Local Action Diagrams

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Introduction

The aim of this research project is to investigate how we can verify **isomorphisms** between *groups acting on infinite trees*, specifically, groups defined by how they affect the local structure of a tree.

An isomorphism is essentially a way of showing that two different objects *"behave in the same way"*. The aim of the project is to develop a computational approach to check whether a particular kind of exists between different groups acting on different trees. We use two different theories to analyse the groups acting on trees:

Bass-Serre Theoery is the primary tool for analysing the action of groups on trees. The main limitation of this theory, is that it is hard to find new examples of groups acting on trees.

Local Action Diagrams are graph based structures that can be used to describe the behaviour of groups acting infinite tress. The Theory of Local Action Diagrams is much newer than Bas-Serre Theory, and is an area of active research.

What are Groups?

A group is a set of items, combined with an operation that behaves "similarly to addition". Mathematically, this means satisfying the following rules:

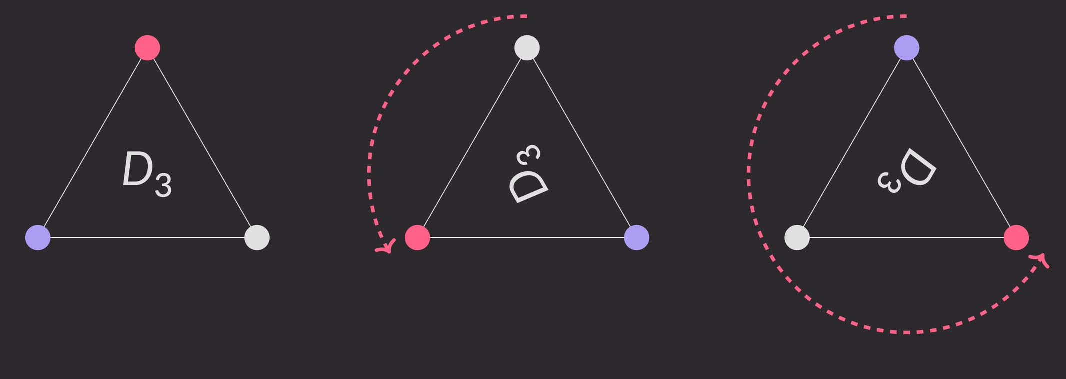
- We can combine any two items $a \star b$ and get a new item c that is also from the group.
- The order that we do operation doesn't matter:
  $$a \star (b \star c) = (a \star b) \star c$$
  This means we don't need to specify in what order the operation is applied. For example the value of:
  $$1 + 2 + 4 + 5$$
  Is the same no matter which + we evaluate first. (In general, we can't swap two items $a \star b \neq b \star a$, only the order of evaluation can be changed.)
- There is a (unique) item e that has no affect:
  $$a \star e = a \qquad e \star a = a$$
  This is like adding 0 to a number: $x + 0 = x$.
- Every item a has an inverse b where:
  $$a \star b = e \qquad b \star a = e$$
  When adding numbers, the inverse of x is $-x$, since $x + (-x) = x - x = 0$.

One example of a group is the integers (whole numbers), written $\mathbb{Z}$, with the addition operation +.

- Adding two whole numbers gives another whole number.
- The order doesn't matter.
- For any whole number n, $0 + n = n + 0 = n$. So 0 is the identity.
- For any n, we have $n + (-n) = 0$ so $-n$ is the inverse of n.

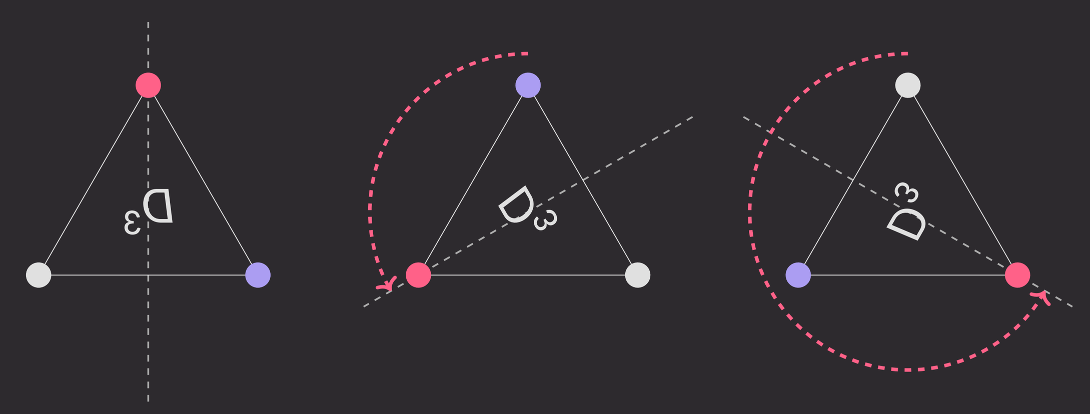
Example: Dihedral Group

Another example of a group comes from the symmetries of regular polygons. If we have an equilateral triangle, there are three different rotations:



r_0^1 (0°) $\quad r_1^1$ (120°) $\quad r_2^1$ (240°)

We can make a group by combining these rotations, the identity is a 0° rotation, 120° and 240° degree rotations are inverses of each other. We also have another three symmetries corresponding to a rotation and a reflection:



$s_0 \quad s_1 = r_1 s_0 \quad s_2 = r_2 s_0$

Transformations are applied right to left, so in the above figure, the reflection is applied first then the rotation.

One interesting property here is that you can create any of the transformations by combining r_1 and s_0:

$$r_0 = r_1^0 \qquad r_1 = r_1^1 \qquad r_2 = r_1^2$$

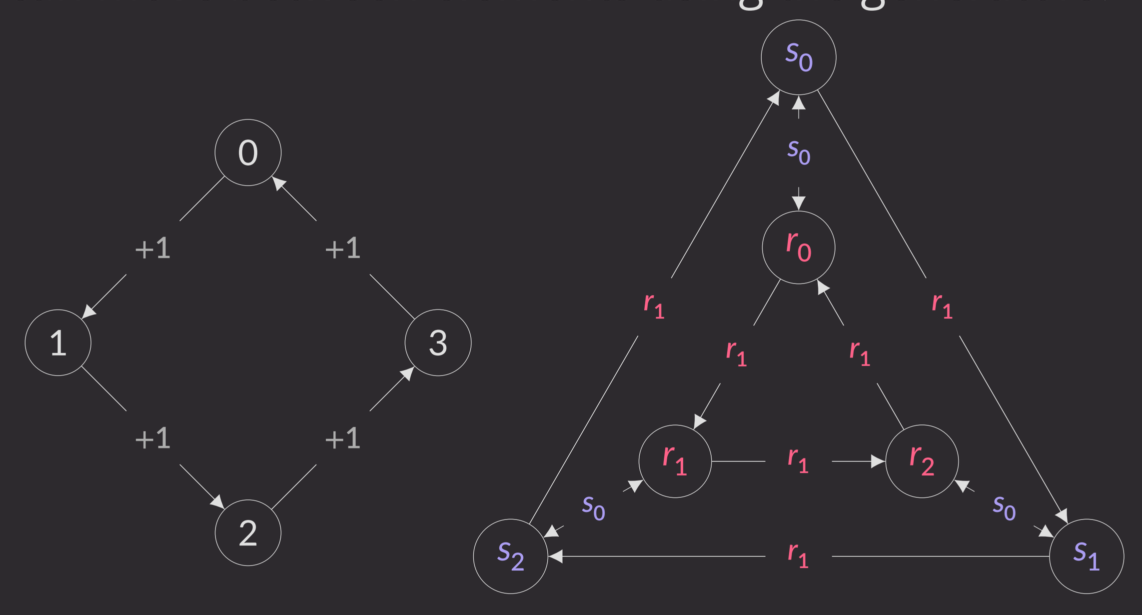
$$s_0 = r_1^0 s_0 \qquad s_1 = r_1^1 s_0 \qquad s_2 = r_1^2 s_0$$

Where x^i means i copies of x, IE $x^2 = xx$. This example is a *dihedral group* called D_3. Since all the combinations of r_1, s_0 produce D_3, we say that these elements *generate* D_3.

Cyclic Groups & Cayley Graph

An even simpler example are the *cyclic* groups, which are generated by a single element. The cycle group C_n consists of n elements, starting with a generator x and repeatedly applying x, we have elements $x^0, x^1, x^2, \ldots, x^{n-1}$. Once, we apply x to x^{n-1} we loop back to x^0 and the cycle repeats.

One way to visualise groups is to draw out how we can move between elements using the generators:



Cayley Graph C_4 $\qquad$ Cayley Graph D_3

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Conclusion