(0,\cdot) = f_0$, whose kinetic energy $\int f(t,v)|v|^2\,dv$ is nonincreasing. Then this kinetic energy is automatically constant in time. Moreover, if

$$M_s(t) \equiv \int f(t,v)|v|^s\,dv,$$

then,

(i) If $\gamma = 0$, then for any $s > 2$,

$$\forall\, t > 0, \qquad \Big[M_s(t) < +\infty \Longleftrightarrow M_s(0) < +\infty\Big],$$

$$M_s(0) < +\infty \Longrightarrow \sup_{t\geq 0} M_s(t) < +\infty.$$

Moreover, under the sole assumption $M_2 < +\infty$, there exists a convex increasing function ϕ, $\phi(|v|) \to \infty$ as $|v| \to \infty$, such that

$$\sup_{t\geq 0} \int f(t,v)\phi(|v|)|v|^2\,dv < +\infty.$$

(ii) If $\gamma > 0$, then for any $s > 2$,

$$\forall\, t_0 > 0, \qquad \sup_{t\geq t_0} M_s(t) < +\infty.$$

Moreover, if $\gamma = 1$ and $N = 3$, and the initial datum f_0 satisfies $\int \exp(\alpha_0|v|^2)\, f_0(v)\,dv < +\infty$ for some $\alpha_0 > 0$, then there exists some $\alpha > 0$ such that

$$\sup_{t\geq 0} \int e^{\alpha|v|^2}\, f(t,v)\,dv < +\infty.$$

(iii) If $\gamma < 0$, then for any $s > 2$,

$$\forall t > 0, \qquad \big[M_s(t) < +\infty \Longleftrightarrow M_s(0) < +\infty\big].$$

Moreover,

a) if $\gamma > -2$, then

$$M_s(0) < +\infty \Longrightarrow \exists C > 0, \quad M_s(t) \leq C(1+t);$$

in particular, for any $\varepsilon > 0$,

$$M_{s/\varepsilon} < +\infty \Longrightarrow \exists C > 0, \quad M_s(t) \leq C(1+t)^{\varepsilon};$$

b) if $\gamma < -2$, then

$$M_s(0) < +\infty \Longrightarrow \exists C > 0,\, \exists \lambda > 0, \quad M_s(t) \leq C(1+t)^{\lambda}$$

($\lambda = 1$ for $s \leq 4$).

Remarks :

1. All the constants in this theorem are explicit.

2. The statement about finiteness of $\int f(t,v)\phi(v)|v|^2\,dv$ in point (i) is interesting because it implies

$$\overline{\lim_{R\to\infty}} \sup_{t\geq 0} \int_{|v|\geq R} f(t,v)|v|^2\,dv = 0;$$

in other words, no energy leaks at infinity. Such an estimate is obvious in situation (ii); it is a seemingly difficult open problem[5] in situation (iii).

3. The range $\gamma \in [-3, 1]$ has been chosen for convenience; it would be possible to adapt most of the proofs to larger values of γ, maybe at the expense of slight changes in the assumptions. Values of γ which would be less than -3 pose a more challenging problem, but do not correspond to any physical example of interest.

The first part of point (i) is due to Truesdell [**274**], while the statement about point (ii) is mainly due to Desvillettes [**170**] and improved by Wennberg [**458**], Wennberg and Mischler [**349**]; the estimate for exponential moments is due to Bobylev [**85**]. As for point (iii), it is proven in Desvillettes [**170**] for $\gamma > -1$, and elements of the proof of the rest can be found in [**446, 444**].

For the Landau equation (with $\Psi(|v-v_*|) = K|v-v_*|^{\gamma+2}$) the very same theorem holds, with the following modification : point (ii) is known to hold only if there exists $s_0 > 2$ such that $M_{s_0}(0) < +\infty$ (see [**182**]). As for point (iii)c), the more precise estimate $\lambda = (s-2)/3$ holds [**429**], at least if the collision kernel is replaced by a mollification which decreases at infinity like $|v - v_*|^\gamma$, but does not present a singularity for $|v - v_*| \simeq 0$.

3. The Grad's cut-off toolbox

We now present several tools which are useful to the study of Boltzmann's equation when the collision kernel satisfies Grad's angular cut-off assumption. This means *at least* that whenever $|v - v_*| \neq 0$,

$$A(|v - v_*|) \equiv \int B(v - v_*, \sigma)\, d\sigma < +\infty. \tag{89}$$

Typical examples are $|v - v_*|^\gamma b(\cos\theta)$, where

$$\int_0^\pi b(\cos\theta)\, \sin^{N-2}\theta\, d\theta < +\infty.$$

We shall mainly insist on two ingredients : the important Q^+ regularity theorem, and the DiPerna-Lions renormalization.

3.1. Splitting. When Grad's assumption holds true, then one can split the Boltzmann collision operator into the so-called "gain" and "loss" terms, and the loss term is then particularly simple. We give this splitting in asymmetric form :

$$Q(g, f) = Q^+(g, f) - Q^-(g, f) = Q^+(g, f) - f(A * g). \tag{90}$$

Clearly, the delicate part in the study is to understand well enough the structure of the complicated integral operator

$$Q^+(g, f) = \int_{\mathbb{R}^N} dv_* \int_{S^{N-1}} d\sigma\, B(v - v_*, \sigma) g(v'_*) f(v').$$

As early as in the thirties, this problem led Carleman to the alternative representation

$$Q^+(g, f) = \int_{\mathbb{R}^N} dv' \int_{E_{vv'}} dv'_* \, \frac{1}{|v - v'|^{N-1}} \tilde{B}\left(2v - v' - v'_*, \frac{v' - v'_*}{|v' - v'_*|}\right) g(v'_*) f(v'), \tag{91}$$

[5] See paragraph 5.3 in chapter 3 for more, and some results.

with $E_{vv'}$ standing for the hyperplane going through v, orthogonal to $v' - v$.

In paragraphs 3.3 and 3.4, we shall expand a little bit on the structure of the Q^+ operator. Before that, we give an easy lemma about the control of Q^+ by means of the entropy dissipation.

3.2. Control of Q^+ by Q^- and entropy dissipation. Using the elementary identity

$$X \leq KY + \frac{1}{\log K}(X - Y) \log \frac{X}{Y}, \qquad K > 1,$$

with $X = f'f'_*$ and $Y = ff_*$, we find, after integration against $B\, dv_*\, d\sigma$,

$$Q^+(f, f) \leq KQ^-(f, f) + \frac{4}{\log K} d(f), \tag{92}$$

where

$$d(f) = \frac{1}{4} \int_{\mathbb{R}^N \times S^{N-1}} (f'f'_* - ff_*) \log \frac{f'f'_*}{ff_*}\, B\, dv_*\, d\sigma$$

is a nonnegative operator satisfying $\int d(f)\, dv = D(f)$, the entropy dissipation functional.

Inequality (92) was first used by Arkeryd [**21**], and has proven very useful in the DiPerna-Lions theory [**192**].

3.3. Dual estimates. Many estimates for Q^+ are best performed in dual formulation, with the help of the pre-postcollisional change of variables. For instance, to bound $\|Q^+(g, f)\|_{L^p(\mathbb{R}^N)}$, it is sufficient to bound

$$\int_{\mathbb{R}^N} Q^+(f, g)\varphi\, dv = \int_{\mathbb{R}^{2N}} dv\, dv_*\, g_* f \left(\int_{S^{N-1}} B\big(|v - v_*|, \cos\theta\big)\varphi(v')\, d\sigma \right)$$

uniformly for $\|\varphi\|_{L^{p'}} \leq 1$. So the meaningful object is the linear operator

$$\varphi \mapsto \int_{S^{N-1}} B\big(|v - v_*|, \cos\theta\big)\varphi(v')\, d\sigma. \tag{93}$$

Pushing the method a little bit, one easily arrives at the following abstract result : let X, Y be two Banach spaces of distributions, equipped with a translation-invariant norm. Assume that the linear operator

$$\mathcal{T} : \varphi \longmapsto \int_{S^{N-1}} B\big(|v|, \cos\theta\big)\ \varphi\left(\frac{v + |v|\sigma}{2}\right)\, d\sigma$$

is bounded (as a linear map) from Y to X. Then, the following estimate on Q^+ holds,

$$\|Q^+(g, f)\|_{Y'} \leq C\|g\|_{L^1}\|f\|_{X'}.$$

Actually, Y' (resp. X') does not really need to be the dual of Y (resp. X), it suffices that $\|Q^+\|_{Y'} = \sup\{\int Q^+ g; \|g\|_Y = 1\}$ (resp. $\int f\psi \leq \|f\|_X\|\psi\|_{X'}$).

As an example of application of this result, consider the simple situation $B(v - v_*, \sigma) = \Phi(|v - v_*|)b(\cos\theta)$, where Φ is bounded and $b(\cos\theta)\, \sin^{N-2}\theta$ is integrable with support in $[0, \pi/2]$. Obviously, $\mathcal{T}$ is bounded $L^\infty \to L^\infty$, and by the change of variables $v \to v'$ (which is valid for fixed σ because we have restricted ourselves to $\theta \in [0, \pi/2]$), one can prove that $\mathcal{T}$ is bounded $L^1 \to L^1$. By interpolation, $\mathcal{T}$ is bounded $L^p \to L^p$ for $1 \leq p \leq \infty$, and therefore we obtain the estimate

$$\|Q^+(g, f)\|_{L^p} \leq C\|g\|_{L^1}\|f\|_{L^p}, \qquad 1 \leq p \leq \infty. \tag{94}$$

A variant of the argument when $B(v-v_*,\sigma)=|v-v_*|^\gamma b(\cos\theta)$, $\gamma>0$, leads to

$$\|Q^+(g,f)\|_{L^p}\le C\|g\|_{L^1_\gamma}\|f\|_{L^p_\gamma},\qquad 1\le p\le\infty, \tag{95}$$

where we use the notation

$$\|f\|_{L^p_s}=\left(\int_{\mathbb{R}^N} f^p(1+|v|^s)^p\,dv\right)^{1/p}. \tag{96}$$

More sophisticated variants of estimates like (95) have been studied by more complicated means in Gustafsson [**269, 270**]. They constitute a first step in the L^p theory for the spatially homogeneous Boltzmann equation with hard potentials and cut-off. These estimates show that, in first approximation, the Q^+ operator resembles a convolution operator. But we shall see in the next paragraph that a stronger property holds.

To conclude this paragraph, we mention that the case where there is a singularity in the kinetic collision kernel for $|v-v_*|\simeq 0$ (soft potentials...) has never been studied very precisely from the point of view of L^p integrability.

3.4. Lions' theorem : the Q^+ regularity. One of the main ideas of Grad [**252**], when developing his linear theory and making clear his assumptions, was that the Q^+ term should be considered as a "perturbation". This may sound strange, but think that the linear counterpart of Q^+ is likely to be an integral operator with some nice kernel, while the linear counterpart of Q^- will contain a multiplicative, noncompact operator. Generally speaking, since Q^+ is more "mixing" than Q^-, we could expect it to have a smoothing effect.

In a nonlinear context, the idea that Q^+ should be smoother than its arguments was made precise by Lions [**307**]. In this paper he proved the following estimate.

Proposition 2. *Let B be a C^∞ collision kernel, compactly supported as a function of $|v-v_*|$, vanishing for $|v-v_*|$ small enough, compactly supported as a function of $\theta\in(0,\pi)$. Then, there is a finite constant C depending only on B and N such that for any $g\in L^1(\mathbb{R}^N)$, $f\in L^2(\mathbb{R}^N)$,*

$$\|Q^+(g,f)\|_{H^{\frac{N-1}{2}}}\le C\|g\|_{L^1}\|f\|_{L^2}, \tag{97}$$

and similarly

$$\|Q^+(g,f)\|_{H^{\frac{N-1}{2}}}\le C\|g\|_{L^2}\|f\|_{L^1}. \tag{98}$$

The proof was based on a duality method quite similar to the one above, and very sophisticated tools about Fourier integral operators; this estimate is actually linked to the regularity theory of the Radon transform. In fact, as a general rule [**406**], operators of the form

$$\mathcal{T}\varphi(x)=\int_{S_x} b(x,y)\varphi(y)\,d\sigma_x(y),$$

where b is a smooth kernel and S_x is a hypersurface varying smoothly with $x\in\mathbb{R}^N$, satisfy an estimate like $\|\mathcal{T}\varphi\|_{H^{(N-1)/2}}\le C\|\varphi\|_{L^2}$, under some nondegeneracy condition[6] about the way S_x varies with x.

[6] In the case of operator (93), S_x is the sphere with diameter $[0,x]$ and σ_x is just the uniform measure on S_x. The nondegeneracy condition is not satisfied at $y=0$ (which is fixed), and this is why Lions' theorem does not apply in presence of frontal collisions, and needs the collision kernel to be vanishing close to $\theta=\pi$. Another degeneracy arises when $v-v_*$ goes to 0, so the Q^+ smoothness also needs vanishing of the collision kernel at zero relative velocity.

Lions' theorem has become one of the most powerful tools in the study of fine properties of the Boltzmann equation with cut-off (see [**101, 307, 308, 349, 429, 393**],...) Wennberg [**459**] gave a simplified proof of this result, with explicit(able) constants, by using Carleman's representation (91). Then, Bouchut and Desvillettes [**98**], and, independently, Lu [**325**] devised an elementary proof, with simple constants, of a slightly weaker result :

$$\|Q^+(g,f)\|_{H^{\frac{N-1}{2}}} \leq C\|g\|_{L^2}\|f\|_{L^2}. \tag{99}$$

However, the qualitative difference between (97) and (99) is significant in some applications : see for instance our a priori estimates in L^p norms for collision kernels which decay at infinity [**429**].

Also a relativistic variant of this result has been established by Andréasson [**14**]. In a more recent paper, Wennberg [**462**] has put both the classical and the relativistic estimates in a unified context of known theorems for the regularity of the Radon transform. He noticed that these two cases are the only ones, for a whole range of parameters, where these theorems apply.

Of course, by Sobolev embedding (and interpolation), Lions' theorem yields refinements of (94) when the collision kernel is smooth. A task which should be undertaken is to use a precise version of Lions' theorem, like the one by Wennberg, to derive nice, improved weighted L^p bounds for realistic collision kernels. This, combined with the methods in [**429**], should enable one to recover the main results of Gustafsson in a much more elegant and explicit way[7].

As a last remark, the fact that the collision kernel vanish at $v - v_* = 0$ is essential in Lions' theorem (not in the weaker version (99)). If this is not the case, then smoothness results similar to the ones obtained by (97) require additional integrability conditions on g. In the case of hard potentials however, it is still possible to prove a weaker gain of smoothness or integrability with respect to f, without further assumptions on g.

3.5. Duhamel formulas and propagation of smoothness. The idea to consider Q^+ as a perturbation can be made precise by the use of Duhamel-type formulas. For instance, in the spatially homogeneous case, the Boltzmann equation can be rewritten as

$$\frac{\partial f}{\partial t} + (Lf)f = Q^+(f,f), \qquad f(0,\cdot) = f_0,$$

where we use the notation $A * f = Lf$. Then the solution can be represented as

$$f(t,v) = f_0(v)e^{-\int_0^t Lf(\tau,v)\,d\tau} + \int_0^t e^{-\int_s^t Lf(\tau,v)\,d\tau} Q^+(f,f)(s,v)\,ds. \tag{100}$$

In the spatially inhomogeneous case, one can write similarly

$$\left(\frac{\partial}{\partial t} + v\cdot\nabla_x\right) f e^{\int_0^t Lf(\tau,x-(t-\tau)v,v)\,d\tau} = Q^+(f,f)e^{\int_0^t Lf(\tau,x-(t-\tau)v,v)\,d\tau},$$

[7] At the time of writing, this task has just been performed by C. Mouhot with the help of the author's advice.

or

$$f(t,x,v) = f_0(x-tv,v)e^{-\int_0^t Lf(\tau,x-(t-\tau)v,v)\,d\tau} + \int_0^t Q^+(f,f)(s,x-(t-s)v,v)e^{-\int_s^t Lf(\tau,x-(t-\tau)v,v)\,d\tau}. \tag{101}$$

By these formulas, one can understand, at least heuristically, the phenomenon of **propagation of regularity and singularities**. Let us illustrate this in the case of the spatially homogeneous Boltzmann equation, starting with formula (100). If $f_0 \in L^2$, then, at least for collision kernels which become very large at infinity, $f(t,\cdot)$ is uniformly bounded in L^2. Now, assume for simplicity that the collision kernel B is sufficiently smooth that estimate (97) applies, then it becomes clear that the second term in the right-hand side of (100) is H^1-smooth (when $N = 3$). Indeed, thanks to the convolution structure, Lf is also smooth, say C^∞ if B is C^∞. On the other hand, the first term on the right-hand side has exactly the same smoothness as f_0. Thus both regularity and singularities are propagated in time. More precise theorems of this kind, for some realistic collision kernels, e.g. hard spheres, are to be found in Wennberg [**459**]. In principle one could actually prove that

$$f(t,v) = G(t,v)f_0(v) + H(t,v), \tag{102}$$

where, at least if B is very smooth, G is a positive C^∞ function of t, v and H is smoother than f_0. This would follow by an iteration of Duhamel's formula, in the spirit of [**349**]. Alternatively, one expects that $f(t,\cdot)$ can be decomposed into the sum of a part which is smooth (at arbitrarily high order) and a part which is just as singular as f_0, but decays exponentially fast.

In the spatially inhomogeneous situation, the same kind of results is expected. In particular, in view of (101) it is believed that singularities of the initial datum are propagated by the characteristics of free transport, $(x,v) \to (x+tv,v)$. Such a result was recently proven by Boudin and Desvillettes for small initial data : they showed the following generalization of (102),

$$f(t,x,v) = G(t,x,v)f_0(x-tv,v) + H(t,x,v),$$

where G and H are not necessarily smooth, but at least possess a fractional Sobolev regularity. Their proof is based on a combination of the Q^+ smoothness and averaging lemmas.

We should make an important point here about our statement above about propagation of singularities. If we consider an initial datum which is very smooth apart from some singular set S, then it is *not* expected that the solution stay very smooth apart from the image S_t of this singular set by the characteristic trajectories. It is possible that the smoothness of the solution deteriorate, even very far from S_t. But one expects that the worst singularities always lie on S_t. And in any case, one always expects that the singular part of the solution be decaying very fast as time goes by.

3.6. The DiPerna-Lions renormalization. The Q^+ smoothing effect cannot be applied directly to the spatially inhomogeneous equation,

$$\frac{\partial f}{\partial t} + v\cdot\nabla_x f = Q(f,f), \tag{103}$$

because of the lack of nice a priori estimates.

As explained above, the idea of renormalization consists in giving a meaningful definition of $\beta'(f)Q(f,f)$ under very weak a priori estimates. Solutions are then defined as follows :

DEFINITION 1. *Let $f \in C([0,T]; L^1(\mathbb{R}^N_x \times \mathbb{R}^N_v)) \cap L^\infty(\mathbb{R}^+, L^1_+(\mathbb{R}^N_x \times \mathbb{R}^N_v))$. It is said to be a* renormalized solution of the Boltzmann equation *if for any nonlinearity $\beta \in C^1(\mathbb{R}^+, \mathbb{R}^+)$, such that $\beta(0) = 0$, $|\beta'(f)| \leq C/(1+f)$, one has*

$$\frac{\partial \beta(f)}{\partial t} + v \cdot \nabla_x \beta(f) = \beta'(f) Q(f,f), \tag{104}$$

in distributional sense.

The DiPerna-Lions renormalization [**190, 192**] achieves this goal by using the splitting (90). First of all,

$$\beta'(f) Q^-(f,f) = f \beta'(f)(A * f). \tag{105}$$

Following paragraph 5.1, let us assume that

$$\sup_{0 \leq t \leq T} \int_{\mathbb{R}^{2N}} f(t,x,v)(1 + |v|^2 + |x|^2)\, dx\, dv < +\infty,$$

and that $A(z) = o(|z|^2)$. Then

$$A * f \in L^\infty\Big([0,T]; L^1(\mathbb{R}^N_x; L^1_{\rm loc}(\mathbb{R}^N_v))\Big).$$

Further assume that the nonlinearity β satisfies

$$0 \leq \beta'(f) \leq \frac{C}{1+f}.$$

Then, $f\beta'(f) \in L^\infty$, and as a consequence (105) is well-defined in $L^1_{\rm loc}$.

As for the renormalized gain term, it can be handled easily because it is non-negative. As a general "rule", when one manages to give sense to all terms but one in some equation, and when this last term has a sign, then the equation automatically yields an a priori estimate. In our situation this is accomplished by integrating equation (104) in *all* variables on $[0,T] \times \mathbb{R}^N_x \times \mathbb{R}^N_v$; using the bounds on mass and energy, one gets

$$\int_0^T \int_{\mathbb{R}^{2N}} \beta'(f) Q^+(f,f)\, dt\, dx\, dv \leq \int_{\mathbb{R}^{2N}} \beta(f_0)\, dx\, dv + \int_0^T \int_{\mathbb{R}^{2N}} \beta'(f) f L f\, dt\, dx\, dv < +\infty,$$

whence

$$\beta'(f) Q^+(f,f) \in L^1\Big([0,T] \times \mathbb{R}^N_x \times \mathbb{R}^N_v\Big).$$

By the way, there is another, more widely known, version of this estimate, based on the entropy dissipation [**192**]; but this variant is more complicated and has the disadvantage to use a symmetric estimate, while the renormalization procedure can also be done from an asymmetric point of view :

$$\beta'(f) Q(g,f) = \beta'(f) Q^+(g,f) - \beta'(f) f L g.$$

On the other hand, the entropy dissipation has been very useful, both in the proof of stability of renormalized solutions, and in certain refinements of the theory by Lions [**308**]. Also, by using the entropy dissipation one can define a renormalized formulation with a stronger nonlinearity, namely $\beta(f) = \sqrt{1+f} - 1$. This is implicit

in [**308**], and has been shown by an elementary method in [**444**, part IV, chapter 3], as an outgrow of the idea of H-solutions explained in paragraph 4.1.

Renormalization and averaging lemmas were the two basic tools in the DiPerna-Lions theorem, which was the very first existence/stability result in the large for the spatially inhomogeneous Boltzmann equation. More precisely, these authors have proven [**192**] that the renormalized Boltzmann equation (104) is stable under weak convergence (a priori estimates of mass, energy and entropy being used). This theorem has remained a singular point in the field, due to the complexity of the proof and the use of technicalities which have no real counterpart for the rest of the theory...

Since then, Lions [**307**] found a simpler proof of existence, using the Q^+ regularity. In fact, he proved that strong compactness propagates in time for the Boltzmann equation with cut-off : this is the analogue of the spatially homogeneous results discussed in the paragraph above. Once strong compactness is established, passing to the limit is almost straightforward.

By the way, it is rather easy to prove the converse property, namely that the sequence of initial data has to be strongly compact if the sequence of solutions is. This theorem is a (very weak) illustration of the principle of propagation of singularities.

3.7. Summary. To conclude this section, we give explicit theorems illustrating the discussion above. As of this date, all of them are best in their category, but not optimal.

We begin with the spatially homogeneous situation.

Theorem 3. *Let $B(v-v_*,\sigma) = |v-v_*|^\gamma b(\cos\theta)$ be a collision kernel for hard potentials, satisfying Grad's cut-off assumption :*

$$\int_0^\pi b(\cos\theta)\,\sin^{N-2}\theta\,d\theta < +\infty, \qquad \gamma > 0,$$

and let f_0 be a probability distribution function with finite second moment,

$$\int_{\mathbb{R}^N} f_0(v)|v|^2\,dv < +\infty.$$

Then,

(i) there exists a unique energy-preserving solution to the Cauchy problem associated to B and f_0. This solution is unique in the class of weak solutions whose energy does not increase;

(ii) if $f_0 \in L^1_{s_1} \cap L^p_s$ for some $p \in [1,+\infty]$, $s_1 > 2$, $1 < s \leq s_1 - \gamma/p$, then $\sup_{t\geq 0} \|f(t,\cdot)\|_{L^p_s} < +\infty$;

(iii) if moreover $\gamma > 1/2$, $s \geq 2$ and $f_0 \in H^1(\mathbb{R}^N)$, $N = 3$, then $\sup_{t\geq 0} \|f(t,\cdot)\|_{H^1} < +\infty$;

(iv) if, on the other hand, $\gamma > 1/2$, $s \geq 2$ and $f_0 \notin H^1(\mathbb{R}^N)$, $N = 3$, then, for all $t \geq 0$, $f(t,\cdot) \notin H^1(\mathbb{R}^N)$. But $f(t,\cdot) = g(t,\cdot) + h(t,\cdot)$ where $\|g(t,\cdot)\|_{L^p_s} = O(e^{-\mu t})$ for some $\mu > 0$, and $\sup_{t\geq 0} \|h(t,\cdot)\|_{H^1} < +\infty$.

Point (i) is due to Mischler and Wennberg [**349**]; point (ii) to Gustafsson [**270**] for $p < \infty$ and to Arkeryd [**20**] for $p = \infty$; as for point (iii), is is due to Wennberg [**459**], point (iv) being an immediate consequence of the proof. A recent work by Mouhot

and the author recovers the conclusion of (ii) under slightly different assumptions on s_1, with the advantage of getting explicit constants; we are working on extending the allowed range of exponents s_1. Further work is in progress to extend also the range of validity of the conclusion of (iii) (arbitrary dimension, more general collision kernels) as well as to treat propagation of H^k smoothness for arbitrary k.

We now turn to the inhomogeneous theory in the small.

THEOREM 4. *Let $B = B(z, \sigma)$ be a collision kernel, $B \in L^\infty(S^{N-1}, W^{1,\infty}(\mathbb{R}^N))$, $N = 3$. Let $f_0(x, v)$ be a nonnegative initial datum satisfying the Maxwellian bound*

$$f_0(x, v) \le C_0 \exp\left(-\frac{|x|^2 + |v|^2}{2}\right),$$

where $C_0 = 1/(81\|B\|_{L^\infty(\mathbb{R}^N; L^1(S^{N-1}))})$. Then,

(i) there exists a global solution to the spatially inhomogeneous Boltzmann equation with collision kernel B and initial datum f_0, and for all $t \in [0, T]$ ($T > 0$), it satisfies a Maxwellian bound of the form

$$f(t, x, v) \le C_T \exp\left(-\frac{|x - vt|^2 + |v|^2}{2}\right), \tag{106}$$

with C_T depending only on T and C_0;

(ii) there exist "smooth" functions R and S in $H^\alpha_{\rm loc}(\mathbb{R}^+ \times \mathbb{R}^N_x \times \mathbb{R}^N_v)$ for any $\alpha < 1/25$, such that

$$f(t, x, v) = f_0(x - vt)R(t, x, v) + S(t, x, v);$$

(iii) if moreover $f_0 \in W^{k,\infty}$ for some $k \in \mathbb{N}$ (or $k = \infty$), then

$$f \in W^{k,\infty}_{\rm loc}(\mathbb{R}^+ \times \mathbb{R}^N_x \times \mathbb{R}^N_v).$$

This theorem is extracted from Boudin and Desvillettes [**101**]. Part (i), inspired from Mischler and Perthame [**348**], is actually an easy variation of more general theorems by Illner and Shinbrot [**278**]. One may of course expect the smoothness of R and S to be better than what this theorem shows ! But this result already displays the phenomenon of propagation of singularities along characteristic trajectories. Moreover, it conveys the feeling that it will be possible to treat propagation of smoothness and singularity for very general situations, as soon as we have solved the open problem of finding nice integrability a priori estimates for large data.

The main idea behind the bound (106) is the following : the left-hand side satisfies the differential inequality

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q(f, f) \le Q^+(f, f),$$

while one can devise a right-hand side of the form

$$g(t, x, v) = C(t) \exp\left(-\frac{|x - vt|^2 + |v|^2}{2}\right)$$

which satisfies the differential inequality

$$\frac{\partial g}{\partial t} + v \cdot \nabla_x g \ge Q^+(g, g),$$

so that it is natural to expect $f \le g$ if $f_0 \le g_0$. Note that since g is a Maxwellian, $Q^+(g,g) = Q^-(g,g)$.

Finally, we consider the DiPerna-Lions theory of renormalized solutions.

THEOREM 5. *Let $B(|v-v_*|, \cos\theta) = \Phi(|v-v_*|)b(\cos\theta)$ be a collision kernel satisfying Grad's angular cut-off, together with a growth condition at infinity :*

$$\int_0^\pi b(\cos\theta)\, \sin^{N-2}\theta\, d\theta < +\infty,$$

$$\Phi \in L^1_{\rm loc}(\mathbb{R}^N); \qquad \Phi(|z|) = o(|z|^2) \text{ as } |z| \to \infty.$$

Let $(f_0^n)_{n\in\mathbb{N}}$ be a sequence of initial data with uniformly bounded mass, energy, entropy,

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{R}^N\times\mathbb{R}^N} f_0^n(x,v)\left[1+|x|^2+|v|^2+\log f_0^n(x,v)\right] dx\, dv < +\infty.$$

Let $f^n(t,x,v)$ be a sequence of solutions[8] *of the Boltzmann equation*

$$\text{(107)} \qquad \begin{cases} \dfrac{\partial f^n}{\partial t} + v\cdot\nabla_x f^n = Q(f^n,f^n), \quad t\ge 0,\ x\in\mathbb{R}^N,\ v\in\mathbb{R}^N, \\ \\ f^n(0,\cdot,\cdot) = f_0^n. \end{cases}$$

Assume that the f^n's satisfy uniform bounds of mass, energy, entropy and entropy dissipation :

$$\text{(108)} \quad \sup_{n\in\mathbb{N}} \sup_{t\in[0,T]} \int_{\mathbb{R}^N\times\mathbb{R}^N} f^n(t,x,v)\left[1+|x|^2+|v|^2+\log f^n(t,x,v)\right] dx\, dv < +\infty.$$

$$\text{(109)} \qquad \sup_{n\in\mathbb{N}} \int_0^T D(f^n(t,x,\cdot))\, dx\, dt < +\infty.$$

Without loss of generality, assume that $f^n \longrightarrow f$ in $L^p([0,T]; L^1(\mathbb{R}^N_x\times\mathbb{R}^N_v))$, $1\le p<\infty$, $T<\infty$. Then,

(i) f is a renormalized solution of the Boltzmann equation. It satisfies global conservation of mass and momentum, and the continuity estimate $f\in C([0,T]; L^1(\mathbb{R}^N_x\times\mathbb{R}^N_v)$;

(ii) moreover, for all $t>0$, $f^n \longrightarrow f$ strongly in L^1 if and only if $f_0^n \longrightarrow f_0$ strongly; in this case the convergence actually holds in $C([0,T], L^1(\mathbb{R}^N_x\times\mathbb{R}^N_v))$;

(iii) if moreover there exists a strong, classical solution of the Boltzmann equation with initial datum f_0, then f coincides with f_0.

COROLLARY 5.1. *Let f_0 be an initial datum with finite mass, energy and entropy :*

$$\int_{\mathbb{R}^N\times\mathbb{R}^N} f_0(x,v)\left[1+|v|^2+|x|^2+\log f_0(x,v)\right] dx\, dv < +\infty.$$

Then there exists a renormalized solution $f(t,x,v)$ of the Boltzmann equation, with $f(0,\cdot,\cdot) = f_0$.

[8] renormalized solutions, or strong solutions, or approximate strong solutions.

Remarks :

1. A typical way of constructing approximate solutions is to solve the equation

$$\frac{\partial f^n}{\partial t} + v \cdot \nabla_x f^n = \frac{Q(f^n, f^n)}{1 + \frac{1}{n} \int f^n \, dv},$$

which is much easier than the "true" Boltzmann equation because the collision operator is sublinear.

2. If the f^n's are strong, approximate solutions, then the bounds (108)–(109) automatically hold, provided that the initial data have sufficient regularity. This remark, combined with the preceding, explains why the corollary follows from the theorem.

3. Point (iii) as stated above is slightly incorrect : for this point it is actually necessary to assume that the f^n's are strong, approximate solutions, or are constructed as limits of strong, approximate solutions.

Point (i) is due to DiPerna and Lions [**192**], points (ii) to Lions [**307, 308**]. For the sake of simplicity, we have stated unnecessarily restrictive assumptions on the collision kernel in points (i) and (ii). Point (iii) was first proven by Lions under an assumption which essentially implies $\Phi \in L^\infty$, then extended by Lu [**324**]. We have not made precise what "classical" means in point (iii) : in Lions' version, g should satisfy the Boltzmann equation almost everywhere on $[0, T] \times \mathbb{R}^N_x \times \mathbb{R}^N_v$, and also satisfy the *dissipative inequalities* introduced in Lions [**308**]. The discussion of dissipative inequalities is subtle and we preferred to skip it; let us only mention that this concept is based on the entropy dissipation inequality, and that it led Lions to a clean proof of local conservation of mass, as well as to the concept of dissipative solutions. In Lu's theorem, much more general collision kernels are included, at the price of slightly more restrictive (but quite realistic) assumptions imposed on the strong solution g. Lu also uses results from the theory of solutions in the small [**63**] to show existence of such strong solutions when $\Phi(|z|) = O(|z|^\gamma)$, $\gamma > -1$, and the initial datum is bounded by a well-chosen, small function. So these results bridge together the theory of renormalized solutions and the theory of solutions in the small.

At the moment, point (ii) is the most direct way towards the corollary. The scheme of the proof is as follows. In a first step, one uses the uniform bounds and the Dunford-Pettis criterion to get weak compactness of the sequence of solutions in L^1. This, combined with the renormalized formulation and averaging lemmas, implies the strong compactness of velocity-averages of f^n. Since the operators $L = A*$ and Q^+ are velocity averaging operators in some sense (remember the Q^+ regularity theorem), one can then prove the strong compactness of Lf^n and $Q^+(f^n, f^n)$. This is combined with a very clever use of Duhamel formulas to prove the strong compactness of the sequence f^n itself, provided that the sequence of initial data f^n_0 is strongly compact. Finally, one easily passes to the limit in the renormalized Q^- operator, and then in the renormalized Q^+ operator by a variant of the dominated convergence theorem which involves the domination of Q^+ by Q^- and (a little bit of) the entropy dissipation, as in paragraph 3.2.

4. The singularity-hunter's toolbox

In this section, we now examine the situation when the collision kernel presents a **nonintegrable angular singularity**. This branch of kinetic theory, very obscure for quite a time, has undergone spectacular progress in the past few years, which is why we shall make a slightly more detailed exposition than in the case of Grad's angular cut-off. The starting point for recent progress was the work by Desvillettes [**171**] on a variant of the Kac model, which was devised to keep some of the structure of the Boltzmann equation without cut-off. The study of these regularizing effects was first developed in the spatially homogeneous theory, and later in the theory of renormalized solutions, in the form of strong compactification.

As explained previously, the main qualitative difference with respect to the situation where Grad's assumption holds, is that one expects immediate regularization of the solution. From the mathematical point of view, the first clear difference is that the splitting

$$Q(f,f) = Q^+(f,f) - Q^-(f,f)$$

is impossible : both terms should be infinite. From the physical point of view, one can argue that when particles collide, there is an overwhelming probability for the change in velocity to be extremely small, hence the density in probability space should spread out, like it does in a diffusion process.

The main analytical idea behind the regularization effect is that $Q(f,f)$ should look like a singular integral operator. As we shall see, it resembles a fractional diffusion operator; this illustrates the physically nontrivial fact that collision processes for long-range interactions are neither purely collisional in the usual sense, nor purely diffusive, but somewhat in between.

There is an important body of work due to Alexandre about the study of the non-cutoff Boltzmann operator, with the help of pseudo-differential formalism [**2, 3, 4, 5, 6, 7, 8, 9**], and on which we shall say almost nothing, the main reason being that most of the results there (some of which have been very important advances at the time of their appearance) can be recovered and considerably generalized by means of the simpler techniques described below.

Generally speaking, there are two faces to singular operators in partial differential equations. On one hand one would like to control them, which means (i) find weak formulations, or (ii) find if, in some situations, they induce compensations due to symmetries. On the other hand, we would like to have (iii) simple estimates expressing the fact that they really are unbounded operators, and that the associated evolution equation does have a regularizing effect. To illustrate these fuzzy considerations, think of the Laplace operator, and the formulas (i) $\int \Delta f \varphi = \int f \Delta \varphi$, (ii) $(\Delta f) * \varphi = f * (\Delta \varphi)$, (iii) $\int f(\Delta f) = -\|f\|^2_{\dot{H}^1}$. Keeping this discussion in mind may help understanding the interest of the weak formulations in paragraph 4.1, the cancellation lemma in paragraph 4.2, and the entropy dissipation estimate in paragraph 4.3, respectively.

Finally, as we already mentioned several times, another singularity problem will come into play : when one is interested in soft potentials, then the *kinetic* collision kernel presents a singularity for $|v - v_*| \simeq 0$. When the strength of this singularity is high, this will entail additional technical difficulties, but it is not clear at the moment that this feature is related to physically relevant considerations.

4.1. Weak formulations. In presence of a nonintegrable singularity, Boltzmann's collision operator is not a bounded operator between weigthed L^1 spaces; it is not even clear that it makes sense almost everywhere. Thus one should look for a distributional definition. The most natural way towards such a definition (both from the mathematical and the physical points of view) is via Maxwell's weak formulations :

$$\int Q(f,f)\varphi\, dv = \int dv\, dv_*\, f f_* \left[\int_{S^{N-1}} B(v-v_*,\sigma)(\varphi'-\varphi)\, d\sigma\right]. \tag{110}$$

As pointed out by Arkeryd [**18**], if φ is a smooth test-function, then $\varphi'-\varphi$ will vanish when $\theta \simeq 0$ (because then $v' \simeq v$), and this may compensate for a singularity in B. This circumstance actually explains why one is able to compute relevant physical quantities, such as the cross-section for momentum transfer, even for non-cutoff potentials...

For the sake of discussion, we still consider the model case $B(v-v_*,\sigma) = |v-v_*|^\gamma b(\cos\theta)$. Using moment estimates, and the formula $\varphi'-\varphi = O(|v-v_*|\theta)$ (which holds true when φ is smooth), Arkeryd [**18**] was able to prove existence of weak solutions for the spatially homogeneous Boltzmann equation as soon as

$$\gamma \geq -1, \qquad \int b(\cos\theta)\theta \sin^{N-2}\theta\, d\theta < \infty.$$

The use of the more symmetric form obtained by the exchange of variables $v \leftrightarrow v_*$ does not a priori seem to help a lot, since one has only

$$|\varphi' + \varphi'_* - \varphi - \varphi_*| \leq C(\varphi)|v-v_*|^2\theta,$$

so there is no gain on the angular singularity.

But, as noted independently by several authors (see for instance [**248, 446**]), an extra order of θ can be gained by integrating in spherical coordinates. More precisely, use the standard parametrization of σ in terms of θ, ϕ ($\phi \in S^{N-2}$), then

$$\left|\int_{S^{N-2}} (\varphi' + \varphi'_* - \varphi - \varphi_*)\, d\phi\right| \leq C(\varphi)|v-v_*|^2\theta^2. \tag{111}$$

This simple remark enables one to extend Arkeryd's results to

$$\gamma \geq -2, \qquad \int b(\cos\theta)(1-\cos\theta)\sin^{N-2}\theta\, d\theta < \infty.$$

In dimension 3, these assumptions are fulfilled by inverse-power forces $1/r^s$ when $s \geq 7/3$ (to compare with $s > 3$ for Arkeryd's original result... the reader may feel that the gain is infinitesimal, but remember that $s = 2$ should be the truly interesting limit exponent !).

However, this point of view, which relies on the $v \longleftrightarrow v_*$ symmetry, is in part misleading. The same control on the angular singularity (but worse in the kinetic singularity) can be obtained without using the symmetry $v \longleftrightarrow v_*$, as shown in Alexandre and Villani [**12**] by the use of more precise computations. When one is only interested in weak solutions in a spatially homogeneous problem, this remark is of no interest, but it becomes a crucial point in spatially inhomogeneous situations, or in the study of fine regularization properties. Here is a precise bound from [**12**].

Introduce the cross-section for momentum transfer, formulas (62) or (63). Then

$$\int_{\mathbb{R}^N_v} Q(g,f)\varphi\, dv \le \frac{1}{4}\|\varphi\|_{W^{2,\infty}} \int_{\mathbb{R}^{2N}} dv\, dv_*\, g_* f |v-v_*|(1+|v-v_*|)M(|v-v_*|). \tag{112}$$

To treat values of γ below -2 with the help of formula (111) and others, it seems that one should require nontrivial a priori estimates like

$$\int_{\mathbb{R}^{2N}} \frac{f(v)f(v_*)}{|v-v_*|^{-(\gamma+2)}} dv\, dv_* < +\infty$$

(the exponent of $|v-v_*|$ in the denominator is positive !). As we shall see in paragraph 4.3, such estimates are indeed available in most cases of interest. But they are by no means easy !

At the time when these extra estimates were not yet available, the search for a treatment of values of γ below -2 led the author [**446**] to introduce a new weak formulation (***H*-solutions**), based on the a priori bound

$$\left| \int Q(f,f)\varphi \right| \le \frac{1}{2} D(f)^{1/2} \left[\int B f f_* (\varphi' + \varphi'_* - \varphi - \varphi_*)^2 \right]^{1/2}.$$

Here D is Boltzmann's entropy dissipation functional (47). This new bound was based on Boltzmann's weak formulation (45), and the elementary estimate

$$D(f) \ge \int_{\mathbb{R}^N \times \mathbb{R}^N \times S^{N-1}} B(v-v_*,\sigma) \Big(\sqrt{f' f'_*} - \sqrt{f f_*} \Big)^2 dv\, dv_*\, d\sigma \tag{113}$$

(which follows from $(X-Y)(\log X - \log Y) \ge 4(\sqrt{X}-\sqrt{Y})^2$). It enabled the author to prove existence of weak solutions under the assumptions

$$\gamma > -4, \qquad \int b(\cos\theta)(1-\cos\theta)\, \sin^{N-2}\theta\, d\theta < \infty,$$

which allow the three-dimensional Coulomb potential as a limit (excluded) case. A main application was the first proof of the Landau approximation[9] for realistic potentials in a spatially homogeneous setting.

This use of the entropy dissipation for the study of grazing collisions had the merit to display some interesting feature : a partial regularity estimate associated to the entropy dissipation. More precisely, finiteness of the entropy dissipation, when the collision kernel is singular, implies a partial smoothness estimate for $f f_*$ in the tensor velocity space $\mathbb{R}^N \times \mathbb{R}^N$. This effect is best seen at the level of the Landau equation : Landau's entropy dissipation can be rewritten as

$$D_L(f) = 2 \int_{\mathbb{R}^N \times \mathbb{R}^N} \left| \Pi(v-v_*)(\nabla - \nabla_*) \sqrt{f f_* \Psi(|v-v_*|)} \right|^2 dv\, dv_*. \tag{114}$$

Recall that $\Pi(v-v_*)$ is the orthogonal projection on $(v-v_*)^\perp$. Eq. (114) is a regularity estimate on the function $f f_*$, but only in the variable $v-v_*$, and only in directions which are orthogonal to $v-v_*$. On the whole, this means $N-1$ directions out of $2N$. At the level of Boltzmann's entropy dissipation, for each

[9] See section 5.

point $(v, v_*) \in \mathbb{R}^{2N}$, these $N-1$ directions are precisely the tangent plane to the $(N-1)$-dimensional manifold

$$S_{vv_*} = \Big\{ (v', v'_*) \quad \text{satisfying (5)} \Big\}.$$

One may conclude to the simple heuristic rule : entropy dissipation yields a smoothness estimate *along collisions*.

These entropy dissipation bounds have a lot of robustness in a spatially homogeneous context, due to the tensorial structure of the entropy dissipation functional. For instance, one can prove that if D_ε is the entropy dissipation functional associated to a Boltzmann operator Q_ε "converging" in a suitable sense to Landau's operator, and D_L is Landau's entropy disipation, and $f^\varepsilon \longrightarrow f$ in weak L^1, then

$$D_L(f) \leq \liminf_{\varepsilon \to 0} D_\varepsilon(f^\varepsilon).$$

On the other hand, precisely because they rely so much on the symmetry $v \leftrightarrow v_*$ and the tensor product structure, these methods turn out to be inadapted to more general problems. More efficient approaches will be presented in the sequel.

4.2. Cancellation lemma. In this and the next two paragraphs, we shall introduce more sophisticated tools for fine surgery on Boltzmann's operator.

As discussed above, integrals such as

$$\int_{\mathbb{R}^N \times S^{N-1}} dv_* \, d\sigma \, B(v - v_*, \sigma)(g' - g) \tag{115}$$

are well-defined for a smooth function g, at least if the collision kernel is not too much singular. When g is not smooth (say L^1), it is not clear at all that such an integral should converge. This is however true with great generality, due to *symmetry effects*. A precise quantitative version was introduced by the author in [**449**] (related estimates are to be found in Desvillettes [**173**] and Alexandre [**6**]). The estimate in [**449**] shows that when the collision kernel B in (115) depends smoothly on $v - v_*$ and presents an nonintegrable angular singularity of order $\nu < 2$, then the integral (115) converges.

The need to cover more singular situations motivated further refinement of this estimate; here we present the sharp version which is proven in [**12**]. It only requires finiteness of the cross-section for momentum transfer, M, and a very weak regularity assumption with respect to the relative velocity variable.

PROPOSITION 6. *Let $B(|z|, \cos\theta)$ be a collision kernel with support in $\theta \in [0, \pi/2]$, and let $\mathcal{S}$ be defined by*

$$\mathcal{S}g \equiv \int_{\mathbb{R}^N \times S^{N-1}} dv_* \, d\sigma \, B(v - v_*, \sigma)(g'_* - g_*)$$

Then, for any $g \in L^1(\mathbb{R}^N_v)$,

$$\mathcal{S}g = g *_v S,$$

where the convolution kernel S is given by

$$S(|z|) = |S^{N-2}| \int_0^{\frac{\pi}{2}} d\theta \, \sin^{N-2}\theta \, \left[\frac{1}{\cos^N(\theta/2)} B\left(\frac{|z|}{\cos(\theta/2)}, \cos\theta \right) - B(|z|, \cos\theta). \right] \tag{116}$$

Recall from paragraph 4.1 that the assumption about the deviation angle is no loss of generality. The proof of this lemma is rather elementary and relies on the change of variables $v_* \to v'_*$, which for fixed $\sigma \in S^{N-1}$ is allowed if the integration domain avoids frontal collisions ($\theta \simeq \pm\pi$).

Here is an easy corollary :

COROLLARY 6.1. *With the notations $z = v - v_*$, $k = (v - v_*)/|v - v_*|$, let*

$$B'(z,\sigma) \equiv \sup_{1<\lambda\leq\sqrt{2}} \frac{|B(\lambda z,\sigma) - B(z,\sigma)|}{(\lambda-1)|z|},$$

$$M'(|z|) = \int_{S^{N-1}} B'(z,\sigma)(1 - k\cdot\sigma)\, d\sigma \tag{117}$$

Then

$$|S(|z|)| \leq C_N \Big[M(|z|) + |z| M'(|z|) \Big],$$

where C_N is a constant depending only on N. In particular, if

$$|z| M'(|z|) \in L^1(\mathbb{R}^N), \tag{118}$$

then $S \in L^1_{loc}(\mathbb{R}^N)$.

What typical collision kernels are allowed by this lemma ? It turns out that the quantity M' measures the regularity of B with respect to the relative velocity variable in a very weak sense. For instance, assumption (118) is satisfied for all potentials of the form

$$B(v - v_*, \sigma) = |v - v_*|^\gamma b(\cos\theta), \qquad \gamma > -N.$$

Of course, this excludes the borderline case where $\gamma = -N$, which may be the most interesting, because in dimension 3 it corresponds to Coulomb interactions... However, now it is *homogeneity* which will save the game. A quick glance at formula (116) may give the impression that if B is homogeneous of degree $-N$ in the relative velocity variable, then $S = 0$!! Of course, this is a trap : S should be defined as a principal value operator, and after a little bit of algebra, one finds the following corollary to Proposition 6 :

COROLLARY 6.2. *If $B(v - v_*, \sigma) = |v - v_*|^{-N} \beta_0(\cos\theta)$, then*

$$S = \lambda\, \delta_0,$$

where δ_0 is the Dirac measure at the origin, and

$$\lambda = -|S^{N-2}||S^{N-1}| \int_0^{\frac{\pi}{2}} \beta_0(\cos\theta) \log\cos(\theta/2)\, \sin^{N-2}\theta\, d\theta.$$

Note that λ is finite as soon as the cross-section for momentum transfer (63) with $b = \beta_0$, is.

The compensation lemma of Proposition 6 has been a crucial tool, 1) to obtain sharp entropy dissipation estimates, see next paragraph; 2) to derive a sharp renormalized formulation for the Boltzmann operator without cut-off, see paragraph 4.6.

4.3. Entropy dissipation estimates. As we have explained in paragraph 4.1, under certain assumptions the entropy dissipation estimate yields a partial regularity bound on ff_* when the collision kernel is singular. But does it imply a true regularity estimate on f itself ?

The first result in this direction is due to Lions [**316**]. He proved that if $B(v - v_*, \sigma) \geq \Phi(|v - v_*|)b(\cos\theta)$, where Φ is smooth and positive, and $\sin^{N-2}\theta\, b(\cos\theta) \geq K\theta^{-1-\nu}$, then for all $R > 0$ there is a constant C_R such that

$$\|\sqrt{f(t,\cdot)}\|^2_{H^s(|v|<R)} \leq C_R\big[D(f)^{1/2} + \|f\|_{L^1}\big], \qquad s < s_0 = \frac{\nu}{2}\left(\frac{1}{1+\frac{\nu}{N-1}}\right). \tag{119}$$

The exponent s_0 here is not optimal. The idea of the proof was clever, and a very unexpected application of the Q^+ regularity in L^2 form (formula (99), with explicit bounds needed). Starting from the entropy dissipation and formula (113), one finds some smoothness-type estimate on $\sqrt{f'f'_*} - \sqrt{ff_*}$ in $\mathbb{R}^N \times \mathbb{R}^N \times S^{N-1}$. Then one integrates this estimate with respect to the variables v_* and σ after multiplication by an artificial, well-chosen collision kernel $\widehat{B}_\varepsilon(v - v_*, \sigma)$. An estimate on $\widehat{Q}_\varepsilon(\sqrt{f}, \sqrt{f})$ follows, where $\widehat{Q}_\varepsilon$ is the Boltzmann operator associated with $\widehat{B}_\varepsilon$. Then one writes

$$\sqrt{f}(\widehat{A}_\varepsilon * \sqrt{f}) = \widehat{Q}^+_\varepsilon(\sqrt{f}, \sqrt{f}) - \widehat{Q}_\varepsilon(\sqrt{f}, \sqrt{f}),$$

where $\widehat{A}_\varepsilon = \int_{S^{N-1}} \widehat{B}_\varepsilon\, d\sigma$. The regularity of $\widehat{Q}^+_\varepsilon$ and the estimates on $\widehat{Q}_\varepsilon(\sqrt{f}, \sqrt{f})$ yield the conclusion in the limit $\varepsilon \to 0$, after quite a bit of intricate computations.

A completely different method [**449**], based on the compensation lemma of last paragraph and on Carleman's representation (91), led the author to a better estimate under more stringent assumption on f : assume that

$$B(v - v_*, \sigma) \geq \Phi_0(|v - v_*|)b_0(\cos\theta), \tag{120}$$

where

$$\Phi_0(|z|) \text{ is continuous }, \quad \Phi_0(|z|) > 0 \text{ if } |z| > 0, \tag{121}$$

and (as usual)

$$b_0(\cos\theta)\sin^{N-2}\theta \geq K\theta^{-(1+\nu)}, \qquad K > 0,\ \nu \geq 0. \tag{122}$$

Further assume that f is positive, then

$$\|\sqrt{f(t,\cdot)}\|^2_{H^{\nu/2}(|v|<R)} \leq C_R\big[D(f) + \|f\|^2_{L^1_2}\big], \tag{123}$$

where C_R depends on f only via $\inf\{f(v);\ |v| \leq R\}$.

Exponent $\nu/2$ is optimal, as shown by a variant of the proof. But the assumption of local bound below for f is too strong. In a joint work with Alexandre, Desvillettes and Wennberg [**10**], we were able to extend the scope of (123), and eventually obtain sharp entropy dissipation estimates. The main difference with the previous argument was the replacement of Carleman's representation for the use of **Fourier transform**. In the next paragraph we shall make this a little bit more precise, for the moment we precisely state our main result in [**10**]. To this date this may be considered as the most general manifestation of the regularizing effects of grazing collisions.

PROPOSITION 7. *Let $f \in L^1_2 \cap L\log L(\mathbb{R}^N)$ be a nonnegative distribution function, and let B be a collision kernel satisfying assumptions* (120)–(122). *Then, for all*

$R > 0$ there is a constant $C = C_R$ such that

$$\|\sqrt{f}\|^2_{H^{\nu/2}(|v|<R)} \leq C_R\big[D(f) + \|f\|^2_{L^1_2}\big], \tag{124}$$

where $\|f\|_{L^1_2} = \int f(v)(1+|v|^2)\,dv$ and C_R (which is explicit) depends only on N, Φ_0, K, ν, R, a lower bound for $\int f\,dv$, and an upper bound for $\int f(1+|v|^2+|\log f|)$.

Remarks :

1. In the limit case $\nu = 0$, one recovers local estimates in $\log H$. As a very general fact, as soon as the function $b_0(\cos\theta)\sin^{N-2}\theta$ is not integrable, i.e. when

$$Z : \theta_0 \longmapsto \int_{\theta_0}^{\pi} b_0(\cos\theta)\,\sin^{N-2}\theta\,d\theta$$

diverges as $\theta_0 \to 0$, then one can estimate $\sqrt{f}$ in $X_{\rm loc}$, where X is the functional space defined in Fourier representation by

$$X = \left\{F \in L^2(\mathbb{R}^N);\quad \int_{|\xi|\geq 1} |\widehat{F}(\xi)|^2 Z\left(\frac{1}{|\xi|}\right)\,d\xi < +\infty\right\}.$$

2. This entropy dissipation estimate is asymmetric, and this is quite in contrast with the estimates which we shall discuss in the study of trend to equilibrium. In fact, the estimate holds just the same for

$$\int Q(g,f)\log f$$

(which is not always a nonnegative expression !), if one imposes $B = \Phi_0 b_0$ instead of $B \geq \Phi_0 b_0$, and replaces $\|f\|_{L^1_2}$ in (124) by $\|f\|_{L^1_2} + \|g\|_{L^1_2}$; then the constant C_R would not depend on f but only on g.

By (124), one can guess a precise heuristic point of view for the regularity properties associated with the non-cutoff Boltzmann operator : if the angular collision kernel is singular of order ν (assumption (122)), and g is a fixed distribution function with finite mass, energy and entropy, then *the linear operator $f \longmapsto Q(g,f)$ "behaves" in the same way as the fractional diffusion operator $-(-\Delta)^{\nu/2}$*. For Maxwellian molecules, the intuition of this result goes back to Cercignani [**138**], who had noticed, thirty years ago, that the eigenvalues of the linearized Boltzmann operator behaved like those of the power 1/4 of the Fokker-Planck operator.

4.4. Boltzmann-Plancherel formula. A key step in the proof of Proposition 7 is the use of Fourier transform. As we said earlier, in the context of Boltzmann equation, it is only for Maxwellian collision kernels that the Fourier transform leads to simple expressions. So one step of the proof (based on a lot of fine surgery) is the reduction to the purely Maxwellian case : $\Phi_0 \equiv 1$ in the previous notations. Then, it all reduces to a sharp estimate from below of expressions of the form

$$\int_{\mathbb{R}^{2N}} dv\,dv_* \int_{S^{N-1}} d\sigma\,g(v_*)\Big[F(v') - F(v)\Big]^2,$$

where g is an approximation of f (say, f multiplied by a smooth cut-off function), and F is an approximation of $\sqrt{f}$. The following Plancherel-type formula, established in [**10**] after the ideas of Bobylev, is the appropriate ingredient.

PROPOSITION 8. *With the notations* $\xi^{\pm} = (\xi \pm |\xi|\sigma)/2$,

$$\int_{\mathbb{R}^{2N}} \int_{S^{N-1}} b(k\cdot\sigma) g_* (F' - F)^2 \, dv \, dv_* \, d\sigma \tag{125}$$

$$= \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} \int_{S^{N-1}} b\left(\frac{\xi}{|\xi|}\cdot\sigma\right) \Big(\widehat{g}(0)|\widehat{F}(\xi)|^2 + \widehat{g}(0)|\widehat{F}(\xi^+)|^2 - 2\Re\, \big(\widehat{g}(\xi^-)\widehat{F}(\xi^+)\overline{\widehat{F}}(\xi)\big)\Big) \, d\xi \, d\sigma,$$

with $\Re$ *standing for real part.*

This general formula can be used in many other regularity problems, in particular to establish Sobolev-regularity estimates for the spatially homogeneous Boltzmann equation without cut-off [**185**].

4.5. Regularization effects. As a consequence of Proposition 7, one can derive some (rather weak) regularization theorems. This is immediately seen in the spatially homogeneous situation. Combining the entropy dissipation estimate

$$\int_0^T D(f(t,\cdot))\, dt + H(f(T,\cdot)) \le H(f_0)$$

with Proposition 7, the a priori bound

$$\sqrt{f} \in L^2\Big([0,T]; H^{\nu/2}_{\rm loc}(\mathbb{R}^N_v)\Big) \tag{126}$$

follows at once.

Weak as it is, this regularization estimate is already useful to the existence theory for singular collision kernels. Indeed, assume that $B(v - v_*, \sigma) = |v - v_*|^\gamma b(\cos\theta)$, where b satisfies the same assumptions as b_0 in (122). As we saw in paragraph 4.1, when γ is very negative one runs into trouble to define relevant weak solutions of the spatially homogeneous Boltzmann equation. But now, with this new entropy dissipation bound, one can get sufficient a priori estimates if the angular singularity is strong enough compared to $-\gamma$. More precisely, if

$$\gamma + \nu + 2 \ge 0, \tag{127}$$

then, by the Hardy-Littlewood-Sobolev and Sobolev inequalities,

$$\begin{aligned} \int f f_* |v - v_*|^{\gamma+2} \, dv \, dv_* \, dt &\le C \|f\|_{L^\infty_t(L^1_v)} \|f\|_{L^1_t(L^q_v)} \\ &\le C \|f_0\|_{L^1} \|\sqrt{f}\|^2_{L^2_t(H^{\nu/2}_v)} \end{aligned}$$

for some well-chosen $q > 1$ (everything being understood in local sense).

It is worth pointing out that inequality (127) always holds for collision kernels coming out from inverse-power forces in dimension 3. We also note that the case which appears the most delicate to treat now, is the one of a collision kernel which is singular in the relative velocity variable but not in the angular variable; soon we shall encounter a similar problem in the spatially inhomogeneous setting.

Other variants of these entropy dissipation estimates lead to (strong) compactness results. For instance, let $(f_n(t,v))_{n\in\mathbb{N}}$ be a sequence of probability distributions,

satisfying

$$\sup_n \left\{ \int_0^T D_n(f_n)\, dt + \sup_{t\in[0,T]} \left(\|f_n\|_{L\log L} + \|f_n\|_{L^1_2} \right) \right\} < +\infty,$$

where D_n is the entropy dissipation associated to a collision kernel B_n, approximating (in almost everywhere sense for instance) a singular collision kernel B. Then, (f_n) is strongly compact in L^1. This holds true even if there is not necessarily a uniform smoothness estimate.

The smoothing effects which we just discussed are rather weak, but using the same kind of techniques one can bootstrap on the regularity again and again, at least if the collision kernel is smooth with respect to the relative velocity variable. The key inequality can be formally written as

$$\frac{d}{dt}\|f\|^2_{H^\alpha} \le -K\|f\|^2_{H^{\alpha+\frac{\nu}{2}}} + C\|f\|^2_{H^\alpha}.$$

This easily leads, after integration on $[0,T]$, to the immediate appearance of the $H^{\alpha+\nu/2}$ norm of f if the H^α norm of f is initially finite – and, by induction, to immediate C^∞ regularization[10].

Such a study is currently worked out by Desvillettes and Wennberg, who have announced a proof of C^∞ instantaneous regularization for solutions to the spatially homogeneous Boltzmann equation, starting from an initial datum which has finite entropy. This result had already been proven in certain particular cases by Desvillettes [**172, 171, 173**], and his student Proutière [**390**], with the use of Fourier-transform techniques.

4.6. Renormalized formulation, or Γ formula. The treatment of regularizing effects for the full, spatially inhomogeneous Boltzmann equation without cut-off requires an additional tool because of the difficulty of defining the collision operator. As we explained in paragraph 1.3, a renormalized formulation of the collision operator, together with entropy dissipation estimates (in the sharp form of Proposition 7), is enough to prove appearance of strong compactness.

For a long time this problem stood open, until Alexandre came up with a very clever idea [**6**]. The implementation of Alexandre's ideas, based on pseudo-differential theory, suffered from intricate computations and the impossibility to cover physically realistic collision kernels. In a joint work [**12**] with Alexandre, we have given a very general definition, based on the use of the cancellation lemma of paragraph 4.2, and the idea of using the asymmetric Boltzmann operator. Here is the renormalized formulation of [**12**], given in asymmetric formulation :

$$\begin{aligned} \beta'(f)Q(g,f) &= \big[f\beta'(f) - \beta(f)\big] \int_{\mathbb{R}^N\times S^{N-1}} dv_*\, d\sigma\, B(v-v_*,\sigma)\,(g'_* - g_*) \\ &+ Q(g,\beta(f)) \\ &- \int_{\mathbb{R}^N\times S^{N-1}} dv_*\, d\sigma\, B g'_* \Gamma(f,f'). \end{aligned} \tag{128}$$

[10] In fact this method is but an adaptation of the "energy method" in the study of parabolic regularity, where similar estimates would hold with the constant ν replaced by 2.

where

$$\Gamma(f,f') = \beta(f') - \beta(f) - \beta'(f)(f'-f). \tag{129}$$

If β is a strictly concave function or strictly convex function, then Γ has a fixed sign. In the context of the study of renormalized solutions, it will be convenient to choose β to be concave (sublinear), satisfying $\beta'(f) \leq C/(1+f)$. Let us explain why each of the three terms in (128) is then well-defined.

For the first one, we may assume $f\beta'(f) - \beta(f) \in L^\infty$, and then this term will satisfy an $L^1_{\rm loc}$ bound as a result of the cancellation lemma of paragraph 4.2.

As for the second term in (128), it can be given a distributional sense, by means of formula (112). The estimate works in a spatially inhomogeneous context because the arguments of the collision operator are g ($\in L^1$, say) and $\beta(f)$ ($\in L^\infty$, say). This is the point where it is very important to have an asymmetric weak formulation !

In the end, the third term is nonnegative as soon as β is concave, and since all other terms are well-defined, it satisfies an a priori estimate in $L^1_{\rm loc}$ for free — just as in the argument for the gain term in the DiPerna-Lions renormalization[11].

Notice that this renormalization procedure can be understood as a "commutator" problem : find a nice expression for

$$\beta'(f)Q(g,f) - Q(g,\beta(f)). \tag{130}$$

Such commutators are widely used in the study of linear diffusion operators. When L is a diffusion operator, then $\beta'(f)Lf - L\beta(f) = -\beta''(f)\Gamma(f)$, where Γ is the "Dirichlet form" associated with L. This justifies our terminology of "Γ formula". As a matter of fact, the renormalization procedure above presents some similarities with the renormalization of parabolic equations by Blanchard and Murat [**76, 77**], and is also very close to the renormalization of the Landau operator given in Lions [**311**].

As an illustration of the drawbacks of "soft" theories, we note that the construction of renormalized solutions with the preceding definition is still an open problem. Instead, one is led to introduce the following, slightly weaker, definition :

DEFINITION 2. *Let $f \in C(\mathbb{R}^+, \mathcal{D}'(\mathbb{R}^N_x \times \mathbb{R}^N_v)) \cap L^\infty(\mathbb{R}^+, L^1_+(\mathbb{R}^N_x \times \mathbb{R}^N_v))$. It is said to be a* renormalized solution of the Boltzmann equation with a defect measure*, if for any nonlinearity $\beta \in C^2(\mathbb{R}^+,\mathbb{R}^+)$ such that $\beta(0) = 0$, $0 \leq \beta'(f) \leq C/(1+f)$, $\beta''(f) < 0$, one has*

$$\frac{\partial}{\partial t}\beta(f) + v \cdot \nabla_x \beta(f) \geq \beta'(f)Q(f,f), \tag{131}$$

in distributional sense, and moreover f satisfies the law of mass-conservation :

$$\forall t \geq 0, \qquad \int_{\mathbb{R}^{2N}} f(t,x,v)\,dx\,dv = \int_{\mathbb{R}^{2N}} f(0,x,v)\,dx\,dv. \tag{132}$$

We insist that this is really a notion of weak solution, not just sub-solution. Indeed, if f were smooth, then the combination of (131) and (132) shows that there is equality in (131). See [**12**], and also DiPerna and Lions [**190**] for similar situations.

Finally, we note that formula (128) is a general tool which finds applications outside the theory of renormalized solutions, for instance in the study of regularity for the spatially homogeneous equation (or even for the spatially inhomogeneous one, if suitable integrability bounds are assumed). In this context, it is convenient to choose β to be convex when studying regularization for initial data which belong

[11] See paragraph 3.6.

to $L\log L$ or L^p spaces, and concave when studying regularization for initial data which are only assumed to be probability measures. In fact, in some sense the Γ formula plays for the Boltzmann equation the same role as integration by parts plays in the energy method for diffusion operators; therefore one should not be surprised of its great utility.

4.7. Summary. The following two theorems summarize our current knowledge of regularizing effects, respectively in the spatially homogeneous setting and in the framework of renormalized solutions. We restrict to the model cases

$$B(v-v_*,\sigma) = \Phi(|v-v_*|)b(\cos\theta), \tag{133}$$

where $\Phi(|z|) > 0$ for $|z| \neq 0$, and b satisfies the usual singularity condition,

$$\sin^{N-2}\theta\, b(\cos\theta) \simeq K\theta^{-(1+\nu)} \qquad \text{as } \theta \to 0. \tag{134}$$

Theorem 9. *Let B satisfy eq.* (133)–(134)*, and let f_0 be a probability density on $\mathbb{R}^N_v$, with bounded energy; f_0 may have a singular part, but should be distinct from a Dirac mass*[12]*. Then,*

(i) if Φ is smooth and bounded from above and below, then there exists a solution $f(t,v)$ to the Boltzmann equation with initial datum f_0, which lies in $C^\infty((0,+\infty)\times\mathbb{R}^N_v)$;

(ii) if $\Phi(|v-v_*|) = |v-v_*|^\gamma$ where $\gamma+\nu > -2$, and f_0 has finite entropy, then there exists a weak solution $f(t,v)$ to the Boltzmann equation with initial datum f_0, such that

$$\sqrt{f} \in L^2_{\rm loc}\big(\mathbb{R}^+; H^{\nu/2}_{\rm loc}(\mathbb{R}^N_v)\big);$$

(iii) if $\Phi(|v-v_*|) = |v-v_*|^\gamma$ where $\gamma+\nu > 0$, then, without further assumptions on f_0 there exists a weak solution $f(t,v)$ to the Boltzmann equation with initial datum f_0, such that

$$\forall t > 0, \quad f(t,\cdot) \in L\log L(\mathbb{R}^N_v).$$

Point (i) of this theorem is work in progress by Desvillettes and Wennberg if one assumes that f_0 has finite entropy. Then, in the case where one only assumes that f_0 has finite mass and energy, work in progress by the author [**440**] shows that the entropy becomes finite for any positive time (actually, one proves estimates in $L^1_{\rm loc}(dt; L^p(\mathbb{R}^N_v))$, for arbitray large p). Key tools in these works are the Plancherel-like formula of paragraph 4.4 and the cancellation lemma of paragraph 4.2.

Related to point (i) are probabilistic works by Fournier [**217**], Fournier and Méléard [**219, 220**] who prove immediate appearance of an L^1 density if the initial datum is not a Dirac mass, and C^∞ smoothness for Maxwellian collision kernels in two dimensions [**217**]. The results by Fournier and Méléard are considerably more restricted because of strong decay assumptions on the initial datum, stringent assumptions on the smoothness of the kinetic collision kernel and restrictions on the strength of the singularity. However, they have the merit to develop on Tanaka's approach[**415**] and to build a stochastic theory of the Boltzmann equation, whose solution is constructed via a complicated nonlinear stochastic jump process. These works constitute a bridge between regularization tools stated here, and Malliavin

[12]Because a Dirac mass is a stationary solution of the spatially homogeneous Boltzmann equation ! so, starting from a Dirac mass does not lead to any regularization.

calculus. They also have applications to the study of stochastic particle systems which are used in many numerical simulations [**177, 221, 223, 222, 256, 257**]. In particular, they are able to study the numerical error introduced in Monte Carlo simulations when replacing a non-cutoff Boltzmann equation by a Boltzmann equation with small cut-off[13].

As for point (ii), it follows from the entropy dissipation estimates in Alexandre, Desvillettes, Villani and Wennberg [**10**] and by now standard computations which can be found for instance in Villani [**446**]. One would expect that when $\gamma > 0$, C^∞ smoothness still holds; current techniques should suffice to prove this, but it remains to be done. Point (iii) is from [**440**].

Uniqueness is still an open problem in this setting, on which the author is currently working. This question is related to smoothing : if one wants to use a classical Gronwall strategy, like in the proof of uniqueness for the spatially homogeneous Landau equation [**182**], then one sees that the key property to prove is that (essentially) the non-cutoff bilinear Boltzmann operator is not only "at least" as singular as the fractional Laplace operator of order ν, but also "at most" as singular as this one, in the sense that it maps L^2 into $H^{-\nu/2}$ (locally). We do hope for rapid progress in this direction !

In the case of the spatially homogeneous Landau equation, then the same regularization results hold true, and are easier to get because the Landau equation already looks like a nonlinear parabolic equation. Hence the smoothing effect can be recovered by standard estimates (only complicated), bootstrap and interpolation lemmas between weighted Sobolev and Lebesgue spaces. It is possible to go all the way to C^∞ smoothness even in cases where Ψ is not so smooth : for instance $\Psi(|v - v_*|) = K|v - v_*|^{\gamma+2}$, $\gamma > 0$. This study was performed in Desvillettes and Villani [**182**]. For this case the authors proved immediate regularization in Schwarz space, and uniqueness of the weak solution, in the class of solutions whose energy is nonincreasing, as soon as the initial datum satisfies $\int f^2(v)(1 + |v|^{2s})\, dv < +\infty$, $2s > 5\gamma + 12 + s$ ($N = 3$). By the way, this uniqueness theorem of a weak solution, building on ideas by Arsen'ev and Buryak [**41**], required some precise Schauder-type estimates for a linear parabolic equation whose diffusion matrix is not uniformly elliptic in the usual sense, and our work has motivated further research in this area [**11**].

We emphasize that the picture is much less complete in the case $\gamma < 0$. In particular, for the Landau equation with Coulomb potential ($\gamma = -3$ in dimension $N = 3$), nothing is known beyond existence of weak solutions (see Villani [**446**] or the remarks in [**10**]).

We now turn to the spatially inhomogeneous setting. It is a striking fact that no theorem of existence of classical small solutions of the Boltzmann equation without cut-off has ever been proven to this day, except maybe for the isolated results in [**9**] which still need further clarification. So we only discuss renormalized solutions.

THEOREM 10. *Assume that the collision kernel B is given by* (133)-(134)*, and* $\Phi(|v - v_*|) = |v - v_*|^\gamma$ *with*

$$0 \le \nu < 2, \quad \gamma \ge -N, \quad \gamma + \nu < 2. \tag{135}$$

[13]Apparently, Monte Carlo methods cannot be directly applied to the study of the non-cutoff Boltzmann equation. The only method which seems able to directly deal with non-cutoff collision kernels, without making some a priori truncation, is the Fourier-based deterministic scheme described in section 4.9.

Let (f^n) be a sequence of solutions[14] *of the Boltzmann equation, satisfying uniform estimates of mass, energy, entropy and entropy dissipation :*

$$\sup_{n\in\mathbb{N}} \sup_{t\in[0,T]} \int_{\mathbb{R}^N\times\mathbb{R}^N} f^n(t,x,v)\left[1+|x|^2+|v|^2+\log f^n(t,x,v)\right] dx\,dv < +\infty. \tag{136}$$

$$\sup_{n\in\mathbb{N}} \int_0^T D(f^n(t,x,\cdot))\,dx\,dt < +\infty. \tag{137}$$

Without loss of generality, assume that $f^n \longrightarrow f$ weakly in $L^p([0,T];L^1(\mathbb{R}^N_x \times \mathbb{R}^N_v))$. Then,

(i) f is a renormalized solution of the Boltzmann equation with a defect measure;

(ii) automatically, $f^n \longrightarrow f$ strongly in L^1.

Corollary 10.1. *Let f_0 be an initial datum with finite mass, energy and entropy :*

$$\int_{\mathbb{R}^N\times\mathbb{R}^N} f_0(x,v)\Big[1+|v|^2+|x|^2+\log f_0(x,v)\Big]\,dx\,dv < +\infty.$$

Then there exists a renormalized solution with a defect measure, $f(t,x,v)$, of the Boltzmann equation, with $f(0,\cdot,\cdot)=f_0$.

This theorem is proven in Alexandre and Villani [**12**], answering positively a conjecture by Lions [**308**]. The result holds in much more generality, for instance it suffices that the angular collision kernel be nonintegrable (no need for a power-law singularity), and the kinetic collision kernel need not either take the particular form of a power-law, if it satisfies some very weak regularity assumption with respect to the relative velocity variable. And also, it is not necessary that the collision kernel split into the product of a kinetic and an angular collision kernel. We mention all these extensions because they are compulsory when one wants to include realistic approximations of the Debye collision kernel, which is not cut-off, but not in product form...

The strategy of proof is the following. First, by Dunford-Pettis criterion, the sequence $(f^n)_{n\in\mathbb{N}}$ is weakly (relatively) compact in L^1. Then, by the renormalized formulation, and the averaging lemmas, one shows that velocity-averages of the f^n's are strongly compact. Then the entropy dissipation regularity estimates yield bounds of regularity in the v variable, outside of a small set and outside of a set where the f^n's are very small. As a consequence, the sequence $(f^n)_{n\in\mathbb{N}}$ can be very well approximated by velocity-averages, and therefore it lies in a strongly compact set (as in [**311**]).

Let us comment on the range of parameters in (135). The assumption $\gamma+\nu<2$ is just a growth condition on the kinetic collision kernel, and is a natural generalization of the assumption $\gamma<2$ in the DiPerna-Lions theorem; by the way, for inverse s-powers in three dimensions, the inequality $\gamma+\nu<1$ always holds true. But now, we see that there are *two* extensions : first, the possibility to choose $\nu\in[0,2)$ (which is the optimal range), secondly, the possibility to have a nonintegrable kinetic collision kernel, provided that the singularity be homogeneous of degree $-N$. This feature allows to deal with Coulomb-like collision kernels in dimension 3.

[14] either renormalized solutions, or renormalized solutions with a defect measure, or approximate solutions, as in Theorem 5.

By the way, a problem which is left open is whether the theorem applies when the collision kernel presents a nonintegrable kinetic singularity of order $-N$ but no angular singularity. Such collision kernels are unrealistic, but sometimes suggested as approximations of Debye collision kernels [**162**]. The renormalized formulation above is able to handle this case (contrary to the DiPerna-Lions renormalization), but without angular singularity the regularizing effect may be lost — or is it implied by the nonintegrable singularity, as some heuristic considerations [**12**] may suggest ?

A result quite similar to Theorem 10 (actually simpler) holds for the Landau equation, see Lions [**311**], and also Alexandre and Villani [**13**].

To this day, no clean implementation of a regularization effect has been done in the framework of spatially inhomogeneous small solutions. Desvillettes and Golse [**176**] have worked on an oversimplified model of the Boltzmann equation without cut-off, for which L^∞ solutions can be constructed for free. For this model equation they prove immediate H^α regularization for some α which is about $1/30$.

In fact, regularization for the spatially inhomogeneous Boltzmann equation without cut-off may be understood as a hypoellipticity problem — with the main problem that the diffusive operator is of nonlocal, nonlinear nature. F. Bouchut has recently communicated to us some very general methods to tackle hypoelliptic transport equations in a Sobolev space setting, via energy-type methods; certainly that kind of tools will be important in the future.

5. The Landau approximation

In this section, we address the questions formulated in paragraph 2.7. In short, how to justify the replacement of Boltzmann's operator by Landau's operator in the case of Debye (= screened Coulomb) potential when the Debye length is very large compared to the Landau length ?

5.1. Structure of the Landau equation. We recall here the structure of the Landau operator, in asymmetric form :

$$Q_L(g,f) = \nabla_v \cdot \left(\int_{\mathbb{R}^N} dv_* \, a(v-v_*) \big[g_*(\nabla f) - f(\nabla g)_* \big] \right), \tag{138}$$

$$a_{ij}(z) = \Psi(|z|) \left[\delta_{ij} - \frac{z_i z_j}{|z|^2} \right], \tag{139}$$

The Landau operator can also be rewritten as a nonlinear diffusion operator,

$$Q_L(g,f) = \nabla \cdot \big(\overline{a} \nabla f - \overline{b} f \big) = \sum_{ij} \overline{a}_{ij} \partial_{ij} f - \overline{c} f, \tag{140}$$

where $b = \nabla \cdot a$, $c = \nabla \cdot b$, or more explicitly

$$b_j = \sum_i \partial_i a_{ij}, \qquad c = \sum_j \partial_j b_j,$$

and

$$\overline{a} = a * g, \qquad \overline{b} = b * g, \qquad \overline{c} = c * g.$$

There is a weak formulation, very similar to Boltzmann's, for instance

$$\int_{\mathbb{R}^N} Q_L(g,f) \varphi = \int_{\mathbb{R}^{2N}} g_* f \, \mathcal{T}_L \varphi \, dv \, dv_*,$$

where

$$\big[\mathcal{T}_L\varphi\big](v,v_*) = -2b(v-v_*)\cdot\nabla\varphi(v) + a(v-v_*) : D^2\varphi(v). \tag{141}$$

Compare this with the following rewriting of Maxwell's weak formulation of the Boltzmann equation :

$$\int_{\mathbb{R}^N} Q_B(g,f)\varphi = \int_{\mathbb{R}^{2N}} g_* f\, \mathcal{T}\varphi\, dv\, dv_*,$$

where

$$\big[\mathcal{T}\varphi\big](v,v_*) = \int_{S^{N-1}} B(v-v_*,\sigma)(\varphi'-\varphi)\, d\sigma. \tag{142}$$

5.2. Reformulation of the asymptotics of grazing collisions. As we explained in paragraph 3.5, one expects that the Boltzmann operator reduce to the Landau operator when the angular collision kernel concentrates on grazing collisions, the total cross-section for momentum transfer being kept finite.

The first rigorous proofs concerned the spatially homogeneous situation : Arsen'ev and Buryak [**41**] for a smooth kinetic collision kernel, Goudon [**248**] for a kinetic singularity of order less than 2, Villani [**446**] for a kinetic singularity of order less than 4. All proofs were based on variants of the weak formulations above, and used the symmetry $v \longleftrightarrow v_*$.

In order to extend these results to the spatially inhomogeneous setting, there was need for a renormalized formulation which would encompass at the same time the Boltzmann and Landau collision operators. This was accomplished with the results about the Boltzmann equation without cut-off in [**12**]. Here is the renormalized formulation of the Landau equation :

$$\begin{aligned}\beta'(f)Q_L(g,f) = -\overline{c}\,\big[f\beta'(f)-\beta(f)\big] + \nabla\cdot\Big[\nabla\cdot\big(\overline{a}\beta(f)\big) - 2\overline{b}\beta(f)\Big] \\ - \frac{\beta''(f)}{\beta'(f)^2}\,\overline{a}\nabla\beta(f)\,\nabla\beta(f).\end{aligned} \tag{143}$$

Again, β stands for a concave nonlinearity, typically $\beta(f) = f/(1+\delta f)$. If one notes that the second term in the right-hand side of (143) can be rewritten as $Q_L(g,\beta(f))$, there is an excellent analogy between this renormalization and the renormalization of the Boltzmann operator which was presented in paragraph 4.6. This is what makes it possible to pass to the limit.

The convergence of the first and second terms in the renormalized formulation can be expressed in terms of the kernels S (appearing in the cancellation lemma) and $\mathcal{T}$. This allows one to cover very general conditions for the asymptotics of grazing collisions, and this generality is welcome to treat such cases as the Debye approximation. Here we only consider a nonrealistic model case.

Let $(B_n)_{n\in\mathbb{N}}$ be a sequence of collision kernels

$$B_n(v-v_*,\sigma) = \Phi(|v-v_*|)\, b_n(\cos\theta), \tag{144}$$

where the kinetic collision kernel Φ satisfies

$$\Phi(|z|) \xrightarrow[|z|\to\infty]{} 0, \tag{145}$$

$$\Phi(|z|), \qquad \sup_{1<\lambda\leq\sqrt{2}} \left[\frac{\Phi(\lambda|z|)-\Phi(|z|)}{\lambda-1}\right] \quad \in L^1_{\rm loc}(\mathbb{R}^N_z), \tag{146}$$

and the family of angular collision kernels $(b_n)_{n\in\mathbb{N}}$ concentrates on grazing collisions, in the sense

$$\begin{cases} \forall \theta_0 > 0, \qquad \sup\limits_{\theta\geq\theta_0} b_n(\cos\theta) \xrightarrow[n\to\infty]{} 0, \\ \\ \displaystyle\int_{S^{N-1}} b_n(k\cdot\sigma)(1-k\cdot\sigma)\,d\sigma \xrightarrow[n\to\infty]{} \mu > 0, \qquad |k|=1. \end{cases} \tag{147}$$

Let S_n be the kernel associated to B_n as in paragraph 4.2, and $\mathcal{T}_n$ be the linear operator associated to B_n as in formula (142). Moreover, let

$$\Psi(|z|) = \frac{\mu}{4(N-1)}|z|^2\Phi(|z|),$$

and let Q_L, $\mathcal{T}_L$ be the associated quantities entering the Landau operator. Then,

$$S_n(|z|) \xrightarrow[n\to\infty]{} (N-1)\nabla\cdot\left(\frac{z}{|z|^2}\Psi(|z|)\right)$$

in weak-measure sense, and

$$\mathcal{T}_n \xrightarrow[n\to\infty]{} \mathcal{T}_L$$

in distributional sense. In this sense one can say that the sequence of Boltzmann kernels Q_n approaches Q_L.

These lemmas are not enough to pass to the limit. It still remains 1) to gain strong compactness in the sequence of solutions to the Boltzmann equation, 2) to pass to the limit in the last term of the renormalized solution. Task 2) is a very technical job, based on auxiliary entropy dissipation estimates and quite intricate computations, from which the reader is unlikely to learn anything interesting. On the other hand, we explain a little bit about the strong compactness.

5.3. Damping of oscillations in the Landau approximation. As we have seen earlier, entropy dissipation bounds for singular Boltzmann kernels entail the appearance of strong compactness, or immediate damping of oscillations. In the case of the Landau equation, this is the same. It turns out that it is also the same if one considers a sequence of solutions of Boltzmann equations in which the collision kernel concentrates on grazing collisions, in the sense of (147). This is a consequence of the following variant of our joint results in [**10**] :

PROPOSITION 11. *Assume that $B_n(v-v_*,\sigma) \geq \Phi_0(|v-v_*|)b_{0,n}(\cos\theta)$, where Φ_0 is continuous, $\Phi(|z|)>0$ for $|z|>0$, and $b_{0,n}$ concentrates on grazing collisions, in the sense of* (147). *Then there exists $\mu'>0$ and a sequence $\alpha(n)\to 0$ such that*

$$\begin{cases} \displaystyle\int_0^{\alpha(n)} \sin^{N-2}\theta\, b_n(\cos\theta)\,(1-\cos\theta)\,d\theta \xrightarrow[n\to\infty]{} \mu' > 0, \\ \\ \displaystyle\int_{\alpha(n)}^{\pi} \sin^{N-2}\theta\, b_n(\cos\theta)\,d\theta \equiv \psi(n) \xrightarrow[n\to\infty]{} +\infty, \end{cases} \tag{148}$$

and there exists $K > 0$ such that

$$\int_0^\pi \sin^{N-2}\theta\, b_n(\cos\theta)\left(\theta^2|\xi|^2 \wedge 1\right) d\theta \geq K \min\left[\psi(n), |\xi|^2\right]. \tag{149}$$

In particular, for any distribution function f, let $F = \chi\sqrt{f}$ be obtained by multiplication of $\sqrt{f}$ with a smooth cut-off function χ, then

$$\int_{|\xi|\geq R} |\widehat{F}(\xi)|^2\, d\xi \leq C \max\left(\frac{1}{\psi(n)}, \frac{1}{R^2}\right)\left[D_n(f) + \|f\|_{L^1_2}\right],$$

where D_n is the entropy dissipation functional associated with B_n, and C depends on f only via a lower bound for $\int f\, dv$ and an upper bound for $\int f(1+|v|^2+|\log f|)\, dv$.

As a consequence of this proposition, strong compactness is automatically gained in the asymptotics of grazing collisions. By the way, this simplifies already existing proofs [**446**] even in the spatially homogeneous setting.

5.4. Summary. Here we give a precise statement from [**13**].

Theorem 12. *Let B_n be a sequence of collision kernels concentrating on grazing collisions, in the sense of* (144)–(147). *Further assume that $\Phi(|z|) > 0$ as $|z| > 0$. Let $(f^n)_{n\in\mathbb{N}}$ be a sequence of renormalized solutions of the Boltzmann equation (with a defect measure)*

$$\frac{\partial f^n}{\partial t} + v\cdot\nabla_x f^n = Q_n(f^n, f^n),$$

satisfying uniform bounds of mass, energy, entropy, entropy dissipation. Without loss of generality, assume that $f^n \longrightarrow f$ in weak L^1. Then, the convergence is automatically strong, and f is a renormalized solution (with a defect measure) of the Landau equation with

$$\Psi(|z|) = \frac{\lambda}{4(N-1)}|z|^2\Phi(|z|).$$

Remark : Theorem 12 allows for kinetic collision kernels with a strong singularity at the origin, but does not allow collision kernels which are unbounded at large relative velocities.

This theorem includes all preceding results in the field, however in a spatially homogeneous situation one could reasonably hope that present-day techniques would yield an explicit rate of convergence (as $n \to \infty$) when Φ is not too singular. On the other hand, when $\Phi(|z|) = 1/|z|^3$, an improvement of this theorem even in the spatially homogeneous setting would require a much deeper understanding of the Cauchy problem for the Landau equation for Coulomb interaction[15].

6. Lower bounds

We conclude this chapter with estimates on the strict positivity of the solution to the Boltzmann equation. Such results are as old as the mathematical theory of the Boltzmann equation, since Carleman himself proved one of them. At the present time, these estimates are limited to the spatially homogeneous setting, and it is a major open problem to get similar bounds in the full, x-dependent framework in satisfactory generality. Therefore, we restrict the ongoing discussion to spatially

[15]See the discussion in paragraph 1.3 of chapter 5.

homogeneous solutions. Even in this situation, more work remains to be done in the non-cutoff case.

6.1. Mixing effects. First consider the case when Grad's angular cut-off is satisfied, and Duhamel's formula (100) applies. Then one is allowed to write

$$f(t,v) \geq \int_0^t e^{-\int_s^t Lf(\tau,v)\,d\tau}\, Q^+(f,f)(s,v)\,ds, \tag{150}$$

$$f(t,v) \geq e^{-\int_0^t Lf(\tau,v)\,d\tau}\, f_0(v), \tag{151}$$

where $Lf = A * f$, $A(z) = \int B(z,\sigma)\,d\sigma$.

As a trivial consequence of (151), if f_0 is strictly positive (resp. bounded below by a Maxwellian), then the same property will be true for $f(t,\cdot)$.

But a much stronger effect holds true : whatever the initial datum, the solution will be strictly positive at later times. Just to get an idea of this effect, assume that A is bounded from above and below, so that

$$\forall s,t \in [0,T], \quad e^{-\int_s^t Lf(\tau,v)\,d\tau} \geq K_T > 0$$

for some constant K_T depending on T. Then, as a consequence,

$$f(t,v) \geq K_T \int_0^t Q^+(f,f)(s,v)\,ds, \qquad 0 \leq t \leq T. \tag{152}$$

Further assume that

$$f_0 \geq \alpha 1_B, \qquad \alpha > 0, \tag{153}$$

where 1_B is the characteristic function of some ball B in velocity space, without loss of generality B is centered on 0. From (151) it follows that

$$f(t,\cdot) \geq \alpha K_T 1_B, \qquad 0 \leq t \leq T.$$

Now, plug this inside (152), to find that

$$f(t,v) \geq \alpha^2 K_T^3 \int_0^t Q^+(1_B,1_B)(s,v)\,ds, \qquad 0 \leq t \leq T.$$

But $Q^+(1_B,1_B)$ is positive and bounded below in all the interior of the ball $(1+\delta)B$ for δ small enough. In particular, there is a positive constant β such that

$$f(t,v) \geq \alpha^2 K_T^3 \beta\, 1_{(1+\delta)B}.$$

By an immediate induction,

$$\forall t > 0,\ \forall v \in \mathbb{R}^N, \quad f(t,v) > 0.$$

Precise estimates of this type have enabled A. Pulvirenti and Wennberg [**392, 393**] to prove optimal (Gaussian-type) bounds from below on f, for the spatially homogeneous Boltzmann equation with Maxwellian or hard potentials. In this respect they have improved on the old results by Carleman [**119**], who obtained a lower bound like $e^{-|v|^{2+\varepsilon}}$ ($\varepsilon > 0$) in the case of hard spheres. Assumption (153) can also be dispended with, by use of the Q^+ regularity.

Also the proofs in [**393**] are sharp enough to prove existence of a uniform (in time) Maxwellian lower bound.

6.2. Maximum principle. The author suggests another explanation for the immediate appearance of strict positivity, which is the **maximum principle for the Boltzmann equation**. The study of this principle is still under progress, so we cannot yet display explicit lower bounds obtained with this method; the most important feature is that it applies in the non-cutoff case. Let us just give an idea of it.

Rewrite the spatially homogeneous Boltzmann equation as

$$\frac{\partial f}{\partial t} = \left(\int_{\mathbb{R}^N \times S^{N-1}} dv_* \, d\sigma \, B f'_* (f' - f) \right) + f \left(\int_{\mathbb{R}^N \times S^{N-1}} dv_* \, d\sigma \, B (f'_* - f_*) \right). \tag{154}$$

We assume that we deal with a C^∞ solution, which is reasonable when the kinetic collision kernel is nice and when there is a nonintegrable angular singularity. The good point about the decomposition (154) is that it is well-defined[16] even in the non-cutoff case.

Assume now, by contradiction, that there is some point (t_0, v_0) $(t_0 > 0)$ such that $f(t_0, v_0) = 0$. Obviously, $\partial f / \partial t = 0$ at (t_0, v_0). Thus the left-hand side of (154), and also the second term on the right-hand side vanish at (t_0, v_0). But, when $v = v_0$, $f' - f \geq 0$, for all v'. Thus the integrand in the first term on the left-hand side of (154) is nonnegative, but the integral vanishes, so $f' = f = 0$, for all v'. This entails that f is identically 0, which is impossible. In other words, we have recovered the weak result that $f(t, \cdot)$ is strictly positive on the whole of $\mathbb{R}^N$ for $t > 0$.

6.3. Summary.

Theorem 13. *Let B be a collision kernel of the form $B(v - v_*, \sigma) = |v - v_*|^\gamma b(\cos\theta)$, where $\gamma \geq 0$. Let f_0 be an initial datum with finite mass and energy, and $f(t, \cdot)$ be a solution of the spatially homogeneous Boltzmann equation. Then,*

(i) if Grad's angular cut-off condition holds, then for any $t_0 > 0$, there exists a Maxwellian distribution $M(v)$ such that for all $t \geq t_0$, $f(t, v) \geq M(v)$;

(ii) if Grad's angular cut-off condition does not hold, and $f(t, v)$ is a C^∞ function on $(0, +\infty) \times \mathbb{R}^N_v$, then for any $t > 0$, $v \in \mathbb{R}^N$, $f(t, v) > 0$.

Point (i) is due to A. Pulvirenti and Wennberg [**392, 393**]. Point (ii) was first proven by Fournier, using delicate probabilistic methods, in the special case of the Kac equation without cut-off [**218**], then also for the two-dimensional Boltzmann equation under technical restrictions [**218**]. Then it was proven in a much simpler way by the author, with the analytical method sketched above. Current work is aiming at transforming this estimate into a quantitative one.

We note that in the case of the Landau equation with Maxwellian or hard potential [**182**], one can prove a theorem similar to that of A. Pulvirenti and Wennberg by means of the standard maximum principle for parabolic equations[17].

[16] by cancellation lemma, for instance, see paragraph 4.2.

[17] Actually, in [**182**] the stated result is not uniform in time, but, as suggested to us by E. Carlen, a uniform bound is easily obtained by tracing back all the constants : since they are uniform for $t \in (\varepsilon, 2\varepsilon)$ and do not depend on the initial datum, it follows that they are uniform in $t > \varepsilon$.

CHAPTER 3

H THEOREM AND TREND TO EQUILIBRIUM

In chapter 1 we have discussed Boltzmann's H theorem, and the natural conjecture that the solution of Boltzmann's equation converges towards statistical equilibrium, which is a global Maxwellian distribution. In this chapter we shall study this problem of trend to equilibrium, and also enlarge a little bit the discussion to models of collisional kinetic theory which are variants of the Boltzmann equation : for instance, Fokker-Planck type equations, or simple models for granular media. The Cauchy problem for these equations is usually not so challenging as for the Boltzmann equation, but the study of trend to equilibrium for these models may be very interesting (both in itself, and to enlighten the Boltzmann case).

As a general fact, one of the main features of many collisional kinetic systems is their tendency to converge to an equilibrium distribution as time becomes large, and very often a thermodynamical principle underlies this property : there is a distinguished Lyapunov functional, or entropy, and the equilibrium distribution achieves the minimum of this functional under constraints imposed by the conservation laws. In section 1 we shall review some of these models. For each example, we shall be interested in the functional of **entropy dissipation**, defined by the equation

$$D(f_0) = -\left.\frac{d}{dt}\right|_{t=0} E[f(t)],$$

where E is the Lyapunov functional, and $(f(t))_{t\geq 0}$ the solution to the equation under study, $f(0) = f_0$. We shall use the denomination "entropy dissipation" even when E is not the usual Boltzmann entropy.

Traditional approaches for the study of trend to equilibrium rely on soft methods, like compactness arguments, or linearization techniques, which ideally yield rates of convergence. In section 2 we briefly review both methods and explain why they cannot yield definitive answers, and should be complemented with other, more constructive methods. This will lead us to discuss entropy dissipation methods, starting from section 3.

In section 4, we expose quantitative versions of the H theorem for the Boltzmann and Landau operators, in the form of some functional inequalities. Then in section 5 we show how these inequalities can be used for the study of the trend to equilibrium for the spatially homogeneous Boltzmann and Landau equations.

Section 6 is devoted to a class of collision models which exhibit a particular gradient structure. Specific tools have been devised to establish variants of the H theorem in this case.

Finally, section 7 deals with the subtle role of the position variable for spatially inhomogeneous models. The construction of this area is only beginning.

Contents

1. A gallery of entropy-dissipating kinetic models

Let us first review some of the basic models and the associated entropy functionals, equilibria, entropy dissipation functionals. We shall not hesitate to copy-cut some of the formulas already written in our introductory chapter.

1.1. Spatially homogeneous models. These models read

$$\frac{\partial f}{\partial t} = Q(f), \qquad t \geq 0, \quad v \in \mathbb{R}^N,$$

where the collision operator Q, linear or not, may be

1) the Boltzmann operator,

$$Q(f) = Q_B(f,f) = \int_{\mathbb{R}^N} dv_* \int_{S^{N-1}} d\sigma\, B(v-v_*,\sigma)\Big(f'f'_* - ff_*\Big); \tag{155}$$

then there are three conservation laws : mass, momentum and energy. Moreover, the natural Lyapunov functional is the H-functional,

$$H(f) = \int_{\mathbb{R}^N} f \log f,$$

and its dissipation is given by the by now familiar functional

$$D(f) = \frac{1}{4} \int dv\, dv_*\, d\sigma\, B(v-v_*,\sigma)\Big(f'f'_* - ff_*\Big) \log \frac{f'f'_*}{ff_*} \geq 0. \tag{156}$$

Define ρ, u, T by the usual formulas (1), then the equilibrium is the Maxwellian

$$M(v) = M^f(v) = \frac{e^{-\frac{|v-u|^2}{2T}}}{(2\pi T)^{N/2}}.$$

Important remark : We shall only consider here the case of the Boltzmann equation with finite temperature. In the case of infinite temperature, almost nothing is known, except for the very interesting recent contribution by Bobylev and Cercignani [**81**].

It should be noted that, since M has the same moments as f up to order 2,

$$H(f) - H(M) = \int_{\mathbb{R}^N} f \log \frac{f}{M},$$

which is nothing but the *Kullback relative entropy* of f with respect to M, and that we shall denote by $H(f|M)$. Generally speaking, the Kullback relative entropy between two probability densities (or more generally two nonnegative distributions) f and g is given by the formula

$$H(f|g) = \int f \log \frac{f}{g}. \tag{157}$$

It is well-known[1] that $H(f|g) \geq 0$ as soon as f and g have the same mass.

[1]The classical proof is to revwrite (157) as $\int f[-\log(g/f)+g/f-1]$ (or as $\int g[(f/g)\log(f/g)-(f/g)+1]$) and to use the inequality $\log X \leq X-1$ (or $X \log X \geq X-1$). Compare with the Cercignani-Lampis trick of eq. (50).

2) the Landau operator,

$$Q(f) = Q_L(f,f) = \nabla_v \cdot \left(\int_{\mathbb{R}^N} dv_* \, a(v-v_*) \big[f_* (\nabla f) - f (\nabla f)_* \big] \right), \tag{158}$$

$$a_{ij}(z) = \Psi(|z|) \left[\delta_{ij} - \frac{z_i z_j}{|z|^2} \right]; \tag{159}$$

in this case there are also three conservation laws, and the natural Lyapunov functional is also the H-functional. Now the entropy dissipation is

$$D_L(f) = \frac{1}{2} \int_{\mathbb{R}^N \times \mathbb{R}^N} f f_* \Psi(|v-v_*|) \left| \Pi(v-v_*) \Big(\nabla (\log f) - [\nabla (\log f)]_* \Big) \right|^2, \tag{160}$$

where $\Pi(z)$ stands for the orthogonal projector onto $z^\perp$. As for the equilibrium state, it is still the same as for the Boltzmann equation.

3) the linear Fokker-Planck operator,

$$Q(f) = Q_{FP}(f) = \nabla_v \cdot \big(\nabla_v f + f v \big). \tag{161}$$

In this case there is only one conservation law, the mass ($\rho = \int f \, dv$), and the natural Lyapunov functional is the *free energy* : this is the sum of the H-functional and the kinetic energy,

$$E(f) = \int_{\mathbb{R}^N} f \log f + \int_{\mathbb{R}^N} f \, \frac{|v|^2}{2} \, dv. \tag{162}$$

Moreover, the entropy dissipation is

$$D_{FP}(f) = \int_{\mathbb{R}^N} f \left| \frac{\nabla f}{f} + v \right|^2 dv,$$

which can be rewritten as the so-called relative *Fisher information* of f with respect to M, thereafter denoted by $I(f|M)$. More generally,

$$I(f|g) = \int_{\mathbb{R}^N} f \left| \nabla \log \frac{f}{g} \right|^2. \tag{163}$$

Compare with the definition of the relative Kullback entropy (157).

4) a coupled Fokker-Planck operator, like

$$\rho^\alpha \nabla_v \cdot \Big[T \nabla_v f + f (v-u) \Big],$$

where $0 \leq \alpha \leq 1$ and ρ, u, T are coupled to f by the usual formulas (1). In this case there are three conservation laws, the natural Lyapunov functional is the H-functional, and the entropy dissipation is

$$\rho^\alpha \int_{\mathbb{R}^N} f \left| \nabla_v \log \frac{f}{M^f} \right|^2 = \rho^\alpha I(f|M^f).$$

The equilibrium is the same as for the Boltzmann operator.

Other couplings are possible : one may decide to couple only T, or only u...

5) some entropy-dissipating model for granular flow, like the one-dimensional model proposed in [**70**],

$$Q(f) = \nabla_v \cdot \Big(f \, \nabla_v (f * U) \Big), \tag{164}$$

where $U(z) = |z|^3/3$. Then there are two conservation laws, mass and momentum; and the natural Lyapunov functional is

$$E(f) = \frac{1}{2}\int_{\mathbb{R}^{2N}} f(v)f(w)U(v-w)\,dv\,dw, \tag{165}$$

while its dissipation is

$$D(f) = \int_{\mathbb{R}^N} f\left|\nabla U * f\right|^2.$$

Moreover the equilibrium is $\rho\delta_u$, i.e. a multiple of the Dirac mass located at the mean velocity.

A particular feature of this model is its gradient flow structure. Generally speaking, models of the form

$$\frac{\partial f}{\partial t} = \nabla\cdot\left(f\,\nabla\frac{\delta E}{\delta f}\right), \tag{166}$$

where E is some energy functional and $\delta E/\delta f$ stands for its gradient with respect to the usual L^2 structure, can be considered as gradient flows [**364, 365**], via geometric and analytical considerations which are strongly linked with the Wasserstein distance[2]. An integration by parts shows that solutions of (166) admit E as a Lyapunov functional, and the dissipation is given by

$$D(f) = \int_{\mathbb{R}^N} f\left|\nabla\frac{\delta E}{\delta f}\right|^2.$$

Falling into this category is in particular the model for granular flow discussed in [**68**], in which one adds up the collision operators (164) and (161).

1.2. Spatially inhomogeneous models. These models can be written in the general form

$$\frac{\partial f}{\partial t} + v\cdot\nabla_x f + F(x)\cdot\nabla_v f = Q(f), \qquad t\geq 0, \quad x\in\mathbb{R}^N, \quad v\in\mathbb{R}^N, \tag{167}$$

where F is the sum of all macroscopic forces acting on the system, and Q is one of the collision operators described in the previous paragraph (*acting only on the velocity variable !*)

In the sequel we shall only consider the situation when the total mass of the gas is finite; without loss of generality it will be normalized to 1. We mention however that the case of infinite mass deserves interest and may be studied in the spirit of [**310**].

If the total mass is finite, then among the forces must be a *confinement* which prevents the system from escaping at infinity, and ensures the existence of a relevant equilibrium state. There are several possibilities :

Potential confinement : assume that the particles interact with the background environment via some fixed potential, $V(x)$. Then the force is just

$$F(x) = -\nabla V(x).$$

[2]The Wasserstein distance is defined by eq. (244). The gradient structure is explained in paragraph 6.1.

The minimum requirement for V to be confining is $e^{-V} \in L^1$. Since V is defined up to an additive constant, one can assume without loss of generality that

$$\int_{\mathbb{R}^N} e^{-V(x)}\, dx = 1.$$

The presence of the confining potential does not harm the conservation of mass, of course; on the other hand, when Q is a Boltzmann-type collision operator (with three conservation laws), it usually destroys the conservations of momentum and energy. Instead, there is conservation of the total mechanical energy,

$$\int_{\mathbb{R}^N \times \mathbb{R}^N} f(x,v) \left[V(x) + \frac{|v|^2}{2} \right] dx\, dv.$$

And as far as the entropy is concerned, it is not changed for Boltzmann-type models : this is still the usual H-functional, the only difference being that now the phase space is $\mathbb{R}^N_x \times \mathbb{R}^N_v$:

$$H(f) = \int_{\mathbb{R}^N \times \mathbb{R}^N} f \log f.$$

This similarity is a consequence of the physical assumption that collisions are localized in space.

For the linear Fokker-Planck equation, the situation is different : to the free energy one has to add the *potential energy*, so the natural Lyapunov functional is

$$E(f) = \int_{\mathbb{R}^N \times \mathbb{R}^N} f \log f + \int_{\mathbb{R}^N \times \mathbb{R}^N} f(x,v) \left[V(x) + \frac{|v|^2}{2} \right] dx\, dv.$$

Box confinement : another possible confinement is when the system is enclosed in a box $X \subset \mathbb{R}^N_x$, with suitable boundary conditions. The most standard case, namely specular reflection, is a limit case of the preceding one : choose $V = +\infty$ outside of X, $V =$ const. within X. When specular reflection is imposed, then the energy conservation is restored (not momentum conservation), and the Lyapunov functional is the same as in the spatially homogeneous case, only integrated with respect to the x variable. For other boundary conditions such as diffusive, the entropy functional should be modified [**141, 143**].

Torus confinement : this is the most convenient case from the mathematical point of view : set the system in the torus $\mathbb{T}^N_x$, so that there are no boundaries. Physicists also use such models for discussing theoretical questions, and numerical analysts sometimes find them convenient.

Additional force terms : many models include other force terms, in particular self-consistent effects described by *mean-field interactions* : typically,

$$F(x) = -\nabla \Phi(x), \qquad \Phi = \phi * \rho, \qquad \rho = \int_{\mathbb{R}^N} f\, dv,$$

where ϕ is an interaction potential between particles. As we already mentioned, from the physical point of view it is not always clear whether interactions should be modelled via collisions, or mean-field forces, or both... A very popular model is the **Vlasov-Fokker-Planck equation**, in which the collision operator is the

Fokker-Planck operator and the forces include both confinement and self-consistent interaction. Let us rewrite the model explicitly :

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + F(x) \cdot \nabla_v f = \nabla_v \cdot \big(\nabla_v f + f v\big),$$

$$F = -\nabla\Big(V + \phi * \rho\Big), \qquad \rho = \int_{\mathbb{R}^N} f \, dv.$$

If the interaction is Coulomb, then one speaks of Vlasov-Poisson-Fokker-Planck model; this case is very singular, but it has a lot of additional structure because (by definition) the potential ϕ is the fundamental solution of the Laplace operator.

In the self-consistent case, one has to add a term of *interaction energy* to the free energy :

$$\frac{1}{2} \int_{\mathbb{R}^N \times \mathbb{R}^N} \rho(x) \rho(y) \phi(x-y) \, dx \, dy.$$

Then the entropy dissipation is unchanged.

Let us now turn to **equilibrium states**. Their classification in a spatially inhomogeneous context is quite a tedious task. Many subcases have to be considered, the dimension of the space comes into play, and also the symmetries of the problem. We are not aware of any systematic treatment; we shall only consider the most typical situations.

- The Boltzmann (or Landau) equation in a box. Then, in dimension $N = 2, 3$ there is a unique steady state which takes the form of a global Maxwellian : $f(x, v) = M(v)$. The mass and temperature of the Maxwellian are determined by the conservation laws, while the mean velocity is 0. This result holds true on the condition that the box be not circular in dimension $N = 2$, or cylindric in dimension $N = 3$ (i.e. with an axis of symmetry). For this one can consult [**254, 167, 143**].
- The Boltzmann (or Landau) equation in a confining potential. Then, the unique steady state has the form $f(x, v) = e^{-V(x)} M(v)$. Again, the mass and temperature of M are determined by the conservation laws, and the mean velocity is 0. This result holds true if the potential V is not quadratic; if it is, then there exist periodic (in time) solutions, which can be considered as stationary even if they are time-dependent. This was already noticed by Boltzmann (see for instance [**143**]).
- The Boltzmann (or Landau) equation in a torus. Then, the unique steady state has the form $f(x, v) = M(v)$ where M is an absolute Maxwellian. The mass and temperature, but also the mean velocity of M are determined by the conservation laws.
- The Fokker-Planck equation in a confining potential. Then, the unique steady state is $f(x, v) = e^{-V(x)} M(v)$, where M is the Maxwellian with unit temperature and zero mean. The mass is of course determined by the conservation law.
- The Vlasov-Fokker-Planck equation in a confining potential. In general there is a unique steady state in this situation, and it takes the form $f(x, v) = \rho_\infty(x) M(v)$, where M is the Maxwellian with unit temperature

and zero mean. The density ρ_∞ is nonexplicit, but solves a nonlinear equation of the form

$$\rho_\infty * \phi = \frac{e^{-(V+\phi*\rho_\infty)}}{\int_{\mathbb{R}^N} e^{-(V+\phi*\rho_\infty)}\, dx}.$$

There are also variational formulations of this problem. In the case of the Vlasov-Poisson-Fokker-Planck equation, a detailed survey of the situation is given by Dolbeault [**197**].

In all the preceding discussion, we have avoided the models for granular collisions, eq. (164). A naive guess would be that the natural Lyapunov functional, in the spatially inhomogeneous case, is obtained by integrating its spatially homogeneous counterpart, eq. (165), in the x variable. This is false ! Because the transport operator $-v \cdot \nabla_x$ may have an influence on the evolution of this functional.

1.3. Related models. The following models are not kinetic models, but have come to be studied by members of the kinetic community because of the unity of methods and problematics.

- The **spatial Fokker-Planck equation**, or Smoluchowski[3] equation [**399**]

$$\frac{\partial \rho}{\partial t} = \nabla_x \cdot \big(\nabla_x \rho + \rho \nabla V(x)\big), \qquad t \geq 0, \quad x \in \mathbb{R}^N. \tag{168}$$

One always assume $e^{-V} \in L^1$, and without loss of generality e^{-V} should be a probability measure, just as ρ. For this equation the natural Lyapunov functional is the free energy, or relative entropy, $H(\rho|e^{-V})$, and the entropy dissipation coincides with the relative Fisher information, $I(\rho|e^{-V})$. For a summary of recent studies concerning the trend to equilibrium for (168), the reader may consult Arnold et al. [**39**], or Markowich and Villani [**330**].

- Equations modelling **porous medium with confinement** :

$$\frac{\partial \rho}{\partial t} = \nabla_x \cdot \big(\nabla_x P(\rho) + \rho \nabla V(x)\big), \qquad t \geq 0, \quad x \in \mathbb{R}^N, \tag{169}$$

where P is a nonlinearity, $P(\rho)$ standing for a pressure term, for instance $P(\rho) = \rho^\gamma$. In this last case (power law), equations like (169), with a quadratic confinement potential, naturally arise as rescaled versions of their counterparts without confinement.

The natural Lyapunov functional for (169) is

$$\int A(\rho)\, dx + \int \rho V(x)\, dx,$$

where $P(\rho) = \rho A'(\rho) - A(\rho)$.

The trend to equilibrium for (169) has been studied independently by Carrillo and Toscani [**131**], Dolbeault and Del Pino [**163**], Otto [**364**] for the power law case, then more generally by Carrillo et al [**129**].

One of the most remarkable features of eq. (168) and (169) is that they have the form of a gradient flow,

$$\frac{\partial \rho}{\partial t} = \nabla \cdot \left(\rho \, \nabla \frac{\delta E}{\delta \rho} \right).$$

[3]There are several types of equations which are called after Smoluchowski !

For a general discussion of the implications, see for instance Otto and Villani [**365**], Markowich and Villani [**330**].

1.4. General comments. In this paragraph we shall informally discuss the features which may help the trend to equilibrium, or on the contrary make it more difficult — both from the physical and from the mathematical point of view.

► First of all, the distribution tails are usually at the origin of the worst difficulties. By distribution tails, we mean how fast the distribution function decreases as $|v| \to \infty$, or $|x| \to \infty$. This is not only a technical point; Bobylev has shown that large tails could be a true obstacle to a good trend to equilibrium for the Boltzmann equation, even in the spatially homogeneous case. More precisely, he proved the following result [**79**]. Consider the spatially homogeneous Boltzmann equation with Maxwell collision kernel (with or without cut-off), and fix the mass, momentum, energy of the initial datum f_0. Let $M(v)$ be the corresponding equilibrium state. Then, for any $\varepsilon > 0$ one can construct an initial datum $f_0 = f_0^\varepsilon$ such that the associated solution $f^\varepsilon(t, v)$ of the Cauchy problem satisfies

$$\forall t \geq 0, \ \|f^\varepsilon(t, \cdot) - M\| \geq K_\varepsilon e^{-\varepsilon t}, \qquad K_\varepsilon > 0.$$

At this point we should make a remark to be honest : an eye observation of a plot of these particular solutions will show hardly any departure from equilibrium, because most of the discrepancy between f^ε and M is located at very high velocities — and because the constant K_ε is rather small. This illustrates the general fact that precise "experimental" information about rates of convergence to equilibrium is very difficult to have, if one wants to take into account distribution tails.

► Moreover, recent studies have shown that the Boltzmann equation, due to its nonlocal nature, is more sensitive to this tail problem than diffusive models like Landau or Fokker-Planck equations. For the latter equations, it is not possible to construct "pathological" solutions as Bobylev; the trend to equilibrium is typically exponential, with a rate which is bounded below. We shall come back to this point, which by the way is also folklore in the study of Markov processes : it is known that jump processes have more difficulties in going to equilibrium than diffusion processes.

► Next, it is clear that the more collisions there are, the more likely convergence is bound to be fast. This is why the size of the collision kernel does matter, in particular difficulties arise in the study of hard potentials because of the vanishing of the collision kernel at zero relative velocities; and also in the study of soft potentials because of the vanishing of the collision kernel for large relative velocities. A common belief is that the problem is worse for soft potentials than for hard. Also note that hard potentials are associated with a good control of the distribution tails, while soft potentials are not.

Studies of the linearized operator show that in principle, one could expect an exponential decay to equilibrium for the spatially homogeneous Boltzmann equation with hard or Maxwellian potentials (under strong control of the distribution tails), while for soft potentials the best that one could hope is decay like $O(e^{-t^\alpha})$ for some $\alpha \in (0, 1)$ (see Caflisch [**111**]). This is of course related to the fact that there is a spectral gap in the first case, not in the second one.

▸ In the case of Boltzmann or Landau models (or some versions of coupled Fokker-Planck), the collision frequency also depends on the density of particles. This of course can be seen via the fact that Boltzmann and Landau operators are quadratic, while the linear Fokker-Planck is not. As a consequence, the trend to equilibrium should be extremely slow at places where the density stays low : typically, very large positions. Therefore, the trend to equilibrium is expected to hold on extremely long scales of times when one considers the Boltzmann equation in a confinement potential, as opposed to the Boltzmann equation in a finite box.

▸ In the x-dependent case, a strong mathematical difficulty arises : the existence of *local equilibria*. These are states which make the entropy dissipation vanish, but are not stationary states. In fact they are in equilibrium with respect to the velocity variable, but not with respect to the position variable; for instance they are local Maxwellians $M_x(v)$, with parameters ρ, u, T depending on x. Of course the trend to equilibrium is expected to be slowed down whenever the system comes close to such a state. We shall discuss this problem in more detail in section 7.

▸ Finally, a gradient flow structure often brings more tools to study the trend to equilibrium. We shall see this in the study of such models as (161) or (164). As we mentioned in paragraph 2.4 of chapter 1, in the case of the spatially homogeneous Boltzmann equation no gradient flow structure has been identified. Moreover, for all the spatially inhomogeneous equations which are considered here, the existence of the local equilibria rules out the possibility of such a structure.

2. Nonconstructive methods

In this section, we briefly review traditional methods for studying the convergence to equilibrium.

2.1. Classical strategy. A preliminary step of (almost) all methods is to identify stationary states by searching for solutions of the functional equation $D(f) = 0$, or more generally $\int_0^T D(f(t))\,dt = 0$. Once uniqueness of the stationary solution has been shown, then weak convergence of the solution towards equilibrium is often an easy matter by the use of compactness tools. Uniqueness may hold within some subclass of functions which is left invariant by the flow.

For instance, in the case of the spatially homogeneous Boltzmann equation, it is easy to prove weak convergence as $n \to \infty$ of $f(n+t,v)_{n\in\mathbb{N}}$ towards the right Maxwellian distribution in weak-$L^p([0,T]\times\mathbb{R}^N)$, as soon as

$$\lim_{R\to\infty} \limsup_{t\to\infty} \int_{|v|\geq R} f(t,v)|v|^2\,dv = 0. \tag{170}$$

Condition (170), thereafter referred to as **"tightness of the energy"**, ensures that there is no leak of energy at large velocities, and that $f(t,\cdot)$ does converge towards the right Maxwellian distribution — and not towards a Maxwellian with too low temperature. In all the sequel, we will assume that the moments of f are normalized, so that the equilibrium distribution is the standard Maxwellian M with zero mean and unit temperature.

As a typical result, under general conditions Arkeryd [**17**] proved that the solution to the spatially homogeneous Boltzmann equation with hard potentials does converge to M, weakly in L^1, as $t \to \infty$. This result is facilitated by the fact that

eq. (170) is very easy to prove for hard potentials, while it is a (seemingly very difficult) open problem for soft potentials.

In the framework of the spatially homogeneous Boltzmann equation with **Maxwellian collision kernel**, other approaches are possible, which do not rely on the entropy dissipation. Truesdell [**274**] was the first one to use such a method : he proved that all spherical moments satisfy closed differential equations, and converge towards corresponding moments of M. This implies weak convergence of $f(t,\cdot)$ towards M.

Also contracting metrics[4] can be used for such a purpose along the ideas of Tanaka [**414, 415**].

A refinement is to prove **strong convergence** of $f(t,\cdot)$ towards M as $t \to \infty$, for instance as a consequence of some uniform (in time) smoothness estimates. The first result of this kind is due to Carleman [**118**] : he proved uniform equicontinuity of the family $(f(t,\cdot))_{t\geq 0}$ when f is the isotropic solution of the spatially homogeneous Boltzmann equation with hard spheres, assuming that the initial datum decays in $O(1/|v|^6)$. As a consequence, he recovered uniform convergence to equilibrium. This method was improved by Gustafsson [**270**] who proved strong L^p convergence for the solution of the spatially homogeneous Boltzmann equation with hard potentials, under an ad hoc L^p assumption on the initial datum.

In the much more general framework of the spatially inhomogeneous Boltzmann equation, by use of the Q^+ regularity, Lions [**308**] proved strong L^1 compactness as $t \to \infty$, say when the system is confined in a torus. This however is not sufficient to prove convergence, because there is no clue of how to prove the spatially-inhomogeneous variant of (170),

$$\lim_{R\to\infty} \limsup_{t\to\infty} \int_{\mathbb{T}^N} dx \int_{|v|\geq R} f(t,v)|v|^2\, dv = 0.$$

At this point we have to recognize that there is, to this date, *no* result of trend to equilibrium in the spatially inhomogeneous context, except in the perturbative framework of close-to-equilibrium[5] solutions : see for instance [**286**] (perturbation setting in whole space) or [**404**] (in a bounded convex domain) — with just one exception : the case of a box with uniform Maxwellian diffuse boundary conditions, which was solved by Arkeryd and Nouri [**35**] in a non-perturbative setting. On the contrary, it is rather easy to prove convergence to equilibrium for, e.g., the spatially inhomogeneous linear Fokker-Planck equation.

Once strong convergence to equilibrium has been established (for instance, in the case of the spatially homogeneous Boltzmann equation with hard potentials), a natural refinement is to ask for a **rate of convergence**. In the "good" cases, a hard work leads to exponential rates of decay thanks to **linearization** techniques and the study of the **spectral gap** of the linearized operator. This strategy was successfully applied by Arkeryd [**23**], Wennberg [**456**] to the spatially homogeneous Boltzmann equation with hard, or Maxwellian potentials. In the spatially inhomogeneous context, it was developed by the Japanese school under the assumption that the initial datum is already extremely close to equilibrium.

[4] See section 2 in chapter 4.

[5] Of course it is not a very satisfactory situation if one is able to prove convergence to equilibrium only when one starts extremely close to equilibrium...

In the case of soft potentials, though there is no spectral gap for the linearized operator, Caflisch [**111**] was able to prove convergence to equilibrium like e^{-t^β} for some exponent $\beta \in (0,1)$ – also under the assumption that the initial datum belong to a very small neighborhood of the equilibrium.

2.2. Why ask for more ? The preceding results, as important as they may be, cannot be considered as a definitive answer to the problem of convergence to equilibrium. There are at least two reasons for that :

1) Non-constructiveness : The spectral gap (when it exists, which is not always the case !) is usually non explicit : for the Boltzmann equation with hard or soft potentials there is only one exception, the spatially homogeneous operator with Maxwellian collision kernel. What is more problematic, nobody knows how to get estimates on its size : usual arguments for proving its existence rely on Weyl's theorem, which asserts that the essential spectrum is invariant under compact perturbation. But this theorem, which is based on a compactness argument, is nonexplicit...

Another problem arises because the natural space for the linearized operator (the space in which it is self-adjoint) is typically $L^2(M^{-1})$, endowed with the norm $\|f\|^2_{L^2(M^{-1})} = \int f^2/M$, which is of course much narrower than the natural spaces for the Cauchy problem (say, Lebesgue or Sobolev spaces with polynomial weights). A new compactness argument is needed [**457**] to prove the existence of a spectral gap in these much larger spaces.

Remark : This problem of functional space arises even for linear equations ! For instance, if one considers the Fokker-Planck equation, then the spectral gap exists in the functional space $L^2(M^{-1})$, but one would like to prove exponential convergence under the sole assumption that the initial datum possess finite entropy and energy.

2) nature of the linearization procedure : In fact, even if linearization may predict an asymptotic rate of convergence, it is by nature unable to yield explicit results. Indeed, it only shows exponential convergence in a very small neighborhood of the equilibrium : a neighborhood in which nonlinearities are negligible in front of the linear terms. It cannot say anything on the time the solution needs to enter such a neighborhood....

This of course does not mean that linearization is in essence a bad method, but that it is a valuable method *only for perturbations of equilibria*.

Entropy dissipation methods have been developed to remedy these problems, and yield explicit estimates of trend to equilibrium in a fully nonlinear context. We note that these methods are not the only effective methods in kinetic theory : other techniques, which have been developed in the particular framework of Maxwellian collision kernels, will be reviewed in chapter 4.

Thus, the ideal mathematical situation, combining the power of both entropy methods and linearization techniques, would be the following. From a starting point which is far from equilibrium, an entropy method applies to show that the solution approaches equilibrium, possibly with a non-optimal rate (maybe not exponential...) After some explicit time, the solution enters a small neighborhood of equilibrium in which linearization applies, and a more precise rate of convergence can be stated.

For this plan to work out, it would seem necessary to 1) refine linearization techniques to have explicit bounds on the spectral gap, 2) establish very strong

a priori estimates, so that convergence in entropy sense imply a much stronger convergence, in a norm well-adapted to linearization — or 2') show that the solution can be decomposed into the sum of an exponentially small part, and a part which is bounded in the sense of this very strong norm.

2.3. Digression. At this point the reader may ask why we insist so much on explicit estimates. This of course is a question of personal mathematical taste. We do believe that estimates on the qualitative behavior of solutions should always be explicit, or at least explicitable, and that a compactness-based argument showing trend to equilibrium cannot really be taken seriously. First because it does not ensure that the result is physically realistic, or at least that it is not unrealistic by many orders of magnitude. Secondly because of the risk that the constants involved be so huge as to get out of the mathematical range which is allowed by the model. For instance, what should we think of a theorem predicting trend to equilibrium like $e^{-10^{-1000}t}$? The corresponding time scale is certainly much larger than the time scale on which the Boltzmann description may be relevant[6].

Of course, asking for realistic estimates may be a formidable requirement, and often one may already be very lucky to get just constructive estimates. Only when no such estimates are known, should one take into account nonexplicit bounds, and they should be considered as rough results calling for improvements. This is why, for instance, we have discussed the results of propagation, or appearance, of strong compactness in the context of the Cauchy problem for renormalized solutions...

3. Entropy dissipation methods

3.1. General principles. The main idea behind entropy dissipation methods is to establish quantitative variants of the mechanism of decreasing of the entropy : in the case of the Boltzmann equation, this is the H theorem. This approach has the merit to stand upon a clear physical basis, and experience has shown its robustness and flexibility.

▸ **Rule 1 :** the "discrepancy" between a distribution function f and the equilibrium f_∞ should not be measured by the L^1 norm, but rather by $E[f|f_\infty] \equiv E(f) - E(f_\infty)$, thereafter called relative entropy by abuse of language. Thus, one should not try to prove that $f(t)$ converges to f_∞ in L^1, but rather show that $E(f(t)) \longrightarrow E(f_\infty)$ as $t \to \infty$, which will be called **"convergence in relative entropy"**. A separate issue is to understand whether convergence in relative entropy implies convergence in some more traditional sense.

▸ **Rule 2 :** one considers as a main object of study the **entropy dissipation functional** D. Of course, the definition of the entropy dissipation relies on the evolution equation; but it is important to consider D as a functional that can be applied to any function, solution or not of the equation.

▸ **Rule 3 :** one tries to quantify the following idea : *if, at some given time t, $f(t)$ is far from f_∞, then $E(f)$ will decrease notably at later times.*

Before turning to less abstract considerations, we comment on the idea to measure the distance in terms of the entropy, rather than, say, in terms of the well-known L^1 distance. A first remark is that there is no physical meaning, in the context of

[6] See the discussion at the end of paragraph 2.4.

kinetic equations, in L^1 distance. Some rather violent words by Truesdell will illustrate this. After proving exponential convergence of all moments in the framework of the spatially homogeneous Boltzmann equation with Maxwell collision kernel, he adds [**274**, p. 116] "Very likely it can be shown that [the solution] itself approaches Maxwellian form, but there is little interest in this refinement." A justification of this opinion is given on p. 112 : "Since apart from the entropy it is only the moments of the distribution function that have physical significance, the result sought is unnecessarily strong". Thus, at the same time that he attacks the relevance of L^1 results, Truesdell implicitly supports entropy results...

A second remark is that, very often, convergence of the entropy implies convergence in L^1 sense. In the case of the H-functional, or more generally when $E(f) - E(f_\infty)$ takes the form of a relative Kullback entropy, this is a well-known result. Indeed, the famous (and elementary) Csiszár-Kullback-Pinsker inequality states that whenever f and g are two probability distributions,

$$\frac{1}{2}\|f-g\|_{L^1}^2 \leq \int f \log \frac{f}{g} = H(f|g).$$

In many other instances, especially when a gradient flow structure is present, the quantity $E(f) - E(f_\infty)$ can also be shown to control some power of the Wasserstein distance[7]. For this see in particular Otto and Villani [**365**]. A basic example is the Talagrand inequality,

$$\frac{1}{2}W(f,M)^2 \leq H(f|M),$$

where M is the zero-mean, unit-temperature Maxwellian distribution. Usually, one can then obtain control of the L^1 norm via some ad hoc interpolation procedure [**130**].

As a final remark, we comment on the entropy dissipation equality itself — say in the case of the Boltzmann equation. As we saw, formally, solutions of the spatially homogeneous Boltzmann equation satisfy the identity

$$\frac{d}{dt}H(f(t,\cdot)) = -D(f),$$

but when is this rigorous ? It was actually proven by Lu [**326**] that this equality always holds for hard potentials with cut-off, under the sole assumptions that the initial datum has finite mass, energy and entropy. In fact, we shall always work under much stronger conditions. Thus, in all the sequel, we shall always consider situations in which *the estimates for the Cauchy problem are strong enough, that the entropy dissipation identity can be made rigorous.* Such is not the case, for instance, in the framework of the DiPerna-Lions theory of renormalized solutions[8].

3.2. Entropy-entropy dissipation inequalities. When trying to implement the preceding general principles, one can be lucky enough to prove an *entropy-entropy dissipation inequality* : this is a functional inequality of the type

$$D(f) \geq \Theta\big(E[f|f_\infty]\big), \tag{171}$$

where $H \mapsto \Theta(H)$ is some continuous function, *strictly positive when* $H > 0$. The main idea is that *"entropy dissipation controls relative entropy"*.

[7] eq. (244) below.

[8] In any case, this theory should be hopelessly excluded from any study of trend to equilibrium until energy conservation, and even local energy conservation, has been proven.

Such an inequality implies an immediate solution to the problem of trend to equilibrium. Indeed, let $f(t)$ be a solution of the evolution equation. Since $D(f(t)) = -(d/dt)E[f(t)|f_\infty]$, it follows that the relative entropy $H(t) \equiv E[f(t,\cdot)|f_\infty]$ satisfies the differential inequality

$$-\frac{d}{dt}H(t) \geq \Theta(H(t)). \tag{172}$$

This implies that $H(t) \longrightarrow 0$ as $t \to +\infty$, and if the function Θ is known with enough details, one can compute an explicit rate of convergence. For instance, a linear bound like

$$D(f) \geq 2\lambda\, E[f|f_\infty]$$

will entail exponential convergence to equilibrium, relative entropy converging to 0 like $e^{-2\lambda t}$. On the other hand, an exponent bigger than 1,

$$D(f) \geq K E[f|f_\infty]^{1+\alpha}, \qquad (K > 0,\, \alpha > 0),$$

will entail "polynomial" rate of convergence to equilibrium, the entropy going down like $O(t^{-1/\alpha})$.

Situations in which the exponent is *lower* than 1 are very rare; in such cases the system converges to equilibrium in finite time. This occurs in certain simple model equations for granular media [**426**].

Very often, one cannot hope for such a strong inequality as (171), but one can prove such an inequality in a restricted class of functions :

$$D(f) \geq \Theta_f\big(E[f|f_\infty]\big), \tag{173}$$

where the explicit form of Θ_f may depend on some features of f such as its size in some (weighted) Lebesgue spaces, its strict positivity, its smoothness, etc. : all kinds of a priori estimates which should be established independently.

In collisional kinetic theory, there are many situations in which entropy-entropy dissipation inequalities *cannot* hold true, in particular for spatially inhomogeneous models when the collisions only involve the velocity variable. As we shall see, in such cases it is sometimes possible to use entropy-entropy dissipation inequalities from spatially homogeneous models.

As a final remark, the interest of entropy-entropy dissipation inequalities is not restricted to proving theorems of trend to equilibrium. Entropy-entropy dissipation inequalities may also in principle be applied in problems of hydrodynamic (as opposed to long-time) limits, yielding rather explicit estimates. For this one may consult the work by Carlen et al. [**124**] on a baby model, the recent paper by Saint-Raymond [**400**] on the hydrodynamic limit for the BGK model, or the study by Berthelin and Bouchut [**74**] on a complicated variant of the BGK model. However, to apply this strategy to more realistic hydrodynamic limits, say starting from the Boltzmann equation, we certainly have to wait for very, very important progress in the field.

3.3. Logarithmic Sobolev inequalities and entropy dissipation. We illustrate the preceding discussion on the simple case of the spatially homogeneous Fokker-Planck equation,

$$\frac{\partial f}{\partial t} = \nabla_v \cdot \big(\nabla_v f + f v\big).$$

Recall that the entropy functional is the Kullback relative entropy of f with respect to the standard Gaussian M,

$$H(f|M) = \int_{\mathbb{R}^N} f \log \frac{f}{M}$$

(equivalently, the additive constant in the free energy has been chosen in such a way that the equilibrium state has zero energy). And the entropy dissipation functional is the relative Fisher information,

$$I(f|M) = \int_{\mathbb{R}^N} f \left| \nabla_v \log \frac{f}{M} \right|^2. \tag{174}$$

The archetype of (171) is the **Stam-Gross logarithmic Sobolev inequality** [**411, 261**]. In an information-theoretical language, this inequality can be written most simply as

$$I(f|M) \geq 2H(f|M). \tag{175}$$

Inequality (175) was first proven, in an equivalent formulation, in a classical paper by Stam[9] [**411**]. The links between the theory of logarithmic Sobolev inequalities and information theory have been pointed out for some time [**45, 120, 165, 16**].

Of course, inequality (175) immediately implies that the solution to the Fokker-Planck equation with initial datum f_0 satisfies

$$H(f(t)|M) \leq e^{-2t} H(f_0|M).$$

This is a complete[10] and satisfactory solution to the problem of trend to equilibrium for the Fokker-Planck equation.

Actually, the interplay between functional inequalities and diffusion equations goes in both directions [**330**]. As was noticed in a famous work by Bakry and Emery [**45**], some properties of trend to equilibrium for the Fokker-Planck equation can be used to *prove* inequalities such as (175). We shall discuss their approach in section 6, together with recent developments.

By the way, as a general rule, logarithmic Sobolev inequalities are stronger than spectral gap inequalities [**261**]. As a typical illustration : if one lets $f = M(1 + \varepsilon h)$ in (175), where $\int Mh = 0$, and then lets ε go to 0, one finds the inequality

$$\int Mh = 0 \Longrightarrow \int M|\nabla h|^2 \geq \int Mh^2, \tag{176}$$

which is the spectral gap inequality for the Fokker-Planck operator. Inequality (176) implies the following estimate for solutions of the Fokker-Planck equation :

$$f_0 \in L^2(M^{-1}) \Longrightarrow \|f(t,\cdot) - M\|_{L^2(M^{-1})} \leq e^{-t} \|f_0 - M\|_{L^2(M^{-1})}.$$

[9] Stam proved the inequality $\mathcal{N}(f) I(f) \geq N$, which is equivalent to (175) by simple changes of variables, in dimension 1. Here $\mathcal{N}$ is the entropy power functional of Shannon, formula (52). The proof of Stam was not completely rigorous, but has been fixed.

[10] The assumption that the initial datum possess finite entropy can even be relaxed by parabolic regularization. For instance, one can prove [**366**] that $H(f(t)|M) = O(1/t)$ as soon as $\int f_0(v)|v|^2\, dv$ is finite.

4. Entropy dissipation functionals of Boltzmann and Landau

In this section, we discuss entropy-entropy dissipation inequalities for functionals (156) and (160).

A common feature of both functionals is **monotonicity** : the Boltzmann entropy dissipation is a nondecreasing function of the collision kernel B, while the Landau entropy dissipation is a nondecreasing function of Ψ. This property makes it possible to only treat algebraically simplified cases where B (resp. Ψ) is "small". As a typical application, if we find a lower bound for D when the collision kernel is Maxwellian, then we shall have a lower bound for all collision kernels whose kinetic part is bounded below. This reduction is interesting because Maxwellian collision kernels do have many additional properties. We shall see some of these properties in a moment, and shall dig more deeply into them in chapter 4. All known lower bounds for the entropy dissipation functionals of Boltzmann or Landau have been obtained from a preliminary study of the Maxwellian case.

As a consequence, it will be natural to define an "over-Maxwellian" collision kernel as a collision kernel B which is bounded below by a Maxwellian collision kernel.

In the sequel, we shall assume without loss of generality that the first moments of the distribution function f are normalized :

$$\int_{\mathbb{R}^N} f(v) \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dv = \begin{pmatrix} 1 \\ 0 \\ N \end{pmatrix}, \tag{177}$$

and denote by M the associated Maxwellian.

4.1. Landau's entropy dissipation. We start with the case of the Landau equation, because its diffusive nature entails better properties of the entropy dissipation functional. Let us first state the main results, then we shall comment on them.

Theorem 14. *Let f be a probability distribution satisfying* (177).

(i) "Over-Maxwellian case" : Let $\Psi(|z|) \geq |z|^2$, and let D_L be the associated entropy dissipation functional, formula (160). *Then there exists a constant $\lambda(f) > 0$, explicit and depending on f only via an upper bound for $H(f)$, such that*

$$D_L(f) \geq \lambda(f)\, I(f|M) \geq 2\lambda(f)\, H(f|M). \tag{178}$$

More precisely, one can choose

$$\lambda(f) = (N-1) \inf_{e \in S^{N-1}} \int_{\mathbb{R}^N} f(v)(v \cdot e)^2\, dv \equiv T_f. \tag{179}$$

(ii) "Soft potentials" : Let $\Psi(|z|) \geq |z|^2(1+|z|)^{-\beta}$, $\beta > 0$. Then, for all $s > 0$, there exists a constant $C_s(f)$, explicit and depending on f only via an upper bound for $H(f)$, such that

$$D_L(f) \geq C_s(f) H(f|M)^{1+\frac{\beta}{s}} F_s^{-\frac{\beta}{s}}, \tag{180}$$

where $F_s = M_{s+2}(f) + J_{s+2}(f)$, and

$$M_{s+2}(f) = \int_{\mathbb{R}^N} f(v)(1+|v|^2)^{\frac{s+2}{2}}\, dv,$$

$$J_{s+2}(f) = \int_{\mathbb{R}^N} \left|\nabla\sqrt{f}\right|^2 (1+|v|^2)^{\frac{s+2}{2}}\, dv.$$

(iii) "Hard potentials" : Let $\Psi(|z|) \geq |z|^{\gamma+2}$, $\gamma > 0$. Then, there exists constants $K_1(f)$, $K_2(f)$, explicit and depending on f only via an upper bound for $H(f)$, such that

$$D_L(f) \geq K_1(f)\, \min\left[I(f|M), I(f|M)^{1+\frac{\gamma}{2}}\right] \tag{181}$$

$$\geq K_2(f) \min\left[H(f|M), H(f|M)^{1+\frac{\gamma}{2}}\right]. \tag{182}$$

Remarks :

1. Note that the constant λ given by (179) has the dimensions of a temperature, and can vanish only if f is concentrated on a line. This is the typical degeneracy of the Landau equation; in particular, the operator in (30) is always strictly elliptic unless f is concentrated on a line. But the finiteness of the entropy prevents such a concentration, and allows one to get a bound from below on $\lambda(f)$. Of course, other estimates are possible : for instance, by use of some L^p, or L^∞, or smoothness bound on f. Or, if f is radially symmetric, then automatically $\lambda(f) = 1$.

2. Also, as we shall see in the next section, it may sometimes be wiser to estimate from below $\lambda(f)$ in terms of the entropy dissipation of f !

3. Further note that the inequalities on the right in (178) and in (182) are nothing but the logarithmic Sobolev inequality (175).

In the preceding theorem, point (i) is the starting point for the remaining cases. It was established in Desvillettes and Villani [**183**] by two different methods. The first one relies on some explicit computations performed in Villani [**443**], which are recalled in formula (30). The second strategy is a variant of Desvillettes' techniques, inspired by a method due to Boltzmann himself [**93**]. It consists in "killing", with a well-chosen operator, the symmetries of the functional D_L which correspond to the equilibrium state[11]. To be just a little bit more precise, one writes

$$D(f) = \int dv\, dv_*\, f f_* |R(v, v_*)|^2,$$

where $R : \mathbb{R}^{2N} \to \mathbb{R}^N$, and one finds a linear operator $T = T(v, v_*) : \mathbb{R}^N \to \mathbb{R}^N$ such that TR is identically 0 if and only if $D = 0$; then

$$D_L(f) \geq \frac{1}{\|T\|^2} \int dv\, dv_*\, f f_* |TR(v, v_*)|^2.$$

A careful choice of the operator T enables a very simple computation of the right-hand side in this inequality.

Point (ii) is proven in [**429**]. The idea is that the vanishing of $\Psi(|v-v_*|)/|v-v_*|^2$ as $|v - v_*| \to \infty$ can be compensated by some good estimates of decay at infinity, in the form of the constant F_{s+2} (which involves both moments and smoothness).

As for point (iii), it is rather easy to get by "perturbation" from point (i), see Desvillettes and Villani [**183**]. The idea is that the contribution of small $|v - v_*|$ is negligible. One writes

$$|v - v_*|^{\gamma+2} \geq \varepsilon^\gamma |v - v_*|^2 - \varepsilon^{\gamma+2},$$

[11] See Boltzmann's argument in paragraph 4.3.

then one estimates *from below* the contribution of $\varepsilon^\gamma |v - v_*|^2$ to the entropy dissipation, and *from above* the contribution of the small constant function $\varepsilon^{\gamma+2}$. A few algebraic tricks [**183**] lead to the estimate (181) without further bounds on the concentration of f : the constant K in this estimate is essentially $\lambda(f)^{1+\gamma/2}$.

Theorem 14 gives explicit and satisfactory answers to the quest of entropy-entropy dissipation estimates for the Landau equation; in the next section we shall see that they can be used efficiently for the study of the trend to equilibrium, at least in the spatially homogeneous situation. However, we should avoid triumphalism : it is abnormal that the exponent in the case of hard potentials (which is $1 + \gamma/2$) be worse than the exponent in the case of soft potentials ($1 + \epsilon$, with ϵ as small as desired, if f has a very good decay and smoothness at infinity). One would expect that for hard potentials, the inequality

$$D_L(f) \geq K(f) H(f|M)$$

hold true.

4.2. Boltzmann's entropy dissipation : Cercignani's conjecture. Now we turn to the more complicated case of the functional (156). Some parts of the following discussion are copied from [**442**].

An old conjecture by Cercignani, formulated at the beginning of the eighties, was that the Boltzmann equation would satisfy a linear entropy-entropy dissipation inequality. We state this conjecture here in a slightly more precise form than the original. There are two forms of it, a weak and a strong.

Cercignani's conjecture : *Let $B \geq 1$ be a collision kernel, and* (156) *be the associated entropy dissipation functional. Let $f(v)$ be a probability distribution on $\mathbb{R}^N$, with unit temperature, and let M be the associated Maxwellian equilibrium. Then,*

(strong version) *there exists $\lambda > 0$, independent of f, such that*

$$D(f) \geq 2\lambda H(f|M). \tag{183}$$

(weak version) *there exists $\lambda(f) > 0$, depending on f only via some estimates of moments, Sobolev regularity, lower bound, such that*

$$D(f) \geq 2\lambda(f) H(f|M). \tag{184}$$

It soon appeared that the strong version of this conjecture had to be false. Indeed, it would have implied a universal exponential rate of convergence for solutions of the spatially homogeneous Boltzmann equation with a collision kernel $B \geq 1$. But, as we mentioned in paragraph 1.4, Bobylev [**79**, p. 224] was able to produce a family of initial data $(f_0^\varepsilon)_{\varepsilon>0}$ with unit mass and temperature, such that the associated solutions of the Cauchy problem (with Maxwellian collision kernel, say $B \equiv 1$) converge to equilibrium slowly, in the sense

$$\forall t \geq 0, \quad \|f^\varepsilon(t,\cdot) - M\| \geq K_\varepsilon e^{-\varepsilon t}, \qquad K_\varepsilon > 0.$$

These initial data are constructed more or less explicitly with the help of the Fourier transform apparatus, and hypergeometric functions.

Later, Wennberg [**461**] produced direct counterexamples to (183), covering the case of hard potentials as well.

Finally, Bobylev and Cercignani [**87**] disproved even the weak version of the conjecture. They exhibited a family of distribution functions for which (184) does not hold for a uniform λ, while these distribution functions do have uniformly bounded L^p or H^k norms (whatever p, k), uniformly bounded moments of order k (whatever k), and are bounded below by a fixed Maxwellian. These counterexamples are obtained by adding a very tiny (but very spread) bump, at very high velocities, to the equilibrium distribution. They again illustrate the principle that distribution tails are the most serious obstacle to a good trend to equilibrium for the Boltzmann equation.

Thus, **Cercignani's conjecture is false**[12]. It may however be that (184) hold true under more stringent assumptions :

- under very strong decay conditions, for instance $f \in L^2(M^{-1})$, as in the linearized theory[13];

- or under an assumption of nonintegrable angular singularity, which may help. This conjecture would be supported by the good behavior of the Landau entropy dissipation.

4.3. Desvillettes' lower bound. The first interesting lower bound for Boltzmann's entropy dissipation functional was obtained by Desvillettes [**166**]. His idea was to go back to Boltzmann's original argument for the identification of cases where the entropy dissipation vanishes. As many proofs of similar results, Boltzmann's proof relies on some well-chosen linear operators which "kill symmetries". Let us sketch this argument (slightly modified) in a nutshell, since it may enlighten a little bit the discussion of the most recent results in the field.

Boltzmann's theorem : *Let $N \geq 2$, and let $f(v)$ be a smooth positive solution of the functional equation*

$$\forall (v, v_*, \sigma) \in \mathbb{R}^N \times \mathbb{R}^N \times S^{N-1}, \qquad f' f'_* = f f_*. \tag{185}$$

Then f is a Maxwellian distribution; in other words there exist constants $\lambda \in \mathbb{R}$, $\mu \in \mathbb{R}^N$ such that

$$\forall v \in \mathbb{R}^N, \qquad \nabla \log f(v) = \lambda v + \mu. \tag{186}$$

Boltzmann's argument. Average (185) over the parameter $\sigma \in S^{N-1}$, to find

$$f f_* = \frac{1}{|S^{N-1}|} \int_{S^{N-1}} (f' f'_*)\, d\sigma. \tag{187}$$

It is easy to convince oneself that the function

$$\int_{S^{N-1}} f' f'_*\, d\sigma \equiv G(v, v_*)$$

[12] To my own surprise, after completion of this review, I discovered that Cercignani's conjecture does hold true when $B(v - v_*, \sigma) \geq 1 + |v - v_*|^2$. This is not in contradiction with the Bobylev-Cercignani counterexamples, because they assume $\int B\, d\sigma \leq C(1 + |v - v_*|^\gamma)$, $\gamma < 2$!

[13] A very recent, deep result by Ball and Barthe about the central limit theorem suggests that there is some hope if f satisfies a Poincaré inequality. This may be the first step towards identifying some "reasonable" conditions for Cercignani's conjecture to be true.

depends only on the sphere $S(v, v_*)$ with diameter $[v, v_*]$. Actually, up to a Jacobian factor $(|v - v_*|/2)^{N-1}$, G is just the mean value of the function $f(w)f(\tilde{w})$ on this sphere, where $\tilde{w}$ stands for the velocity on S which is diametrically symmetric to w. The spheres $S(v, v_*)$ are in turn parametrized by only $N + 1$ parameters, say $(v + v_*)/2$ and $|v - v_*|$; or, equivalently, by the physical variables

$$\text{(188)} \qquad \begin{cases} m = v + v_* \qquad \text{[total momentum of colliding particles]}; \\ \\ e = \dfrac{|v|^2}{2} + \dfrac{|v_*|^2}{2} \qquad \text{[total kinetic energy of colliding particles]}. \end{cases}$$

Thus we shall abuse notations by writing $G(v, v_*) = G(m, e)$.

Now, introduce the linear differential operator $T = (v - v_*) \wedge (\nabla - \nabla_*)$ (or, which amounts to the same, $\Pi(v - v_*)(\nabla - \nabla_*)$, where $\Pi(v - v_*)$ is the orthogonal projection on $(v - v_*)^\perp$). Its kernel consists precisely of those functions that depend only on m and e. If we apply this operator to the equation

$$\log f f_* = \log G(m, e),$$

we find

$$(v - v_*) \wedge \big[\nabla \log f - (\nabla \log f)_*\big] \equiv 0.$$

In words, for all v, v_* there exists a real number λ_{v,v_*} such that

$$\text{(189)} \qquad \nabla \log f(v) - \nabla \log f(v_*) = \lambda_{v,v_*}(v - v_*).$$

This functional equation, set in $\mathbb{R}^N$, $N \geq 2$, implies the conclusion at once. □

Remark : The very last part of the proof, starting from (189), is exactly what one needs to identify cases of equality for Landau's entropy dissipation functional. This can make us suspect a deep connection between the entropy dissipations of Boltzmann and Landau. We shall soon see that there is indeed a hidden connection.

With the help of the open mapping theorem, Desvillettes was able to produce a "quantitative" version of Boltzmann's argument, leading to the

Theorem 15. *Let $B \geq 1$, and let D be the associated entropy dissipation functional* (47)*. Let f be a nonnegative density on $\mathbb{R}^N$, with finite mass and energy. Without loss of generality, assume that the first moments of f are normalized by* (177)*. Then, for all $R > 0$ there is a constant $K_R > 0$, depending only on R, such that*

$$D(f) \geq K_R \inf_{m \in \mathcal{M}} \int_{|v| \leq R} \big|\log f - \log m\big| \, dv,$$

where $\mathcal{M}$ is the space of all Maxwellian distributions.

Note that the quantity on the right is always positive for some $R > 0$ if f is not Maxwellian. Several variants were obtained, with better estimates and simpler proofs, and recently Desvillettes [**175**] found a way to avoid the use of the open mapping theorem, and get explicit constants. Also Wennberg [**455**] extended the result to hard potentials.

Although Desvillettes' result is rather weak, it was important as the very first of its kind. Subsequent developments were partly motivated by the search for stronger estimates.

4.4. The Carlen-Carvalho theorem. At the beginning of the nineties, Carlen and Carvalho [**121, 122**] made a crucial contribution to the subject by using the tools of information theory and logarithmic Sobolev inequalities. They proved that there always exists an entropy-entropy dissipation inequality for Boltzmann's collision operator as soon as one has some (very weak) control on the decay at infinity and smoothness of the distribution function. In their general result, decay at infinity of a distribution function f is measured by the decay of

$$\chi_f : R \longmapsto \int_{|v|\geq R} f(v)|v|^2\, dv$$

as $R \uparrow \infty$, while the smoothness is measured by the decay of

$$\psi_f : \lambda \longmapsto H(f) - H(S_\lambda f)$$

as $\lambda \downarrow 0$. Here $(S_t)_{t\geq 0}$ is as usual the semigroup generated by the Fokker-Planck operator; sometimes it is called the adjoint Ornstein-Uhlenbeck semigroup.

Carlen and Carvalho's general theorem [**121**] can be stated as follows :

THEOREM 16. *Let $B(v - v_*, \sigma) \geq 1$ be a collision kernel. Let χ_0, ψ_0 be two continuous functions, decreasing to 0 as $R \uparrow +\infty$ and $\lambda \downarrow 0$ respectively. Let then f be a probability distribution function with unit mass and temperature, and let M be the associated Maxwellian distribution. Assume that*

$$\chi_f \leq \chi_0, \qquad \psi_f \leq \psi_0. \tag{190}$$

Then, there exists a continuous function $\Theta = \Theta_{\chi,\psi}$, strictly increasing from 0, depending on f only via χ_0 and ψ_0, such that

$$D(f) \geq \Theta(H(f|M)).$$

Remarks :

1. This result crucially uses the special properties of Maxwellian collision kernels, explained in chapter 4.

2. The result in [**121**] is stated for a collision kernel which is bounded below in ω-representation[14]. Recent works have shown that this assumption can be relaxed (see the references in chapter 4).

The main ideas behind the proof of Theorem 16 are 1) the reduction to Maxwellian collision kernel by monotonicity, 2) the inequality[15]

$$D(f) \geq H(f) - H(Q^+(f,f)) \geq 0,$$

which holds true for a Maxwellian collision kernel $b(\cos\theta)$ such that $\int b(\cos\theta)\, \sin^{N-2}\theta\, d\theta = 1$, and 3) show that when f satisfies (190) and $H(f) - H(M) \geq \varepsilon$, then f lies in a compact set of probability measures on which $H - H(Q^+)$ attains its minimum value.

One of the key ingredients is a study of the Fisher information functional $I(f) = \int |\nabla f|^2/f$, and the representation formula

$$H(Q^+(f,f)) - H(M) = \int_0^{+\infty} \big[I(Q^+(S_\lambda f, S_\lambda f)) - I(S_\lambda f)\big]\, d\lambda. \tag{191}$$

[14] See paragraph 4.6 in chapter 1.

[15] See paragraph 3.2 in chapter 4.

This formula and related estimates are explained in chapter 4. A crucial point is to bound below the integrand in (191), for λ positive enough, by the method of Carlen [**120**].

We note that there is no assumption of lower bound on f in the Carlen-Carvalho theorem, though they actually use lower bounds in their estimates. There is no contradiction, because Maxwellian lower bounds are automatically produced by the semigroup (S_λ). However, these lower bounds are rather bad, and so are the resulting estimates. Better bounds can be obtained if the probability density f is bounded below by some Maxwellian distribution.

In a companion paper [**122**], Carlen and Carvalho showed how to extend their method to physically realistic cases like the hard-spheres kernel, $B(v - v_*, \sigma) = |v - v_*|$, and gave a recipe for computing the function Θ.

These results were the first entropy dissipation estimates which would find interesting and explicit applications to the Boltzmann equation, see section 5. More importantly, they set new standards of quality, and introduced new tools in the field. However, the Carlen-Carvalho entropy-entropy dissipation inequalities are not very satisfactory because the function Θ is quite intricate, and usually very flat near the origin.

4.5. Cercignani's conjecture is almost true. As we mentioned earlier, the "linear" entropy-entropy dissipation inequality conjectured by Cercignani ($\Theta(H) = \text{const.}H$) is in general false. Nevertheless, it was proven a few years ago by Toscani and Villani [**428**] that one can choose $\Theta(H) = \text{const.}H^{1+\varepsilon}$, with ε as small as desired. Here is a precise statement from [**428**]. We use the usual notation

$$\|f\|_{L^1_s} = \int_{\mathbb{R}^N} f(v)(1 + |v|^2)^{s/2}\, dv$$

and its natural extension

$$\|f\|_{L^1_s \log L} = \int_{\mathbb{R}^N} f(v) \log(1 + f(v))(1 + |v|^2)^{s/2}\, dv.$$

Theorem 17. [16] *(i) "Over-Maxwellian case" : Let $B \geq 1$ be a collision kernel, and D be the associated entropy dissipation functional, eq.* (156)*. Let f be a probability density on $\mathbb{R}^N$ with unit temperature, and let M be the associated Maxwellian equilibrium. Let $\varepsilon > 0$ be arbitrary, and assume that for some $\delta > 0$, $A, K > 0$,*

$$\|f\|_{L^1_{4+2/\varepsilon+\delta}},\ \|f\|_{L^1_{2+2/\varepsilon+\delta} \log L} < +\infty,$$

$$f(v) \geq K e^{-A|v|^2}. \tag{192}$$

Then, there exists a positive constant $C_\varepsilon(f)$, depending only on N, ε, δ, $\|f\|_{L^1_{4+2/\varepsilon+\delta}}$, $\|f\|_{L^1_{2+2/\varepsilon+\delta} \log L}$, A and K, such that

$$D(f) \geq C_\varepsilon(f) H(f|M)^{1+\varepsilon}. \tag{193}$$

As an example (choosing $\delta = 1$), the following more explicit constant works :

$$D(f) \geq K T_f F_\varepsilon^{-\varepsilon} H(f|M)^{1+\varepsilon}, \tag{194}$$

[16] All the results in this theorem have been improved in recent work by the author.

where K is an absolute constant (not depending on f), T_f is the "temperature" given by (179)*, and*

$$F_\varepsilon = \left(\log\frac{1}{K} + A\right) \|f\|^2_{L^1_{5+2/\varepsilon}} \|f\|_{L^1_{3+2/\varepsilon}} \log L.$$

(ii) "Soft potentials" : Assume now that

$$B(v - v_*, \sigma) \geq (1 + |v - v_*|)^{-\beta}, \qquad \beta > 0.$$

Then, for all $\varepsilon > 0$, eq. (194) *still holds with*

$$F_\varepsilon = \left(\log\frac{1}{K} + A\right) \|f\|^2_{L^1_{5+(2+\beta)/\varepsilon}} \|f\|_{L^1_{3+(2+\beta)/\varepsilon}} \log L.$$

(iii) "Hard potentials" : Assume now that

$$B(v - v_*, \sigma) \geq |v - v_*|^\gamma, \qquad \gamma > 0.$$

Assume moreover that $f \in L^p_\kappa$, for some $p > 1$, and κ large enough. Then, there exists $\alpha > 1$, $C > 0$, depending on N, γ, p, κ, $\|f\|_{L^p_\kappa}$, and on A, K in (192)*, such that*

$$D(f) \geq CH(f|M)^\alpha.$$

Thus Cercignani's conjecture is "almost" true, in the sense that any power of the relative entropy, arbitrarily close to 1, works for point (i), provided that f decays fast enough and satisfies a Gaussian lower bound estimate.

This theorem is remindful of some results in probability theory, about modified logarithmic Sobolev inequalities for jump processes, see Miclo [**345**]. Even if the situation considered in this reference is quite different, and if the methods of proof have nothing in common, the results present a good analogy. From the physical point of view, this is not surprising, because the Boltzmann equation really models a (nonlinear) jump process.

Let us briefly comment on the assumptions and conclusions.

1. The main improvement lies in the form of the entropy-entropy dissipation inequality, which is both much simpler and much stronger.

2. The lower bound assumption can be relaxed into $f(v) \geq Ke^{-A|v|^p}$ for some $p > 2$, provided that more moments are included in the estimate.

3. Strictly speaking, this theorem is not stronger than the Carlen-Carvalho theorem, because the assumptions of decay at infinity are more stringent. On the other hand, it does not require any smoothness condition.

As regards the proof, it is completely different from that of the Carlen-Carvalho theorem, and relies strongly on Theorem 14, point (i). Since this is quite unexpected, we shall give a brief explanation in the next paragraph.

Once again, the result for hard potentials is not so good as it should be, because the power in point (iii) cannot be chosen arbitrarily close to 1. We have hope to fix this problem by improving the error estimates for small relative velocities which were sketched in [**428**][17].

[17] As this review goes to print, we just managed to prove the desired result, at the expense of very strong smoothness estimates (in all Sobolev spaces).

4.6. A sloppy sketch of proof. In this survey, we have chosen to skip all proofs, or even sketches of proof. We make an exception for Theorem 17 because of its slightly unconventional character, and also because of its links with Boltzmann's original argument[18] about cases of equality in the entropy dissipation — with ideas of information theory coming into play. Of course, we shall only try to give a flavor of the proof, and not go into technical subtleties, which by the way are extremely cumbersome. Also we only consider point (i), and set $B = 1$, or rather $B = |S^{N-1}|^{-1}$, so that $\int B\, d\sigma = \fint d\sigma = 1$. Thus the functional to estimate from below is

$$D(f) = \frac{1}{4} \int_{\mathbb{R}^{2N}} dv\, dv_* \fint_{S^{N-1}} d\sigma\, (f' f'_* - f f_*) \log \frac{f' f'_*}{f f_*}.$$

The three main ingredients in Theorem 17 are

- a precise study of symmetries for the Boltzmann collision operator, and in particular the fact that the entropy dissipation can be written as a functional of the tensor product $f \otimes f$;
- a regularization argument à la Stam;
- our preliminary estimate for the Landau entropy dissipation, Theorem 14.

Stam's argument : At the end of the fifties, Stam [**411**] had the clever idea to prove the so-called Shannon-Stam inequality, conjectured by Shannon :

$$H\big(\sqrt{\alpha} X + \sqrt{1-\alpha} Y\big) \leq \alpha H(X) + (1-\alpha) H(Y), \tag{195}$$

actually equivalent to (52), as a consequence of the Blachman-Stam inequality, which he introduced on that occasion :

$$I\big(\sqrt{\alpha} X + \sqrt{1-\alpha} Y\big) \leq \alpha I(X) + (1-\alpha) I(Y). \tag{196}$$

In inequalities (195) and (196), X and Y are arbitrary independent random variables on $\mathbb{R}^N$, and one writes $H(X) = H(f) = \int f \log f$, $I(X) = I(f) = \int |\nabla f|^2 / f$ whenever f is the law of X.

Stam found out that (196) is essentially an infinitesimal version of (195) under heat regularization. Think that I is nothing but the entropy dissipation associated to the heat equation... A modern presentation of Stam's argument is found in Carlen and Soffer [**125**] : these authors replace H and I by their relative counterparts with respect to the standard Gaussian M, and obtain (195) by integrating (196) along the adjoint Ornstein-Uhlenbeck semigroup $(S_t)_{t \geq 0}$. More explicitly, since $I(f|M)$ is the derivative of $H(f|M)$ along regularization by S_t, and since also $S_t f \to M$ as $t \to \infty$, one can write

$$H(f|M) = \int_0^{+\infty} I(S_t f | M)\, dt.$$

The strategy in [**428**] is inspired from this point of view : we would like to start from

$$D(f) = \int_0^{+\infty} \left[-\frac{d}{dt} D(S_t f) \right] dt.$$

This identity is formally justified because $S_t f \longrightarrow M$ as $t \to \infty$, and $D(M) = 0$. Then one can hope that for some reason, the derivative $-dD/dt$ will be easier to handle that the entropy dissipation functional D. This is the case in the proof of the

[18] See paragraph 4.3.

Shannon-Stam inequality, and also here in the framework of the Boltzmann entropy dissipation.

It actually turns out, rather surprisingly, that

$$D(f) \gtrsim \frac{K}{R^2} \int_0^{+\infty} D_L(S_t f)\, dt, \tag{197}$$

with $K > 0$, and R a typical size for the velocity. In other words, the entropy dissipation for the Landau equation is a kind of differential version of the entropy dissipation for the Boltzmann equation ! Admit for a while (197), and combine it with the result of Theorem 14, in the form

$$\begin{aligned} D_L(S_t f) &\geq (N-1) T_{S_t f}\, I(S_t f | M) \\ &\geq (N-1) T_f\, I(S_t f | M). \end{aligned}$$

It follows that

$$D(f) \geq C(f) \int_0^{+\infty} I(S_t f | M)\, dt = C(f)\, H(f|M);$$

which is the statement in Cercignani's conjecture.

Of course, we know that Cercignani's conjecture is false, which means that (197) cannot rigorously hold true. A precise variant is established in [**428**]. The technical problem which prevents (197) is the presence of large velocities, as one could expect. Controlling the contribution of large velocities to the entropy dissipation is the most technical point in the proof presented in [**428**]. It means for instance establishing quantitative bounds on the tails of the entropy dissipation, like

$$\int_0^{+\infty} dt \int_{|X| \geq R} |X|^2 (S_t F - S_t G) \log \frac{S_t F}{S_t G}\, dX \leq \frac{C_s}{R^s}$$

for arbitrary probability densities $F(X)$ and $G(X)$ in $\mathbb{R}^{2N}$, where C_s is a constant depending on s and on suitable estimates on F and G (moments, lower bound...).

In the next two pages, we shall skip all these technicalities and present a sketch of proof of (197) under the absurd assumption that *all velocities are bounded*, just to give the reader an idea of the kernel of the proof.

Sloppy sketch of proof for (197). First we introduce the adjoint Ornstein-Uhlenbeck semigroup (S_t), and we try to compute $(-d/dt) D(S_t f)$. At first sight this seems an impossible task to perform in practice, due to the number of occurrences of f in the entropy dissipation functional, and the complicated arguments v', v'_*. But a first observation will help : $D(f)$ is actually a functional of the *tensor product* $f \otimes f = f f_*$. And it is easily checked that the following diagram is commutative, with $\mathcal{T}$ standing for tensorization,

$$\begin{array}{ccc} f & \xrightarrow{\mathcal{T}} & F = f f_* \\ \Big\downarrow S_t & & \Big\downarrow S_t \\ S_t f & \xrightarrow{\mathcal{T}} & S_t F. \end{array} \tag{198}$$

(Here we use the same symbol for the semigroups S_t in $L^1(\mathbb{R}^N)$ and in $L^1(\mathbb{R}^{2N})$.)

This enables to replace in computations $(S_t f)(S_t f)_*$ by $S_t(f f_*)$. One could hope that, similarly,

$$\begin{array}{ccc} F = f f_* & \longrightarrow & f' f'_* \\ \Big\downarrow S_t & & \Big\downarrow S_t \\ S_t F & \longrightarrow & S_t(f' f'_*) \end{array} \tag{199}$$

is commutative. This is false ! The point is that the angular variable σ is not intrinsic to the problem. To remove this flaw, we integrate with respect to the parameter σ. Since

$$(x, y) \mapsto (x - y) \log \frac{x}{y}$$

is a jointly convex function of its arguments, by Jensen's inequality

$$D(f) \geq \overline{D}(f) \equiv \frac{1}{4} \int_{\mathbb{R}^{2N}} dv\, dv_* \left(f f_* - \fint d\sigma\, f' f'_* \right) \log \frac{f f_*}{\fint d\sigma\, f' f'_*}. \tag{200}$$

Now it is true, even if not immediate at all[19], that

$$\begin{array}{ccccc} f & \xrightarrow{\mathcal{T}} & F = f f_* & \xrightarrow{\mathcal{A}} & G = \int d\sigma f' f'_* \\ \Big\downarrow S_t & & \Big\downarrow S_t & & \Big\downarrow S_t \\ S_t f & \xrightarrow{\mathcal{T}} & S_t F & \xrightarrow{\mathcal{A}} & S_t G \end{array} \tag{201}$$

with $\mathcal{A}$ standing for the averaging operation over the sphere, is an entirely commutative diagram. This actually is a consequence of the fact that (S_t) is a *gaussian* regularization semigroup.

This suggests to work with $\overline{D}$ instead of D, and to write $\overline{D}(S_t f)$ in the form $\overline{D}(S_t F, S_t G)$, with the abuse of notations

$$\overline{D}(F, G) = \frac{1}{4} \int_{\mathbb{R}^{2N}} (F - G) \log \frac{F}{G}\, dX \qquad X = (v, v_*) \in \mathbb{R}^{2N}.$$

After these preliminaries, it is not hard to compute

$$-\frac{d}{dt} D(S_t F, S_t G) = \frac{1}{4} \int_{\mathbb{R}^{2N}} (S_t F + S_t G) \left| \frac{\nabla(S_t F)}{S_t F} - \frac{\nabla(S_t G)}{S_t G} \right|^2 dX. \tag{202}$$

Here, of course,

$$\nabla = \left[\nabla_v, \nabla_{v_*} \right]$$

is the gradient in $\mathbb{R}^N_v \times \mathbb{R}^N_{v_*}$.

Under suitable assumptions one can also prove that $t \mapsto \overline{D}(S_t f)$ is a continuous function as $t \to 0$, and goes to 0 as $t \to +\infty$. Then

$$\begin{aligned} \overline{D}(f) &= \frac{1}{4} \int_0^{+\infty} dt \int_{\mathbb{R}^{2N}} (S_t F + S_t G) \left| \frac{\nabla(S_t F)}{S_t F} - \frac{\nabla(S_t G)}{S_t G} \right|^2 dX \\ &\geq \frac{1}{4} \int_0^{+\infty} dt \int_{\mathbb{R}^{2N}} S_t F \left| \frac{\nabla(S_t F)}{S_t F} - \frac{\nabla(S_t G)}{S_t G} \right|^2 dX. \end{aligned} \tag{203}$$

Since $S_t G$ is a very complicated object, we would like to get rid of it. Recall from Boltzmann's original argument that $S_t G$, being an average on spheres with

[19] A weaker property, sufficient for the argument, is that S_t preserves the class of functions which only depend on $v + v_*$ and $|v|^2 + |v_*|^2$.

diameter $[v, v_*]$, does not depend on all of the variables v, v_*, but only upon the *reduced variables* $m = v + v_*$, $e = |v|^2/2 + |v_*|^2/2$. Accordingly, we shall abuse notations and write $S_t G(v, v_*) = S_t G(m, e)$.

Now comes the key point : there is a **conflict of symmetries** between $S_t G$, which only depends on a low-dimensional set of variables, and $S_t F$, which is a tensor product. In Boltzmann's argument, the Maxwellian distribution pops out because it is the only probability distribution which is compatible with both symmetries.

Here these different structures of $S_t F$ and $S_t G$ reflect at the level of their respective gradients :

$$\frac{\nabla(S_t F)}{S_t F} = \left[\frac{\nabla(S_t f)}{S_t f}, \frac{(\nabla S_t f)_*}{(S_t f)_*} \right]; \tag{204}$$

$$\frac{\nabla(S_t G)}{S_t G} = \frac{1}{S_t G} \left[\nabla_m S_t G + v \frac{\partial S_t G}{\partial e}, \nabla_m S_t G + v_* \frac{\partial S_t G}{\partial e} \right]. \tag{205}$$

In particular, $\nabla(S_t G)$ always lies (pointwise) in the kernel of the linear operator

$$P : [A, B] \in \mathbb{R}^{2N} \longmapsto \Pi(v - v_*)[A - B] \in \mathbb{R}^N,$$

where $\Pi(z)$ is the orthogonal projection upon $z^\perp$. Of course $\|P\| = \sqrt{2}$ as a linear operator[20], and so

$$\left| \frac{\nabla(S_t F)}{S_t F} - \frac{\nabla(S_t G)}{S_t G} \right|^2 \geq \frac{1}{\|P\|^2} \left| P \left(\frac{\nabla S_t F}{S_t F} \right) \right|^2 = \frac{1}{2} \left| \Pi(v - v_*) \left[\frac{\nabla(S_t f)}{S_t f} - \frac{(\nabla S_t f)_*}{(S_t f)_*} \right] \right|^2. \tag{206}$$

By combining (200), (203) and (206),

$$D(f) \geq \frac{1}{8} \int_0^{+\infty} dt \int_{\mathbb{R}^{2N}} (S_t f)(S_t f)_* \left| \Pi(v - v_*) \left[\frac{\nabla S_t f}{S_t f} - \frac{(\nabla S_t f)_*}{(S_t f)_*} \right] \right|^2 dv\, dv_*.$$

The reader may have recognized a familiar object in the integrand of the right-hand side. Actually, apart from a factor $|v - v_*|^2$, it is precisely the integrand in the *Landau entropy dissipation*, computed for $S_t f$! If we now use our absurd assumption of boundedness of all relative velocities, in the form $|v - v_*| \leq R$, we get

$$D(f) \geq \frac{1}{8R^2} \int_0^{+\infty} dt \int_{\mathbb{R}^{2N}} (S_t f)(S_t f)_* |v - v_*|^2 \left| \Pi(v - v_*) \left[\frac{\nabla S_t f}{S_t f} - \frac{(\nabla S_t f)_*}{(S_t f)_*} \right] \right|^2 dv\, dv_*$$

$$= \frac{1}{8R^2} \int_0^{+\infty} dt\, D_L(S_t f).$$

□

[20] In contrast with the linear operator appearing in Boltzmann's proof, which was unbounded.

4.7. Remarks. We shall point out a few remarks about the preceding argument. First of all, in the course of the rigorous implementation, it is quite technical to take into account error terms due to large velocities. One has to study the time-evolution of expressions like $\int \varphi(X)(S_tF - S_tG)\log(S_tF/S_tG)$. But the calculations are considerably simplified by a striking "algebraic" property : a *local* (not integrated) version of (202) holds true. Let

$$h(F,G) = (F-G)\log\frac{F}{G}, \quad j(F,G) = (F+G)\left|\frac{\nabla F}{F} - \frac{\nabla G}{G}\right|^2.$$

Then, one can check that

$$\left.\frac{d}{dt}\right|_{t=0} [S_t, h] = j,$$

in the sense that for all (smooth) probability distributions F and G,

$$\left.\frac{d}{dt}\right|_{t=0} \Big(S_t h(F,G) - h\big(S_tF, S_tG\big)\Big) = j(F,G).$$

This property is somewhat reminiscent of the Γ calculus used for instance in Bakry and Emery [**45**] and Ledoux [**294**]. It yields another bridge between entropy dissipation inequalities and the theory of logarithmic Sobolev inequalities.

Our second remark concerns the use of the Fokker-Planck semigroup regularization. As we have seen, the main point above was to estimate from below the negative of the time-derivative of $D(f)$ along the semigroup $(S_t)_{t\geq 0}$. As was already understood by Carlen and Carvalho, and even a long time ago by McKean [**341**] in the framework of the Kac model, this estimate has to do with the behavior of the *Fisher information* $I(f) = \int |\nabla f|^2/f$ along the Boltzmann semigroup. Note that $I(f)$ is the dissipation of the H-functional along the semigroup $(S_t)_{t\geq 0}$. As we shall explain in chapter 4, the semigroup (B_t), generated by the spatially homogeneous Boltzmann equation with Maxwell collision kernel, commutes with (S_t), and it follows that

$$-\left.\frac{d}{dt}\right|_{t=0} D(S_tf) = -\left.\frac{d}{dt}\right|_{t=0} I(B_tf). \tag{207}$$

We shall see in chapter 4 that the right-hand side of (207) is always nonnegative; this could be considered as an a priori indication that the functional D behaves well under Fokker-Planck regularization.

Actually, in the simpler case of the Kac model[21], McKean [**341**, section 7, lemma d)] used relation (207) the other way round ! He proved directly, with a very simple argument based on Jensen's inequality, that the left-hand side of (207) is nonnegative for the Kac model. His argument can be transposed to the Boltzmann equation with Maxwell collision kernel in dimension 2, and also to the case where the collision kernel is constant in ω-representation : see [**428**, section 8].

As a third remark, we insist that the above argument, besides being rather intricate, is certainly not a final answer to the problem. The use of the average over σ seems crucial to its implementation, while for some applications it would be desirable to have a method which works directly for arbitrary Maxwellian collision kernels $b(\cos\theta)$. There is no clue of how to modify the argument in order to tackle

[21] eq. (21).

the problem of Cercignani conjecture (with exponent 1) for very strongly decaying distribution functions. It also does not manage to recover spectral gap inequalities for Maxwellian collision kernels, which are known to be true. Applied to simpler models than Boltzmann's equation, it yield results which are somewhat worse than what one can prove by other, elementary means ! However, in terms of lower bounds for Boltzmann's entropy dissipation, at the moment this is by far the best that we have.

Our final remark concerns the problem of solving (53). As mentioned in paragraph (2.5), many authors have worked to prove, under increasing generality, that these solutions are Maxwellian distributions. The problem with Boltzmann's proof was that it needed C^1 smoothness. However, as suggested by Desvillettes, the use of the gaussian semigroup (S_t) (or just the simple heat regularization) allows one to save Boltzmann's argument : let f be a L^1 solution of (53) with finite energy; without loss of generality f has unit mass, zero mean and unit temperature. Average (53) over σ to get

$$ff_* = G(m, e)$$

as in formula (187). Then apply the semigroup (S_t) to find

$$(S_t f)(S_t f)_* = S_t G(m, e).$$

Since $S_t f$ is C^∞ for $t > 0$, Boltzmann's proof applies and $S_t f$ is a Gaussian, which has to be M by identification of first moments. Since this holds true for any $t > 0$, by weak continuity $f = M$.

5. Trend to equilibrium, spatially homogeneous Boltzmann and Landau

As we already explained, in principle the trend to equilibrium is an immediate consequence of an entropy-entropy dissipation inequality and of suitable a priori estimates. However, there are some interesting remarks to make about the implementation.

5.1. The Landau equation. By theorem 14, one obtains at once convergence to equilibrium for the spatially homogeneous Landau equation

- with explicit exponential rate if $\Psi(|z|) \geq K|z|^2$;
- with explicit polynomial rate if $\Psi(|z|) \geq K|z|^{\gamma+2}$, $\gamma > 0$.

These results hold in the sense of relative entropy, but also in any Sobolev space, thanks to the regularization results which we discussed in chapter 2 and standard interpolation inequalities.

An interesting feature is that the rate of convergence given by the entropy-entropy dissipation inequality is likely to improve as time becomes large, by a "feedback" effect. Indeed, when f approaches equilibrium, then the constant T_f in (179) will approach the equilibrium value $T_M = 1$. In the case $\Psi(|z|) = |z|^2$, this enables one to recover an asymptotically *optimal* rate of convergence [**183**].

The case of soft potentials ($\gamma < 0$) is more problematic, because the moment estimates are not uniform in time — and neither are the smoothness estimates which enter the constant F_{s+2} in Theorem 14.

The fact that we do not have any uniform moment estimate for some moment of order $s > 2$ may seem very serious. It is not clear that condition (170) should be satisfied. Compactness-based methods spectacularly fail in such a situation.

However, and this is one of the greatest strengths of the entropy method, it is not necessary that the constant F_{s+2} be uniformly bounded. Instead, it is sufficient to have some estimate showing that it does not grow too fast, say in $O(t^\alpha)$ for α small enough. With this idea in mind, Toscani and Villani [**429**] prove the following theorem :

THEOREM 18. *Let $\Psi(|z|) = |z|^2\Phi(|z|)$, where $\Phi(|z|)$ is smooth, positive and decays like $|z|^{-\beta}$ at infinity, $0 < \beta < 3$. Let f_0 be an initial datum with unit mass and temperature, and let M be the associated Maxwellian distribution. Assume that f_0 is rapidly decreasing, in the sense that*

$$\forall s > 0, \qquad \|f_0\|_{L^2_s} < +\infty.$$

Then, for all $\varepsilon > 0$ there exists $s_0 > 0$ and a constant $C_\varepsilon(f_0)$, depending only on ε, N, Φ and $\|f_0\|_{L^2_{s_0}}$, such that the unique smooth solution of the spatially homogeneous Landau equation with initial datum f_0 satisfies

$$H\big(f(t,\cdot)|M\big) \leq C_\varepsilon(f_0)\, t^{-1/\varepsilon}.$$

We note that this theorem does not cover the interesting case $\beta = 3$ (Coulomb potential in dimension 3) : the proof in [**429**] just fails for this limit exponent. Including this case would be a significant improvement. We also note that this theorem deals with a smooth Ψ, while realistic Ψ's would present a singularity at the origin. This singularity cannot harm the entropy-entropy dissipation inequality, but may entail serious additional difficulties in getting the right a priori estimates[22].

5.2. A remark on the multiple roles of the entropy dissipation. Numerical applications for the constant T_f appearing in (179) are very disappointing (say, 10^{-20}...) This is because the entropy is quite bad at preventing concentration. Much better estimates are obtained via L^∞ bounds for instance (which can be derived from regularization).

Another possibility is to use the *entropy dissipation* as a control of concentration for f. The idea is the following : if the entropy dissipation is low (which is the bad situation for trend to equilibrium), then the distribution function cannot be concentrated too much close to a hyperplane, because the entropy dissipation measures some smoothness. As a consequence, T_f cannot be too small. More explicitly, say if $\Psi(|z|) \geq |z|^2$, then [**183**, section 5]

$$T_f \geq \frac{(N-1)^2}{N + \frac{D_L(f)}{N}}.$$

By re-injecting this inequality in the proof of Theorem 14, one finds the following improvement (still under the assumption $\Psi(|z|) \geq |z|^2$)

$$D_L(f) \geq \sqrt{2N(N-1)^2 H(f|M) + \frac{N^4}{4}} - \frac{N^2}{2}.$$

This in turns implies exponential convergence to equilibrium with realistic bounds, which we give explicitly as an illustration.

THEOREM 19. *Let $\Psi(|z|) \geq |z|^2$, and let f_0 be a probability distribution on $\mathbb{R}^N$, with zero mean velocity and unit temperature. Let M be the associated Maxwellian*

[22] See the discussion in paragraph 1.3 of chapter 5.

distribution. Let $f(t,\cdot)$ be a classical solution of the Landau equation with initial datum f_0. Then, for all time $t \geq 0$,

(208)

$$\|f(t,\cdot) - M\|_{L^1} \leq \left(\frac{\sqrt{NC_0}}{N-1} e^{\frac{1}{N^2} C_0} \right) e^{-\frac{(N-1)^2 t}{N}} + \left(\frac{C_0}{\sqrt{N}(N-1)} e^{\frac{2}{N^2} C_0} \right) e^{-\frac{2(N-1)^2 t}{N}},$$

where

$$C_0 = \sqrt{2N(N-1)^2 H(f_0|M) + \frac{N^4}{4}} - \frac{N^2}{2}.$$

This estimate shows that satisfactory bounds can sometimes be obtained by cleverly combining all elements at our disposal !

5.3. The Boltzmann equation. Once again, one has to separate between over-Maxwellian collision kernel, hard potentials or soft potentials. In order to apply Theorem 17, we need

- Moment estimates. they hold true for hard potentials without any assumption on the initial datum, and for Maxwellian collision kernels if a sufficient number of moments are finite at the initial time; we have discussed all this in chapter 2. In the case of soft potentials, these estimates are only established locally in time, but in some situations one can control the growth well enough.

- Lower bound estimates. Uniform such bounds were proven by A. Pulvirenti and Wennberg for Maxwellian collision kernels or hard potentials *with cut-off*. In the case of soft potentials, uniform bounds are an open problem. But, still under the cut-off assumption, local (in time) bounds are very easy to obtain as a consequence of Duhamel's formula (100), if the initial datum satisfies a lower bound assumption. Such a crude bound as

$$f(t,v) \geq K e^{-A_t |v|^2}, \qquad A_t = (1+t)$$

is sufficient in many situations [**428**, section 4]. The case of non-cutoff collision kernels is still open.

- L^p estimates. In the model case of hard potentials with Grad's cut-off assumption, such estimates are a consequence of the studies of Arkeryd and Gustafsson, as discussed in section 3 of chapter 2. For instance, if the initial datum lies in L^∞ with suitable polynomial decay, then the solution will be bounded, uniformly in time. Also the case of Maxwell collision kernel can be treated in the same way. However, when the collision kernel decays at infinity, things become more intricate. The search for robust estimates led the authors in [**428**] to a new way of controlling L^p norms by the Q^+ smoothness[23], moment estimates, and a lot of interpolation.

On the other hand, in the case of non-cutoff collision kernels, L^p estimates are obtained via Sobolev estimates and regularizing effect.

Here we see that many of the estimates which we discussed in chapter 2 can be combined to yield a qualitative theorem for solutions of the Boltzmann equation : trend to equilibrium with some explicit rate. Since the general panorama of a priori estimates for the spatially homogeneous Boltzmann equation is not completely

[23]For simplicity, kinetic collision kernels $\Phi(|v - v_*|)$ considered in [**429**] were smooth and bounded from above and below. The authors had forgotten that in such a case the Q^+ smoothness could not apply directly, because $\Phi(0) > 0$. The proof is however easy to fix by treating separately relative velocities which are close to 0; this has been done recently by Mouhot.

settled yet, we do not have a general theorem. Let us give one which encompasses the few cases that can be treated completely. We put very strong conditions on the initial datum so that a unified result can be given for different kinds of collision kernels.

Theorem 20. *Let $B(v-v_*,\sigma)=\Phi(|v-v_*|)b(\cos\theta)$ be a collision kernel satisfying Grad's angular cut-off, let f_0 be an initial datum with unit mass and temperature, and let $f(t,\cdot)$ be a strong solution of the Boltzmann equation with initial datum f_0. Assume that f_0 lies in L^∞ and decays at infinity like $O(|v|^{-k})$ for any $k\geq 0$. Assume moreover that $f_0(v)\geq Ke^{-A|v|^2}$ for some $A,K>0$. Then*

(i) If $\Phi\equiv 1$, then $H(f|M)=O(t^{-\infty})$;

(ii) If $\Phi(|v-v_*|)=|v-v_*|^\gamma$, $\gamma>0$, then $H(f|M)=O(t^{-\kappa})$ for some *$\kappa>0$;*

(iii) If $\Phi(|v-v_*|)$ is bounded, strictly positive and decays at infinity like $|v-v_*|^{-\beta}$, with $0<\beta<2$, then $H(f|M)=O(t^{-\infty})$.

Moreover, all the constants in these estimates are explicitly computable.

Of course, $O(t^{-\infty})$ means $O(t^{-\kappa})$ for *any* $\kappa>0$. For parts (i) and (ii), see [**428**]; for part (iii), see [**429**]. Also we insist that result (i) also holds when Φ is bounded from above and below. In chapter 4, we shall see that the structure of the particular case $\Phi\equiv 1$ allows a better result, in the form of an explicit exponential rate of convergence.

5.4. Infinite entropy. We conclude this section with an interesting remark due to Abrahamsson [**1**]. Of course, it seems intuitive that entropy dissipation methods require an assumption of finiteness of the entropy. This is not true ! In some situations one can decompose the solution of the spatially homogeneous Boltzmann equation into a part with infinite entropy, but going to zero in L^1 sense, and a part with finite entropy, on which the entropy dissipation methods can be applied. If the estimates are done with enough care, this results in a theorem of convergence with explicit rate, even when the initial datum has infinite entropy. It is important here to have a good control of the entropy-entropy dissipation inequality which is used, in terms of the initial datum.

With this technique, Abrahamsson [**1**] was able to prove convergence to equilibrium in L^1 for the spatially homogeneous Boltzmann equation with hard spheres, assuming only that the initial datum has finite mass and energy. On this occasion he used the Carlen-Carvalho theorem, and also some iterated Duhamel formulas in the a priori estimates.

Note that this problem mainly arises for cut-off collision kernels, because for most kernels with an angular singularity, entropy becomes finite for any positive time [**440**] by regularization effects.

This ends our review of applications of entropy methods in the spatially homogeneous Boltzmann and Landau equations. Before discussing spatially inhomogeneous models, we shall briefly consider another class of spatially homogeneous systems, characterized by their gradient flow structure.

6. Gradient flows

This section is a little bit outside the main stream of our review, but reflects active trends of research in kinetic theory, and may enlighten some of the considerations

appearing here and there in this chapter. The main application to Boltzmann-like equations is Theorem 21 below, for simple models of granular flows.

6.1. Metric tensors. As we explained before, several equations in kinetic theory have a gradient flow structure : they can be written

$$\frac{\partial f}{\partial t} = \nabla_v \cdot \left(f \, \nabla_v \frac{\delta E}{\delta f} \right) \tag{209}$$

for some energy functional $f \longmapsto E(f)$, which we shall always call the entropy for consistency. Typical examples are the Fokker-Planck equation, for which

$$E(f) = \int_{\mathbb{R}^N} f \log f + \frac{1}{2} \int_{\mathbb{R}^N} f|v|^2 \, dv;$$

or the model from [**68**] for granular flow,

$$\frac{\partial f}{\partial t} = \nabla_v \cdot \Big(f(f * \nabla U) \Big) + \sigma \Delta_v f + \theta \nabla_v \cdot (fv), \tag{210}$$

with $U(v) = |v|^3/3$, and $\sigma, \theta > 0$; then

$$E(f) = \frac{1}{2} \int_{\mathbb{R}^2} f(v) f(w) U(v-w) \, dv \, dw + \sigma \int_{\mathbb{R}} f \log f + \frac{\theta}{2} \int_{\mathbb{R}} f|v|^2 \, dv. \tag{211}$$

Among examples outside kinetic theory, we have also mentioned the heat equation, the spatial Fokker-Planck equation, the porous medium equation...

Generally speaking, a gradient flow is an equation of the form

$$\frac{dX}{dt} = - \operatorname{grad} E(X(t)). \tag{212}$$

Underlying the definition of the gradient operator is that of a Riemannian metric tensor on some "manifold" in which the unknown X lives. Thus, to explain why (209) is a gradient flow, we first have to explain how to define a meaningful metric tensor on the "manifold" of all probability measures. Of course this is a formal point of view, because infinite-dimensional Riemannian geometry usually does not make much sense, even if it is sometimes enlightening, as the well-known works by Arnold [**40**] in fluid mechanics illustrate.

In our context, the relevant metric tensor is defined as follows. Let f be a probability density (assume that f is smooth and positive, since this is a formal definition). Let $\partial f / \partial s$ be a "tangent vector" : formally, this just means some function with vanishing integral. Then define

$$\left\| \frac{\partial f}{\partial s} \right\|^2 = \inf \left\{ \int f|u|^2 \, dv; \quad \frac{\partial f}{\partial s} + \nabla_v \cdot (fu) = 0 \right\}. \tag{213}$$

The infimum in (213) is taken over all vector fields u on $\mathbb{R}^N$ such that the linear transport equation $\partial f / \partial s + \nabla_v \cdot (fu) = 0$ is satisfied. By polarization, formula (213) defines a metric tensor, and then one is allowed to all the apparatus of Riemannian geometry (gradients, Hessians, geodesics, etc.), at least from the formal point of view. Then, an easy computation shows that (209) is the gradient flow associated with the energy E, on the "manifold" of all (smooth, positive) probability measures endowed with this Riemannian structure.

The metric tensor defined by (213) has been introduced and studied extensively by Otto [**364**]. One of its important features is that the associated *geodesic distance*

is nothing but the Wasserstein distance on probability measures[24]. This is part of the whole area of mass transportation, whose connections with partial differential equations are reviewed in Villani [**452**].

6.2. Convergence to equilibrium. A general property of gradient flows is that they make the entropy decrease. From formula (212) one sees that

$$\frac{d}{dt}E(X(t)) = -\big\|\text{grad } E(X(t))\big\|^2.$$

And the equilibrium positions of (212) are the critical points for E : typically, minima. In all the cases which we consider, there is a unique minimizer for E, which is therefore the only equilibrium state.

Now, it is a general, well-known fact that the rate of convergence to equilibrium for a gradient flow is very much connected to the (uniform, strict...) *convexity* of the energy functional. A typical result is the following : *assume that the energy E is uniformly convex, in the sense that its Hessian is bounded below by some positive multiple of the identity tensor,*

$$\text{Hess } E \geq \lambda \text{Id}.$$

Then, E admits a unique minimizer X_∞, and the gradient flow (212) *satisfies the linear entropy-entropy dissipation inequality*

$$\|\text{grad } E\| \geq 2\lambda\Big[E(X) - E(X_\infty)\Big]. \tag{214}$$

A possible strategy to prove (214) is to go to the *second derivative* of the entropy functional with respect to time. From (212) and the definition of the Hessian,

$$-\frac{d}{dt}\big\|\text{grad } E(X(t))\big\|^2 = 2\Big\langle \text{Hess}\,(E)\cdot\nabla E, \nabla E\Big\rangle. \tag{215}$$

The functional which just appeared in the right-hand side is the **dissipation of entropy dissipation**. Therefore, the assumption of uniform positivity of the Hessian implies

$$\frac{d}{dt}\big\|\text{grad } E(X(t))\big\|^2 \leq -2\lambda\big\|\text{grad } E(X(t))\big\|^2.$$

Integrating this inequality in time, one easily arrives at (214) if everything is well-behaved[25].

This remark shows that the trend to equilibrium for eq. (209) can in principle be studied via the properties of convexity of the underlying energy E. But the right notion of convexity is no longer the usual one : it should be adapted to the definition (213). This concept is known as *displacement convexity*, and was first studied by McCann [**338, 340**], later by Otto [**364**], Otto and Villani [**365**].

Definition 3. *Let f_0, f_1 be two (smooth, positive) probability measures on $\mathbb{R}^N$. By a classical theorem of Brenier* [**103, 339**] *and others, there exists a unique gradient of convex function, $\nabla\varphi$, such that*

$$\nabla\varphi \# f_0 = f_1,$$

[24] See formula (244) below.

[25] There are also other, simpler derivations of (214) based on Taylor formula. The above procedure was chosen because this is precisely a way to understand the famous Bakry-Emery method for logarithmic Sobolev inequalities.

meaning that the image measure[26] *of* f_0 *by the mapping* $\nabla\varphi$ *is the measure* f_1*. Let us define the interpolation* $(f_s)_{0\le s\le 1}$ *between* f_0 *and* f_1 *by*

$$f_s = \Big[(1-s)\,\mathrm{Id} + s\nabla\varphi\Big]\#f_0.$$

Then, the functional E *is said to be* displacement convex *if whatever* f_0*,* f_1*,*

$$s \mapsto E(f_s) \qquad \text{is convex on } [0,1].$$

It is furthermore said to be uniformly displacement convex *with constant* $\lambda > 0$ *if whatever* f_0*,* f_1*,*

$$\frac{d^2}{ds^2}E(f_s) \ge \lambda W(f_0,f_1)^2 \qquad (0\le s\le 1),$$

where W *stands for the Wasserstein distance*[27] *between* f_0 *and* f_1*.*

Remarks :

1. To get a feeling of this interpolation procedure, note that the interpolation between δ_a and δ_b is $\delta_{(1-s)a+sb}$, instead of $(1-s)\delta_a + s\delta_b$. Further note that the preceding definition reduces to the usual definition of convexity if the interpolation $(f_s)_{0\le s\le 1}$ is replaced by the linear interpolation.

2. Let us give some examples. The functionals $\int f\log f$, or $\int f^p$ $(p\ge 1)$ are displacement convex as one of the main results of McCann [**340**]. The functional $\int fV$ is displacement convex if and only if the potential V is convex. Moreover, if V is uniformly convex with constant λ, the functional $\int fV$ is uniformly displacement convex, with the same constant.

Another interesting example is the case of functionals like $\int_{\mathbb{R}^{2N}} f(v)f(w)U(v-w)\,dv\,dw$. Such a functional is never convex in the usual sense, except for some very peculiar potentials (power laws...). On the other hand, it is displacement convex as soon as U is convex.

Among the results in Otto and Villani [**365**], we mention the following statement. If E is uniformly displacement convex, with constant $\lambda > 0$, then the associated gradient flow system (209) satisfies a linear entropy-entropy dissipation inequality of the form

$$\int_{\mathbb{R}^N} f\left|\nabla\frac{\delta E}{\delta f}\right|^2 dv \ge 2\lambda\big[E(f) - E(f_\infty)\big],$$

where f_∞ is the unique minimizer of the energy. This is not a true theorem, because the proof is formal, but this is a general principle which can be checked on each example of interest. A standard strategy of proof goes via the second derivative of the entropy, as we sketched above. In the context of the linear Fokker-Planck equation (168), this strategy of taking the second derivative of the entropy is known as the **Bakry-Emery strategy**, and goes back to the mid-eighties.

To check the assumption of uniform displacement convexity, it is in principle sufficient to compute the Hessian of the entropy. When this is done, one immediately obtains the dissipation of entropy dissipation via formula (215). This calculation is however very intricate, as one may imagine. This is where Bakry and Emery [**45**] need their so-called Γ_2 calculus, which is a set of formal computation rules involving

[26] By definition of the image measure : for all bounded continuous function h, $\int (h\circ\nabla\varphi)f_0 = \int hf_1$. If φ is C^2, then for all v, $f_0(v) = f_1(\nabla\varphi(v))\det(D^2\varphi)(v)$.

[27] See formula (244) below.

linear diffusion operators and commutators. On the other hand, the formalism developed by Otto [**364**], Otto and Villani [**365**] enables simpler formal computations, and can be adapted to nonlinear cases such as granular flows [**130**].

Just to give an idea of the complexity of the computations, and why it is desirable to have efficient formal calculus here, let us reproduce below the dissipation of entropy dissipation which is associated to the gradient flow for equation (210) :

(216)

$$\begin{aligned} DD(f) = \; & 2\sigma \int_{\mathbb{R}} f(v) \left[\frac{\partial}{\partial v} \xi(v) \right]^2 dv \\ & + 2\theta \int_{\mathbb{R}} f(v) |\xi(v)|^2 \, dv \\ & + \int_{\mathbb{R}^2} f(v) f(w) \Big\langle D^2 U(v-w) \cdot [\xi(v) - \xi(w)], [\xi(v) - \xi(w)] \Big\rangle \, dv \, dw, \end{aligned}$$

where

$$\xi(v) = \frac{\partial}{\partial v} \left[\sigma \log f(v) + \theta \frac{|v|^2}{2} + U * f \right], \qquad U(v) = |v|^3/3.$$

6.3. A survey of results. Let us now review some results of trend to equilibrium which were obtained via the considerations above, or which can be seen as related. A survey paper on this subject is Markowich and Villani [**330**].

The first partial differential equation to be treated in this way was the spatial Fokker-Planck equation,

$$\frac{\partial \rho}{\partial t} = \nabla_x \cdot \big(\nabla_x \rho + \rho \nabla V(x) \big).$$

The classical paper by Bakry and Emery [**45**] shows that the solution to this equation converges exponentially fast, in relative entropy sense, to the equilibrium e^{-V} (assuming $\int e^{-V} = 1$), at least if V is uniformly convex with constant λ. The decrease of the entropy is like $e^{-2\lambda t}$. Underlying entropy-entropy dissipation inequalities are known under the name of logarithmic Sobolev inequalities, and have become very popular due to their relationship with many other fields of mathematics (concentration of measure, hypercontractivity, information theory, spin systems, particle systems... see the review in [**16**]).

The Bakry-Emery strategy, and the corresponding proof of the Stam-Gross logarithmic Sobolev inequality, were recently re-discovered by Toscani [**422, 423**] in the case of the kinetic Fokker-Planck equation. Instead of Γ_2 calculus, Toscani generalized a lemma by McKean [**341**] to compute the second derivative of the entropy functional. Though this work was mainly a re-discovery of already known results, it had several merits. First, it suggested a more physical way of understanding the Bakry-Emery proof, in terms of entropy dissipation and dissipation of entropy dissipation. Also, Toscani directly studied the Fokker-Planck equation $\partial_t f = \Delta f + \nabla \cdot (fv)$, while previous authors mainly worked on the adjoint form, $\partial_t h = \Delta h - v \cdot \nabla h$. Last but not least, his paper made these methods and techniques popular among the kinetic community, which began to work on this subject : see in particular the recent synthesis by Arnold, Markowich, Toscani and Unterreiter [**39**].

Then, these results were generalized to the porous medium equation with drift,

$$\frac{\partial \rho}{\partial t} = \Delta_x \rho^m + \nabla_x \cdot (\rho x).$$

It was found that when $m \geq 1 - 1/N$, solutions to this equation converge exponentially fast (with relative entropy decreasing like e^{-2t}) towards a probability density known as Barenblatt-Pattle profile. These results were obtained independently by Otto [**364**], Carrillo and Toscani [**131**], and Del Pino and Dolbeault [**163**]. All three papers established nonlinear analogues of the logarithmic Sobolev inequalities. The paper by Otto made the link with mass transportation problem. Various generalizations of all these results can be found in [**129**].

Let us now come back to kinetic, Boltzmann-like systems and display recent results about the equations for granular media suggested in [**70, 68**]. These results were proven by Carrillo, McCann and Villani [**130**] by using the ideas above, and in particular those of gradient flows, Wasserstein distance and Bakry-Emery strategy. The theorem which we state is slightly more general : the dimension is arbitrary, and the interaction potential is not necessarily cubic.

THEOREM 21. *Let U be a convex, symmetric potential on $\mathbb{R}^N$, and $\sigma, \theta \geq 0$. Let*

$$E(f) = \sigma \int f \log f + \theta \int f(v) \frac{|v|^2}{2}\, dv + \frac{1}{2} \int_{\mathbb{R}^N \times \mathbb{R}^N} f(v) f(w) U(v-w)\, dv\, dw.$$

Let moreover

$$\xi = \nabla \left(\sigma \log f + \theta \frac{|v|^2}{2} + U * f \right).$$

We consider the equation

$$\frac{\partial f}{\partial t} = \nabla \cdot (f \xi), \tag{217}$$

for which E is a Lyapunov functional, whose time-derivative is given by the negative of

$$D(f) = \int f |\xi|^2\, dv.$$

Let f_0 be an initial datum, and $f(t)$ be the solution[28] *to* (217)*. Let moreover f_∞ be the unique minimizer of E (if $\theta = 0$, the unique minimizer of E which has the same mean velocity as f_0), and let*

$$E(f|f_\infty) = E(f) - E(f_\infty).$$

Then,

(i) If $\theta > 0$, then $D(f) \geq 2\theta E(f|f_\infty)$ for all probability distribution f, and $E(f(t)|f_\infty) \leq e^{-2\theta t} E(f_0|f_\infty)$.

(ii) If $\theta = 0$ and U is uniformly convex, $D^2 U \geq \lambda$, then $D(f) \geq 2\lambda E(f|f_\infty)$ for all f with the same mean velocity as f_∞, and $E(f(t)|f_\infty) \leq e^{-2\lambda t} E(f_0|f_\infty)$.

(iii) If $\theta = 0$ and U is strictly convex, in the sense

$$D^2 U(z) \geq K(|z|^\alpha \wedge 1), \qquad K, \alpha > 0, \tag{218}$$

[28] We assume that U is sufficiently well-behaved that existence of a unique "nice" solution to (217) is guaranteed. This is quite a weak assumption.

then $D(f) \geq CE(f|f_\infty)^\beta$ for some positive constants C, β, and for any f with the same mean velocity as f_∞, and $E(f(t)|f_\infty)$ converges to 0 at least in $O(t^{-\kappa})$ for some $\kappa > 0$.

(iv) If $\theta = 0$, U is strictly convex in the sense of (218)*, and moreover $\sigma > 0$, then for all f with the same mean velocity as f_∞, one has $D(f) \geq \lambda(f)E(f|f_\infty)$ for some $\lambda(f) > 0$ which only depends on an upper bound for $E(f)$. Moreover, $E(f(t)|f_\infty) \leq e^{-\lambda_0 t}E(f_0|f_\infty)$, for some positive constant λ_0 which only depends on an upper bound for $E(f_0)$.*

Remarks :

1. The assumptions on the mean velocity reflect an important physical feature : in the cases in which they are imposed, the entropy is translation-invariant.

2. The motivation to study strictly convex, but not uniformly convex potentials comes from the physical model where $U(z) = |z|^3/3$. Then, lack of uniformity may come from small values of z. This difficulty is of the same type than in the study of hard potentials for the Boltzmann equations; this is not surprising since the model in [**70**] can be seen as a limit regime for some Boltzmann-type equation with inelastic *hard spheres*.

3. In general, point (iii) cannot be improved into exponential decay : when $\sigma = \theta = 0$ and $U(z) = |z|^3/3$, then the decay of the energy is $O(1/t)$ and this is optimal.

4. We note that part (iv) of this theorem is the most surprising, because in this case the energy functional is not uniformly displacement convex; yet there is a "linear" entropy-entropy dissipation inequality (not uniform in f). This result raises hope that the entropy-entropy dissipation inequalities described in section 4 for the Landau equation with hard potentials may be improved into inequalities of linear type.

To conclude this section, we mention that the trend to equilibrium for equation (217) has been studied by Malrieu [**329**] with the Γ_2 calculus of Bakry and Emery. Even though his results are much more restrictive (only $\sigma, \theta > 0$) and the constants found by Malrieu are not so good, on this occasion he introduced several interesting ideas about particle systems and his work provided yet another connection between the kinetic and probabilistic communities[29].

7. Trend to equilibrium, spatially inhomogeneous systems

We now turn to the study of spatially inhomogeneous kinetic systems like the ones presented in paragraph 1.2. We first make several remarks.

1. Once again, we are mainly interested in explicit results, and wish to cover situations which are not necessarily perturbations of the equilibrium. Thus we do not want to use linearization tools, and focus on entropy dissipation methods.

2. For many of the spatially inhomogeneous models which we have introduced, the entropy and the entropy dissipation functionals are just the same as in the spatially homogeneous case, up to integration in x. Also, the transport part does not contribute to the entropy dissipation. Thus, one may think, the same entropy-entropy dissipation inequalities which we already used for the spatially homogeneous

[29] In a more recent, quite clever work, Malrieu was able to remove the condition $\theta > 0$.

case will apply to the spatially inhomogeneous case. This is completely false, as we shall explain ! And the obstruction is not a technical subtlety, but stands for a good physical reason.

3. Nevertheless, it is plainly irrelevant to ask for an x-dependent version of the entropy-entropy dissipation inequalities presented in section 4, since the entropy dissipation does not make the x variable play any role.

4. The boundary conditions, and the global geometry of the spatial domain, are extremely important in this study. In this respect the problem of trend to equilibrium departs notably from the problem of the hydrodynamic limit, which fundamentally is a local problem.

5. To work on the trend to equilibrium, one should deal with well-behaved solutions, satisfying *at least* global conservation laws. In the sequel, we shall even assume that we deal with *very* well-behaved solutions, for which all the natural estimates of decay, smoothness and positivity are satisfied. Of course, for such equations as the Boltzmann or Landau equation, nobody knows how to construct such solutions under general assumptions... Therefore, the results dealing about these equations will be conditional, in the sense that they will depend on some strong, independent regularity results which are not yet proven. It is however likely that such regularity bounds can be obtained with present-day techniques in certain particular situations, like weakly inhomogeneous solutions [**32**]. We wish to insist that even if we assume extremely good a priori estimates, the problem of convergence to equilibrium remains interesting and delicate, both from the mathematical and from the physical point of view !

7.1. Local versus global equilibrium. When studying the trend to equilibrium in a spatially dependent context, a major obstacle to overcome is the existence of **local equilibrium states**, i.e. distribution functions which are in equilibrium with respect to the velocity variable, but not necessarily with respect to the position variable. For instance, for the Boltzmann or Landau equation, a local equilibrium is a local Maxwellian,

$$f(x,v) = \rho(x) \frac{e^{-\frac{|v-u(x)|^2}{2T(x)}}}{(2\pi T(x))^{N/2}}. \tag{219}$$

For the linear Fokker-Planck equation, a local equilibrium is a distribution function of the form

$$f(x,v) = \rho(x) \frac{e^{-\frac{|v|^2}{2}}}{(2\pi)^{N/2}} \equiv \rho(x) M(v). \tag{220}$$

Local equilibria are not equilibrium distributions in general, but *they make the entropy dissipation vanish*. This shows that there is no hope to find an entropy-entropy dissipation inequality for the full x-dependent system.

If the system ever happens to be in local equilibrium state at some particular time t_0 , then the entropy dissipation will vanish at t_0, and it is a priori not clear that the entropy functional could stay (almost) constant for some time, before decreasing again. This may result in a strong slowing-down of the process of trend to equilibrium. This difficulty has been known for a long time (even to Boltzmann ! as pointed out to us by C. Cercignani), and is discussed with particular attention by

Grad [**254**], Truesdell [**430**, p. 166–172] and, in the different but related context of hydrodynamic limits for particle systems, Olla and Varadhan [**362**].

On the other hand, whenever the solution happens to coincide with some local equilibrium state $M_{\rm loc}$ at some time, then the combined effect of transport and confinement will make it go out of local equilibrium again, unless $M_{\rm loc}$ satisfies some symmetry properties which ensure that it is a stationary state. In fact, in most situations these symmetry properties select *uniquely* the stationary state among the class of all local equilibria.

Note the fundamental difference with the problem of hydrodynamical limit : in the latter, one wishes to prove that the solution stays as much as possible close to local equilibrium states, while here we wish to prove that if the solution ever happens to be very close to local equilibrium, then this property will *not* be preserved at later times.

Thus, one can see the trend to equilibrium for spatially inhomogeneous systems as the result of a negociation between collisions on one hand, transport and confinement on the other : by dissipating entropy, collisions want to push the system close to local equilibrium, but transport and confinement together do not like local equilibria — except one. This is why transport phenomena, even if they do not contribute in entropy dissipation, play a crucial role in selecting the stationary state. Our problem is to understand whether these effects can be quantified.

An answer to this question was recently obtained by the author in a series of collaborations with Desvillettes. In the sequel, we shall explain it on a simple case : the linear Fokker-Planck equation, with potential confinement,

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f - \nabla V(x) \cdot \nabla_v f = \nabla_v \cdot (\nabla_v f + f v). \tag{221}$$

The trend to equilibrium for this model was studied by Bouchut and Dolbeault [**100**] with the help of compactness tools, so no explicit rate of convergence was given. Also, Talay [**413**] proved exponential decay with a probabilistic method (based on general theorems about recurring Markov chains) which does not seem to be entirely constructive. Other probabilistic approaches have been proposed to study this model, but they also strongly depend on the possibility to interpret (221) as the evolution equation for the law of the solution of some stochastic differential equation with particular properties[30].

In the sequel, we shall explain how the entropy method of Desvillettes and Villani [**184**] leads to polynomial (as opposed to exponential), but fully explicit estimates. This method is robust in the sense that it can be generalized to smooth solutions of nonlinear equations, in particular Boltzmann or Vlasov-Poisson-Fokker-Planck equations.

With respect to the Boltzmann equation, the model (221) has several pedagogical advantages. First, one can prove all the a priori estimates which are needed in the implementation of the method. What is more important, the local equilibrium only depends on one parameter (the density), instead of three (density, mean velocity and temperature). This entails significant simplifications in the computations and intermediate steps, which however remain somewhat intricate.

[30] A much more complete, very satisfactory study was recently performed by Hérau and Nier; see also the references provided in their work.

7.2. Local versus global entropy: discussion on a model case. To use entropy methods in a spatially dependent context, the main idea is to work *at the same time* at the level of local and global equilibria; i.e., estimate simultaneously how far f is from being in local equilibrium and how far it is from being in global equilibrium.

1) One first introduces the **local equilibrium associated with** f, i.e. the one with the same macroscopic parameters as f. For instance, in the case of the linear Fokker-Planck equation, the local equilibrium is just (220), with $\rho(t,x) = \int f(t,x,v)\,dv$. In the case of the Boltzmann equation, the local equilibrium is the local Maxwellian (219), with ρ, u, T given by (1).

How close f is from local or global equilibrium will naturally be measured by relative entropies. Thus one defines $H_{\rm glo}$ to be the relative entropy of f with respect to the global equilibrium, and $H_{\rm loc}$ to be the relative entropy of f with respect to the associated local equilibrium. In the Fokker-Planck (resp. Boltzmann) case, $H_{\rm loc}$ is $H(f|\rho M)$ (resp. $H(f|M^f)$); note that this is an integral over $\mathbb{R}^N_x \times \mathbb{R}^N_v$ now. Then, one looks for a system of differential inequalities satisfied by $H_{\rm glo}$ and $H_{\rm loc}$.

2) The first equation is given by an entropy-entropy dissipation estimate of the same type that the ones we discussed in section 4. We just have to apply this inequality pointwise in x. For instance, for the linear Fokker-Planck equation,

$$D(f) = -\frac{d}{dt} H_{\rm glo} = \int_{\mathbb{R}^N} I(f|\rho M)\,dx = \int_{\mathbb{R}^N} \rho\, I\left(\frac{f}{\rho}\middle| M\right)\,dx, \tag{222}$$

and by the logarithmic Sobolev inequality (175),

$$D(f) \geq 2\int_{\mathbb{R}^N_x} \rho\, H\left(\frac{f}{\rho}\middle| M\right)\,dx = 2H(f|\rho M) = 2H_{\rm loc} \tag{223}$$

(check the last-but-one equality to be convinced !). Note that the symbol H is used above in two different meanings : relative entropy of two probability distributions of the v variable, relative entropy of two probability distributions of the x and v variables.

Similarly, if we have nice uniform a priori bounds for the solution of the Boltzmann equation, it will follow from our discussion in section 4 that

$$-\frac{d}{dt} H_{\rm glo} \geq K H_{\rm loc}^{\alpha}, \tag{224}$$

for some constants $K > 0$, $\alpha > 1$.

In a spatially homogeneous context, this inequality would be essentially sufficient to conclude by Gronwall's lemma. Here, we need to keep much more information from the dynamics in order to recover a control on how the positivity of $H_{\rm glo}$ forces $H_{\rm loc}$ to go up again if it ever vanishes.

3) To achieve this, we now look for a differential inequality involving the time-behavior of $H_{\rm loc}$. We start with a heuristic discussion. At a time when the entropy dissipation would vanish, then both the local relative entropy and its time derivative would vanish, since the relative entropy is always nonnegative. Therefore, one can only hope to control from below the *second* time derivative of the local relative entropy ! Taking into account the first differential inequality about $H_{\rm glo}$ and $H_{\rm loc}$, this more or less resembles to considering the *third* derivative of the entropy at an inflexion point.

It is easy to compute $(d^2/dt^2)H_{\rm loc}$ at a time t_0 when f happens to be in local equilibrium. For instance, in the case of the linear Fokker-Planck equation, we have the remarkably simple formula

$$\left.\frac{d^2}{dt^2}\right|_{t=t_0} H(f|\rho M) = \int_{\mathbb{R}^N_x} \rho \left| \frac{\nabla \rho}{\rho} + \nabla V \right|^2 dx \equiv I_x(\rho|e^{-V}). \tag{225}$$

Here I_x is the Fisher information, applied to functions of the x variable. We do not describe here the corresponding results for the Boltzmann equation, which are of the same nature but much, much more complicated [**181**]. Here we shall continue the discussion only for the Fokker-Planck equation, and postpone the Boltzmann case to the end of the next paragraph.

If V is well-behaved, the logarithmic Sobolev inequality, applied in the x variable, yields

$$\left.\frac{d^2}{dt^2}\right|_{t=t_0} H(f|\rho M) \geq K H(\rho|e^{-V}) \tag{226}$$

for some positive constant K depending only on V. This is the piece of information that was lacking ! Indeed, for the linear Fokker-Planck equation,

$$H_{\rm glo} = H_{\rm loc} + H(\rho|e^{-V});$$

thus eq. (226) turns into

$$\left.\frac{d^2}{dt^2}\right|_{t=t_0} H(f|\rho M) \geq K H_{\rm glo} - K H_{\rm loc}. \tag{227}$$

Note that the use of the logarithmic Sobolev inequality in the x variable is the precise point where the geometry of the boundary conditions (here replaced by a confinement potential) comes into play. The fact that this effect can be quantified by a functional inequality is very important for the method; see the remarks in the end of the chapter for the analogous properties in the Boltzmann case.

Of course, the preceding calculations only apply at a time t_0 where f happens to be in local equilibrium — which is a very rare event. Therefore, one establishes a quantitative variant of (227), in the form

$$\frac{d^2}{dt^2} H(f|\rho M) \geq \frac{K}{2}(H_{\rm glo} - H_{\rm loc}) - J(f|\rho M), \tag{228}$$

where $J(f|\rho M)$ is a complicated functional which vanishes only if $f = \rho M$:

$$\begin{aligned} \frac{1}{4} J(f|\rho M) = &\int_{\mathbb{R}^N} \frac{(\rho u)^2}{\rho}\, dx + \frac{1}{4}\int_{\mathbb{R}^N} \frac{|\nabla_x \cdot (\rho u)|^2}{\rho}\, dx + \int_{\mathbb{R}^N} \frac{|\nabla_x \cdot (\rho u \otimes u)|^2}{\rho}\, dx \\ &+ \int_{\mathbb{R}^N} \frac{|\nabla_x [\rho(T-1)]|^2}{\rho}\, dx + \int_{\mathbb{R}^N} \frac{|\nabla_x \cdot S|^2}{\rho}\, dx + I_v(f|\rho M) \\ &+ \frac{1}{2} I_v(f|\rho\, M)^{1/2}\, I_x(f|\rho\, M)^{1/2}. \end{aligned} \tag{229}$$

Here ρ, u, T are the usual macroscopic fields, and S is the matrix defined by the equation

$$\rho(x)\, u(x) \otimes u(x) + \rho(x)\, T(x) I_N + S(x) = \int_{\mathbb{R}^N} f(x,v) v \otimes v\, dv, \tag{230}$$

4) The next step of the program is to control J in terms of $H_{\rm loc}$, in order to have a *closed system of differential inequalities on* $H_{\rm loc}$ *and* $H_{\rm glo}$. This is done by some ad hoc nonlinear **interpolation** procedure, which yields

$$\frac{d^2}{dt^2} H(f|\rho M) \geq \frac{K}{2}\, H(f|f_\infty) - C_\varepsilon(f)\, H(f|\rho M)^{1-\varepsilon}.$$

Here ε is an arbitrary positive number in $(0,1)$ and $C_\varepsilon(f)$ is a constant depending on f via *moment bounds*, *smoothness bounds*, and *positivity estimates* on f.

All these bounds have to be established explicitly and uniformly in time, which turns out to be quite technical but feasible [**184**] (see also Talay [**413**]); then the constant $C_\varepsilon = C_\varepsilon(f)$ can be taken to be independent of t.

In the case of the Boltzmann equation, it is possible to perform a similar interpolation procedure; the only missing step at the moment is establishing the a priori bounds.

5) Summing up, for solutions of the Fokker-Planck equation we have obtained the system of differential inequalities

$$\text{(231)} \qquad \begin{cases} -\dfrac{d}{dt} H_{\rm glo} \geq 2 H_{\rm loc} \\ \\ \dfrac{d^2}{dt^2} H_{\rm loc} \geq \dfrac{K}{2}\, H_{\rm glo} - C_\varepsilon\, H_{\rm loc}^{1-\varepsilon}. \end{cases}$$

The last, yet not the easiest step, consists in proving that the differential system (231) alone implies that $H_{\rm glo}$ converges to 0 like $O(t^{-\kappa})$. Since there is apparently no comparison principle hidden behind this system, one has to work by hand... The bound established in Desvillettes and Villani is

$$H_{\rm glo} = O(t^{1-1/\varepsilon}),$$

which is presumably optimal. Thus, the global entropy converges to 0 with some explicit rate, which was our final goal.

7.3. Remarks on the nature of convergence. Solutions to (231) do have a tendency to *oscillate*, at least for a certain range of parameters. In fact, were it not for the *positivity* of relative entropies, system (231) would not imply convergence to 0 at all ! We expect "typical" solutions of (231) to decrease a lot for small times, and then converge to 0 more slowly as $t \to +\infty$, with some mild oscillations in the slope. This kind of behavior is completely different from what one can prove in the context of spatially homogeneous kinetic equations[31]. We think that it may reflect the physical nature of approach to equilibrium for spatially inhomogeneous systems. As time becomes large and the system approaches global equilibrium, it is more and more likely to "waste time" fighting against local equilibria... And this may result in oscillations in the entropy dissipation.

But examination of a particular, "integrable" case, suggests that 1) these oscillations may be present only when the confinement potential is strong enough, 2) the decay should be exponential. This case corresponds to the quadratic confinement potential, $V(x) = \omega^2|x|^2/2 + C$. For this particular shape of the potential, the

[31] The rate of convergence typically improves as $t \to +\infty$, see paragraph 5.1.

Fokker-Planck equation can be solved in semi-explicit form [**399**, chapter 10], and the rate of decay is governed by the quantities $\exp(-\lambda t)$, where

$$\begin{cases} \lambda = \dfrac{1 - \sqrt{1 - 4\omega^2}}{2}, & \omega^2 \leq \dfrac{1}{4} \\ \\ \lambda = \dfrac{1 \pm i\sqrt{4\omega^2 - 1}}{2}, & \omega^2 > \dfrac{1}{4} \end{cases} \tag{232}$$

(in ref. [**399**], these equations are established only in dimension 1). Thus the decay is always exponential, the rate being given by the real part of λ. When the confinement is very tiny, then the convergence is very slow (think that there is no trend to equilibrium when there is no confinement); when the confinement becomes stronger then the rate increases up to a limit value $1/2$. For stronger confinements, the rate does not improve, but complex exponentials appear in the asymptotics of the solution. Note that in the same situation, the rate of convergence for the spatially homogeneous equation would be equal to 1.

Another integrable case is when there is no confinement potential, but $x \in \mathbb{T}^N$, the N-dimensional torus. Then the decay is always exponential and the rate depends on the size of the periodic box. In dimension 1 of space, it is maximal (equal to 1) when the side of the box has length $\leq 2\pi$ [**152**].

It is yet an open problem to generalize the above considerations to nonintegrable cases, and to translate them at the level of entropy dissipation methods. In our opinion, these examples show that a lot of work remains to be done to get an accurate picture of the convergence, even in very simplified situations[32].

7.4. Summary and informal discussion of the Boltzmann case. We now sum up the state of the art concerning the application of entropy dissipation methods to spatially inhomogeneous systems. The following theorem is the main result of Desvillettes and Villani [**184**].

Theorem 22. *Let $M(v)$ denote the standard Maxwellian probability distribution on $\mathbb{R}^N$ with zero mean velocity and unit temperature. Let V be a smooth confining potential on $\mathbb{R}^N$, behaving quadratically at infinity :*

$$V(x) = \omega^2 \frac{|x|^2}{2} + \Phi(x),$$

where $\omega > 0$ and $\Phi \in \bigcap_{k \geq 0} H^k(\mathbb{R}^N)$. Assume without loss of generality that $\int e^{-V(x)}\, dx = 1$, and let

$$f_\infty(x, v) = e^{-V(x)} M(v)$$

denote the unique global equilibrium of the Fokker-Planck equation

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f - \nabla V(x) \cdot \nabla_v f = \nabla_v \cdot (\nabla_v f + f v).$$

Let $f_0 = f_0(x, v)$ be a probability density such that f_0 / f_∞ is bounded from above and below, and let $f(t) = f(t, x, v)$ be the unique solution of the Fokker-Planck equation with initial datum f_0. Then, for all $\varepsilon > 0$ there exists a constant $C_\varepsilon(f_0)$, explicitly computable and depending only on V, f_0 and ε, such that

$$H(f(t) | f_\infty) \leq C_\varepsilon(f_0)\, t^{-1/\varepsilon}.$$

[32]For progress on these questions, the recent work by Hérau and Nier is recommended.

We already pointed out several shorthands of this result : in particular, the convergence should be exponential. We consider it as a major open problem in the field to compute the optimal rate of decay, in relative entropy, as a function of the confinement potential V.

Let us now turn to nonlinear situations. The following result was recently obtained by the author in collaboration with Desvillettes [**181**].

THEOREM 23. *Let B be a smooth collision kernel, bounded from above and below. Let $f_0 = f_0(x,v)$ be a smooth probability density on $\Omega_x \times \mathbb{R}^3_v$, where Ω is a smooth bounded, connected open subset of $\mathbb{R}^3$ with no axis of symmetry, and let $f(t) = f(t,x,v)$ be a smooth solution of the Boltzmann equation*

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q(f,f), \qquad t \geq 0,\ x \in \Omega,\ v \in \mathbb{R}^3,$$

with specular reflection boundary condition. Let moreover $f_\infty(x,v)$ be the unique global equilibrium compatible with the total mass and kinetic energy of f_0. Assume that all the moments of f are uniformly bounded in time, that f is bounded in all Sobolev spaces, uniformly in time, and that it satisfies a lower bound estimate,

$$f(t,x,v) \geq \rho_0\, e^{-A_0|v|^p}$$

for some $p \geq 2$, $\rho_0 > 0$, $A_0 > 0$, uniformly in time. Then, for all $\varepsilon > 0$ there is a constant C_ε, depending only on (finitely many of) the requested a priori bounds, such that

$$H(f(t)|f_\infty) \leq C_\varepsilon t^{-1/\varepsilon}.$$

We do not display here the system of differential inequalities — much, much more complicated than (231) — which underlies this result, and we refer to [**181**] for more information. An unexpected feature was revealed by this study : not only is the entropy dissipation process indeed slowed down when the distribution function becomes a local Maxwellian state, but also, it is much more slowed down for some particular local Maxwellian states, in particular those of the form $\rho(x)M(v)$ i.e. with constant temperature and zero velocity. More precisely, the entropy dissipation vanishes in time up to order 4 (instead of 2) when going through such a Maxwellian state.

From the mathematical point of view, this entails a spectacular complication of the arguments, and the need for at least three differential inequalities : apart from the behavior of the global entropy, one studies at the same time the departure of f with respect to M^f and the departure of f with respect to ρM.

From the physical point of view, this additional degeneracy could be interpreted as an indication that the relaxation of the density typically holds on a longer time-scale, than the relaxation of the temperature and the local velocity — although we should be cautious about this.

The proof of Theorem 23 combines the general method of Desvillettes and Villani [**184**] with the entropy dissipation results of Toscani and Villani [**428**]. In the computations, the natural functionals $H(f|M^f)$, $H(f|\rho M)$ were traded for the simpler substitutes $\|f - M^f\|^2_{L^2}$, $\|f - \rho M\|^2_{L^2}$ which enable one to weaken significantly the assumptions of theorem 23. Of course, these assumptions are still very strong, even though rather natural after our discussions in chapter 2.

The influence of the shape of the box is quantified by the values of several "geometric" constants related to it. One of these constants is the Poincaré constant $P(\Omega)$, defined in a scalar setting by

$$\|\nabla_x F\|^2_{L^2(\Omega)} \geq P(\Omega) \left\| F - \int_\Omega F \right\|^2_{L^2(\Omega)},$$

whenever F is a real-valued function on Ω, and in a vector setting by

$$\|\nabla_x u\|^2_{L^2(\Omega)} \geq P(\Omega)\, \|u\|^2_{L^2(\Omega)},$$

whenever u is a vector field in Ω, tangent to $\partial\Omega$. Another constant which appears in the proof is what we call **Grad's number**, defined in [**254**] :

$$G(\Omega) = \inf_{\omega_0 \in S^{N-1}} \left\{ \int_\Omega \sum_{ij} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)^2 ; \quad \nabla \cdot v = 0,\ \nabla \wedge v = \omega_0,\ v \cdot n = 0 \text{ on } \partial\Omega \right\},$$

n standing for the unit normal on $\partial\Omega$. The number $G(\Omega)$ is strictly positive if and only if Ω has no axis of symmetry. This number contributes to a lower bound for the constant $K(\Omega)$ in a variant of the Korn inequality which reads

$$\left\| \frac{\nabla_x u + {}^T\nabla_x u}{2} \right\|^2_{L^2(\Omega)} \geq K(\Omega) \|\nabla_x u\|^2_{L^2(\Omega)}. \tag{233}$$

The Korn inequality, of paramount importance in elasticity theory [**201**], is naturally needed to establish the system of differential inequalities which we use to quantify the trend to equilibrium. Our proof of (233) is partly inspired by Grad [**254**].

The whole thing adapts to the case of the torus, or to the bounce-back boundary condition, with significant simplifications. On the other hand, in the case of domains with an axis of symmetry, additional global conservation laws (angular momentum) have to be taken into account, and the case of a *spherical* domain also has to be separated from the rest. These extensions are discussed by Grad [**254**], but have not yet been transformed in a quantitative variant along the lines above.

Remark : As we have seen in chapter 2, if the initial datum is not very smooth and if the Boltzmann collision kernel satisfies Grad's cut-off assumption, then the solution of the Boltzmann equation is not expected to be very smooth. But in this case, as we discussed in section 3.5 of chapter 2, one can hope for a theorem of propagation of singularities in which a vanishingly small (as $t \to \infty$) singular part could be isolated from a very smooth remainder, and, as in [**1**], the entropy dissipation strategy would still apply.

Theorem 23 certainly calls for lots of improvement and better understanding. Yet, it already shows that, in theory, *entropy dissipation methods are able to reduce the problem of trend to equilibrium for the full Boltzmann equation, to a problem of uniform a priori estimates on the moments, smoothness and strict positivity of its solutions.* Moreover, it shows that there is no need for stronger a priori estimates than the ones which are natural in a nonlinear setting : in particular, no estimates in $L^2(M^{-1})$ are needed. We hope that these results will also provide a further motivation for the improvement of known a priori bounds.

CHAPTER 4

MAXWELL COLLISIONS

In this chapter we focus on the Boltzmann collision operator when the collision kernel only depends on the deviation angle :

$$B(v - v_*, \sigma) = b(\cos\theta). \tag{234}$$

As recalled in chapter 1, the modelling of Maxwell molecules, i.e. more or less fictitious particles interacting via repulsive forces in $1/r^5$, in three dimensions of space, leads to a collision kernel B which satisfies (234). By extension, we shall call Maxwellian collision kernel *any* collision kernel of the form (234).

Assumption (234) entails a number of particular properties. The gain part of the Boltzmann collision operator

$$Q^+(g,f) = \int_{\mathbb{R}^N} dv_* \int_{S^{N-1}} d\sigma\, b(\cos\theta) g(v'_*) f(v'), \qquad \cos\theta = \left\langle \frac{v - v_*}{|v - v_*|}, \sigma \right\rangle \tag{235}$$

shares many features with the (more symmetric) **rescaled convolution operator**,

$$g \star f \equiv g_{\frac{1}{2}} * f_{\frac{1}{2}}, \tag{236}$$

where the rescaling operation is defined by

$$f_\lambda(v) = \frac{1}{\lambda^{N/2}} f\left(\frac{v}{\sqrt{\lambda}}\right). \tag{237}$$

Note that if X and Y are independent random variables with respective law f and g, then $f \star g$ is the law of $(X+Y)/\sqrt{2}$. Therefore, with the analogy between the Q^+ and $\star$ operations in mind, the theory of the spatially homogeneous Boltzmann equation with Maxwellian collision kernel resembles that of rescaled sums of independent random variables.

In the sequel, we shall insist on some peculiar topics which illustrate the originality of Maxwellian collision kernels : in section 1 the Wild sum representation, which is an appealing semi-explicit representation formula for solutions in terms of iterated Q^+ operators; in section 2, the existence and applications of several contracting probability metrics compatible with the Boltzmann equation. Finally, in section 3, we describe some interesting connections with information theory.

For more standard issues concerning the Cauchy problem or the qualitative behavior of solutions, the best reference is the long synthesis paper by Bobylev [**79**], entirely based on the use of Fourier transform, which also reviews many contributions by various authors.

Before embarking on this study, we recall that besides its specific interest, the study of Maxwellian collision kernels is often an important step in the study of more general properties of the Boltzmann operator[1].

[1]See for instance paragraph 4.4 in chapter 2 or paragraphs 4.3 to 4.6 in chapter 3.

Contents

1. Wild sums

If the collision kernel is Maxwellian and Grad's cut-off assumption is satisfied, then one can assume without loss of generality that for some (and thus any) unit vector k,

$$\int_{S^{N-1}} b(k\cdot\sigma)\,d\sigma = |S^{N-2}| \int_0^\pi b(\cos\theta)\,\sin^{N-2}\theta\,d\theta = 1. \tag{238}$$

Then the spatially homogeneous Boltzmann equation can be rewritten as

$$\frac{\partial f}{\partial t} = Q^+(f,f) - f. \tag{239}$$

However, when the collision kernel is nonintegrable, then one can only write the general form

$$\frac{\partial f}{\partial t} = \int_{\mathbb{R}^N} dv_* \int_{S^{N-1}} d\sigma\, b(\cos\theta)\big[f'f'_* - f f_*\big]. \tag{240}$$

As was noticed by Wild [**464**], given any initial datum f_0, eq. (239) can be solved recursively in terms of iterated Q^+ operators (this is nothing but a particularly simple iterated Duhamel formula, if one considers (239) as a perturbation of $\partial_t f = -f$). One finds

$$f(t,\cdot) = e^{-t}\sum_{n=1}^{\infty}(1-e^{-t})^{n-1}Q_n^+(f_0), \tag{241}$$

where the n-linear operator Q_n^+ is defined recursively by

$$Q_1^+(f_0) = f_0,$$

$$Q_n^+(f_0) = \frac{1}{n-1}\sum_{k=1}^{n-1} Q^+\Big(Q_k^+(f_0), Q_{n-k}^+(f_0)\Big).$$

The sum (241) can also be rewritten

$$f(t,v) = \sum_{n=1}^{\infty} e^{-t}(1-e^{-t})^{n-1}\left[\sum_{\gamma\in\Gamma(n)}\alpha(\gamma)Q_\gamma^+(f_0)\right], \tag{242}$$

where $\Gamma(n)$ stands for the set of all binary graphs with n leaves, each node having zero or two "children", and $Q_\gamma^+(f_0)$ is naturally defined as follows : if γ has two subtrees γ_1 and γ_2 (starting from the root), then $Q_\gamma^+(f_0) = Q^+(Q_{\gamma_1}^+(f_0), Q_{\gamma_2}^+(f_0))$. Moreover, $\alpha(\gamma)$ are combinatorial coefficients. Wild sums and their combinatorial contents are discussed with particular attention by McKean [**341**], and more recently by Carlen, Carvalho and Gabetta [**126**].

It follows from the Wild representation that a solution of the Boltzmann equation (239) can be represented as a convex combination, with time-dependent weights, of terms of the form

$$f_0,\quad Q^+(f_0,f_0),\quad Q^+(Q^+(f_0,f_0),f_0),\quad Q^+(Q^+(f_0,f_0),Q^+(Q^+(f_0,f_0),Q^+(f_0,f_0))),$$

etc.

This representation is rather intuitive because it more or less amounts to count collisions : the f_0 term takes into account particles which have undergone no collision since the initial time, the term in $Q^+(f_0,f_0)$ corresponds to particles which have undergone only one collision with a particle which had never collided before,

$Q^+(f_0, Q^+(f_0, f_0))$ to particles which have twice undergone a collision with some particle having undergone no collision before.....

This point of view is also interesting in numerical simulations : in a seemingly crude truncation procedure, one can replace (241) by

$$\widetilde{f}(t,v) = e^{-t} \sum_{n=1}^{N_0} (1-e^{-t})^{n-1} Q_n^+(f_0) + (1-e^{-t})^{N_0} M,$$

where M is the Maxwellian distribution with same first moments as f_0. Later in this chapter, we shall explain why such a truncation is rather natural, how it is related to the problem of trend to equilibrium and how it can be theoretically justified[2].

2. Contracting probability metrics

In this section, probability metrics are just metrics defined on a subset of the space of probability measures on $\mathbb{R}^N$.

We call a probability metric d *nonexpansive* along solutions of equation (240) if, whenever $f(t,\cdot)$ and $g(t,\cdot)$ are two solutions of this equation, then

$$d\big(f(t), g(t)\big) \leq d(f_0, g_0). \tag{243}$$

We also say that d is *contracting* if equality in (243) only holds for stationary solutions, i.e. (in the case of finite kinetic energy) when f_0, g_0 are Maxwellian distributions.

2.1. The Wasserstein distance. In his study of the Boltzmann equation for Maxwellian molecules, Tanaka [**414, 353, 415**] had the idea to use the Wasserstein (or Monge-Kantorovich) distance of order 2,

$$W(f,g) = \inf\left\{ \sqrt{E|X-Y|^2}; \quad \mathrm{law}(X) = f,\ \mathrm{law}(Y) = g \right\}. \tag{244}$$

Here the infimum is taken over all random variables X, Y with respective law f and g. It is always assumed that f and g have finite moments of order 2, which ensures that $W(f,g) < +\infty$.

In analytical terms, W can be rewritten as

$$W(f,g) = \inf\left\{ \sqrt{\int_{\mathbb{R}^N \times \mathbb{R}^N} |v-w|^2 \, d\pi(v,w)}\ ; \quad \pi \in \Pi(f,g) \right\},$$

where $\Pi(f,g)$ stands for the set of probability distributions π on $\mathbb{R}^N \times \mathbb{R}^N$ which admit f and g as marginals, More explicitly, $\pi \in \Pi(f,g)$ if and only if

$$\forall (\varphi, \psi) \in C_0(\mathbb{R}^N) \times C_0(\mathbb{R}^N), \quad \int_{\mathbb{R}^N \times \mathbb{R}^N} \big[\varphi(v) + \psi(w)\big] d\pi(v,w) = \int_{\mathbb{R}^N} f\varphi + \int_{\mathbb{R}^N} g\psi.$$

The Wasserstein distance and its variants are also known under the names of Fréchet, Höffding, Gini, Hutchinson, Tanaka, Monge or Kantorovich distances.

The infimum in (244) is finite as soon as f and g have finite second moments. Moreover, it is well-known that convergence in Wasserstein sense is equivalent to the

[2]This is the purpose of Theorem 24 (v).

conjunction of weak convergence in measure sense, and convergence of the second moment :

$$W(f^n, f) \xrightarrow[n\to\infty]{} 0 \iff \begin{cases} \forall \varphi \in C_0(\mathbb{R}^N), \quad \displaystyle\int f^n \varphi \, dv \xrightarrow[n\to\infty]{} \int f \varphi \, dv \\ \displaystyle\int f^n |v|^2 \, dv \xrightarrow[n\to\infty]{} \int f |v|^2 \, dv. \end{cases} \tag{245}$$

Tanaka's theorem [**415**] states that whenever f, g are two probability measures with the same mean, and b is normalized by (238), then

$$W\big(Q^+(f,f), Q^+(g,g)\big) \le W(f,g). \tag{246}$$

Tanaka's representation (74) entered the proof of this inequality, which is formally similar to a well-known inequality for rescaled convolution, $W(f \star f, g \star g) \le W(f,g)$. Thanks to the Wild formula, Tanaka's theorem implies that W is a nonexpansive (in fact contractive) metric, say when restricted to the set of zero-mean probability measures. On this subject, besides Tanaka's papers one may consult [**391**].

As a main application, Tanaka proved theorems of convergence to equilibrium for equation (240) without resorting to the H theorem. In fact, convergence to equilibrium follows almost for free from the contractivity property, since the distance to equilibrium, $W(f(t), M)$, has to be decreasing (unless f is stationary). So one can prove that

$$W\big(f(t), M\big) \xrightarrow[t\to\infty]{} 0$$

as soon as f_0 has just finite energy, not necessarily finite entropy.

Since that time, Tanaka's theorem has been largely superseded : more convenient metrics have been found, and entropy methods have become so elaborate as to be able to cover cases where the entropy is infinite[3]. Yet, Tanaka's theorem reminds us that Boltzmann's H theorem is not the only possible explanation for convergence to equilibrium. The next paragraphs will confirm this.

2.2. Toscani's distance. Since Bobylev's work, it was known that the Fourier transform provides a powerful tool for the study of the spatially homogeneous Boltzmann equation with Maxwellian collision kernel. To measure discrepancies in Fourier space, Toscani introduced the distance

$$d_2(f,g) = \sup_{\xi \in \mathbb{R}^N} \frac{|\widehat{f}(\xi) - \widehat{g}(\xi)|}{|\xi|^2}. \tag{247}$$

The supremum in (247) is finite as soon as f and g have finite second moments, and the same mean velocity :

$$\int_{\mathbb{R}^N} f(v) v \, dv = \int_{\mathbb{R}^N} g(v) v \, dv, \qquad \int_{\mathbb{R}^N} (f+g)|v|^2 \, dv < +\infty.$$

Also, convergence in d_2 sense is equivalent to convergence in Wasserstein sense (245). It turns out [**427**] that, under the normalization (238),

$$d_2\big(Q^+(f,f), Q^+(g,g)\big) \le d_2(f,g), \tag{248}$$

[3]Recall paragraph 5.4 in chapter 3.

with equality only if f, g are Maxwellian distributions. As a consequence, d_2 is a contracting probability metric along solutions of (239) (when one restricts to probability measures with some given mean).

As shown by Toscani and the author [**427**], this contracting property remains true for eq. (240) with a singular collision kernel. As a main consequence, the Cauchy problem associated with (240) admits at most one solution. This uniqueness theorem holds under optimal assumptions : it only requires finiteness of the energy and of the cross-section for momentum transfer (63).

As other applications of the d_2 distance, we mention

- a simple proof of weak convergence to equilibrium under an assumption of finite energy only;

- some partial results for the non-existence of nontrivial eternal solutions [**450**, Annex II, Appendix];

- some explicit estimates of rate of convergence in the central limit theorem [**427**], by refinement of the inequality $d_2(f \star f, M) \leq d_2(f, M)$.

2.3. Other Fourier-based metrics. Other useful Fourier metrics are defined by

$$d_s(g,f) = \sup_{\xi \in \mathbb{R}^N} \frac{|\widehat{f}(\xi) - \widehat{g}(\xi)|}{|\xi|^s}, \qquad s > 2. \tag{249}$$

They are well-defined only when f, g have the same moments up to high enough order. For instance, one cannot directly compare f to the associated Maxwellian distribution M^f in distance d_4 unless $\int f v_i v_j v_k \, dv = \int M^f v_i v_j v_k \, dv$ for all i, j, k. But this drawback is easily fixed by subtracting from $\widehat{f}$ a well-chosen Taylor polynomial.

The interest of using exponents s greater than 2 comes from the fact that the distances d_s become "more and more contracting" as s is increased, and this entails better properties of decay to equilibrium. As soon as $s > 2$, one can prove [**225**] exponential decay to equilibrium in distance d_s, if the initial datum has a finite moment of order s. If one only assumes that the initial datum has a finite moment of order 2, then the method also yields exponential decay in some distance of the form

$$d_\phi(f,g) = \sup_{\xi \in \mathbb{R}^N} \frac{|\widehat{f}(\xi) - \widehat{g}(\xi)|}{|\xi|^2 \phi(\xi)},$$

for some well-chosen function ϕ with $\phi(0) = 0$.

By taking larger values of s, one improves the rate of convergence in d_s metric; in particular, the choice $s = 4$ yields the *optimal* rate of convergence [**128**], which is the spectral gap of the linearized Boltzmann operator[4]. This exponent 4 is related to the fact that the linearized Boltzmann operator admits Hermite polynomials as eigenfunctions, and the lowest eigenvalues are obtained for 4th-degree spherical Hermite polynomials.

Of course, this result of optimal convergence is obtained in quite a weak sense; but, by interpolation, it also yields strong convergence in, say, L^1 sense if one has very strong (uniform in time) smoothness and decay bounds at one's disposal. Such bounds were established in [**128**], thanks to an inequality which can be seen as

[4]Recall that when the collision kernel is Maxwellian, then the spectrum of the linearized operator can be computed explicitly [**79**, p. 135].

reminiscent of the Povzner inequalities, but from the point of view of smoothness, i.e. with moments in Fourier space, instead of velocity space[5] :

$$\|Q^+(f,f)\|^2_{H^m} \leq \frac{1}{2}\|f\|^2_{H^m} + C_m, \qquad m \in \mathbb{N}. \tag{250}$$

This inequality holds true at least when f is close enough to M^f in relative entropy sense.

After establishing the optimal decay to equilibrium in d_4 distance on one hand, and the uniform smoothness bound in H^m on the other hand, Carlen, Gabetta and Toscani [**128**] had no difficulty in interpolating between both partial results to prove convergence to equilibrium in L^1 at exponential rate. The interpolation can be made at the price of an arbitarily small deterioration in the rate of convergence if m is very large. A precise theorem will be given in paragraph 4.1.

Remark : This theorem of exponential trend to equilibrium with explicit rate is at the moment restricted to Maxwellian collision kernels. This is because only in this case are nice contracting probability metrics known to exist. Also note that the spectral gap of the linearized collision operator is known only for Maxwellian collision kernel. Accordingly, a lower bound on the spectral gap is known only when the collision kernel is bounded below by a Maxwellian collision kernel.

The preceding problem of trend to equilibrium takes its roots on the very influencial 1965 work by McKean [**341**]. In this paper, he studied Kac's equation (21), and at the same time proved exponential convergence to equilibrium, with rate about 0.016, suggested the central limit theorem for Maxwellian molecules[6] and established the decrease of the Fisher information[7]. The value 0.016 should be compared to the optimal rate 0.25, which is obtained, up to an arbitrarily small error, in [**128**]. McKean's results have inspired research in the area until very recently, as the rest of this chapter demonstrates. Another related early work was Grünbaum [**262**].

2.4. The central limit theorem for Maxwell molecules. Once again, let us consider the Boltzmann equation with Maxwellian collision kernel, fix an initial datum f_0 with unit mass, zero mean and unit temperature, and denote by M the corresponding Maxwellian. Recall from section 1 that the solution to the Boltzmann equation with initial datum f_0 can be written as the sum of a Wild series, which is a convex combination of iterated Q^+ operators acting on f_0.

Let us be interested in the behavior of the terms of the Wild series as $t \to +\infty$. It is obvious that the terms of low order have less and less importance as t becomes large, and in the limit the only terms which matter are those which take into account a large number of collisions.

But the action of Q^+ is to decrease the distance to equilibrium; for instance, one has the inequality

$$d_2(Q^+(f,g),M) \leq \max\Big[d_2(f,M), d_2(g,M)\Big],$$

[5]It is not rare in the theory of the spatially homogeneous Boltzmann equation that smoothness estimates and decay estimates bear a formal resemblance, and this may be explained by the fact that the Fourier transform of the spatially homogeneous Boltzmann equation is a kind of Boltzmann equation, see eq. (77)...

[6]See next paragraph.

[7]See paragraph 3.1.

as a variant of the inequalities discussed in paragraph 2.2. Therefore we can expect that terms of high order in the Wild series will be very close to M, and this may be quantified into a statement that f approaches equilibrium as time becomes large.

This however is not true for *all* terms of high order in the Wild series, but only for those terms $Q^+_\gamma(f_0)$ such that the corresponding tree γ is *deep* enough (no leaves of small height). Intuitively, small depth means that all particles involved have collided sufficiently many times. For instance, one would expect[8]

$$Q^+(Q^+(Q^+(f_0,f_0),Q^+(f_0,f_0)),Q^+(Q^+(f_0,f_0),Q^+(f_0,f_0)))$$

to be rather close to M, but not

$$Q^+(Q^+(Q^+(Q^+(Q^+(Q^+(f_0,f_0),f_0),f_0),f_0),f_0),f_0),f_0)$$

(think that even $Q^+(f_0,M)$ is not very close to M...). Thus, to make the argument work, McKean [**341**] had to perform an exercise in combinatorics of trees, and show that the combined weight of "deep enough" trees approaches 1 as time becomes large. These ideas were implemented in a very clean, and more or less optimal way, by Carlen, Carvalho and Gabetta [**126**], thanks to Fourier-defined probability metrics. A precise result will be given in paragraph 4.1.

3. Information theory

3.1. The Fisher information. Among the most important objects in information theory are the Shannon entropy and the Fisher information. Up to a change of sign, Shannon's entropy[9] is nothing but the Boltzmann H-functional. As for the Fisher information, it is defined as

$$I(f) = \int_{\mathbb{R}^N} \frac{|\nabla f|^2}{f} = 4 \int_{\mathbb{R}^N} |\nabla \sqrt{f}|^2 \tag{251}$$

(compare with the relative Fisher information (174)). The Fisher information is always well-defined in $[0,+\infty]$, be it via the L^2 square norm of the distribution $\nabla\sqrt{f}$ or by the convexity of the function $(x,y) \mapsto |x|^2/y$. It is a convex, isotropic functional, lower semi-continuous for weak and strong topologies in distribution sense.

Fisher [**216**] introduced this object as part of his theory of "efficient statistics". The Fisher information measures the localization of a probability distribution function, in the following sense. Let $f(v)$ be a probability density on $\mathbb{R}^N$, and (X_n) a family of independent, identically distributed random variables, with law $f(\cdot - \theta)$, where θ is unknown and should be determined by observation. A *statistic* is a random variable $\widehat{\theta} = \widehat{\theta}(X_1, \ldots, X_n)$, which is intended to give a "best guess" of θ. In particular, $\widehat{\theta}$ should converge towards θ with probability 1 as $n \to \infty$; and also one often imposes (especially when n is not so large) that $\widehat{\theta}$ be unbiased, which means $E\widehat{\theta} = \theta$, independently of n. Now, the Fisher information measures the best possible rate of convergence of $\widehat{\theta}$ towards θ in the sense of mean quadratic error, as $n \to \infty$. More explicitly, if $\widehat{\theta}$ is unbiased, then

$$\text{Var}\,(\widehat{\theta}) \geq \frac{N^2}{nI(f)}. \tag{252}$$

[8] Draw the corresponding trees !

[9] See the references in paragraph 2.4 of chapter 1.

Inequality (252) is called the Cramér-Rao inequality, but for an analyst it is essentially a variant of the Heisenberg inequality, which is not surprising since a high Fisher information denotes a function which is very much "localized"... In fact, the standard Heisenberg inequality in $\mathbb{R}^N$ can be written

$$\forall v_0 \in \mathbb{R}^N, \qquad I(f) \int_{\mathbb{R}^N} f(v)|v - v_0|^2 \, dv \geq N^2 \left(\int_{\mathbb{R}^N} f \, dv \right)^2 .$$

For given mass and energy, the Fisher information takes its minimum value for Maxwellian distributions — just as the entropy. And for given covariance matrix, it takes its minimum value for Gaussian distributions. This makes it plausible that the Fisher information may be used in problems such as the long-time behavior of solutions to the Boltzmann equation, or the central limit theorem.

The idea of an information-theoretical proof of the central limit theorem was first implemented by Linnik [**305**] in a very confuse, but inspiring paper. His ideas were later put in a clean perspective by Barron [**59**] and others, see the references in [**156**]. The same paper by Linnik also inspired McKean [**341**] and led to the introduction of the Fisher information in kinetic theory[10].

3.2. Stam inequalities for the Boltzmann operator. As one of the key remarks made by McKean [**341**], the Fisher information is a Lyapunov functional for the Kac model (21). We already mentioned in paragraph 4.7 of chapter 3 that his argument can be adapted to the 2-dimensional Boltzmann equation. Also Toscani [**421**] gave a direct, different proof of this 2-dimensional result.

A more general result goes via Stam-type inequalities. The famous Blachman-Stam and Shannon-Stam inequalities[11] [**411, 75, 125**] admit as particular cases

$$I(f \star f) \leq I(f), \qquad H(f \star f) \leq H(f).$$

These inequalities are central in information theory [**156, 165**]. Their counterparts for the Boltzmann equation are

$$I\big(Q^+(f,f)\big) \leq I(f), \qquad H\big(Q^+(f,f)\big) \leq H(f), \tag{253}$$

where Q^+ is defined by (235), and the collision kernel has been normalized by (238).

Inequalities (253) immediately entail (by Wild sum representation when (238) holds, by approximation in the general case — or by an ad hoc application of the definition of convexity) that the Fisher information and the H-functional are Lyapunov functionals along the semigroup generated by the Boltzmann equation with Maxwellian collision kernel. In particular, this gives a new proof of the H theorem in this very particular situation. This remark is not so stupid as it may seem, because this proof of the H theorem is robust under time-discretization, and also applies for an explicit Euler scheme[12] — apparently this is the only situation in which the entropy can be shown to be nonincreasing for an explicit Euler scheme.

Inequalities (253) were proven in dimension 2 by Bobylev and Toscani [**83**], and in arbitrary dimension, but for constant collision kernel (in ω-representation), by Carlen and Carvalho [**121**]. Finally, the general case was proven by the author

[10] McKean used the denomination "Linnik functional" for the Fisher information, which is why this terminology was in use in the kinetic community for some time.

[11] Inequalities (196) and (195), respectively.

[12] Also one may dream a little bit and imagine that this remark could end up with new lower bounds for the entropy dissipation !

in [**445**]. Apart from rather classical ingredients, the proof relied on a new representation formula for ∇Q^+ in the Maxwellian case :

$$(254)\quad \nabla Q^+(f,f) = \frac{1}{2}\int dv_*\, d\sigma\, b(k\cdot\sigma)\Big(f'_*(I+P_{\sigma k})(\nabla f)' + f'(I-P_{\sigma k})(\nabla f)'_*\Big),$$

where $k = (v-v_*)/|v-v_*|$, $I_N : \mathbb{R}^N \to \mathbb{R}^N$ stands for the identity map and $P_{\sigma k} : \mathbb{R}^N \to \mathbb{R}^N$ is the linear mapping defined by

$$P_{\sigma k}(x) = (k\cdot\sigma)x + (\sigma\cdot x)k - (k\cdot x)\sigma.$$

Formula (254) was obtained by an integration by parts on S^{N-1}, which crucially used the assumption of Maxwellian collision kernel. In the non-Maxwellian case, we could only obtain an inequality weaker by a factor 2 : $I(Q^{\pm}(f,f)) \leq 2\|A\|_{L^\infty} I(f)$, where $A(z) = \int_{S^{N-1}} B(z,\sigma)\, d\sigma$.

Just as in the well-known Stam proof, the first inequality in (253) implies the second one via adjoint Ornstein-Uhlenbeck regularization. To be explicit, if $(S_t)_{t\geq 0}$ stands for the semigroup associated to the Fokker-Planck equation, and if f has unit mass, zero mean, unit temperature, then

$$H\left(Q^+(f,f)\right) - H(M) = \int_0^{+\infty} \Big[I\left(Q^+(S_t f, S_t f)\right) - I(M)\Big]\, dt.$$

Underlying this formula is of course the commutation between S_t and Q^+, $S_t Q^+(f,f) = Q^+(S_t f, S_t f)$, which follows from Bobylev's lemma[13] for instance.

Remarks :

1. The decreasing property of the Fisher information also holds for solutions of the Landau equation with Maxwell molecules. This can be seen by asymptotics of grazing collisions, but a direct proof is also possible, and gives much better quantitative results [**451**]. In particular, one can prove that the Fisher information converges exponentially fast to its equilibrium value.

2. Still for the Landau equation, this decreasing property was compared with results from numerical simulations by Buet and Cordier [**105**]. In many situations, they observed a decreasing behavior even for non-Maxwell situations, e.g. Coulomb potential. They also suggested that this decreasing phenomenon was associated with entropic properties of the code : enforcing the decrease of the entropy in the numerical scheme would have a stabilizing effect which prevents the Fisher information to fluctuate.

3.3. Consequence : decreasing of the Carlen-Carvalho ψ functional. In [**121**], Carlen and Carvalho introduced the function

$$\psi : \lambda \longmapsto H(f) - H(S_\lambda f)$$

to measure the smoothness of the distribution function f, in the context of entropy-entropy dissipation inequalities. Here $(S_t)_{t\geq 0}$ is as usual the adjoint Ornstein-Uhlenbeck semigroup, i.e. the semigroup generated by the Fokker-Planck operator (161). Note that if the moments of f are normalized by (177), then ψ is a nonnegative function. One of the main points in the Carlen-Carvalho theorems[14] was to obtain a control of ψ for λ close to 0.

[13] See paragraph 4.8 in chapter 1.

[14] Recall paragraph 4.4 in chapter 3.

We now claim that for Maxwell collision kernel, **the function ψ is pointwise nonincreasing** : for each value of λ, $H(f) - H(S_\lambda f)$ is nonincreasing as a function of time. To see this, recall that in the case of Maxwellian collision kernel, the Boltzmann semigroup (B_t) commutes with (S_t), as a consequence of Bobylev's lemma[15]. Therefore, the time-derivative of $H(f) - H(S_\lambda f)$ is

$$D(S_\lambda f) - D(f).$$

To prove that $D(S_\lambda f) - D(f)$ is nonpositive, we just have to prove

$$\frac{d}{d\lambda} D(S_\lambda f) \leq 0,$$

But[16], by commutation property, the dissipation along (S_t), of the dissipation of entropy along (B_t), is also the dissipation along (B_t), of the dissipation of entropy along (S_t). Hence,

$$\left.\frac{d}{d\lambda}\right|_{\lambda=0} D(S_\lambda f) = \left.\frac{d}{dt}\right|_{t=0} I(B_t f) \leq 0.$$

This proves the claim.

4. Conclusions

4.1. Summary. We now summarize most of our discussion about specific properties of the Boltzmann equation with Maxwellian collision kernel in a single theorem. As usual, we set

$$\|f\|_{L^1_s(\mathbb{R}^N_v)} = \int_{\mathbb{R}^N} f(v)(1+|v|^2)^{s/2}\, dv.$$

Theorem 24. *Let $b(\cos\theta)$ be a nonnegative collision kernel, satisfying finiteness of the cross-section for momentum transfer,*

$$\int_0^\pi b(\cos\theta)(1-\cos\theta)\, \sin^{N-2}\theta\, d\theta < +\infty,$$

and let $f_0 \in L^1_2(\mathbb{R}^N_v)$ be an initial datum with finite mass and energy. Without loss of generality, assume that f_0 has unit mass, zero mean velocity and unit temperature. Then,

(i) there exists a unique (weak) solution $(f(t))_{t\geq 0}$ to the spatially homogeneous Boltzmann equation with initial datum f_0;

(ii) the quantities $H(f(t))$, $I(f(t))$, $d_2(f(t), M)$, $W(f(t), M)$ are nonincreasing as functions of t;

(iii) $d_2(f(t), M)$ and $W(f(t), M)$ converge to 0 as $t \to +\infty$, and also $H(f(t)) - H(M)$ if $H(f_0) < +\infty$.

(iv) Assume that the collision kernel satisfies Grad's angular cut-off assumption, i.e.

$$\int_0^\pi b(\cos\theta)\, \sin^{N-2}\theta\, d\theta < +\infty.$$

Let λ be the spectral gap of the linearized Boltzmann operator. Then, for all $\epsilon > 0$, there exists $s > 0$ and $k \in \mathbb{N}$ such that, if $f_0 \in L^1_{2+s} \cap H^k(\mathbb{R}^N)$, then there exists a

[15] See paragraph 4.8 in chapter 1.

[16] We already made this remark in paragraph 4.4 of chapter 3.

constant $C < +\infty$, explicit and depending on f only via $\|f_0\|_{L^1_{2+s}}$ and $\|f_0\|_{H^k}$, such that

$$\forall t \geq 0, \quad \big\|f(t) - M\big\|_{L^1} \leq Ce^{-(\lambda-\epsilon)t}.$$

(v) Assume that the collision kernel satisfies Grad's angular cut-off, and is normalized by (238)*. Consider the Wild representation of $f(t)$, and for any $N_0 \geq 1$ let $f_{N_0}(t)$ be the truncation of the series at order N_0 (take formula* (241) *and throw away all terms starting from the one in $e^{-t}(1-e^{-t})^{N_0}$). Define*

$$g_{N_0}(t) = f_{N_0}(t) + (1-e^{-t})^{N_0} M.$$

Assume that $f_0 \in L^1_{2+\delta} \cap H^{2+\delta}(\mathbb{R}^{N_0})$ for some $\delta > 0$. Then, there exist constants $C < +\infty$ and $\alpha > 0$, depending on f only via $\|f_0\|_{L^1_{2+\delta}}$ and $\|f\|_{H^{2+\delta}}$, such that

$$\forall N_0 \geq 1,\ \forall t \geq 0, \quad \|f(t) - g_{N_0}(t)\|_{L^1} \leq C\frac{(1-e^{-t})^{N_0}}{{N_0}^{\alpha}}.$$

Also, $\|f(t) - M\|_{L^1}$ converges exponentially fast to 0.

Part (i) of this theorem is from Toscani and Villani [**427**]. As for part (ii), the statement about W_2 is due to Tanaka [**415**], the one about d_2 is in Toscani and Villani [**427**], the one about I is from Villani [**445**]. Point (iii) is due to Tanaka for W, the same proof applies for d_2. Actually Tanaka's proof was given only under additional moment assumptions; for the general result one needs tightness of the energy, which is proven in [**225**]. The statement about the entropy is more delicate : in addition to the tightness of the energy, it requires the monotonicity property of the function ψ, as described in paragraph 3.3. These two estimates make it possible to use the main result in Carlen and Carvalho [**121**] and conclude that the relative entropy satisfies a closed differential equation which implies its convergence to 0 at a computable rate. This argument is explicitly written in Carlen, Carvalho and Wennberg [**127**] in the particular case when the collision kernel is constant in ω-representation.

Next, point (iv) is the main result of Carlen, Gabetta and Toscani [**128**], while point (v) is the main result of Carlen, Carvalho and Gabetta [**126**]. Point (v) can be seen as a bound (essentially optimal) on the error which is performed when replacing the solution of the Boltzmann equation by a truncation of the Wild sum. Note that the result of exponential convergence in (v) is much more general than the one in (iv), but the rate of convergence is a priori worse.

We note that the proof of point (iv) uses the last part of (iii). Indeed, the convergence to equilibrium is shown to be exponential only in a certain neighborhood of the equilibrium[17]; to make the constant C explicit it remains to estimate the time needed to enter such a neighborhood, which is what entropy methods are able to do.

In conclusion, one can say that the theory of spatially homogeneous Maxwell molecules is by now essentially complete. The links between information theory and kinetic theory have been completely clarified in the last years, this being due in large part to the contributions by Carlen and coworkers. Among the few questions

[17] In particular because the bound (250) is only proven when f is close enough to M.

still open, we mention the classification of all nontrivial eternal solutions[18] — which certainly can be attacked more efficiently in the Maxwellian case, thanks to the many additional tools available, as demonstrated by the advances made by Bobylev and Cercignani [**81**] — and the problems which are mentioned in the next two paragraphs. Also, it would be extremely interesting to know how point (iv) above generalizes to a spatially inhomogeneous setting, even from the formal point of view, and even assuming on the solutions all the smoothness one can dream of.

4.2. A remark on sub-additivity. An interesting problem is the classification of all Lyapunov functionals for the Boltzmann equation. Recall the following result by McKean[19] [**342**] : for the Kac model, the entropy, or H-functional is, up to addition of an affine function or multiplication by a constant, the only Lyapunov functional of the form $\int A(f)$. McKean also conjectured that the Fisher information would be the only Lyapunov functional of the form $\int A(f, \nabla f)$.

A related problem is to classify all functionals J, say convex and isotropic, which satisfy $J(Q^+) \leq J$, in a way similar to (253), under the normalization (238). Such functionals are particular Lyapunov functionals for the Boltzmann equation (240). In dimension 2 of velocity space, Bobylev and Toscani [**83**] have obtained the following **sufficient condition** : for all probability distributions f and g on $\mathbb{R}^2$, and for all $\lambda \in [0,1]$,

$$J(f_\lambda * g_{1-\lambda}) \leq \lambda J(f) + (1-\lambda) J(g). \tag{255}$$

Whatever the dimension, this criterion is satisfied by all functionals that we have encountered so far : H, I, $d_2(\cdot, M)$, $W(\cdot, M)^2$. However, nobody knows if it is sufficient in dimension higher than 2.

As a consequence of our remarks on the Landau equation with Maxwellian collision kernel [**443**] and the asymptotics of grazing collisions, any Lyapunov functional J has to satisfy (255) in the particular case when f is radially symmetric and g is the Maxwellian distribution with zero mean, and same energy as f.

4.3. Remark : McKean's conjectures. In his seminal 1965 work [**341**], McKean also formulated several conjectures. Even though they all seem to be false, they have triggered interesting developments. Let us mention two of these conjectures.

- **the "super-H theorem"** postulates that the entropy is a completely monotone function of time :

$$dH/dt \leq 0, \quad (d^2H)/(dt^2) \geq 0, \quad \dots \quad (-1)^n (d^nH)/(dt^n) \geq 0.$$

For some time this was a popular subject among a certain group of physicists. This conjecture is however false, as shown by Lieb [**303**] with a very simple argument. Strangely, for the particular Bobylev-Krook-Wu explicit solutions, this "theorem" holds true for $n \leq 101$ and breaks downs afterwards [**361**].

[18] See paragraph 2.9 in chapter 1.

[19] This result is somewhat reminiscent of the axiomatic characterization of entropy by Shannon, see for instance [**156**, pp.42-43] and references therein.

- the "McKean conjecture", strictly speaking. Let M_δ stand for the Maxwellian distribution with zero mean and δ temperature, and consider the formal expansion

$$H(f * M_\delta) = -\sum_{n=0}^{\infty} \frac{I_n(f)}{n!} \left(\frac{\delta}{2}\right)^n , \tag{256}$$

so that $I_0(f) = -H(f)$, $I_1(f) = I(f)$, $I_2(f) = -\sum_{ij} \int f[\partial_{ij}(\log f)]^2$, etc. Knowing that $dI_0/dt \geq 0$ and $dI_1/dt \leq 0$, McKean conjectured the more general inequality $(-1)^n dI_n/dt \geq 0$. This conjecture seems to be false in view of the formal study realized by Ledoux [**295**] for the Fokker-Planck equation.

Keeping in mind that the entropy measures volume in infinite dimension, the successive terms in (256) could be seen as infinite-dimensional analogues of the mixed volumes arising in convex geometry. It is not even clear that they have alternate signs for $n \geq 1$... However, this conjecture has inspired a few works in kinetic theory, see for instance Gabetta [**224**], or the discussion of the Kac model in Toscani and Villani [**428**, section 7]. These ideas have also been used by Lions and Toscani [**323**] to establish certain strengthened variants of the central limit theorem.

CHAPTER 5

OPEN PROBLEMS AND NEW TRENDS

The goal of this chapter is to present some of the main open problems in collisional kinetic theory, then to discuss some of the new questions arising in two developing branches of the field : the study of granular media on one hand, quantum kinetic theory on the other. Other choices could have included semiconductors (whose modelling is very important for industrial applications), modelling of biological interactions (in which problems have not been very clearly identified up to now), the study of aerosols and sprays (which naturally involve the coupling of kinetic equations with fluid mechanics), etc.

Also, we only discuss problems associated with the qualitative behavior of solutions, and do not come back on less traditional issues like those which were presented in paragraph 2.9 of chapter 1.

Selecting "important" problems is always dangerous because of subjectivity of the matter, and changes in mathematical trends and fashions. To illustrate this, let us quote Kac himself [**283**, p.178, footnote 5] : "Since the master equation[1] is truly descriptive of the physical situation, and since existence and uniqueness of the solutions of the master equation are almost trivial, the preoccupation with existence and uniqueness theorems for the Boltzmann equation appears to be unjustified on grounds of physical interest and importance."

[1]See paragraph 1.5 below.

Contents

1. Open problems in classical collisional kinetic theory

1.1. Strong solutions in a spatially inhomogeneous setting. The theory of the Cauchy problem for Boltzmann-like equations is by now fairly advanced under the assumption of spatial homogeneity. For instance, in the case of hard potentials, it seems reasonable to expect that this theory will soon be completed with the help of already existing tools. On the other hand, essentially nothing is known concerning the general, spatially inhomogeneous case in a non-perturbative context. Progress is badly needed on the following issues :

- moment estimates,
- regularity estimates (propagation of regularity/singularity, regularization),
- strict positivity and lower bounds.

This lack of a priori estimates is a limiting factor in many branches of the field. A priori estimates would enable one to

- prove uniqueness of solutions, and energy conservation;
- perform a simple treatment of boundary conditions (walls, etc.);
- give estimates of speed of convergence to equilibrium along the lines presented in chapter 3 (for this one needs uniform estimates as times goes to $+\infty$).
- justify the linearization procedure which is at the basis of so many practical applications of kinetic theory, see for instance [**148**].

Such a priori estimates also would be very useful, even if not conclusive, to

- prove the validity of the Landau approximation in plasma physics, viewed as a large-time correction to the Vlasov-Poisson equation;
- prove the validity of the fluid approximation to the Boltzmann equation. For this, local conservation laws seem to be the minimum one can ask for in order to prove the hydrodynamic limit[2].

Even in the more modest framework of solutions in the small, where smooth solutions can often be built, many gaps remain, like the treatment of singular collision kernels, or the derivation of uniform smoothness estimates as $t \to +\infty$.

1.2. Derivation issues. As already mentioned, Lanford's theorem is limited to a perturbative framework (small solutions), and concerns only the hard-sphere interaction. The treatment of more general interactions is almost completely open, and would be of considerable interest. As an oustanding problem in the field is of course the formidable task of rigorously deriving collisional kinetic equations for Coulomb interaction.

As for the problem of extending Lanford's theorem to a nonperturbative setting, one of the main difficulties is certainly related to the fact that there is no good theory for the Cauchy problem in the large — but we expect much, much more obstacles to overcome here !

[2]This is what the author believed, until very recently some proofs of hydrodynamic limit appeared [**54, 53, 240, 245**], covering situations in which local conservation laws are not known to hold for fixed Knudsen number, but are asymptotically recovered in the limit when the Knudsen number goes to 0 !!

1.3. Role of the kinetic singularity. Let us consider a Boltzmann collision kernel, say of the form

$$B(v - v_*, \sigma) = \Phi(|v - v_*|)\, b(\cos\theta).$$

In chapter 2 we have seen how the properties of the Boltzmann equation depend on whether b is integrable or singular. On the other hand, what remains unclear even in the spatially homogeneous case, is the influence of the kinetic collision kernel Φ. When Φ is singular, does it induce blow-up effects, and in which sense ? Does it help or harm regularizing effects induced by an angular singularity ?

The most important motivation for this problem comes from the modelling of Coulomb collisions in plasma physics : the collision kernel given by the Rutherford formula presents a singularity like $|v - v_*|^{-3}$ in dimension $N = 3$. Here are two questions which arise naturally.

1) Consider the Boltzmann equation with truncated Rutherford collision kernel, of the form $|v - v_*|^{-3} b(\cos\theta) 1_{\theta \geq \varepsilon}$, which is physically unrealistic but used in certain modelling papers [**162**]. This collision kernel presents a nonintegrable (borderline) kinetic singularity. Does it entail that the equation induces smoothness, or even just compactifying effects ? Some formal arguments given in Alexandre and Villani [**12**] may support a positive answer, but the situation seems very intricate. As explained in [**12**, section 5], the geometry of the problem is "dual", in some sense, to the one which appears in the study of nonintegrable, borderline angular singularities... If the answer is negative, this suggests that such collision kernels should be used with a lot of care !

2) On the other hand, consider the Landau approximation for Coulomb collisions, or Landau-Coulomb equation, which is more realistic from the physical point of view. Does this equation have smooth solutions ?

In the spatially homogeneous situation, the Landau-Coulomb equation can be rewritten as

$$\frac{\partial f}{\partial t} = \sum_{ij} \overline{a}_{ij} \frac{\partial^2 f}{\partial v_i \partial v_j} + 8\pi f^2, \qquad t \geq 0,\ v \in \mathbb{R}^3 \tag{257}$$

where

$$\overline{a}_{ij} = \frac{1}{|v|} \left[\delta_{ij} - \frac{v_i v_j}{|v|^2} \right] * f.$$

If f is smooth, then the matrix $(\overline{a}_{ij})$ is locally positive definite, but bounded, and (257) is reminiscent of the nonlinear heat equation

$$\frac{\partial f}{\partial t} = \Delta f + f^2 \tag{258}$$

which has been the object of a lot of studies [**468**] and generically blows up in finite time, say in L^∞ norm. The common view about (258) is that the diffusive effects of the Laplace operator are too weak to compensate for blow-up effects induced by the quadratic source term. And if the diffusion matrix $(\overline{a}_{ij})$ is bounded, this suggests that (257) is no more diffusive than (258).

Weak solutions to (257) have been built in Villani [**446**]; they satisfy the a priori estimate $\sqrt{f} \in L^2_t(H^1_v)$ locally. This estimate is however, to the best of the knowledge of the author, compatible with known a priori estimates for (258).

These considerations may suggest that blow-up in finite time may occur for solutions of (257). If there is blow-up, then other questions will arise : how good is the Landau approximation at a blow-up time ? What happens to blow-up if the physical scales are such that the Landau effects should only be felt as $t \to +\infty$?

However, blow-up has never been reported by numerical analysts. And after seeing some numerical simulations by F. Filbet, the author has changed his mind on the subject, to become convinced that blow-up should indeed not occur. All this calls for a wide clarification.

1.4. Improved entropy-entropy dissipation estimates. In this paragraph we only consider very nice distribution functions, say smooth and rapidly decaying, bounded below by a fixed Maxwellian. We saw in chapter 3 that such probability distributions satisfy entropy-entropy dissipation inequalities of the form

$$D(f) \geq K H(f|M^f)^\alpha,$$

where M^f is the Maxwellian equilibrium associated with f, K and α are positive constants, H is the relative entropy functional, and D is the entropy dissipation for either Boltzmann or Landau's equation. In several places do our results call for improvement :

▸ **Landau equation with hard potentials :** In the case of the Landau equation,

$\alpha = 1$ is admissible when $\Psi(|z|) \geq |z|^2$;
$\alpha = 1 + \varepsilon$ (ε arbitrarily small) is admissible when $\Psi(|z|) \geq |z|^{2+\gamma}$, $\gamma < 0$.

It is natural to conjecture that also $\alpha = 1$ be admissible for hard potentials ($\gamma > 0$). More generally, this should be true when $\Psi(|z|) = |z|^2 \psi(|z|)$ with ψ continuous and uniformly positive for $|z| \geq \delta > 0$. This conjecture is backed by the spectral analysis of the linearized Landau operator [**161**], and also by the similar situation appearing in Carrillo, McCann and Villani [**130**] in the study of entropy-entropy dissipation inequalities for variants of granular media models. At the moment, the best available exponent for hard potentials is $\alpha = 1 + 2/\gamma$, from Desvillettes and Villani [**183**].

▸ **Boltzmann equation with hard potentials :** In the case of the Boltzmann equation, $\alpha = 1 + \varepsilon$ is admissible for Maxwellian or soft potentials. It is accordingly natural to think that $\alpha = 1 + \varepsilon$ is also admissible[3] for hard potentials. Recall that Cercignani's conjecture ($\alpha = 1$) is false in most cases according to Bobylev and Cercignani [**87**].

▸ **Cercignani's conjecture revisited ?** Counterexamples in [**87**] leave room for Cercignani's conjecture to hold true in two situations of interest :

- when the collision kernel is non cut-off and presents an angular singularity. This would be plausible since grazing collisions behave better with respect to large velocities, as the example of the Landau equation demonstrates[4];
- when $f \in L^p((M^f)^{-1})$ for some $p \geq 1$. Of special interest are the cases $p = 1$ (Cf. Bobylev's estimate for hard spheres, in Theorem 1 (ii)); $p = 2$ (natural space for linearization) and $p = \infty$ (when f/M^f is bounded from

[3] As this review goes to print, the author just managed to prove precisely this result, under the assumption that the density be bounded in all Sobolev spaces.

[4] Similar results in the theory of linear Markov processes would also be interesting.

above). Maybe a Maxwellian bound from below is also needed for proving such theorems.

As we mentioned when discussing Cercignani's conjecture in chapter 3, about this topic one also has to make the connection with the recent Ball-Barthe result about the central limit theorem.

1.5. Approach to equilibrium for Kac's master equation. A related topic is the **Kac spectral gap problem** and its **entropy dissipation variant**. This subject is a little bit in digression with respect to those which we discussed so far, but we wish to explain it briefly because of its intimate (and not well-known) connections with Cercignani's conjecture. These connections were brought to our attention by E. Carlen.

In his famous paper [**283**], Kac introduced a stochastic model which he believed to be a way of understanding the spatially homogeneous Boltzmann equation. His equation models the behavior of n particles interacting through binary elastic collisions occurring at random Poissonnian times, with collision parameter σ randomly chosen on the sphere. It reads

$$\frac{\partial f_n}{\partial t} = \frac{1}{\binom{n}{2}} \sum_{i<j} \fint_{S^{N-1}} d\sigma\, B(v_i - v_j, \sigma) \big[\mathcal{A}_\sigma^{ij} f_n - f_n\big], \tag{259}$$

where the summation runs over all pairs of distinct indices (i, j) in $\{1, \dots, n\}$, and f_n is a symmetric probability distribution on the manifold (actually a sphere) of codimension $N + 1$ in $(\mathbb{R}^N)^n$ defined by the relations

$$\sum_{i=1}^n |v_i|^2 = 2nE > 0, \qquad \sum_{i=1}^n v_i = nV \in \mathbb{R}^N.$$

We use the notation $\fint$ for the normalized integral on the sphere, $|S^{N-1}|^{-1} \int$. Moreover the linear operator $\mathcal{A}_\sigma^{ij}$ represents the result of the collision of the spheres with indices i and j,

$$\mathcal{A}_\sigma^{ij} f_n(v_1, \dots, v_n) = f(v_1, \dots, v_i', \dots, v_j', \dots, v_n),$$

$$\begin{cases} v_i' = \dfrac{v_i + v_j}{2} + \dfrac{|v_i - v_j|}{2}\sigma, \\ \\ v_j' = \dfrac{v_i + v_j}{2} - \dfrac{|v_i - v_j|}{2}\sigma. \end{cases}$$

As explained by Kac, the spatially homogeneous Boltzmann equation can be recovered, at least formally, as the equation governing the evolution of the one-particle marginal of f_n in the limit $n \to +\infty$. In this limit, time has to be sped up by a factor n. See [**283, 412, 256**] for a study of this and related subjects.

A simplified version, which is commonly called **Kac's master equation**, is given by

$$\frac{\partial f_n}{\partial t} = L_n f_n = \frac{1}{\binom{n}{2}} \sum_{i<j} \fint_0^{2\pi} d\theta \left[f_n \circ R_\theta^{ij} - f_n\right], \tag{260}$$

where f_n is a probability distribution on the sphere in $\mathbb{R}^n$, defined by

$$\sum v_i^2 = 2nE. \tag{261}$$

Moreover,

$$R_\theta^{ij} v = (v_1, \dots, v_i', \dots, v_j', \dots, v_n),$$

where

$$(v_i', v_j') = R_\theta(v_i, v_j)$$

is obtained from (v_i, v_j) by a rotation of angle θ in the (i, j) plane. Without loss of generality, we set $E = 1/2$ in (261), so that the sphere has radius $\sqrt{n}$. With this choice, the image measure of the uniform probability measure on the sphere, under projection onto some axis of coordinate, becomes the standard Gaussian measure as $n \to \infty$ (Poincaré's lemma, actually due to Maxwell). Moreover, we shall use the uniform probability measure on the sphere as reference measure, so that probability distributions are normalized by

$$\fint_{\sqrt{n}S^{n-1}} f_n \, d\sigma = 1.$$

Among the problems discussed by Kac is that of establishing an asymptotically sharp lower bound on the spectral gap λ_n of L_n as $n \to +\infty$. Recently, Diaconis and Saloff-Coste [**189**] proved $\lambda_n^{-1} = O(n^3)$, then Janvresse [**281**] proved Kac's conjecture that $\lambda_n^{-1} = O(n)$; she used Yau's so-called martingale method. Finally, a complete solution was given very recently by Carlen, Carvalho and Loss [**123**], who managed to *compute* the spectral gap by a quite unexpected method (also by induction on the dimension). This work also extends to eq. (259) if the collision kernel B is Maxwellian.

Since time should be sped up by a factor n in the limit $n \to \infty$, the corresponding evolution equation will satisfy estimates like

$$\|f_n(t, \cdot) - 1\|_{L^2(\sqrt{n}S^{n-1})} \leq e^{-\lambda t} \|f_n(0, \cdot) - 1\|_{L^2(\sqrt{n}S^{n-1})} \tag{262}$$

for some $\lambda > 0$, which can be chosen uniform as $n \to \infty$ according to Janvresse's theorem. Here 1 is the equilibrium state, i.e. the density of the uniform probability measure on the sphere $\sqrt{n}S^{n-1}$. Inequality (262) conveys a feeling of uniform trend to equilibrium as $n \to \infty$, which was Kac's goal.

However, it is not very clear in which sense (262) is a uniform estimate. Since all the functions f_n's are defined on different spaces, one should be careful in comparing them. In particular, think that if f_n satisfies the chaos property, then $\|f_n\|_{L^2}$ is roughly of order C^n for some constant $C > 0$ (which in general is not related to the L^2 (or $L^2(M^{-1})$) norm of the limit one-particle marginal f, see [**283**, eq. (6.44)]). And $\|f_n - 1\|_{L^2}^2 = \|f_n\|_{L^2}^2 - 1$ is also of order C^n. Having this in mind, it would be natural to compare distances in dimension n by the quantity $\|\cdot\|_{L^2}^{1/n}$. But if we do so, we find

$$\|f_n(t, \cdot) - 1\|_{L^2(S^{n-1})}^{1/n} \leq e^{-\frac{\lambda}{n} t} \|f_n(0, \cdot) - 1\|_{L^2(S^{n-1})}^{1/n},$$

which does not behave well in the limit[5] !

[5]Or should the relevant scaling be $f_n \simeq (1 + h/\sqrt{n})^{\otimes n}$, which would mean that we are interested in fluctuations of the equilibrium state ? This ansatz formally leads to $\|f_n - 1\|_{L^2}^2 = O(n)$...

A way to circumvent the difficulty would be to compare all first marginals, which all live in $L^1(\mathbb{R})$, and prove that under some precise conditions on the sequence (f_n),

$$\exists \lambda > 0,\ \forall n \geq 1, \qquad \|P_1 f_n(t,\cdot) - M\|_{L^2(M^{-1})} \leq C e^{-\lambda t}.$$

Now, a problem which looks more natural and more interesting in this context is the problem of the *entropy-entropy dissipation estimate* for Kac's master equation. Again, we state this problem assuming without loss of generality that $E = 1/2$, so that $\sum v_i^2 = n$ in (261) and we use the uniform probability measure as reference measure for the definition of the entropy :

$$H(f_n) = \fint_{\sqrt{n}S^{n-1}} f_n \log f_n \, d\sigma.$$

Note that $H(1) = 0$.

Problem : *Find K_n optimal such that for all symmetric probability distribution f_n on $\sqrt{n}S^{n-1}$,*

$$-\int (L_n f_n) \log f_n \geq K_n H(f_n). \tag{263}$$

If $K_n^{-1} = O(n)$, then (263) entails the following entropy estimate for solutions of the Kac equation :

$$H(f_n(t,\cdot)) - H(1) \leq e^{-\mu t}\big[H(f_n(0,\cdot)) - H(1)\big],$$

for some $\mu > 0$. Since $H(f_n)$ typically is $O(n)$, this would lead to the satisfactory estimate

$$\frac{H(f_n(t,\cdot)) - H(1)}{n} \leq e^{-\mu t} \left[\frac{H(f_n(0,\cdot)) - H(1)}{n}\right], \tag{264}$$

and also one-particle marginals of all f_n's could be compared easily as a consequence an adequate chaos assumption for $f_n(0,\cdot)$.

But from the counterexamples due to Bobylev and Cercignani [**87**], one expects that $K_n^{-1} = O(n)$ is impossible. Indeed, by passing to the limit as $n \to \infty$ in (263), under a chaos assumption, one would have a proof of Cercignani's entropy dissipation conjecture for Kac's model, which should be false (although this has never been checked explicitly)... On the other hand, the author [**450**, Annex III, Appendix B] was able to prove $K_n^{-1} = O(n^2)$ by the same method as in paragraph 4.6 of chapter 3. This leads to two open questions :

- What is the optimal estimate ?
- Does an estimate like $O(n)$ hold for a well-chosen sub-class of probability distributions ?

1.6. Influence of the space variable on the equilibration rate. Let us now consider trend to equilibrium in a spatially inhomogeneous context.

▸ **Diffusive models.** The strategy of Desvillettes and Villani, exposed in Chapter 3, shows trend to equilibrium like $O(t^{-\infty})$ for many entropy-dissipating systems when good smoothness a priori estimates are known. However, an *exponential* rate of convergence would be expected, at least for the linear Fokker-Planck, or Landau equation. The following issues would be of particular interest :

- admitting that the solution of the linear Fokker-Planck equation with confinement potential V goes to equilibrium in relative entropy like $O(e^{-\alpha t})$, what is the optimal value of α and how does it depend on V ?[6] Can one obtain this result by an entropy method ?
- admitting that the solution of the Landau equation in a box (periodic, or with appropriate boundary condition) goes to equilibrium in relative entropy like $O(e^{-\alpha t})$, what is the optimal value of α and how does it depend on the boundary condition or the size of the box ? Can one devise an entropy method to obtain exponential decay ?

▸ **Boltzmann equation.** When Boltzmann's equation with a Maxwellian collision kernel is considered, then one can compute the spectral gap of the linearized operator. In the spatially homogeneous case, as we have seen in chapter 4, this spectral gap essentially governs the rate of decay to equilibrium, even in a non-linearized setting. At the moment, entropy methods seem unable to predict such a result[7], but a clever use of contracting probability metrics saves the game. Now, how does all this adapt to a spatially inhomogeneous context and how is the rate of convergence affected by the box, boundary conditions, etc. ? Of course, in a preliminary investigation one could take for granted all the a priori bounds that one may imagine : smoothness, decay, positivity...

This concludes our survey of open problems for the classical theory of the Boltzmann equation. As the reader has seen, even if the field is about seventy years old, a lot remains to be done ! Now we shall turn to many other problems, arising in less classical contexts which have become the object of extensive studies only recently.

2. Granular media

Over the last years, due to industrial application and to the evolution of the trends in theoretical physics, a lot of attention was given to the modelling of granular material (sand, powders, heaps of cereals, grains, molecules, snow, or even asteroids...). The literature on the subject has grown so fast that some journals are now entirely devoted to it ! And also the number of involved physicists has become extremely large. Among the main motivations are the understanding of how granular material behaves under shaking, how flows are evolving or how to prevent them, how to facilitate mixing, how to prevent violent blow-up of a silo, or avalanches, how matter aggregates in a newborn solar system, etc.

One popular model for these studies is a kinetic description of a system of particles interacting like hard spheres, but with some *energy loss*: the collisions are said to be *inelastic*. This may be due to friction (because the surface of these particles is often rough) or to other causes – just think of a ball falling on the floor, which usually bounces back to a smaller height. Many studies have been based on variants of the Boltzmann or Enskog equations which allow energy loss. This subject leads to huge difficulties in the modelling; see for instance the nice review done by Cercignani [**146**] a few years ago. Also some mathematical contributions have started to develop, most notably the works by Pulvirenti and collaborators [**70, 71, 72, 68, 69, 395**]. Here we point out some of the most fundamental mathematical issues in the field. Since

[6] On this problem see the recent progress by Hérau and Nier.

[7] Because of the obstruction to Cercignani's conjecture...

the physical literature is considerable, we only give a very restricted choice of physicists' contributions. Thanks are due to E. Caglioti for explaining us a lot about the subject and providing references.

2.1. Derivation issues : problems of separation of scales. The separation between microscopic and macroscopic scales in the study of granular media is not at all so clear as in the classical situation, and this results in many problems when it comes to derivation of the relevant equations.

To illustrate this, we mention the astonishing numerical experiments described in [**237**] : a gas of inelastic particles is enclosed in a one-dimensional box with specular reflection (elastic wall) on one end, Maxwellian re-emission (heating wall) on the other. It is found that, basically, just *one* particle keeps all the energy, while all the others remain slow and stay close to the elastic wall. In other words, the wall is unable to heat the gas, and moreover it is plainly impossible to define meaningful macroscopic quantities !

► **Enskog equation.** At the basis of the derivation of the Boltzmann equation, be it formal or more rigorous, is the localization of collisions : the length scale for interaction is much smaller than the length scale for spatial fluctuations of density. The Boltzmann-Grad limit $n \to \infty$, $r \to 0$, $nr^2 \to 1$ (n = number of particles, r = radius of particles, dimension = 3) is a way to formalize this for a gas of elastic hard spheres.

On the other hand, in the case of granular media, the size of particles is generally not negligible in front of the typical spatial length. This is why many researchers use an Enslog-like equation, with *delocalized collisions* : for instance,

(265)
$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = r^2 \int_{\mathbb{R}^3} dv_* \int_{S^2} d\sigma \, |v - v_*| \Big[J\, G(x, x + r\sigma; \rho) f(t, x, \tilde{v}) f(t, x - r\sigma, \tilde{v}_*) - G(x, x - r\sigma; \rho) f(t, x, v) f(t, x - r\sigma, v_*) \Big].$$

(See Cercignani [**146**] and Bobylev et al. [**86**]). Here $r > 0$ is the radius of particles, v, v_* are *post*-collisional velocities and $\tilde{v}, \tilde{v}_*$ are *pre*-collisional velocities, given by the formulas

(266)
$$\begin{cases} \tilde{v} = \dfrac{v + v_*}{2} - \dfrac{1 - e}{4e}(v - v_*) + \dfrac{1 + e}{4e}|v - v_*|\sigma, \\ \\ \tilde{v}_* = \dfrac{v + v_*}{2} + \dfrac{1 - e}{4e}(v - v_*) - \dfrac{1 + e}{4e}|v - v_*|\sigma. \end{cases}$$

Moreover e is an **inelasticity parameter** : when $e = 1$ the preceding equations are the usual equations of elastic collisions, while the case $e = 0$ correspond to sticky particles. In general e may depend on $|v - v_*|$ but we shall take it constant to simplify. Finally, to complete the explanation of (265), J is the Jacobian associated to the transformation (266),

(267)
$$J = \frac{1}{e^2} \frac{|v - v_*|}{|\tilde{v} - \tilde{v}_*|},$$

and G is the famous but rather mysterious *correlation function* which appears in the Enskog equation. Roughly speaking, G relates the 2-particle probability density

with the 1-particle probability density, as follows :

$$f_2(t,x,v,y,w) = G\big(x,y;\rho(t,x),\rho(t,y)\big) f(t,x,v) f(t,y,w),$$

where $\rho(t,x) = \int f\,dv$. In the Boltzmann case this term did not appear because of the *chaos* assumption... Exactly which function G should be used is not clear and a little bit controversial : see Cercignani [**146**] for a discussion.

If the reader finds the complexity of (265) rather frightening, we should add that we did not take into account variables of internal rotation, which may possibly be important in some situations since the particles are not perfectly spherical; see [**146**] for the corresponding modifications.

There are no clear justifications for eq. (265), even at a formal level. In the limit of rarefied gases with a chaos assumption, or in the spatially homogeneous case, one formally recovers an inelastic Boltzmann equation, which is more simple : the collision operator just reads, with obvious notations,

$$Q_e(f,f) = \int_{\mathbb{R}^3} dv_* \int_{S^2} d\sigma\, |v-v_*| \big(J\tilde f \tilde f_* - f f_*\big). \tag{268}$$

This collision operator also has a nice weak formulation,

$$\int Q_e(f,f)\varphi = \int_{\mathbb{R}^3\times\mathbb{R}^3\times S^2} |v-v_*|\, f f_* [\varphi' - \varphi]\, dv\, dv_*\, d\sigma, \tag{269}$$

where the post-collisional velocities v', v'_* (with respect to v, v_* taken as pre-collisional velocities) are

$$\begin{cases} v' = \dfrac{v+v_*}{2} + \dfrac{1-e}{4}(v-v_*) + \dfrac{1+e}{4}|v-v_*|\sigma, \\[2ex] v'_* = \dfrac{v+v_*}{2} - \dfrac{1-e}{4}(v-v_*) - \dfrac{1+e}{4}|v-v_*|\sigma. \end{cases} \tag{270}$$

As in the elastic case, formula (269) can be symmetrized once more by exchange of v and v_*. Note that $\tilde v, \tilde v_*$ do not coincide with v', v'_* because collisions are not reversible. Also note that

$$\int Q_e(f,f)\, dv = 0, \qquad \int Q_e(f,f)\, v\, dv = 0,$$

but

$$\int Q_e(f,f)|v|^2\, dv \le 0.$$

Now let us enumerate some more models. Starting from (268), many variants can be obtained by

- reducing the dimension of phase space by considering only 2-dimensional, or even 1-dimensional models;
- adding a little bit of diffusion, which is presumably realistic in most situations (heat bath, shaking....);
- adding some drift term $\theta \nabla_v \cdot (fv)$, to model some linear friction acting on the system;

- change the "hard-sphere-like" collision kernel $|v - v_*|$ for $|v - v_*|^\gamma$ with, say $-1 \leq \gamma \leq 1$. This provides an equation with some inelastic features, and which may be easier to study... The dimensional homogeneity of the equation can be preserved by multiplying the collision operator by a suitable power of the temperature. For instance, Bobylev, Carrillo and Gamba [**86**] have performed a very detailed study of the case $\gamma = 0$ ("pseudo-Maxwellian collision kernel") along the general lines of the theory developed by Bobylev for elastic Maxwellian collisions;
- only retain grazing collisions by an asymptotic procedure similar to the Landau approximation[8] [**424, 426**]. Actually, some physicists mention that grazing collisions do occur very frequently in granular material [**236**]... The operator which pops out of this limit procedure is of the form

$$Q_L(f, f) + \nabla_v \cdot \big[f \nabla_v (f * U) \big],$$

where $U(z)$ is proportional to $|z|^3$ (or more generally to $|z|^{\gamma+2}$), and Q_L is an elastic Landau operator with $\Psi(|z|)$ proportional to $|z|^3$ (or $|z|^{\gamma+2}$). In particular, in dimension 1 of phase space, this elastic Landau operator vanishes and the resulting collision operator is just a nonlinear friction operator

$$\nabla_v \cdot \Big[f \nabla_v (f * W) \Big].$$

The resulting evolution equation is

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \nabla_v \cdot \Big[f \nabla_v (f * W) \Big], \tag{271}$$

with $U(z) = |z|^3$. The very same equation can also be obtained as the mean-field limit of a one-dimensional system of particles colliding inelastically, as first suggested (so it seems) by McNamara and Young [**343**], and rigorously proven by Benedetto, Caglioti and Pulvirenti [**70**]. Note that (271) is a Boltzmann-like, not an Enskog-like equation !

Of course, we have only presented some of the most mathematically oriented models. In the physical literature one encounters dozens of other equations, derived from physical or phenomenological principles, which we do not try to review.

► **Hydrodynamics.** Another topic where the separation of scales is problematic is the hydrodynamic limit. This subject is important for practical applications, but nobody really knows, *even at a formal level*, what hydrodynamic equations should be used. Due to the possibility of long-range correlations, there generally seems to be no clear separation of scales between the kinetic and hydrodynamic regimes [**236**]. The Chapman-Enskog expansion works terribly bad : each term of the series should be of order 1 ! Resummation methods have been tried to get some meaningful fluid equations [**236**]. From the mathematical viewpoint this procedure is rather esoteric, which suggests to look for alternative methods...

To add to the confusion, due to the tendency of granular media to cluster, it is not clear what should be considered as local equilibrium ! See the discussion about Homogeneous Cooling States in paragraph 2.4 below.

Also, to be honest we should add that even in the most favorable situation, i.e. a simplified model like (271), with some additional diffusion term to prevent clustering

[8] See paragraph 2.7 in chapter 1.

as much as possible, and under the assumption of separation of scales, then limit hydrodynamic equations can be written formally [**69**], but rigorous justification is also an open problem, mainly due to the absence of Lyapunov functional...

2.2. Spatial inhomogeneities. In the classical, elastic theory of the Boltzmann equation, spatial homogeneity is a mathematical ad hoc assumption. However, it is not so unrealistic from the physical viewpoint in the sense that it is supposed to be a stable property : weakly inhomogeneous initial data should lead to weakly inhomogeneous solutions (recall [**32**]).

On the other hand, in the case of granular media, some physicists think that severe inhomogeneities may develop from weakly inhomogeneous states, particularly because of the possibility of collapse by loss of energy [**198**]. Also, numerical experiments seem to indicate the unstability of a homogeneous description (see the references and comments in [**70**, section 4]). Besides shedding more doubts on mathematical studies based on the assumption of spatial homogeneity, these remarks raise a very interesting mathematical challenge, namely prove that weak inhomogeneity may break down in finite time for some realistic initial configurations. It is not very clear whether this study should be performed with a Boltzmann-like, or an Enskog-like equation...

Some related considerations about blow-up : first of all, there are initial configurations of n inelastic particles which lead to collapse in finite time, and they are not exceptional (some nice examples are due to Benedetto and Caglioti [**67**]). Next, for the inelastic Boltzmann equation, there is no entropy functional to prevent dramatic collapse, as in the elastic case. Even the DiPerna-Lions theory cannot be adapted to inelastic Boltzmann equations, so the Cauchy problem in the large is completely open in this case !

We further note that Benedetto, Caglioti and Pulvirenti prove that there is no blow-up for equation (271) when $W(z) = \lambda|z|^3$ with λ very small; in this case eq. (271) can be treated as a perturbation of vacuum. Of course it is precisely when λ is of order 1 that one could expect blow-up effects. Also, Benedetto and Pulvirenti [**73**] study the one-dimensional Boltzmann equation for a gas of inelastic particles with a velocity-dependent inelasticity parameter $e = e(|v - v_*|)$ behaving like $(1 + a|v - v_*|^\gamma)^{-1}$ for some $a, \gamma > 0$, and they show, by an adaptation of the one-dimensional techniques of Bony [**95**], that blow-up does not occur in that situation.

2.3. Trend to equilibrium. For the moment, and in spite of our remarks in the previous paragraph, we restrict to the spatially homogeneous setting (or say that we are only interested in trend to local equilibrium, in some loose sense). We consider two cases :

1) the inelastic Boltzmann collision operator alone; then, because of energy loss, the equilibrium is a Dirac mass at some mean velocity.

2) the inelastic Boltzmann collision operator together with some diffusion. Then there is a nontrivial, smooth stationary state (see Gamba, Panferov and Villani [**228**]). It is not explicit but some qualitative features can be studied about smoothness, tails, etc. The same topic has also been recently studied by Carrillo and Illner in the case of the pseudo-Maxwellian collision kernel.

In both cases entropy methods do not seem to apply because no relevant Lyapunov functional has been identified (apart from the energy). What is worse, in the

second case, even uniqueness of the stationary state is an open problem. The study of trend to equilibrium is therefore open.

However, such a study was successfully performed for two simplified models :

- in the pseudo-Maxwellian variant of the inelastic Boltzmann operator, with or without diffusion [**86, 80**]. Then the behavior of all moments can be computed, and trend to equilibrium can be studied from the relaxation of moments;

- in the simplified model considered by Benedetto, Caglioti and Pulvirenti [**70**], or its diffusive variant. In this case, the appearance of a new Lyapunov functional, with an interaction energy

$$\frac{1}{2}\int_{\mathbb{R}\times\mathbb{R}} f(v)f(w)\frac{|v-w|^3}{3}\,dv\,dw$$

enables a study of rate of convergence by entropy dissipation methods[9].

In the spatially inhomogeneous case, the situation is still worse. In the non-diffusive case, all states of the form $\rho_\infty(x)\delta_0(v)$, for instance, are global equilibria. But are there some preferred profiles ρ_∞ ? Even for the simplified equation (271) or its diffusive variants, the Lyapunov functional which worked fine in the spatially homogeneous context, now fails to have any particular behavior. At a more technical level, the method of Desvillettes and Villani [**184**] cannot be applied because of the non-smoothness of the equilibrium distribution. On the whole, trend to equilibrium for granular media is a really challenging problem.

2.4. Homogeneous Cooling States. For most physicists, a Dirac mass is not a relevant steady state, and the role played in the classical theory by Maxwellian distributions should rather be held by particular solutions which "attract" all other solutions[10]. They often agree to look for these particular solutions in the self-similar form

$$f_S(t,v) = \frac{1}{\alpha^N(t)}F\left(\frac{v-v_0}{\alpha(t)}\right) \qquad t\geq 0,\ v\in\mathbb{R}^N. \tag{272}$$

In the sequel, we set $v_0 = 0$, i.e. restrict the discussion to centered probability distributions.

A solution of the spatially homogeneous inelastic Boltzmann equation which takes the form (272) is called a Homogeneous Cooling State (HCS). Though the existence of HCS is often taken for granted, it is in general a considerable act of faith. Sometimes HCS are considered under some scaling where also the elasticity coefficient e goes to 1 as $t\to+\infty$... For physical studies of these questions one can consult [**104, 238**].

For the pseudo-Maxwellian variant of the inelastic Boltzmann operator, Bobylev, Carrillo and Gamba [**86**] have constructed self-similar distribution functions which capture the behavior of all moments of all solutions (having finite moments of all orders), but, except for some particular values of the restitution coefficient e these distribution functions are *not* nonnegative ! In this class of functions, HCS exist only in the following weakened sense : for any integer n_0, there exists a self-similar solution

[9] See the discussion in paragraph 6.3 of chapter 3.

[10] By the way, in the classical setting it was once conjectured that the Bobylev-Krook-Wu explicit solutions would attract all solutions of the spatially homogeneous Boltzmann equation with Maxwellian molecules. But this has been shown to be false... except, in some sense, in the unphysical regime of negative times [**81**] !

of the inelastic Boltzmann equation which gives the right behavior for all moments of order $\leq n_0$ of all solutions. Thus, for some time it was believed that HCS did not exist for this model. But the picture is more subtle! In fact, as was suggested by Ernst, it is possible to construct nonnegative, self-similar solutions with only *slow decay* at infinity (inverse polynomial). Both the nonrealistic solutions with all moments finite, and the realistic solutions on the other hand, constitute two distinct one-parameter families within a two-parameter family of self-similar solutions. For exceptional values of the restitution coefficient, these families cross each other, and precisely for those values, the Bobylev-Carrillo-Gamba solutions cease to have fast decay.

For model (271), in a spatially homogeneous setting, i.e.

$$\frac{\partial f}{\partial t} = \nabla_v \cdot \Big[f \nabla_v (f * U) \Big], \tag{273}$$

with $U(z) = |z|^3/3$, then HCS do exist and are just made of a combination of two Dirac masses : up to some change of scales,

$$f_S(t,v) = \frac{1}{2} \Big[\delta_{-\frac{1}{2t}} + \delta_{\frac{1}{2t}} \Big].$$

This solution is obtained via the search for steady states to the rescaled equation

$$\frac{\partial f}{\partial t} = \nabla_v \cdot \Big(f \nabla_v (f * W) \Big) - \nabla_v \cdot (fv). \tag{274}$$

One finds that a distinguished steady state is $F_S = (\delta_{-\frac{1}{2}} + \delta_{\frac{1}{2}})/2$.

It is also true [**70**] that this HCS is a better approximation to solutions of (273), than just the Dirac mass (at least if the initial datum has no singular part). This means that solutions of (274) do converge towards the steady state F_S. But this approximation is in general quite bad ! It was shown by Caglioti and Villani [**116**] that the improvement in the rate of convergence is essentially no better than logarithmic in time. For instance, if W stands for the Wasserstein distance (244), then solutions of (273) satisfy

$$\int_1^{+\infty} W\big(f(t), f_S(t)\big)\, dt = +\infty.$$

Since also $W(f(t), \delta_0) = O(1/t)$, this shows that the improvement in the rate of convergence cannot be $O(\log^{1+\varepsilon} t)$, for any $\varepsilon > 0$ — which means negligible by usual standards[11].

The preceding considerations cast further doubts about the mathematical relevance of HCS, which however are at the basis of several hydrodynamical equations for granular media. Further clarification is still badly needed; an attempt is done in [**86**] for the simplified pseudo-Maxwellian model.

This concludes our brief discussion of the kinetic theory of granular media. In the sequel, we shall enter a completely different physical world.

[11] or maybe should one use an even weaker notion of distance to measure the rate of convergence ??

3. Quantum kinetic theory

Very recently, Lu [**328**], Escobedo, and Mischler [**210, 211**] have begun to apply the techniques of the modern theory of the spatially homogeneous Boltzmann equation, to quantum kinetic models, thus opening up the path to a promising new direction of research. This is part of a general trend which has become increasingly active over last years : the mathematical derivation and study of quantum statistical models. This also coincides with a time when the interest of physicists in Bose condensation is enhanced by the possibility of experiments with very cold atoms. Most of the explanations which follow come from discussions with Escobedo and Mischler, and also from Lu's paper [**328**].

First of all, we should clarify the meaning of quantum kinetic theory : it does not rest on the traditional quantum formalism (wave function, Wigner transforms, etc.). Instead, it is rather a classical description of interacting particles with quantum features. This approach was initiated by the physicists Nordheim, and Uehling and Uhlenbeck, in the thirties.

Thus, the basic equation still looks just like a Boltzmann equation :

$$\frac{\partial f}{\partial t} + v(p) \cdot \nabla_x f = Q(f,f), \qquad t \geq 0,\ x \in \mathbb{R}^3,\ p \in \mathbb{R}^3. \tag{275}$$

Here p stands for the impulsion of the particle and v is the corresponding velocity : $v(p) = \nabla_p E(p)$, where $E(p)$ is the energy corresponding to the impulsion p. The unknown is a time-dependent probability density on the phase space of positions and impulsions. When dealing with massive particles, we shall consider a non-relativistic setting (to simplify) and identify v with p. On the other hand, when dealing with photons, which have no mass, we assume that the energy is proportional to $|p|$, and the velocity to $p/|p|$.

Now, all the quantum features are encoded at the level of the collision operator Q in (275). One traditionally considers three types of particles :

⋆ **fermions**, which satisfy Pauli's exclusion principle. In this case the transition from state p' to state p is easier if $f(p)$ is low. Accordingly, the "Boltzmann-Fermi" collision operator reads

$$Q_F(f,f) = \int_{\mathbb{R}^3} dp_* \int_{S^2} d\sigma\, B(v - v_*, \sigma) \Big[f' f'_* (1 + \varepsilon f)(1 + \varepsilon f_*) - f f_* (1 + \varepsilon f')(1 + \varepsilon f'_*) \Big], \tag{276}$$

where ε is a *negative* constant[12]. Up to change of units, we shall assume that $\varepsilon = -1$. Moreover, as in the classical case, $f' = f(p')$ and so on, and p', p'_* are given by the formulas

$$\begin{cases} p' = \dfrac{p + p_*}{2} + \dfrac{|p - p_*|}{2}\sigma, \\ \\ p'_* = \dfrac{p + p_*}{2} - \dfrac{|p - p_*|}{2}\sigma. \end{cases} \tag{277}$$

[12] In physical units, ε should be $-(h/m)^3/g$, where h is Planck's constant, m the mass of a particle and g the so-called "statistical weight" of this species of particles. For the derivation of (276) see Chapman and Cowling [**154**, chapter 17].

Eq. (275) with $Q = Q_F$ will be called the **Boltzmann-Fermi equation**. It is supplemented with the a priori bound

$$0 \leq f \leq 1 \quad [= -1/\varepsilon];$$

⋆ **bosons**, which, on the contrary to fermions, do like to cluster. The collision operator, Q_B, is just the same as (276), but now $\varepsilon = +1$. The corresponding equation will be called the Boltzmann-Bose equation.

⋆ **photons**, which are mass-free particles exchanging energy. Usually they are considered only in interaction with bosons or fermions. For instance, here is the Boltzmann-Compton model :

$$(278)\quad Q_C(f,f) = \int_0^\infty b(k,k')\Big[f'(k^2+f)e^{-k} - f(k'^2+g')e^{-k'}\Big]\,dk', \qquad t \geq 0,\, k \geq 0.$$

Here the phase space is just $\mathbb{R}_+$, the space of energies, because the distribution of photons is assumed to be spatially homogeneous and isotropic. Thus the corresponding evolution equation is just

$$(279)\qquad \frac{\partial f}{\partial t} = Q_C(f,f), \qquad t \geq 0,\; k \geq 0.$$

We quote from Escobedo and Mischler [**211**] : eq. (279) describes the behavior of a low-energy, spatially homogeneous, isotropic photon gas interacting with a low-temperature electron gas with Maxwellian distribution of velocities, via Compton scattering. This model will be called Boltzmann-Compton.

We now survey some of the main problems in the field.

3.1. Derivation issues. The derivation of equations like Boltzmann-Fermi or Boltzmann-Bose is not a tidy business (see Chapman and Cowling [**154**]...). Therefore, the expected range of applicability and the precise form of the equations are not so clear.

A better understood situation is the linear setting : description of the effect of a lattice of quantum scatterers on a density of quantum particles. Not only is the exact equation well understood, but also a theoretical basis, in the spirit of Lanford's theorem, can be given. Starting from the many-body Schrödinger equation as microscopic equation, Erdös and Yau [**205, 206**] have been able to retrieve the expected linear Boltzmann equation as a macroscopic description. Related works are performed in [**132, 133, 134**]... In the sequel we do not consider these issues and restrict the discussion to the nonlinear equations written above. Here are a few problematic issues about them.

▸ **Collision kernels.** It seems, nobody really knows what precise form of the cross-section, or equivalently of the collision kernel $B(v-v_*,\sigma)$ in (275) (or $b(k,k')$ in (278)) should be used — except in some particular cases with photon interaction... Some formulas can be found in [**154**] but they are not very explicit. This makes it difficult to give an interpretation of some of the mathematical results, as we shall see. It would be desirable to identify some model collision kernels playing the same role as the ones associated with inverse-power interactions in the classical theory. According to certain physicists, it would be not so bad to understand the case of a simple hard-sphere collision kernel.

► **Grazing collisions.** Some variants of these equations are obtained by a grazing collision asymptotics. To this class belong the quantum Landau equation (see Lemou [**298**] and references therein), or the well-known Kompaneets equation[13] [**290**],

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial k}\left[k^2\frac{\partial f}{\partial k} + (k^2 - 2k)f + f^2\right] \equiv \frac{\partial F}{\partial k}, \qquad t \geq 0,\ k \geq 0 \tag{280}$$

A flux condition must be added at the boundary :

$$\lim_{k\to 0} F(k) = 0. \tag{281}$$

Eq. (280) describes the same kind of phenomena as the Boltzmann-Compton equation, and can in fact be obtained from it by an asymptotic procedure similar to the one leading from the Boltzmann to the Landau equation (see Escobedo and Mischler [**211**]) under some assumptions on the initial datum. However, the validity of this approximation cannot be universally true, because the Kompaneets equation has some strange "blow-up" properties : Escobedo, Herrero and Velazquez [**208**] have shown that the flux condition (281) may break down in finite time for arbitrarily small initial data.

Also the long-time behavior of the Kompaneets equation can be non-conventional; this is consistent with the remark by Caflisch and Levermore [**114**] that for large enough mass there are no stationary states...

Besides physical interest, all these considerations illustrate the fact that the asymptotics of grazing collisions may destroy (or create ?) some important features of the models.

► **Consistency with classical mechanics.** All these quantum models involve the Planck constant as a parameter; of course when one lets the Planck constant go to 0 (which would in fact be the formal consequence of a change in physical scales, from microscopic to macroscopic), one expects to recover the Boltzmann-like equations of classical mechanics. This can be justified in some cases, see for instance [**196**].

► **Hydrodynamics.** Physicists expect that some hydrodynamic limit of the Boltzmann-Bose equation leads to the Gross-Pitaevski, or Ginzburg-Landau, model (based on a cubic nonlinear Schrödinger equation) for the evolution of the Bose condensate[14] part. Again, this would need clarification... We note however that the justification of this limit would look more interesting if the derivation of the Boltzmann-Bose equation was first put on a more rigorous basis.

3.2. Trend to equilibrium. Equations such as Boltzmann-Fermi, Boltzmann-Bose or Boltzmann-Compton all satisfy entropy principles, and equilibrium states are entropy minimizers[15] :

1) for the Boltzmann-Fermi equation, the entropy is

$$H_{BF}(f) = \int \big[f\log f - (1-f)\log(1-f)\big]$$

Equilibrium states are of the form

$$\mathcal{F}(p) = \frac{1}{e^{\alpha|p-p_0|^2+\beta}+1} \qquad (\alpha > 0,\ \beta \in \mathbb{R}) \tag{282}$$

[13] This equation is often written with $k^2 f$ as unknown.

[14] See the next paragraph.

[15] Under the constraint $0 \leq f \leq 1$ for the Boltzmann-Fermi model.

or

$$\mathcal{F}(p) = 1_{|p-p_0|\leq R} \qquad (R > 0).$$

A state like (282) is called a **Fermi-Dirac distribution**. Here p_0 is the mean impulsion.

2) for the Boltzmann-Bose equation, the entropy is

$$H_{BB}(f) = \int [f \log f - (1+f)\log(1+f)]$$

(here $\varepsilon = +1$) and the shape of equilibrium states depends on the temperature. There is a critical *condensation temperature* T_c such that the equilibrium state $\mathcal{B}$ takes the form

$$\mathcal{B}(p) = \frac{1}{e^{\alpha|p-p_0|^2+\beta} - 1} \qquad (\alpha > 0,\ \beta \geq 0) \quad \text{when } T \geq T_c, \tag{283}$$

$$\mathcal{B}(p) = \frac{1}{e^{\alpha|p-p_0|^2} - 1} + \mu\delta_{p_0} \qquad (\alpha > 0,\ \mu > 0) \quad \text{when } T < T_c. \tag{284}$$

These distributions are called **Bose-Einstein distributions**. The singular part of (284) is called a **Bose condensate**.

3) Finally, for the Boltzmann-Compton equation, the entropy is given by

$$H_{BC}(f) = \int_0^\infty \left[(k^2+f)\log(k^2+f) - f\log f - kf - k^2\log(k^2)\right] dk,$$

and, according to Caflisch and Levermore [**114**], the minimizers are of the form

$$\mathcal{B}(k) = \frac{1}{e^{k+\lambda} - 1} + \alpha\delta_0,$$

where λ and α are nonnegative numbers, at least one of them being 0. For $\lambda > 0$ this is a Bose distribution, for $\lambda = 0$ it is called a **Planck distribution**.

As in the Boltzmann case, these distributions, obtained by a minimization principle, also coincide with the probability distributions which make the collision operator vanish. There are some technicalities associated with the fact that singular measures should be allowed : they have recently been clarified independently by Escobedo and Mischler, and by Lu.

Now, let us consider the problem of convergence to equilibrium, in a spatially homogeneous setting for simplicity. As far as soft methods (compactness and so on) are concerned, Pauli's exclusion principle facilitates things a great deal because of the additional L^∞ bound. Therefore, convergence to equilibrium in a (very) weak sense is not very difficult [**211**]. However, no constructive result in this direction has ever been obtained, neither has any entropy-entropy dissipation inequality been established.

In the Bose case, this is an even more challenging problem since also soft methods fail, due to the lack of a priori bounds. The entropy is now sublinear and fails to prevent concentration, which is consistent with the fact that condensation may occur in the long-time limit. Actually, as soon as $T < T_c$, a given solution cannot stay within a weakly compact set of L^1 as $t \to +\infty$... Lu [**328**] has attacked this problem with the well-developed tools of modern spatially homogeneous theory, and proven that

- when the temperature is very large ($T \gg T_c$), solutions of the spatially homogeneous Boltzmann-Bose equation are weakly compact in L^1 as $t \to +\infty$, and converge weakly towards a Bose distribution of the form (283);

- when the temperature is very low ($T < T_c$), solutions are not weakly compact in L^1, but converge to equilibrium in the following *extremely* weak sense [**328**] : if (t_n) is a sequence of times going to infinity, then from $f(t_n, \cdot)$ one can extract a subsequence converging in biting-weak L^1 sense towards a Bose distribution of the form (284). In this theorem, not only is biting-weak L^1 convergence a very weak notion (weaker than distributional convergence), but also the limit may depend on the sequence (t_n).

Furthermore, it is not known whether weak L^1 compactness as $t \to \infty$ holds true when T is greater than T_c, but not so large.

Lu's theorem is proven for isotropic homogeneous solutions. Isotropy should not be a serious restriction, but seems compulsory to the present proof. What is more, Lu's work relies on a *strong cut-off assumption* for the kernel B : essentially,

$$B(|v - v_*|, \cos\theta) \leq C\theta\big(|v - v_*|^3 \wedge |v - v_*|\big), \qquad C > 0, \tag{285}$$

where θ is as usual the deviation angle. This assumption enables a very good control of the Q^+ part, but may do some harm for other, yet to be found, a priori estimates.

3.3. Condensation in finite time. Physical experiments with very cold atoms have recently become possible, and have aroused a lot of interest. For instance, a few years ago it became possible to experimentally create and study Bose condensates. Among other phenomena, physicists report the formation of a condensate in finite time. However, Lu has proven that there is no finite time clustering for the Boltzmann-Bose equation studied in [**328**]. This seems to leave room for two possibilities, both of which may lead to exciting new research directions :

- either the Boltzmann-Bose equation should be discarded for a more precise model when trying to model Bose condensation;
- or the Bose condensation is excluded by the strong cut-off assumption (285), which penalizes interactions with $v \simeq v_*$ (supposedly very important in condensation effects). On this occasion we strongly feel the need to have a better idea of what collision kernels would be physically realistic. Proving the possibility (or genericity) of finite-time condensation for "bigger" collision kernels (say $B \equiv 1$?) would be a mathematical and physical breakthrough for the theory of the Boltzmann-Bose model.

3.4. Spatial inhomogeneities. So far we have only considered spatially homogeneous quantum Boltzmann equations, now what happens for spatially inhomogeneous data ? Due to the additional L^∞ bound, the Boltzmann-Fermi model seems easier to study than the classical Boltzmann equation; in particular existence results can be obtained without too much difficulty [**196, 309**]. The situation is completely different for the Boltzmann-Bose model, since one would like to consider singular measures as possible data. A completely new mathematical theory would have to be built ! A particularly exciting problem would be the understanding of the space-time evolution for the Bose condensate.

It was communicated to us by Lu that for small initial data in the whole of $\mathbb{R}^3$, one can prove that Bose-Einstein condensation never occurs... This should be taken as a clue that the underlying mathematical phenomena are very subtle.

BIBLIOGRAPHICAL NOTES

General references : Standard references about the kinetic theory of rarefied gases and the Boltzmann equation are the books by Boltzmann [**93**], Carleman [**119**], Chapman and Cowling [**154**], Uhlenbeck and Ford [**433**], Truesdell and Muncaster [**430**], Cercignani [**141, 148**], Cercignani, Illner and Pulvirenti [**149**], as well as the survey paper by Grad [**250**]. The book by Cercignani et al., with a very much mathematically oriented spirit, may be the best mathematical reference for nonspecialists. The book by Uhlenbeck and Ford is a bit outdated, but a pleasure to read. There is no up-to-date treatise which would cover the huge progress accomplished in the theory of the Boltzmann equation over the last ten years.

For people interested in more applied topics, and practical aspects of modelling by the Boltzmann equation, Cercignani [**148**] is highly recommended. We may also suggest the very recent book by Sone [**407**], which is closer to numerical simulations.

The book by Glassey [**233**] is a good reference for the general subject of the Cauchy problem in kinetic theory (in particular for the Vlasov-Poisson and Vlasov-Maxwell equations, and for the Boltzmann equation near equilibrium). Also the notes by Bouchut [**96**] provide a compact introduction to the basic tools of modern kinetic theory, like characteristics and velocity-averaging lemmas, with applications.

To the best of our knowledge, there is no mathematically-oriented exposition of the kinetic theory of plasma physics. Among physicists' textbooks, Balescu [**46**] certainly has the most rigorous presentation. The very clear survey by Decoster [**160**] gives an accurate view of theoretical problems arising nowadays in applied plasma physics.

There are many, many general references about equilibrium and non-equilibrium statistical physics; for instance [**49, 227**]. People who would like to know more about information theory are advised to read the marvelous book by Cover and Thomas [**156**]. A well-written and rather complete survey about logarithmic Sobolev inequalities and their links with information theory is [**16**] (in french).

Historical references : The founding papers of modern kinetic theory were those of James Clerk Maxwell [**335, 336**] and Ludwig Boltzmann [**92**]. It is very impressive to read Maxwell's paper [**335**] and see how he made up all computations from scratch ! The book [**93**] by Boltzmann has been a milestone in kinetic theory.

References about the controversy between Boltzmann and his peers can be found in [**149**, p. 61], or Lebowitz [**293**]. Some very nice historical anecdotes can also be found in Balian [**49**].

Certainly the two mathematicians who have most contributed to transform the study of the Boltzmann equation into a mathematical field are Torsten Carleman in the thirties, and Harold Grad after the Second World War.

Derivation of the Boltzmann equation : For this subject the best reference is certainly Cercignani, Illner and Pulvirenti [**149**, chapters 2 and 4]. A pedagogical discussion of slightly simplified problems is performed in Pulvirenti [**394**]. Another excellent source is the classical treatise by Spohn [**410**] about large particle systems. These authors explain in detail why reversible microdynamics and irreversible macrodynamics are not contradictory — a topic which was first developed in the famous work by Ehrenfest and Ehrenfest [**202**], and later in the delightful book by Kac [**284**]. Further information on the derivation of macroscopic dynamics from microscopic equations can be found in Kipnis and Landim [**287**].

Hydrodynamic limits : A very nice review of rigorous results about the transition from kinetic to hydrodynamic models is Golse [**239**]. No prerequisite in either kinetic theory, or fluid mechanics is assumed from the reader. Note the discussion about ghost effects, which is also performed in Sone's book [**407**]. The important advances which were accomplished very recently by several teams, in particular Golse and co-workers, were reviewed by the author in [**441**].

There is a huge probabilistic literature devoted to the subject of hydrodynamic limits for particle systems, starting from a vast program suggested by Morrey [**352**]. Entropy methods were introduced into this field at the end of the eighties, see in particular the founding works by Guo, Papanicolaou and Varadhan (the GPV method, [**263**]), and Yau [**466**]. For a review on the results and methods, see the notes by Varadhan [**439**], the recent survey by Yau [**467**] or again, the book by Kipnis and Landim [**287**].

Mathematical landmarks : Here are some of the most influential works in the mathematical theory of the Boltzmann equation.

The very first mathematical steps are due to Carleman [**118, 119**] in the thirties. Not only was Carleman the very first one to state and solve mathematical problems about the Boltzmann equation (Cauchy problem, H theorem, trend to equilibrium), but he was also very daring in his use of tools from pure mathematics of the time.

In the seventies, the remarkable work by Lanford [**292**] showed that the Boltzmann equation could be rigorously derived from the laws of reversible mechanics, along the lines first suggested by Grad [**249**]. This ended up a very old controversy and opened new areas in the study of large particle systems. Yet much remains to be understood in the Boltzmann-Grad limit.

At the end of the eighties, the classical paper by DiPerna and Lions [**192**] set up new standards of mathematical level and dared to attack the problem of solutions in the large for the full Boltzmann equation, which to this date has still received no satisfactory answer. A synthetical review of this work can be found in Gérard [**231**].

Finally, we also mention the papers by McKean [**341**] in the mid-sixties, and Carlen and Carvalho [**121**] in the early nineties, for their introduction of information theory in the field, and the enormous influence that they had on research about the trend to equilibrium for the Boltzmann equation.

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