Robert Christie

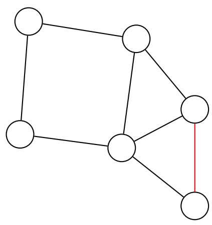
# MATHS 326

**S1 2024**

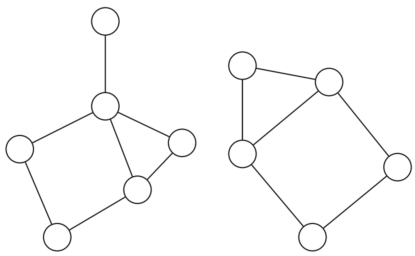
Assignment 1
Due: 20-03-2024

## Q1

(a) Call the graph $G$, and apply deletion contraction theorem to the edge $e$ in red:

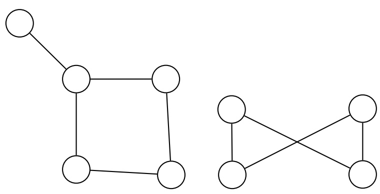


Thus, $P_G(x) = P_{G-e}(x) - P_{G_e}(x)$ This produces two graphs, $G - e$ and ${}_e$:



$G - e$ $\quad$ $G_e$

Notice that $G - e$ can be constructed by adding a vertex to $H = G_e$. Applying deletion contraction to $H$:



$H - e$ $\quad$ $H_e$

We see that $H_e = C_4$, and $H - e$ is $C_4$ with an additional vertex $v$ added, which can take any color except that of its neighbor. Thus:

$$P_{H-e}(x) = (x-1)C_4(x)$$
$$P_{H_e}(x) = C_4(x)$$

Using this with the deletion contraction gives the chromatic polynomial of $H$:

$$\begin{aligned} P_H(x) &= P_{H-e}(x) - P_{H_e}(x) \\ &= (x-2)C_4(x) \\ &= (x-2)((x-1)^4 + x - 1) \end{aligned}$$

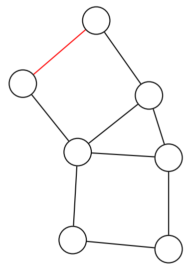
Since $G - e$, is just $H$ with an additional vertex $v$ with a single neighbor, we find:

$$P_{G-e}(x) = (x-1)H$$

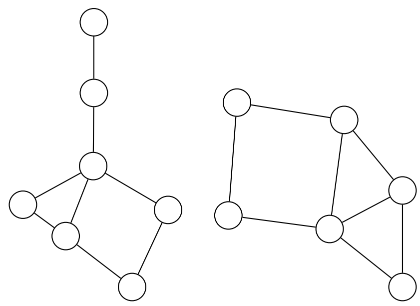
From deletion contraction we can use our results to determine :

$$\begin{aligned} P_G(x) &= P_{G-e}(x) - P_{G_e}(x) \\ &= (x-1)H - H \\ &= (x-2)H \\ &= (x-2)^2((x-1)^4 + x - 1) \end{aligned}$$

(b) We apply deletion contraction theorem to $G$ on the edge in red:



This produces graphs:



$G - e$ $\quad$ $G_e$

Notice that $G_e$ is the graph from part a, while $G-e$ can be constructed by adding two vertices to the graph $H$ from part a. Notice that the added vertices have only one neighbor each meaning they can take on $(x-1)$ colors for each existing coloring of $H$. Thus, the chromatic polynomials are:

$$\begin{aligned}P_{G-e} &= (x-1)^2 P_H(x)\\ &= (x-1)^2(x-2)((x-1)^4+x-1)\\ P_{G_e}(x) &= P_{G_a}(x)\\ &= (x-2)^2((x-1)^4+x-1)\end{aligned}$$

Finally we use these results in the deletion contraction theorem applied to $G$:

$$\begin{aligned}P_G(x) &= (x-1)^2(x-2)((x-1)^4+x-1)\\ &\quad -(x-2)^2((x-1)^4+(x-1))\\ &= \left((x-1)^2-x+2\right)(x-2)\left((x-1)^4+x-1\right)\end{aligned}$$

# Q2

(a) First we show that: $P_G(x)$ contains an $x$ term, implies $G$ is connected.

We prove the contrapositive: $G$ is disconnected implies that $P_G(x)$ does not contain a nonzero $x$ term.

Consider any disconnected graph $G$ which can be expressed as the disjoint union of $k \geq 2$ subgraphs $H_1, \ldots, H_k$.

By *Lemma 1.7*:

$$P_G(x) = \prod_{i=1}^{k} P_{H_i}(x)$$

Since $P_{H_i}(x)$ are chromatic polynomials, they do not contain a constant term. Thus, the product is of terms:

$$\begin{aligned}& P_G(x)\\ =\ & (x^{n_1}+\cdots+c_{2,1}x^2+c_{1,1}x)\\ & \times\\ & \vdots\\ & \times\\ & (x^{n_k}+\cdots+c_{2,k}x^2+c_{1,k}x)\\ =\ & \text{H.O.T.} + (c_{1,1}\ldots c_{1,k})x^k\end{aligned}$$

Thus, there is no non-zero $x$ term in $P_G(x)$.

Now we show that: $G$ is connected, implies $P_G(x)$ contains a non-zero $x$ term.

Apply strong induction on $m = |E|$ for connected graphs $G = (V, E)$.

**Base case:**

For $|E| = 0$, the only connected graph is a single vertex with $P_G(x) = x$.

**Induction Step "$1, \ldots, m \implies m+1$":**

Consider a graph with $|E| = m+1$, apply deletion contraction, thus:

$$P_G(x) = P_{G-e}(x) - P_{G_e}(x)$$

Where $G_e$ has $m$ or fewer edges. Clearly $G_e$ is connected, as any path from $u, v$ in $G_e$ can be constructed from an existing path in $G$.

Since $G_e$ is connected with $\leq m$ edges, the induction hypothesis applies and $P_{G_e}$ contains a non-zero $x$ term.

Since $G$ has $n$ vertices, so does $G-e$ so $P_{G-e}(x)$ is of degree $n$ while $G_e$ has $n-1$ vertices so $P_{G_e}(x)$ has degree $n-1$.

Fo a chromatic polynomial, the $x^n$ has coefficient 1 and the signs alternate, since $P_{G-e}$ and $P_{G_e}$ differ in degree by 1 they have opposite signs for the $x$ term.

So by deletion contraction for some $c_a \geq 0, c_b > 0$:

$$\begin{aligned}P_G(x) &= P_{G-e}(x) - P_{G_e}(x)\\ &= (x^n+\cdots \pm c_a x)\\ &\quad -(x^{n-1}+\cdots \mp c_b x)\\ &= x^n+\cdots \pm (c_a+c_b)x\end{aligned}$$

As $c_a + c_b > 0$, $P_G(x)$ must also contain a non-zero $x$ term.

Thus, we have shown that $G$ is connected if and only if $P_G(x)$ contains a non-zero $x$ term.

(b) When $G$ is connected, *Corollary 1.15* still applies and so G is a tree if and only if $P_G(x) = x(x-1)^{n-1}$. So the new statement holds in this case.

When $G$ is disconnected, $G$ is not a graph as graphs are connected by definition. However, $P_G(x)$ cannot contain a non-zero $x$ term by (a). Again the statement holds.

Since all simple graphs are either connected or disconnected, the new statement holds in all cases.

## Q3

(a) Expanding $P(x) = (x-1)^4$ results in a polynomial with constant term $(-1)^4 = 1$, so by *Theorem 1.14*, $P(x)$ is not the chromatic polynomial of any graph as it has a non-zero constant term.

(b) Let $P(x) = x^6+6x^5+7x^3-2x$. As $x^3$ and $-2x$ have different signs but are 2 terms apart the signs of coefficients do not alternate. Thus, by *Theorem 1.14*, $P(x)$ is not the chromatic polynomial of any graph.

(c) Assume $P(x) = x^4 - 3x^3 + 4x^2 - 2x$ is a chromatic polynomial for some $G$. We can deduce the following using *Theorem 1.14*:

- Since it has degree 4, it corresponds to a graph of 4 vertices.
- The $x^{n-1} = x^3$ term has a coefficient of $-3$, so $G$ must have 3 edges.
- The $x$ term has a non-zero coefficient, thus $G$ is connected.

We can't have a cycle in $G$ as this requires all 3 edges and doesn't connect all vertices. As $G$ is connected and acyclic it is a tree.

By (2), $G$ is a tree if and only if it has chromatic polynomial $P_G(x) = x(x-1)^{n-1}$. Now see that:

$$P_G(2) = 2 \neq P(2) = 2^4 - 3 \cdot 2^3 + 4 \cdot 2^2 - 2 \cdot 2 = 4$$

As this resulted in a contraction, $P$ is not the chromatic polynomial for any graph $G$.

## Q4

Let $G = (V, E)$ be any planar simple graph.

When $|V| \leq 2$, $G$ has at most 1 edge so the statement holds.

For the case with $3 \neq n \leq 11$ vertices and $m$ edges. We use proof by contraction:

By *Theorem 1.19*, we know that $m \leq 3n - 6$.

Assume all vertices of $G$ have degree at least 5. By the handshaking lemma:

$$5n \leq \sum_{v \in V} \deg(v) = 2\,|E| = 2m$$

Thus, $2.5n \leq m \leq 3n - 6$. We can only find such $m$ when:

$$\begin{aligned} & 2.5n \leq 3n - 6 \\ \iff & 6 \leq 0.5n \\ \iff & 12 \leq n \end{aligned}$$

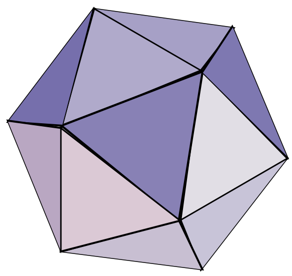
This is a contraction for $n < 12$, so the assumption must be false and for graphs with $n < 12$, there exists a vertex with degree at most 4.

To show the bound is sharp we look for a graph on 12 vertices such that $\forall_{v \in V} \deg v \geq 5$. We derived the bound:

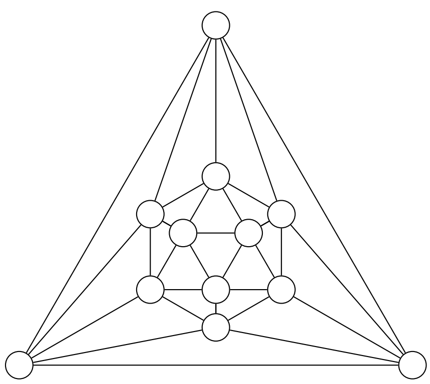
$$2.5n = 30 \leq m \leq 3n - 6 = 30$$

So we have exactly $m = 30$ edges. Since $5 \cdot 12 = 2 \cdot 30$ each vertex has degree exactly 5.

The regular icosahedron is a Platonic solid with 12 vertices and 30 edges where each vertex is adjacent to 5 edges. This is similar to our graph requirements.



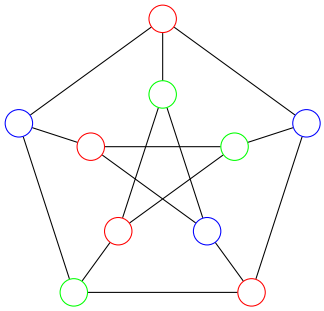
In the 2D projection above, the 3 vertices of the back most face can be scaled up to remove all edge-edge intersections in the protection. We can consider the edges and vertices in the projection as the edges and vertices of a planar embedding of a graph. This produces the graph below:



This is a planar simple 5-regular graph on 12 vertices demonstrating the bound is sharp.

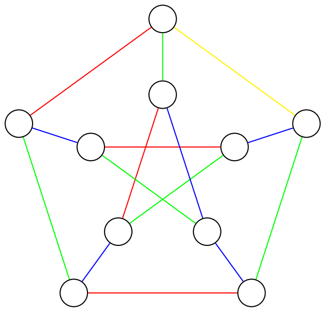
## Q5

(a) We can create a 3-coloring of the Petersen graph:



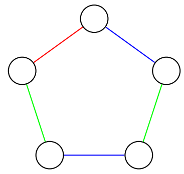
Therefore, $\chi(P) \leq 3$, however, since $P$ contains an odd cycle of length 5, we also have $\chi(P) \geq 3$, hence $\chi(P) = 3$.

We can construct the following proper edge coloring of $P$:

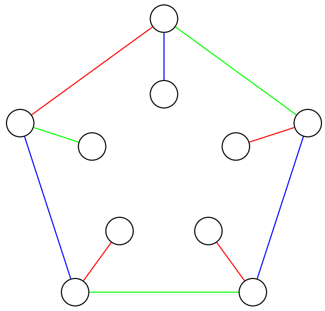


Thus $\chi_e(P) \leq 4$. Now consider a three coloring of $P$. We make the following observations:

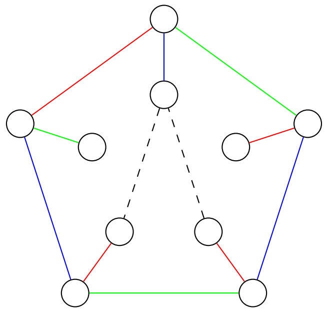
- The cycle $C_5$ has chromatic index 3.
- If we take a subgraph of three edges where no pair is adjacent, then there a 6 vertices in the subgraph by definition of edge adjacency.
- Since $C_5$ has only 5 vertices, we cannot find 3 adjacent edges such that no pair is adjacent. Thus, a coloring of $C_5$ can use each color at most twice. A 3 edge-coloring of $C_5$ must use each color at least once as no 2 edge-coloring exists.
- We must use two colors twice and one color once in order to color all 5 edges if each color is used either once or twice.
- WLOG, let $r$ be the color used once. The remaining colors $g, b$ must alternate on the remaining vertices.



- As only three colors are available, when edges are added to produce the following graph, their coloring is determined by the existing edges in the cycle.



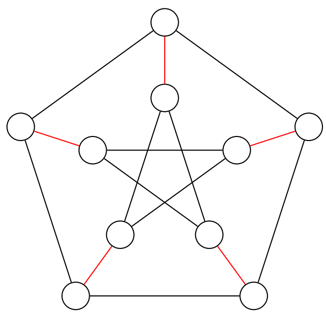
- Consider the subgraph of $P$ created by adding the two dashed edges from the Petersen graph to the previous subgraph:



Both edges are adjacent to $r$ and $b$ edges already, so they must both be $g$. However, since they are adjacent they cannot receive the same colors resulting in a contraction. Thus $P$ is not 3 edge-colorable and $\chi_e(P) > 3$.

Since $3 < \chi_e(P) \leq 4$, the chromatic index of the Petersen graph $\chi_e(P) = 4$.

(b) Contract the edges of $P$ show in red below:



This produces $K_5$ so by Wagner's theorem, $P$ must be non-planar.

# Q6

Claim: A subgraph of a bipartite graph is bipartite.

Claim: If $H$ is bipartite, then $\alpha(G) \geq \frac{1}{2}m_H$.