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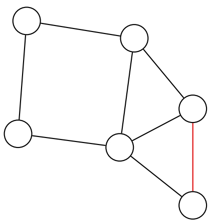
MATHS 326

S2 2023

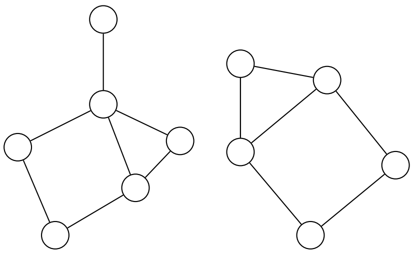
Assignment 4
Due: 20-03-2024

Q1

(a) Call the graph G, and apply deletion contraction theorem to the edge e in red:

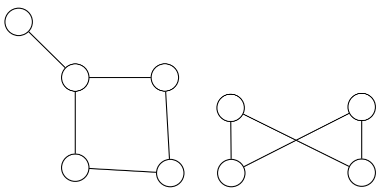


Thus, $P_G(x) = P_{G-e}(x) - P_{G_e}(x)$ This produces two graphs, $G - e$ and $_e$:



$G - e$ G_e

Notice that $G-e$ can be constructed by adding a vertex to $H = G_e$. Applying deletion contraction to H:



$H - e$ H_e

We see that $H_e = C_4$, and $H-e$ is C_4 with an additional vertex v added, which can take any color except that of its neighbor. Thus:

$$P_{H-e}(x) = (x-1)C_4(x)$$
$$P_{H_e}(x) = C_4(x)$$

Using this with the deletion contraction gives the chromatic polynomial of H:

$$\begin{aligned} P_H(x) &= P_{H-e}(x) - P_{H_e}(x) \\ &= (x-2)C_4(x) \\ &= (x-2)((x-1)^4 + x - 1) \end{aligned}$$

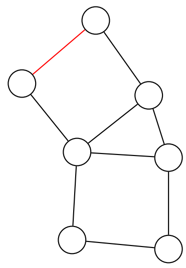
Since $G - e$, is just H with an additional vertex v with a single neighbor, we find:

$$P_{G-e}(x) = (x-1)H$$

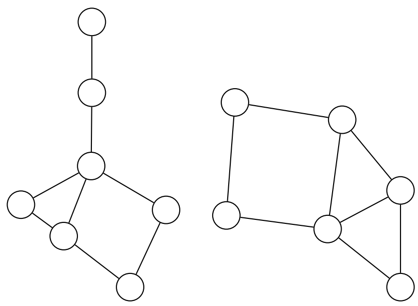
From deletion contraction we can use our results to determine :

$$\begin{aligned} P_G(x) &= P_{G-e}(x) - P_{G_e}(x) \\ &= (x-1)H - H \\ &= (x-2)H \\ &= (x-2)^2((x-1)^4 + x - 1) \end{aligned}$$

(b) We apply deletion contraction theorem to G on the edge in red:



This produces graphs:



$G - e$ G_e

Notice that G_e is the graph from part a, while $G-e$ can be constructed by adding two vertices to the graph H from part a. Notice that the added vertices have only one neighbor each meaning they can take on $(x - 1)$ colors for each existing coloring of H. Thus, the chromatic polynomials are:

$$\begin{aligned} P_{G-e} &= (x-1)^2 P_H(x) \\ &= (x-1)^2(x-2)((x-1)^4 + x - 1) \\ P_{G_e}(x) &= P_{G_a}(x) \\ &= (x-2)^2((x-1)^4 + x - 1) \end{aligned}$$

Finally we use these results in the deletion contraction theorem applied to G:

$$\begin{aligned} P_G(x) = &(x-1)^2(x-2)((x-1)^4 + x - 1) \\ &- (x-2)^2((x-1)^4 + (x-1)) \\ &= \left((x-1)^2 - x + 2\right)(x-2)\left((x-1)^4 + x - 1\right) \end{aligned}$$

Q2

(a) First we show that G is connected implies $P_G(x)$ contains a non-zero x term. Apply induction on the number of vertices n of a connected graph G.

- Base case: $n = 1$, so $P_G(x) = x$.
- Induction step: Apply induction on the number of edges m:
  - Base case: Spanning tree of G, so $P_G(x) = x(x-1)^{n-1}$.
  - Induction step: Consider some G with $n+1$ vertices and $m+1$ edges, apply deletion contraction to some edge e. As $G - e$ has m vertices it contains a non-zero x term by the induction hypothesis for the edge induction. The graph G_e will have n vertices and has a non-zero x term by the induction hypothesis for induction on the vertices. Since $P_{G-e}(x), P_{G_e}(x)$ have degree $n+1, n$ and signs alternate as they are chromatic polynomials, the x terms must have different signs. Therefore:

    $$P_G(x) = P_{G-e}(x) - P_{G_e}(x)$$

    Will also have a non-zero x term as the x term in $P_{G-e}(x)$ and $-P_{G_e}(x)$ now have the same sign.

Now we prove the other direction, that is, if $P_G(x)$ contains a non-zero x term, then G is connected:

A disconnected G consists of multiple connected subgraphs $C_1, \ldots, C_k$, each subgraph can be colored independently, thus:

$$P_G(x) = \prod_{i=1}^{k} P_{C_i}(x)$$

Since each C_i is connected, by the first implication, $P_{C_i}(x)$ has a non-zero x. As $P_{C_i}(x)$ is a chromatic polynomial, the constant term must be zero. Thus, when we take the product of $P_{C_i}(x)$ the lowest degree term is x^k and so $P_G(x)$ has a coefficient of zero for the x term.

(b)

Q3

(a) Expanding $(x-1)^4$ would result in a polynomial with constant term of $(-1)^4 = 1$, thus the polynomial is not the chromatic polynomial for any graph as it has a non-zero constant term.

(b) Notice that the x^3 term is positive but the x term is negative, thus the signs do not alternate otherwise the terms would have the same sign.

(c) Assume the polynomial is a chromatic polynomial $P_G(x)$. We can deduce the following:

- Since it has degree 4, it corresponds to a graph of 4 vertices.
- The $x^{n-1} = x^3$ term has a coefficient of -3, so G must have 3 edges.
- The x term has a non coefficient, thus G is connected.

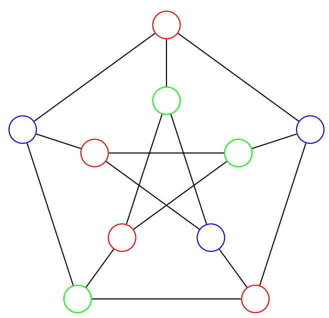
The only connected graph with 3 edges and 4 vertices is P_4, which has chromatic polynomial:

$$P_{P_4}(x) = x(x-1)^3 = x^4 - 3x^3 + 3x^2 - x \neq P_G(x)$$

Thus $P_G(x)$ is not a chromatic polynomial of any graph.

Q5

1. We can create a 3-coloring of the Petersen graph:



Therefore, $\chi(P) \leq 3$, however, since P contains an odd cycle of length 5, we also have $\chi(P) \geq 3$, hence $\chi(P) = 3$.

2. A 4-edge-coloring of the Petersen graph,