Robert Christie

MATHS 326

S1 2024

Assignment 4
Due: 24-05-2024

Q1

(a) We are given $r = 4, k = 11, \lambda = 2$, assume that these are parameters for a balanced design, by *Theorem 4.10*:

$$bk = vr \implies 11b = 4v$$

$$\lambda(v-1) = r(k-1) \implies 2(v-1) = 4(11-1) \implies 2v - 2 = 40 \implies v = 21$$

Contradiction as this implies $b = \frac{4 \cdot 21}{11}$ which is not an integer as neither 4 nor 21 have a prime factor of 11. Thus, no balanced block design has these parameters.

(b) We are given $b = 30, r = 6, k = 5$, assume that these are parameters for a balanced design, by *Theorem 4.10*:

$$bk = vr \implies 30 \cdot 5 = 6v \implies v = 25$$

$$\lambda(v-1) = r(k-1) \implies \lambda(24) = 6(4) \implies \lambda = 1$$

Using *Construction 4.24* with the field $\mathbb{F} = \mathbb{Z}_5$, we can construct an affine plane of order 5. From the proof of *Theorem 4.38* an affine plane of order $n = 5$ with is a balanced block design with parameters:

$$(v, b, r, k, \lambda) = (n^2, n^2 + n, n + 1, n, 1) = (25, 30, 6, 5, 1)$$

Thus we can construct affine plane which is a BIBD satisfying the given parameters.

(c) We are given $v = 46, b = 10, \lambda = 2$, assume that these are parameters for a balanced design, by *Theorem 4.10*:

$$bk = vr \implies 10k = 46r \implies k = 4.6r$$

$$\lambda(v-1) = r(k-1) \implies 2(46-1) = r(k-1) \implies 0 = 4.6r^2 - r - 90$$

Solving for possible values of r using the quadratic equation:

$$r = \frac{1 \pm \sqrt{1657}}{4.6}$$

Which has no integer solutions as $40^2 < 1657 < 41^2$, hence $\sqrt{1657}$ is irrational.

This is a contradiction so no balanced block design has these parameters.

Q2

Assume that there is a BIBD for $v = b = 40$ with parameters (v, b, r, k, λ). Then since $vb = rk$ we have $k = r$. Thus:

$$\lambda(v-1) = r(k-1) \implies 39\lambda = r(r-1) = k(k-1)$$

Since the design is incomplete, $r, k, \lambda \leq 39$:

Since we have $\lambda = k(k-1)/39$:

$$k - \lambda \in \{1, 2, 4, 9, 16, 25, 36\}$$

$$(39k - k(k-1))/39 \in \{1, 2, 4, 9, 16, 25, 36\}$$

We can factorise $39 = 3 \cdot 13$ and $\lambda = \lambda_1\lambda_2$. This gives the following cases:

Case	r	$r-1$	
A	$39\lambda_1$	λ_2	$39\lambda_1 = \lambda_2 + 1$
B	$13\lambda_1$	$3\lambda_2$	$13\lambda_1 = 3\lambda_2 + 1$
C	$3\lambda_1$	$13\lambda_2$	$3\lambda_1 = 13\lambda_2 + 1$
D	λ_1	$39\lambda_2$	$\lambda_1 = 39\lambda_2 + 1$

Solving for λ in each of these cases:

Case A

$$\begin{aligned} \lambda_1 = 1 &\implies \lambda_2 = 38 \\ \lambda_1 \geq 2 &\implies \lambda > 39 \end{aligned}$$

$$\implies \lambda \in \{38\}$$

Case B

$$\begin{aligned} \lambda_1 = 1 &\implies \lambda_2 = 4 \\ \lambda_1 = 2 &\implies \lambda_2 = 25/3 \notin \mathbb{Z} \\ \lambda_1 \geq 3 &\implies \lambda > 39 \end{aligned}$$

$$\implies \lambda \in \{4\}$$

Case C

$$\begin{aligned} \lambda_2 = 1 &\implies \lambda_1 = 14/3 \notin \mathbb{Z} \\ \lambda_2 = 2 &\implies \lambda_1 = 9 \\ \lambda_2 = 3 &\implies \lambda_1 = 40/3 \notin \mathbb{Z} \\ \lambda_2 \geq 3 &\implies \lambda > 39 \end{aligned}$$

$$\implies \lambda \in \{18\}$$

Case D

$$\lambda_2 \geq 1 \implies \lambda > 39$$

$$\implies \lambda \in \varnothing$$

So we must have $\lambda \in \{4, 18, 38\}$.

Q3

(a) CLAIM: Every pair of blocks has at most 1 common vertex. (FALSE!!!)

(b) By (a), since $\lambda = 2$:

$$v \leq \binom{7}{2} + 1 = 22$$

By *Theorem 4.10*:

$$\lambda(v-1) = r(k-1) \implies v = \frac{7(k-1)}{2} + 1$$

To have $v \in \mathbb{Z}$, k must be odd, also $k > 1$, searching for possible values of v:

$$\begin{aligned}
k = 3 &\implies v = 8 \implies b = \frac{8 \cdot 7}{3} \notin \mathbb{Z} \\
k = 5 &\implies v = 15 \implies b = 21 \\
k = 7 &\implies v = 22 \implies b = 22 \\
k = 9 &\implies v > 22
\end{aligned}$$

So we either have $v = 15$ or $v = 22$.

For $v = 22$, notice that since $r = k$, we have $b = v$ and v is even. So by even case of the Bruck-Ryser-Chowla Theorem, $k - \lambda = 5$ should be a perfect square. Since 5 is not a perfect square, there is no BIBD with these parameters.

So if a BIBD does have $\lambda = 2$, $r = 7$, $k > 1$, it must have $v = 15$.

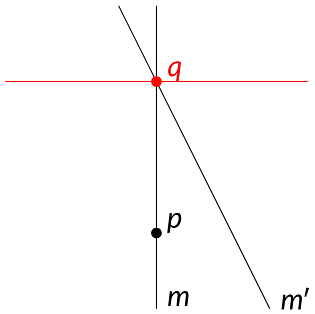
Q4

First we verify that the construction can be performed. By P3, there are at least 4 points, and by P1, any distinct pair of these points is on a unique line. Thus, there is a line ℓ to remove.

We check the axioms for an affine plane hold for the plane constructed by removing ℓ:

- A1: Any two points in the construction already existed on some unique line m in the projective plane, we have $m \neq \ell$ otherwise we would have removed the points. Hence, the line m is present in the construction. As no other lines have become incident with the points so m is the unique line incident with both points.
- Consider any point p and line m' in the constructed plane such that m' is not incident on p.

  Clearly m' is distinct from ℓ, so by P2, m and ℓ have a unique common point q. Now by P1, q and p lie on a unique line m. Since $p \in m$ but $p \notin \ell$ $m \neq \ell$ and so m is present in the constructed plane. Since q is the only common point of m and m', and is not present in the constructed plane, $m \cap m' = \emptyset$.

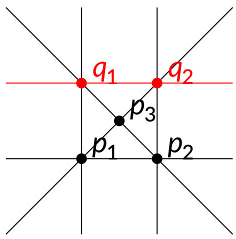
  It remains to show that m is unique. Assume that there is another line k such that $m \neq k$ while k is also incident on p and has $k \cap m' = \emptyset$.

  Since $k \neq m$, k cannot be incident on q, by P2 k and m' have a unique common point q', since this point cannot be q, and q is the only point on both m' and ℓ, q' is not on ℓ.

  Therefore, q' is in the constructed plane, meaning that $k \cap m' \neq \emptyset$. This is a contradiction, therefore m is unique.
- A3: By P3, the projective plane contains 4 points no 3 of which are collinear. If ℓ is not incident on any of these points, they are in the constructed plane and A3 is satisfied. Otherwise:

  Assume that we have 4 points no 3 of which are collinear, then ℓ is incident on at most 2 points. First address the case where ℓ is incident on 2 points.

  WLOG, let ℓ be incident on q_1, q_2 but no p_1, p_2, since each pair of points is on a unique line, and no three points are collinear, we can make the following construction:

  Where p_3 is the unique common point of the unique lines a_1, a_2 through q_1, p_2 and q_2, p_1 respectively. The lines b_1, b_2 containing q_1, p_1 and q_2, p_2 must also have a unique common point $p_4 \notin \{q_1, q_2, p_1, p_2\}$ (otherwise there would be 3 collinear points). Note that a_1, a_2, b_1, b_2 must be distinct, if any were the same line, then three of q_1, q_2, p_1, p_2 must be collinear.

  We claim that no three points of p_1, p_2, p_3, p_4 are collinear.

Assume for contradiction that 3 points are collinear, the 3 points must lie on the same line. If any two points were on the same line m, then m would share two common points with one of a_1, a_2, b_1, b_2, hence $m \in \{a_1, a_2, b_1, b_2\}$. However, this means that m would share two common points with a second line $m' \in \{a_1, a_2, b_1, b_2\} \setminus m$ meaning $m = m'$ and the 4 lines aren't distinct.

The points p_1, p_2, p_3, p_4 aren't on ℓ, and so the set of 4 points no 3 of which are collinear exists in the constructed plane.

Case where only a single point q of the 4 is on ℓ.

Q5

Consider the set $\{L_1, \dots, L_6\}$ of order 7 Latin squares with entries $(L_k)_{ij} = i + kj \bmod 7$. Verifying that this is a set of Latin squares:

$$\begin{aligned} & (L_k)_{ij} = (L_k)_{ij'} \\ \implies & i + kj = i + kj' \\ \implies & \quad kj = kj' \\ \implies & \quad j = j' \qquad \text{Divide by } k \end{aligned}$$

$$\begin{aligned} & (L_k)_{ij} = (L_k)_{i'j} \\ \implies & i + kj = i' + kj \\ \implies & \quad i = i' \end{aligned}$$

Note that we can divide by k in mod 7 as $1, \dots, 6$ are not zero divisors.

Assume that $k \neq k'$, and $(i, j) \neq (i', j')$, now for the sake of contradiction assume that:

$$\begin{aligned} \Big((L_k)_{ij}, (L_{k'})_{ij}\Big) &= \Big((L_k)_{i'j'}, (L_{k'})_{i'j'}\Big) \\ \Big(i + kj, i + k'j\Big) &= \Big(i' + kj', i' + k'j'\Big) \\ \Big(0, 0\Big) &= \Big((i - i') + k(j - j'), (i - i') + k'(j - j')\Big) \\ (i - i') + k(j - j') &= (i - i') + k'(j - j') \\ k(j - j') &= k'(j - j') \\ 0 &= (k - k')(j - j') \end{aligned}$$

However, the only zero divisor in mod 7 is 0, thus either $k - k' = 0$, or $j - j' = 0$. Since we assumed $k - k' = 0$, we must have $j = j'$. However, we have that:

$$\Big(0, 0\Big) = \Big((i - i') + k(j - j'), (i - i') + k'(j - j')\Big) \implies 0 = i - i' \implies i = i'$$

This is a contradiction. Thus, $\{L_1, \dots, L_6\}$ are a set of 6 MOLS of order 7.

Q6

(a) The square was completed in the following order:

- The gray cells were given.
- The blue must be some permutation of $3, 4, 5$ and can be re-ordered by interchanging rows, order chosen WLOG.
- The violet cells must also be some permutation of $3, 4, 5$ distinct from the ordering of the blue cells. There are two options, the other failed to complete the square.
- The cyan cell, had to be either 1 or 2, since no remaining cells are constrained by a 1 or 2, the choice is made WLOG.
- Each cell with a single remaining possibility was filled until the square was complete.

1	2	3	4	5
2	1	5	3	4
3	4	1	5	2
4	5	2	1	3
5	3	4	2	1

(b) A Latin square of order 5 has an orthogonal mate if and only if it contains 5 disjoint traversals. Assume that there exist 5 disjoint traversals of some completion.

Consider the top left 2×2 region, each cell must be in a different traversal:

The region contains the following cells:

1	2
2	1

If two cells are in the same traversal, they cannot be in the same row/column, thus they must be diagonal (in the 2×2 region). All diagonal entries are the same, so they cannot be part of the same traversal.

Thus, each cell in the 2×2 region is part of a distinct traversal.

Label the traversals A, B, C, D, E, WLOG we can fix the traversal that each of the cells in the 2×2 region are part of. Since each traversal appears once in each row/column, we can deduce which traversals the cells in each region must be assigned to:

A	B	C, D, E
C	D	A, B, E
B, D, E	A, C, E	A, A, B, B C, C, D, D, E

Notice that to be a traversal E should contain both cells with values 1 and 2, however none of the E's in first two rows/columns could contain a 1 or 2. This only leaves a single E in the bottom right 3×3 region. So it is impossible for the E traversal to contain both a 1 and 2.

Therefore, it is impossible for any completion to contain 5 distinct traversals and thus no completion has an orthogonal mate.