$\mathbb{R}^{\omega_1}$ is a Non-Metrisable Product Space

Robert Christie

MATHS 350

August 19, 2024

Basic Definitions

By $\mathbb{R}^{\omega_1}$, we mean the product space $(X, \mathcal{T})$ where $X = \prod_{\alpha \in \omega_1} \mathbb{R}$ and:

$$\mathcal{T} \quad \text{is generated by basis} \quad \mathcal{B} = \left\{ \prod_{\alpha \in \omega_1} U_\alpha \ : \ \begin{array}{c} \text{Each } U_\alpha \text{ open in } \mathbb{R} \\ \text{Only finitely many } U_\alpha \neq \mathbb{R} \end{array} \right\}$$

Where ω_1 is the first uncountable ordinal and $\mathbb{R}$ has the usual topology.

Definition (2.1)

Suppose $(X, \mathcal{T})$ is a topological space. If there exists a metric d on X such that the metric topology on X induced by d is $\mathcal{T}$, then X is said to be metrisable.

First Countable

Any metrisable space $(X, \mathcal{T})$ is **first countable**: For any $x \in X$, there is a countable neighbourhood basis at x.

Example (4.2)

(1) Any metrisable space is first countable.

(2) $\mathbb{R}$ is separable, first countable and second countable. $\mathbb{Q}$ is a countable dense subset.

Metric Spaces are First Countable

//TODO: Should we prove metric spaces are first countable?

Proof Outline

1. If we can show that $\mathbb{R}^{\omega_1}$ is not first countable, then $\mathbb{R}^{\omega_1}$ cannot be metrisable.

Proof Outline

1. If we can show that $\mathbb{R}^{\omega_1}$ is not first countable, then $\mathbb{R}^{\omega_1}$ cannot be metrisable.
2. To show the space is not first countable, we want to show any countable family of neighbourhoods is not a neighbourhood basis for some point of the space.

Proof Outline

1. If we can show that $\mathbb{R}^{\omega_1}$ is not first countable, then $\mathbb{R}^{\omega_1}$ cannot be metrisable.
2. To show the space is not first countable, we want to show any countable family of neighbourhoods is not a neighbourhood basis for some point of the space.
3. We will show that for any countable family of open sets, there is a projection π_β where every member's projection is $\mathbb{R}$.

Proof Outline

1. If we can show that $\mathbb{R}^{\omega_1}$ is not first countable, then $\mathbb{R}^{\omega_1}$ cannot be metrisable.
2. To show the space is not first countable, we want to show any countable family of neighbourhoods is not a neighbourhood basis for some point of the space.
3. We will show that for any countable family of open sets, there is a projection π_β where every member's projection is $\mathbb{R}$.
4. We then show how this prevents a countable family from been a neighbourhood basis.

Countable Families: Setup

- Let $(X, \mathcal{T})$ be $\mathbb{R}^{\omega_1}$ with the product topology. Let I be a countable indexing set, consider any countable family $\{U_\alpha\}_{\alpha \in I} \subset \mathcal{T}$ of **open non-empty** sets.

Countable Families: Setup

- Let $(X, \mathcal{T})$ be $\mathbb{R}^{\omega_1}$ with the product topology. Let I be a countable indexing set, consider any countable family $\{U_\alpha\}_{\alpha \in I} \subset \mathcal{T}$ of **open non-empty** sets.
- As each U_α is open and non-empty, $U_\alpha = \cup_{b \in B} b$ where $\varnothing \neq B \subset \mathcal{B}$.

Countable Families: Setup

- Let $(X, \mathcal{T})$ be $\mathbb{R}^{\omega_1}$ with the product topology. Let I be a countable indexing set, consider any countable family $\{U_\alpha\}_{\alpha \in I} \subset \mathcal{T}$ of **open non-empty** sets.
- As each U_α is open and non-empty, $U_\alpha = \cup_{b \in B} b$ where $\varnothing \neq B \subset \mathcal{B}$.
- Take some $b' \in B \neq \varnothing$. Consider any arbitrary $\beta \in \omega_1$, then $b' \subseteq U_\alpha$ so:

$$\pi_\beta(b') \subseteq \pi_\beta(U_\alpha) \subseteq \mathbb{R}$$

Countable Families: Setup

- Let $(X, \mathcal{T})$ be $\mathbb{R}^{\omega_1}$ with the product topology. Let I be a countable indexing set, consider any countable family $\{U_\alpha\}_{\alpha \in I} \subset \mathcal{T}$ of **open non-empty** sets.
- As each U_α is open and non-empty, $U_\alpha = \cup_{b \in B} b$ where $\emptyset \neq B \subset \mathcal{B}$.
- Take some $b' \in B \neq \emptyset$. Consider any arbitrary $\beta \in \omega_1$, then $b' \subseteq U_\alpha$ so:

$$\pi_\beta(b') \subseteq \pi_\beta(U_\alpha) \subseteq \mathbb{R}$$

- Since $\pi_\beta(b') \neq \mathbb{R}$ for only finitely many β, the remaining β have $\pi_\beta(b') = \mathbb{R}$ so $\mathbb{R} \subseteq \pi_\beta(U_\alpha) \subseteq \mathbb{R}$ hence $\pi_\beta(U_\alpha) = \mathbb{R}$. So $\pi_\beta(U_\alpha) \neq \mathbb{R}$ for at most finitely many β.

Countable Families: Contradiction

Assume for any $\beta \in \omega_1$, there is some $\alpha \in I$ where $\pi_\beta(U_\alpha) \neq \mathbb{R}$, also note that $\pi_\beta(U_\alpha)$ is the i-th projection where $\pi_\gamma(U_\alpha) \neq \mathbb{R}$ for some finite $i \in \mathbb{N}$. Define a function:

$$\begin{aligned} f : \omega_1 &\to I \times \mathbb{N} \\ \beta &\mapsto (\alpha, i) \end{aligned}$$

Countable Families: Contradiction

Assume for any $\beta \in \omega_1$, there is some $\alpha \in I$ where $\pi_\beta(U_\alpha) \neq \mathbb{R}$, also note that $\pi_\beta(U_\alpha)$ is the i-th projection where $\pi_\gamma(U_\alpha) \neq \mathbb{R}$ for some finite $i \in \mathbb{N}$. Define a function:

$$\begin{aligned} f\ : \omega_1 &\to I \times \mathbb{N} \\ \beta &\mapsto (\alpha, i) \end{aligned}$$

Take any $\beta, \beta' \in \omega_1$ where $f(\beta) = f(\beta')$. Then, $(\alpha, i) = f(\beta) = f(\beta')$ corresponds to some $\alpha \in I$ where $\pi_\beta(U_\alpha)$ and $\pi_{\beta'}(U_\alpha)$ are both the i-th component of U_α where $\pi_\gamma(U_\alpha) \neq \mathbb{R}$. Thus, $\beta = \beta'$ and so f is injective.

Countable Families: Contradiction

Assume for any $\beta \in \omega_1$, there is some $\alpha \in I$ where $\pi_\beta(U_\alpha) \neq \mathbb{R}$, also note that $\pi_\beta(U_\alpha)$ is the i-th projection where $\pi_\gamma(U_\alpha) \neq \mathbb{R}$ for some finite $i \in \mathbb{N}$. Define a function:

$$\begin{aligned} f : \omega_1 &\to I \times \mathbb{N} \\ \beta &\mapsto (\alpha, i) \end{aligned}$$

Take any $\beta, \beta' \in \omega_1$ where $f(\beta) = f(\beta')$. Then, $(\alpha, i) = f(\beta) = f(\beta')$ corresponds to some $\alpha \in I$ where $\pi_\beta(U_\alpha)$ and $\pi_{\beta'}(U_\alpha)$ are both the i-th component of U_α where $\pi_\gamma(U_\alpha) \neq \mathbb{R}$. Thus, $\beta = \beta'$ and so f is injective.

Since f is injective, we have that $|\omega_1| \leq |I \times \mathbb{N}|$. However, the Cartesian product of countable sets is countable, so we have a contraction.

Countable Families: Contradiction

Assume for any $\beta \in \omega_1$, there is some $\alpha \in I$ where $\pi_\beta(U_\alpha) \neq \mathbb{R}$, also note that $\pi_\beta(U_\alpha)$ is the i-th projection where $\pi_\gamma(U_\alpha) \neq \mathbb{R}$ for some finite $i \in \mathbb{N}$. Define a function:

$$\begin{aligned} f : \omega_1 &\to I \times \mathbb{N} \\ \beta &\mapsto (\alpha, i) \end{aligned}$$

Take any $\beta, \beta' \in \omega_1$ where $f(\beta) = f(\beta')$. Then, $(\alpha, i) = f(\beta) = f(\beta')$ corresponds to some $\alpha \in I$ where $\pi_\beta(U_\alpha)$ and $\pi_{\beta'}(U_\alpha)$ are both the i-th component of U_α where $\pi_\gamma(U_\alpha) \neq \mathbb{R}$. Thus, $\beta = \beta'$ and so f is injective.

Since f is injective, we have that $|\omega_1| \leq |I \times \mathbb{N}|$. However, the Cartesian product of countable sets is countable, so we have a contraction.

Therefore, there is a $\beta \in \omega_1$ where all $\alpha \in I$ have $\pi_\beta(U_\alpha) = \mathbb{R}$.

Consequences

Now assume that $\mathbb{R}^{\omega_1}$ is first countable, so for $x = 0 \in \mathbb{R}^{\omega_1}$ we have a countable neighbourhood basis $\mathcal{M} = \{M_\alpha\}_{\alpha \in I}$ where I is a countable indexing set. Now define a family of open sets $\{U_\alpha\}_{\alpha \in I}$ where:

$$U_\alpha := \bigcup_{\substack{U \in \mathcal{T} \\ U \subseteq M_\alpha}} U$$

Note each $U_\alpha \neq \varnothing$ as M_α is a neighbourhood and $U_\alpha \subseteq M_\alpha$.

Consequences

Now assume that $\mathbb{R}^{\omega_1}$ is first countable, so for $x = 0 \in \mathbb{R}^{\omega_1}$ we have a countable neighbourhood basis $\mathcal{M} = \{M_\alpha\}_{\alpha \in I}$ where I is a countable indexing set. Now define a family of open sets $\{U_\alpha\}_{\alpha \in I}$ where:

$$U_\alpha := \bigcup_{\substack{U \in \mathcal{T} \\ U \subseteq M_\alpha}} U$$

Note each $U_\alpha \neq \varnothing$ as M_α is a neighbourhood and $U_\alpha \subseteq M_\alpha$. From the previous result, there must be a $\beta \in \omega_1$ where $\pi_\beta(U_\alpha) = \mathbb{R}$ for any $\alpha \in I$. Consider the open neighbourhood of 0 given by:

$$N := \prod_{\gamma \in \omega_1} N_\gamma \qquad N_\gamma := \begin{cases} (-1, 1) & \gamma = \beta \\ \mathbb{R} & \gamma \neq \beta \end{cases}$$

Consequences: Continued

$$N := \prod_{\gamma\in\omega_1} N_\gamma \qquad N_\gamma := \begin{cases} (-1,1) & \gamma = \beta \\ \mathbb{R} & \gamma \neq \beta \end{cases}$$

If $\mathcal{M}$ were a neighbourhood basis then, there would be a $M_\alpha \subseteq N$, since images preserve inclusion, we expect $\pi_\beta(U_\alpha) \subseteq \pi_\beta(M_\alpha) \subseteq \pi_\beta(N)$. But for any $\alpha \in I$ we have $\pi_\beta(U_\alpha) = \mathbb{R}$, while $\pi_\beta(N) = (-1,1)$ so $U_\alpha \not\subseteq N$. This is a contradiction and so $\mathbb{R}^{\omega_1}$ is not first countable.

Conclusion

Since $\mathbb{R}^{\omega_1}$ is not first countable, it cannot be metrisable.

- Since $|\omega_1| \leq |\mathbb{R}|$ depending on the continuum hypothesis, this means the space $\mathbb{R}^{\mathbb{R}}$ is not metrisable.
- This is equivalent to the space of functions $\mathbb{R} \to \mathbb{R}$ not being metrisable.
- This is why we must restrict functions in the Lebesgue space to only those with finite integrals.