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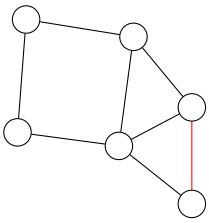
MATHS 326

S1 2024

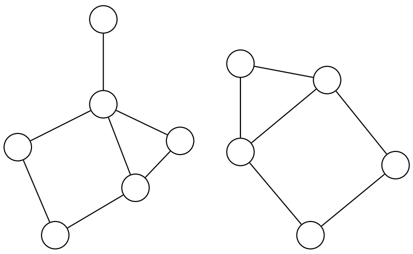
Assignment 1
Due: 22-03-2024

Q1

(a) Call the graph G, and apply deletion contraction theorem to the edge e in red:

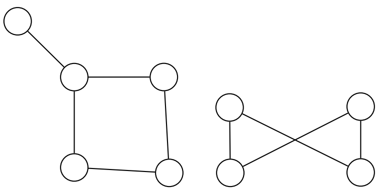


Thus, $P_G(x) = P_{G-e}(x) - P_{G_e}(x)$ This produces two graphs, $G - e$ and G_e:



$G - e$ $\qquad$ G_e

Notice that $G - e$ can be constructed by adding a vertex to $H = G_e$. Applying deletion contraction to H:



$H - e$ $\qquad$ H_e

We see that $H_e = C_4$, and $H - e$ is C_4 with an additional vertex v added, which can take any colour except that of its neighbour. Thus:

$$\begin{aligned} P_{H-e}(x) &= (x-1)C_4(x) \\ P_{H_e}(x) &= C_4(x) \end{aligned}$$

Using this with the deletion contraction gives the chromatic polynomial of H:

$$\begin{aligned} P_H(x) &= P_{H-e}(x) - P_{H_e}(x) \\ &= (x-2)C_4(x) \\ &= (x-2)((x-1)^4 + x - 1) \end{aligned}$$

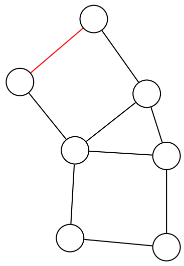
Since $G - e$, is just H with an additional vertex v with a single neighbor, we find:

$$P_{G-e}(x) = (x-1)H$$

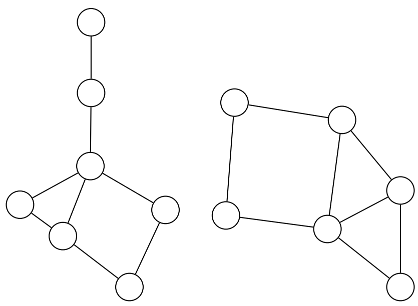
From these results, we finish the deletion contraction of G:

$$\begin{aligned} P_G(x) &= P_{G-e}(x) - P_{G_e}(x) \\ &= (x-1)H - H \\ &= (x-2)H \\ &= (x-2)^2((x-1)^4 + x - 1) \end{aligned}$$

(b) Let G be the given graph, apply deletion contraction theorem on the edge in red:



This produces graphs:



$G - e$ $\qquad$ G_e

Notice that G_e is the graph from part (a), while $G-e$ can be constructed by adding two vertices to the graph H from the working for part (a). Notice that both the added vertices have only one neighbour each, so they can have any of

$(x-1)$ colours for each existing colouring of H. Thus, the chromatic polynomials are:

$$\begin{aligned} P_{G-e} &= (x-1)^2 P_H(x) \\ &= (x-1)^2(x-2)((x-1)^4 + x - 1) \\ P_{G_e}(x) &= P_{G_a}(x) \\ &= (x-2)^2((x-1)^4 + x - 1) \end{aligned}$$

Finally we use these results in the deletion contraction theorem applied to G:

$$\begin{aligned} P_G(x) &= (x-1)^2(x-2)((x-1)^4 + x - 1) \\ &\quad - (x-2)^2((x-1)^4 + (x-1)) \\ &= \left((x-1)^2 - x + 2\right)(x-2)\left((x-1)^4 + x - 1\right) \end{aligned}$$

Q2

(a) First we prove that: $P_G(x)$ contains an x term, implies G is connected.

We prove the contrapositive: G is disconnected implies that $P_G(x)$ does not contain a non-zero x term.

Consider any disconnected graph G which can be expressed as the disjoint union of $k \geq 2$ subgraphs $H_1, \dots, H_k$.

By *Lemma 1.7*:

$$P_G(x) = \prod_{i=1}^{k} P_{H_i}(x)$$

Since $P_{H_i}(x)$ are chromatic polynomials, they do not contain a constant term. Thus:

$$\begin{aligned} & P_G(x) \\ = \; & (x^{n_1} + \dots + c_{2,1}x^2 + c_{1,1}x) \\ & \times \\ & \vdots \\ & \times \\ & (x^{n_k} + \dots + c_{2,k}x^2 + c_{1,k}x) \\ = \; & \text{H.O.T.} + (c_{1,1} \dots c_{1,k})x^k \end{aligned}$$

Which does not contain a non-zero x.

Now we prove the inverse: G is connected, implies $P_G(x)$ contains a non-zero x term.

Apply strong induction on $m = |E|$ for connected graphs $G = (V, E)$.

Base case:

For $|E| = 0$, the only connected graph is a single vertex with $P_G(x) = x$.

Induction Step "$1, \dots, m \implies m + 1$":

Consider a graph with $|E| = m + 1$, apply deletion contraction, thus:

$$P_G(x) = P_{G-e}(x) - P_{G_e}(x)$$

Where G_e has m or fewer edges. Clearly G_e is connected, as any path from u to v in G_e can be constructed from an existing path in G.

Since G_e is connected with $\leq m$ edges, the induction hypothesis applies and P_{G_e} contains a non-zero x term.

Since G has n vertices, so does $G - e$ so $P_{G-e}(x)$ is of degree n while G_e has $n-1$ vertices so $P_{G_e}(x)$ has degree $n - 1$.

For a chromatic polynomial, the x^n has coefficient 1 and the signs alternate, since P_{G-e} and P_{G_e} differ in degree by 1 they have opposite signs for the x term.

So by deletion contraction for some $c_a \geq 0, c_b > 0$:

$$\begin{aligned} P_G(x) &= P_{G-e}(x) - P_{G_e}(x) \\ &= (x^n + \dots \pm c_a x) \\ &\quad - (x^{n-1} + \dots \mp c_b x) \\ &= x^n + \dots \pm (c_a + c_b)x \end{aligned}$$

As $c_a + c_b > 0$, $P_G(x)$ must also contain a non-zero x term.

Thus, we have shown that G is connected if and only if $P_G(x)$ contains a non-zero x term.

(b) When G is connected, *Corollary 1.15* still applies and so G is a tree if and only if $P_G(x) = x(x-1)^{n-1}$. So the new statement holds in this case.

When G is disconnected, G is not a tree as trees are connected by definition. However, $P_G(x)$ also cannot contain a non-zero x term by (a). So the statement holds.

Since all simple graphs are either connected or disconnected, the new statement is true.

Q3

(a) Expanding $P(x) = (x-1)^4$ results in a polynomial with constant term $(-1)^4 = 1$, so by *Theorem 1.14*, $P(x)$ is not the chromatic polynomial of any graph as it has a non-zero constant term.

(b) Let $P(x) = x^6+6x^5+7x^3-2x$. As x^3 and $-2x$ have different signs but are 2 terms apart the signs of coefficients do not alternate. Thus, by *Theorem 1.14*, $P(x)$ is not the chromatic polynomial of any graph.

(c) Assume $P(x) = x^4 - 3x^3 + 4x^2 - 2x$ is a chromatic polynomial for some G. We can deduce the following using *Theorem 1.14*:

- Since P has degree 4, G has 4 vertices.
- The $x^{n-1} = x^3$ term has a coefficient of -3, so G must have 3 edges.
- The x term has a non-zero coefficient, thus G is connected.

We can't have a cycle in G as the smallest cycle requires all 3 edges, and so we can't connect all vertices. As G is connected and acyclic, it is a tree.

By part (2b), G is a tree if and only if it has chromatic polynomial:

$$P_G(x) = x(x-1)^{n-1} = x(x-1)^3$$

. Now see that:

$$P_G(2) = 2 \neq P(2) = 2^4 - 3\cdot 2^3 + 4\cdot 2^2 - 2\cdot 2 = 4$$

Thus, $P(x)$ is not $P_G(x)$.

As this resulted in a contraction, P is not the chromatic polynomial for any graph G.

Q4

Let $G = (V, E)$ be any planar simple graph.

When $|V| \leq 2$, G has at most 1 edge so the statement holds.

For the case with $3 \leq n \leq 11$ vertices and m edges. We use proof by contraction:

By *Theorem 1.19*, we know that $m \leq 3n - 6$.

Assume all vertices of G have degree at least 5. By the handshaking lemma:

$$5n \leq \sum_{v\in V} \deg(v) = 2|E| = 2m$$

Thus, $2.5n \leq m \leq 3n-6$. We can only find such m when:

$$\begin{aligned} & 2.5n \leq 3n - 6 \\ \iff & 6 \leq 0.5n \\ \iff & 12 \leq n \end{aligned}$$

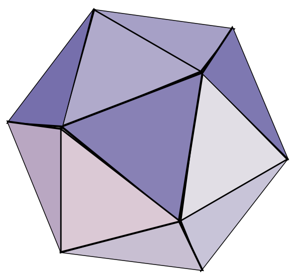
This is a contraction for $n < 12$, so the assumption must be false and for graphs with $n < 12$, there exists a vertex with degree at most 4.

To show the bound is sharp we look for a graph on 12 vertices such that $\forall_{v\in V} \deg v \geq 5$. We derived the bound:

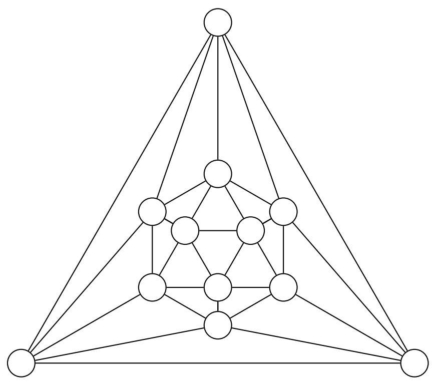
$$2.5n = 30 \leq m \leq 3n - 6 = 30$$

So we have exactly $m = 30$ edges. Since $5\cdot 12 = 2\cdot 30$ each vertex has degree exactly 5.

The regular icosahedron is a Platonic solid with 12 vertices and 30 edges where each vertex is adjacent to 5 edges. This is similar to our graph requirements.



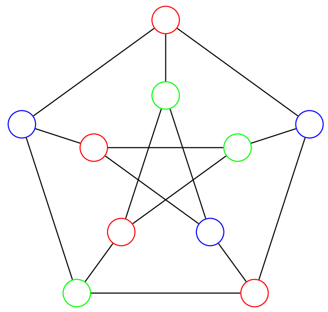
In the 2D projection above, the 3 vertices of the back most hidden face can be scaled up to remove all edge-edge intersections in the protection. We can define a graph from the edges and vertices of the icosahedron and consider the modified projection a planar embedding of this graph:



This is a planar simple 5-regular graph on 12 vertices demonstrating the bound is sharp as each vertex has degree $5 > 4$.

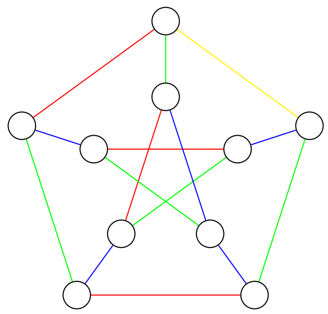
Q5

(a) We can create a 3-colouring of the Petersen graph:



Therefore, $\chi(P) \leq 3$, however, since P contains an odd cycle of length 5, we also have $\chi(P) \geq 3$, hence $\chi(P) = 3$.

The following proper edge 4-colouring of P exists:



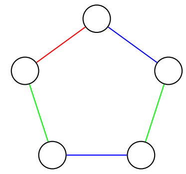
Thus $\chi_e(P) \leq 4$.

Next we show that no edge 3-colouring exists:

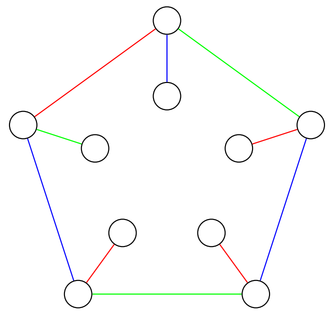
- The cycle C_5 (a subgraph of P) has chromatic index 3. So no edge 2-colouring exists.
- If we take any graph with 3 edges where no pair is adjacent, then there a 6 vertices in the graph.

  Since C_5 has only 5 vertices, we cannot find a subgraph with 3 edges such that no pair is adjacent.

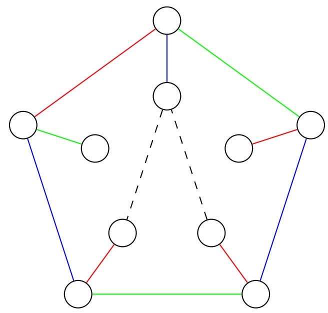
  Since we can't find 3 non-adjacent edges to give the same colour, each colour is used at most twice in an edge-colouring of C_5.
- We must use 2 colours twice and 1 colour once in order to colour all 5 edges of C_5 as each colour is used either once or twice.
- WLOG, let r be the colour used once. The remaining colours g, b must alternate on the remaining edges.



- When edges are added to C_5 to produce the following subgraph of P, their colouring is fully determined by the existing edges in the cycle.



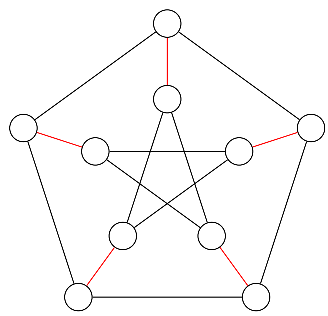
- Consider the subgraph of P created by adding the two dashed edges to the previous subgraph:



  Both edges are adjacent to r and b edges already, so they must both be g. However, since they are adjacent they cannot receive the same colours. Thus, we have a contraction, so P is not 3 edge-colourable and $\chi_e(P) > 3$.

Since $3 < \chi_e(P) \leq 4$, the chromatic index of the Petersen graph $\chi_e(P) = 4$.

(b) Contract the edges of P show in red below:



This produces K_5 so by Wagner's theorem, P must be non-planar.

Q6

First we prove that for a simple graph G: G is bipartite implies every subgraph H of G satisfies $\alpha(H) \geq m_H/2$:

Let G be a bipartite graph. By *Theorem 1.6*, G contains no odd cycle. So any subgraph H of G also contains no odd cycle, so by *Theorem 1.6*, $H = (V, E)$ is bipartite. Therefore, its vertices can be written as the disjoint union $V = V_1 \uplus V_2$ so that every edge in H has one endpoint in each of V_1, V_2.

Let V_i, V_j be the largest and smallest of the two vertex sets respectively (or V_1, V_2 if the sets have the same size). Clearly V_i must contain at least $m_H/2$ vertices, otherwise:

$$|V_j| \leq |V_i| < m_H/2 \implies |V_1| + |V_2| < m_H = |V|$$

By the definition of a bipartite graph, no two vertices in V_i are adjacent, thus V_i is a set of independent vertices and so $\alpha(H) \geq V_i \geq m_H/2$.

Now we prove the inverse to show equivalence:

Consider any odd cycle graph C_{2k+1} with vertex set V. Assume that $\alpha(C_{2k+1}) \geq |V|/2$. There must be an independent set U containing at least $|V|/2$ vertices. Since $|V|$ is odd $|U| > |V|/2$.

Every $v \in V$ is adjacent to exactly two other vertices. Therefore, every $u \in U$ is adjacent to two vertices $w_1, w_2 \in V \setminus U$. Let W be the count of how many distinct $\{u, w_i\}$ edges exist, $W = 2|U|$. Since each w_i vertex appears in at most 2 edges, there at least $W/2$ such vertices and so:

$$|V \setminus U| \geq \frac{W}{2} = |U|$$

Therefore:

$$|U| + |V \setminus U| = 2|U| > |V|$$

Which is a contraction as U and $V \setminus U$ are disjoint subsets of V, so we should have:

$$|U| + |V \setminus U| = |V|$$

As our assumption was false:

$$\alpha(C_{2k+1}) < |V|/2$$

For any odd cycle graph C_{2k+1}.

Now let G be a graph where every subgraph H of G satisfies $\alpha(H) \geq m_H/2$. Thus, G contains no subgraph H with $\alpha(H) < m_H/2$. By the previous working, G contains no odd cycles and so by *Theorem 1.6*, G is bipartite.

Thus, we have shown that G is bipartite if and only if every subgraph H of G satisfies $\alpha(H) \geq m_H/2$.