Give appropriate justifications for your answers. Remember that the work you submit must be your own work, in your own words!

1. Show that Dijkstra's algorithm does not necessarily find shortest paths if both positive and negative edge weights are allowed.

2. Prove that if all edge weights of a weighted graph are different, then G has a unique minimum spanning tree.

3. Let G be a weighted graph with positive edge weights. Define a new weight function on the edges of G by replacing the weight of each edge by its square. Prove or disprove the following statements.

(a) A path is a shortest path between its endpoints before changing the weights if and only if it is a shortest path between its endpoints after changing the weights.

(b) A tree is an MST before changing the weights if and only if it is a MST after changing the weights.

4. For a graph G, let $o(G)$ be the number of connected components of G containing an odd number of vertices. Show that a tree T has a perfect matching if and only if $o(T - v) = 1$ for each vertex v of T.

5. Call an edge in a graph *unmatchable* if it is not contained in a perfect matching.

(a) Show that a regular class 1 graphs cannot contain an unmatchable edg.

(b) Give an example of a regular class 2 graph with no unmatchable edge.

6. Let $G = (V, E)$ be a bipartite graph, let $M \subseteq E$ be a matching, and let $I \subseteq V$ be an independent set. Prove that $|M| + |I| \leq |V|$, and equality holds if and only if M is a maximum matching and I is a maximum independent set.

7. Two players, Alice and Bob play the following game on a simple connected graph G. First, Alice picks a vertex v of G. Then, starting with Bob, players take turns choosing an edge e according to the following rule: e was not previously chosen and the set of all chosen edges (including e) forms a path containing v. The first player who cannot choose an edge according to this rule loses.

Show that Bob has a winning strategy if and only if G contains a perfect matching.