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MATHS 326

S1 2024

Assignment 4
Due: 24-05-2024

Q1

(a) We are given $r = 4, k = 11, \lambda = 2$, assume that these are parameters for a balanced design, by *Theorem 4.10*:

$$bk = vr \implies 11b = 4v$$

$$\lambda(v-1) = r(k-1) \implies 2(v-1) = 4(11-1) \implies 2v - 2 = 40 \implies v = 21$$

Contradiction as this implies $b = \frac{4 \cdot 21}{11}$ which is not an integer as neither 4 nor 21 have a prime factor of 11. Thus, no balanced block design has these parameters.

(b) We are given $b = 30, r = 6, k = 5$, assume that these are parameters for a balanced design, by *Theorem 4.10*:

$$bk = vr \implies 30 \cdot 5 = 6v \implies v = 25$$

$$\lambda(v-1) = r(k-1) \implies 2(46-1) = r(k-1) \implies 0 = 4.6r^2 - r - 90$$

(c) We are given $v = 46, b = 10, \lambda = 2$, assume that these are parameters for a balanced design, by *Theorem 4.10*:

$$bk = vr \implies 10k = 46r \implies k = 4.6r$$

$$\lambda(v-1) = r(k-1) \implies 2(46-1) = r(k-1) \implies 0 = 4.6r^2 - r - 90$$

Solving for possible values of r using the quadratic equation:

$$r = \frac{1 \pm \sqrt{1657}}{4.6}$$

Which has no integer solutions as $40^2 < 1657 < 41^2$, hence $\sqrt{1657}$ is irrational.

This is a contradiction so no balanced block design has these parameters.

Q2

Assume that there is a BIBD for $v = b = 40$ with parameters (v, b, r, k, λ). Then since $vb = rk$ we have $k = r$. Thus:

$$\lambda(v-1) = r(k-1) \implies 39\lambda = r(r-1) = k(k-1)$$

Since the design is incomplete, $r, k, \lambda \leq 39$:
Since we have $\lambda = k(k-1)/39$:

$$k - \lambda \in \{1, 2, 4, 9, 16, 25, 36\}$$

$$(39k - k(k-1))/39 \in \{1, 2, 4, 9, 16, 25, 36\}$$

We can factorise $39 = 3 \cdot 13$ and $\lambda = \lambda_1\lambda_2$. This gives the following cases:

Case	r	$r-1$	
A	$39\lambda_1$	λ_2	$39\lambda_1 = \lambda_2 + 1$
B	$13\lambda_1$	$3\lambda_2$	$13\lambda_1 = 3\lambda_2 + 1$
C	$3\lambda_1$	$13\lambda_2$	$3\lambda_1 = 13\lambda_2 + 1$
D	λ_1	$39\lambda_2$	$\lambda_1 = 39\lambda_2 + 1$

Case A

$$\lambda_1 = 1 \implies \lambda_2 = 38$$
$$\lambda_1 \geq 2 \implies \lambda > 39$$

$$\implies \lambda \in \{38\}$$

Case B

$$\lambda_1 = 1 \implies \lambda_2 = 4$$
$$\lambda_1 = 2 \implies \lambda_2 = 25/3 \notin \mathbb{Z}$$
$$\lambda_1 \geq 3 \implies \lambda > 39$$

$$\implies \lambda \in \{4\}$$

Case C

$$\lambda_2 = 1 \implies \lambda_1 = 14/3 \notin \mathbb{Z}$$
$$\lambda_2 = 2 \implies \lambda_1 = 9$$
$$\lambda_2 = 3 \implies \lambda_1 = 40/3 \notin \mathbb{Z}$$
$$\lambda_2 \geq 3 \implies \lambda > 39$$

$$\implies \lambda \in \{18\}$$

Case D

$$\lambda_2 \geq 1 \implies \lambda > 39$$

$$\implies \lambda \in \emptyset$$

Q3

(a)

Q4

We check that the axioms hold:

- Trivial.
- CL: Given a point p and a line l not through p, there is exactly one line l' through p with $l \cap l' \neq \emptyset$.
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Q5

Consider $L_{ij} = i + kj$ for $k \in \{0, \dots, 6\}$

Q6

(a)

(b) Assume that there exist 5 disjoint traversals of some completion.

Consider the top left 2×2 region, each cell must be in a different traversal:

The region contains the following cells:

1	2
2	1

If two cells are in the same traversal, they cannot be in the same row/column, thus they must be diagonal (in the 2×2 region). All diagonal entries are the same, so they cannot be part of the same traversal.

Thus, each cell in the 2×2 region is part of a distinct traversal.

Label the traversals A, B, C, D, E, WLOG we can fix the traversal that each of the cells in the 2×2 region are part of. Since each traversal appears once in each row/column, we can deduce which traversals the cells in each region must be assigned to:

<table>
<tr><td>A</td><td>B</td><td>C, D, E</td></tr>
<tr><td>C</td><td>D</td><td>A, B, E</td></tr>
<tr><td>B, D, E</td><td>A, C, E</td><td>A, A, B, B
C, C, D, D, E</td></tr>
</table>

Now consider the relative positions of the E traversals in top right 2×3 region:

3	4	5
5	3	4