$\mathbb{R}^{\omega_1}$ is a Non-Metrisable Product Space

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MATHS 350

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Research Topic

Research Topic (2.4)

~~Let ω_1 have the discrete topology.~~ Prove that $\mathbb{R}^{\omega_1}$ is not metrisable.

By $\mathbb{R}^{\omega_1}$, we mean the product space $(X, \mathcal{T})$ where $X = \prod_{\alpha \in \omega_1} \mathbb{R}$ and:

$$\mathcal{T} \quad \text{is generated by basis} \quad \mathcal{B} = \left\{ \prod_{\alpha \in \omega_1} U_\alpha \ : \ \begin{array}{c} \text{Each } U_\alpha \text{ open in } \mathbb{R} \\ \text{Only finitely many } U_\alpha \neq \mathbb{R} \end{array} \right\}$$

Where ω_1 is the first uncountable ordinal and $\mathbb{R}$ has the usual topology. By the definition of the Cartesian product, X is the set of all functions $\omega_1 \to \mathbb{R}$.

Definition (2.1)

Suppose $(X, \mathcal{T})$ is a topological space. If there exists a metric d on X such that the metric topology on X induced by d is $\mathcal{T}$, then X is said to be metrisable.

Research Topic

Hello and welcome to my talk. The research topic that I choose was to: "*Let ω_1 have the discrete topology. Prove that $\mathbb{R}^{\omega_1}$ is not metrisable.*"

Where ω_1 refers to the first uncountable ordinal, and so by $\mathbb{R}^{\omega_1}$, we mean the product of uncountably many copies of $\mathbb{R}$ with the product topology. Where each $\mathbb{R}$ has the usual topology.

Since we have an infinite product, the product topology is actually not the same as the box topology. And can be generated by the arbitrary unions of a basis set consisting of products of opens sets in $\mathbb{R}$, where all but a finite number of components are the whole space $\mathbb{R}$.

Metrisable simply means that there exists a metric for $\mathbb{R}^{\omega_1}$, where the metric topology induces the same topology as the product topology.

You'll notice this definition doesn't actually use the topology of ω_1 anywhere, so the first part of the question is a red herring.

First Countable

Any metrisable space $(X, \mathcal{T})$ is **first countable**: For any $x \in X$, there is a countable neighbourhood basis at x.

Example (4.2)

(1) Any metrisable space is first countable.

(2) $\mathbb{R}$ is separable, first countable and second countable. $\mathbb{Q}$ is a countable dense subset.

First Countable

One way that we can show a space is not metrisable, is to show that it does not satisfy some property that a metric space would.

One such property is *first countability*. An example given in the course, is that all metric spaces are first countable, meaning any point has a countable neighbourhood basis.

Since ω_1 is uncountable, this seemed like a good property to check.

Metric Spaces are First Countable

Let (X, d) be a metric space. Consider any $x \in X$, define:

$$\mathcal{M} = \left\{B_{1/n}(x)\right\}_{n \in \mathbb{N}^+}$$

Since $\mathbb{N}$ is countable, and $x \in B_\epsilon(x)$ which is open for $\epsilon > 0$, we have a countable family of open neighbourhoods for x.

Now consider any neighbourhood N of x, by definition there is an open U where $x \in U \subseteq N$.

Since $x \in U$ which is open, there is a $B_\epsilon(x) \subset U$, let:

$$n = \left\lceil \frac{1}{\epsilon} \right\rceil \in \mathbb{N}$$

Then $\frac{1}{n} \leq \epsilon$ so:

$$B_{1/n}(x) \subseteq B_\epsilon(x) \subseteq U \subseteq N$$

Hence $\mathcal{M}$ is a countable neighbourhood basis at x.

Metric Spaces are First Countable

First we give a proof for this example.

Start by letting (X, d) be a metric space. Consider any point $x \in X$.

We define family $\mathcal{M}$ indexed by the natural numbers, consisting of open balls around x of radius $1/n$.

Thus, each element of $\mathcal{M}$ is an open set containing x, and thus an open neighbourhood of x.

Now take any neighbourhood N of x. By definition of an open set in a metric space, there is an ϵ-ball about x contained in U.

So rounding $1/\epsilon$ up to the next natural number $n \in \mathbb{N}$, then $1/n \leq \epsilon$ so $B_{1/n}(x)$ is an element of $\mathcal{M}$ and is contained in $B_\epsilon(x)$ which is contained in U contained in N.

Thus, $\mathcal{M}$ is a neighbourhood basis at x as there is a basis element contained in any neighbourhood of x.

Proof Outline

1. If we can show that $\mathbb{R}^{\omega_1}$ is not first countable, then $\mathbb{R}^{\omega_1}$ cannot be metrisable.

Proof Outline

Our proof structure will go like this:

First, we want to show $\mathbb{R}^{\omega_1}$ is not first countable, and thus not metrisable.

Then to show our space isn't first countable, we can show that any countable family of neighbourhoods is not a neighbourhood basis. Therefore, no point can have a neighbourhood basis.

To show this, we first show that for any countable family of non-empty open sets, there is component β where the β-projection of every component is $\mathbb{R}$.

Lastly, we then show how this prevents a countable family from forming a neighbourhood basis.

Proof Outline

1. If we can show that $\mathbb{R}^{\omega_1}$ is not first countable, then $\mathbb{R}^{\omega_1}$ cannot be metrisable.
2. To show the space is not first countable, we want to show any countable family of neighbourhoods is not a neighbourhood basis for some point of the space.

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3. We will show that for any countable family of open sets, there is a projection π_β where every member's projection is $\mathbb{R}$.

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Countable Families: Setup

- Let $(X, \mathcal{T})$ be $\mathbb{R}^{\omega_1}$ with the product topology. Let I be a countable indexing set, consider any countable family $\{U_\alpha\}_{\alpha \in I} \subset \mathcal{T}$ of **open non-empty** sets.

Countable Families: Setup

To get started, let (X, T) be $\mathbb{R}^{\omega_1}$ with the product topology. And I be a countable indexing set for a family of non-empty open sets.

Since each member of the family is non-empty and open, the member is a non-empty union of basis elements.

Considering some basis element b' in the union, and an arbitrary component β in ω_1. Clearly $b' \subseteq U_\alpha$. Since the image under a function preserves inclusion, the β-projection of b' is contained in the projection of U_α which is contained in $\mathbb{R}$.

There are only a finite number of β where the projection of b' is not $\mathbb{R}$. So for the rest, by double inclusion, the project of our U_α is $\mathbb{R}$. Therefore, there are only finitely many projections of U_α that are not $\mathbb{R}$.

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- As each U_α is open and non-empty, $U_\alpha = \bigcup_{b \in B} b$ where $\varnothing \neq B \subset \mathcal{B}$.
- Take some $b' \in B \neq \varnothing$. Consider any arbitrary $\beta \in \omega_1$, then $b' \subseteq U_\alpha$ so:

$$\pi_\beta(b') \subseteq \pi_\beta(U_\alpha) \subseteq \mathbb{R}$$

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$$\pi_\beta(b') \subseteq \pi_\beta(U_\alpha) \subseteq \mathbb{R}$$

- Since $\pi_\beta(b') \neq \mathbb{R}$ for only finitely many β, the remaining β have $\pi_\beta(b') = \mathbb{R}$ so $\mathbb{R} \subseteq \pi_\beta(U_\alpha) \subseteq \mathbb{R}$ hence $\pi_\beta(U_\alpha) = \mathbb{R}$. So $\pi_\beta(U_\alpha) \neq \mathbb{R}$ for at most finitely many β.

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Contradiction Part 1

First notice that by definition, since I is countable, either there is a bijection $\lambda : I \to \mathbb{N}$, or I is finite in which case we can define an injective $\lambda : I \to \mathbb{N}$.

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Now, by definition, since I is countable, either it is finite, and we can choose an injective function $\lambda : I \to \mathbb{N}$. Or I is infinite and so there is a bijection $\lambda : I \to \mathbb{N}$.

Now, for the sake of contradiction assume that for any $\beta \in \omega_1$, there is an $\alpha \in I$ where $\pi_\beta(U_\alpha) \neq \mathbb{R}$. Then there is a non-empty A_β consisting of the image under λ of all such indices α.

Now for any β, we can use the well-ordering of $\mathbb{N}$ to take the least element A_β, since λ is injective and A_β is in the image of λ, there is a unique element of I which λ maps to the least element of A_β.
Using this we define a function $a : \omega_1$ to I. This allows us to consistently choose the α in the assumption as a function of β.

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Assume for any $\beta \in \omega_1$, there is some $\alpha \in I$ where $\pi_\beta(U_\alpha) \neq \mathbb{R}$. There is a non-empty subset $A_\beta \subseteq \mathbb{N}$ given by:

$$A_\beta = \left\{\lambda(\alpha) : \alpha \in I, \pi_\beta(U_\alpha) \neq \mathbb{R}\right\}$$

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Since $\mathbb{N}$ Is well-ordered and λ injective, we let the function $a : \omega_1 \to I$ be defined by letting $a(\beta) \in I$ be the unique value where $\lambda(a(\beta)) = (\min A_\beta)$.

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Contradiction Part 2

Now for any $\beta \in \omega_1$, let $\alpha = a(\beta)$, so $\pi_\beta(U_\alpha) \neq \mathbb{R}$. There is a finite non-empty set $\Gamma_\alpha \subset \omega_1$ where:

$$\Gamma_\alpha = \left\{\gamma \in \omega_1 \, : \, \pi_\gamma(U_\alpha) \neq \mathbb{R}\right\}$$

We must have $\beta \in \Gamma_\alpha$.

Contradiction Part 2

Now considering an arbitrary β, we use this function to choose an α. We know that $\pi_\beta(U_\alpha) \neq \mathbb{R}$ from the definition of A_β on the previous slide. We now define a new set Γ_α consists of all the indices in ω_1 where the projection of $U_\alpha \neq \mathbb{R}$. This set is clearly non-empty as it must contain β, and we have shown previously that it must be finite as there are only finitely many such projections.

Now since ω_1 is ordered, we can order the finite set Γ_α, and so β is the i-th element of Γ_α for some finite natural number i. Since there is exactly one choice of i dependent completely on β, we define a function $f : \omega_1 \to I \times \mathbb{N}$ by:

$$f(\beta) = (\alpha, i) = (a(\beta), i)$$

We now show that the function we have defined is injective. Consider any two $\beta, \beta' \in \omega_1$, where $f(\beta) = f'(\beta)$, then both β, β' are mapped to the same α, i. Thus, both β, β' are the i-th component of Γ_α. Hence, they are equal and f is injective.

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$$f(\beta) = (\alpha, i) = (a(\beta), i)$$

To show injectivity, assume $f(\beta) = f(\beta') = (\alpha, i)$, then both β, β' are the i-th element of Γ_α so $\beta = \beta'$.

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Now considering an arbitrary β, we use this function to choose an α. We know that $\pi_\beta(U_\alpha) \neq \mathbb{R}$ from the definition of A_β on the previous slide. We now define a new set Γ_α consists of all the indices in ω_1 where the projection of $U_\alpha \neq \mathbb{R}$. This set is clearly non-empty as it must contain β, and we have shown previously that it must be finite as there are only finitely many such projections.

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Contradiction Part 3

Since $f : \omega_1 \to I \times \mathbb{N}$ is injective, we have that $|\omega_1| \leq |I \times \mathbb{N}|$. However, the Cartesian product of two countable sets is countable, so we have a contraction.

Since our assumption that *for any* $\beta \in \omega_1$*, there is a* U_α *with* $\pi_\beta(U_\alpha) \neq \mathbb{R}$ was false, there is a $\beta \in \omega_1$ where all $\alpha \in I$ have $\pi_\beta(U_\alpha) = \mathbb{R}$.

Contradiction Part 3

Now since we have show that $f : \omega_1 \to I \times \mathbb{N}$ is injective, we have show that by definition $|\omega_1| \leq |I \times \mathbb{N}|$. However, the Cartesian product of two countable sets is countable, this gives us a contradiction since ω_1 is by definition uncountable and cannot be smaller than a countable cardinality.

Therefore, the assumption that any β has a U_α where the β-projection is not $\mathbb{R}$, must be false. So there is a β where all of our U_α have β-projection of $\mathbb{R}$.

Consequences for Neighbourhoods

Now assume that $\mathbb{R}^{\omega_1}$ is first countable, so for $x = 0 \in \mathbb{R}^{\omega_1}$ we have a countable neighbourhood basis $\mathcal{M} = \{M_\alpha\}_{\alpha \in I}$ where I is a countable indexing set. Now define a family of open sets $\{U_\alpha\}_{\alpha \in I}$ where:

$$U_\alpha := \bigcup_{\substack{U \in \mathcal{T} \\ U \subseteq M_\alpha}} U$$

Note each $U_\alpha \neq \varnothing$ as M_α is a neighbourhood and $U_\alpha \subseteq M_\alpha$.

Consequences for Neighbourhoods

We now use this result to show that $\mathbb{R}^{\omega_1}$ is not first countable using another proof-by-contraction.

First, assume that the space is first countable, we choose to look at the point 0 in $\mathbb{R}^{\omega_1}$, where we expect to have a countable basis of neighbourhoods $\mathcal{M}$, indexed by some countable set I.

We can define a family of open U_α's, with the same indexing set: for each neighbourhood M_α, we take the union over all open sets contained in M_α. By definition of a neighbourhood, this is non-empty, so the arbitrary union gives us a non-empty open set for each neighbourhood.

The previous result tells us that there is a $\beta \in \omega_1$ where the β-projection of every member U_α of the family is $\mathbb{R}$.

We can now construct a neighbourhood N where every component is $\mathbb{R}$ except the β-component which is the open interval between -1 and 1. This is an open set since it is an element of the basis we chose for the topology on $\mathbb{R}^{\omega_1}$, it also contains 0, so it is an open neighbourhood of 0.

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Note each $U_\alpha \neq \varnothing$ as M_α is a neighbourhood and $U_\alpha \subseteq M_\alpha$. From the previous result, there must be a $\beta \in \omega_1$ where $\pi_\beta(U_\alpha) = \mathbb{R}$ for any $\alpha \in I$. Consider the open neighbourhood of 0 given by:

$$N := \prod_{\gamma\in\omega_1} N_\gamma \qquad N_\gamma := \begin{cases} (-1,1) & \gamma = \beta \\ \mathbb{R} & \gamma \neq \beta \end{cases}$$

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Consequences: Continued

$$N := \prod_{\gamma \in \omega_1} N_\gamma \qquad N_\gamma := \begin{cases} (-1,1) & \gamma = \beta \\ \mathbb{R} & \gamma \neq \beta \end{cases}$$

If $\mathcal{M}$ were a neighbourhood basis then, there would be a $M_\alpha \subseteq N$, since images preserve inclusion, we expect:

$$\pi_\beta(U_\alpha) \subseteq \pi_\beta(M_\alpha) \subseteq \pi_\beta(N)$$

But for any $\alpha \in I$ we have $\pi_\beta(U_\alpha) = \mathbb{R}$, while $\pi_\beta(N) = (-1,1)$ so $U_\alpha \not\subseteq N$.

This is a contradiction, and so $\mathbb{R}^{\omega_1}$ is not first countable. Therefore, $\mathbb{R}^{\omega_1}$ is not metrisable.

Consequences: Continued

Now if the family $\mathcal{M}$ was a neighbourhood basis, we expect that there is an $M_\alpha \subseteq N$. Since function images preserve inclusion, we expect that the β-projection will also satisfy this inclusion.

However, we know that the β-projection of M_α is $\mathbb{R}$ while the β-projection of N is the interval $(-1,1)$. Clearly $\mathbb{R}$ is not a subset or equal to this interval.

Since this is a contraction, so our assumption that $\mathbb{R}^{\omega_1}$ was first countable must be false.

Thank You

Since we have shown that $\mathbb{R}^{\omega_1}$ is not first countable, it is also not metrisable, and we have answered the research topic.

Thank you.