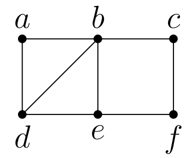
1. (a) Label the vertices of the graph as follows:



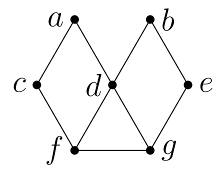
A colouring of G can be obtained by first colouring the cycle $bcfe$, then colouring d and then colouring a.

There are $P_{C_4}(x)$ ways to colour the cycle with x colours; by Exercise 1.13 (solved in the lectures) $P_{C_4}(x) = (x-1)^4 + x - 1$. Since b and e receive different colours, there are $(x-2)$ colours to choose from for d in the second step, and since b and d receive different colours there are $(x-2)$ colours to choose from for a in the final step.

Hence $P_G(x) = ((x-1)^4 + x - 1)(x-2)^2$.

(Alternatively, use deletion contraction) [3 marks]

(b) Label the graph as below:



Use deletion contraction on the edge a, d. Note that $G_{\{a,d\}}$ is the graph from part (a). In order to determine $P_{G-\{a,d\}}$ we proceed similarly to part (a): First colour the cycle $bdge$, then colour the vertex f, then c, and finally a.

There are $P_{C_4}(x) = (x-1)^4 + x - 1$ possibilities for the first step, $(x-2)$ for the colour of f (because d and g have different colours), $(x-1)$ ways of choosing the colour of c, and $(x-1)$ ways of choosing the colour of a.

Therefore $P_{G-\{a,d\}}(x) = ((x-1)^4 + x - 1)(x-2)(x-1)^2$. Finally,

$$\begin{aligned} P_G(x) &= P_{G-\{a,d\}}(x) - P_{G_{\{a,d\}}}(x) \\ &= ((x-1)^4 + x - 1)(x-2)(x-1)^2 - ((x-1)^4 + x - 1)(x-2)^2 \\ &= x^7 - 9x^6 + 35x^5 - 75x^4 + 93x^3 - 63x^2 + 18x. \end{aligned}$$

[3 marks]

2. (a) We prove this statement by induction on the number of edges of G.

Base case: If G has no edges and n vertices, then $P_G(x) = x^n$. The coefficient of x is non-zero if and only if $n = 1$, and the empty graph on n vertices is connected if $n = 1$ and disconnected otherwise.

Induction step: Let e be an arbitrary edge of G. By the deletion-contraction formula, we know that $P_G(x) = P_{G-e}(x) - P_{G_e}(x)$.

If G is not connected, then neither $G-e$ nor G_e are connected. By the inductive hypothesis, the coefficients of x in $P_{G-e}(x)$ and $P_{G_e}(x)$ are both 0, and therefore the same is true for $P_G(x)$.

If G is connected, then G_e is also connected (but $G-e$ may or may not be connected). By the inductive hypothesis, the coefficient of x in $P_{G_e}(x)$ is non-zero. Moreover, if the coefficient of x in $P_{G-e}(x)$ is non-zero, then it has the opposite sign of the coefficient of x in $P_{G_e}(x)$. Hence the coefficient of x in $P_G(x)$ is non-zero. [4 marks]

(b) If G is a tree, then $P_G(x) = x(x-1)^{n-1}$ by Lemma 1.8, so we only need to show the converse implication.

Let G be a graph with chromatic polynomial $P_G(x) = x(x-1)^{n-1}$. We note that

$$x(x-1)^{n-1} = x^n - (n-1)x^{n-1} + \binom{n-1}{2}x^{n-2} - \cdots + (-1)^{n-2}(n-1)x^2 + (-1)^{n-1}x.$$

In particular, the coefficient of x is non-zero, so G is connected, the degree is n, so G has n vertices, and the coefficient of x^{n-1} is $n-1$, so G has $n-1$ edges. Therefore G must be a tree. [2 marks]

3. (a) By Theorem 1.14 (5), the constant term in any chromatic polynomial is 0, but the constant term in $(x-1)^4$ is 1. [1 marks]

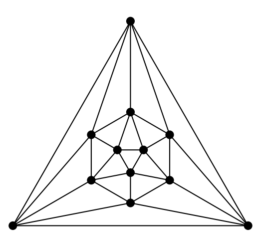
(b) By Theorem 1.14 (6) the coefficients of a chromatic polynomial alternate in sign. Since the sign of the coefficient of x^6 is positive, the sign of the coefficient of x^3 would have to be negative (which it isn't). [1 marks]

(c) If this were the chromatic polynomial of a graph G, then G would be a graph with 4 vertices and 3 edges. Moreover the coefficient of x is not zero, so G would have to be connected (by problem 2a) and therefore a tree. But $x^4 - 3x^3 + 4x^2 - 2x \neq x(x-1)^3$. [2 marks]

4. Let G be a planar graph with p vertices and q edges with minimal degree 5. we will show that $p \geq 12$.

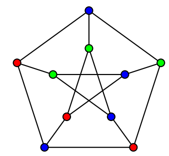
First note that by the handshaking lemma $2q = \sum_{v \in V} \deg(v)$. Since all degrees are ≥ 5 this implies that $2q \geq 5p$. By Theorem 1.19 from the lecture notes we know that $q \leq 3p - 6$. Combining the two, we obtain $5p \leq 6p - 12$ which implies $p \geq 12$.

To see that the bound is sharp consider the graph below:



The graph is planar (because no two edges in the drawing cross), has 12 vertices, and every vertex is incident to exactly 5 edges. [5 marks]

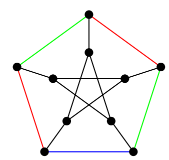
5. (a) We note that P contains an odd cycle (e.g. the 5-cycle around the outside), so by Theorem 1.6 we know that $\chi(P) > 2$. Moreover, $\chi(P) \leq 3$ since we can properly 3-colour the vertices as below:



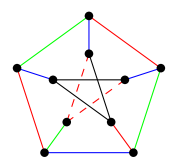
hence $\chi(P) = 3$.

Next we determine $\chi_e(P)$. Note that the P is 3-regular, so by Vizings Theorem $\chi_e(P) \in \{3, 4\}$. We show that there is no 3-edge colouring of P.

Assume that there was one, then all 3 colours must appear on the outer cycle. No colour can appear 3 times on this cycle, so one appears once, and the other two appear twice. Without loss of generality assume that the colouring of the outer cycle looks as follows:



Then the colours on the 'spokes' must be as below, and both of the two red dashed edges would have to receive the same colour.



Hence it is impossible to properly 3-edge colour P, so $\chi_e(P) = 4$. [4 marks]

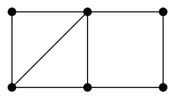
(b) If we contract all the 'spokes' in the Petersen graph, we obtain K_5, hence P is not planar. [1 marks]

6. We first show that if G is bipartite, then $\alpha(H) \geq \frac{1}{2}m_H$. Let G be a bipartite graph, and let H be a subgraph of G. Then H is bipartite as well, and each of the two partite classes forms an independent set. One of the partite classes contains at least half of the vertices of H, so $\alpha(H) \geq \frac{1}{2}m_H$.

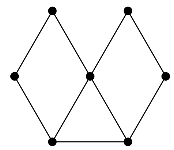
For the converse implicationlet G be a graph which is not bipartite. By Theorem 1.6, G contains an odd cycle; let H be this odd cycle. By the pigeonhole principle, every set of $\geq \frac{1}{2}m_H$ must contain two consecutive vertices, so $\alpha(H) < \frac{1}{2}m_H$. [4 marks]

All working should be complete and **your own work, written in your own words**.

1. Determine the chromatic polynomials of the graphs drawn below:



(a)



(b)

2. Let G be a graph and let $P_G(x)$ be its chromatic polynomial.

(a) Prove that the coefficient of x in $P_G(x)$ is non-zero if and only if G is connected.

(b) Use part (a) to show that the word 'connected' can be ommited in the statement of Corollary 1.15. In other words, prove that a simple graph on n vertices is a tree if and only if its chromatic polynomial is $x(x-1)^{n-1}$.

3. Prove that none of the following polynomials is the chromatic polynomial of a graph.

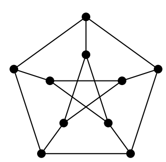
(a) $(x-1)^4$

(b) $x^6 - 6x^5 + 7x^3 - 2x$

(c) $x^4 - 3x^3 + 4x^2 - 2x$

4. Prove that every planar graph with less then 12 vertices has a vertex of degree at most 4. Show that this is sharp by finiding a planar graph on 12 vertices which has no such vertex.

5. Let P be the Petersen graph (drawn below).



(a) Determine the chromatic number and the edge chromatic number of P.

(b) Using Wagner's theorem or otherwise, show that P is not planar.

6. Prove that a graph G is bipartite if and only if every subgraph H of G satisfies $\alpha(H) \geq \frac{1}{2}m_H$, where m_H denotes the number of vertices of H.

1. Let $G = K_3$ with vertices x, y, z and edge weights $w(\{x, y\}) = 1$, $w(\{x, z\}) = 2$, and $w(\{y, z\}) = -2$. Starting at x, Dijkstra's algorithm will set $\delta(y) = 1$ and $\delta(z) = 2$ in the first iteration and choose y as the next 'current vertex'. In the next iteration it will mark y as visited and not update $\delta(y)$ after that. Hence at the end of the algorithm $\delta(y)$ will be equal to 1, but the path from x to y via z has length 0.

2. Assume for a contradiciton that there were 2 different minimal spanning trees T_1 and T_2. Let e be an edge contained in T_1 but not in T_2. Let P be the (unique) path in T_2 connecting the two endpoints of e. there is at least one edge f in P with one endpoint in each of the two connected components of $T_1 - e$ (because the first vertex of P lies in one connected component and the last vertex in the other). So $T_1 - e + f$ is a spanning tree, and since T_1 is a minimal spanning tree we conclude that $w(e) \leq w(f)$. Conversely, e connects the two connected components of $T_2 - f$ (because there are paths in $T_2 - f$ connecting the endpoints of e to the enpoints of f). So $T_2 - f + e$ is a spanning tree, and since T_2 is a minimal spanning tree we conclude that $w(f) \leq w(e)$. But this means that $w(e) = w(f)$,so not all edge weights are different.

3. (a) This statement is false. Consider e.g. the graph C_4 with edge weights 1, 1, 1, and 2. Then the path whose only edge is the edge of weight 2 is a shortest path, but after squaring all weights this path will have weight 4, and the path using the other 3 edges will still have weight 3.

(b) This statement is true.

We claim that if T is a MST with respect to some weight function w, and the order in Kruskal's algorithm is chosen such that edges of T come before edges with the same weight that are not contained in T, then the algorithm returns T. Assume that this was not the case, and let e be the first edge outside of T added by the algorithm. Then e connects two different connected components of the previously constructed graph. Let f be an edge of T connecting these two connected components. Since f was not added before e was considered, we know that $w(f) > w(e)$. Note that e connects the two connected components of $T - f$ (because there are paths in $T - f$ connecting the endpoints of e to the enpoints of f). So $T - f + e$ is a spanning tree. But $w(T - f + e) = w(T) - w(f) + w(e) < w(T)$ contradicting the fact that T was a MST.

Since an order for the edges can be used in Kruskal's algorithm for the original weights if and only if it can be used for the squared weights (it only has to be non-decreasing), we conclude that every MST T for the original weights can be the outcome of Kruskal's algorithm for square weights, and therefore is a minimal spanning tree for the squared weights, and conversely, every MST T for the original weights can be the outcome of Kruskal's algorithm for square weights, and therefore is a minimal spanning tree for these weights.

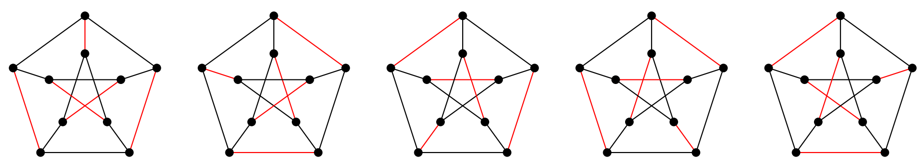
4. For the forward implication, let T be a tree and let M be a perfect matching of T. Let $e = \{u, v\}$ be the edge incident to v contained in M, and note that $M \setminus \{e\}$ is a matching in $T - v$ in which

u is the only unmatched vertex. Hence every connected component of $T - v$ except the one containing u has a perfect matching, and thus has an even number of vertices. The connected component which contains u has a matching in which all vertices except u are matched and therefore has an odd number of vertices. So $o(T - v) = 1$.

For the converse implication, assume that $o(T - v) = 1$ for every vertex v of T. We claim that the set M consisting of all edges such that both connected components of $G - e$ are is a perfect matching. To prove this claim, let v be a vertex of T. If an edge $e = \{u, v\}$ is in M, then u lies in a connected component of $G - e$ (which is also a connected component of $G - v$) with an odd number of vertices. Since $o(T - v) = 1$, at most one edge of M is incident to v. Conversely, if the connected component of $G - v$ containing u has odd size, then the connected component $G - u$ containing v is the union of some even sized connected component of $G - v$ and v, so this has odd size as well. Hence there is an edge of M incident to v. We have thus proved that every vertex v is incident to exactly one edge of M, so M is a perfect matching.

5. (a) Let G be a Δ-regular class 1 graph, and consider a proper edge colouring with Δ edges. every vertex is incident to exactly one edge of each colour, so every colour class forms a perfect matching. Since every edge is contained in some colour class, each edge is contained in some perfect matching, and thus no edge is unmatchable.

(b) By Assignment 1, Question 5a we know that the Petersen graph P is class 2. Below are 5 perfect matchings (all of them are 'rotations' of the first one) of P so that every edge is contained in at least one of them, hence there is no unmatchable edge in P



6. By Kőnig's Theorem (Theorem 2.12 in the lecture notes), the size $m(G)$ of a maximum matching in a bipartite graph is the size $c(G)$ of a minimum vertex cover. By Exercise 1.26 from the lectures, $|V| = c(G) + \alpha(G)$ where $\alpha(G)$ is the size of a maximum independent set.

If M is a matching and I is an independent set then $|M| \leq m(G)$ and $|I| \leq \alpha(G)$, and thus $|M| + |I| \leq m(G) + \alpha(G) = |V|$. Equality holds if and only if $|M| = m(G)$ and $|I| = \alpha(G)$, which (by definition) is the case if and only if M is a maximum matching and I is a maximum independent set.

7. Let P_k denote the path consisting of all edges after Alice's k-th move and let Q_k denote the path consisting of all edges after Bob's k-th move.

Claim: If G has a perfect matching, then Bob has a strategy to win the game.

It is enough to show that Bob is able to choose another edge whenever Alice chooses an edge. Let M be a perfect matching. We will show by induction that Bob can play so that Q_k is an alternating path which starts and ends in a matching edge.

In the first step this can be achieved by choosing a matching edge incident to v.

For the induction step, assume that Q_k is an alternating path which starts and ends in a matching edge. If Alice cannot make another move, then Bob has won and does not need another move. If Alice can make another move, then she must pick an edge $e = \{u, w\}$ where

u is the first or last vertex of Q_k. As Q_k starts and ends in a matching edge we know that $e \notin M$. Moreover, since Q_k is an alternating path, all matching edges incident to vertices in Q_k are contained in Q_k and thus Bob can choose the unique matching edge incident to w to ensure that Q_{k+1} has the desired property.

Claim: If G has no perfect matching, then Alice has a strategy to win the game (and thus Bob cannot have a winning strategy).

It is enough to show that Alice is able to choose another edge whenever Bob chooses an edge. Let M be a maximum matching. We will show by induction that Alice can play so that P_k is the union of two alternating paths (possibly of length 0) both starting in the same unmatched vertex v, and ending in a matching edge unless the length is 0.

In the first step this can be achieved by choosing v to be an unmatched vertex; such a vertex exists because M is not a perfect matching.

For the induction step, assume that P_k is as claimed. If Bob cannot make another move, then Alice has won and does not need another move. If Alice can make another move, then she must pick an edge $e = \{u, w\}$ where u is the last vertex of one of the two alternating paths. As this path ends in a matching edge, or only consists of the unmatched vertex v we know that $e \notin M$. Moreover, v must be matched because otherwise we would have found an augmenting path, contradicting the fact that M is a maximum matching. All matching edges incident to vertices in P_k are contained in P_k and thus Alice can choose the unique matching edge incident to w to ensure that P_{k+1} has the desired property.

Give appropriate justifications for your answers. Remember that the work you submit must be your own work, in your own words!

1. Show that Dijkstra's algorithm does not necessarily find shortest paths if both positive and negative edge weights are allowed.

2. Prove that if all edge weights of a weighted graph are different, then G has a unique minimum spanning tree.

3. Let G be a weighted graph with positive edge weights. Define a new weight function on the edges of G by replacing the weight of each edge by its square. Prove or disprove the following statements.

(a) A path is a shortest path between its endpoints before changing the weights if and only if it is a shortest path between its endpoints after changing the weights.

(b) A tree is an MST before changing the weights if and only if it is a MST after changing the weights.

4. For a graph G, let $o(G)$ be the number of connected components of G containing an odd number of vertices. Show that a tree T has a perfect matching if and only if $o(T - v) = 1$ for each vertex v of T.

5. Call an edge in a graph *unmatchable* if it is not contained in a perfect matching.

(a) Show that a regular class 1 graphs cannot contain an unmatchable edg.

(b) Give an example of a regular class 2 graph with no unmatchable edge.

6. Let $G = (V, E)$ be a bipartite graph, let $M \subseteq E$ be a matching, and let $I \subseteq V$ be an independent set. Prove that $|M| + |I| \leq |V|$, and equality holds if and only if M is a maximum matching and I is a maximum independent set.

7. Two players, Alice and Bob play the following game on a simple connected graph G. First, Alice picks a vertex v of G. Then, starting with Bob, players take turns choosing an edge e according to the following rule: e was not previously chosen and the set of all chosen edges (including e) forms a path containing v. The first player who cannot choose an edge according to this rule loses.

Show that Bob has a winning strategy if and only if G contains a perfect matching.

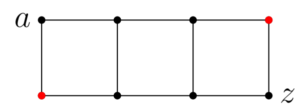
1. (a) We note that the function $g(x) = p_x$ is harmonic on $V \setminus \{a, z\}$ and that the constant function mapping $h(x) = 1$ is harmonic on all of V and thus also on $V \setminus \{a, z\}$.

Let $\beta = f(z)$ and $\alpha = f(a) - f(z)$. Since $p_a = 1$ and $p_z = 0$, we see that $f(a) = \alpha p_a + \beta = \alpha g(a) + \beta h(a)$, and $f(z) = \alpha p_z + \beta = \alpha g(z) + \beta h(z)$.

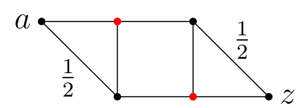
Since f is harmonic on $V \setminus \{a, z\}$, the superposition principle implies that $f(x) = \alpha g(x) + \beta h(x) = \alpha p_x + \beta$ for all x.

(b) By part (a), the set $\{g(x), h(x)\}$ from part (a) generates the vector space of functions which are harmonic on $V \setminus \{a, z\}$. If the two functions were linearly dependent, then there would be some $\alpha \in \mathbb{R}$ such that $\alpha g(x) = h(x)$ for all $x \in V$. But $g(z) = 0$ and $h(z) = 1$. Hence $\{g(x), h(x)\}$ forms a basis, and thus the dimension is 2.

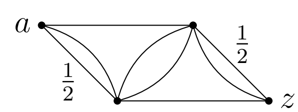
2. We iteratively apply network transformations.



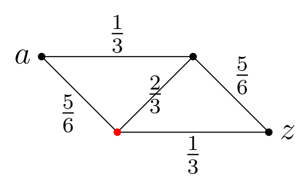
Applying the series law to both vertices marked in red above gives the following graph (all edge weights not explicitly given are equal to 1).



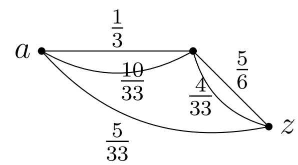
Applying the star-triangle law to the two vertices drawn in red gives the following graph where all unlabelled edges have conductance $\frac{1}{3}$ (due to the conductance formula in the star triangle law).



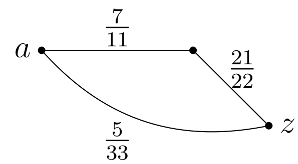
Next apply the parallel law to each pair of parallel edges.



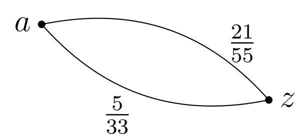
Apply the star triangle law to the vertex drawn in red to get



Apply the parallel law to all pairs of parallel edges to get the following graph:



Applying the series law at the only remaining vertex $\notin \{a, z\}$ gives



Finally, applying the parallel law to the pair of parallel edges gives a network consisting of vertices a and z and a single edge between them. This edge has conductance $\frac{8}{15}$. Since the transformations did not change the effective resistance, we have that $\mathcal{R}(a \leftrightarrow z) = \frac{15}{8}$.

3. No. For instance consider the graph obtained from a path a, x, y, z by adding 4 vertices a_1, a_2, a_3, a_4 incident to a and y, and 4 vertices z_1, z_2, z_3, z_4 incident to z and x.

To compute the voltage, note that $v(a_i) = \frac{1}{2}(v(a)+v(y))$, so the voltage at every a_i is the same. Similarly $v(z_i) = \frac{1}{2}(v(z) + v(x))$, so the voltage at every z_i is the same.

Hence the system defining the voltage can be simplified as follows:

$$\begin{aligned} 6v(x) &= v(y) + 4v(z_i) \\ 2v(a_i) &= v(y) \\ 6v(y) &= 1 + v(x) + 4v(a_i) \\ 2v(z_i) &= 1 + v(x) \end{aligned}$$

Solving this system gives $v(x) = \frac{3}{5}$, $v(y) = \frac{2}{5}$, $v(a_i) = \frac{1}{5}$ and $v(z_i) = \frac{4}{5}$, thus the voltage along the path a, x, y, z are not monotonic.

4. Let a and z be the endvertices of e. We know (Theorem 3.20) that $\mathbb{P}(e \in T) = \mathcal{R}_G(a \leftrightarrow z)$ and $\mathbb{P}(e \in T') = \mathcal{R}_H(a \leftrightarrow z)$, where $\mathcal{R}_G$ and $\mathcal{R}_H$ denote the effective resistance in G and H respectively.

Moreover, by Lemma 3.29, $\mathcal{R}_H(a \leftrightarrow z) = \mathcal{E}_H(i')$ where i' is a unit current flow from a to z in H, and $\mathcal{E}_H$ denotes the energy of this flow in H (it is not important here, but note that the energy in H and G are the same for this flow). Since H is a subgraph of G, the flow i' is still a flow of unit strength from a to z in G.

Let i be the unit current flow from a to z in G. Then by Thomson's principle $\mathcal{E}_G(i') \geq \mathcal{E}_G(i)$ where $\mathcal{E}_G$ denotes the energy in G. By Lemma 3.29 we know that $\mathcal{R}_G(a \leftrightarrow z) = \mathcal{E}_G(i)$, and the desired result follows.

5. (a) Let $\mathcal{E}_G$ and $\mathcal{E}_{G-e}$ denote the energy of flows in G and $G-e$, and let i and i' be current flows of unit strength in G and $G-e$, respectively. Note that $\mathcal{E}_{G-e}(i') = \mathcal{E}_G(i')$ because the weights of edges in G and $G-e$ are the same.

Thus (by Thomson's principle) $\mathcal{R}_{G-e}(a \leftrightarrow z) = \mathcal{E}_{G-e}(i') = \mathcal{E}_G(i') \geq \mathcal{E}_G(i) = \mathcal{R}_G(a \leftrightarrow z)$.

(b) Let x and y be the endpoints of e. We claim that $v(x) = v(y)$ where v is a voltage with pre-defined values at a and z. Note that this clearly implies $i(x,y) = 0$, and thus i is a flow from a to z in $G-e$, which in turn implies that $\mathcal{E}_G(i') \leq \mathcal{E}_G(i)$.

Let H be the subgraph of G containing all vertices which lie on some path from a to z. If a connected component of $G-H$ has more than one neighbour in H, then we can find a path from a to z using vertices of this connected component. Thus every connected component has a unique neighbour in H.

Let C be a connected component of $G-H$ and let G_C be the subgraph induced by this connected component and its unique neighbour x_c in H. Since a and z lie in H, and the only vertex in G_C which has neighbours outside G_c is x_c, the voltage must be harmonic in every vertex of G_C except x_c. By the uniqueness principle, the voltage must be constant on G_C. Since this applies to every connected component C, we conclude that $v(x) = v(y)$.

(c) Let P be a shortest path from a to z. Inductively apply (a), removing all edges $\notin P$ one by one we see that $\mathcal{R}_G(a \leftrightarrow z) \leq \mathcal{R}_P(a \leftrightarrow z)$. Inductively applying the series law shows that $\mathcal{R}_P(a \leftrightarrow z) = d(a,z)$.

Finally by Theorem 3.18, we have $\text{Comm}(a,z) = 2\sum_{e\in E} c(e)\mathcal{R}_G(a \leftrightarrow z) \leq 2|E|\, d(a,z)$.

(d) Let i,i', and i'' be unit current flows from a to b for the weights c, c', and c'', respectively. Let $\mathcal{E}$, $\mathcal{E}'$, and $\mathcal{E}''$ denote the energies of flows with respect to the corresponding weights.

Then

$$\begin{aligned}
\mathcal{R}_{\frac{r+r'}{2}}(a \leftrightarrow b) &= \mathcal{E}''(i'') && \text{by Lemma 3.29} \\
&= \sum_{\substack{x\in V \\ y\in N(X)}} i''(x,y)^2 \frac{r(\{x,y\}) + r'(\{x,y\})}{2} && \text{by definition} \\
&= \sum_{\substack{x\in V \\ y\in N(X)}} i''(x,y)^2 \frac{r(\{x,y\})}{2} + \sum_{\substack{x\in V \\ y\in N(X)}} i''(x,y)^2 \frac{r'(\{x,y\})}{2} \\
&= \frac{1}{2}\mathcal{E}(i'') + \frac{1}{2}\mathcal{E}'(i'') && \text{by definition} \\
&\geq \frac{1}{2}\mathcal{E}(i) + \frac{1}{2}\mathcal{E}'(i') && \text{Thomson's principle} \\
&= \frac{1}{2}\mathcal{R}_r(a \leftrightarrow b) + \frac{1}{2}\mathcal{R}_r(a \leftrightarrow b) && \text{by Lemma 3.29}
\end{aligned}$$

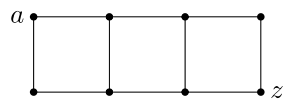
Give appropriate justifications for your answers.

1. Let $G = (V, E, c)$ be a weighted graph and let a and z be vertices of G. Let p_x denote the probability that a random walk started at x reaches a before z.

(a) Prove that if $f \colon V \to \mathbb{R}$ is harmonic on $V \setminus \{a, z\}$, then there are constants $\alpha, \beta \in \mathbb{R}$ such that $f(x) = \alpha p_x + \beta$.

(b) Using part (a) or otherwise, determine the dimension of the vector space of functions which are harmonic on $V \setminus \{a, z\}$.

2. Compute $\mathcal{R}(a \leftrightarrow z)$ in the graph drawn below.



3. Let v be a voltage function defined on the vertices of a finite (unweighted) graph by $v(a) = 0$ and $v(z) = 1$ and harmonic elsewhere. Must the voltages of the vertices along every shortest path between a and z be non-decreasing?

4. Let H be a connected subgraph of a connected graph G and let e be an edge of H. Let T be a random spanning tree of H and let T' be a random spanning tree of G. Show that $\mathbb{P}[e \in T] \geq \mathbb{P}[e \in T']$.

5. Let G be a graph, let a and z be vertices, and let e be an edge. Let $\mathcal{R}_G$ and $\mathcal{R}_{G-e}$ denote the effective resistance in G and $G - e$, respectively

(a) Prove that $\mathcal{R}_G(a \leftrightarrow z) \leq \mathcal{R}_{G-e}(a \leftrightarrow z)$.

(b) Prove that if e does not lie on any path from a to z, then $\mathcal{R}_G(a \leftrightarrow z) = \mathcal{R}_{G-e}(a \leftrightarrow z)$.

(c) Using part (a) or otherwise, prove that $\mathrm{Comm}(a \leftrightarrow z) \leq 2|E| \cdot d(a, z)$, where $d(a, z)$ denotes the length of a shortest path from a to z in G.

6. Show that the effective resistance is concave as a function of the individual resistances in the sense that

$$\frac{1}{2}(\mathcal{R}_r(a \leftrightarrow z) + \mathcal{R}_{r'}(a \leftrightarrow z)) \leq \mathcal{R}_{\frac{r+r'}{2}}(a \leftrightarrow z),$$

where $\mathcal{R}_r$, $\mathcal{R}_{r'}$, and $\mathcal{R}_{\frac{r+r'}{2}}$ denote effective resistances in the same graph with respect to the different weight functions $c = \frac{1}{r}$, $c' = \frac{1}{r'}$ and $c'' = \frac{2}{r+r'}$, respectively.

Hint: consider current flows of unit strength.

1. (a) We use Theorem 4.10 to work out the remaining parameters. By (ii), $2(v-1) = 4(11-1)$, so $v = 21$. By (i), $b \cdot 11 = 21 \cdot 4$, but $\frac{84}{11}$ is not an integer, so no such design can exist.

(b) Such a design exists: there is an affine plane of order 5 because 5 is a prime, and this affine plane is a $(25, 30, 6, 5, 1)$ design.

(c) We again use Theorem 4.10 to work out the remaining parameters. By (ii), $2 \cdot 45 = 10(k-1)$, so $k = 10$. By (i), $b \cdot 10 = 46 \cdot 10$, so $b = 46$. We note that $v = b$, and v is even, but $k - \lambda = 8$ which is not a square, so by (iv) no such design exists.

2. Since the design is symmetric, we have $b = v = 40$, and we further know that $k = r$. Theorem 4.10 (ii) now tells us that $\lambda \cdot 39 = k(k-1)$, so 39 must divide $k(k-1)$. This means that 3 divides $k(k-1)$, and thus $k \mod 3$ is either 0 or 1, and that 13 divides $k(k-1)$, and thus $k \mod 13$ is either 0 or 1. The only values for $k < 40$ that satisfy these conditions are $k = 13$, $k = 27$ and $k = 39$.

Since $r = k$ and $b = v$ we can determine the corresponding values of λ.

If $k = 13$, then $\lambda = \frac{13 \cdot 12}{39} = 4$.

If $k = 27$, then $\lambda = \frac{27 \cdot 26}{39} = 18$.

If $k = 39$, then $\lambda = \frac{39 \cdot 38}{39} = 38$.

3. (a) By Fisher's inequality we know that $b \geq v$, and therefore by Theorem 4.10 (i), $k = \frac{br}{v} \geq r$. By Theorem 4.10 (ii), we have that $2(v-1) = r(k-1) \leq r(r-1)$ and thus $v \leq \frac{r(r-1)}{2} + 1 = \binom{r}{2} + 1$.

(b) Note that by Theorem 4.10 (ii) we have $2(v-1) = 7(k-1)$, so $v - 1$ must be divisible by 7. Together with $v \leq \binom{7}{2} + 1$ from (a) this only leaves the options $v = 1$, $v = 8$, $v = 15$, and $v = 22$.

If $v = 1$, then $k = 1$, which contradicts the assumption $k > 1$.

If $v = 8$, then $k = 3$. By Theorem 4.10 (i) we have that $b = \frac{vr}{k} = \frac{56}{3}$ which is not an integer, so no such design can exist.

If $v = 22$, then $k = 7$. By Theorem 4.10 (i) $b = \frac{vr}{k} = 22$. Since $b = v$ and v is even, $k - \lambda$ would have to be a square number for such a design to exist, but $k - \lambda = 5$.

4. Let (P, L) be the points and lines of the projective plane, and let $l_\infty \in L$. We show that (A1), (A2), and (A3) are satisfied for the incidence structure with point set $P' = P \setminus l_\infty$ and set of blocks $L' = L \setminus \{l_\infty\}$ (i.e. remove all points in l_∞ from P, and remove the line l_∞ from L).

For (A1), note that by $(P1)$ each pair of points $p, q \in P'$ of lies on a unique line $l_{pq} \in L$. If $l_{pq} \notin L'$, then $l_{pq} = l_\infty$, and therefore neither p nor q are contained in P'.

For (A2), let $l \in L'$ be a line and let $p \in P'$ be a point not on l. Let $q = l \cap l_\infty$, and note that q is the only point of l which is not contained in P'. Hence, the line in L' corresponding to k

is disjoint from the line corresponding to l if and only if $k \cap l = q$. There is a unique such line through p, namely the line l_{pq}.

Finally, for (A3), let p_1, p_2, p_3, p_4 be four points in P such that no three of them lie on a common line (these exist by (P3)). If none of the points lie on l_∞, then all of them are contained in P' and therefore witness (A3).

Now assume that at least one of the points lies on l_∞. Note that l_∞ contains at most two of the four points, so without loss of genarality $p_4 \in l_\infty$ and p_1 and p_2 do not lie on l_∞. Let l_ij denote the unique line containing p_i and p_j, and let $q_1 = l_{14} \cap l_{23}$ and $q_2 = l_{24} \cap l_{13}$.

Now l_{14} contains p_1 and q_1, but not p_2 or q_2 (otherwise it would coincide with l_{12} or l_{13}). Similarly, l_{13} contains p_1 and q_2, but not p_2 or q_1, l_{24} contains p_2 and q_2, but not p_1 or q_1, and l_{23} contains p_2 and q_1, but not p_1 or q_2. This shows that no 3 of the points p_1, p_2, q_1, q_2 lie on a common line. Finally note that $p_1, p_2 \in P'$ by assumption, and if $q_i \in l_\infty$, then we would have $l_\infty = l_{i4}$, a contradiction.

5. First, construct an affine plane of order 7 using Construction 4.24 from the course notes. Next carry out Construction 4.59 with $C_0 = \{L_{0,b} \mid b \in \mathbb{Z}_7\}$, and $C_n = \{V_a \mid a \in \mathbb{Z}_7\}$.

For each parallel class $P_a := \{L_{a,b} \mid b \in \mathbb{Z}_7\}$ this gives a Latin square M_a; the entry $(M_a)_{i,j}$ at position (i,j) is the unique b such that $L_{a,b}$ contains the point $(i,j) = V_i \cap L_{0,j}$. In other words, this is the unique b for which $ai + b = j$, so $(M_a)_{i,j} = j - ai$.

It follows from Exercise 4.60 that this gives a set of 6 MOLS.

Alternatively, we can check by hand that this is a set of 6 MOLS as follows. The element $x \in \mathbb{Z}_7$ appears as the $((x - ai) \mod 7)$-th entry of the i-th row of M_a, so each row contains every entry exactly once. Moreover, since 7 is a prime number, for each $a \in \mathbb{Z}_7 \setminus \{0\}$ there is an inverse element a^{-1} satisfying $aa^{-1} \equiv 1 \mod 7$; the element $x \in \mathbb{Z}_7$ appears as the $(a^{-1}(x - j)$ mod 7)-th entry of the j-th column of M_a, so each column contains every entry exactly once. Therefore each M_a is a Latin square.

Finally we need to check that M_a and $M_{a'}$ are orthogonal for $a \neq a'$. Let (x, y) be a pair of elements in $\mathbb{Z}_7$. We need to show that there are i, j such that $(M_a)_{i,j} = x$ and $(M_{a'})_{i,j} = y$. This corresponds to a solution of

$$j - ai = x \qquad j - a'i = y.$$

This system can be solved in $\mathbb{Z}_7$, giving $i = (y - x)(a - a')^{-1}$, and $j = x - a(y - x)(a - a')^{-1}$

6. (a) There are many possible completions. For instance:

1	2	3	4	5
2	1	5	3	4
3	4	1	5	2
4	5	2	1	3
5	3	4	2	1

(b) Take any completion of the latin rectangle. We will refer to the 2×2 sub-square B in the top left corner as the 'top left block', to the 3×2 rectangle below B as the 'bottom left block' and to the 2×3 rectangle to the right of B as the top right block.

Let T be a transversal. If T contains no elements in B, then it must contain 2 elements from the top right block (to include elements in the first two rows), and 2 elements in the bottom left block (to include elements in the first two columns). But all elements in the top right and bottom left block are either 3 ,4, or 5, so at most three of them can be included in any transversal.

Thus every transversal contains at least one element of B, and since B only has 4 elements, there can be at most 4 disjoint transversals. By Theorem 3.63 this means that the latin square has no orthogonal mate.

Give appropriate justifications for your answers.

1. For each of the following parameter values either determine the remaining parameters and give an example of a (v, b, r, k, λ)-BIBD with these parameters, or show that no such BIBD exists.

(a) $r = 4$, $k = 11$, $\lambda = 2$.

(b) $b = 30$, $r = 6$, $k = 5$.

(c) $v = 46$, $r = 10$, $\lambda = 2$.

2. Consider a symmetric balanced design with parameters (v, b, r, k, λ) with $v = 40$. Show that $\lambda \in \{4, 18, 38\}$.

(You do not have to construct BIBDs with these parameters.)

3. Consider a BIBD with parameters (v, b, r, k, λ).

(a) Show that if $\lambda = 2$, then $v \leq \binom{r}{2} + 1$.

(b) Show that if $\lambda = 2$, $r = 7$, and $k > 1$, then $v = 15$.
(Again, you do not have to construct BIBDs with these parameters.)

4. Prove that Construction 4.31 in the course notes works; in other words, show that if a line ℓ and all points incident to ℓ are removed from a projective plane, the result is an affine plane.

5. Construct a set of 6 MOLS of order 7 (you do not need to write them out in full, it suffices to give a formula for the entry in position i, j and show that this is a set of MOLS).

6. Consider the following partially filled latin square

1	2	3	4	5
2	1	5	3	4

(a) Find a completion.

(b) Show that there is no completion which has an orthogonal mate.
Hint: use Theorem 4.63.

1. (a) We concatenate the codewords $\psi(f) = 0010$, $\psi(a) = 11$, $\psi(d) = 10$, $\psi(e) = 0011$, and $\psi(d) = 10$ to get the encoded message 00101110001110.

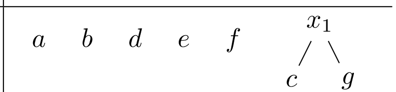
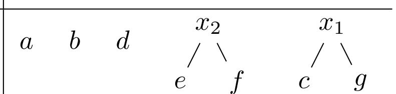
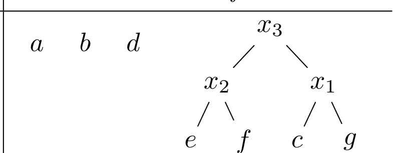
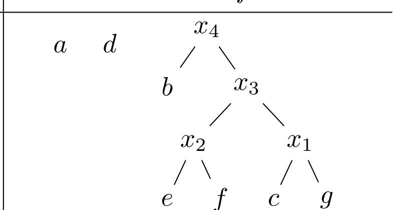
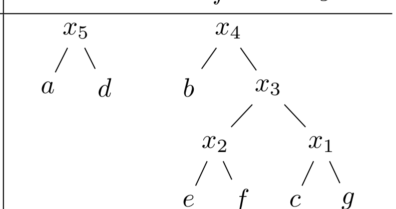
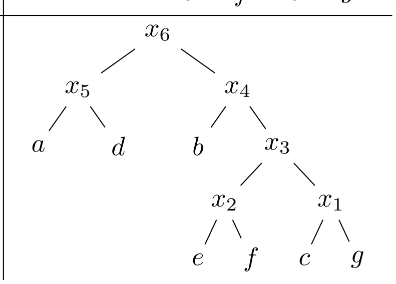
(b) We read the encoded message from left to right, inserting a comma every time we have read a complete codeword $0001, 11, 0010, 0011$. Since $0001 = \psi(c)$, $11 = \psi(a)$, $0010 = \psi(f)$, and $0011 = \psi(e)$, this decodes to $cafe$.

(c) There are many sets of frequencies that work. A very simple one is to assign probability 2^{-n} to every letter with codeword length n, but I will use a slightly different set of frequencies which will be more useful in part (d).

Let us set $p_a = p_d = 0.3$, $p_b = 0.2$, $p_c = p_g = 0.04$, and $p_e = p_f = 0.06$.

We start the construction of the Huffman tree with a set of isolated vertices, In each step of the construction of the Huffman tree, we take the two letters q_1, q_2 with the lowest frequencies, replace them by a new letter r which has frequency $p_r = p_{q_1} + p_{q_2}$, and attach nodes labelled q_1 and q_2.

The step-by-step construction looks as follows:

	alphabet + frequencies	current forest
replace c and g by x_1	a b d e f x_1 0.3 0.2 0.3 0.06 0.06 0.08	
replace e and f by x_2	a b d x_2 x_1 0.3 0.2 0.3 0.12 0.08	
replace x_1 and x_2 by x_3	a b d x_3 0.3 0.2 0.3 0.2	
replace b and x_3 by x_4	a d x_4 0.3 0.3 0.4	
replace a and d by x_5	x_5 x_4 0.6 0.4	
replace x_4 and x_5 by x_6	x_6 1	

This tree corresponds to the given code with the convention that left child means 1, and right child means 0 in the codeword.

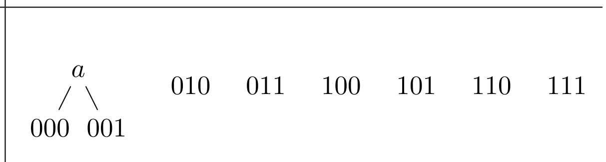
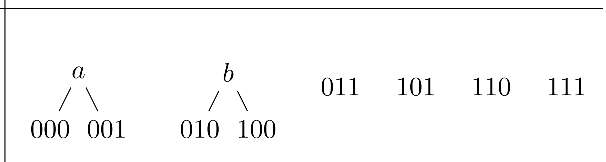
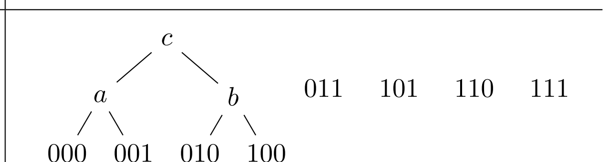
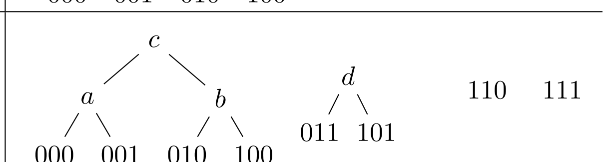
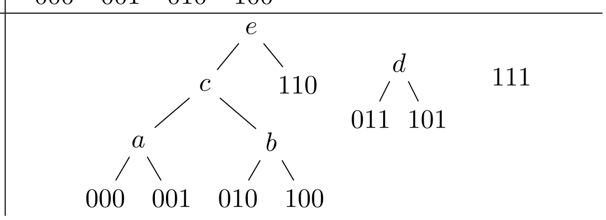
(d) Consider the frequencies $p_a = p_d = 0.3$, $p_b = 0.2$, $p_c = p_e = 0.04$, and $p_f = p_g = 0.06$. Constructing the Huffman code with these frequencies leads to the same codeword lengths as in part (b) because we only swapped the frequencies of the letters e and g whose codewords had the same length. Hence the codeword lengths of the given code ψ are the same as in a Huffman code with these letter frequencies, so ψ is optimal.

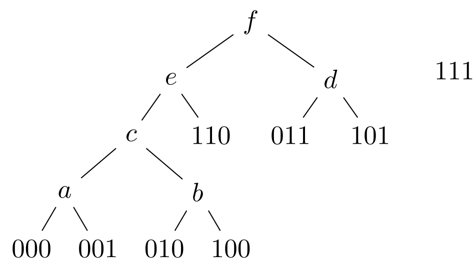
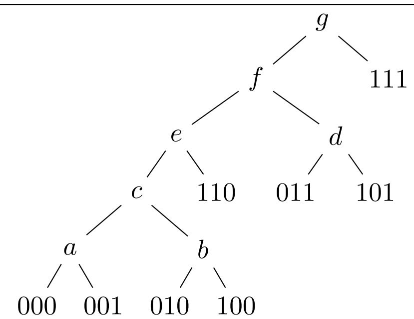
To see that ψ is not a Huffman code, recall that in a Huffman code the lowest frequency letters are always siblings. The letters with the lowest frequencies are c and e, but they are not siblings in the tree corresponding to the code ψ because their codewords differ before the last bit.

2. (a) Since the individual bits are independent, the probability of a codeword is simply the product of the probabilities of the individual bits. This gives

$$p_{000} = \frac{1}{4}\cdot\frac{1}{4}\cdot\frac{1}{4} = \frac{1}{64} \qquad p_{100} = \frac{3}{4}\cdot\frac{1}{4}\cdot\frac{1}{4} = \frac{3}{64}$$
$$p_{001} = \frac{1}{4}\cdot\frac{1}{4}\cdot\frac{3}{4} = \frac{3}{64} \qquad p_{101} = \frac{3}{4}\cdot\frac{1}{4}\cdot\frac{3}{4} = \frac{9}{64}$$
$$p_{010} = \frac{1}{4}\cdot\frac{3}{4}\cdot\frac{1}{4} = \frac{3}{64} \qquad p_{110} = \frac{3}{4}\cdot\frac{3}{4}\cdot\frac{1}{4} = \frac{9}{64}$$
$$p_{011} = \frac{1}{4}\cdot\frac{3}{4}\cdot\frac{3}{4} = \frac{9}{64} \qquad p_{110} = \frac{3}{4}\cdot\frac{3}{4}\cdot\frac{3}{4} = \frac{27}{64}$$

(b) As in problem 1(c), we inductively construct the Huffman tree:

	letter frequencies	current forest
replace 000 and 001 by a	a 4/64, 010 3/64, 011 9/64, 100 3/64, 101 9/64, 110 9/64, 111 27/64	
replace 010 and 100 by b	a 4/64, b 6/64, 011 9/64, 101 9/64, 110 9/64, 111 27/64	
replace a and b by c	c 10/64, 011 9/64, 101 9/64, 110 9/64, 111 27/64	
replace 011 and 101 by d	c 10/64, d 18/64, 110 9/64, 111 27/64	
replace c and 110 by e	e 19/64, d 18/64, 111 27/64	

replace d and e by f	f 111 $37/64$ $27/64$	
replace f and 111 by g	g 1	

With the convention that left child means 1, and right child means 0 in the codeword this gives the following code:

$$\psi(000) = 11111 \qquad \psi(001) = 11110 \qquad \psi(010) = 11101 \qquad \psi(011) = 101$$
$$\psi(100) = 11100 \qquad \psi(101) = 100 \qquad \psi(110) = 110 \qquad \psi(111) = 0$$

(c) The average codeword length is given by

$$\sum_{x\in\mathbb{Z}_2^3} p_x \cdot \ell(\psi(x)) = \frac{1}{64}\cdot 5 + \frac{3}{64}\cdot 5 + \frac{3}{64}\cdot 5 + \frac{9}{64}\cdot 3 + \frac{3}{64}\cdot 5 + \frac{9}{64}\cdot 3 + \frac{9}{64}\cdot 3 + \frac{27}{64}\cdot 1 = \frac{79}{32} \approx 2.469$$

Compared to the original encoding where every codeword uses 3 bits, on average we save $\frac{17}{32}$ bits per code word (this is $\approx 17.7\%$).

3. (a) A codeword in Fitingof's code consists of the prefix which has fixed length $\lceil \log_2(14)\rceil = 4$ and the suffix, which has length $\lceil \log_2 \binom{14}{w}\rceil$, where w is the Hamming weight of the element of $\mathbb{Z}_2^{14}$.

The value of $\binom{14}{w}$, and thus also the codeword length is minimised when $w = 0$ or $w = 14$, and $\binom{14}{0} = \binom{14}{14} = 1$. The codeword for an element of Hamming weight 0 or 14 has $4 + \lceil \log_2 1\rceil = 4$ bits.

The value of $\binom{14}{w}$, and thus also the codeword length is maximised when $w = 7$, and $\binom{14}{7} = 3432$. The codeword for an element of Hamming weight 7 has $4 + \lceil \log_2 3432\rceil = 4 + 12 = 16$ bits.

(b) the codeword is shorter than 14 bits if and only if $\lceil \log_2 \binom{14}{w}\rceil < 10$, or equivalently $\binom{14}{w} \leq 2^9 = 512$. We see that $\binom{14}{3} = 364 < 2^9$ and $\binom{14}{4} = 1001 > 2^9$. Using the fact that $\binom{n}{k} = \binom{n}{n-k}$ and monotonicity of the binomial coefficients for $k \leq \frac{n}{2}$, we conclude that the codeword has fewer than 14 bits if and only if the hamming weight is either at most 3, or at least 11.

(c) We first determine the prefix. As noted in (a), the prefix length is $\lceil \log_2(14)\rceil = 4$. The Hamming weight of the string 00100000001100 is 3. Hence the prefix is 0011 (the binary representation of 3 using 4 bits).

Next we determine the suffix. Suffix length is $\lceil \log_2 \binom{14}{3} = 9$. The 1s in the bitstring are in positions 3, 11, and 12, hence we compute the ordinal number of this string as

$$\binom{14-12}{1} + \binom{14-11}{2} + \binom{14-3}{3} = 2 + 3 + 165 = 170.$$

The binary representation of 170 is 10101010, so the suffix id 010101010 (because suffix length is 9).

Finally, we obtain the codeword by concatenating prefix and suffix as 0011010101010.

(d) the first 4 bits must be the prefix of the first codeword. Since 0011 is 3 in decimal, this means that the codeword has Hammming weight 3, and therefore the suffix has length $\lceil \log_2 \binom{14}{3} = 9$.

Thus the suffix is 000001100, which translates to 12 in decimal. We need to solve

$$\binom{14-n_3}{1} + \binom{14-n_2}{2} + \binom{14-n_1}{3} = 12.$$

By Proposition 4.30 in the lecture notes, $14 - n_1$ is equal to the largest x for which $\binom{x}{3} \leq 12$. We compute $\binom{5}{3} = 10$ and $\binom{6}{3} = 10$ and conclude that $14 - n_1 = 5$, and thus $n_1 = 9$.

Next, $14 - n_2$ can be found as the maximal x for which $\binom{x}{2} \leq 12 - \binom{14-n_1}{3} = 2$. Since $\binom{2}{2} = 1$ and $\binom{3}{2} = 3$ we see that $14 - n_2 = 2$, and hence $n_2 = 12$.

Finally $14 - n_2$ can be found as the maximal x for which $\binom{x}{2} \leq 2 - \binom{14-n_2}{2} = 1$. Since $\binom{1}{1} = 1$ we conclude that $n_2 = 13$.

Hence the decoded string has 1s in positions 9, 12, and 13, and thus the first block decodes to 00000000100110.

This is a practice assignment and will not be worth any marks. You are still welcome to submit solutions if you want feedback on them.

1. Suppose that a text source generated from the alphabet $X = \{\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}, \mathrm{e}, \mathrm{f}, \mathrm{g}\}$ is encoded via the code $\psi : X \to W(\mathbb{Z}_2)$ defined as follows:

$$\phi(\mathrm{a}) = 11, \quad \phi(\mathrm{b}) = 01, \quad \phi(\mathrm{c}) = 0001, \quad \phi(\mathrm{d}) = 10, \quad \phi(\mathrm{e}) = 0011, \quad \phi(\mathrm{f}) = 0010, \quad \phi(\mathrm{g}) = 0000.$$

(a) Encode the word *faded*.

(b) Decode the codeword 00011100100011.

(c) Find a set of frequencies (probabilities) for the letters in the source that would produce this code as a Huffman code. Justify your answer by constructing the corresponding binary tree using Huffman's method.

(d) Find a set of frequencies for the letters such that the code is optimal, but not a Huffman code.

2. We want to compress a bitstring from a source which generates (independent) random bits with frequencies $p_0 = \frac{1}{4}$ and $p_1 = \frac{3}{4}$ by grouping it into substrings of length 3, and then constructing a Huffman code for these substrings (you may assume that the length of the whole bitstring is divisible by 3, so that the grouping into substrings actually works out without remainder).

(a) For each $x \in \mathbb{Z}_2^3$ compute the frequency p_x of x.

(b) Construct a Huffman code $\psi \colon \mathbb{Z}_2^3 \to W(\mathbb{Z})$ with respect to these frequencies.

(c) What is the average length of a codeword with respect to this Huffman code? How much space do we save compared to the original encoding?

3. Let $\psi : \mathbb{Z}_2^{14} \to W(\mathbb{Z}_2)$ be Fitingof's code.

(a) What are the maximal and minimal length of $\psi(\mathbf{x})$ for $\mathbf{x} \in \mathbb{Z}_2^{14}$?

(b) For which Hamming weights $w(\mathbf{x})$ is the encoded codeword $\psi(\mathbf{x})$ shorter than 14 bits?

(c) Encode the word $\mathbf{x} = 00100000001100$.

(d) A file which was compressed using the code ψ starts as follows:

$$00110000011000110\ldots$$

Determine the first 14 bits of the decoded file, in other words, decode the first 14-bit block.