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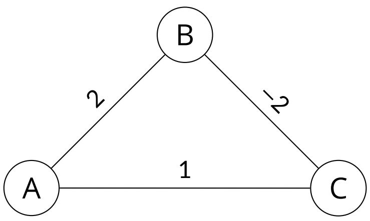
MATHS 326

S1 2024

Assignment 2
Due: 20-04-2024

Q1

Consider the following weighted graph:



Applying Dijkstra's Algorithm starting from $v = A$:

- Initial stage: We set:

$$R = \varnothing, \quad \delta(v) = \begin{cases} 0 & v = A \\ \infty & v = B \\ \infty & v = C \end{cases}$$

  Also set A to be the current vertex.

- Iteration 1: Add A to R, and update δ giving:

$$R = \{A\}, \quad \delta(v) = \begin{cases} 0 & v = A \\ 2 & v = B \\ 1 & v = C \end{cases}$$

  Then we set C as the current vertex as it has the smallest δ of the unvisited vertices.

- Iteration 2: We add C to R, and update δ giving:

$$R = \{A, C\}, \quad \delta(v) = \begin{cases} 0 & v = A \\ -1 & v = B \\ 1 & v = C \end{cases}$$

  Now that C has been visited, the algorithm will not alter $\delta(C)$ any further.

Thus, the final value of $\delta(C) = 1$. However, the actual shortest path from A to C is A—B—C with length 0. So Dijkstra's algorithm doesn't always find shortest paths if positive and negative edge weights are allowed.

Q2

- Assume that M is not unique, so there exists some other minimum spanning tree M'.
- Since both are MST, M and M' have the same weight.
- Since each edge weight is unique, M and M' must differ at at-least two points.
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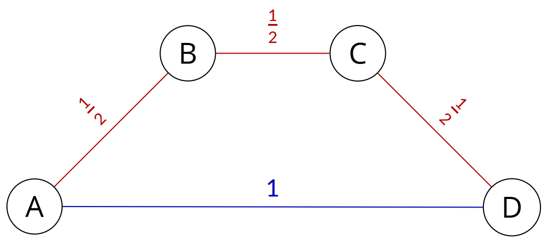
Q3

(a) Counter example: Consider the following graph:



In the original weighting, the shortest path from A to D is given by the blue path with weight 1. Replacing each edge weight with the square would make the red path shortest with weight $\frac{3}{4}$, the blue path still has length 1.

(b) Proof: Let $G = (V, E, w)$ be a weight graph with original weighting function $w : E \to \mathbb{R}^+$. Since a minimum spanning tree M of G is a forest with $p = |V|$ vertices and $q = |E|$ edges and $c = 1$ connect components, By *lemma 0.1*, $p - q = c$ so $q = p - c = p - 1$.

1=>2: Let M be a MST in w. If M is not an MST in

2=>1:

Q4

1=>2:

Assume a tree $T = (V, E)$ has a perfect matching. For some $v \in V$, there is some $u \in V$ matched to v. This splits T into $n \geq 1$ connected components.

One of these components C_1 must contain u, there must be no other edge from C_1 to V in T otherwise T would contain a cycle. As T has a perfect matching, $C_1 - u$ must have a perfect matching so C_1 has an odd number of vertices.

For each remaining connected C_i did contain a vertex matched to v. Thus, C_i has a perfect matching and so C_i has an even number of vertices.

2=>1:

Claim: $T - v$ must contain exactly one odd connected component as $o(T - v) = 1$ (by hypothesis) call this component $\Theta(v)$.

Claim: Since T was connected, each connected component must contain a vertex u adjacent to v in T.

Claim: There is exactly one such vertex u in each component. (Otherwise, if there were two vertices then there must be a cycle in T which is a contradiction)

Let M be a set. For every $v \in V$, there is some u from $\Theta(v)$ that it is adjacent to v in T, add $e = v, u$ to M.

Claim: The edges in M are mutually disjoint.

Proof by contradiction:

- Assume that two edges in M are not disjoint, so some $u, v \in V$ were both paired to some w.
- Let G_u be the subgraph of T induced by all the vertices of the even connected components of $T - u$ and u. Notice that G_u is connected as T was connected, and the only edges removed from $T - u$ contain u so every connected component was connected to u in T. Define G_v in the same way.
- For any subgraph $G \in \{G_u, G_v\}$, only u/v could be connected to a vertex outside of G,
- These subgraphs are disjoint since $\Theta(u)$ contains w which is still connected to v in G_v, thus G_v is a subgraph of $\Theta(u)$ which is not contained in G_u.
- Notice that both G_u and G_v contain an odd number of vertices as they contain the even connected components in $T - u$ or $T - v$ as well as u or v respectfully.
- Now consider $T - w$, both G_u and G_v are connected components in $T - w$ as they are not contained in any larger connected as the only edge in T leaving either subgraph contained w.

  with odd numbers of vertices so $o(T - w) \geq 2$ which is a contradiction.

Therefore, M is a matching by definition, however M is also a perfect matching as it contains every $v \in V$.

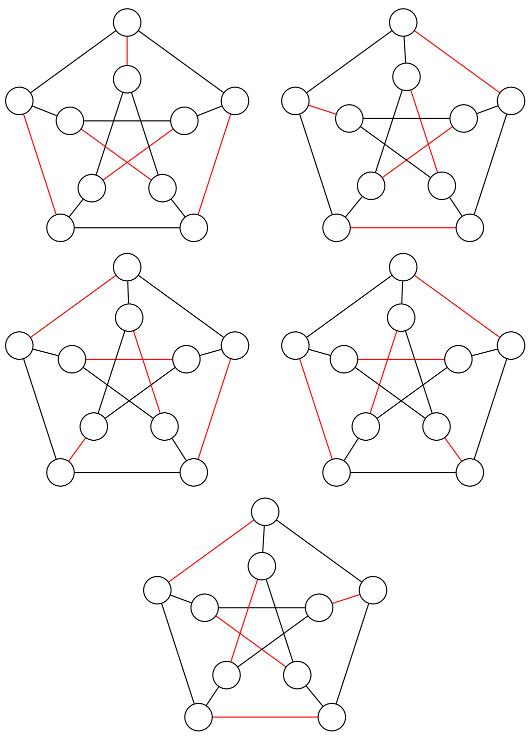
Q5

(a) Since $G = (V, E)$ is regular, G is d-regular for $d = \Delta(G)$. Since G is class 1, there is a d-edge colouring $\lambda : E \to C$ with $|C| = d$. For any edge e_0 of G, consider the matching give by:

$$M = \left\{e \in E : \lambda(e) = \lambda(e_0)\right\}$$

By definition of edge-colouring, the edges of M are mutually disjoint and therefore a valid matching. Since each degree has degree d, there are d edges with distinct colours including an edge with colour $\lambda(e_0)$. Thus, every vertex must be contained in M making M a perfect matching so any e_0 is matchable.

(b) The Petersen graph is 3-regular and was shown to have chromatic index 4 in *Assignment 1*, thus it is regular and class 2. Consider the following matchings on the Petersen graph where the red edges are the matched edges:



Since every edge is contained in at least one of the matchings above, no edge is unmatchable.

Q6

Since $G = (V, E)$ is bipartite, then $V = V_1 \uplus V_2$, WLOG assume $|V_1| \geq |V_2|$.

Q7