## TAYLOR POLYNOMIALS

**Definition.** For a sufficiently smooth function $f : \mathbb{R} \to \mathbb{R}$, the $K$-th degree Taylor Polynomial of $f(x)$ centred about $a$ is given by:

$$P_k(x) = \sum_{j=0}^{k} \frac{f^{(j)}(a)}{j!}(x - a)$$

**Example.** Calculate the linear approximation to $f(x) = \sin(x)$ centred at $a = 0$.

$$P_0(x) = f(0) = 0$$
$$P_1(x) = f(0) + f'(0)(x - 0) = x$$

**Definition.** Let $S$ be a surface given by $f(x, y, z) = 0$ where $f : \mathbb{R}^3 \to \mathbb{R}$ is continuously differentiable. Then the degree one Taylor Polynomial at $(a, b, c)$ is:

$$f(a,b,c) + \begin{bmatrix} \frac{\partial f}{\partial x}(a,b,c) & \frac{\partial f}{\partial y}(a,b,c) & \frac{\partial f}{\partial z}(a,b,c) \end{bmatrix} \cdot \begin{bmatrix} x-a \\ y-b \\ z-c \end{bmatrix}$$

When $(a, b, c)$ is on the surface $S$, then the tangent plane at $(a, b, c)$ is:

$$\frac{\partial f}{\partial x}(a,b,c)(x-a) + \frac{\partial f}{\partial y}(a,b,c)(y-b) + \frac{\partial f}{\partial z}(a,b,c)(z-c)$$

Where $Df(\mathbf{a})$ is the **total derivative** (or jacobian) of $f$ at $\mathbf{a}$.

$$Df(a) = \begin{bmatrix} \frac{\partial f}{\partial x_1}(\mathbf{a}), \dots, \frac{\partial f}{\partial x_m}(\mathbf{a}) \end{bmatrix}$$

And $Hf(\mathbf{a})$ is the **Hessian** of $f$ at $\mathbf{a}$, a matrix with:

$$[Hf(a)]_{i,j} = \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{a})$$

**Definition.** For $f : \mathbb{R}^n \to \mathbb{R}$, the degree $1, 2$ Taylor polynomials centred at $a$ are:

$$P_1(\mathbf{r}) = f(\mathbf{a})Df(\mathbf{a})(\mathbf{r} - \mathbf{a})$$
$$P_2(\mathbf{r}) = f(\mathbf{a})Df(\mathbf{a})(\mathbf{r} - \mathbf{a}) + \frac{1}{2}(\mathbf{r} - \mathbf{a})^T Hf(\mathbf{a})(\mathbf{r} - \mathbf{a})$$

## VECTOR DERIVATIVES

**Definition.** The derivative of $r(t)$ where $r : \mathbb{R} \to \mathbb{R}^n$ with respect to $t$ is done component wise:

$$r'(t) = \begin{bmatrix} r_1'(t) \\ \vdots \\ r_n'(t) \end{bmatrix}$$

**Example.** For functions $u, v, w : \mathbb{R} \to \mathbb{R}^n$ and $f : \mathbb{R} \to \mathbb{R}$ with $\alpha, \beta \in \mathbb{R}$:

$$\begin{array}{rl} \alpha u + \beta v & \alpha u' + \beta v' \\ u \cdot v & u' \cdot v + u \cdot v' \\ u \times v & u' \times v + u \times v' \\ u \cdot (v \times w) & u' \cdot v \times w + u \cdot v' \times w + u \cdot v \times w' \\ u(f(t)) & f'(t)\dfrac{du}{df} = f'(t)u'(f(t)) \end{array}$$

## CURVILINEAR COORDINATES

A coordinate parametrisation for a new system $\mathbf{u} = [u_1, \dots, u_n]$ from the standard coordinates on $\mathbb{R}^n$:

$$\xi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$
$$\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \mapsto \begin{bmatrix} \xi_1(\mathbf{u}) \\ \vdots \\ \xi_n(\mathbf{u}) \end{bmatrix}$$

We can find a basis for $\mathbf{u}$ from the columns of $D\xi$,

$$\frac{\partial \xi}{\partial u_1}, \dots, \frac{\partial \xi}{\partial u_n} \qquad e_{u_j} = \frac{\partial \xi}{\partial u_j} \left\| \frac{\partial \xi}{\partial u_j} \right\|^{-1}$$

Note that when the basis are not linearly independent, our coordinates are not well-behaved. (Cylindrical/polar when $r = 0$, Spherical when $x, y = 0$.) These points are **coordinate singularities**.

### Polar

$$\xi : \begin{bmatrix} r \\ \theta \end{bmatrix} \mapsto \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix} \qquad \frac{de_r}{d\theta} = e_\theta, \quad \frac{de_\theta}{d\theta} = -e_r$$
$$dA = r\,dr\,d\theta$$

### Cylindrical

$$\xi : \begin{bmatrix} r \\ \theta \\ z \end{bmatrix} \mapsto \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ z \end{bmatrix} \qquad \begin{array}{lll} \frac{\partial e_r}{\partial r} = 0 & \frac{\partial e_r}{\partial \theta} = e_\theta & \frac{\partial e_r}{\partial z} = 0 \\ \frac{\partial e_\theta}{\partial r} = 0 & \frac{\partial e_\theta}{\partial \theta} = -e_r & \frac{\partial e_\theta}{\partial z} = 0 \\ \frac{\partial e_z}{\partial r} = 0 & \frac{\partial e_z}{\partial \theta} = 0 & \frac{\partial e_z}{\partial z} = 0 \end{array}$$
$$dA = r\,dr\,d\theta\,dz$$

### Spherical

$$\xi : \begin{bmatrix} r \\ \theta \\ \phi \end{bmatrix} \mapsto \begin{bmatrix} r\cos\theta\sin\phi \\ r\sin\theta\sin\phi \\ r\cos\phi \end{bmatrix}$$

$$\begin{array}{lll} \frac{\partial e_r}{\partial r} = 0 & \frac{\partial e_r}{\partial \theta} = \sin\phi e_\theta & \frac{\partial e_r}{\partial z} = e_\phi \\ \frac{\partial e_\theta}{\partial r} = 0 & \frac{\partial e_\theta}{\partial \theta} = -\sin\phi e_r - \cos\phi e_\phi & \frac{\partial e_\theta}{\partial z} = 0 \\ \frac{\partial e_z}{\partial r} = 0 & \frac{\partial e_z}{\partial \theta} = \cos\phi e_\theta & \frac{\partial e_z}{\partial z} = -e_r \end{array}$$
$$dV = r^2|\sin\phi|\,dr\,d\theta\,d\phi$$

## PARAMETRISED CURVES

Continuous map $c : I \to \mathbb{R}^n$, $I \subseteq \mathbb{R}$.

- **Closed curve/loop**, $I = [t_0, t_1]$ and $c(t_0) = c(t_1)$.
- **Simple**, $c$ is injective for non-endpoint $t \in I$.
- **Regular**, differentiable and $\forall_{t \in I} c'(t) \neq 0$.
- **Incremental Arc length**: $ds = \sqrt{c' \cdot c'}dt$
- **Arc length**: $s(t_0, t_1) = \int_{t_0}^{t_1} ds$
- **Path Integral**: $\int_s f \circ c(t) ds$

## MANIFOLDS

Multidimensional integrals are linear wrt integrands, and additive wrt. Regions of integration.

**Theorem** (Generalised Fubini)**.** *Generalises to higher dimensions.*

$$\iint_R f(x,y)dA = \int_{y_1}^{y_2} \left[ \int_{x_1(y)}^{x_2(y)} f(x,y)dx \right] dy$$
$$\iint_R f(x,y)dA = \int_{x_1}^{x_2} \left[ \int_{y_1(x)}^{y_2(x)} f(x,y)dy \right] dx$$

*Type 1: y from a to b, x from $g_1(y)$ to $g_2(y)$.*
*Type 2: x from a to b, y from $h_1(x)$ to $h_2(x)$.*

**Definition.** A **parametrised $n$-manifold** in $\mathbb{R}^m$ is a continuous map:

$$\mathbf{R} : S \subseteq \mathbb{R}^n \to \mathbb{R}^m$$

- If $n = 1$ curve, $n = 2$ surface, $n = m - 1$ hyper-surface.
- Input $u_1, \dots, u_n$ are curvilinear coordinates on the manifold.
- Sometimes the image is referred to as the manifold.
- Tangent vectors are columns of

$$D\mathbf{R} = \left[ \frac{\partial \mathbf{R}}{\partial u_1}, \dots, \frac{\partial \mathbf{R}}{\partial u_n} \right]$$

  If linearly independent gives us moving frame.
- If lin-indpt for all $\mathbf{u}$, $\mathbf{R}$ is **regular** or **immersed**.
- Regularity can be checked using $\mathtt{Null}(D\mathbf{R}) = 0$ or $n$-volume of paralleletope spanned by tangents.
- Calculated with **Gram determinant**:

$$\sqrt{\det((D\mathbf{R})^T(D\mathbf{R}))}$$

- Or $|\det D\mathbf{R}|$ when $n = m$.
- Multiply this by $du_1, \dots, du_n$ to get $dM$ infinitesimal $n$-volume element of the manifold.

For a surface $R(u, v)$ in $\mathbb{R}^3$, $dA = \left\| \frac{\partial \mathbf{R}}{\partial u} \times \frac{\partial \mathbf{R}}{\partial v} \right\|$ du dv. Otherwise, for a surface in higher dimensions we can use $dA = \left\| \frac{\partial \mathbf{R}}{\partial u} \right\| \left\| \frac{\partial \mathbf{R}}{\partial v} \right\| \sin\theta\, du\, dv$.

## VECTOR FIELDS

### Simply Connected Regions

A region $U \subseteq \mathbb{R}^n$ is **simply connected** if:

- $U$ is path connected, any two points can be joined by a continuous path in $U$.
- Any loop in $U$ can be continuously contracted to a point while remaining in $U$.

### Gradient

For $f : \mathbb{R}^n \to \mathbb{R}$, the total derivative matrix is a row vector and the gradient of $f$ is:

$$\operatorname{grad} f = \nabla f = (Df)^T$$

For $\mathbf{F} : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ a scalar potential $f : U \to \mathbb{R}$ has $\nabla f = \mathbf{F}$. Only **conservative** vector fields have potential functions.

Necessary condition for conservative $(\forall_{i\neq j})\ \frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}$

To find a potential, $f = \int F_1 dx$, this has an arbitrary function $G(y)$ instead of a constant of integration. Try to solve $F_2 = \frac{\partial f}{\partial y}$ for $G'(y)$, if possible find $\int G'(y)dy$.

The **path integral** of a vector field **F** along and **oriented** curve $C$ parametrised by $\mathbf{r} : [t_0, t_1] \to \mathbb{R}^n$ is:

$$\int_C F_1 dx_1 + \cdots + F_n dx_n = \int_C F \cdot d\mathbf{r} = \int_{t_0}^{t_1} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt$$

This can be turned into a path integral of a scalar function with respect to arc length. (Note that curve orientation matters unlike with path integrals of scalar fields).

The **FTC for line integrals**: Let $C$ be a (piecewise) smooth curve in $\mathbb{R}^n$ with parametrisation $\mathbf{r} : [t_0, t_1] \to \mathbb{R}^n$, and $f$ a continuously differentiable function.

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(t_1)) - f(\mathbf{r}(t_0))$$

If **F** is a vector field with continuous partial derivatives defined on an open simply conected region $U \subseteq \mathbb{R}^n$. Then **F** is conservative if and only if:

$$(\forall_{i\neq j})\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}$$

### Curl

For $\mathbf{F}(x, yz)$ a vector field on $\mathbb{R}^3$. The **curl** of **F** is:

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{bmatrix} \frac{\partial \mathbf{F}_3}{\partial y} - \frac{\partial \mathbf{F}_2}{\partial z} \\ \frac{\partial \mathbf{F}_1}{\partial z} - \frac{\partial \mathbf{F}_3}{\partial x} \\ \frac{\partial \mathbf{F}_2}{\partial x} - \frac{\partial \mathbf{F}_1}{\partial y} \end{bmatrix}$$

If $\operatorname{curl} \mathbf{F} = 0$, then **F** is **curl-free** or **irrotational**. The curl of a gradient is zero $0 = \operatorname{curl}(\nabla f)$ where $f$ is a scalar function.

Give a vector field $\mathbf{v}(x, y, z)$ defined on a region $U \subseteq \mathbb{R}^3$, a **vector potential** for **v** is a vector field **F** such that $\mathbf{v} = \operatorname{curl} \mathbf{F}$.

A vector potential is determined up to an arbitrary conservative vector field since $\operatorname{curl}(\mathbf{F} + \nabla f) = \operatorname{curl} \mathbf{F} + 0$

### Green's Theorem

For $R$, a closed bounded region in $\mathbb{R}^2$ with boundary $\partial R$ consisting of piecewise smooth curves.

- The **induced positive orientation** is given by orienting all boundary curves such that $R$ lies to the left of the curve direction.
- Green's theorem (Like FC for double integrals): Let **F** be a smooth vector field, then:

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} dx dy\right) = \oint_{\partial R} F_1 dx + F_2 dy$$

We can use an **F** with $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1$ to compute areas.

### Stokes' Theorem

A surface $S$ in $\mathbb{R}^3$ is **orientable** if we can find a continuous unit normal vector. If we have a parametrisation $r(u, v)$, then normalising $\partial ru \times \partial rv$ gives us an orientation.

If the surface has boundaries $\partial S$ consisting of piecewise smooth curves, then choosing a normal vector induces an orientation on the boundary. Walking along the boundary oriented head in the direction of $n$, the surface should be on your left.

A surface is closed if it is compact (closed and bounded), and $\partial S = \emptyset$. Orientation is canonically given by orienting $n$ to point outwards.

The surface integral of a vector field **F** on an oriented surface $S$ parametrised by $\mathbf{r} : U \to \mathbb{R}^3$, with a unit normal vector field $\mathbf{n} = \frac{T_u \times T_v}{\|T_u \times T_v\|}$ is given by:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n} dA = \iint_U \mathbf{F}(\mathbf{r}) \cdot T_u \times T_v \, du\, dv$$

The integrand $\mathbf{F} \cdot d\mathbf{S}$ is the **flux density**, the integral is the **total flux** of **F** across **S**.

**Stokes' theorem** for $S$ a compact oriented surface in $\mathbb{R}^3$, where $\partial S$ consists of piecewise smooth curves with induced positive orientations. Then for any smooth vector field **F**:

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \oint_{\partial S} F_1 dx + F_2 dy + F_3 dz$$

The theorem still applies when $\partial S = \emptyset$. Integral is 0.

### Divergence

The **divergence** of $\mathbf{v}(x_1, \dots, x_n) = [v_1, \dots, v_n]^T$, a vector field on $\mathbb{R}^n$ is:

$$\nabla \cdot \mathbf{v} = \div \mathbf{v} = \sum_{i=1}^{n} \frac{\partial v_i}{\partial x_i}$$

- If $\operatorname{div} \mathbf{v}(x) > 0$, then $x$ is a source.
- If $\operatorname{div} \mathbf{v}(x) < 0$, then $x$ is a sink.

$$\nabla(fg) = f\nabla g + (\nabla f)g$$
$$\operatorname{curl}(f\mathbf{F}) = f \operatorname{curl} \mathbf{F} + \nabla f \times \mathbf{F}$$
$$\operatorname{div}(f\mathbf{v}) = f \operatorname{div} \mathbf{v} + \nabla f \cdot \mathbf{v}$$

Where $C^\infty(U)$ is the smooth scalar functions and $\mathfrak{X}(U)$ the smooth vector function of a region $U \subseteq \mathbb{R}^3$. We have a co-chain complex:

$$C^\infty(U) \xrightarrow{\text{grad}} \mathfrak{X}(U) \xrightarrow{\text{curl}} \mathfrak{X}(U) \xrightarrow{\text{div}} C^\infty(U)$$

Where composing two adjacent operators gives zero.

### Vector Potential

A vector potential **F** for a vector field **v** on region $U \subseteq \mathbb{R}^3$:

$$\mathbf{v} = \operatorname{curl} \mathbf{F}$$

A vector potential is determined up to an arbitrary scalar potential since:

$$\operatorname{curl}(\mathbf{F} + \nabla f) = \operatorname{curl} \mathbf{F} + 0$$

Since $\operatorname{div} \circ \operatorname{curl} = 0$, we must have $\operatorname{div} \mathbf{v} = 0$, this is sufficient for a potential to exist if $U$ has no **cavities** (sphere can be contracted to a point).

### Divergence Theorem

For $V$ a compact (bounded, closed) volume region in $\mathbb{R}^3$, with boundary $\partial V$ of piecewise smooth surfaces with the positive orientation. Then for a smooth vector field **v**:

$$\iiint_V \operatorname{div} \mathbf{v} dV = \oiint_{\partial V} v \cdot d\mathbf{V}$$

## USEFUL STUFF

If $S$ is a upwards-oriented surface given by a graph $z = f(x, y)$, where $f : U \to \mathbb{R}^3$ is a function on region $U \subseteq \mathbb{R}^2$:

$$dS = \left[-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right]^T$$