Robert Christie

MATHS 340

S1 2023

Assignment 2

Due: ??-??-2023

Q1

(a) .

(b) .

(c)

$$
\begin{aligned}
&0 \le x \le 1 \\
&0 \le y \le 1 \\
&x^2 \le z \le x \iff \sqrt{x^2} \le \sqrt{z} \wedge z \le x
\end{aligned}
$$

Thus we can write the bounds as:

$$
\begin{aligned}
&0 \le z \le 1 \\
&0 \le y \le 1 \\
&z \le x \le \sqrt{z}
\end{aligned}
$$

As our triple integral is "nice", we can rewrite it with these new bounds:

$$\int_0^1 \int_0^1 \int_z^{\sqrt{z}} f(x, y, z) \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

Q2

(a) First we find integrals for the surface area of each component body:

- The box has 6 sides, each with area:

$$\int_0^4 \int_0^4 1 \mathrm{d}u\mathrm{d}v$$

- We parametrise the surface as:

$$r(u, v) = \begin{bmatrix} e^u \cos v \\ e^u \sin v \\ u \end{bmatrix}$$

Where $u \in [0,1]$ and $v \in [0,2\pi]$. Giving a surface integral:

$$\begin{aligned}
&\int_0^{2\pi}\int_0^1 \left\| \frac{\partial r(u,v)}{\partial u} \times \frac{\partial r(u,v)}{\partial v} \right\| \mathrm{d}u\mathrm{d}v \\
&= \int_0^{2\pi}\int_0^1 \left\| \begin{bmatrix} e^u \cos v \\ e^u \sin v \\ 1 \end{bmatrix} \times \begin{bmatrix} -e^u \sin v \\ e^u \cos v \\ 0 \end{bmatrix} \right\| \mathrm{d}u\mathrm{d}v \\
&= \int_0^{2\pi}\int_0^1 \left\| \begin{bmatrix} -e^u \cos v \\ -e^u \sin v \\ e^{2u}\cos^2 v + e^{2u}\sin^2 v \end{bmatrix} \right\| \mathrm{d}u\mathrm{d}v \\
&= \int_0^{2\pi}\int_0^1 \sqrt{e^{2u} + e^{4u}}\, \mathrm{d}u\mathrm{d}v \\
&= \int_0^{2\pi}\int_0^1 e^u\sqrt{1 + e^{2u}}\, \mathrm{d}u\mathrm{d}v
\end{aligned}$$

- The top of the box must have a hole for the tube, therefore we must remove an area correspond to the bottom of the tube, a disk with radius $e^0 = 1$:

$$-\int_0^1 \int_0^{2\pi} r \mathrm{d}r$$

- The bottom of the half sphere connects the top of the tube to the sphere, thus it is a ring with inner radius e^1 and outer radius 4. Again using polar coordinates:

$$\int_e^4 \int_0^{2\pi} r \mathrm{d}r$$

- We parametrise the surface of the sphere in spherical coordiates, $[\rho,\theta,\phi]^T$ for $\rho = 4$, $\theta \in [0,2\pi]$, $\phi \in [0,\frac{\pi}{2}]$, the area element of the sphere is given by $\rho^2 \sin\phi \mathrm{d}\theta\mathrm{d}\phi$. Hence we have an integral for the surface area of:

$$\int_0^{\frac{\pi}{2}} \int_0^{2\pi} 4^2 \sin\phi \mathrm{d}\theta\mathrm{d}\phi$$

Thus the surface area of R is given by:

$$A = 6\int_0^4\int_0^4 1\mathrm{d}u\mathrm{d}v + \int_0^{2\pi}\int_0^1 e^u\sqrt{1+e^{2u}}\,\mathrm{d}u\mathrm{d}v - \int_0^1\int_0^{2\pi} r\mathrm{d}r + \int_e^4\int_0^{2\pi} r\mathrm{d}r + \int_0^{\frac{\pi}{2}}\int_0^{2\pi} 4^2\sin\phi\mathrm{d}\theta\mathrm{d}\phi$$

(b) We find three separate integrals for the mass of each region of R:

- For the box we simply integrate ρ over its bound in Cartesian coordinates:

$$\int_{-4}^0 \int_{-2}^2 \int_{-2}^2 \rho(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

- For the tube, we integrate in cylindrical coordinates where the volume element is given by $\mathrm{d}V = r\mathrm{d}r\mathrm{d}\theta\mathrm{d}z$, giving an integral of:

$$\int_0^1 \int_0^{2\pi} \int_0^{e^z} \rho(r\cos\theta, r\sin\theta, z) r\mathrm{d}r\mathrm{d}\theta\mathrm{d}z$$

- Lastly for the sphere, we integrate in Cartesian coordinates:

$$\int_{-4}^4 \int_{-\sqrt{4^2-x^2}}^{\sqrt{4^2-x^2}} \int_1^{1+\sqrt{4^2-x^2-y^2}} \rho(x,y,z)\mathrm{d}z\mathrm{d}y\mathrm{d}x$$

Thus we have a total mass of:

$$m = \int_{-4}^{0}\int_{-2}^{2}\int_{-2}^{2} \rho(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z$$
$$+ \int_{0}^{1}\int_{0}^{2\pi}\int_{0}^{e^z} \rho(r\cos\theta, r\sin\theta, z) r\mathrm{d}r\mathrm{d}\theta\mathrm{d}z$$
$$+ \int_{-4}^{4}\int_{-\sqrt{4^2-x^2}}^{\sqrt{4^2-x^2}}\int_{1}^{1+\sqrt{4^2-x^2-y^2}} \rho(x,y,z)\mathrm{d}z\mathrm{d}y\mathrm{d}x$$

Q3

(a) Looking at the gradient field, we see that a factor of $2x$ and $2y$ indicating the inside was differentiated using the chain rule, and a reciprocal indicating that the outer function could have been $\ln z$, with derivative z^{-1}. Thus, as a guess we consider $\phi(x,y) = \log(x^2+y^2)$ defined on $\mathbb{R}^2 \setminus \{0\}$, which is continuous and differentiable over its domain. Checking its gradient:

$$\nabla\phi(x,y) = \begin{bmatrix} \frac{\partial\phi(x,y)}{\partial x} \\ \frac{\partial\phi(x,y)}{\partial y} \end{bmatrix} = \frac{2}{x^2+y^2}\begin{bmatrix} x \\ y \end{bmatrix}$$

As $\mathbb{R}^2 \setminus \{0\} \subset \{(x,y) | y = \alpha x \; \alpha \in \mathbb{R}\}$, ϕ is defined here and one has $\nabla\phi = F$. Hence, ϕ is a potential which satisfies the question.

(b) In (a), we have already found a potential that satisfies the question, thus it exists.