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MATHS 340

S2 2023

Assignment 3

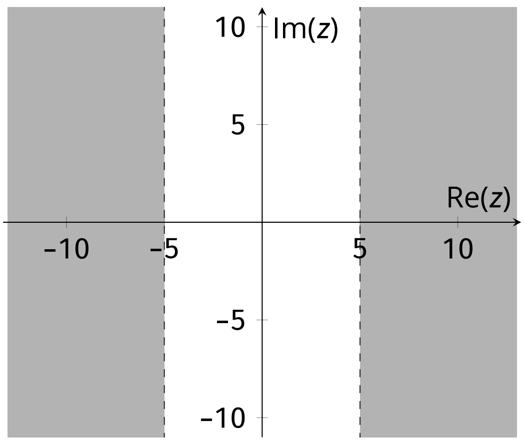
Due: 26-09-2023

Q1

(a) Let $z = x + iy$, then:

$$|z + \bar{z}| = |2x|$$

Hence the region corresponds to $|x| > 5$ (Due to the strict inequality boundary points are not included.)



(b) Let $z = re^{i\theta}$. Note that at $z = 0$, the region's existence is not well-defined. However, in the limit as $z \to 0$, $\frac{1}{|z|} \to +\infty$, while $|\bar{z}| \to 0$, so we redefine the region R as:

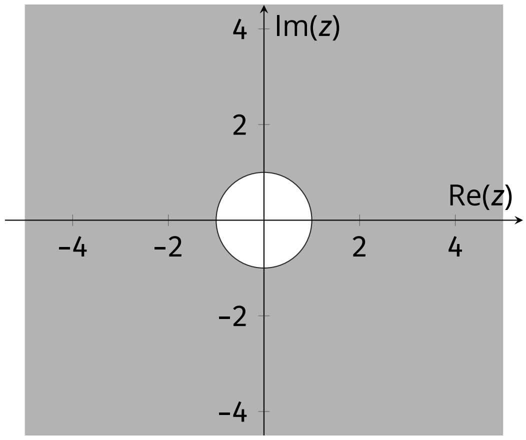
$$R = \left\{ z \in \mathbb{C} \setminus \{0\} \;\middle|\; \frac{1}{|z|} \leq |\bar{z}| \right\} \subset \mathbb{C}$$

Since $\bar{z} = re^{-i\theta}$, $|z| = |\bar{z}| = r$, and:

$$\frac{1}{r} \leq r \iff 1 \leq r^2 \iff 1 \leq r$$

Thus the region is given by:

$$R = \left\{ re^{i\theta} \in \mathbb{C} \setminus \{0\} \;\middle|\; 1 \leq r \right\}$$

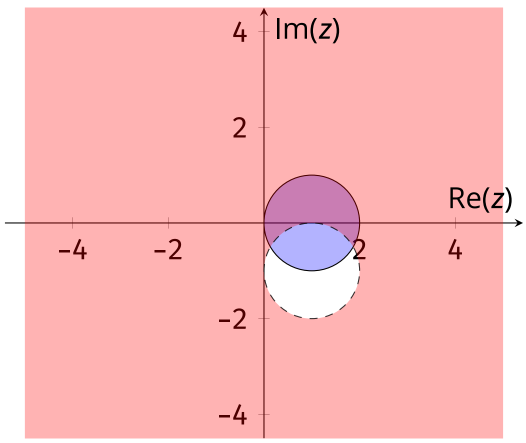


(c) Let $z = x + yi$, then:

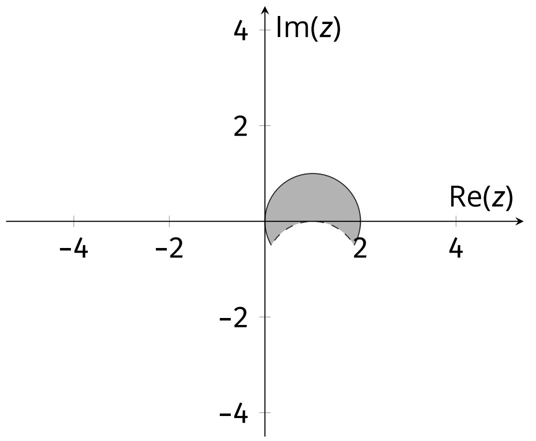
$$R_1 = \left\{ x + yi \in \mathbb{C} \;\middle|\; \sqrt{(x-1)^2 + y^2} \leq 1 \right\}$$
$$R_2 = \left\{ x + yi \in \mathbb{C} \;\middle|\; \sqrt{(x-1)^2 + (y+1)^2} > 1 \right\}$$

Thus we see that R_1, is a closed circle of radius 1 about $z = 1 + 0i$ and R_2 is all of $\mathbb{C}$ excluding a circle of radius 1 around $z = 1 - 1i$, the boundary of R_2 is open.

Plotting R_1 (blue) and R_2 (red):



Thus, plotting the intersection of the regions:



Note that the points where the two boundaries meet are not included in the region as they are closer that 1 to $1 - i$ and thus not part of R_2 and hence not present in $R_1 \cap R_2$.

Q2

(a) As:

$$\sinh x = -i\sin(ix)$$
$$\cosh x = \cos(ix)$$

We may choose $w \geq 0$, for which $e^{-w} \leq 1$, we have:

$$|\sin(iw)| = |i\sinh(w)| = \left|\frac{e^w - e^{-w}}{2}\right| > \frac{e^w}{2} - 1$$
$$|\cos(iw)| = |\cosh(w)| = \left|\frac{e^w + e^{-w}}{2}\right| > \frac{e^w}{2}$$

See that for $w \geq 0$:

$$\begin{aligned} &\ln(1998) \leq w \\ \implies &1998 \leq e^w \\ \implies &1000 \leq \frac{e^w}{2} - 1 \leq \frac{e^w}{2} \end{aligned}$$

Thus for the inequalities will hold for $w \geq 8 \geq \ln(1998)$. So we choose $y = 8$ and $z = i8$.

(b) Unlike on the numbers $|z|^2 \neq z^2$, even though the numbers have large absolute values (and therefore their squares have large absolute value), when we consider their modulus they may be pointing in near opposite directions.

The algebra confirms this:

$$\begin{aligned} &\sin^2(iw) + \cos^2(iw) \\ &= (i\sinh(w))^2 + (\cosh(w))^2 \\ &= \cosh^2(w) - \sinh^2(w) \\ &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 \\ &= \left(\frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2}\right) \\ &= (e^x)(e^{-x}) \\ &= 1 \end{aligned}$$

(c) See that:

$$e^{iz} = e^{-y+ix} = e^{-y}e^{ix}$$

Therefore:

$$|e^{iz}| = |e^{-y}e^{ix}| = |e^{-y}||e^{ix}| = e^{-y}$$

So we can write the inequality as:

$$y \geq c \implies e^{-y} \leq e^{-c}$$

If we let $a = -y \in \mathbb{R}$ and $b = -c \in \mathbb{R}$, then we can equivalently write:

$$a \leq b \implies e^a \leq e^b$$

Which is clearly true as e^x for $x \in \mathbb{R}$ is strictly increasing.

(d) For any H we can choose $y \geq 0$ and a point $z = 0 + iy \in H$, notice that since:

$$\sinh x = -i\sin(ix)$$
$$\cosh x = \cos(ix)$$

We must have (as in part a):

$$|\sin(iy)| = |i\sinh(y)| = \left|\frac{e^y - e^{-y}}{2}\right| > \frac{e^y}{2} - 1$$
$$|\cos(iy)| = |\cosh(y)| = \left|\frac{e^y + e^{-y}}{2}\right| > \frac{e^y}{2}$$

Therefore, for any bound M that we choose, if we consider $z = 0 + iy$ where $y = \ln(2M+2)$, then:

$$|\sin(iy)|, |\cos(iy)| > \frac{e^y}{2} = M + 1$$

Thus neither function is bounded for any H.

Q3

(a) Since $\ln(1) = \ln(r) + i(\theta_0 + 2\pi k)$ where $r = 1$ and $\theta_0 = 0$, we see that for this branch $k = 0$.

Now for $z = \frac{1}{2} + i\frac{\sqrt{3}}{2}$:

$$r = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = 1$$

$$\theta_0 = \arctan\left(\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}\right) = \frac{\pi}{3}$$

Travelling counter-clockwise from $\theta = 0$ to $\theta = \frac{\pi}{3}$ would cross the branch, so we must travel clockwise, hence the point is located at $\theta = -\frac{5\pi}{3}$

$$\ln\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = \ln(r) + i(\theta + 2\pi \cdot 0) = -\frac{5i\pi}{3}$$

(b) Again $\ln(1) = \ln(r) + i(\theta_0 + 2\pi k)$ where $r = 1$ and $\theta_0 = 0$, we see that for this branch $k = 0$.

Now for $z = \frac{1}{2} + i\frac{\sqrt{3}}{2}$:

$$r = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = 1$$

$$\theta_0 = \arctan\left(\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}\right) = \frac{\pi}{3}$$

Thus:

$$\ln\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = \ln(r) + i(\theta_0 + 2\pi k) = \frac{i\pi}{3}$$