TAYLOR POLYNOMIALS

Definition. For a sufficiently smooth function $f : \mathbb{R} \to \mathbb{R}$, the K-th degree Taylor Polynomial of $f(x)$ centred about a is given by:

$$P_k(x) = \sum_{j=0}^{k} \frac{f^{(j)}(a)}{j!}(x-a)$$

Example. Calculate the linear approximation to $f(x) = \sin(x)$ centred at $a = 0$.

$$P_0(x) = f(0) = 0$$
$$P_1(x) = f(0) + f'(0)(x-0) = x$$

Definition. Let S be a surface given by $f(x,y,z) = 0$ where $f : \mathbb{R}^3 \to \mathbb{R}$ is continuously differentiable. Then the degree one Taylor Polynomial at (a,b,c) is:

$$f(a,b,c) + \begin{bmatrix} \frac{\partial f}{\partial x}(a,b,c) & \frac{\partial f}{\partial y}(a,b,c) & \frac{\partial f}{\partial z}(a,b,c) \end{bmatrix} \cdot \begin{bmatrix} x-a \\ y-b \\ z-c \end{bmatrix}$$

When (a,b,c) is on the surface S, then the tangent plane at (a,b,c) is:

$$\frac{\partial f}{\partial x}(a,b,c)(x-a) + \frac{\partial f}{\partial y}(a,b,c)(y-b) + \frac{\partial f}{\partial z}(a,b,c)(z-c)$$

Where $Df(\mathbf{a})$ is the **total derivative** (or jacobian) of f at $\mathbf{a}$.

$$Df(a) = \left[\frac{\partial f}{\partial x_1}(\mathbf{a}), \dots, \frac{\partial f}{\partial x_m}(\mathbf{a})\right]$$

And $Hf(\mathbf{a})$ is the **Hessian** of f at $\mathbf{a}$, a matrix with:

$$[Hf(a)]_{i,j} = \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{a})$$

Definition. For $f : \mathbb{R}^n \to \mathbb{R}$, the degree $1, 2$ Taylor polynomials centred at a are:

$$P_1(\mathbf{r}) = f(\mathbf{a})Df(\mathbf{a})(\mathbf{r}-\mathbf{a})$$
$$P_2(\mathbf{r}) = f(\mathbf{a})Df(\mathbf{a})(\mathbf{r}-\mathbf{a}) + \frac{1}{2}(\mathbf{r}-\mathbf{a})^T Hf(\mathbf{a})(\mathbf{r}-\mathbf{a})$$

VECTOR DERIVATIVES

Definition. The derivative of $r(t)$ where $r : \mathbb{R} \to \mathbb{R}^n$ with respect to t is done component wise:

$$r'(t) = \begin{bmatrix} r_1'(t) \\ \vdots \\ r_n'(t) \end{bmatrix}$$

Example. For functions $u, v, w : \mathbb{R} \to \mathbb{R}^n$ and $f : \mathbb{R} \to \mathbb{R}$ with $\alpha, \beta \in \mathbb{R}$:

$$\begin{aligned}
\alpha u + \beta v &\quad \alpha u' + \beta v' \\
u \cdot v &\quad u' \cdot v + u \cdot v' \\
u \times v &\quad u' \times v + u \times v' \\
u \cdot (v \times w) &\quad u' \cdot v \times w + u \cdot v' \times w + u \cdot v \times w' \\
u(f(t)) &\quad f'(t)\frac{du}{df} = f'(t)u'(f(t))
\end{aligned}$$

CURVILINEAR COORDINATES

A coordinate parametrisation for a new system $\mathbf{u} = [u_1, \dots, u_n]$ from the standard coordinates on $\mathbb{R}^n$:

$$\begin{aligned} \xi : \ & \mathbb{R}^n \longrightarrow \mathbb{R}^n \\ & \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \mapsto \begin{bmatrix} \xi_1(\mathbf{u}) \\ \vdots \\ \xi_n(\mathbf{u}) \end{bmatrix} \end{aligned}$$

We can find a basis for $\mathbf{u}$ from the columns of $D\xi$,

$$\frac{\partial \xi}{\partial u_1}, \dots, \frac{\partial \xi}{\partial u_n} \qquad e_{u_j} = \frac{\partial \xi}{\partial u_j} \left\| \frac{\partial \xi}{\partial u_j} \right\|^{-1}$$

Note that when the basis are not linearly independent, our coordinates are not well-behaved. (Cylindrical/polar when $r = 0$, Spherical when $x, y = 0$.) These points are **coordinate singularities**.

Polar

$$\xi : \begin{bmatrix} r \\ \theta \end{bmatrix} \mapsto \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix} \qquad \frac{de_r}{d\theta} = e_\theta, \quad \frac{de_\theta}{d\theta} = -e_r$$

$$dA = r\,dr\,d\theta$$

Cylindrical

$$\xi : \begin{bmatrix} r \\ \theta \\ z \end{bmatrix} \mapsto \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ z \end{bmatrix} \qquad \begin{matrix} \frac{\partial e_r}{\partial r} = 0 & \frac{\partial e_r}{\partial \theta} = e_\theta & \frac{\partial e_r}{\partial z} = 0 \\ \frac{\partial e_\theta}{\partial r} = 0 & \frac{\partial e_\theta}{\partial \theta} = -e_r & \frac{\partial e_\theta}{\partial z} = 0 \\ \frac{\partial e_z}{\partial r} = 0 & \frac{\partial e_z}{\partial \theta} = 0 & \frac{\partial e_z}{\partial z} = 0 \end{matrix}$$

$$dA = r\,dr\,d\theta\,dz$$

Spherical

$$\xi : \begin{bmatrix} r \\ \theta \\ \phi \end{bmatrix} \mapsto \begin{bmatrix} r\cos\theta\sin\phi \\ r\sin\theta\sin\phi \\ r\cos\phi \end{bmatrix}$$

$$\begin{matrix} \frac{\partial e_r}{\partial r} = 0 & \frac{\partial e_r}{\partial \theta} = \sin\phi e_\theta & \frac{\partial e_r}{\partial z} = e_\phi \\ \frac{\partial e_\theta}{\partial r} = 0 & \frac{\partial e_\theta}{\partial \theta} = -\sin\phi e_r - \cos\phi e_\phi & \frac{\partial e_\theta}{\partial z} = 0 \\ \frac{\partial e_z}{\partial r} = 0 & \frac{\partial e_z}{\partial \theta} = \cos\phi e_\theta & \frac{\partial e_z}{\partial z} = -e_r \end{matrix}$$

$$dV = r^2 |\sin\phi| dr\,d\theta\,d\phi$$

PARAMETRISED CURVES

Continuous map $c : I \to \mathbb{R}^n$, $I \subseteq \mathbb{R}$.

- **Closed curve/loop**, $I = [t_0, t_1]$ and $c(t_0) = c(t_1)$.
- **Simple**, c is injective for non-endpoint $t \in I$.
- **Regular**, differentiable and $\forall_{t \in I} c'(t) \neq 0$.
- **Incremental Arc length**: $ds = \sqrt{c' \cdot c'}dt$
- **Arc length**: $s(t_0, t_1) = \int_{t_0}^{t_1} ds$
- **Path Integral**: $\int_s f \circ c(t) ds$

MANIFOLDS

Multidimensional integrals are linear wrt integrands, and additive wrt. Regions of integration.

Theorem (Generalised Fubini)**.** *Generalises to higher dimensions.*

$$\iint_R f(x,y)dA = \int_{y_1}^{y_2} \left[\int_{x_1(y)}^{x_2(y)} f(x,y)dx \right] dy$$
$$\iint_R f(x,y)dA = \int_{x_1}^{x_2} \left[\int_{y_1(x)}^{y_2(x)} f(x,y)dy \right] dx$$

Type 1: y from a to b, x from $g_1(y)$ to $g_2(y)$.
Type 2: x from a to b, y from $h_1(x)$ to $h_2(x)$.

Definition. A **parametrised n-manifold** in $\mathbb{R}^m$ is a continuous map:

$$\mathbf{R} : S \subseteq \mathbb{R}^n \to \mathbb{R}^m$$

- If $n = 1$ curve, $n = 2$ surface, $n = m - 1$ hyper-surface.

- Input $u_1, \dots, u_n$ are curvilinear coordinates on the manifold.
- Sometimes the image is referred to as the manifold.
- Tangent vectors are columns of

$$D\mathbf{R} = \left[\frac{\partial \mathbf{R}}{\partial u_1}, \dots, \frac{\partial \mathbf{R}}{\partial u_n}\right]$$

  If linearly independent gives us moving frame.
- If lin-indpt for all $\mathbf{u}$, $\mathbf{R}$ is **regular** or **immersed**.
- Regularity can be checked using $\mathtt{Null}(D\mathbf{R}) = 0$ or n-volume of paralleletope spanned by tangents.
- Calculated with **Gram determinant**:

$$\sqrt{\det((D\mathbf{R})^T(D\mathbf{R}))}$$

- Or $|\det D\mathbf{R}|$ when $n = m$.
- Multiply this by $du_1, \dots, du_n$ to get dM infinitesimal n-volume element of the manifold.

For a surface $R(u, v)$ in $\mathbb{R}^3$, $dA = \left\|\frac{\partial \mathbf{R}}{\partial u} \times \frac{\partial \mathbf{R}}{\partial v}\right\| \mathrm{du}\,\mathrm{dv}$. Otherwise, for a surface in higher dimensions we can use $dA = \left\|\frac{\partial \mathbf{R}}{\partial u}\right\| \left\|\frac{\partial \mathbf{R}}{\partial v}\right\| \sin\theta \, du \, dv$.

VECTOR FIELDS

Simply Connected Regions

A region $U \subseteq \mathbb{R}^n$ is **simply connected** if:

- U is path connected, any two points can be joined by a continuous path in U.
- Any loop in U can be continuously contracted to a point while remaining in U.

Gradient

For $f : \mathbb{R}^n \to \mathbb{R}$, the total derivative matrix is a row vector and the gradient of f is:

$$\operatorname{grad} f = \nabla f = (Df)^T$$

For $\mathbf{F} : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ a scalar potential $f : U \to \mathbb{R}$ has $\nabla f = \mathbf{F}$. Only **conservative** vector fields have potential functions.

Necessary condition for conservative $(\forall_{i \neq j})\ \frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}$

To find a potential, $f = \int F_1 dx$, this has an arbitrary function $G(y)$ instead of a constant of integration. Try to solve $F_2 = \frac{\partial f}{\partial y}$ for $G'(y)$, if possible find $\int G'(y)dy$.

The **path integral** of a vector field $\mathbf{F}$ along and **oriented** curve C parametrised by $\mathbf{r} : [t_0, t_1] \to \mathbb{R}^n$ is:

$$\int_C F_1 dx_1 + \dots + F_n dx_n = \int_C F \cdot d\mathbf{r} = \int_{t_0}^{t_1} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt$$

This can be turned into a path integral of a scalar function with respect to arc length. (Note that curve orientation matters unlike with path integrals of scalar fields).

The **FTC for line integrals**: Let C be a (piecewise) smooth curve in $\mathbb{R}^n$ with parametrisation $\mathbf{r} : [t_0, t_1] \to \mathbb{R}^n$, and f a continuously differentiable function.

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(t_1)) - f(\mathbf{r}(t_0))$$

If $\mathbf{F}$ is a vector field with continuous partial derivatives defined on an open simply conected region $U \subseteq \mathbb{R}^n$. Then $\mathbf{F}$ is conservative if and only if:

$$(\forall_{i \neq j}) \frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}$$

Curl

For $\mathbf{F}(x, yz)$ a vector field on $\mathbb{R}^3$. The **curl** of $\mathbf{F}$ is:

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{bmatrix} \frac{\partial \mathbf{F}_3}{\partial y} - \frac{\partial \mathbf{F}_2}{\partial z} \\ \frac{\partial \mathbf{F}_1}{\partial z} - \frac{\partial \mathbf{F}_3}{\partial x} \\ \frac{\partial \mathbf{F}_2}{\partial x} - \frac{\partial \mathbf{F}_1}{\partial y} \end{bmatrix}$$

If $\operatorname{curl} \mathbf{F} = 0$, then $\mathbf{F}$ is **curl-free** or **irrotational**. The curl of a gradient is zero $0 = \operatorname{curl}(\nabla f)$ where f is a scalar function.

Give a vector field $\mathbf{v}(x, y, z)$ defined on a region $U \subseteq \mathbb{R}^3$, a **vector potential** for $\mathbf{v}$ is a vector field $\mathbf{F}$ such that $\mathbf{v} = \operatorname{curl} \mathbf{F}$.

A vector potential is determined up to an arbitrary conservative vector field since $\operatorname{curl}(\mathbf{F} + \nabla f) = \operatorname{curl} \mathbf{F} + 0$

Divergence

Green's Theorem

For R, a closed bounded region in $\mathbb{R}^2$ with boundary ∂R consisting of piecewise smooth curves.

- The **induced positive orientation** is given by orienting all boundary curves such that R lies to the left of the curve direction.
- Green's theorem (Like FC for double integrals): Let $\mathbf{F}$ be a smooth vector field, then:

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} dxdy\right) = \oint_{\partial R} F_1 dx + F_2 dy$$

We can use an $\mathbf{F}$ with $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1$ to compute areas.

Stokes' Theorem

A surface S in $\mathbb{R}^3$ is **orientable** if we can find a continuous unit normal vector. If we have a parametrisation $r(u, v)$, then normalising $\partial ru \times \partial rv$ gives us an orientation.

If the surface has boundaries ∂S consisting of piecewise smooth curves, then choosing a normal vector induces an orientation on the boundary. Walking along the boundary oriented head in the direction of n, the surface should be on your left.

A surface is closed if it is compact (closed and bounded), and $\partial S = \emptyset$. Orientation is canonically given by orienting n to point outwards.

The surface integral of a vector field $\mathbf{F}$ on an oriented surface S parametrised by $\mathbf{r} : U \to \mathbb{R}^3$, with a unit normal vector field $\mathbf{n} = \frac{T_u \times T_v}{\|T_u \times T_v\|}$ is given by:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n} dA = \iint_U \mathbf{F}(\mathbf{r}) \cdot T_u \times T_v \, du \, dv$$

The integrand $\mathbf{F} \cdot d\mathbf{S}$ is the **flux density**, the integral is the **total flux** of $\mathbf{F}$ across $\mathbf{S}$.

Stokes' theorem for S a compact oriented surface in $\mathbb{R}^3$, where ∂S consists of piecewise smooth curves with induced positive orientations. Then for any smooth vector field $\mathbf{F}$:

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \oint_{\partial S} F_1 dx + F_2 dy + F_3 dz$$

The theorem still applies when $\partial S = \emptyset$. Integral is 0.