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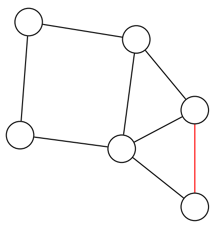
MATHS 326

S1 2024

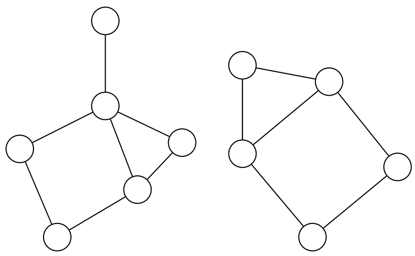
Assignment 1
Due: 20-03-2024

Q1

(a) Call the graph G, and apply deletion contraction theorem to the edge e in red:

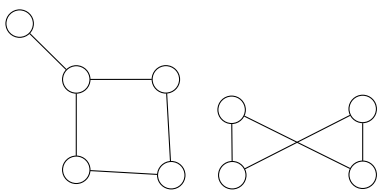


Thus, $P_G(x) = P_{G-e}(x) - P_{G_e}(x)$ This produces two graphs, $G - e$ and $_e$:



$G - e$ $\quad$ G_e

Notice that $G-e$ can be constructed by adding a vertex to $H = G_e$. Applying deletion contraction to H:



$H - e$ $\quad$ H_e

We see that $H_e = C_4$, and $H - e$ is C_4 with an additional vertex v added, which can take any color except that of its neighbor. Thus:

$$P_{H-e}(x) = (x-1)C_4(x)$$
$$P_{H_e}(x) = C_4(x)$$

Using this with the deletion contraction gives the chromatic polynomial of H:

$$\begin{aligned} P_H(x) &= P_{H-e}(x) - P_{H_e}(x) \\ &= (x-2)C_4(x) \\ &= (x-2)((x-1)^4 + x - 1) \end{aligned}$$

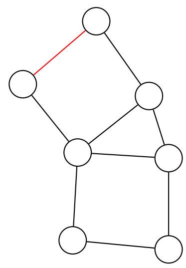
Since $G - e$, is just H with an additional vertex v with a single neighbor, we find:

$$P_{G-e}(x) = (x-1)H$$

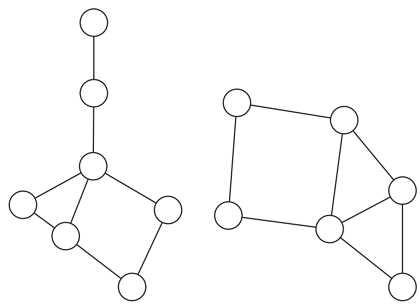
From deletion contraction we can use our results to determine :

$$\begin{aligned} P_G(x) &= P_{G-e}(x) - P_{G_e}(x) \\ &= (x-1)H - H \\ &= (x-2)H \\ &= (x-2)^2((x-1)^4 + x - 1) \end{aligned}$$

(b) We apply deletion contraction theorem to G on the edge in red:



This produces graphs:



$G - e$ $\quad$ G_e

Notice that G_e is the graph from part a, while $G-e$ can be constructed by adding two vertices to the graph H from part a. Notice that the added vertices have only one neighbor each meaning they can take on $(x - 1)$ colors for each existing coloring of H. Thus, the chromatic polynomials are:

$$\begin{aligned}P_{G-e} &= (x-1)^2 P_H(x)\\ &= (x-1)^2(x-2)((x-1)^4+x-1)\\ P_{G_e}(x) &= P_{G_a}(x)\\ &= (x-2)^2((x-1)^4+x-1)\end{aligned}$$

Finally we use these results in the deletion contraction theorem applied to G:

$$\begin{aligned}P_G(x) &= (x-1)^2(x-2)((x-1)^4+x-1)\\ &- (x-2)^2((x-1)^4+(x-1))\\ &= \big((x-1)^2 - x + 2\big)(x-2)\big((x-1)^4+x-1\big)\end{aligned}$$

Q2

(a) First we show that if G is connected, then $P_G(x)$ contains a non-zero x term:

Apply strong induction on $m = |E|$ for connected graphs $G = (V, E)$.

- Base case:
  For $|E| = 0$, the only connected graph is a single vertex with $P_G(x) = x$.
- Induction Step ("$m \implies m+1$"):
  Consider a graph with $|E| = m+1$, apply deletion contraction, thus:

  $$P_G(x) = P_{G-e}(x) - P_{G_e}(x)$$

  Note that both $G - e$ and G_e have m or fewer edges and therefore contain an x term in their chromatic polynomials by the induction hypothesis.

  Note that if G has n vertices, so does $G - e$ while G_e has $n-1$ vertices. Hence, $P_{G-e}(x)$ is of degree n and $P_{G_e}(x)$ degree $n-1$. Since the x^n term has positive coefficient and the signs alternate, the two polynomials have opposite signs on the x term. Applying this observation to the deletion contraction equation:

  $$\begin{aligned}P_G(x) &= P_{G-e}(x) - P_{G_e}(x)\\ &= (x^n + \cdots \pm c_a x)\\ &- (x^{n-1} + \cdots \mp c_b x) \quad \text{For } c_a, c_b \in \mathbb{R}^+\\ &= x^n + \cdots \pm (c_a + c_b)x\end{aligned}$$

  Note that $c_a + c_b > 0$ therefore $P_G(x)$ contains a non-zero x term.

(b) Now we show that if $P_G(x)$ contains an x term, then G is connected. We prove the contrapositive G is disconnected implies that $P_G(x)$ does not contain a nonzero x term.

Consider any disconnected graph G which can be expressed as the disjoint union of $k \geq 2$ subgraphs $H_1, \dots, H_k$.

By *Lemma 1.7*:

$$P_G(x) = \prod_{i=1}^{k} P_{H_i}(x)$$

Since $P_{H_i}(x)$ are chromatic polynomials, they do not contain a constant term. Thus, the product is of terms:

$$\begin{aligned}&P_G(x)\\ =\ &(x^{n_1} + \cdots + c_{2,1}x^2 + c_{1,1}x)\\ &\times\\ &\vdots\\ &\times\\ &(x^{n_k} + \cdots + c_{2,k}x^2 + c_{1,k}x)\\ =\ &\text{H.O.T.} + (c_{1,1} \dots c_{1,k})x^k\end{aligned}$$

Thus, there is no x term in $P_G(x)$.

(c) Consider the two cases:

- G is connected:
  As G is connected, *Corollary 1.15* still applies and so G is a tree if and only if $P_G(x) = x(x-1)^{n-1}$. Thus, the new statement holds in this case.
- G is disconnected and *Corollary 1.15* cannot be applied:
  LHS of the equivalence is false as G does not satisfy the definition of a tree.

  Since G is disconnected, it does not contain an non-zero x term in its chromatic polynomial. However:

  $$x(x-1)^{n-1} = \cdots \pm x$$

  Does contain a non-zero x term meaning it is not the chromatic polynomial of G. Hence the RHS of the equivalence is also false.

  Since $F \iff F$, the equivalence also holds in this case.

These cases cover all possibilities and so "connected" can be omitted from the Corollary.

Q3

(a) Expanding $(x-1)^4$ would result in a polynomial with constant term of $(-1)^4 = 1$, thus the polynomial is not the chromatic polynomial for any graph as it has a non-zero constant term.

(b) Notice that the x^3 term is positive but the x term is negative, thus the signs do not alternate otherwise the terms would have the same sign.

(c) Assume the polynomial is a chromatic polynomial $P_G(x)$. We can deduce the following:

- Since it has degree 4, it corresponds to a graph of 4 vertices.
- The $x^{n-1} = x^3$ term has a coefficient of -3, so G must have 3 edges.
- The x term has a non coefficient, thus G is connected.

The only connected graph with 3 edges and 4 vertices is P_4, which has chromatic polynomial:

$$P_{P_4}(x) = x(x-1)^3 = x^4 - 3x^3 + 3x^2 - x \neq P_G(x)$$

Thus $P_G(x)$ is not a chromatic polynomial of any graph.

Q4

Consider $n = 11$. Thus, $m \leq 27$ by theorem 1.19.

Consider a graph of $n \leq 11$ vertices and m edges. By *Theorem 1.19*, we know that $m \leq 3n - 6$. Assume the statement does not hold, then all vertices have degree at least 5. By the handshaking lemma:

$$5n = 5|V| \leq \sum_{v \in V} deg(v) = 2|E| = 2m$$

Thus, $2.5n \leq m \leq 3n - 6$. This is only possible for:

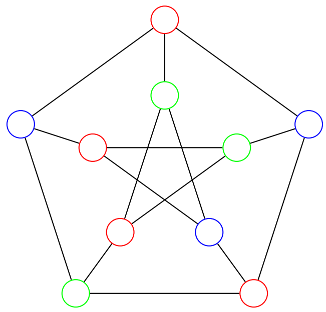
$$\begin{aligned} & 2.5n \leq 3n - 6 \\ \iff & 6 \leq 0.5n \\ \iff & 12 \leq n \end{aligned}$$

Thus, we have a contraction when $n < 12$, thus the assumption was false and the original statement was true.

To show that this bound is sharp, consider

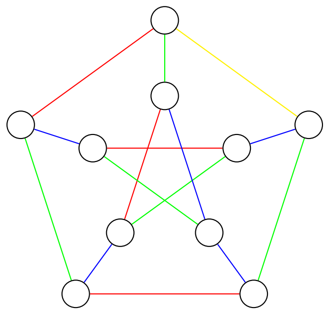
Q5

(a) We can create a 3-coloring of the Petersen graph:



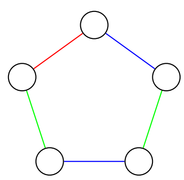
Therefore, $\chi(P) \leq 3$, however, since P contains an odd cycle of length 5, we also have $\chi(P) \geq 3$, hence $\chi(P) = 3$.

We can construct the following proper edge coloring of P:

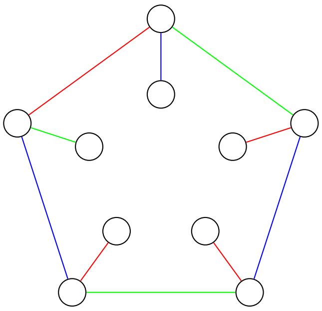


Thus $\chi_e(P) \leq 4$. Now consider a three coloring of P. We make the following observations:

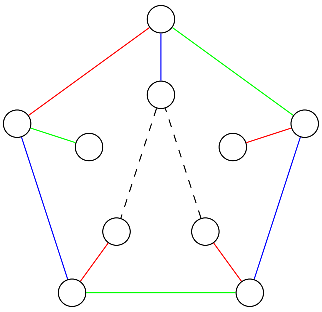
- The cycle C_5 has chromatic index 3.
- If we take a subgraph of three edges where no pair is adjacent, then there a 6 vertices in the subgraph by definition of edge adjacency.
- Since C_5 has only 5 vertices, we cannot find 3 adjacent edges such that no pair is adjacent. Thus, a coloring of C_5 can use each color at most twice. A 3 edge-coloring of C_5 must use each color at least once as no 2 edge-coloring exists.
- We must use two colors twice and one color once in order to color all 5 edges if each color is used either once or twice.
- WLOG, let r be the color used once. The remaining colors g, b must alternate on the remaining vertices.



- As only three colors are available, when edges are added to produce the following graph, their coloring is determined by the existing edges in the cycle.



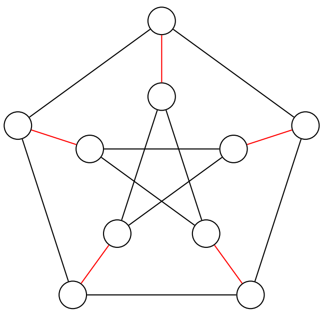
- Consider the subgraph of P created by adding the two dashed edges from the Petersen graph to the previous subgraph:



Both edges are adjacent to r and b edges already, so they must both be g. However, since they are adjacent they cannot receive the same colors resulting in a contraction. Thus P is not 3 edge-colorable and $\chi_e(P) > 3$.

Since $3 < \chi_e(P) \leq 4$, the chromatic index of the Petersen graph $\chi_e(P) = 4$.

(b) Contract the edges of P show in red below:



This produces K_5 so by Wagner's theorem, P must be non-planar.

Q6

Claim: A subgraph of a bipartite graph is bipartite.

Claim: If H is bipartite, then $\alpha(G) \geq \frac{1}{2} m_H$.