

Q1

(a) For any $x \in V \setminus \{a, z\}$, then we have:

$$p_x = \sum_{y \in N(x)} \frac{c(\{x, y\})p_y}{\pi(x)}$$

As the probability of reaching a before z , is the sum of probabilities from each neighbour, weighted by the chance of reaching that neighbour.

This also means that as a function $x \mapsto p_x$, on $V \setminus \{a, z\}$, p_x is harmonic. We also know that $p_a = 1$ and $p_z = 0$. Meaning that this is an instance of the discrete Dirichlet problem, hence $x \mapsto p_x$ must be the unique solution.

Now consider any $f : V \rightarrow \mathbb{R}$ harmonic on $V \setminus \{a, z\}$. Consider a function $g : V \rightarrow \mathbb{R}$ given by $g(x) = f(a)p_x + f(z)$. We know $g(x)$ is harmonic as it is a linear combination of harmonic functions (on $V \setminus \{a, z\}$).

As $p_a = 1$ and $p_z = 0$, $\alpha = g(a) = f(a)$ and $\beta = g(z) = f(z)$, we have that f, g are solutions to the same discrete Dirichlet problem. Thus, $f = g$ as the solution is unique. Hence, we can write:

$$f(x) = f(a)p_x + f(z) = \alpha p_x + \beta$$

(b) We can rewrite:

$$f(x) = f(a)p_x + f(z) = \alpha p(x) + \beta q(x)$$

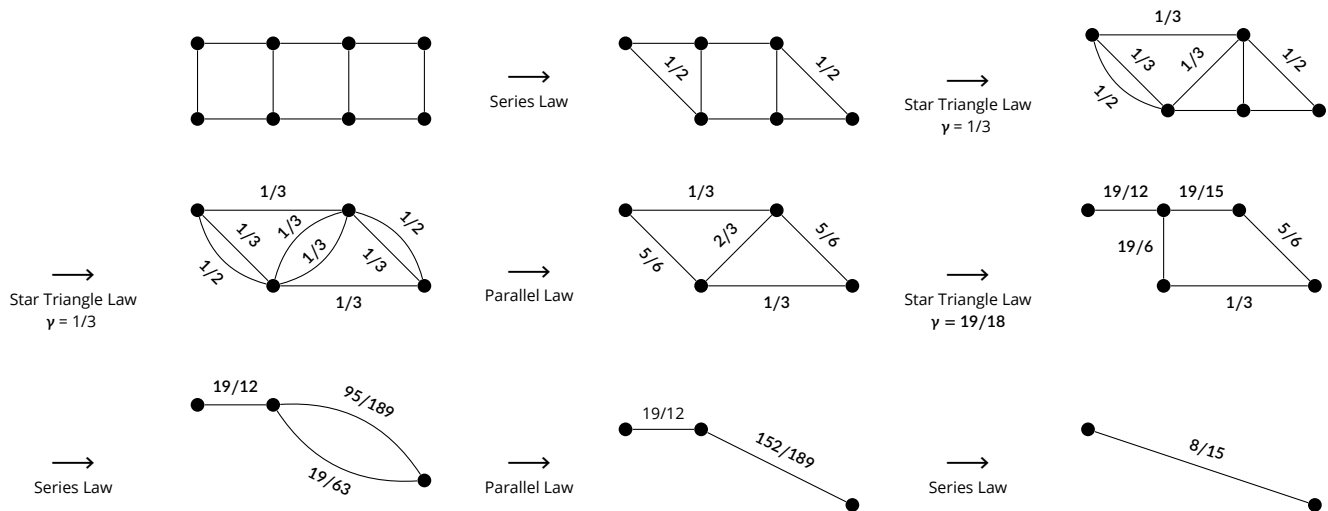
Where $p(x) = p_x$ and $q(x) = 1$ are both harmonic functions on $V \setminus \{a, z\}$. By (a), $\{p, q\}$ spans the vector space of function harmonic on $V \setminus \{a, z\}$. We also see that:

$$\alpha p + \beta q = 0 \implies \begin{matrix} \alpha p(a) + \beta q(a) = \alpha + \beta = 0 \\ \alpha p(z) + \beta q(z) = \beta = 0 \end{matrix} \implies \alpha = \beta = 0$$

Since the set $\{p, q\}$ is linearly independent and spanning it is a basis with cardinality 2. Hence, the dimension of the vector space is also 2.

Q2

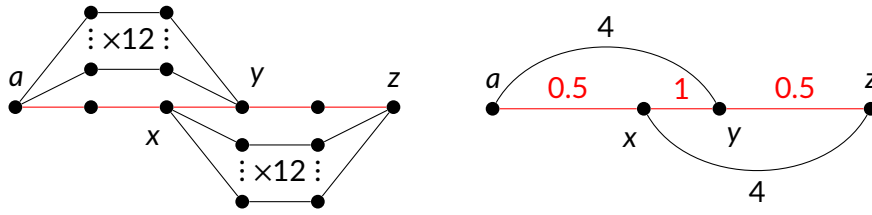
Applying the series, parallel, and star triangle laws to calculate $\mathcal{C}(a \leftrightarrow z)$ for the given graph G , non-unit conductances are labelled:



Thus, $\mathcal{C}(a \leftrightarrow z) = 8/15$ and $\mathcal{R}(a \leftrightarrow z) = 15/8 = 1.875$.

Q3

Counter Example: Consider the following graph and shortest path shown in red. Let v be a voltage function with $v(a) = 0$ and $v(z) = 1$. This graph can be reduced to a weighted graph using the Series/Parallel Laws:



Where the weights correspond to conductance, this preserves the effective conductance between the any pair of a, x, y, z are preserved, thus, $v(x), v(y)$ are also preserved.

From Ohm's Law:

$$\begin{aligned}
 i(a, x) &= c(\{a, x\}) [v(a) - v(x)] = -\frac{1}{2}v(x) \\
 i(x, z) &= c(\{x, z\}) [v(x) - v(z)] = 4v(x) - 4 \\
 i(a, y) &= c(\{a, y\}) [v(a) - v(y)] = -4v(y) \\
 i(y, z) &= c(\{y, z\}) [v(y) - v(z)] = \frac{1}{2}v(y) - \frac{1}{2} \\
 i(x, y) &= c(\{x, y\}) [v(x) - v(y)] = v(x) - v(y)
 \end{aligned}$$

Applying Kirchhoff's Node Law at x, y :

$$\begin{aligned}
 0 &= \sum_{z \in N(x)} i(z, x) = i(a, x) + i(y, x) + i(z, x) \\
 &= -\frac{1}{2}v(x) - [v(x) - v(y)] - [4v(x) - 4] \\
 v(y) &= \frac{1}{2}v(x) + v(x) + 4v(x) - 4 \\
 &= \frac{11}{2}v(x) - 4 \\
 0 &= \sum_{z \in N(y)} i(z, y) = i(a, y) + i(x, y) + i(z, y) \\
 &= [-4v(y)] + [v(x) - v(y)] - \left[\frac{1}{2}v(y) - \frac{1}{2} \right] \\
 v(x) &= 4.5v(y) - \frac{1}{2}
 \end{aligned}$$

Solving this system of two linear equations gives:

$$\frac{74}{95} = v(x) > v(y) = \frac{27}{95}$$

Q4

Let $H = (V_H, E_H)$ be a connected subgraph of $G = (V_G, E_G)$. Let T_H, T_G be random spanning trees of H, G respectively. For $e \in E_H$:

$$\begin{aligned}\mathbb{P}[e \in T_H] &= \mathcal{R}(x \leftrightarrow y) && \text{By Theorem 3.20} \\ &= \epsilon(i)\end{aligned}$$

For some unit strength current flow i from x to y on H . Extend i to i_G on G by defining:

$$i_G(x, y) = \begin{cases} i(x, y) & \text{When } \{x, y\} \in E_H \\ 0 & \text{Otherwise} \end{cases}$$

Clearly i_G is a flow (not necessarily a current flow) as it still satisfies Kirchhoff's node law, moreover it also has unit strength. So by Thompson's principle, a unit current flow i' from x to y has:

$$\epsilon(i) = \sum_{e=\{x,y\} \in E_H} i(x, y)^2 r(e) = \sum_{e=\{x,y\} \in E_G} i_G(x, y)^2 r(e) = \epsilon(i_G) \geq \epsilon(i') = \mathcal{R}_G(x \leftrightarrow y)$$

Thus, we have shown:

$$\mathbb{P}[e \in T_H] = \mathcal{R}(x \leftrightarrow y) = \epsilon(i) = \epsilon(i_G) \geq \epsilon(i') = \mathcal{R}_G(x \leftrightarrow y) = \mathbb{P}[e \in T_G]$$

Q5

- (a) Let $G = (V, E)$ be a connected graph, and i' a current flow from a to z of unit strength on $G - e = (V, E')$. Extend i' to a flow θ on G with $\theta(x, y) = \theta(y, x) = 0$ for $\{x, y\} = e$ and $\theta(u, v) = i'(u, v)$ for $\{u, v\} \in E'$. So:

$$\epsilon(\theta) = \frac{1}{2} \sum_{\substack{u \in V \\ v \in N_G(x)}} \theta(u, v)^2 r(\{u, v\}) = \frac{1}{2} \sum_{\substack{u \in V \\ v \in N_{G-e}(x)}} i'(u, v)^2 r(\{u, v\}) + \theta(x, y)^2 r(\{x, y\}) = \frac{1}{2} \sum_{\substack{u \in V \\ v \in N_{G-e}(x)}} i'(u, v)^2 r(\{u, v\}) = \epsilon(i')$$

So for a unit strength current flow i from a to z on G , $\epsilon(i) \leq \epsilon(i')$ by Thompson's Principle. Thus:

$$\mathcal{R}_G(a \leftrightarrow b) = \epsilon(i) \leq \epsilon(i') = \mathcal{R}_{G-e}(a \leftrightarrow b)$$

- (b) Consider any current flow i from a to z on $G = (V, E)$. We claim that $i(x, y) \neq 0$ implies that we can construct a path from a to z that contains both of x, y . Proof:

Let P be a path with vertices $x_1 = x, x_2 = y$. Consider the algorithm where we repeatedly apply the following step until both endpoints of P are in $\{a, z\}$:

Let x_1, \dots, x_{k+1} be the vertices of P . If $x_{k+1} \notin \{a, z\}$, by Kirchhoff's node law, there is some $w \in N(x_{k+1})$ with $i(x_{k+1}, w) > 0$, attempt to append w to P . Otherwise, if $x_{k+1} \in \{a, z\}$ and $x_1 \notin \{a, z\}$, by Kirchhoff's node law there is some $w' \in N(x_1)$ such that $i(w', x_1) > 0$. Attempt to extend P by prepending w' .

On a finite G , the algorithm always succeeds or fails. To fail, adding w or w' to P did not produce a path. Meaning P already contained w or w' . Hence, we have a cycle consisting entirely of edges with $i(x_i, x_{i+1}) > 0$ and Kirchhoff's cycle law is not satisfied. Since i is a current flow, this is a contradiction and the algorithm must always construct a path between a and z containing the original edge. (a to a or z to z aren't paths).

(c) For any unweighed $G = (V, E)$, the conductance function $c : E \rightarrow \mathbb{R}$ is given by $c(e) = 1$. So by Theorem 3.18 of the course book:

$$\text{Comm}(a \leftrightarrow z) = 2 \left(\sum_{e \in E} c(e) \right) \mathcal{R}(a \leftrightarrow z) = 2 |E| \mathcal{R}(a \leftrightarrow z)$$

Now consider some fixed shortest path P between a and z with length $d(a, z)$. Apply (a) to remove every edge of G not in P , by (a) this does not decrease the effective resistance between a and z . IE, if G' is the graph obtained, then:

$$\mathcal{R}_G(a \leftrightarrow z) \leq \mathcal{R}_{G'}(a \leftrightarrow z)$$

Since each non-endpoint vertex of P is of degree 2 in G' , applying the series law $\mathcal{R}_{G'}(a \leftrightarrow z)$ is the sum of the resistances in G . Thus, $\mathcal{R}_{G'}(a \leftrightarrow z) = d(a, z)$ as the edges are of unit resistance. Thus:

$$\text{Comm}(a \leftrightarrow z) = 2 \left(\sum_{e \in E} c(e) \right) \mathcal{R}(a \leftrightarrow z) = 2 |E| \mathcal{R}(a \leftrightarrow z) \leq 2 |E| d(a, z)$$

Q6

Consider a unit current flow i'' on the resistances $\frac{r+r'}{2}$ from a to z . Then:

$$\begin{aligned} \mathcal{R}_{\frac{r+r'}{2}}(a \leftrightarrow z) &= \varepsilon_{\frac{r+r'}{2}}(i'') && \text{By Lemma 3.29} \\ &= \frac{1}{2} \sum_{\substack{x \in V \\ y \in N(x)}} i''(x, y)^2 \frac{r(\{x, y\}) + r'(\{x, y\})}{2} \\ &= \frac{1}{4} \sum_{\substack{x \in V \\ y \in N(x)}} i''(x, y)^2 r(\{x, y\}) + \frac{1}{4} \sum_{\substack{x \in V \\ y \in N(x)}} i''(x, y)^2 r'(\{x, y\}) \\ &= \frac{1}{2} [\varepsilon_r(i'') + \varepsilon_{r'}(i'')] \\ &\geq \frac{1}{2} [\varepsilon_r(i) + \varepsilon_{r'}(i')] && \text{By Thompson's Principle} \\ &= \frac{1}{2} [\mathcal{R}_r(a \leftrightarrow z) + \mathcal{R}_{r'}(a \leftrightarrow z)] && \text{By Lemma 3.29} \end{aligned}$$

Where i and i' are unit strength current flows on r and r' respectively and Thompson's Principle is applicable as i'' also is a unit strength flow (but not necessarily a current flow) as the strength of a flow is independent of resistance.