TAYLOR POLYNOMIALS

Definition. For a sufficiently smooth function $f : \mathbb{R} \to \mathbb{R}$, the K-th degree Taylor Polynomial of $f(x)$ centred about a is given by:

$$P_k(x) = \sum_{j=0}^{k} \frac{f^{(j)}(a)}{j!}(x-a)$$

Example. Calculate the linear approximation to $f(x) = \sin(x)$ centred at $a = 0$.

$$P_0(x) = f(0) = 0$$
$$P_1(x) = f(0) + f'(0)(x-0) = x$$

Definition. Let S be a surface given by $f(x,y,z) = 0$ where $f : \mathbb{R}^3 \to \mathbb{R}$ is continuously differentiable. Then the degree one Taylor Polynomial at (a,b,c) is:

$$f(a,b,c) + \begin{bmatrix} \frac{\partial f}{\partial x}(a,b,c) & \frac{\partial f}{\partial y}(a,b,c) & \frac{\partial f}{\partial z}(a,b,c) \end{bmatrix} \cdot \begin{bmatrix} x-a \\ y-b \\ z-c \end{bmatrix}$$

When (a,b,c) is on the surface S, then the tangent plane at (a,b,c) is:

$$\frac{\partial f}{\partial x}(a,b,c)(x-a) + \frac{\partial f}{\partial y}(a,b,c)(y-b) + \frac{\partial f}{\partial z}(a,b,c)(z-c)$$

Where $Df(\mathbf{a})$ is the **total derivative** (or jacobian) of f at $\mathbf{a}$.

$$Df(a) = \left[\frac{\partial f}{\partial x_1}(\mathbf{a}), \dots, \frac{\partial f}{\partial x_m}(\mathbf{a})\right]$$

And $Hf(\mathbf{a})$ is the **Hessian** of f at $\mathbf{a}$, a matrix with:

$$[Hf(a)]_{i,j} = \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{a})$$

Definition. For $f : \mathbb{R}^n \to \mathbb{R}$, the degree $1,2$ Taylor polynomials centred at a are:

$$P_1(\mathbf{r}) = f(\mathbf{a})Df(\mathbf{a})(\mathbf{r}-\mathbf{a})$$
$$P_2(\mathbf{r}) = f(\mathbf{a})Df(\mathbf{a})(\mathbf{r}-\mathbf{a}) + \frac{1}{2}(\mathbf{r}-\mathbf{a})^T Hf(\mathbf{a})(\mathbf{r}-\mathbf{a})$$

VECTOR DERIVATIVES

Definition. The derivative of $r(t)$ where $r : \mathbb{R} \to \mathbb{R}^n$ with respect to t is done component wise:

$$r'(t) = \begin{bmatrix} r_1'(t) \\ \vdots \\ r_n'(t) \end{bmatrix}$$

Example. For functions $u, v, w : \mathbb{R} \to \mathbb{R}^n$ and $f : \mathbb{R} \to \mathbb{R}$ with $\alpha, \beta \in \mathbb{R}$:

$$\begin{array}{rl} \alpha u + \beta v & \alpha u' + \beta v' \\ u \cdot v & u' \cdot v + u \cdot v' \\ u \times v & u' \times v + u \times v' \\ u \cdot (v \times w) & u' \cdot v \times w + u \cdot v' \times w + u \cdot v \times w' \\ u(f(t)) & f'(t)\dfrac{du}{df} = f'(t)u'(f(t)) \end{array}$$

For $f : \mathbb{R}^n \to \mathbb{R}$, the total derivative matrix is a row vector and the gradient of f is:

$$\text{grad } f = \nabla f = (Df)^T$$

CURVILINEAR COORDINATES

A coordinate parametrisation for a new system $\mathbf{u} = [u_1, \dots, u_n]$ from the standard coordinates on $\mathbb{R}^n$:

$$\xi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$
$$\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \mapsto \begin{bmatrix} \xi_1(\mathbf{u}) \\ \vdots \\ \xi_n(\mathbf{u}) \end{bmatrix}$$

We can find a basis for $\mathbf{u}$ from the columns of $D\xi$,

$$\frac{\partial \xi}{\partial u_1}, \dots, \frac{\partial \xi}{\partial u_n} \qquad e_{u_j} = \frac{\partial \xi}{\partial u_j}\left\| \frac{\partial \xi}{\partial u_j} \right\|^{-1}$$

Note that when the basis are not linearly independent, our coordinates are not well-behaved. (Cylindrical/polar when $r = 0$, Spherical when $x, y = 0$.) These points are **coordinate singularities**.

Polar

$$\xi : \begin{bmatrix} r \\ \theta \end{bmatrix} \mapsto \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix} \qquad \frac{de_r}{d\theta} = e_\theta, \quad \frac{de_\theta}{d\theta} = -e_r$$

Cylindrical

$$\xi : \begin{bmatrix} r \\ \theta \\ z \end{bmatrix} \mapsto \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ z \end{bmatrix} \qquad \begin{array}{lll} \frac{\partial e_r}{\partial r} = 0 & \frac{\partial e_r}{\partial \theta} = e_\theta & \frac{\partial e_r}{\partial z} = 0 \\ \frac{\partial e_\theta}{\partial r} = 0 & \frac{\partial e_\theta}{\partial \theta} = -e_r & \frac{\partial e_\theta}{\partial z} = 0 \\ \frac{\partial e_z}{\partial r} = 0 & \frac{\partial e_z}{\partial \theta} = 0 & \frac{\partial e_z}{\partial z} = 0 \end{array}$$

Spherical

$$\xi : \begin{bmatrix} r \\ \theta \\ \phi \end{bmatrix} \mapsto \begin{bmatrix} r\cos\theta\sin\phi \\ r\sin\theta\sin\phi \\ r\cos\phi \end{bmatrix}$$

$$\begin{array}{lll} \frac{\partial e_r}{\partial r} = 0 & \frac{\partial e_r}{\partial \theta} = \sin\phi e_\theta & \frac{\partial e_r}{\partial z} = e_\phi \\ \frac{\partial e_\theta}{\partial r} = 0 & \frac{\partial e_\theta}{\partial \theta} = -\sin\phi e_r - \cos\phi e_\phi & \frac{\partial e_\theta}{\partial z} = 0 \\ \frac{\partial e_z}{\partial r} = 0 & \frac{\partial e_z}{\partial \theta} = \cos\phi e_\theta & \frac{\partial e_z}{\partial z} = -e_r \end{array}$$

CURVES

SURFACES

VOLUMES