Robert Christie

MATHS 340

S1 2023

Assignment 1
Due: 1-08-2023

Q1

(a) We can see that the surface S, is defined implicitly $f(x, y, z) = 0$ where $f : \mathbb{R}^3 \to \mathbb{R}$ is continuously differentiable (as it is a polynomial). Thus, we can see that as $(a, b, c) = (1, 2, 1)$ is a surface point, the tangent plane is given by:

$$\begin{aligned}
0 &= \frac{\partial f}{\partial x}(a, b, c)(x - a) + \frac{\partial f}{\partial y}(a, b, c)(y - b) + \frac{\partial f}{\partial z}(a, b, c)(z - c) \\
&= 2a(x - a) + 4b(y - b) - 10c(z - c) \\
&= 2(x - 1) + 8(y - 2) - 10(z - 1) \\
&= 2x + 8y - 10z - 8
\end{aligned}$$

(b) MATLAB code for producing plot:

```
syms x y z
f = x.^2 + 2*y.^2 - 5*z.^2 - 4; % Implicit eq for surface
x0 = [1;2;1];                   % Point for tangent plane

% Implicit function for tangent plane
t = subs(jacobian(f, [x, y, z]), [x, y, z], x0.')*([x;y;z]-x0);

% Shadding and colouring
colormap(bone); lighting gouraud; shading interp;
fimplicit3(f);  % Plot implicit surface
hold on         % Overlay both surfaces
fimplicit3(t);  % Plot the x0 tangent plane (Twice for colouring trick)
fimplicit3(t, "FaceColor", [0.9, 0.1, 0.1], "FaceAlpha", 0.5);

% Label the plot
xlabel('x'); ylabel('y'); zlabel('z');
title("Surface x^2 + 2y^2 - 5z^2 = 4 with [1 2 1]^T tangent plane")
view([-238.75 6.89])
```

Resulting plot shown in Figure 1.

(c) We are considering a surface defined by a function $f(x, y, z) = 0$, evaluating $f(0, 0, 1) = -9 \neq 0$, thus the point does not lie on the surface. Since there is no surface at this point to be tangent to, it does not make sense to ask for the tangent plane at this point.

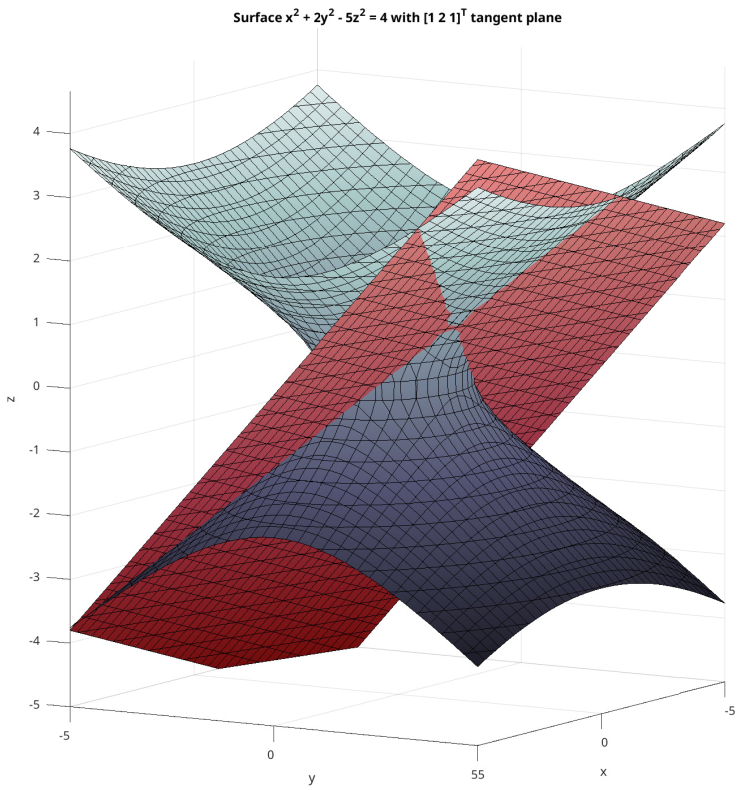

Figure 1: Plot of the tangent plane for the point $[1, 2, 1]^T$ for the surface implicitly defined by $x^2 + 2y^2 - 5z^2 = 4$

Q2

(a) Consider the function $\zeta : \begin{bmatrix} u \\ v \end{bmatrix} \mapsto \begin{bmatrix} x \\ y \end{bmatrix}$. From the definitions given we know:

$$\zeta : \begin{bmatrix} u \\ v \end{bmatrix} \mapsto \begin{bmatrix} ve^{u} \\ ve^{-u} \end{bmatrix}$$

Thus we find that:

$$D\zeta = \left[\begin{array}{c|c} \frac{\partial \zeta}{\partial u} & \frac{\partial \zeta}{\partial v} \end{array} \right] = \begin{bmatrix} ve^{u} & e^{u} \\ -ve^{-u} & e^{-u} \end{bmatrix}$$

The columns of $D\zeta$ give us a moving frame of basis vectors, which we normalise to give unit vectors:

$$e_u = \frac{\begin{bmatrix} ve^{u} \\ -ve^{-u} \end{bmatrix}}{\sqrt{v^2 e^{2u} + v^2 e^{-2u}}} = \frac{\begin{bmatrix} e^{u} \\ -e^{-u} \end{bmatrix}}{\sqrt{e^{2u} + e^{-2u}}} = \begin{bmatrix} \sqrt{\frac{e^{2u}}{e^{2u}+e^{-2u}}} \\ -\sqrt{\frac{e^{-2u}}{e^{2u}+e^{-2u}}} \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{1}{\frac{e^{2u}+e^{-2u}}{e^{2u}}}} \\ -\sqrt{\frac{1}{\frac{e^{2u}+e^{-2u}}{e^{-2u}}}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{1+e^{-4u}}} \\ \frac{-1}{\sqrt{1+e^{4u}}} \end{bmatrix}$$

$$e_v = \frac{\begin{bmatrix} e^{u} \\ e^{-u} \end{bmatrix}}{\sqrt{e^{2u} + e^{-2u}}} = \begin{bmatrix} \sqrt{\frac{e^{2u}}{e^{2u}+e^{-2u}}} \\ \sqrt{\frac{e^{-2u}}{e^{2u}+e^{-2u}}} \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{1}{\frac{e^{2u}+e^{-2u}}{e^{2u}}}} \\ \sqrt{\frac{1}{\frac{e^{2u}+e^{-2u}}{e^{-2u}}}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{1+e^{-4u}}} \\ \frac{1}{\sqrt{1+e^{4u}}} \end{bmatrix}$$

Note that care was taken to preserve signs, the signs are not affected by division of v as $v = \sqrt{xy} > 0$, or by changing $e^{\pm u}$ to $\sqrt{e^{\pm 2u}}$ since $e^{u} > 0$. Writing the results explicitly in terms of the standard Cartesian basis vectors:

$$e_u = e_x \frac{e_x}{\sqrt{1 + e^{-4u}}} - \frac{e_y}{\sqrt{1 + e^{4u}}} \qquad e_v = e_x \frac{e_x}{\sqrt{1 + e^{-4u}}} + \frac{e_y}{\sqrt{1 + e^{4u}}}$$

(b) We can show this by substituting and simplifying:

$$e_u + e_v = \begin{bmatrix} \frac{1}{\sqrt{1+e^{-4u}}} \\ \frac{-1}{\sqrt{1+e^{4u}}} \end{bmatrix} + \begin{bmatrix} \frac{1}{\sqrt{1+e^{-4u}}} \\ \frac{1}{\sqrt{1+e^{4u}}} \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{1+e^{-4u}}} \\ \frac{1-1}{\sqrt{1+e^{4u}}} \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{1+e^{-4u}}} \\ 0 \end{bmatrix} = \frac{2}{\sqrt{1+e^{-4u}}} e_x$$

Similarly for $e_u - e_v$:

$$e_u - e_v = \begin{bmatrix} \frac{1}{\sqrt{1+e^{-4u}}} \\ \frac{-1}{\sqrt{1+e^{4u}}} \end{bmatrix} - \begin{bmatrix} \frac{1}{\sqrt{1+e^{-4u}}} \\ \frac{1}{\sqrt{1+e^{4u}}} \end{bmatrix} = \begin{bmatrix} \frac{0}{\sqrt{1+e^{-4u}}} \\ \frac{-1-1}{\sqrt{1+e^{4u}}} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{-2}{\sqrt{1+e^{4u}}} \end{bmatrix} = \frac{-2}{\sqrt{1+e^{4u}}} e_y$$

(c) First we find the partial derivatives of the unit bases vectors:

$$\begin{aligned}
\frac{\partial e_u}{\partial u} &= \frac{\partial}{\partial u} \begin{bmatrix} \frac{1}{\sqrt{1+e^{-4u}}} \\ \frac{-1}{\sqrt{1+e^{4u}}} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\partial}{\partial u} \left(1+e^{-4u}\right)^{-\frac{1}{2}} \\ -\frac{\partial}{\partial u} \left(1+e^{4u}\right)^{-\frac{1}{2}} \end{bmatrix} \\
&= \begin{bmatrix} \frac{-1}{2} \left(1+e^{-4u}\right)^{\frac{-3}{2}} (0 - 4e^{-4u}) \\ -\frac{-1}{2} \left(1+e^{4u}\right)^{\frac{-3}{2}} (0 + 4e^{4u}) \end{bmatrix} \\
&= 2 \begin{bmatrix} e^{-4u} \left(1+e^{-4u}\right)^{\frac{-3}{2}} \\ e^{4u} \left(1+e^{4u}\right)^{\frac{-3}{2}} \end{bmatrix} \\
&= \frac{e^{-4u}}{1+e^{-4u}}(e_u + e_v) - \frac{e^{4u}}{1+e^{4u}}(e_u - e_v) \\
&= \frac{e_u + e_v}{1+e^{4u}} - \frac{e_u - e_v}{1+e^{-4u}}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial e_v}{\partial u} &= \frac{\partial}{\partial u} \begin{bmatrix} \frac{1}{\sqrt{1+e^{-4u}}} \\ \frac{1}{\sqrt{1+e^{4u}}} \end{bmatrix} \\
&= 2 \begin{bmatrix} e^{-4u} \left(1+e^{-4u}\right)^{\frac{-3}{2}} \\ -e^{4u} \left(1+e^{4u}\right)^{\frac{-3}{2}} \end{bmatrix} \\
&= \frac{e^{-4u}}{1+e^{-4u}}(e_u + e_v) + \frac{e^{4u}}{1+e^{4u}}(e_u - e_v) \\
&= \frac{e_u + e_v}{1+e^{4u}} + \frac{e_u - e_v}{1+e^{-4u}}
\end{aligned}$$

$$\frac{\partial e_u}{\partial v} = \frac{\partial}{\partial v} \begin{bmatrix} \frac{1}{\sqrt{1+e^{-4u}}} \\ \frac{-1}{\sqrt{1+e^{4u}}} \end{bmatrix} = 0 \qquad \frac{\partial e_v}{\partial v} = \frac{\partial}{\partial v} \begin{bmatrix} \frac{1}{\sqrt{1+e^{-4u}}} \\ \frac{1}{\sqrt{1+e^{4u}}} \end{bmatrix} = 0$$

Finding the velocity in terms of e_u, e_v by deriving $R(t)$:

$$\begin{aligned}
R'(t) &= \frac{dR(t)}{dt} \\
&= \frac{d}{dt}\Big[u(t) \cdot e_u(u(t), v(t))\Big] + \frac{d}{dt}\Big[v(t) \cdot e_v(u(t), v(t))\Big] \\
&= \left[u' e_u + u \cdot \frac{d}{dt}\Big[e_u(u(t), v(t))\Big]\right] + \left[v' e_v + v \cdot \frac{d}{dt}\Big[e_v(u(t), v(t))\Big]\right] && \text{Product Rule} \\
&= \left[u' e_u + u \cdot \left[u' \frac{\partial e_u}{\partial u} + v' \frac{\partial e_u}{\partial v}\right]\right] + \left[v' e_v + v \cdot \left[u' \frac{\partial e_v}{\partial u} + v' \frac{\partial e_v}{\partial v}\right]\right] && \text{Chain Rule} \\
&= u' e_u + uu' \left(\frac{e_u + e_v}{1+e^{4u}} - \frac{e_u - e_v}{1+e^{-4u}}\right) + v' e_v + vu' \left(\frac{e_u + e_v}{1+e^{4u}} + \frac{e_u - e_v}{1+e^{-4u}}\right) && \text{Substituting previous results}
\end{aligned}$$

Q3

(a) First we choose α, β such for $t \in [0, \frac{\pi}{2}]$:

$$\begin{aligned} 4 &= (\alpha \cos t)^2 + 2(\beta \sin t)^2 \\ &= \alpha^2 \cos^2 t + 2\beta^2 \sin^2 t \end{aligned}$$

See that by choosing $\alpha = 2$ and $\beta = \sqrt{2}$, the equation is satisfied for all $t \in [0, \frac{\pi}{2}]$:

$$x^2 + 2y^2 = 2^2 \cos^2 t + 2(\sqrt{2})^2 \sin^2 t = 4(\cos^2 t + \sin^2 t) = 4$$

To show that the curves are the same, we use the fact that $0 \leq \sin t, \cos t \leq 1$ since $t \in [0, \frac{\pi}{2}]$. Hence, the identity $\sin^2 t + \cos^2 t = 1 \implies \sin t = \sqrt{1 - \cos^2 t}$. Using this to rewrite $r(t)$:

$$r(t) = \begin{bmatrix} 2\cos t \\ \sqrt{2} \sin t \end{bmatrix} = \begin{bmatrix} 2\cos t \\ \sqrt{2}\sqrt{1 - \cos^2 t} \end{bmatrix} = \begin{bmatrix} 2\cos t \\ \sqrt{2 - 2\cos^2 t} \end{bmatrix} = \begin{bmatrix} 2\cos t \\ \sqrt{\frac{4-(2\cos t)^2}{4}} \end{bmatrix} = \begin{bmatrix} x \\ \sqrt{\frac{4-x^2}{2}} \end{bmatrix}$$

Where $x \in [0, 2]$. Now we rearrange the implicit equation for the wire, note that $x, y \geq 0$:

$$x^2 + 2y^2 = 4 \implies y^2 = \frac{4 - x^2}{2} \implies y = \sqrt{\frac{4 - x^2}{2}}$$

Note that we have $y, x \geq 0$, and $x^2 + 2y^2 = 4 \implies x^2 \leq 4 \implies x \leq 2$. Thus, the solution to the implicit equation is exactly $\left\{[x, \frac{4-x^2}{2}]^T : x \in [0, 2]\right\}$ which is the same as we found from $r(t)$. Hence, the two curves are the same.

(b) We find the mass of the wire by integrating its linear density along its length:

$$m = \int_0^{\frac{\pi}{2}} \rho(r(t)) \|r'(t)\| dt$$

Where:

$$\begin{aligned} \rho(t) &= (\alpha \cos t)(\beta \sin t) \\ r'(t) &= \begin{bmatrix} -\alpha \sin t \\ \beta \cos t \end{bmatrix} \\ \|r'(t)\| &= \sqrt{\alpha^2 \sin^2 t + \beta^2 \cos^2 t} \\ &= \sqrt{4 \sin^2 t + 2\cos^2 t} \end{aligned}$$

By substituting values, then applying the identity $\sin^2 \theta + \cos^2 \theta = 1$ and pulling the constants out of the integral:

$$m = \int_0^{\frac{\pi}{2}} \left[2\sqrt{2} \sin t \cos t\right] \sqrt{4\sin^2 t + 2\cos^2 t}\, dt = 4 \int_0^{\frac{\pi}{2}} [\sin t \cos t] \sqrt{\sin^2 t + 1t}\, dt$$

Applying u-substitution with $u(t) = \sin^2 t + 1t$, hence $\frac{du}{dt} = 2 \sin t \cos t$, therefore $dt = \frac{1}{2 \sin t \cos t} du$. Thus:

$$\begin{aligned} m &= 4 \int_{u(0)}^{u(\frac{\pi}{2})} \sin t \cos t \sqrt{u} \frac{1}{2 \sin t \cos t} du \\ &= 2 \int_{u(0)}^{u(\frac{\pi}{2})} \sqrt{u}\, du && \text{Cancelling and extracting constant} \\ &= 2 \int_1^2 \sqrt{u}\, du && \text{Evaluating bounds} \\ &= 2 \left[\frac{2}{3} 2^{\frac{3}{2}} - \frac{2}{3} 1^{\frac{3}{2}}\right] && \text{Solving and substituting bounds} \\ &= \frac{8\sqrt{2} - 4}{3} && \text{Simplifying} \end{aligned}$$