Some few examples of exponential integrators and their stability

1 Introduction

We consider the following VP equation

$$\partial_t \hat{f} + ikv\hat{f} - G[f] = 0,$$

where $\hat{f}$ is the space Fourier transform of f and $G[f] = -\widehat{E\partial_v f}$ denotes the space Fourier transform of the nonlinear term $-E\partial_v f$. We rewrite the equation using Duhamel formula as

$$\hat{f}(t^{n+1}) = e^{-ikvh}\hat{f}(t^n) + \int_{t^n}^{t^{n+1}} e^{ikv(t-t^{n+1})} G[f](t)dt.$$

which will be discretized using exponential integrators. To do so, we shall use the classical notations $\varphi_\ell := \varphi_\ell(ikvh)$

exprk22
Let us write the exprk22 scheme

$$\begin{aligned} k_1 &= e^{ikvh}\hat{f}^n + h\varphi_1 G[f^n] \\ \hat{f}^{n+1} &= e^{ikvh}\hat{f}^n + h\Big[(\varphi_1 - \varphi_2)G[f^n] + \varphi_2 G[k_1]\Big]. \end{aligned}$$

Cox-Matthews
Let us write the Cox-Matthews scheme

$$\begin{aligned} k_1 &= e^{ikvh/2}\hat{f}^n + \frac{h}{2}\varphi_{1,2}G[f^n] \\ k_2 &= e^{ikvh/2}\hat{f}^n + \frac{h}{2}\varphi_{1,3}G[k_1] \\ k_3 &= e^{ikvh}\hat{f}^n + \frac{h}{2}\varphi_{1,3}[\varphi_{0,3} - 1]G[f^n] + h\varphi_{1,3}G[k_2] \\ \hat{f}^{n+1} &= e^{ikvh}\hat{f}^n + h\Big[(\varphi_1 - 3\varphi_2 + 4\varphi_3)G[f^n] + (2\varphi_2 - 4\varphi_3)(G[k_1] + G[k_2]) + (-\varphi_2 + 4\varphi_3)G[k_3]\Big]. \end{aligned}$$

Here, we denote $\varphi_{1,3} = \varphi_{1,2} = \varphi_1(ikvh/2)$ and $\varphi_{0,3} = e^{ikvh/2}$.

Krogstad
Let us write the Krogstad scheme

$$k_1 = e^{ikvh/2}\hat{f}^n + \frac{h}{2}\varphi_{1,2}G[f^n]$$

$$
\begin{aligned}
k_2 &= e^{ikvh/2}\hat{f}^n + h[\frac{1}{2}\varphi_{1,3} - \varphi_{2,3}]G[f^n] + h\varphi_{2,3}G[k_1] \\
k_3 &= e^{ikvh}\hat{f}^n + h[\varphi_{1,4} - 2\varphi_{2,4}]G[f^n] + 2h\varphi_{2,4}G[k_2] \\
\hat{f}^{n+1} &= e^{ikvh}\hat{f}^n + h\Big[(\varphi_1 - 3\varphi_2 + 4\varphi_3)G[f^n] + (2\varphi_2 - 4\varphi_3)(G[k_1] + G[k_2]) + (-\varphi_2 + 4\varphi_3)G[k_3]\Big].
\end{aligned}
$$

Here, we denote $\varphi_{\ell,3} = \varphi_\ell(ikvh/2)$ and $\varphi_{\ell,4} = \varphi_\ell(ikvh)$.

Hochbruck-Ostermann

Let us write the Hochbruck-Ostermann scheme

$$
\begin{aligned}
k_1 &= e^{ikvh/2}\hat{f}^n + \frac{h}{2}\varphi_{1,2}G[f^n] \\
k_2 &= e^{ikvh/2}\hat{f}^n + h[\frac{1}{2}\varphi_{1,3} - \varphi_{2,3}]G[f^n] + h\varphi_{2,3}G[k_1] \\
k_3 &= e^{ikvh}\hat{f}^n + h[\varphi_{1,4} - 2\varphi_{2,4}]G[f^n] + h\varphi_{2,4}G[k_1] + h\varphi_{2,4}G[k_2] \\
k_4 &= e^{ikvh/2}\hat{f}^n + h[\frac{1}{2}\varphi_{1,5} - 2a_{5,2} - a_{5,4}]G[f^n] + ha_{5,2}(G[k_1] + G[k_2]) + h[\frac{1}{4}\varphi_{2,5} - a_{5,2}]G[k_3] \\
\hat{f}^{n+1} &= e^{ikvh}\hat{f}^n + h\Big[(\varphi_1 - 3\varphi_2 + 4\varphi_3)G[f^n] + (-\varphi_2 + 4\varphi_3)G[k_3] + (4\varphi_2 - 8\varphi_3)G[k_4]\Big].
\end{aligned}
$$

Here, we denote $\varphi_{\ell,2} = \varphi_\ell(ikvh/2)$, $\varphi_{\ell,3} = \varphi_\ell(ikvh/2)$, $\varphi_{\ell,5} = \varphi_\ell(ikvh/2)$ and $\varphi_{\ell,4} = \varphi_\ell(ikvh)$. Moreover, $a_{5,2} = \frac{1}{2}\varphi_{2,5} - \varphi_{3,4} + \frac{1}{4}\varphi_{2,4} - \frac{1}{2}\varphi_{3,5}$ and $a_{5,4} = \frac{1}{4}\varphi_{2,5} - a_{5,2}$.