1. (a) We use Theorem 4.10 to work out the remaining parameters. By (ii), $2(v-1) = 4(11-1)$, so $v = 21$. By (i), $b \cdot 11 = 21 \cdot 4$, but $\frac{84}{11}$ is not an integer, so no such design can exist.

(b) Such a design exists: there is an affine plane of order 5 because 5 is a prime, and this affine plane is a $(25, 30, 6, 5, 1)$ design.

(c) We again use Theorem 4.10 to work out the remaining parameters. By (ii), $2 \cdot 45 = 10(k-1)$, so $k = 10$. By (i), $b \cdot 10 = 46 \cdot 10$, so $b = 46$. We note that $v = b$, and v is even, but $k - \lambda = 8$ which is not a square, so by (iv) no such design exists.

2. Since the design is symmetric, we have $b = v = 40$, and we further know that $k = r$. Theorem 4.10 (ii) now tells us that $\lambda \cdot 39 = k(k-1)$, so 39 must divide $k(k-1)$. This means that 3 divides $k(k-1)$, and thus $k \mod 3$ is either 0 or 1, and that 13 divides $k(k-1)$, and thus $k \mod 13$ is either 0 or 1. The only values for $k < 40$ that satisfy these conditions are $k = 13$, $k = 27$ and $k = 39$.

Since $r = k$ and $b = v$ we can determine the corresponding values of λ.

If $k = 13$, then $\lambda = \frac{13 \cdot 12}{39} = 4$.

If $k = 27$, then $\lambda = \frac{27 \cdot 26}{39} = 18$.

If $k = 39$, then $\lambda = \frac{39 \cdot 38}{39} = 38$.

3. (a) By Fisher's inequality we know that $b \geq v$, and therefore by Theorem 4.10 (i), $k = \frac{br}{v} \geq r$. By Theorem 4.10 (ii), we have that $2(v-1) = r(k-1) \leq r(r-1)$ and thus $v \leq \frac{r(r-1)}{2} + 1 = \binom{r}{2} + 1$.

(b) Note that by Theorem 4.10 (ii) we have $2(v-1) = 7(k-1)$, so $v-1$ must be divisible by 7. Together with $v \leq \binom{7}{2} + 1$ from (a) this only leaves the options $v = 1$, $v = 8$, $v = 15$, and $v = 22$.

If $v = 1$, then $k = 1$, which contradicts the assumption $k > 1$.

If $v = 8$, then $k = 3$. By Theorem 4.10 (i) we have that $b = \frac{vr}{k} = \frac{56}{3}$ which is not an integer, so no such design can exist.

If $v = 22$, then $k = 7$. By Theorem 4.10 (i) $b = \frac{vr}{k} = 22$. Since $b = v$ and v is even, $k - \lambda$ would have to be a square number for such a design to exist, but $k - \lambda = 5$.

4. Let (P, L) be the points and lines of the projective plane, and let $l_\infty \in L$. We show that (A1), (A2), and (A3) are satisfied for the incidence structure with point set $P' = P \setminus l_\infty$ and set of blocks $L' = L \setminus \{l_\infty\}$ (i.e. remove all points in l_∞ from P, and remove the line l_∞ from L).

For (A1), note that by $(P1)$ each pair of points $p, q \in P'$ of lies on a unique line $l_{pq} \in L$. If $l_{pq} \notin L'$, then $l_{pq} = l_\infty$, and therefore neither p nor q are contained in P'.

For (A2), let $l \in L'$ be a line and let $p \in P'$ be a point not on l. Let $q = l \cap l_\infty$, and note that q is the only point of l which is not contained in P'. Hence, the line in L' corresponding to k

is disjoint from the line corresponding to l if and only if $k \cap l = q$. There is a unique such line through p, namely the line l_{pq}.

Finally, for (A3), let p_1, p_2, p_3, p_4 be four points in P such that no three of them lie on a common line (these exist by (P3)). If none of the points lie on l_∞, then all of them are contained in P' and therefore witness (A3).

Now assume that at least one of the points lies on l_∞. Note that l_∞ contains at most two of the four points, so without loss of genarality $p_4 \in l_\infty$ and p_1 and p_2 do not lie on l_∞. Let l_ij denote the unique line containing p_i and p_j, and let $q_1 = l_{14} \cap l_{23}$ and $q_2 = l_{24} \cap l_{13}$.

Now l_{14} contains p_1 and q_1, but not p_2 or q_2 (otherwise it would coincide with l_{12} or l_{13}). Similarly, l_{13} contains p_1 and q_2, but not p_2 or q_1, l_{24} contains p_2 and q_2, but not p_1 or q_1, and l_{23} contains p_2 and q_1, but not p_1 or q_2. This shows that no 3 of the points p_1, p_2, q_1, q_2 lie on a common line. Finally note that $p_1, p_2 \in P'$ by assumption, and if $q_i \in l_\infty$, then we would have $l_\infty = l_{i4}$, a contradiction.

5. First, construct an affine plane of order 7 using Construction 4.24 from the course notes. Next carry out Construction 4.59 with $C_0 = \{L_{0,b} \mid b \in \mathbb{Z}_7\}$, and $C_n = \{V_a \mid a \in \mathbb{Z}_7\}$.

For each parallel class $P_a := \{L_{a,b} \mid b \in \mathbb{Z}_7\}$ this gives a Latin square M_a; the entry $(M_a)_{i,j}$ at position (i, j) is the unique b such that $L_{a,b}$ contains the point $(i, j) = V_i \cap L_{0,j}$. In other words, this is the unique b for which $ai + b = j$, so $(M_a)_{i,j} = j - ai$.

It follows from Exercise 4.60 that this gives a set of 6 MOLS.

Alternatively, we can check by hand that this is a set of 6 MOLS as follows. The element $x \in \mathbb{Z}_7$ appears as the $((x - ai) \mod 7)$-th entry of the i-th row of M_a, so each row contains every entry exactly once. Moreover, since 7 is a prime number, for each $a \in \mathbb{Z}_7 \setminus \{0\}$ there is an inverse element a^{-1} satisfying $aa^{-1} \equiv 1 \mod 7$; the element $x \in \mathbb{Z}_7$ appears as the $(a^{-1}(x - j)$ mod 7)-th entry of the j-th column of M_a, so each column contains every entry exactly once. Therefore each M_a is a Latin square.

Finally we need to check that M_a and $M_{a'}$ are orthogonal for $a \neq a'$. Let (x, y) be a pair of elements in $\mathbb{Z}_7$. We need to show that there are i, j such that $(M_a)_{i,j} = x$ and $(M_{a'})_{i,j} = y$. This corresponds to a solution of

$$j - ai = x \qquad j - a'i = y.$$

This system can be solved in $\mathbb{Z}_7$, giving $i = (y - x)(a - a')^{-1}$, and $j = x - a(y - x)(a - a')^{-1}$

6. (a) There are many possible completions. For instance:

1	2	3	4	5
2	1	5	3	4
3	4	1	5	2
4	5	2	1	3
5	3	4	2	1

(b) Take any completion of the latin rectangle. We will refer to the 2×2 sub-square B in the top left corner as the 'top left block', to the 3×2 rectangle below B as the 'bottom left block' and to the 2×3 rectangle to the right of B as the top right block.

Let T be a transversal. If T contains no elements in B, then it must contain 2 elements from the top right block (to include elements in the first two rows), and 2 elements in the bottom left block (to include elements in the first two columns). But all elements in the top right and bottom left block are either 3 ,4, or 5, so at most three of them can be included in any transversal.

Thus every transversal contains at least one element of B, and since B only has 4 elements, there can be at most 4 disjoint transversals. By Theorem 3.63 this means that the latin square has no orthogonal mate.