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MATHS 326

S1 2024

Assignment 3
Due: 10-05-2024

Q1

(a) For any $x \in V \setminus \{a,z\}$, then we have:

$$p_x = \sum_{y \in N(x)} \frac{c(\{x,y\})p_y}{\pi(x)}$$

As the probability of reaching a before z, is the sum of probabilities from each neighbour, weighted by the chance of reaching that neighbour.

This also means that as a function $x \mapsto p_x$, on $V \setminus \{a,z\}$, p_x is harmonic. We also know that $p_a = 1$ and $p_z = 0$. Meaning that this is an instance of the discrete Dirchlet problem, hence $x \mapsto p_x$ must be the unique solution.

Now consider any $f : V \to \mathbb{R}$ harmonic on $V \setminus \{a,z\}$. Consider a function $g : V \to \mathbb{R}$ given by $g(x) = f(a)p_x + f(z)$. We know $g(x)$ is harmonic as it is a linear combination of harmonic functions (on $V \setminus \{a,z\}$).

As $p_a = 1$ and $p_z = 0$, $\alpha = g(a) = f(a)$ and $\beta = g(z) = f(z)$, we have that f, g are solutions to the same discrete Dirchlet problem. Thus, $f = g$ as the solution is unique. Hence, we can write:

$$f(x) = f(a)p_x + f(b) = \alpha p_x + \beta$$

(b) We can rewrite:

$$f(x) = f(a)p_x + f(b) = \alpha p(x) + \beta q(x)$$

Where $p(x) = p_x$ and $q(x) = 1$ are both harmonic functions on $V \setminus a, z$. By (a), $\{p, q\}$ spans the vector space of function harmonic on $V \setminus \{a,z\}$. We also see that:

$$\alpha p + \beta q = 0 \implies \begin{matrix} \alpha p(a) + \beta q(a) = \alpha + \beta = 0 \\ \alpha p(z) + \beta q(z) = \beta = 0 \end{matrix} \implies \alpha = \beta = 0$$

Thus, the set $\{p, q\}$ is linearly independent and spanning, thus it is a basis with cardinality 2 hence, the dimension of the vector space is also 2.

Q2

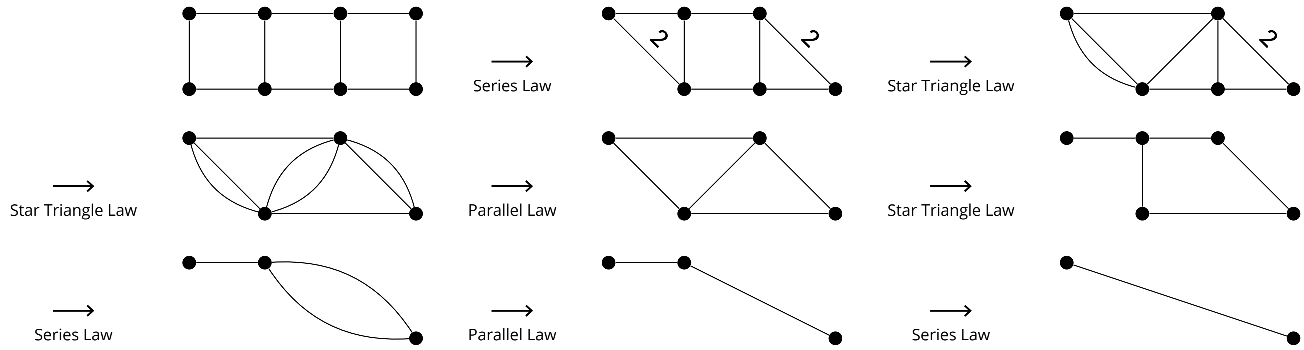
Restating the Star-Triangle law for resistance:

Consider a start with centre x with edges to y_0, y_1, y_2. Then:

$$\begin{aligned} \gamma &= \frac{c(x,y_0)c(x,y_1)c(x,y_2)}{c(x,y_0)+c(x,y_1)+c(x,y_2)} \\ &= \frac{1}{r(x,y_0)r(x,y_1)r(x,y_2)\left[1/r(x,y_0)+1/r(x,y_1)+1/r(x,y_2)\right]} \\ &= \frac{1}{r(x,y_0)r(x,y_1)+r(x,y_1)r(x,y_2)+r(x,y_2)r(x,y_0)} \end{aligned}$$

So $\{y_i, y_{i+1}\}$ where indices are taken mod 3, has resistance:

$$
\begin{aligned}
r(y_i, y_{i+1}) &= \frac{1}{\gamma c(x, y_{i+2})} \\
&= \frac{1}{r(x, y_{i+2}) r(x, y_0) r(x, y_1) r(x, y_2) \left[1/r(x, y_0) + 1/r(x, y_1) + 1/r(x, y_2)\right]} \\
&= r(x, y_{i+2}) \left[\frac{1}{r(x, y_{i+0}) r(x, y_{i+1})} + \frac{1}{r(x, y_{i+1}) r(x, y_{i+2})} + \frac{1}{r(x, y_{i+2}) r(x, y_{i+0})} \right] \\
&= \frac{r(x, y_{i+2})}{r(x, y_{i+0}) r(x, y_{i+1})} + \frac{1}{r(x, y_{i+1})} + \frac{1}{r(x, y_{i+0})}
\end{aligned}
$$



Q3

Assume that there is a shortest path from a to z where the voltage increases between some pair of vertices. Let $v_1, \dots, v_k, \dots, v_n$ with $v_1 = a$ and $v_n = z$ be vertices of the path such that v_k is the first vertex such that $v(v_k) > v(v_{k-1})$.

Q4

Q5

(a) Let i be a current flow from a to z of unit strength on $G = (V, E)$, and i' a current flow from a to z of unit strength on $G - e = (V, E')$. Extend i' to a flow on G by letting $i'(x, y) = i'(y, x) = 0$ where $\{x, y\} = e$. The energy of the flow i' is the same on both G and $G - e$:

$$
\varepsilon(i') = \frac{1}{2} \sum_{\substack{u \in V \\ v \in N_G(x)}} \theta(u, v)^2 r(\{u, v\}) = \frac{1}{2} \sum_{\substack{u \in V \\ v \in N_{G-e}(x)}} \theta(u, v)^2 r(\{u, v\}) + \theta(x, y)^2 r(\{x, y\}) = \frac{1}{2} \sum_{\substack{u \in V \\ v \in N_{G-e}(x)}} \theta(u, v)^2 r(\{u, v\})
$$

Since on $G - e$, i' is a current flow a to z of unit strength $\mathcal{R}_{G-e}(a \leftrightarrow b) = \varepsilon(i')$. So on G, $\mathcal{R}_G(a \leftrightarrow b) = \varepsilon(i) \leq \varepsilon(i')$ as i, i' are flows a to z of the same strength and i is a current flow while i' may not be. Thus:

$$
\mathcal{R}_G(a \leftrightarrow b) = \varepsilon(i) \leq \varepsilon(i') \leq \mathcal{R}_{G-e}(a \leftrightarrow b)
$$

(b)

Q6

Consider current flows i and i' of unit strength on the weights r and r'. Then, $\mathcal{R}_r = \varepsilon(i)$ and $\mathcal{R}_{r'} = \varepsilon(i')$.