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MATHS 340

S2 2023

Assignment 2
Due: 22-08-2023

Q1

For each question, we apply Fubini's theorem to express to iterated integrals as triple integrals over region defined by their bounds. We then rewrite the bounds though a series of equivalent sets of inequalities. Lastly applying Fubini's theorem again to get iterated integrals in the correct order.

(a)

$$\begin{cases} 0 \le z \le 1 \\ 0 \le x \le \ln 2 \\ e^x \le y \le 2 \end{cases} \iff \begin{cases} 0 \le z \le 1 \\ 0 \le x \le \ln 2 \\ x \le \ln y \le \ln 2 \\ y \le 2 \end{cases} \iff \begin{cases} 0 \le z \le 1 \\ 0 \le x \le \ln 2 \\ x \le \ln y \le \ln 2 \\ e^0 \le e^x \le y \le 2 \end{cases} \iff \begin{cases} 0 \le z \le 1 \\ 1 \le y \le 2 \\ 0 \le x \le \ln y \end{cases}$$

$$\therefore \int_0^1 \int_0^{\ln 2} \int_{e^x}^2 f(x,y,z)\,dy\,dx\,dz = \int_0^1 \int_1^2 \int_0^{\ln y} f(x,y,z)\,dx\,dy\,dz$$

(b)

$$\begin{cases} 0 \le x \le 2 \\ 0 \le z \le 3 - \frac{3}{2}x \\ 0 \le y \le 5 - \frac{5}{3}z - \frac{5}{2}x \end{cases} \iff \begin{cases} 0 \le x \\ x \le 2 \\ 0 \le z \\ z \le 3 - \frac{3}{2}x \\ 0 \le y5 - \frac{5}{3}z - \frac{5}{2}x \end{cases} \iff \begin{cases} 0 \le z \\ z \le 3 \\ 0 \le x \\ x \le 2 - \frac{2}{3}z \\ 0 \le y \le 5 - \frac{5}{3}z - \frac{5}{2}x \end{cases}$$

$$\iff \begin{cases} 0 \le z \le 3 \\ 0 \le x \le 2 - \frac{2}{3}z \\ 0 \le y \\ y \le 5 - \frac{5}{3}z \\ x \le 2 - \frac{2}{3}z - \frac{2}{5}y \end{cases} \iff \begin{cases} 0 \le z \le 3 \\ 0 \le y \le 5 - \frac{5}{3}z \\ 0 \le x \le 2 - \frac{2}{3}z - \frac{2}{5}y \end{cases}$$

$$\therefore \int_0^2 \int_0^{3-\frac{3}{2}x} \int_0^{5-\frac{5}{3}z-\frac{5}{2}x} f(x,y,z)\,dy\,dz\,dx = \int_0^3 \int_0^{5-\frac{5}{3}z} \int_0^{2-\frac{2}{3}z-\frac{2}{5}y} f(x,y,z)\,dx\,dy\,dz$$

(c)

$$\begin{cases} 0 \le x \le 1 \\ 0 \le y \le 1 \\ x^2 \le z \le x \iff \sqrt{x^2} \le \sqrt{z} \land z \le x \end{cases} \iff \begin{cases} 0 \le z \le 1 \\ 0 \le y \le 1 \\ z \le x \le \sqrt{z} \end{cases}$$

$$\therefore \int_0^1 \int_0^1 \int_{x^2}^x f(x,y,z)\,dz\,dy\,dx = \int_0^1 \int_0^1 \int_z^{\sqrt{z}} f(x,y,z)\,dx\,dy\,dz$$

Q2

(a) First we find parametric integrals for the surface area of each component body:

- **The square prism** R_{Prism}

6 square sides that can be parametrised in terms of $u \times v \in [0,4] \times [0,4]$ in the form:

$$r(u,v) = ue_u + ve_v + r_0$$

Where $\{e_u, e_v\} \subset \{e_x, e_y, e_z\}$ and $r_0 \in \mathbb{R}^3$, thus the area element:

$$\begin{aligned} dA &= \left\| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right\| \, du\, dv \\ &= \left\| e_u \times e_v \right\| \, du\, dv \\ &= du\, dv \end{aligned}$$

Integrating over each parametrised surface gives:

$$A_{\text{Face}} = \iint_{R_{\text{Face}}} 1 \, dA = \int_0^4 \int_0^4 1 \, du\, dv$$

To connect the prism and tube, the top face must have a cutout, the surface to be removed can be parametrised in terms of $u \times v \in [0,1] \times [0, 2\pi]$:

$$r(u,v) = \begin{bmatrix} u\cos v \\ u \sin v \\ 0 \end{bmatrix}$$

Giving an area element:

$$dA = \left\| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right\| \, du\, dv = \left\| \begin{bmatrix} \cos v \\ \sin v \\ 0 \end{bmatrix} \times \begin{bmatrix} -u\sin v \\ u\cos v \\ 0 \end{bmatrix} \right\| = u\, du\, dv$$

Thus the surface integral over the removed section is:

$$\int_0^{2\pi} \int_0^1 u \, du\, dv$$

Hence, the total area of the square prism is given by:

$$A_{\text{Prism}} = 6 \int_0^4 \int_0^4 1 \, du\, dv - \int_0^{2\pi} \int_0^1 u\, du\, dv$$

- **Tube** R_{Tube}

We parametrise the surface in terms of $u \times v \in [0,1] \times [0,2\pi]$:

$$r(u,v) = \begin{bmatrix} e^u \cos v \\ e^u \sin v \\ u \end{bmatrix}$$

Thus the area element is given by:

$$\begin{aligned} dA &= \left\| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right\| \, du\, dv = \left\| \begin{bmatrix} e^u \cos v \\ e^u \sin v \\ 1 \end{bmatrix} \times \begin{bmatrix} -e^u \sin v \\ e^u \cos v \\ 0 \end{bmatrix} \right\| \, du\, dv \\ &= \left\| \begin{matrix} -e^u \cos v \\ -e^u \sin v \\ e^{2u} \cos^2 v + e^{2u} \sin^2 v \end{matrix} \right\| \, du\, dv = \left\| \begin{matrix} -e^u \cos v \\ -e^u \sin v \\ e^{2u} \end{matrix} \right\| \, du\, dv \\ &= \sqrt{e^{2u} + e^4 u} \, du\, dv = \sqrt{e^u \sqrt{1 + e^{2u}}} \, du\, dv \end{aligned}$$

Integrating over the surface of the tube:

$$\begin{aligned} A_{\text{Tube}} &= \iint_{R_{\text{Tube}}} 1 \, dA \\ &= \int_0^{2\pi} \int_0^1 e^u \sqrt{1 + e^{2u}} \, du\, dv \end{aligned}$$

- **Half sphere $R_{\text{H-sphere}}$**

We can parametrise the curved surface of the half sphere in terms of $u \times v \in [0, \frac{\pi}{2}] \times [0, 2\pi]$:

$$r(u, v) = \begin{bmatrix} 4\sin u \cos v \\ 4 \sin u \sin v \\ 1 + 4\cos u \end{bmatrix}$$

Note that:

$$\|a \times b\| = \|a\| \|b\| |\sin\theta| = \sqrt{\|a\|^2 \|b\|^2 \sin^2\theta} = \sqrt{\|a\|^2\|b\|^2(1 - \cos^2\theta)} = \sqrt{(a \cdot a)(b \cdot b) - (a \cdot b)^2}$$

The area element is:

$$\begin{aligned} dA = \left\| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right\| &= 16 \left\| \begin{bmatrix} \cos u \cos v \\ \cos u \sin v \\ -\sin u \end{bmatrix} \times \begin{bmatrix} -\sin u \sin v \\ \sin u \cos v \\ 0 \end{bmatrix} \right\| \\ &= 16\sqrt{(\sin^2 u + \cos^2 u)(\sin^2) - (\sin^2 u + \cos^2 u)^2} \\ &= 16\sqrt{\sin^2 u - 1} \\ &= 16 \cos u \end{aligned}$$

Integrating over the surface gives:

$$\int_0^{2\pi} \int_0^{\frac{\pi}{2}} 16 \cos u \, du \, dv$$

The bottom of the sphere is a region of the $z = 1$ plane between the tubes top at $(r, z) = (e, 1)$, and the bottom of the sphere at $(r, z) = (4, 1)$. Parametrising this surface in terms of $u \times v = [e, 4] \times [0, 2\pi]$:

$$r(u, v) = \begin{bmatrix} u \cos v \\ u \sin v \\ 1 \end{bmatrix}$$

Note that $\frac{\partial r}{\partial u}$ and $\frac{\partial r}{\partial v}$ are the same as for the parametrisation of the region removed from the prism so the area element is the same $dA = u \, du \, dv$. Thus, the area of the bottom of the half sphere is.

$$\int_0^{2\pi} \int_e^4 u \, du, dv$$

The union of these two non-overlapping parametrised surfaces gives the half sphere, thus we integrate over both surfaces:

$$A_{\text{H-sphere}} = \iint_{R_{\text{H-sphere}}} = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} 16 \cos u \, du \, dv + \int_0^{2\pi} \int_e^4 u \, du, dv$$

Thus the total area of R is:

$$\begin{aligned} A &= A_{\text{Prism}} + A_{\text{Tube}} + A_{\text{H-sphere}} \\ &= 6 \int_0^4 \int_0^4 1 \, du \, dv - \int_0^{2\pi} \int_0^1 u \, du \, dv + \int_0^{2\pi} \int_0^1 e^u \sqrt{1 + e^{2u}} \, du \, dv + \int_0^{2\pi} \int_0^{\frac{\pi}{2}} 16 \cos u \, du \, dv + \int_0^{2\pi} \int_e^4 u \, du \, dv \end{aligned}$$

(b) Now consider the volume of the body R, we split R into three sections and integrate over each of them. Note that for all three we use the parametrisation:

$$r(u,v,w) = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

Thus the volume element is $dV = dx\,dy\,dz$ and the bounds for the parametrisation are simply the bounds of the body.

- Integrating ρ over the prism:

$$M_{\text{Prism}} = \int_{-4}^{0}\int_{-2}^{2}\int_{-2}^{2} \rho(u,v,w)\,du\,dv\,dw$$

- For the bounds of the tube are given by:

$$\begin{cases} 0 \le w \le 1 & \\ -e^{w} \le v \le e^{w} & \text{As } y \in [-r,r] \\ -\sqrt{e^{2w}-v^2} \le u \le \sqrt{e^{2w}-v^2} & \text{As } x^2+y^2=r^2 \end{cases}$$

  Giving a parametric integral:

$$M_{\text{Tube}} = \int_{0}^{1}\int_{-e^{w}}^{e^{w}}\int_{-\sqrt{e^{2w}-v^2}}^{\sqrt{e^{2w}-v^2}} \rho(u,v,w)\,du\,dv\,dw$$

- For the sphere we have:

$$\begin{cases} 1 \le w \le 1+4 & \text{Bounded below by } z=1 \text{ and limited by } r=4 \\ -\sqrt{4^2-(w-1)^2} \le v \le \sqrt{4^2-(w-1)^2} & \text{Limited by } x^2+y^2+z^2 \le r \\ -\sqrt{4^2-v^2-(w-1)^2} \le u \le \sqrt{4^2-v^2-(w-1)^2} & \text{Limited by } x^2+y^2+z^2 \le r \end{cases}$$

  Integrating ρ within these bounds gives the mass of the half sphere section:

$$\int_{1}^{5}\int_{-4}^{4}\int_{-\sqrt{4^2-v^2-w^2}}^{\sqrt{4^2-v^2-w^2}} \rho(u,v,w)\,du\,dv\,dw$$

The union of these non-overlapping regions gives is body R, so the sum of the parametric integrals for each region gives the integral of ρ over R:

$$\begin{aligned} M &= M_{\text{Prism}} + M_{\text{Tube}} + M_{\text{H-sphere}} \\ &= \int_{-4}^{0}\int_{-2}^{2}\int_{-2}^{2} \rho(u,v,w)\,du\,dv\,dw \\ &+ \int_{0}^{1}\int_{-e^{w}}^{e^{w}}\int_{-\sqrt{e^{2w}-v^2}}^{\sqrt{e^{2w}-v^2}} \rho(u,v,w)\,du\,dv\,dw \\ &+ \int_{1}^{5}\int_{-4}^{4}\int_{-\sqrt{4^2-v^2-w^2}}^{\sqrt{4^2-v^2-w^2}} \rho(u,v,w)\,du\,dv\,dw \end{aligned}$$

Where $\rho(x,y,z) = x^2\cos^2(y)(10-z)$ as given in the question.

Q3

(a) Partition R into two disjoint regions ($R = R_1 \cup R_2$ and $R_1 \cap R_2 = \emptyset$):

$$R_1 = \{(x, y) \mid y < \alpha x \ \alpha \in \mathbb{R}\} \qquad R_2 = \{(x, y) \mid y > \alpha x \ \alpha \in \mathbb{R}\}$$

Each of these regions is simply connected as they are convex subsets of $\mathbb{R}^2$ — any two paths between the same end points and be transformed between continuously. We can also see that the regions are simply connected as any loop can be contracted to a point. We see that:

$$\begin{aligned}
&\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \\
&= \frac{\partial}{\partial x}\left[\frac{2y}{x^2 + y^2}\right] - \frac{\partial}{\partial y}\left[\frac{2x}{x^2 + y^2}\right] \\
&= \left\{\frac{-4xy}{(x^2 + y^2)^2}\right\} - \left[\frac{-4xy}{(x^2 + y^2)^2}\right] \\
&= 0
\end{aligned}$$

Thus on each of R_1 and R_2 regions, F satisfies the *Criterion for Conservative Vector Fields in* $\mathbb{R}^2$, therefore there exists potentials $\phi_1 : R_1 \to \mathbb{R}$ and $\phi_2 : R_2 \to \mathbb{R}$ which satisfy $\nabla\phi_{1/2}(x, y) = F(x, y)$ for all $(x, y) \in R_{1/2}$ respectively.

Now consider the potential $\phi : R \to \mathbb{R}$ defined piecewise by:

$$\phi(x, y) = \begin{cases} \phi_1(x, y) & (x, y) \in R_1 \\ \phi_2(x, y) & (x, y) \in R_2 \end{cases}$$

As R_1 and R_2 are disjoint and do not share a closed boundary, the constructed potential ϕ is still differentiable everywhere in R. See that ϕ satisfies $\nabla\phi(x, y) = F(x, y)$ for all $(x, y) \in R$. Hence, $\nabla\phi = F$.

(b) Looking at the gradient field, we see that a factor of $2x$ and $2y$ indicating the inside was differentiated using the chain rule, and a reciprocal indicating that the outer function could have been $\ln z$, with derivative z^{-1}.

Thus, as a guess we consider $\phi(x, y) = \ln(x^2 + y^2)$ defined on $\mathbb{R}^2 \setminus \{0\}$, which is continuous and differentiable over its domain. Checking its gradient:

$$\nabla\phi(x, y) = \begin{bmatrix} \frac{\partial\phi(x,y)}{\partial x} \\ \frac{\partial\phi(x,y)}{\partial y} \end{bmatrix} = \frac{2}{x^2 + y^2}\begin{bmatrix} x \\ y \end{bmatrix}$$

Thus such a potential must exist as we have found it.