Robert Christie

MATHS 340

S2 2023

Assignment 4
Due: 17-10-2023

Q1

The function f is differentiable at z the following limit:

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

Exists, consider Δz approaching along a straight line with angle θ.

$$\begin{aligned}
&\lim_{r\to 0} \frac{f(z+re^{i\theta}) - f(z)}{re^{i\theta}} \\
&= \lim_{r\to 0} \frac{\frac{1}{\overline{z}+re^{-i\theta}} - \frac{1}{\overline{z}}}{re^{i\theta}} \\
&= \lim_{r\to 0} \frac{\frac{\overline{z}}{(\overline{z}+re^{-i\theta})\overline{z}} - \frac{(\overline{z}+re^{-i\theta})}{(\overline{z}+re^{-i\theta})\overline{z}}}{re^{i\theta}} \quad \text{as } \overline{z} \neq 0 \\
&= \lim_{r\to 0} \frac{\overline{z} - (\overline{z}+re^{-i\theta})}{re^{i\theta}(\overline{z}+re^{-i\theta})\overline{z}} \\
&= \lim_{r\to 0} \frac{-re^{-i\theta}}{re^{i\theta}\overline{z}^2 + r^2\overline{z}} \\
&= \lim_{r\to 0} \frac{-e^{-2i\theta}}{\overline{z}^2 + re^{-i\theta}\overline{z}}
\end{aligned}$$

Notice that this limit will depend on θ as the denominator:

$$\overline{z}^2 + re^{-i\theta}\overline{z} \to \overline{z}^2$$

While the numerator depends on θ. Hence, the limit does not exist for any $z \in \mathbb{C}\setminus\{0\}$ meaning the function is differentiable nowhere on $\mathbb{C}$.

To verify this, we can write

$$\begin{aligned}
f(x+iy) &= \frac{1}{x-iy} = \frac{x+iy}{x^2+y^2} \\
&= \frac{x}{x^2+y^2} + \frac{iy}{x^2+y^2} \\
&= u(x,y) + iv(x,y)
\end{aligned}$$

Using the product rule:

$$\frac{\partial u}{\partial x} = \frac{1-2x}{(x^2+y^2)^2} \neq \frac{\partial v}{\partial y} = \frac{1-2y}{(x^2+y^2)^2} \quad \text{When } x \neq y$$

$$\frac{\partial u}{\partial y} = \frac{-2y}{(x^2+y^2)^2} \neq -\frac{\partial v}{\partial x} = -\frac{-2x}{(x^2+y^2)^2} \quad \text{When } x \neq -y$$

So $f(z)$ does not satisfy the Cauchy-Riemann equations for any $z \in \mathbb{C}\setminus\{0\}$ and is therefore not differentiable anywhere on $z \in \mathbb{C}\setminus\{0\}$.

Q2

Note that f is analytic except when $z = 0$ or $z^3 = 1$. Of the three solutions to z^3 only $z = 1$ lies within C.

Since C is a piecewise smooth closed curve oriented anticlockwise, and $f(z)$ is analytic on and inside C except at 2 isolated points $z = 0$ and $z = 1$, we can apply the residue theorem:

$$\int_C f(z)dz = 2\pi i(\mathrm{Res}_0 f + \mathrm{Res}_1 f)$$

As $z = 0$ is a pole of order $m = 3$:

$$\begin{aligned}
\mathrm{Res}_0 f &= \frac{1}{(m-1)!}\lim_{z\to 0}\left(\frac{d^{m-1}}{dz^{m-1}}[(z-0)^m f(z)]\right) \\
&= \frac{1}{2}\lim_{z\to 0}\left(\frac{d^2}{dz^2}\left[z^3\frac{1}{z^3(1-z^3)}\right]\right) \\
&= \frac{1}{2}\lim_{z\to 0}\left(\frac{d}{dz}\left[3x^2(1-x^3)\right]\right) \\
&= \frac{1}{2}\lim_{z\to 0}\left(\frac{6z}{(1-z^3)^2} + \frac{18z^4}{(1-z^3)^3}\right) \\
&= 0
\end{aligned}$$

We have a first order pole at $z = 1$ as we can write $(1-z^3) = (1-z)(z^2+z+1)$. Thus, the residue at $z = 1$ is:

$$\begin{aligned}
\mathrm{Res}_1 f &= \frac{1}{(m-1)!}\lim_{z\to 1}\left(\frac{d^{m-1}}{dz^{m-1}}[(z-1)^m f(z)]\right) \\
&\lim_{z\to 1}\left((z-1)\frac{1}{z^3(1-z^3)}\right) \\
&\lim_{z\to 1}\left((z-1)\frac{1}{z^3(1-z)(z^2+z+1)}\right) \\
&\lim_{z\to 1}\left(\frac{-1}{z^3(z^2+z+1)}\right) \\
&= \frac{-1}{1(1+1+1)} \\
&= -\frac{1}{3}
\end{aligned}$$

Thus:

$$\int_C f(z)dz = -\frac{2\pi i}{3}$$

Q3

(a) We know the Taylor series of e^z about $z = 0$ which converges to e^z is:

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \dots$$

From lectures we also know that the Laurent series for $\frac{1}{\sin z}$ in the region $0 < |z| < \pi$ is:

$$z^{-1} + z\left(\frac{1}{3!}\right) + z^3\left(-\frac{1}{5!} + \left(\frac{1}{3!}\right)^2\right) + \dots$$

Multiplying these series to the third term gives:

$$z^{-1} \cdot 1 \quad + \quad z^{-1} \cdot z \quad + \quad z^{-1} \cdot \frac{z^2}{2} \quad + \quad \frac{z}{3!} \cdot \frac{z^0}{1}$$
$$= \frac{1}{z} + 1 + \frac{2z}{3}$$

(b) We use the series

$$(1 - a)^{-1} = 1 + a + a^2 + a^3 + \dots \quad \text{For } |a| < 1$$

Let $\omega = z - 1$ so $z = \omega + 1$, where $|\omega| > 1 \implies |\omega^{-1}| < 1$. Thus:

$$\begin{aligned}
g(z) &= \frac{z-2}{z(z-1)} \\
&= \frac{\omega - 1}{(\omega + 1)\omega} \\
&= \frac{\omega - 1}{(1 + \frac{1}{\omega})\omega^2} \\
&= \frac{\omega - 1}{\omega^2}\left(1 - \left(-\frac{1}{\omega}\right)\right)^{-1} \\
&= \left[\frac{1}{\omega} - \frac{1}{\omega^2}\right]\left[1 - \frac{1}{\omega} + \frac{1}{\omega^2} - \frac{1}{\omega^3} + \dots\right] \\
&= \left[\omega^{-1} - 1\right] + [-1 + \omega] + \left[\omega - \omega^2\right] + \left[-\omega^2 + \omega^3\right] + \dots \\
&= \frac{1}{\omega} - 2 + 2\omega - 2\omega^2 + 2\omega^3 + \dots
\end{aligned}$$

Thus, the first three terms in the Laurent series of $g(z)$ about $z = 1$ in the region $|z - 1 > 1$ are:

$$1(z-1)^{-1} - 2(z-1)^{-2} + 2(z-1)^{-3}$$

Q4

$$\begin{aligned}
&\int_0^{\pi} \frac{1}{6 + \cos(t)}dt \\
&= -\int_{2\pi}^{\pi} \frac{1}{6 + \cos(u)}du \quad u = 2\pi - t, dt = -du \\
&= \int_{\pi}^{2\pi} \frac{1}{6 + \cos(t)}dt
\end{aligned}$$

Therefore:

$$\begin{aligned}
&\int_0^{\pi} \frac{1}{6 + \cos(t)}dt \\
&= \frac{1}{2}\int_0^{2\pi} \frac{1}{6 + \cos(t)}dt
\end{aligned}$$

Now let be C an anticlockwise unit circle in the complex plane. So $z = e^{it}$ is on the circle. Hence:

$$\cos t = \frac{z + \frac{1}{z}}{2}$$

And:

$$dz = ie^{it}dt = izdt \implies dt = \frac{dz}{iz}$$

Thus, the integral can be written as:

$$\begin{aligned}
&\int_0^{\pi} \frac{1}{6 + \cos(t)}dt \\
&= \frac{1}{2}\int_0^{2\pi} \frac{1}{6 + \cos(t)}dt \\
&= \frac{1}{2}\int_C \frac{1}{iz\left(6 + \frac{z + z^{-1}}{2}\right)}dz \\
&= \int_C \frac{1}{i(12 + z^2 + 1)}dz \\
&= -i\int_C \frac{1}{z^2 + 12z + 1}dz
\end{aligned}$$

The integrand is not analytic at the roots of $z^2 + 12z + 1$, using the quadratic formula:

$$\begin{aligned}
&\frac{-12 \pm \sqrt{12^2 - 4}}{2} \\
&= \frac{-12 \pm 2\sqrt{35}}{2} \\
&= -6 \pm \sqrt{35}
\end{aligned}$$

Thus f is not analytic at the single isolated point in C, $z = \sqrt{35} - 6 = a$. Which will be a simple pole as we can factorise the denominator into linear factors:

$$(z + 6 - \sqrt{35})(z + 6 + \sqrt{35})$$

Thus, the residue at the pole in C is:

$$\begin{aligned}
\mathrm{Res}_a f &= \frac{1}{(m-1)!}\lim_{z\to a}\left(\frac{d^{m-1}}{dz^{m-1}}\left[(z-a)^m f(z)\right]\right)\\
&= \lim_{z\to a}\left((z-a)f(z)\right)\\
&= \lim_{z\to a}\left(\frac{z+6-\sqrt{35}}{(z+6-\sqrt{35})(z+6+\sqrt{35})}\right)\\
&= \lim_{z\to a}\left(\frac{1}{(z+6+\sqrt{35})}\right)\\
&= \frac{1}{\sqrt{35}-6+6+\sqrt{35}}\\
&= \frac{1}{2\sqrt{35}}
\end{aligned}$$

Since C is a piecewise smooth closed curve oriented anticlockwise, and $f(z)$ is analytic on and inside C except at 1 isolated points $z = a$, we can apply the residue theorem:

$$-i\left[\int_C \frac{1}{z^2+12z+1}dz\right] = -i\left[2\pi i\frac{1}{2\sqrt{35}}\right] = \frac{\pi}{\sqrt{35}}$$

Thus we have found:

$$\int_0^{\pi} \frac{1}{1+6\cos t}dt = \frac{\pi}{\sqrt{35}}$$