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MATHS 340

S2 2023

Assignment 1
Due: 1-08-2023

Q1

(a) We can see that the surface S, is defined implicitly $f(x, y, z) = 0$ where $f : \mathbb{R}^3 \to \mathbb{R}$ is continuously differentiable (as it is a polynomial). Thus, we can see that as $(a, b, c) = (1, 2, 1)$ is a surface point, the tangent plane is given by:

$$\begin{aligned} 0 &= \frac{\partial f}{\partial x}(a,b,c)(x-a) + \frac{\partial f}{\partial y}(a,b,c)(y-b) + \frac{\partial f}{\partial z}(a,b,c)(z-c) \\ &= 2a(x-a) + 4b(y-b) - 10c(z-c) \\ &= 2(x-1) + 8(y-2) - 10(z-1) \\ &= 2x + 8y - 10z - 8 \end{aligned}$$

(b) MATLAB code for producing plot:

```
syms x y z
f = x.^2 + 2*y.^2 - 5*z.^2 - 4; % Implicit eq for surface
x0 = [1;2;1];                   % Point for tangent plane

% Implicit function for tangent plane
t = subs(jacobian(f, [x, y, z]), [x, y, z], x0.')*([x;y;z]-x0);

% Shadding and colouring
colormap(bone); lighting gouraud; shading interp;
fimplicit3(f);  % Plot implicit surface
hold on         % Overlay both surfaces
fimplicit3(t);  % Plot the x0 tangent plane (Twice for colouring trick)
fimplicit3(t, "FaceColor", [0.9, 0.1, 0.1], "FaceAlpha", 0.5);

% Label the plot
xlabel('x'); ylabel('y'); zlabel('z');
title("Surface x^2 + 2y^2 - 5z^2 = 4 with [1 2 1]^T tangent plane")
view([-238.75 6.89])
```

Resulting plot shown in Figure 1.

(c) We are considering a surface defined by a function $f(x, y, z) = 0$, evaluating $f(0, 0, 1) = -9 \neq 0$, thus the point does not lie on the surface, and it does not make sense to ask for the tangent plane at this point.

We could still use f to find a plane at this point from the partial derivatives of f, however, this plane is not determined by the surface, it is determined by the choice of implicit function f, any plane passing through this point could match this definition by choosing a different f that gives the same surface.

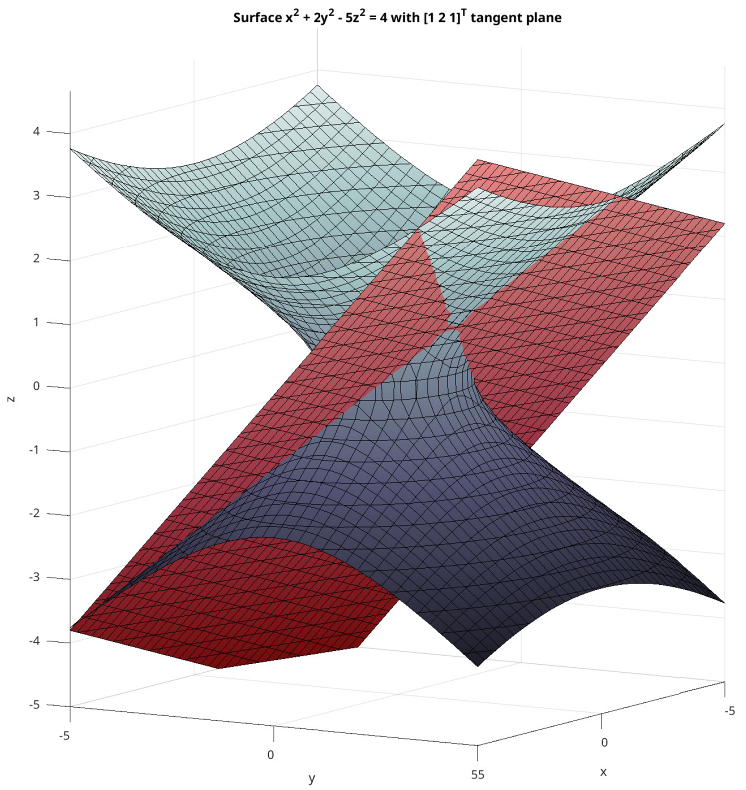

Figure 1: Plot of the tangent plane for the point $[1, 2, 1]^T$ for the surface implicitly defined by $x^2 + 2y^2 - 5z^2 = 4$

Q2

(a) Consider the function $\zeta : \begin{bmatrix} u \\ v \end{bmatrix} \mapsto \begin{bmatrix} x \\ y \end{bmatrix}$. From the definitions given we know:

$$\zeta : \begin{bmatrix} u \\ v \end{bmatrix} \mapsto \begin{bmatrix} ve^u \\ ve^{-u} \end{bmatrix}$$

Thus we find that:

$$D\zeta = \left[\begin{array}{c|c} \frac{\partial \zeta}{\partial u} & \frac{\partial \zeta}{\partial v} \end{array} \right] = \begin{bmatrix} ve^u & e^u \\ -ve^{-u} & e^{-u} \end{bmatrix}$$

The columns of $D\zeta$ give us a moving frame of basis vectors, which we normalise to give unit vectors:

$$e_u = \frac{\begin{bmatrix} ve^u \\ -ve^{-u} \end{bmatrix}}{\sqrt{v^2e^{2u} + v^2e^{-2u}}} = \frac{\begin{bmatrix} e^u \\ -e^{-u} \end{bmatrix}}{\sqrt{e^{2u} + e^{-2u}}} = \begin{bmatrix} \sqrt{\frac{e^{2u}}{e^{2u}+e^{-2u}}} \\ -\sqrt{\frac{e^{-2u}}{e^{2u}+e^{-2u}}} \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{1}{\frac{e^{2u}+e^{-2u}}{e^{2u}}}} \\ -\sqrt{\frac{1}{\frac{e^{2u}+e^{-2u}}{e^{-2u}}}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{1+e^{-4u}}} \\ \frac{-1}{\sqrt{1+e^{4u}}} \end{bmatrix}$$

$$e_v = \frac{\begin{bmatrix} e^u \\ e^{-u} \end{bmatrix}}{\sqrt{e^{2u} + e^{-2u}}} = \begin{bmatrix} \sqrt{\frac{e^{2u}}{e^{2u}+e^{-2u}}} \\ \sqrt{\frac{e^{-2u}}{e^{2u}+e^{-2u}}} \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{1}{\frac{e^{2u}+e^{-2u}}{e^{2u}}}} \\ \sqrt{\frac{1}{\frac{e^{2u}+e^{-2u}}{e^{-2u}}}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{1+e^{-4u}}} \\ \frac{1}{\sqrt{1+e^{4u}}} \end{bmatrix}$$

(b) See that:

$$e_u + e_v = \begin{bmatrix} \frac{1}{\sqrt{1+e^{-4u}}} \\ \frac{-1}{\sqrt{1+e^{4u}}} \end{bmatrix} + \begin{bmatrix} \frac{1}{\sqrt{1+e^{-4u}}} \\ \frac{1}{\sqrt{1+e^{4u}}} \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{1+e^{-4u}}} \\ \frac{1-1}{\sqrt{1+e^{4u}}} \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{1+e^{-4u}}} \\ 0 \end{bmatrix} = \frac{2}{\sqrt{1+e^{-4u}}} e_x$$

Now for $e_u - e_v$ we see that:

$$e_u - e_v = \begin{bmatrix} \frac{1}{\sqrt{1+e^{-4u}}} \\ \frac{-1}{\sqrt{1+e^{4u}}} \end{bmatrix} - \begin{bmatrix} \frac{1}{\sqrt{1+e^{-4u}}} \\ \frac{1}{\sqrt{1+e^{4u}}} \end{bmatrix} = \begin{bmatrix} \frac{0}{\sqrt{1+e^{-4u}}} \\ \frac{-1-1}{\sqrt{1+e^{4u}}} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{-2}{\sqrt{1+e^{4u}}} \end{bmatrix} = \frac{-2}{\sqrt{1+e^{4u}}} e_y$$

(c) We find the partial derivatives of the unit vectors:

$$\begin{aligned}
\frac{\partial e_u}{\partial u} &= \frac{\partial}{\partial u} \begin{bmatrix} \frac{1}{\sqrt{1+e^{-4u}}} \\ \frac{-1}{\sqrt{1+e^{4u}}} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\partial}{\partial u} \left(1+e^{-4u}\right)^{-\frac{1}{2}} \\ -\frac{\partial}{\partial u} \left(1+e^{4u}\right)^{-\frac{1}{2}} \end{bmatrix} \\
&= \begin{bmatrix} \frac{-1}{2} \left(1+e^{-4u}\right)^{\frac{-3}{2}} (0 - 4e^{-4u}) \\ -\frac{-1}{2} \left(1+e^{4u}\right)^{\frac{-3}{2}} (0 + 4e^{4u}) \end{bmatrix} \\
&= 2 \begin{bmatrix} e^{-4u} \left(1+e^{-4u}\right)^{\frac{-3}{2}} \\ e^{4u} \left(1+e^{4u}\right)^{\frac{-3}{2}} \end{bmatrix} \\
&= \frac{e^{-4u}}{1+e^{-4u}} (e_u + e_v) - \frac{e^{4u}}{1+e^{4u}} (e_u - e_v) \\
&= \frac{e_u + e_v}{1+e^{4u}} - \frac{e_u - e_v}{1+e^{-4u}}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial e_v}{\partial u} &= \frac{\partial}{\partial u} \begin{bmatrix} \frac{1}{\sqrt{1+e^{-4u}}} \\ \frac{1}{\sqrt{1+e^{4u}}} \end{bmatrix} \\
&= 2 \begin{bmatrix} e^{-4u} \left(1+e^{-4u}\right)^{\frac{-3}{2}} \\ -e^{4u} \left(1+e^{4u}\right)^{\frac{-3}{2}} \end{bmatrix} \\
&= \frac{e^{-4u}}{1+e^{-4u}} (e_u + e_v) + \frac{e^{4u}}{1+e^{4u}} (e_u - e_v) \\
&= \frac{e_u + e_v}{1+e^{4u}} + \frac{e_u - e_v}{1+e^{-4u}}
\end{aligned}$$

$$\frac{\partial e_u}{\partial v} = \frac{\partial}{\partial v} \begin{bmatrix} \frac{1}{\sqrt{1+e^{-4u}}} \\ \frac{-1}{\sqrt{1+e^{4u}}} \end{bmatrix} = 0 \qquad \frac{\partial e_v}{\partial v} = \frac{\partial}{\partial v} \begin{bmatrix} \frac{1}{\sqrt{1+e^{-4u}}} \\ \frac{1}{\sqrt{1+e^{4u}}} \end{bmatrix} = 0$$

Finding the velocity in terms of e_u, e_v:

$$\begin{aligned}
R'(t) &= \frac{dR(t)}{dt} \\
&= \frac{d}{dt}\Big[u(t) \cdot e_u(u(t), v(t))\Big] + \frac{d}{dt}\Big[v(t) \cdot e_v(u(t), v(t))\Big] \\
&= \left[u'(t)e_u + u(t) \cdot \frac{d}{dt}\Big[e_u(u(t), v(t))\Big]\right] + \left[v'(t)e_v + v(t) \cdot \frac{d}{dt}\Big[e_v(u(t), v(t))\Big]\right] \\
&= \left[u'(t)e_u + u(t) \cdot \left[u'(t) \cdot \frac{\partial e_u}{\partial u} + v'(t) \cdot \frac{\partial e_u}{\partial v}\right]\right] + \left[v'(t)e_v + v(t) \cdot \left[u'(t) \cdot \frac{\partial e_v}{\partial u} + v'(t) \cdot \frac{\partial e_v}{\partial v}\right]\right] \\
&= u'e_u + uu' \left(\frac{e_u + e_v}{1+e^{4u}} - \frac{e_u - e_v}{1+e^{-4u}}\right) + v'e_v + vu' \left(\frac{e_u + e_v}{1+e^{4u}} + \frac{e_u - e_v}{1+e^{-4u}}\right)
\end{aligned}$$

Q3

(a) First we choose α, β such for $t \in [0, \frac{\pi}{2}]$:

$$\begin{aligned} 4 &= (\alpha \cos t)^2 + 2(\beta \sin t)^2 \\ &= \alpha^2 \cos^2 t + 2\beta^2 \sin^2 t \end{aligned}$$

See that by choosing $\alpha = 2$ and $\beta = \sqrt{2}$, the equation is satisfied for all $t \in [0, \frac{\pi}{2}]$:

$$\begin{aligned} 4 &= 2^2 \cos^2 t + 2(\sqrt{2})^2 \sin^2 t \\ &= 4(\cos^2 t + \sin^2 t) \\ &= 4 \end{aligned}$$

Thus $\left\{r(t) : t \in [0, \frac{\pi}{2}]\right\} \subseteq \left\{(x, y) \in \mathbb{R}^2 : x^2 + 2y^2 = 4 \wedge x, y \geq 0\right\}$.

(b) The mass of the wire will be given by:

$$m = \int_0^{\frac{\pi}{2}} \rho(r(t)) \|r'(t)\| dt$$

Where:

$$\begin{aligned} \rho(t) &= (\alpha \cos t)(\beta \sin t) \\ r'(t) &= \begin{bmatrix} -\alpha \sin t \\ \beta \cos t \end{bmatrix} \\ \|r'(t)\| &= \sqrt{\alpha^2 \sin^2 t + \beta^2 \cos^2 t} \\ &= \sqrt{4 \sin^2 t + 2 \cos^2 t} \end{aligned}$$

By subtitling values, then applying the identity $\sin^2 \theta + \cos^2 \theta = 1$ and pulling the constants out of the integral:

$$\begin{aligned} m &= \int_0^{\frac{\pi}{2}} \left[2\sqrt{2} \sin t \cos t\right] \sqrt{4 \sin^2 t + 2 \cos^2 t}\, dt \\ &= 4 \int_0^{\frac{\pi}{2}} [\sin t \cos t] \sqrt{\sin^2 t + 1t}\, dt \end{aligned}$$

Now we use u-substitution with $u(t) = \sin^2 t + 1t$, hence $\frac{du}{dt} = 2 \sin t \cos t$, therefore $dt = \frac{du}{2 \sin t \cos t}$. Thus:

$$\begin{aligned} m &= 4 \int_{u(0)}^{u(\frac{\pi}{2})} [\sin t \cos t] \sqrt{u} \frac{du}{2 \sin t \cos t} \\ &= 2 \int_{u(0)}^{u(\frac{\pi}{2})} \sqrt{u}\, du \\ &= 2 \int_1^2 \sqrt{u}\, du \\ &= 2 \left[\frac{2}{3} 2^{\frac{3}{2}} - \frac{2}{3} 1^{\frac{3}{2}}\right] \\ &= \frac{8\sqrt{2} - 4}{3} \end{aligned}$$