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MATHS 326

S1 2024

Assignment 3
Due: 10-05-2024

Q1

(a) For any $x \in V \setminus \{a, z\}$, then we have:

$$p_x = \sum_{y \in N(x)} \frac{c(\{x, y\})p_y}{\pi(x)}$$

As the probability of reaching a before z, is the sum of probabilities from each neighbour, weighted by the chance of reaching that neighbour.

This also means that as a function $x \mapsto p_x$, on $V \setminus \{a, z\}$, p_x is harmonic. We also know that $p_a = 1$ and $p_z = 0$. Meaning that this is an instance of the discrete Dirchlet problem, hence $x \mapsto p_x$ must be the unique solution.

Now consider any $f : V \to \mathbb{R}$ harmonic on $V \setminus \{a, z\}$. Consider a function $g : V \to \mathbb{R}$ given by $g(x) = f(a)p_x + f(z)$. We know $g(x)$ is harmonic as it is a linear combination of harmonic functions (on $V \setminus \{a, z\}$).

As $p_a = 1$ and $p_z = 0$, $\alpha = g(a) = f(a)$ and $\beta = g(z) = f(z)$, we have that f, g are solutions to the same discrete Dirchlet problem. Thus, $f = g$ as the solution is unique. Hence, we can write:

$$f(x) = f(a)p_x + f(b) = \alpha p_x + \beta$$

(b) We can rewrite:

$$f(x) = f(a)p_x + f(b) = \alpha p(x) + \beta q(x)$$

Where $p(x) = p_x$ and $q(x) = 1$ are both harmonic functions on $V \setminus a, z$. By (a), $\{p, q\}$ spans the vector space of function harmonic on $V \setminus \{a, z\}$. We also see that:

$$\alpha p + \beta q = 0 \implies \begin{matrix} \alpha p(a) + \beta q(a) = \alpha + \beta = 0 \\ \alpha p(z) + \beta q(z) = \beta = 0 \end{matrix} \implies \alpha = \beta = 0$$

Since the set $\{p, q\}$ is linearly independent and spanning it is a basis with cardinality 2. Hence, the dimension of the vector space is also 2.

Q2

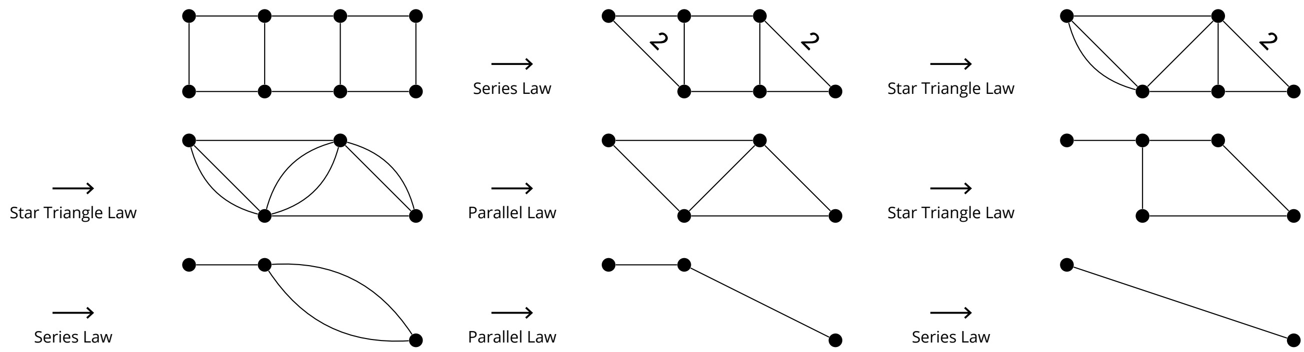
Restating the Star-Triangle law for resistance:

Consider a start with centre x with edges to y_0, y_1, y_2. Then:

$$\begin{aligned}
\gamma &= \frac{c(x,y_0)c(x,y_1)c(x,y_2)}{c(x,y_0)+c(x,y_1)+c(x,y_2)} \\
&= \frac{1}{r(x,y_0)r(x,y_1)r(x,y_2)\left[1/r(x,y_0)+1/r(x,y_1)+1/r(x,y_2)\right]} \\
&= \frac{1}{r(x,y_0)r(x,y_1)+r(x,y_1)r(x,y_2)+r(x,y_2)r(x,y_0)}
\end{aligned}$$

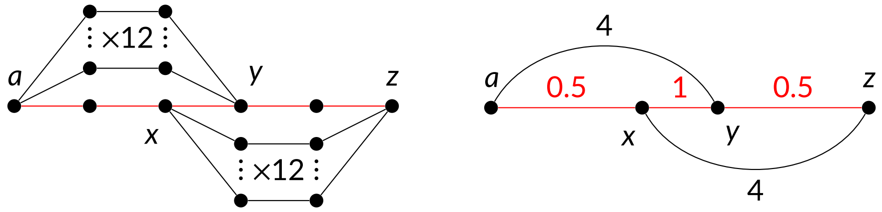
So $\{y_i, y_{i+1}\}$ where indices are taken mod 3, has resistance:

$$\begin{aligned}
r(y_i, y_{i+1}) &= \frac{1}{\gamma c(x, y_{i+2})} \\
&= \frac{1}{r(x,y_{i+2})r(x,y_0)r(x,y_1)r(x,y_2)\left[1/r(x,y_0)+1/r(x,y_1)+1/r(x,y_2)\right]} \\
&= r(x,y_{i+2})\left[\frac{1}{r(x,y_{i+0})r(x,y_{i+1})}+\frac{1}{r(x,y_{i+1})r(x,y_{i+2})}+\frac{1}{r(x,y_{i+2})r(x,y_{i+0})}\right] \\
&= \frac{r(x,y_{i+2})}{r(x,y_{i+0})r(x,y_{i+1})}+\frac{1}{r(x,y_{i+1})}+\frac{1}{r(x,y_{i+0})}
\end{aligned}$$



Q3

Counter Example: Consider the following graph and shortest path shown in red. Let v be a voltage function with $v(a) = 0$ and $v(z) = 1$. This graph can be reduced to a weighted graph using the Series/Parallel Laws:



Where the weights correspond to conductance, this preserves the effective conductance between the any pair of a, x, y, z are preserved, thus, $v(x), v(y)$ are also preserved.
From Ohm's Law:

$$\begin{aligned}
i(a,x) = c(\{a,x\})\,[v(a) - v(x)] &= -\frac{1}{2}v(x) \\
i(x,z) = c(\{x,z\})\,[v(x) - v(z)] &= 4v(x) - 4 \\
i(a,y) = c(\{a,y\})\,[v(a) - v(y)] &= -4v(y) \\
i(y,z) = c(\{y,z\})\,[v(y) - v(z)] &= \frac{1}{2}v(y) - \frac{1}{2} \\
i(x,y) = c(\{x,y\})\,[v(x) - v(y)] &= v(x) - v(y)
\end{aligned}$$

Applying Kirchhoff's Node Law at x, y:

$$\begin{aligned}
0 &= \sum_{z \in N(x)} i(z,x) = i(a,x) + i(y,x) + i(z,x) \\
&= -\frac{1}{2}v(x) - [v(x) - v(y)] - [4v(x) - 4] \\
v(y) &= \frac{1}{2}v(x) + v(x) + 4v(x) - 4 \\
&= \frac{11}{2}v(x) - 4 \\
0 &= \sum_{z \in N(y)} i(z,y) = i(a,y) + i(x,y) + i(z,y) \\
&= [-4v(y)] + [v(x) - v(y)] - \left[\frac{1}{2}v(y) - \frac{1}{2}\right] \\
v(x) &= 4.5v(y) - \frac{1}{2}
\end{aligned}$$

Solving this system of two linear equations gives:

$$\frac{74}{95} = v(x) > v(y) = \frac{27}{95}$$

Q4

Let $H = (V_H, E_H)$ be a connected subgraph of $G = (V_G, E_G)$. Let T_H, T_G be random spanning trees of H, G respectively. For $e \in E_H$:

$$\begin{aligned} \mathbb{P}[e \in T_H] &= \mathcal{R}(x \leftrightarrow y) && \text{By Theorem 3.20} \\ &= \epsilon(i) \end{aligned}$$

For some unit strength current flow i from x to y on H. Extend i to i_G on G by defining:

$$i_G(x,y) = \begin{cases} i(x,y) & \text{When } \{x,y\} \in E_H \\ 0 & \text{Otherwise} \end{cases}$$

Clearly i_G is a flow (not necessarily a current flow) as it still satisfies Kirchhoff's node law, moreover it also has unit strength. So by Thompson's principle, a unit current flow i' from x to y has:

$$\epsilon(i) = \sum_{e=\{x,y\}\in E_H} i(x,y)^2 r(e) = \sum_{e=\{x,y\}\in E_G} i_G(x,y)^2 r(e) = \epsilon(i_G) \geq \epsilon(i') = \mathcal{R}_G(x \leftrightarrow y)$$

Thus, we have shown:

$$\mathbb{P}[e \in T_H] = \mathcal{R}(x \leftrightarrow y) = \epsilon(i) = \epsilon(i_G) \geq \epsilon(i') = \mathcal{R}_G(x \leftrightarrow y) = \mathbb{P}[e \in T_G]$$

Q5

(a) Let $G = (V, E)$ be a connected graph, and i' a current flow from a to z of unit strength on $G - e = (V, E')$. Extend i' to a flow θ on G with $\theta(x,y) = \theta(y,x) = 0$ for $\{x,y\} = e$ and $\theta(u,v) = i(u,v)$ for $\{u,v\} \in E'$. So:

$$\varepsilon(\theta) = \frac{1}{2}\sum_{\substack{u\in V \\ v\in N_G(x)}} \theta(u,v)^2 r(\{u,v\}) = \frac{1}{2}\sum_{\substack{u\in V \\ v\in N_{G-e}(x)}} i'(u,v)^2 r(\{u,v\}) + \theta(x,y)^2 r(\{x,y\}) = \frac{1}{2}\sum_{\substack{u\in V \\ v\in N_{G-e}(x)}} i'(u,v)^2 r(\{u,v\}) = \varepsilon(i')$$

So for a unit strength current flow i from a to z on G, $\varepsilon(i) \leq \varepsilon(i')$ by Thompson's Principle. Thus:

$$\mathcal{R}_G(a \leftrightarrow b) = \varepsilon(i) \leq \varepsilon(i') = \mathcal{R}_{G-e}(a \leftrightarrow b)$$

(b) Consider any current flow i from a to z on $G = (V, E)$. We claim that $i(x,y) \neq 0$ implies that we can construct a path from a to z that contains both of x, y. Proof:

> Let P be a path with vertices $x_1 = x, x_2 = y$. Consider the algorithm where we repeatedly apply the following step until both endpoints of P are in $\{a, z\}$:
>
> Let $x_1, \dots, x_{k+1}$ be the vertices of P. If $x_{k+1} \notin \{a, z\}$, by Kirchhoff's node law, there is some $w \in N(x_{k+1})$ with $i(x_{k+1}, w) > 0$, attempt to append w to P. Otherwise, if $x_{k+1} \in \{a, z\}$ and $x_1 \notin \{a, z\}$, by Kirchhoff's node law there is some $w' \in N(x_1)$ such that $i(w', x_1) > 0$. Attempt to extend P by prepending w'.
>
> On a finite G, the algorithm always succeeds or fails. To fail, adding w or w' to P did not produce a path. Meaning P already contained w or w'. Hence, we have a cycle consisting entirely of edges with $i(x_i, x_{i+1}) > 0$ and Kirchhoff's cycle law is not satisfied. Since i is a current flow, this is a contradiction and the algorithm must always construct a path between a and z containing the original edge. (a to a or z to z aren't paths).

(c) For any unweighed $G = (V, E)$, the conductance function $c : E \to \mathbb{R}$ is given by $c(e) = 1$. So by Theorem 3.18 of the course book:

$$\text{Comm}(a \leftrightarrow z) = 2\left(\sum_{e\in} c(e)\right)\mathcal{R}(a \leftrightarrow z) = 2\,|E|\,\mathcal{R}(a \leftrightarrow z)$$

Now consider some fixed shortest path P between a and z with length $d(a,z)$. Apply (a) to remove every edge of G not in P, by (a) this does not decrease the effective resistance between a and z. IE, if G' is the graph obtained, then:

$$\mathcal{R}_G(a \leftrightarrow z) \leq \mathcal{R}_{G'}(a \leftrightarrow z)$$

Since each non-endpoint vertex of P is of degree 2 in G', applying the series law $\mathcal{R}_{G'}(a \leftrightarrow z)$ is the sum of the resistances in G. Thus, $\mathcal{R}_{G'}(a \leftrightarrow z) = d(a,z)$ as the edges are of unit resistance. Thus:

$$\text{Comm}(a \leftrightarrow z) = 2\left(\sum_{e\in} c(e)\right)\mathcal{R}(a \leftrightarrow z) = 2\,|E|\,\mathcal{R}(a \leftrightarrow z) \leq 2\,|E|\,d(a,z)$$

Q6

Consider a unit current flow i'' on the resistances $\frac{r+r'}{2}$ from a to z. Then:

$$\begin{aligned}
\mathcal{R}_{\frac{r+r'}{2}}(a \leftrightarrow z) &= \varepsilon_{\frac{r+r'}{2}}(i'') && \text{By Lemma 3.29}\\
&= \frac{1}{2}\sum_{\substack{x\in V\\ y\in N(x)}} i''(x,y)^2\,\frac{r(\{x,y\}) + r'(\{x,y\})}{2}\\
&= \frac{1}{4}\sum_{\substack{x\in V\\ y\in N(x)}} i''(x,y)^2 r(\{x,y\}) + \frac{1}{4}\sum_{\substack{x\in V\\ y\in N(x)}} i''(x,y)^2 r'(\{x,y\})\\
&= \frac{1}{2}\left[\varepsilon_r(i'') + \varepsilon_{r'}(i'')\right]\\
&\geq \frac{1}{2}\left[\varepsilon_r(i) + \varepsilon_{r'}(i')\right] && \text{By Thompson's Principle}\\
&= \frac{1}{2}\left[\mathcal{R}_r(a \leftrightarrow z) + \mathcal{R}_{r'}(a \leftrightarrow z)\right] && \text{By Lemma 3.29}
\end{aligned}$$

Where i and i' are unit strength current flows on r and r' respectively and Thompson's Principle is applicable as i'' also is a unit strength flow (but not necessarily a current flow) as the strength of a flow is independent of resistance.