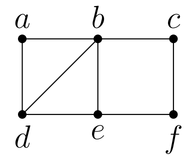
1. (a) Label the vertices of the graph as follows:



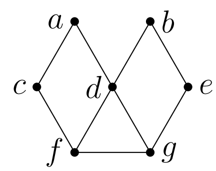
A colouring of G can be obtained by first colouring the cycle $bcfe$, then colouring d and then colouring a.

There are $P_{C_4}(x)$ ways to colour the cycle with x colours; by Exercise 1.13 (solved in the lectures) $P_{C_4}(x) = (x-1)^4 + x - 1$. Since b and e receive different colours, there are $(x-2)$ colours to choose from for d in the second step, and since b and d receive different colours there are $(x-2)$ colours to choose from for a in the final step.

Hence $P_G(x) = ((x-1)^4 + x - 1)(x-2)^2$.

(Alternatively, use deletion contraction) [3 marks]

(b) Label the graph as below:



Use deletion contraction on the edge a, d. Note that $G_{\{a,d\}}$ is the graph from part (a). In order to determine $P_{G-\{a,d\}}$ we proceed similarly to part (a): First colour the cycle $bdge$, then colour the vertex f, then c, and finally a.

There are $P_{C_4}(x) = (x-1)^4 + x - 1$ possibilities for the first step, $(x-2)$ for the colour of f (because d and g have different colours), $(x-1)$ ways of choosing the colour of c, and $(x-1)$ ways of choosing the colour of a.

Therefore $P_{G-\{a,d\}}(x) = ((x-1)^4 + x - 1)(x-2)(x-1)^2$. Finally,

$$\begin{aligned} P_G(x) &= P_{G-\{a,d\}}(x) - P_{G_{\{a,d\}}}(x) \\ &= ((x-1)^4 + x - 1)(x-2)(x-1)^2 - ((x-1)^4 + x - 1)(x-2)^2 \\ &= x^7 - 9x^6 + 35x^5 - 75x^4 + 93x^3 - 63x^2 + 18x. \end{aligned}$$

[3 marks]

2. (a) We prove this statement by induction on the number of edges of G.

Base case: If G has no edges and n vertices, then $P_G(x) = x^n$. The coefficient of x is non-zero if and only if $n = 1$, and the empty graph on n vertices is connected if $n = 1$ and disconnected otherwise.

Induction step: Let e be an arbitrary edge of G. By the deletion-contraction formula, we know that $P_G(x) = P_{G-e}(x) - P_{G_e}(x)$.

If G is not connected, then neither $G-e$ nor G_e are connected. By the inductive hypothesis, the coefficients of x in $P_{G-e}(x)$ and $P_{G_e}(x)$ are both 0, and therefore the same is true for $P_G(x)$.

If G is connected, then G_e is also connected (but $G-e$ may or may not be connected). By the inductive hypothesis, the coefficient of x in $P_{G_e}(x)$ is non-zero. Moreover, if the coefficient of x in $P_{G-e}(x)$ is non-zero, then it has the opposite sign of the coefficient of x in $P_{G_e}(x)$. Hence the coefficient of x in $P_G(x)$ is non-zero. [4 marks]

(b) If G is a tree, then $P_G(x) = x(x-1)^{n-1}$ by Lemma 1.8, so we only need to show the converse implication.

Let G be a graph with chromatic polynomial $P_G(x) = x(x-1)^{n-1}$. We note that

$$x(x-1)^{n-1} = x^n - (n-1)x^{n-1} + \binom{n-1}{2}x^{n-2} - \cdots + (-1)^{n-2}(n-1)x^2 + (-1)^{n-1}x.$$

In particular, the coefficient of x is non-zero, so G is connected, the degree is n, so G has n vertices, and the coefficient of x^{n-1} is $n-1$, so G has $n-1$ edges. Therefore G must be a tree. [2 marks]

3. (a) By Theorem 1.14 (5), the constant term in any chromatic polynomial is 0, but the constant term in $(x-1)^4$ is 1. [1 marks]

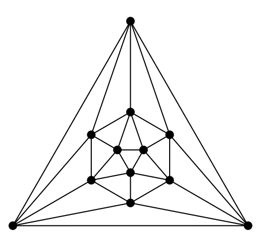
(b) By Theorem 1.14 (6) the coefficients of a chromatic polynomial alternate in sign. Since the sign of the coefficient of x^6 is positive, the sign of the coefficient of x^3 would have to be negative (which it isn't). [1 marks]

(c) If this were the chromatic polynomial of a graph G, then G would be a graph with 4 vertices and 3 edges. Moreover the coefficient of x is not zero, so G would have to be connected (by problem 2a) and therefore a tree. But $x^4 - 3x^3 + 4x^2 - 2x \neq x(x-1)^3$. [2 marks]

4. Let G be a planar graph with p vertices and q edges with minimal degree 5. we will show that $p \geq 12$.

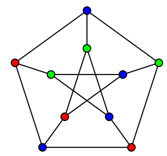
First note that by the handshaking lemma $2q = \sum_{v \in V} \deg(v)$. Since all degrees are ≥ 5 this implies that $2q \geq 5p$. By Theorem 1.19 from the lecture notes we know that $q \leq 3p - 6$. Combining the two, we obtain $5p \leq 6p - 12$ which implies $p \geq 12$.

To see that the bound is sharp consider the graph below:



The graph is planar (because no two edges in the drawing cross), has 12 vertices, and every vertex is incident to exactly 5 edges. [5 marks]

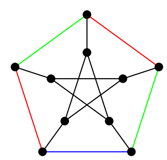
5. (a) We note that P contains an odd cycle (e.g. the 5-cycle around the outside), so by Theorem 1.6 we know that $\chi(P) > 2$. Moreover, $\chi(P) \le 3$ since we can properly 3-colour the vertices as below:



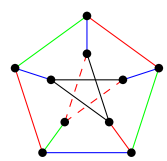
hence $\chi(P) = 3$.

Next we determine $\chi_e(P)$. Note that the P is 3-regular, so by Vizings Theorem $\chi_e(P) \in \{3, 4\}$. We show that there is no 3-edge colouring of P.

Assume that there was one, then all 3 colours must appear on the outer cycle. No colour can appear 3 times on this cycle, so one appears once, and the other two appear twice. Without loss of generality assume that the colouring of the outer cycle looks as follows:



Then the colours on the 'spokes' must be as below, and both of the two red dashed edges would have to receive the same colour.



Hence it is impossible to properly 3-edge colour P, so $\chi_e(P) = 4$. [4 marks]

(b) If we contract all the 'spokes' in the Petersen graph, we obtain K_5, hence P is not planar. [1 marks]

6. We first show that if G is bipartite, then $\alpha(H) \ge \frac{1}{2}m_H$. Let G be a bipartite graph, and let H be a subgraph of G. Then H is bipartite as well, and each of the two partite classes forms an independent set. One of the partite classes contains at least half of the vertices of H, so $\alpha(H) \ge \frac{1}{2}m_H$.

For the converse implicationlet G be a graph which is not bipartite. By Theorem 1.6, G contains an odd cycle; let H be this odd cycle. By the pigeonhole principle, every set of $\ge \frac{1}{2}m_H$ must contain two consecutive vertices, so $\alpha(H) < \frac{1}{2}m_H$. [4 marks]