The Banach-Mazur Game

Gaining some intuition on Baire Categories

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The Baire Category Definitions

Let $(X, \mathcal{T})$ be a topological space and $M \subseteq X$.

Definition 1 (Rare). M is **rare** if the closure $\bar{M}$ has an empty interior.

Definition 2 (Meager). M is said to be **meager** *(or of the first category in X)* if M is the union of countably many *rare* sets in X.

Definition 3 (Non-Meager). M is said to be **non-meager** *(or of the second category)* in X if M is not meager in X.

> *Can you think of examples of meager and non-meager sets of* $\mathbb{R}$ *with the usual topology?*

Examples of sets of the first and second categories.

In $\mathbb{R}$, the following sets are *non-meager*:

1. $\mathbb{R}$ i.e The Real Line

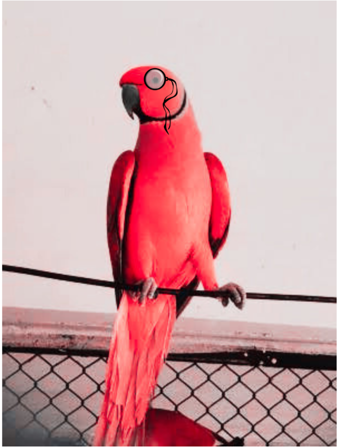
In $\mathbb{R}$ the following sets are *meager*:

1. $\mathbb{N}$ i.e. the Natural Numbers
2. $\mathbb{Q}$ i.e the Rational Numbers

Meet Alice and Bob

Alice and Bob are immortal beings with infinite time.

(a) Alice



(b) Bob

Figure: Bob is (clearly) Evil!!!

The Banach-Mazur Game

Let's play the game!

Let X be a set with some Topology $\mathcal{T}$ and let $\mathcal{W} \subset \mathcal{T}$ be the set of all nonempty open sets of $\mathcal{T}$. Alice and Bob take turns choosing sets.

- Bob goes first and chooses a non-empty open set $B_1 \subset X$.
- Then Alice chooses a non-empty open set $A_1 \subset B_1$
- Then it is now Bob's turn again so he chooses a new non-empty open subset of the set Alice chose and we continue on ...

Alice wins if:

$$\bigcap_{k=1}^{\infty} A_k \neq \emptyset$$

Otherwise Bob Wins.

Example: The Banach-Mazur Game on $\mathbb{R}$

Choose some $X \subseteq \mathbb{R}$.
Here it is equivalent to Alice and Bob choosing nesting open intervals of $X \subseteq \mathbb{R}$ and Alice wins if the set of all these **nested open intervals** has a common point with X.

Mazur observed:
If the complement of the set X itself is of the first Baire Category on $\mathbb{R}$ then Alice wins!

Winning Strategies

When can Bob win the game.

Theorem Let X be non-meager in itself, then X admits no winning strategies for Bob.

Proof. Suppose for contradiction that X admits a winning strategy for Bob, call this strategy T.
Note T is based on all the sets that have been chosen before it. Let B_1 be the first set chosen by Bob under this strategy. Consider the sets directly from which Bob chooses (Alice's turn before and construct a sequence of tuples. For all $n \in \mathbb{N}$ define $\gamma_n := (A_{n-1}, B_n)$. If we let $A_0 = B_0 = X$ this is well defined.

Winning Strategies

I don't want to let Bob win the Game.

Theorem Let X be non-meager in itself, then X admits no winning strategies for Bob.

Proof. (continued) Additionally, since $(\gamma_n)_{n\in\mathbb{N}}$ is a part of a valid sequence of moves in the game we have:

1. For all $n \geq 2$ and $(A_{n-1}, B_n) \in \gamma_n$ there is some $(A_{n-2}, B_{n-1}) \in \gamma_{n-1}$ such that $A_{n-1} \subset B_{n-1}$.
2. for all $n \geq 2$ and $(A_{n-1}, B_n) \in \gamma_n$ there are some $(A_{i-1}, B_i) \in \gamma_i$ for all $i < n$ such that the sequence $(B_1, A_1, ..., B_{n-1}, A_{n-1})$ are all all valid moves in the game by turn $n-1$.

Let the sets B_i of $(\gamma_n)_{n\in\mathbb{N}}$ be chosen to be maximal under these conditions under the strategy T.

Winning Strategies

Bob wont win :)

Theorem Let X be non-meager in itself, then X admits no winning strategies for Bob.

Proof.(continued) Let
$W_n := \bigcup\{B_i : (A_{i-1}, B_i) \in \gamma_i, \forall i \leq n\}$
Now we aim to show that W_n is dense in B_1.
Consider as a base case that $W_1 = B_1$ is clearly dense in B_1. Suppose true for all $1 \leq k \leq n$ then if W_{n+1} is not dense in B_1 there is some open $U \in B_1$ such that $U \cap W_{n+1} = \emptyset$. However since W_n is dense in B_1 by inductive hypothesis we have that so there is some $\gamma_n = (A_{n-1}, B_n)$ so that $B_n \cap B_1 \neq \emptyset$ and so there is a sequence (by condition 2) such that $(B_1, A_1, ..., B_{n-1}, A_{n-1})$ and $B_n = T(B_1, A_1, ..., B_{n-1}, A_{n-1})$

Winning Strategies

I wont let Bob win!!!

Theorem Let X be non-meager in itself, then X admits no winning strategies for Bob.

Proof.(continued) Let $A'_n = B_n \cap X$ and let $B'_{n+1} = T(B_1, A_1, ..., B_n, A'_n)$. But then if we let the sequence $(\gamma'_n)_{n\in\mathbb{N}}$ be defined to contain these new sets we have a sequence of sets larger than (γ_n),so the constructed W_n are all dense in B_1. However, since there is a winning strategy for Bob, we have:
$\bigcap_{n=1}^{\infty} W_n = \bigcap_{n=1}^{\infty} B_n = \emptyset$.
Thus, the we have a collection of dense sets whose intersection is empty. Taking the converse of this statement gives us that X is meager! $\square$

not today BOB

Figure: Bob Losing

The End

Questions? Comments?