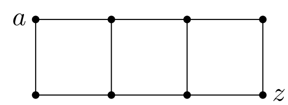
Give appropriate justifications for your answers.

1. Let $G = (V, E, c)$ be a weighted graph and let a and z be vertices of G. Let p_x denote the probability that a random walk started at x reaches a before z.

(a) Prove that if $f : V \to \mathbb{R}$ is harmonic on $V \setminus \{a, z\}$, then there are constants $\alpha, \beta \in \mathbb{R}$ such that $f(x) = \alpha p_x + \beta$.

(b) Using part (a) or otherwise, determine the dimension of the vector space of functions which are harmonic on $V \setminus \{a, z\}$.

2. Compute $\mathcal{R}(a \leftrightarrow z)$ in the graph drawn below.



3. Let v be a voltage function defined on the vertices of a finite (unweighted) graph by $v(a) = 0$ and $v(z) = 1$ and harmonic elsewhere. Must the voltages of the vertices along every shortest path between a and z be non-decreasing?

4. Let H be a connected subgraph of a connected graph G and let e be an edge of H. Let T be a random spanning tree of H and let T' be a random spanning tree of G. Show that $\mathbb{P}[e \in T] \geq \mathbb{P}[e \in T']$.

5. Let G be a graph, let a and z be vertices, and let e be an edge. Let $\mathcal{R}_G$ and $\mathcal{R}_{G-e}$ denote the effective resistance in G and $G - e$, respectively

(a) Prove that $\mathcal{R}_G(a \leftrightarrow z) \leq \mathcal{R}_{G-e}(a \leftrightarrow z)$.

(b) Prove that if e does not lie on any path from a to z, then $\mathcal{R}_G(a \leftrightarrow z) = \mathcal{R}_{G-e}(a \leftrightarrow z)$.

(c) Using part (a) or otherwise, prove that $\mathrm{Comm}(a \leftrightarrow z) \leq 2|E| \cdot d(a, z)$, where $d(a, z)$ denotes the length of a shortest path from a to z in G.

6. Show that the effective resistance is concave as a function of the individual resistances in the sense that

$$\frac{1}{2}(\mathcal{R}_r(a \leftrightarrow z) + \mathcal{R}_{r'}(a \leftrightarrow z)) \leq \mathcal{R}_{\frac{r+r'}{2}}(a \leftrightarrow z),$$

where $\mathcal{R}_r$, $\mathcal{R}_{r'}$, and $\mathcal{R}_{\frac{r+r'}{2}}$ denote effective resistances in the same graph with respect to the different weight functions $c = \frac{1}{r}$, $c' = \frac{1}{r'}$ and $c'' = \frac{2}{r+r'}$, respectively.

Hint: consider current flows of unit strength.