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MATHS 326

S1 2024

Assignment 4
Due: 24-05-2024

Q1

(a) We are given $r = 4, k = 11, \lambda = 2$, assume that these are parameters for a balanced design, by *Theorem 4.10*:

$$bk = vr \implies 11b = 4v$$

$$\lambda(v-1) = r(k-1) \implies 2(v-1) = 4(11-1) \implies 2v - 2 = 40 \implies v = 21$$

Contradiction as this implies $b = \frac{4 \cdot 21}{11}$ which is not an integer as neither 4 nor 21 have a prime factor of 11. Thus, no balanced block design has these parameters.

(b) We are given $b = 30, r = 6, k = 5$, assume that these are parameters for a balanced design, by *Theorem 4.10*:

$$bk = vr \implies 30 \cdot 5 = 6v \implies v = 25$$

$$\lambda(v-1) = r(k-1) \implies \lambda(24) = 6(4) \implies \lambda = 1$$

Using *Construction 4.24* with the field $\mathbb{F} = \mathbb{Z}_5$, we can construct an affine plane of order 5.

From the proof of *Theorem 4.38*, an affine plane of order $n = 5$ with is a balanced block design with parameters:

$$(v, b, r, k, \lambda) = (n^2, n^2 + n, n + 1, n, 1) = (25, 30, 6, 5, 1)$$

Thus we can construct affine plane which is a BIBD satisfying the given parameters.

(c) We are given $v = 46, b = 10, \lambda = 2$, assume that these are parameters for a balanced design, by *Theorem 4.10*:

$$bk = vr \implies 10k = 46r \implies k = 4.6r$$

$$\lambda(v-1) = r(k-1) \implies 2(46-1) = r(k-1) \implies 0 = 4.6r^2 - r - 90$$

Solving for possible values of r using the quadratic equation:

$$r = \frac{1 \pm \sqrt{1 + 4 \cdot 4.6 \cdot 90}}{2 \cdot 4.6} = \frac{1 \pm \sqrt{1657}}{2 \cdot 4.6}$$

Which has no integer solutions as $40^2 < 1657 < 41^2$, hence 1657 is not a perfect square and therefore $\sqrt{1657}$ must be irrational.

Since r must be an integer, this is a contradiction so no balanced block design has these parameters.

Q2

Assume that there is a BIBD for $v = b = 40$ with parameters (v, b, r, k, λ). Then since $vr = bk$ we have $k = r$. Thus:

$$\lambda(v-1) = r(k-1) \implies 39\lambda = r(r-1) = k(k-1)$$

Since the design is incomplete, $r, k, \lambda \leq 39$. We can factorise $39 = 3 \cdot 13$ and $\lambda = \lambda_1 \lambda_2$. This gives the following cases:

Case	r	$r-1$	
A	$39\lambda_1$	λ_2	$39\lambda_1 = \lambda_2 + 1$
B	$13\lambda_1$	$3\lambda_2$	$13\lambda_1 = 3\lambda_2 + 1$
C	$3\lambda_1$	$13\lambda_2$	$3\lambda_1 = 13\lambda_2 + 1$
D	λ_1	$39\lambda_2$	$\lambda_1 = 39\lambda_2 + 1$

Solving for λ in each of these cases:

Case A

$$\lambda_1 = 1 \implies \lambda_2 = 38$$
$$\lambda_1 \geq 2 \implies \lambda > 39$$
$$\implies \lambda \in \{38\}$$

Case B

$$\lambda_1 = 1 \implies \lambda_2 = 4$$
$$\lambda_1 = 2 \implies \lambda_2 = 25/3 \notin \mathbb{Z}$$
$$\lambda_1 \geq 3 \implies \lambda > 39$$
$$\implies \lambda \in \{4\}$$

Case C

$$\lambda_2 = 1 \implies \lambda_1 = 14/3 \notin \mathbb{Z}$$
$$\lambda_2 = 2 \implies \lambda_1 = 9$$
$$\lambda_2 = 3 \implies \lambda_1 = 40/3 \notin \mathbb{Z}$$
$$\lambda_2 \geq 3 \implies \lambda > 39$$
$$\implies \lambda \in \{18\}$$

Case D

$$\lambda_2 \geq 1 \implies \lambda > 39$$
$$\implies \lambda \in \varnothing$$

So we must have $\lambda \in \{4, 18, 38\}$.

Q3

(a) Since $\lambda = 2$, by *Theorem 4.10*:

$$\lambda(v-1) = r(k-1) \implies v - 1 = \frac{r(k-1)}{2}$$

Therefore:

$$v \leq \binom{r}{2} + 1 = \frac{r(r-1)}{2} + 1 \iff v - 1 \leq \frac{r(r-1)}{2} \iff \frac{r(k-1)}{2} \leq \frac{r(r-1)}{2} \iff r \leq k$$

Thus, it suffices to show that $r \leq k$. Assume for contradiction that $r > k$. Then by:

$$vr = bk \implies v < b$$

However, since the design is incomplete, $v > k$ so by *Theorem 4.10*, we expect $b \geq v$. Thus, we have a contradiction.

(b) By (a), since $\lambda = 2$:

$$v \leq \binom{7}{2} + 1 = 22$$

By *Theorem 4.10*:

$$\lambda(v-1) = r(k-1) \implies v = \frac{7(k-1)}{2} + 1$$

To have $v \in \mathbb{Z}$, k must be odd, also $k > 1$, searching for possible values of v:

$$\begin{aligned}
k = 3 &\implies v = 8 \implies b = \frac{8 \cdot 7}{3} \notin \mathbb{Z} \\
k = 5 &\implies v = 15 \qquad \implies b = 21 \\
k = 7 &\implies v = 22 \qquad \implies b = 22 \\
k = 9 &\implies v > 22
\end{aligned}$$

So we either have $v = 15$ or $v = 22$.

For $(v, k) = (22, 7)$, notice that since $r = k$, we have $b = v$ and v is even. So by even case of the Bruck-Ryser-Chowla Theorem, $k - \lambda = 5$ should be a perfect square. Since 5 is not a perfect square, there is no BIBD with these parameters.

So if a BIBD does have $\lambda = 2$, $r = 7$, $k > 1$, it must have $v = 15$.

Q4

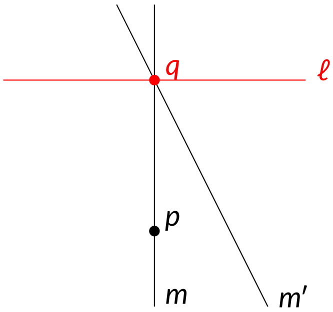
First we verify that the construction can be performed. Assume that we have a projective plane. By axiom P3, there are at least 4 points, and by P1, any distinct pair of these points is on a unique line. Thus, there is a line ℓ to remove.

We check the axioms for an affine plane hold for the plane constructed by removing a line ℓ:

- A1: Any two distinct points in the constructed plane already existed on some unique line m in the projective plane, we have $m \neq \ell$ otherwise we would have removed the points. Hence, m is present in the constructed plane. As no other lines have become incident with the points, m is the unique line incident with both points.

- Consider any point p and line m' in the constructed plane such that m' is not incident on p.

  Clearly m' is distinct from ℓ, so by P2, m and ℓ have a unique common point q. Now by P1, q and p lie on a unique line m. Since $p \in m$ but $p \notin \ell$ we must have $m \neq \ell$ and so m is present in the constructed plane. Since q is the only common point of m and m', and is not present in the constructed plane, $m \cap m' = \varnothing$.

  It remains to show that m is unique. Assume that there is another line k such that $m \neq k$ while k is also incident on p and has $k \cap m' = \varnothing$.

  Since $k \neq m$, k cannot be incident on q as any two points are on a unique common line. By P2 k and m' have a unique common point q', since this point cannot be q, and q is the only point on both m' and ℓ, q' is not on ℓ.

  Therefore, q' is in the constructed plane, meaning that $k \cap m' \neq \varnothing$. This is a contradiction, therefore m is unique.

- A3: By P3, the projective plane contains 4 points no 3 of which are collinear. If ℓ is not incident on any of these points, they are in the constructed plane and A3 is satisfied. Otherwise:

  Assume that we have 4 points no 3 of which are collinear, then ℓ is incident on at most 2 points.

  WLOG, let ℓ be incident on q_1 and possibly q_2 but not p_1, p_2, since each pair of points is on a unique line, and no three points are collinear.

  Let p_3 be the unique common point of the unique lines a_1, a_2 through q_1, p_1 and q_2, p_2 respectively. The lines b_1, b_2 respectively containing q_1, p_2 and q_2, p_1 must also have a unique common point p_4.

  Let $Q = \{q_1, q_2, p_1, p_2\}$ and $P = \{p_1, p_2, p_3, p_4\}$.

  Claim: The points p_3, p_4 are distinct and $p_3, p_4 \notin Q$.

  > First assume $p_3 = p_4$, then a_1, b_1 both contain q_1 and $p_3 = p_4$, since the line containing two points is unique, $a_1 = b_1$. Hence, q_1, p_1, p_2 are all on the line $a_1 = b_1$ and thus collinear. This is a contradiction so $p_3 \neq p_4$.
  >
  > Next assume that at least one of $p_3, p_4 \in Q$, WLOG (by interchanging p_1, p_2), let $p_3 \in Q$.
  >
  > We have that q_1, p_1 on a_1 and q_2, p_2 on a_2. By definition p_3 is on both a_1 and a_2. Thus, some a_i now contains 3 distinct points in Q, thus there are 3 collinear points in Q4. Contradiction so $p_3, p_4 \notin Q$.

  Claim: No three distinct points in P are collinear.

  > Assume that 3 points in P are collinear. Then the collinear points lie on some line m.
  >
  > However, there two lines x, y from $\{a_1, a_2, b_1, b_2\}$ that each contain at least distinct pairs of the points. Since a unique line contains any two points, both $x = m = y$. Since x, y contain distinct pairs of points from Q, at least three points must now be on the same line $m = x = y$ and therefore be collinear which is a contradiction.

  Claim: Neither of $\{p_3, p_4\}$ are on ℓ.

  > Next assume that at least one of $p_3, p_4 \in \ell$, WLOG (by interchanging p_1, p_2), let $p_3 \in \ell$.
  >
  > Since ℓ contains q_1, q_2, p_3 it is the unique line containing any pair those points. Thus, $a_1 = \ell = a_2$.
  >
  > However, if $\ell = a_1 = a_2$ then q_1, q_2, p_1 are all on ℓ and therefore collinear. This is a contradiction so $p_3 \notin \ell$ and $p_4 \notin \ell$.

  Thus, there are 4 points $\{p_1, p_2, p_3, p_4\}$ such that no 3 are collinear, and these points will be present in the constructed plane as no point is incident with ℓ. Therefore, A3 will be satisfied for the constructed plane.

Hence, we have shown that the construction is an affine plane by definition.

Q5

Consider the set $\{L_1, \dots, L_6\}$ of order 7 Latin squares with entries $(L_k)_{ij} = i + kj \bmod 7$. Verifying that this is a set of Latin squares:

$$\begin{aligned} &(L_k)_{ij} = (L_k)_{ij'} \\ \implies &i + kj = i + kj' \\ \implies &kj = kj' \\ \implies &j = j' \qquad \text{Divide by } k \end{aligned}$$

$$\begin{aligned} &(L_k)_{ij} = (L_k)_{i'j} \\ \implies &i + kj = i' + kj \\ \implies &i = i' \end{aligned}$$

Note that we can divide by k in mod 7 as $1, \dots, 6$ are not zero divisors. So we have shown that in the same row/column, only a single cell holds each value.

Assume that $L_k \neq L_{k'}$ aren't orthogonal, and therefore for $(i, j) \neq (i', j')$:

$$\begin{aligned} &\Big((L_k)_{ij}, (L_{k'})_{ij}\Big) = \Big((L_k)_{i'j'}, (L_{k'})_{i'j'}\Big) \\ \implies &\Big(i + kj, i + k'j\Big) = \Big(i' + kj', i' + k'j'\Big) \\ \implies &\Big(0, 0\Big) = \Big((i - i') + k(j - j'), (i - i') + k'(j - j')\Big) \\ \implies &(i - i') + k(j - j') = (i - i') + k'(j - j') \\ \implies &k(j - j') = k'(j - j') \\ \implies &0 = (k - k')(j - j') \end{aligned}$$

However, the only zero divisor in mod 7 is 0, thus either $k - k' = 0$, or $j - j' = 0$. Since we assumed $L_k \neq L_{k'} \implies k - k' \neq 0$, we must have $j = j'$. Therefore:

$$\Big(0, 0\Big) = \Big((i - i') + k(j - j'), (i - i') + k'(j - j')\Big) \implies 0 = i - i' \implies i = i'$$

This is a contradiction, so each distinct pair $L_k, L_{k'}$ must be orthogonal by definition. This also verifies that there are 6 distinct Latin squares since a Latin square is not orthogonal with itself.

Thus, $\{L_1, \dots, L_6\}$ are a set of 6 MOLS of order 7.

Q6

(a) The square was completed in the following order:

- The gray cells were given.
- The blue must be some permutation of $3, 4, 5$ and can be re-ordered by interchanging rows, order chosen WLOG.
- The violet cells must also be some permutation of $3, 4, 5$ distinct from the ordering of the blue cells. There are two options, the other failed to complete the square.
- The cyan cell, had to be either 1 or 2, since no remaining cells are constrained by a 1 or 2, the choice is made WLOG.
- Each cell with a single remaining possibility was filled until the square was complete.

1	2	3	4	5
2	1	5	3	4
3	4	1	5	2
4	5	2	1	3
5	3	4	2	1

(b) By *Theorem 4.63*, a Latin square of order 5 has an orthogonal mate if and only if it contains 5 disjoint traversals. Assume that there exist 5 disjoint traversals of some completion.

Consider the top left 2×2 region:

The region contains the following cells:

1	2
2	1

If two cells are in the same traversal, they cannot be in the same row/column, thus they must be diagonal (in the 2×2 region). All diagonal entries are the same, so they cannot be part of the same traversal.

Thus, each cell in the 2×2 region is part of a distinct traversal.

Label the traversals A, B, C, D, E, WLOG we can fix the traversal that each of the cells in the 2×2 region are part of. Since each traversal appears once in each row/column, we can deduce which traversals the cells in each region must be assigned to:

A	B	C, D, E
C	D	A, B, E
B, D, E	A, C, E	A, A, B, B C, C, D, D, E

Notice that to be a traversal E should contain both cells with values 1 and 2, however none of the E's in first two rows/columns could contain a 1 or 2. This only leaves a single E in the bottom right 3×3 region. So it is impossible for the E traversal to contain both a 1 and 2.

Therefore, it is impossible for any completion to contain 5 distinct traversals and thus no completion has an orthogonal mate.