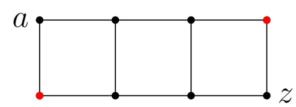
1. (a) We note that the function $g(x) = p_x$ is harmonic on $V \setminus \{a, z\}$ and that the constant function mapping $h(x) = 1$ is harmonic on all of V and thus also on $V \setminus \{a, z\}$.

Let $\beta = f(z)$ and $\alpha = f(a) - f(z)$. Since $p_a = 1$ and $p_z = 0$, we see that $f(a) = \alpha p_a + \beta = \alpha g(a) + \beta h(a)$, and $f(z) = \alpha p_z + \beta = \alpha g(z) + \beta h(z)$.

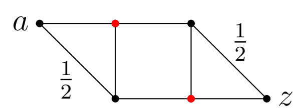
Since f is harmonic on $V \setminus \{a, z\}$, the superposition principle implies that $f(x) = \alpha g(x) + \beta h(x) = \alpha p_x + \beta$ for all x.

(b) By part (a), the set $\{g(x), h(x)\}$ from part (a) generates the vector space of functions which are harmonic on $V \setminus \{a, z\}$. If the two functions were linearly dependent, then there would be some $\alpha \in \mathbb{R}$ such that $\alpha g(x) = h(x)$ for all $x \in V$. But $g(z) = 0$ and $h(z) = 1$. Hence $\{g(x), h(x)\}$ forms a basis, and thus the dimension is 2.

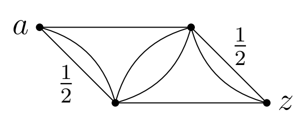
2. We iteratively apply network transformations.



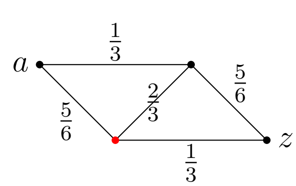
Applying the series law to both vertices marked in red above gives the following graph (all edge weights not explicitly given are equal to 1).



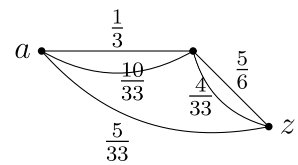
Applying the star-triangle law to the two vertices drawn in red gives the following graph where all unlabelled edges have conductance $\frac{1}{3}$ (due to the conductance formula in the star triangle law).



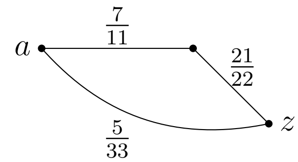
Next apply the parallel law to each pair of parallel edges.



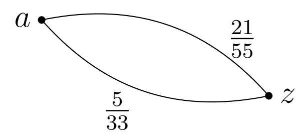
Apply the star triangle law to the vertex drawn in red to get



Apply the parallel law to all pairs of parallel edges to get the following graph:



Applying the series law at the only remaining vertex $\notin \{a, z\}$ gives



Finally, applying the parallel law to the pair of parallel edges gives a network consisting of vertices a and z and a single edge between them. This edge has conductance $\frac{8}{15}$. Since the transformations did not change the effective resistance, we have that $\mathcal{R}(a \leftrightarrow z) = \frac{15}{8}$.

3. No. For instance consider the graph obtained from a path a, x, y, z by adding 4 vertices a_1, a_2, a_3, a_4 incident to a and y, and 4 vertices z_1, z_2, z_3, z_4 incident to z and x.

To compute the voltage, note that $v(a_i) = \frac{1}{2}(v(a)+v(y))$, so the voltage at every a_i is the same. Similarly $v(z_i) = \frac{1}{2}(v(z) + v(x))$, so the voltage at every z_i is the same.

Hence the system defining the voltage can be simplified as follows:

$$\begin{aligned}
6v(x) &= v(y) + 4v(z_i) \\
2v(a_i) &= v(y) \\
6v(y) &= 1 + v(x) + 4v(a_i) \\
2v(z_i) &= 1 + v(x)
\end{aligned}$$

Solving this system gives $v(x) = \frac{3}{5}$, $v(y) = \frac{2}{5}$, $v(a_i) = \frac{1}{5}$ and $v(z_i) = \frac{4}{5}$, thus the voltage along the path a, x, y, z are not monotonic.

4. Let a and z be the endvertices of e. We know (Theorem 3.20) that $\mathbb{P}(e \in T) = \mathcal{R}_G(a \leftrightarrow z)$ and $\mathbb{P}(e \in T') = \mathcal{R}_H(a \leftrightarrow z)$, where $\mathcal{R}_G$ and $\mathcal{R}_H$ denote the effective resistance in G and H respectively.

Moreover, by Lemma 3.29, $\mathcal{R}_H(a \leftrightarrow z) = \mathcal{E}_H(i')$ where i' is a unit current flow from a to z in H, and $\mathcal{E}_H$ denotes the energy of this flow in H (it is not important here, but note that the energy in H and G are the same for this flow). Since H is a subgraph of G, the flow i' is still a flow of unit strength from a to z in G.

Let i be the unit current flow from a to z in G. Then by Thomson's principle $\mathcal{E}_G(i') \geq \mathcal{E}_G(i)$ where $\mathcal{E}_G$ denotes the energy in G. By Lemma 3.29 we know that $\mathcal{R}_G(a \leftrightarrow z) = \mathcal{E}_G(i)$, and the desired result follows.

5. (a) Let $\mathcal{E}_G$ and $\mathcal{E}_{G-e}$ denote the energy of flows in G and $G - e$, and let i and i' be current flows of unit strength in G and $G - e$, respectively. Note that $\mathcal{E}_{G-e}(i') = \mathcal{E}_G(i')$ because the weights of edges in G and $G - e$ are the same.

Thus (by Thomson's principle) $\mathcal{R}_{G-e}(a \leftrightarrow z) = \mathcal{E}_{G-e}(i') = \mathcal{E}_G(i') \geq \mathcal{E}_G(i) = \mathcal{R}_G(a \leftrightarrow z)$.

(b) Let x and y be the endpoints of e. We claim that $v(x) = v(y)$ where v is a voltage with pre-defined values at a and z. Note that this clearly implies $i(x, y) = 0$, and thus i is a flow from a to z in $G - e$, which in turn implies that $\mathcal{E}_G(i') \leq \mathcal{E}_G(i)$.

Let H be the subgraph of G containing all vertices which lie on some path from a to z. If a connected component of $G-H$ has more than one neighbour in H, then we can find a path from a to z using vertices of this connected component. Thus every connected component has a unique neighbour in H.

Let C be a connected component of $G - H$ and let G_C be the subgraph induced by this connected component and its unique neighbour x_c in H. Since a and z lie in H, and the only vertex in G_C which has neighbours outside G_c is x_c, the voltage must be harmonic in every vertex of G_C except x_c. By the uniqueness principle, the voltage must be constant on G_C. Since this applies to every connected component C, we conclude that $v(x) = v(y)$.

(c) Let P be a shortest path from a to z. Inductively apply (a), removing all edges $\notin P$ one by one we see that $\mathcal{R}_G(a \leftrightarrow z) \leq \mathcal{R}_P(a \leftrightarrow z)$. Inductively applying the series law shows that $\mathcal{R}_P(a \leftrightarrow z) = d(a, z)$.

Finally by Theorem 3.18, we have $\mathrm{Comm}(a, z) = 2\sum_{e\in E} c(e)\mathcal{R}_G(a \leftrightarrow z) \leq 2|E|\, d(a, z)$.

(d) Let i,i', and i'' be unit current flows from a to b for the weights c, c', and c'', respectively. Let $\mathcal{E}$, $\mathcal{E}'$, and $\mathcal{E}''$ denote the energies of flows with respect to the corresponding weights.

Then

$$\begin{aligned}
\mathcal{R}_{\frac{r+r'}{2}}(a \leftrightarrow b) &= \mathcal{E}''(i'') && \text{by Lemma 3.29}\\
&= \sum_{\substack{x\in V\\ y\in N(X)}} i''(x,y)^2 \frac{r(\{x,y\}) + r'(\{x,y\})}{2} && \text{by definition}\\
&= \sum_{\substack{x\in V\\ y\in N(X)}} i''(x,y)^2 \frac{r(\{x,y\})}{2} + \sum_{\substack{x\in V\\ y\in N(X)}} i''(x,y)^2 \frac{r'(\{x,y\})}{2} \\
&= \frac{1}{2}\mathcal{E}(i'') + \frac{1}{2}\mathcal{E}'(i'') && \text{by definition}\\
&\geq \frac{1}{2}\mathcal{E}(i) + \frac{1}{2}\mathcal{E}'(i') && \text{Thomson's principle}\\
&= \frac{1}{2}\mathcal{R}_r(a \leftrightarrow b) + \frac{1}{2}\mathcal{R}_r(a \leftrightarrow b) && \text{by Lemma 3.29}
\end{aligned}$$