New Concepts in Adaptive Control Using Multiple Models

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***Abstract*—The concept of using multiple models to cope with transients which arise in adaptive systems with large parametric uncertainties was introduced in the 1990s. Both switching between multiple fixed models, and switching and tuning between fixed and adaptive models was proposed, and the stability of the resulting schemes was established. In all cases, the number of models needed is generally large (c^n where n is the dimension of the parameter vector and c an integer), and the models do not "cooperate" in any real sense. In this paper, a new approach is proposed which represents a significant departure from past methods. First, it requires $(n+1)$ models (in contrast to c^n) which is significantly smaller, when "n" is large. Second, while each of the $(n+1)$ models chosen generates an estimate of the plant parameter vector, the new approach provides an estimate which depends on the collective outputs of all the models, and can be viewed as a time-varying convex combination of the estimates. It is then shown that control based on such an estimate results in a stable overall system. Further, arguments are given as to why such a procedure should result in faster convergence of the estimate to the true value of the plant parameter as compared to conventional adaptive controllers, resulting in better performance. Simulation studies are included to practically verify the arguments presented, and demonstrate the improvement in performance.**

***Index Terms*—Adaptive control, multiple models.**

I. Introduction

ADAPTIVE control theory, dealing with the control of linear time-invariant systems with unknown parameters, has been studied since the 1960s, and an extensive literature currently exists in this area [1]–[6]. It is now generally accepted that when parametric errors are small classical adaptive control assures both stability and robustness.

When parametric errors are large, it has been observed over the years that the transient response of adaptive systems is oscillatory, and numerous efforts have been made to improve the performance in such cases. One such effort, involving multiple models, was introduced in the 1990s. During this period, both fixed models [7], [8] and fixed and adaptive models [9]–[11] were proposed for improving the transient response. In [7] and [8], a supervisor controller switches into feedback a sequence of linear set-point controllers from a family of candidate controllers. In contrast to the above, both fixed and adaptive models were used in [9]–[11] for the identification of the plant, and later for its control. Based on an index of performance, one of the models is chosen at any instant as the "best" and used at that instant to determine the control input. At the same time an adaptive model is initiated from the same point in parameter space, and the process is continued. The qualitative explanation provided is that the index of performance would, in general, choose the model "closest" to the plant in some sense, and consequently adaptation would commence from that model, resulting in improved performance. Switching to the closest model implies fast response in the adaptive context, and tuning from that model improves the system response on a slower time scale (i.e., incrementally as in classical adaptive control). Extensive simulation studies have demonstrated that the methods proposed perform satisfactorily when no limits are placed on the number of models that can be used [12]–[17].

From a practical standpoint, the methods proposed above suffer from two major drawbacks. First, it is found that the number of models needed to assure that at least one of the fixed models is sufficiently close to the plant in parameter space is generally quite large and grows exponentially with the dimension of the unknown parameter vector. Second, the various models do not cooperate in any real sense to make the decision concerning the location of the unknown plant parameter vector. In particular, the performance indices of the different models are merely used to locate a model close to the plant. As a consequence, resources (i.e., the data available at the different models) are not used efficiently. In spite of these shortcomings, the methods are found to perform satisfactorily when the plant is time-invariant and the number of models that can be used is sufficiently large.

It is well known that the demands of a rapidly advancing technology are the prime movers of new theoretical advances. In numerous areas such as medicine, neuroscience, finance, and national security, classes of problems are arising where decisions have to be made in the presence of large parametric uncertainty and ambient noise, or rapid variations in parameters. The adaptive methods that are currently available are generally inadequate to deal with such problems. The adaptive procedure used in the paper is based collectively on all the models so that the resources are used efficiently. Further, from a practical stand point very few models $(n+1)$ are needed for the identification process as compared to c^n needed for current methods. The objective of this paper, consequently, is to set up a general framework based on multiple models, propose a new method for identifying the plant using the combined information provided by all

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the models, present arguments to justify why the latter results in faster identification, and finally demonstrate that such a procedure results in a stable overall system.

II. Multiple Models for Adaptive Control (State Variables Accessible)

In this section, we consider the adaptive control of a linear time-invariant (LTI) plant using multiple adaptive models, when the state variables of the plant are accessible. To facilitate the introduction of the principal concepts contained in this paper, we start our discussions with a relatively simple adaptive control problem whose solution can be found in any standard text on adaptive control. After stating the problem, we provide in quick succession the adaptive solution using a single identification model, the solutions based on current approaches using a finite number N of adaptive models, a discussion of the creation of an arbitrary number of virtual adaptive models, and finally the new approach introduced in this paper using second level adaptation, for control purposes. This sets the stage for the consideration of the more general problem of adaptive control in Section III, when only the inputs and outputs of the plant are accessible. Throughout the paper, for the sake of completeness, well known arguments in adaptive literature are included in the discussions, but details are omitted to conserve space.

A. Statement of the Adaptive Control Problem

An LTI plant Σ_p is described by the state equations

$$\Sigma_p: \quad \dot{x}_p(t) = A_p x_p(t) + bu(t) \tag{1}$$

where $x_p(\cdot): \mathbb{R}^+ \to \mathbb{R}^n$, $u(\cdot): \mathbb{R}^+ \to \mathbb{R}$. $A_p \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$ are in companion form. The elements of the last row of the matrix A_p are $[a_{p(1)}, a_{p(2)}, \ldots, a_{p(n)}] = \theta_p^T$ and are assumed to be unknown. $b = [0, \ldots, 0, 1]^T$. A reference model Σ_m is described by the differential equation

$$\Sigma_m: \quad \dot{x}_m(t) = A_m x_m(t) + br(t) \tag{2}$$

where $r(\cdot): \mathbb{R}^+ \to \mathbb{R}$ is a known bounded piecewise continuous reference signal. The matrix A_m is also in companion form, is stable, and has the last row θ_m^T. Assuming that $\theta_p \in \mathcal{S}_\theta$ where $\mathcal{S}_\theta$ is a compact set in parameter space, the objective is to determine the input $u(\cdot)$ to the plant such that $\lim_{t\to\infty}[x_p(t) - x_m(t)] = 0$.

B. Single Identification Model

Assuming that an indirect approach is used to control Σ_p, an identification model Σ_I is set up which is described by the differential equation

$$\Sigma_I: \quad \dot{x}_I(t) = A_m x_I(t) + [A_I(t) - A_m]\, x_p(t) + bu(t) \tag{3}$$

where $A_I(t)$ is a matrix in companion form, whose last row $\theta_I^T(t) = [a_{I(1)}(t), a_{I(2)}(t), \ldots, a_{I(n)}(t)]$ (the estimate of the plant parameters) can be adjusted adaptively. Defining $\tilde{\theta}_I(t) \triangleq \theta_I(t) - \theta_p$ and $e_I(t) \triangleq x_I(t) - x_p(t)$, the error equation can be written as

$$\dot{e}_I(t) = A_m e_I(t) + b\tilde{\theta}_I^T(t) x_p(t). \tag{4}$$

$e_I(t)$ will be referred to as the identification error and $\tilde{\theta}_I(t)$ as the parameter error. Using a Lyapunov function candidate $V(e_I, \tilde{\theta}_I) = e_I^T P e_I + \tilde{\theta}_I^T \tilde{\theta}_I$, where P is the positive definite matrix solution of the Lyapunov equation $A_m^T P + P A_m = -Q$, $Q = Q^T > 0$, it follows directly from well known results in adaptive control that the adaptive law

$$\dot{\theta}_I(t) = -e_I^T(t) P b x_p(t) \tag{5}$$

results in $\dot{V}(e_I, \tilde{\theta}_I) = -e_I^T Q e_I \le 0$. This assures the boundedness of both the identification error $e_I(t)$ and the parameter error $\tilde{\theta}_I(t)$ (and hence $\theta_I(t)$). To assure the stability of the plant, and hence the boundedness of $x_p(t)$, feedback control is used so that

$$u(t) = -k^T(t) x_p(t) + r(t) \tag{6}$$

where $k(t) = \theta_I(t) - \theta_m$. Equations (1), (2), and (6) yield the (control) error equation $\dot{e}_c(t) = A_m e_c(t) + b\tilde{k}^T(t) x_p(t)$, where $e_c(t) \triangleq x_p(t) - x_m(t)$, $\tilde{k}(t) \triangleq k(t) - k^*$ and $k^* \triangleq \theta_p - \theta_m$. Following the same arguments as before, it follows that $e_c \in \mathcal{L}^2 \cap \mathcal{L}^\infty$ which assures the boundedness of $x_p(t)$ and $\dot{e}_c(t)$. From Barbalat's lemma it follows that $\lim_{t\to\infty} e_c(t) = 0$. Or the state $x_p(t)$ of the plant follows the state $x_m(t)$ of the reference model asymptotically.

C. Identification Using N Adaptive Models

In adaptive control it is well known that the designer can use an arbitrary number of models to identify the plant, but only one controller to control it. It therefore follows that N identification models $\Sigma_1, \Sigma_2, \ldots, \Sigma_N$ (with the same structure as Σ_I defined in Section II-B) can be set up to provide N estimates of the parameter vector. The model $\Sigma_i (i \in \Omega = \{1, 2, \ldots, N\})$ includes the parameter estimate $\theta_i(t)$ which can be updated adaptively, i.e.,

$$\Sigma_i: \quad \begin{array}{l} \dot{x}_i(t) = A_m x_i(t) + [A_i(t) - A_m]\, x_p(t) + bu(t) \\ x_i(t_0) = x_p(t_0) \end{array}. \tag{7}$$

Comment 2.3.1: The N adaptive models are consequently described by identical differential equations with the same initial state as the plant but with different initial values of the parameter vectors (i.e., $\theta_i(t_0)$). The former condition is realizable since it is assumed that the plant states are accessible.

From the above assumptions it follows that the identification errors $e_i(t) = x_i(t) - x_p(t)$ satisfy the error differential equations

$$\begin{aligned} \dot{e}_i(t) &= A_m e_i(t) + b\tilde{\theta}_i^T(t) x_p(t) \\ \theta_i(t_0) &= \theta_{it_0} \text{ and } e_i(t_0) = 0 \quad i \in \Omega. \end{aligned} \tag{8}$$

Consequently, we have $e_i(t) = \int_{t_0}^t \Phi(t,\tau) b \tilde{\theta}_i^T(\tau) x_p(\tau) d\tau$, where $\Phi(t,\tau) = e^{A_m(t-\tau)}$ is the transition matrix of $\dot{e}_i = A_m e_i$, and $\tilde{\theta}_i(t) = \theta_i(t) - \theta_p$.

D. Control Using Multiple Models

Assuming that N models are operating in parallel, the question arises as to how the information obtained is to be used to control the system at every instant. This becomes particularly relevant when the plant is unstable. Using classical theory, any one of the estimates can be used to stabilize the system. In [9]–[11], it was suggested that different performance indices could be used to compare the different estimates and provide a basis for the choice of the control parameter vector. For example, if an index of the form $J_i(t) = \int_{t_0}^{t} \|e_i(\tau)\|^2 d\tau (i = 1, 2, \ldots, N)$ is chosen and one of the models Σ_j is considered to be the "best" at any instant according to this criteria (i.e., $J_j(t) = \min_i J_i(t)$), it can, in turn, be used to select the controller parameter. It was shown in [11] that if a finite dwell time is used, switching between different parameters results in the stability of the overall system and the asymptotic convergence of the control error to zero. As in classical adaptive control (Section II-B), the parameter estimates need not converge to the plant parameter vector θ_p but do so if the reference input is persistently exciting.

Comment 2.4.1: The efficacy of the control depends upon how rapidly (and how accurately) the plant parameter can be estimated. When the number of models N is small and the region of uncertainty $\mathcal{S}_\theta$ is large, the improvement in the transient behavior of the system, over that realized using a single model, may not be significant. Additional properties of the multiple models need to be exploited, and this is treated in the following section.

E. Preliminaries for the New Approach

Thus far, we have defined N identification models Σ_i which are used to estimate the unknown parameter vector θ_p of the plant. In the following discussions other identification models such as "time-invariant virtual models" and "time-varying virtual models" are used. The former are defined primarily for theoretical purposes, while the latter include the model proposed in this paper for adaptively controlling the unknown plant. To avoid confusion, the different models are first defined in this section.

Definitions: A set $\mathcal{K}$ in a linear space $\mathcal{L}$ is called convex if the line segment ab is in $\mathcal{K}$ for any elements $a, b \in \mathcal{K}$ i.e., $x = (1-\lambda)a + \lambda b \in \mathcal{K}$ for any pair (a, b) and $\lambda \in [0, 1]$ [18].

Lemma: Let $\mathcal{K}$ be a convex subset of $\mathcal{L}$. Then every convex combination of $a_1, a_2, \ldots, a_m \in \mathcal{K}$ is also an element of $\mathcal{K}$ [18].

Primary Models: We shall refer to the identification models $\Sigma_i (i \in \Omega = \{1, 2, \ldots, N\})$ as the primary models. The parameter associated with model Σ_i is $\theta_i(t)$, and the latter is adjusted using standard adaptive laws (5). If the initial values of $\theta_i(t)$ at time t_0 are $\theta_i(t_0)$, the region of uncertainty $\mathcal{S}_\theta$ of the plant parameter vector θ_p lies in their convex hull (i.e., $\mathcal{S}_\theta \subset \mathcal{K}(t_0)$, where $\mathcal{K}(t_0)$ is the convex hull of $\theta_i(t_0) (i \in \Omega)$).

Virtual Models: Given $\{\theta_i(t_0)(i \in \Omega)\}$, any element of the convex hull $\mathcal{K}(t_0)$ of $\{\theta_i(t_0)(i \in \Omega)\}$ can be expressed as

$$\theta_0(t_0) = \sum_{i=1}^{N} \beta_i \theta_i(t_0) \tag{9}$$

where $\beta_i \geq 0$ are constants and $\sum_{i=1}^{N} \beta_i = 1$. We refer to $\theta_0(t_0)$ as a time-invariant virtual model. Thus, any element of the convex hull $\mathcal{K}(t_0)$ (corresponding to a specific choice of the constants β_i), is also a virtual model. The property of a virtual model that is relevant for our purposes can be described as follows.

Let every primary model be adjusted using the standard adaptive law (5) $\dot{\theta}_i(t) = -e_i^T(t) P b x_p(t)$. Since $\theta_0(t) - \theta_p = \sum_{i=1}^{N} \beta_i \theta_i(t) - \theta_p = \sum_{i=1}^{N} \beta_i [\theta_i(t) - \theta_p] = \sum_{i=1}^{N} \beta_i \tilde{\theta}_i(t)$, if an additional identification model Σ_0 is set up as $\Sigma_0 : \dot{x}_0(t) = A_m x_0(t) + [A_0(t) - A_m] x_p(t) + bu(t)$ with parameter vector $\theta_0(t)$ as the last row of $A_0(t)$ and initial condition $\theta_0(t_0) = \sum_{i=1}^{N} \beta_i \theta_i(t_0)$, it follows that the adaptive law governing the adjustment of $\theta_0(t)$ can be derived as

$$\begin{aligned} \dot{\theta}_0(t) &= \sum_{i=1}^{N} \dot{\theta}_i(t) \beta_i = -\sum_{i=1}^{N} x_p(t) b^T P e_i(t) \beta_i \\ &= -x_p(t) b^T P \int_{t_0}^{t} \Phi(t, \tau) b \left[\sum_{i=1}^{N} \beta_i \tilde{\theta}_i^T(\tau) \right] x_p(\tau) d\tau \\ &= -x_p(t) b^T P e_0(t) \end{aligned} \tag{10}$$

where $e_0(t) = x_0(t) - x_p(t)$ is the identification error of model Σ_0. The law is observed to be identical to that of the N primary models. Therefore, at every instant, with constant β_i, the evolution of $\theta_0(t)$ is governed by the same adaptive law as the primary models. This implies that every element of the convex hull $\mathcal{K}(t_0)$ corresponds to a virtual model, and when adjusted adaptively using the same laws as the primary models, can be expressed as a fixed convex combination of the latter. The set of all virtual models starting in $\mathcal{K}(t_0)$ at time t_0 lie in $\mathcal{K}(t)$ at time t.

Comment 2.5.1: As stated above, a time-invariant virtual model is no different from any primary model and consequently has convergence properties identical to those of the primary models. We use the terms "primary" and "virtual" to distinguish between the N models which are used in the final adaptive procedure and the others which are merely used in the analysis.

From the above analysis it follows that the transformation $\tilde{\theta}_i(t) = \Psi(t, t_0, \tilde{\theta}_i(t_0))$ satisfies the property $\Psi(t, t_0, \sum_{i=1}^{m} \lambda_i \tilde{\theta}_i(t_0)) = \Psi(t, t_0, \tilde{\theta}_j(t_0)) = \tilde{\theta}_j(t) = \sum_{i=1}^{m} \lambda_i \Psi(t, t_0, \tilde{\theta}_i(t_0))$, i.e., $\Psi : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a convex map with respect to its last variable. Further, from the adaptive laws it follows that $\dot{\tilde{\theta}}_i(t) = -e_i^T(t) P b x_p(t) = \int_{t_0}^{t} [-x_p(t) b^T P \Phi(t, \tau) b x_p^T(\tau)] \tilde{\theta}_i(\tau) d\tau = \int_{t_0}^{t} \Pi(t, \tau) \tilde{\theta}_i(\tau) d\tau$, where $\Pi(t, \tau) = -x_p(t) x_p^T(\tau)(b^T P \Phi(t, \tau) b)$. Therefore, $\tilde{\theta}_i(t) = \int_{t_0}^{t} \int_{t_0}^{\mu} \Pi(\mu, \tau) \tilde{\theta}_i(\tau) d\tau d\mu$.

The Plant Parameter Vector: The unknown plant parameter vector $\theta_p \in \mathcal{S}_\theta$ belongs to $\mathcal{K}(t_0)$ and consequently can be considered as a time-invariant virtual model. In addition, if a virtual model with initial condition $\theta_0(t_0) = \theta_p$ is adjusted using the same adaptive law as the primary models, $\theta_0(t)$ is a constant, i.e., $\theta_0(t) = \theta_0(t_0) = \theta_p$. This implies that θ_p is a fixed point in $\mathcal{K}(t)$, which is useful for the derivation of the convex hull property introduced in the following section.

F. Convex Hull Property of Multiple Adaptive Models

The discussion in Section II-E concerning virtual models defined as the convex combination of the N primary models leads to the principal theorem in this section, which is central to the developments in the rest of the paper. Let the initial values of the parameter vectors $\theta_i(t_0)$ be chosen such that $\mathcal{S}_\theta \subset \mathcal{K}(t_0)$, where $\mathcal{K}(t_0)$ is the convex hull of the set $\{\theta_i(t_0)\}$, i.e., $\theta_p \in \mathcal{K}(t_0)$.

Comment 2.6.1: If $\theta_p \in \mathbb{R}^n$, $N = n + 1$ is sufficient to satisfy the above condition (i.e., $\mathcal{S}_\theta \subset \mathcal{K}(t_0)$). In practice N

can be chosen to be greater than $(n+1)$ for convenience or for efficiency.

Theorem 1: If N adaptive identification models described in (7) are adjusted using adaptive laws (5) with initial conditions $\theta_i(t_0)$ and initial states $x_i(t_0) = x_p(t_0)$, and if the plant parameter vector θ_p lies in the convex hull $\mathcal{K}(t_0)$ of $\theta_i(t_0)(i \in \Omega)$, then θ_p lies in the convex hull $\mathcal{K}(t)$ of $\theta_i(t)(i \in \Omega)$ for all $t \geq t_0$.

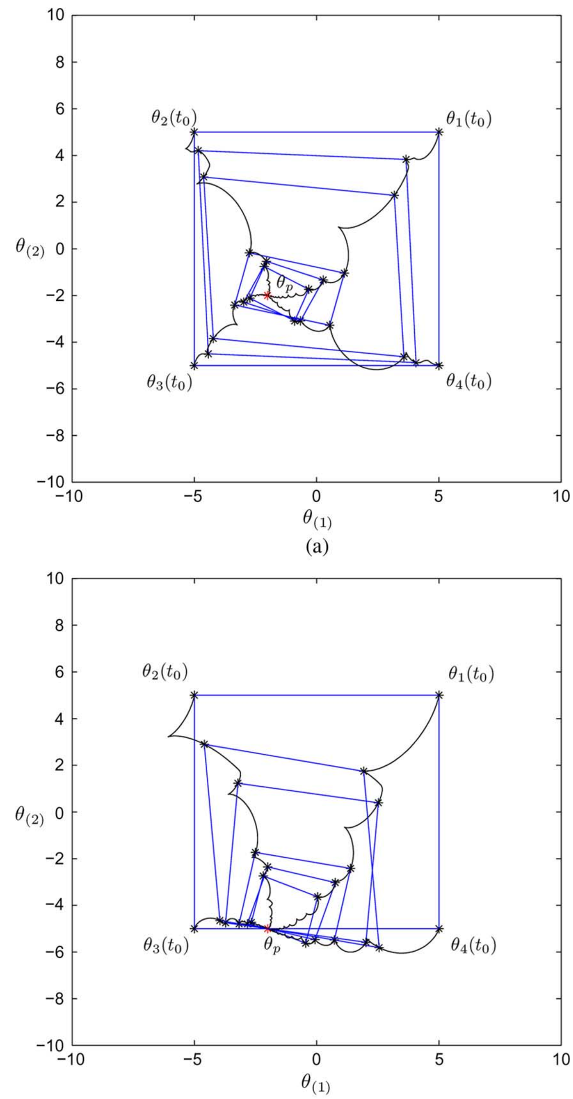
Proof: Since θ_p lies in the convex hull of $\theta_i(t_0)(i \in \Omega)$, it follows that it satisfies the equation $\theta_p = \sum_{i=1}^{N} \alpha_i \theta_i(t_0), \sum_{i=1}^{N} \alpha_i = 1, \alpha_i \geq 0$. As stated earlier assuming that a virtual model $\theta_0(t)$ is initiated with $\theta_0(t_0) = \theta_p$, it follows that $\theta_0(t) = \theta_p$ for all $t \geq t_0$ if it is adjusted adaptively using (10). Therefore, $\theta_p = \theta_0(t_0) = \theta_0(t) = \sum_{i=1}^{N} \alpha_i \theta_i(t)$ for all $t \geq t_0$.

Comment 2.6.2: The proof of Theorem 1 is seen to follow directly from the linearity of the error (8).

From Theorem 1 it follows that if θ_p lies in the convex hull $\mathcal{K}(t_0)$, it also lies in the convex hull $\mathcal{K}(t)$. If it lies outside the convex hull, it remains outside the convex hull, and if it lies on the boundary of the convex hull, it will remain on the boundary of the convex hull, for all $t \geq t_0$. A crucial assumption made in deriving the above results is that all the identification models have initial conditions $x_i(t_0) = x_p(t_0)$ so that $e_i(t_0) = 0$. Since it was assumed that $x_p(t)$ is accessible, all models can be chosen to satisfy this condition. Extension of the results to the general case when only inputs and outputs are accessible is treated in Section III.

Example 1: In view of the importance of the convex hull property of multiple adaptive models given in Theorem 1, Example 1 is included below to illustrate it. A stable second order plant with $\theta_p = [-2, -2]^T$ is identified by four adaptive models $\Sigma_i(i = 1, 2, 3, 4)$. The initial values of the parameters $\theta_i(t_0)$ are chosen as $\theta_1(t_0) = [5, 5]^T$, $\theta_2(t_0) = [-5, 5]^T$, $\theta_3(t_0) = [-5, -5]^T$, and $\theta_4(t_0) = [5, -5]^T$, which are the vertices of a square $\mathcal{K}(t_0)$, within which θ_p lies. In Fig. 1(a), the evolution of the convex hull as a function of time is indicated, and it is seen that the parameter vector θ_p is contained in the convex hull at every instant. If θ_p is chosen as $[-2, -5]^T$, it lies on the boundary of $\mathcal{K}(t_0)$ at the initial time t_0. In this case it is seen from Fig. 1(b) that it continues to remain on the boundary for all $t \geq t_0$. It is worth noting that the convex hull $\mathcal{K}(t)$ is not necessarily a subset of $\mathcal{K}(t_0)$ for all $t \geq t_0$. This implies that we do not have a nested set of convex hulls for any arbitrary monotonic sequence $\{t_0, t_1, \ldots\}$.

The primary question of interest to us is to determine if faster methods of estimating θ_p exist which are stable and result in improved performance. Towards this end, we consider time-varying virtual models (later referred to as second level adaptive models) in which the coefficients β_i in (9) are not constants but vary with time. To understand the rationale for using time-varying virtual models, consider a finite number of primary models being used to identify a plant. It was shown in [10] that switching between the different models with a finite dwell time, and controlling the plant using the current model would also result in stability. Since all the elements of $\mathcal{K}(t)$ can be considered as stable time-invariant virtual models, it appears possible to generate time-varying virtual models which are stable and which may also result in faster convergence.



(a)

(b)

Fig. 1. Convex hull property of multiple adaptive models. (a) Interior case; (b) boundary case.

Since $\theta_p \in \mathcal{K}(t_0)$, it follows that $\theta_p = \sum_{i=1}^{N} \alpha_i \theta_i(t_0)$ for some constant values α_i. Further, since θ_p is a fixed point of $\mathcal{K}(t)$, it follows that $\theta_p = \sum_{i=1}^{N} \alpha_i \theta_i(t)$. This implies that $\alpha_i(i = 1, 2, \ldots, N)$ and hence θ_p can be determined by observing the evolution of $\theta_i(t)$. Or equivalently, the identification problem which involves the estimation of θ_p can be transformed into the problem of estimating α_i.

Demonstrating how α_i can be estimated, how it is used to determine the control input, proving the stability of the overall system, and providing arguments as to why this would result in better performance are the principal questions to be addressed, and, as stated in the introduction, are the main objectives of this paper. These are discussed in Sections II-G–II-H, respectively.

G. Second-Level Adaptation

Speed, accuracy, stability, and robustness are the features sought after in any efficient adaptive system. If the indirect approach is used, this depends upon the speed and accuracy with which the unknown plant parameter vector θ_p can be determined. In Section II-F it was shown that the plant parameter vector θ_p can be expressed as $\theta_p = \sum_{i=1}^{N} \alpha_i \theta_i(t_0) = \sum_{i=1}^{N} \alpha_i \theta_i(t), t \geq t_0$, for $\sum_{i=1}^{N} \alpha_i = 1$ and $\alpha_i \geq 0$. Subtracting θ_p from both sides, we have $\sum_{i=1}^{N} \alpha_i \tilde{\theta}_i(t) = 0$. Further, considering the identification error (8) of the primary models, and using linearity as well as the fact that the initial state errors are zero, it follows that $\sum_{i=1}^{N} \alpha_i e_i(t) = 0$. This can be expressed in matrix form as

$$[e_1(t), e_2(t), \ldots, e_N(t)]\,\alpha = E(t)\alpha = 0 \tag{11}$$

where $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_N]^T$ and the columns of the $(n \times N)$ matrix $E(t)$ are the identification errors $e_i(t)$ of the primary models. In Section II-F, it was also stated that the minimum value that can be chosen for N (so that the convex hull of $\{\theta_i(t_0)\}(i \in \Omega)$ contains $\mathcal{S}_\theta$) is $(n+1)$. In the following discussion, we use $N = n+1$ so that the vector α in (11) is unique. The determination of the unknown parameter vector θ_p is central to the adaptive control problem. From (11) it is seen that this is also equivalent to the determination of the vector α. Expressing $\alpha \in \mathbb{R}^{n+1}$ as $\alpha = [\bar{\alpha}^T, \alpha_{n+1}]^T$, where $\bar{\alpha} \in \mathbb{R}^n$ and $\bar{\alpha} = [\alpha_1, \alpha_2, \ldots, \alpha_n]$ it follows from the convexity condition that the scalar $\alpha_{n+1} = 1 - \sum_{i=1}^{n} \alpha_i$. This permits (11) to be written as

$$M(t)\bar{\alpha} = \ell(t). \tag{12}$$

Estimation of $\bar{\alpha}$: As in most classical adaptive control problems, $\bar{\alpha}$ can be computed algebraically using (12) or adaptively from a set of differential equations derived from it. In the latter case an estimation model is set up as

$$M(t)\hat{\bar{\alpha}}(t) = \hat{\ell}(t) \tag{13}$$

where $\hat{\bar{\alpha}}(t)$ is the estimate of $\bar{\alpha}$, and is obtained using the adaptive law

$$\dot{\hat{\bar{\alpha}}}(t) = -M^T(t)M(t)\hat{\bar{\alpha}}(t) + M^T(t)\ell(t). \tag{14}$$

This assures the boundedness of $\hat{\bar{\alpha}}(t)$ [1]. At every instant $\hat{\bar{\alpha}}(t)$ is used to compute the estimate $\hat{\theta}_p(t)$ of θ_p, and in turn used to determine the feedback control parameters.

Comment 2.7.1: Since first level adaptation involves the real identification models and the estimates $\theta_i(t)$, we refer to the above as second level adaptation.

In the following section it is shown that the use of time-varying coefficients $\hat{\alpha}_i(t)$ as described above will assure the stability of the overall system.

H. Stability of Control Using Time-Varying Virtual Models (Second-Level Adaptation)

From the discussions in the previous section, any convex combination of the primary models $\theta_i(t)(i \in \Omega)$ is a virtual model. Depending on whether the convex combination is time-invariant or time-varying, we will refer to the virtual models as time-invariant virtual models or time-varying virtual models, respectively. As shown earlier, a time-invariant virtual model has the same dynamical property as a primary model.

In Section II-D, it was stated, using results in [11], that switching between a finite number of primary models (with a finite dwell time) does not affect the stability of the overall system. It directly follows that switching between a finite number of time-invariant virtual models (with a finite dwell time) will also be stable. Since the convex hull $\mathcal{K}(t)$ is a set of uncountably many time-invariant virtual models, the question naturally arises whether switching within this set of time-invariant virtual models will also be stable. Furthermore, the second level adaptation used in the previous section, results in a time-varying virtual model (i.e., a time-varying convex combination of the primary models), and the results presented in this section demonstrate that models so generated satisfy the conditions needed to assure the stability of the overall system. This is stated as Theorem 2.

Theorem 2: If the assumptions in Theorem 1 are satisfied, and the control signal is generated algebraically based on the plant parameter estimate

$$\bar{\theta}(t) = \mathrm{Proj}_{\bar{\theta}(t)\in\mathcal{S}_\theta} \left\{ \sum_{i=1}^{N} \alpha_i(t)\theta_i(t) \right\} \tag{15}$$

for bounded piecewise differentiable $\alpha_i(t)$ which satisfy the condition $\sum_{i=1}^{N} \alpha_i(t) = 1$, then the overall system is asymptotically stable.

Before proceeding to present the proof of Theorem 2, the following comments concerning the identification models are relevant. In the description of the models given in (7) it is well known that A_m can be any stable matrix. In (7), they were chosen to be identical to the reference model. While proving the stability of the overall system with time-varying coefficients the models are modified (in the spirit of adaptive observer design) so that they are "more stable" than the reference model (i.e., their eigenvalues are to the left of those of A_m).

Proof: Let N identification models $\Sigma_i (i \in \Omega)$ be used to estimate the unknown parameter vector θ_p. The models are seen to be similar to those given in (7) but contain an additional term $\lambda e_i(t)$

$$\dot{x}_i(t) = A_m x_i(t) + [A_i(t) - A_m]\, x_p(t) + bu(t) - \lambda e_i(t) \tag{16}$$

where $\lambda > 0$.

As before, A_m is assumed to be in companion form. The error equations can be derived as

$$\begin{aligned} \tilde{\Sigma}_i: \quad \dot{e}_i(t) &= A_m e_i(t) + b\tilde{\theta}_i^T(t)x_p(t) - \lambda e_i(t) \\ &= (A_m - \lambda I)e_i(t) + b\tilde{\theta}_i^T(t)x_p(t). \end{aligned} \tag{17}$$

Comment 2.8.1: Comparing (8) and (17), it follows that the additional term $\lambda e_i(t)$ in (16) has no effect on the convex hull property derived earlier.

A constant vector k^* exists such that $A_p - bk^{*^T} = A_m$, i.e., $k^* = \theta_p - \theta_m$. Further, the control input is chosen as $u(t) = -\bar{k}^T(t)x_p(t) + r(t)$ where $\bar{k}(t)$ is chosen algebraically as $\bar{k}(t) = \bar{\theta}(t) - \theta_m$ where $\bar{\theta}(t)$ is an estimate of θ_p (as defined later). It is shown in what follows that if $\bar{\theta}(t) \in \mathcal{S}_\theta$ is any piecewise differentiable estimate of the form $\bar{\theta}(t) = \mathrm{Proj}_{\bar{\theta}(t)\in\mathcal{S}_\theta}\left\{\sum_{i=1}^{N}\alpha_i(t)\theta_i(t)\right\}$ with $\sum_{i=1}^{N}\alpha_i(t) = 1$ and $\alpha_i(t)$ bounded, where Proj is the projection operator, then the overall system is stable, and the control error $e_c(t)$ tends to zero.

From the equation of the plant, and the choice of the control input it follows (since $\bar{k}(t)$ is bounded) that

$$\left|\frac{x_p^T \dot{x}_p}{1+x_p^T x_p}\right| = \left|\frac{x_p^T(A_p - b\bar{k}^T)x_p + x_p^T br}{1+x_p^T x_p}\right| \leq \lambda_0. \tag{18}$$

for some constant λ_0 which can be computed. As in standard proofs of stability of adaptive systems [1], we define the variable $\xi_i \triangleq e_i/\sqrt{1+x_p^T x_p}$ and choose the Lyapunov function candidate $V_i(\xi_i, \tilde{\theta}_i)$ as $V_i(\xi_i, \tilde{\theta}_i) = \xi_i^T P \xi_i + \tilde{\theta}_i^T \tilde{\theta}_i$. It then follows that the time-derivative of V_i along any trajectory can be expressed as

$$\dot{V}_i(\xi_i, \tilde{\theta}_i) = -\frac{e_i^T Q e_i}{(1+x_p^T x_p)} + 2\left[\frac{e_i^T P b \tilde{\theta}_i^T x_p}{(1+x_p^T x_p)} + \tilde{\theta}_i^T \dot{\tilde{\theta}}_i\right] - \frac{2e_i^T P e_i}{(1+x_p^T x_p)}\left[\frac{x_p^T \dot{x}_p}{(1+x_p^T x_p)} + \lambda\right]. \tag{19}$$

By using the normalized adaptive law

$$\dot{\theta}_i = \dot{\tilde{\theta}}_i = -\frac{e_i^T P b x_p}{1+x_p^T x_p} \tag{20}$$

the second term of (19) can be made zero, and if $\lambda > \lambda_0$ the third term is nonpositive so that V_i is a Lyapunov function. From the above discussion it follows that $\xi_i = e_i/\sqrt{1+x_p^T x_p} \in \mathcal{L}^2 \cap \mathcal{L}^\infty$ and $\tilde{\theta}_i \in \mathcal{L}^\infty$ and $\dot{\tilde{\theta}}_i \in \mathcal{L}^2 \cap \mathcal{L}^\infty$. As in the classical proofs of stability, the proof in this case is also by contradiction. Since the plant is linear and the parameter vector $\bar{k}(t)$ is bounded, it follows that $\|x_p(t)\|$ can grow at most exponentially. By definition, since $e_i(t) = \xi_i(t)\sqrt{1+x_p^T(t)x_p(t)}$, $e_i(t) = \mathcal{O}(\sup_{\tau \leq t}\|x_p(\tau)\|)$.

Further, since $\dot{e}_i(t) = [A_m - \lambda I]e_i(t) + b\tilde{\theta}_i^T(t)x_p(t)$, it follows that $e_i(t) = o(\sup_{\tau \leq t}\|x_p(\tau)\|)$, $\tilde{\theta}_i^T x_p = o(\sup_{\tau \leq t}\|x_p(\tau)\|)$.Since the control error e_c is described by the equation $\dot{e}_c = A_m e_c + b\tilde{\theta}^T x_p$ where $\tilde{\theta}$ is a linear combination of $\tilde{\theta}_i (i = 1, 2, \ldots, N)$ with coefficients $\alpha_i(t)$ and the $\alpha_i(t)$ are bounded piecewise differentiable, it follows that $e_c(t) = x_m(t) - x_p(t) = o(\sup_{\tau \leq t}\|x_p(\tau)\|)$ which is a contradiction. Hence, $x_p(t)$ is bounded and the overall system is stable. Using standard arguments, it can then be shown that $\lim_{t\to\infty} e_c(t) = 0$.

Comment 2.8.2: From the foregoing discussion, any bounded piecewise differentiable convex combination of the N estimates results in a control parameter vector which assures stability. This decouples the stability and performance issues, and $\alpha_i(t)$ can be chosen primarily to improve performance.

Example 2: We conclude this section by considering simulation studies of the adaptive control of an unstable fourth order system using both conventional (first level) adaptation and second level adaptation. The unstable plant to be controlled is described by (1) where $n = 4$ and $\theta_p = [-4, 0, 5, 0]^T$ is unknown. The poles of the plant are located at -2, -1, 1, and 2, respectively. The reference model is stable with $\theta_m = [-24, -50, -35, -10]^T$ and has poles at -4, -3, -2, and -1. θ_p is known to belong to the set $\mathcal{S}_\theta$ where $\mathcal{S}_\theta = [-10, 10] \times [-10, 10] \times [-10, 10] \times [-10, 10] \in \mathbb{R}^4$. For convenience, the convex hull $\mathcal{K}(t_0)$ which contains $\mathcal{S}_\theta$ is chosen to be the same as $\mathcal{S}_\theta$, i.e., $\mathcal{K}(t_0) = \mathcal{S}_\theta$. Classical adaptive control using a single model, and multiple-model based second level adaptive control described in this section, are shown in Fig. 2(a) and (b), respectively. In both cases, the output of the plant and the reference model are included in the same plot and the output error as well as the control signal used are shown separately in both case. Although only plots of one run is given in this example, additional simulations have shown that the system cannot be controlled satisfactorily by a single identification model using the classical approach even with extensive tuning of adaptive gains. On the other hand, control

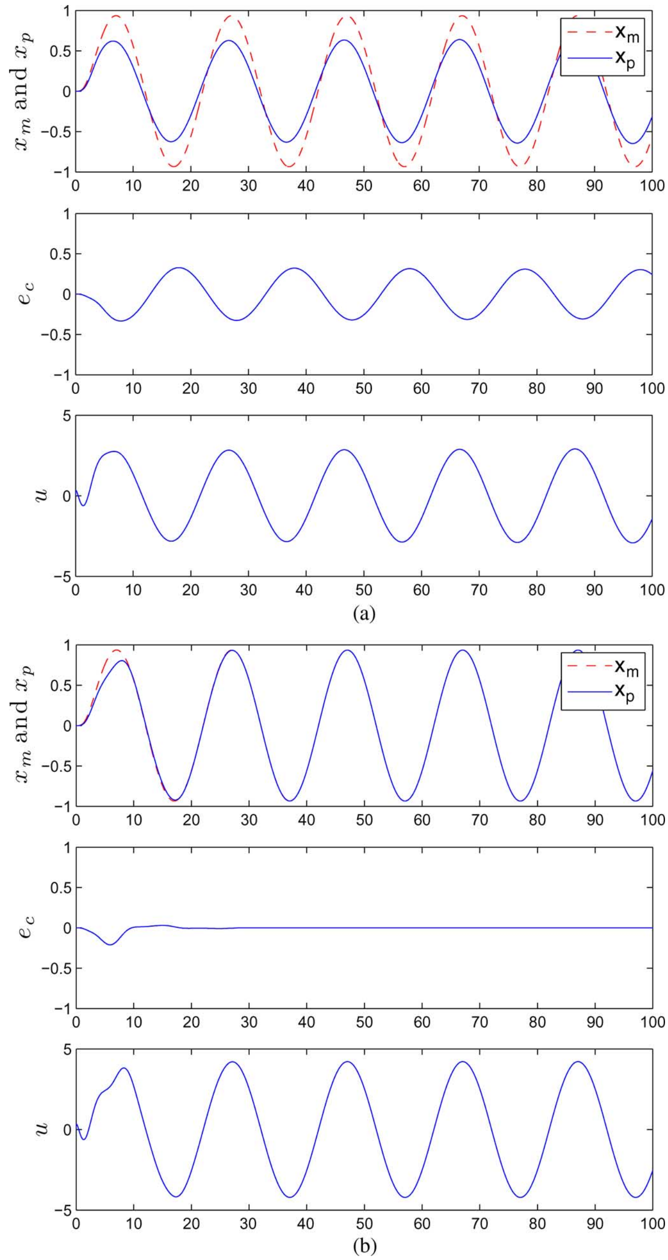


Fig. 2. Comparison of first-level and second-level adaptation for a fourth-order system. (a) Classical adaptive control using a single model; (b) adaptive control using second level adaptation.

using second level adaptation results in significantly improved performance.

I. Comparison of First Level and Second Level Adaptation

As described in earlier sections, first level adaptation corresponds to classical adaptive control. $N = n+1$ primary models are used to identify the unknown plant, and any one of them (i.e., the parameter estimates) can be used to control the plant in a stable fashion. In second level adaptation, the trajectories of the N estimates are used to identify the plant and control it. In Section II-H, why control using such a procedure results in global stability was described. In this section, we attempt to answer the question as to why second level adaptation should result in better performance.

We start our discussion with a simulation study, so that the reader is familiar with the precise nature of the improvement in performance. Following this, both qualitative and quantitative arguments are presented to account for the improved performance.

Example 3: As in Example 1, the unknown plant parameter vector θ_p was chosen to be $[-2,-2]^T$. Four primary models $\theta_i(t)(i=1,2,3,4)$ were used to estimate θ_p (note that 3 are theoretically adequate). Second level adaptation can be initialized at any point in $\mathcal{K}(t_0)$, and in the simulation studies four different initial values (i.e., $[5,5]^T$, $[0,0]^T$, $[4,4]^T$, and $[4,-4]^T$) were chosen, the first coinciding with a primary model, the second representing the center of the region of uncertainty, and the remaining two within $\mathcal{K}(t_0)$ but located in different regions. Each simulation corresponding to a different initial condition represents a typical realization of control for the overall system using second level adaptation. Four different initial conditions were chosen to emphasize that the improvement in performance is independent of the location of the initial condition (Fig. 4).

The trajectories of the parameter estimates of the primary models are indicated in solid lines, and those of second level adaptation in dotted lines in Fig. 4. The response of the overall system for first level and second level adaptation are shown for a typical case (initial condition $[4,4]^T$) in Fig. 3, where $x_m(t)$, $x_p(t)$, and the control error $e_c(t)$ are plotted as functions of time. Second-level adaptation is seen to be significantly better than that of the first level both in speed and accuracy. The evolution of parameters using the two methods is shown in Fig. 4(a) and (b) over 5 units of time. During this period second level estimates have already converged (almost) to θ_p, while first level estimates are seen to be quite poor. In Fig. 4(c) and (d), where trajectories are plotted over an interval of 50 time units, parameter estimates of both methods are observed to have converged to the true value θ_p. However, in all cases, second-level adaptation, in addition to being much faster, is also seen to be smoother and independent of the nature of the input to the plant.

From the extensive simulation studies carried out (of which typical samples were included in Example 3) it is quite evident that second level adaptation is, in general, far superior to first level adaptation, particularly when the initial parametric errors are large. In what follows we attempt to provide both qualitative and quantitative explanations for the empirically observed results.

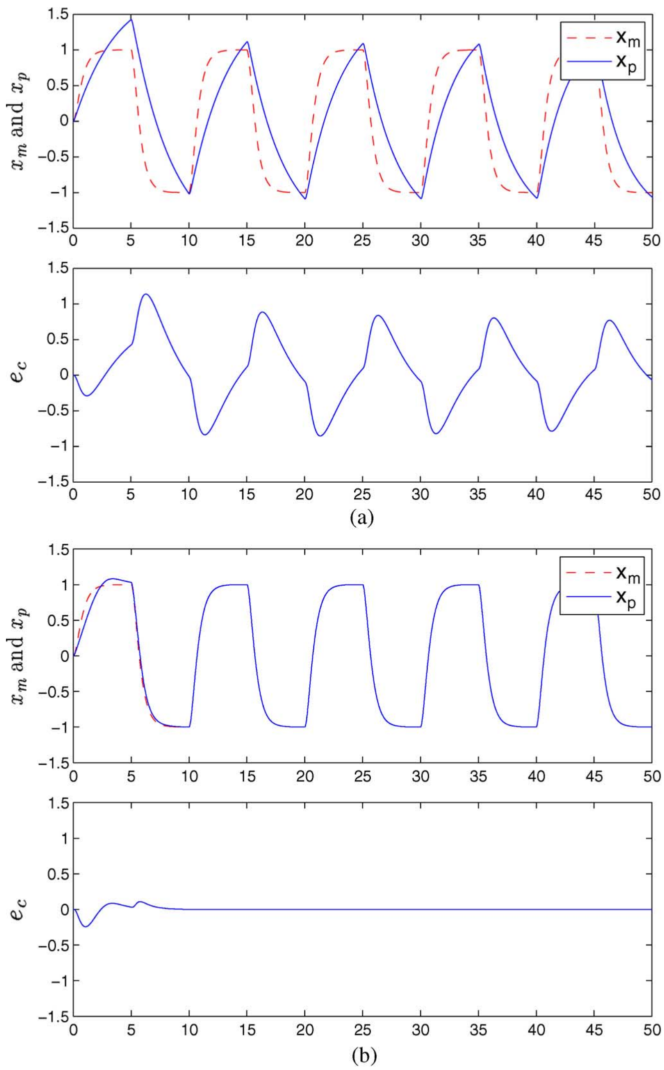


Fig. 3. Comparison of first-level and second-level adaptation for a second-order system. (a) Classical adaptive control using a single model; (b) adaptive control using second level adaptation.

A Qualitative Explanation: Before considering second-level adaptation in a general context, for ease of explanation, we first consider the following two coupled systems described by first order differential equations to convey the principal ideas involved:

$$\begin{aligned}\dot{x}_1 &= -\gamma x_1, \quad \gamma > 0\\ \dot{x}_2 &= -x_1 x_2, \quad x_1(0),\ x_2(0) > 0.\end{aligned} \tag{21}$$

Two different runs with $\gamma = 0.05$ and $\gamma = 2$ are shown in Fig. 5(a) and (b), respectively.

The convergence of $x_1(t)$ (the output of a primary model) to zero is governed by the value of γ. If $\gamma \ll 1$, convergence is slow and $x_1(t)$ is almost a constant for $t \approx 0$. In such a case $x_2(t)$ tends to zero rapidly. In the case when $\gamma \gg 1$, $x_1(t)$ tends to zero rapidly, which results in the slow convergence of $x_2(t)$ to zero. If our interest is in the output $y(t)$ where $y(t) = x_1(t)x_2(t)$, it converges to zero rapidly in both cases.

In the adaptive control problem $x_1(t)$ corresponds to the parametric identification error $\tilde{\theta}_i(t)$ of the primary models, $x_2(t)$ to the parametric error $\tilde{\alpha}_i(t)$ of the second-level adaptive model,

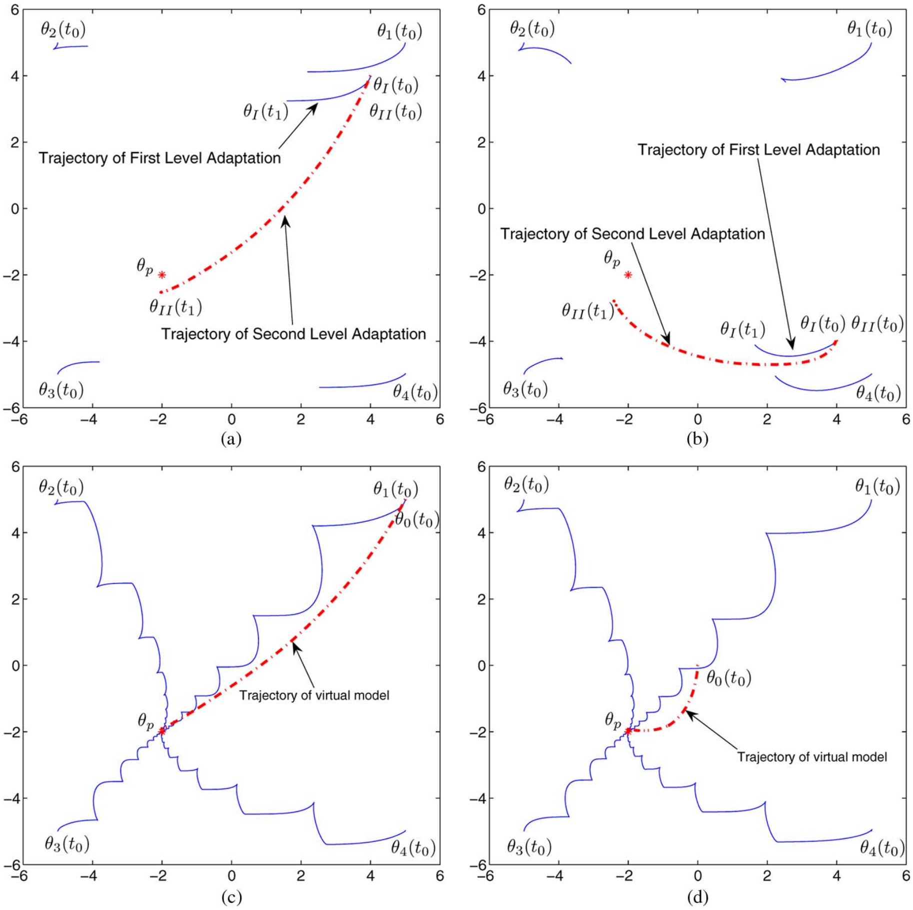


Fig. 4. Trajectories using first- and second-level adaptation. (a) Comparison of first- and second-level adaptation; (b) comparison of first- and second-level adaptation; (c) virtual model starting from the boundary; (d) virtual model starting from the interior.

so that the parametric identification error of the time-varying virtual model is $\tilde{\theta}(t) = \sum_{i=1}^{N} \alpha_i(t)\theta_i(t) - \sum_{i=1}^{N} \alpha_i\theta_i(t) = \sum_{i=1}^{N} \tilde{\alpha}_i(t)\theta_i(t) - \sum_{i=1}^{N} \tilde{\alpha}_i(t)\theta_p = \sum_{i=1}^{N} \tilde{\alpha}_i(t)\tilde{\theta}_i(t)$, i.e., $x_1(t)x_2(t)$ in the above example. In both cases, the parameter error of the time-varying virtual model is seen to converge rapidly to zero.

The ideas described above can be extended to the general adaptive control problem. As stated in the introduction, our interest in multiple models arises when parametric errors are large and the convergence of the primary models is slow. This implies that the matrices $M(t)$ in (12) tend to zero slowly, but from (14) describing the evolution of $\hat{\alpha}(t)$, this implies that $\hat{\alpha}(t)$ tends to its desired value rapidly. These are entirely consistent with the behaviors of the systems observed in Examples 2 and 3. The above comments naturally raise the question as to how second level adaptation would perform when the convergence of the primary models are rapid. Since the virtual model is merely a convex combination of the primary models, identification and control are rapid in this case also. Thus, whether the primary models are slow or fast, second-level adaptation is invariably fast.

More specifically, from (14) we have the following homogeneous error equation:

$$\dot{\tilde{\alpha}}(t) = -M^T(t)M(t)\tilde{\alpha}(t) \tag{22}$$

The qualitative justification of the rapid convergence of second level adaptation can now be explained as follows: if the convergence of the primary models are slow, so that the error $e_i(t)$ over an interval following the initial time t_0 is large, the minimum eigenvalue $\lambda_{[t_0,t)}$ of $M^T(t)M(t)$ is large. This accounts for the fast convergence of $\tilde{\alpha}(t)$ to zero. An alternative quantitative explanation can also be given in terms of the identification error.

A Quantitative Explanation: Recall that the right-hand side of (14) can be considered as the negative gradient of the function $1/2\|M(t)\hat{\alpha}(t) - \ell(t)\|^2$ with respect to $\hat{\alpha}$. It then follows that

$$\begin{aligned}\frac{1}{2}\left\|M(t)\hat{\alpha}(t) - \ell(t)\right\|^2 &= \frac{1}{2}\left\|\sum_{i=1}^{N}\hat{\alpha}_i(t)e_i(t)\right\|^2 \\ &\le \frac{1}{2}\left\|\sum_{i=1}^{N}\hat{\alpha}_i(t_0)e_i(t)\right\|^2 \\ &= \frac{1}{2}\left\|e_I(t)\right\|^2\end{aligned} \tag{23}$$

where $e_I(t)$ is the identification error of a model initialized with $\theta_I(t_0) = \sum_{i=1}^{N}\hat{\alpha}_i(t_0)\theta_i(t_0)$ and adjusted using the standard adaptive law (5). On the other hand, let a virtual model with identification error $e_{II}(t)$ be initialized with $\theta_{II}(t_0) = \sum_{i=1}^{N}\hat{\alpha}_i(t_0)\theta_i(t_0)$ and adjusted algebraically using the equation $\theta_{II}(t) = \sum_{i=1}^{N}\hat{\alpha}_i(t)\theta_i(t)$, it follows that $\dot{e}_{II}(t) = A_m e_{II}(t) + b\tilde{\theta}_{II}^T(t)x_p(t) = A_m e_{II}(t) + b\sum_{i=1}^{N}\hat{\alpha}_i(t)\tilde{\theta}_i^T(t)x_p(t)$. Since by (8) $\dot{e}_i(t) =$

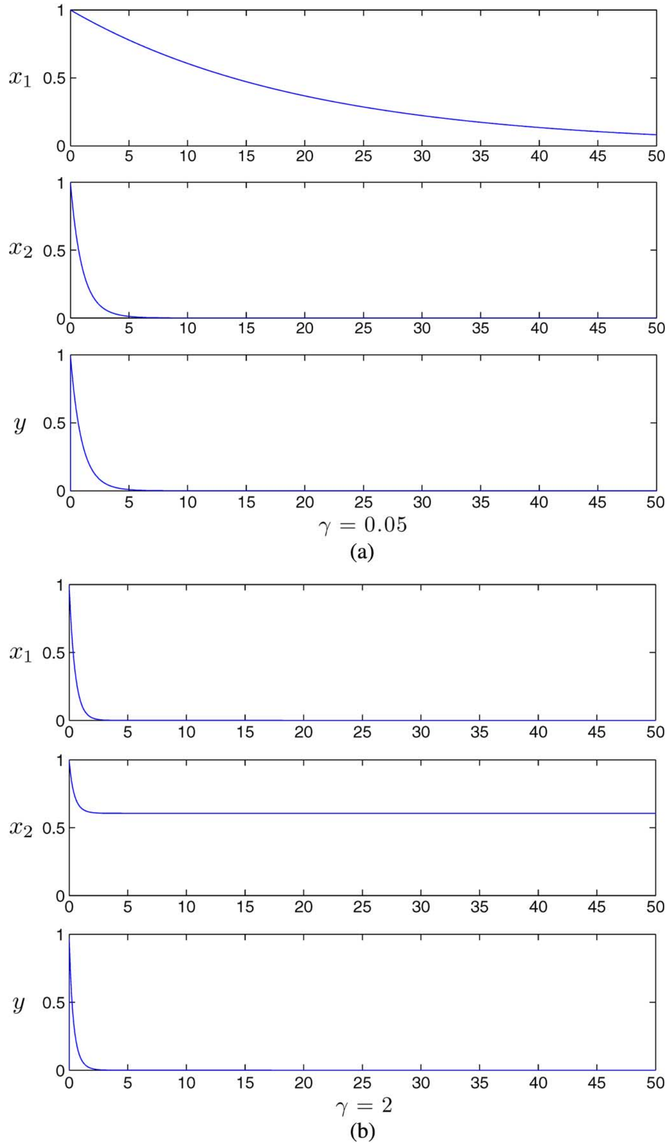


Fig. 5. Faster convergence of second-level adaptation. (a) Slow first level fast second level; (b) fast first level slow second level.

$A_m e_i(t) + b\tilde{\theta}_i^T(t)x_p(t)$, if we define $\bar{e}(t) \triangleq \sum_{i=1}^N \hat{\alpha}_i(t)e_i(t)$ and $\epsilon(t) \triangleq e_{II}(t) - \bar{e}(t)$, using standard arguments from adaptive control theory, based on the fact that $\dot{\hat{\alpha}} \in \mathcal{L}^2$, it follows that $\epsilon(t) = o(\sup_{\tau \le t} \|e_{II}(\tau)\|)$, or $e_{II}(t) \approx \bar{e}(t)$ for comparison purposes. Since, from (8) $1/2\|\bar{e}(t)\|^2 \le 1/2\|e_I(t)\|^2$, it follows that $1/2\|e_{II}(t)\|^2 \le 1/2\|e_I(t)\|^2$. This implies that if the $\mathcal{L}_2$ norm $J_i(t) = \int_{t_0}^t \|e_i(\tau)\|^2 d\tau$ of the identification error is chosen as the performance index, second-level adaptation results in improved performance.

III. Multiple Models for Adaptive Control (Input Output Accessible)

In the analysis carried out in the previous section, it was assumed that the state variables of the plant and models are accessible, so that the initial values of all the models could be set equal to that of the plant. (i.e., the initial values of the error equations are zero). This assumption simplified the analysis substantially. However, when only the input and the output of the plant are accessible, this assumption is no longer valid, and the initial conditions of the plant will play a role in the analysis. In this section, using a more general parameterization of the plant [1] it is shown that essentially the same results as those included in Section II can be derived. Both the stability of the overall system and the effect of nonzero initial conditions on the convex hull property derived earlier are examined. In particular, it is shown that all the conclusions drawn in Section II hold asymptotically even in this case.

A. MRAC Problem

The well known Model Reference Adaptive Control (MRAC) problem, solved in 1980 [1], may be stated as follows: a linear time-invariant single-input-single-output plant Σ_p has a transfer function $W_p(s) = k_p Z_p(s)/R_p(s) = k_p(s^{n-1} + b_2 s^{n-2} + \cdots + b_n)/s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_n$. The plant satisfies the standard assumptions [1], i.e., 1) the sign of k_p, 2) the order n, and 3) the relative degree n^* are known and $Z_p(s)$ is Hurwitz (or the plant is minimum phase). An SISO stable reference model Σ_m is described by the transfer function $W_m(s)$, so that its input–output relation is $W_m(s)r = y_m$, where $W_m(s) = k_m/R_m(s)$, and $R_m(s)$ is a Hurwitz polynomial of degree n^*. The control objective is to determine a bounded input $u(\cdot)$ to the plant such that all the signals in the system are bounded and $\lim_{t\to\infty} e_c(t) = \lim_{t\to\infty}(y_m(t) - y_p(t)) = 0$.

B. Identification Using a Single Adaptive Model

While this problem was solved in 1980 [1], the highlights of the results are presented briefly to provide a framework within which the approach based on multiple identification models can be discussed.

It is well known that parameterization plays a critical role in all adaptive identification and control problems. In the present case, using adaptive observer theory, an equivalent non-minimal representation of the plant is given as [1]

$$\begin{aligned} \dot{\omega}_1^* &= \Lambda\omega_1^* + lu \qquad \omega_1^* \in \mathbb{R}^{n-1} \\ \dot{\omega}_2^* &= \Lambda\omega_2^* + ly_p \qquad \omega_2^* \in \mathbb{R}^{n-1} \end{aligned} \tag{24}$$

and two $2n$ dimensional regression vectors ω^* and $\bar{\omega}^*$ are defined as $\omega^* \triangleq (u, \omega_1^{*T}, y_p, \omega_2^{*T})^T$, $\bar{\omega}^* \triangleq W_m(s)I_{2n\times 2n}\{\omega^*\}$ to represent both the plant and the reference model in a convenient form. If the plant is represented by the parameter vector p, where $p = (\beta_0^*, \beta_1^{*T}, \alpha_0^*, \alpha_1^{*T})^T$, y_p can be expressed as $y_p = p^T\bar{\omega}^*$. Given the input u and the output y_p of the plant, $\bar{\omega}^*$ can be generated using the state representation

$$\begin{aligned} \dot{x}_I^* &= A_I x_I^* + d_I y_p + b_I u \\ \bar{\omega}^* &= C_I x_I^* \end{aligned} \tag{25}$$

where $x_I^* \in \mathbb{R}^{2(n^*+n-1)}$, and all the parameters of the system are known. Thus, the parameterization reduces the plant to a convenient form so that the unknown parameter vector p can be easily estimated (error model 1 [1]). To accomplish this, using an identical model [i.e., (25)] with zero initial condition, a vector $\bar{\omega}$ is generated and the estimate of the plant output is expressed as $\hat{y}_p = \hat{p}^T\bar{\omega}$. The identification error can therefore be written as $e = \hat{y}_p - y_p = \tilde{p}^T\bar{\omega} + p^T\tilde{\bar{\omega}}$, $\tilde{p} = \hat{p} - p$, where $p^T\tilde{\bar{\omega}}(t) = p^T C_I e^{A_I(t-t_0)}\tilde{x}_I(t_0) \le ce^{-\gamma(t-t_0)}$ which tends to zero exponentially fast and c and γ are positive constants depending upon Λ. Since the choice of Λ is at the discretion of the designer, it follows that Λ can be chosen such that γ is as large as desired. Therefore, since $\bar{\omega}^*$ and $\bar{\omega}$ are derived from the same stable model with different initial conditions, they differ by exponentially decaying terms, and hence are equivalent. Hence, the adaptive law for adjusting $\hat{p}(t)$ is chosen as $\dot{\hat{p}} = -e\bar{\omega}/1 + \bar{\omega}^T\bar{\omega}$.

C. Identification Using Multiple Adaptive Models

The convenient parameterization defined earlier permits N different models to be set up with the same basic structure but with different initial values in the $2n$-dimensional parameter space. Defining the j^{th} model by the equation

$$\hat{y}_j = \hat{p}_j^T\bar{\omega}\hat{p}_j(t_0) = \hat{p}_{jt_0} \tag{26}$$

yields the adaptive laws

$$\dot{\tilde{p}}_j = \dot{\hat{p}}_j = -\frac{e_j\bar{\omega}}{1+\bar{\omega}^T\bar{\omega}} \quad e_j = \hat{y}_j - y_p \tag{27}$$

and using the Lyapunov function $V_j(\tilde{p}_j) = \tilde{p}_j^T\tilde{p}_j$ assures that $\hat{p}_j \in \mathcal{L}^\infty$. However, since $y_p(t)$ is not known to be bounded, the boundedness of $\bar{\omega}$, $\hat{y}_j$ and e_j cannot be concluded. This leads to the control problem. Before considering the latter and demonstrating the stability of the overall system, we describe the modification in the "convex hull property" of the parameter estimates, since $\tilde{p}_j$ have been demonstrated to be bounded.

D. Multiple Adaptive Models and the Convex Hull Property

As in Section II, we wish to establish a relation between the estimates $\hat{p}_j$ of the N identification models and the plant parameter vector $p \in \mathbb{R}^{2n}$. In view of the simple form of the identification model (i.e., $\hat{y}_j(t) = \hat{p}_j^T(t)\bar{\omega}(t)$) such an extension also follows relatively easily.

Since $e_j = \hat{y}_j - y_p = \tilde{p}_j^T\bar{\omega} + p^T\tilde{\bar{\omega}}$, it follows that

$$\dot{\tilde{p}}_j = -\frac{e_j\bar{\omega}}{1+\bar{\omega}^T\bar{\omega}} = -\frac{\bar{\omega}\bar{\omega}^T}{1+\bar{\omega}^T\bar{\omega}}\tilde{p}_j - \frac{\bar{\omega}p^T\tilde{\bar{\omega}}}{1+\bar{\omega}^T\bar{\omega}}. \tag{28}$$

Denoting the transition matrix of (28) as $\bar{\Phi}(t,t_0)$

$$\tilde{p}_j(t) = \bar{\Phi}(t,t_0)\tilde{p}_j(t_0) - \int_{t_0}^{t} \bar{\Phi}(t,\tau)\frac{\bar{\omega}(\tau)p^T\tilde{\bar{\omega}}(\tau)}{1+\bar{\omega}^T(\tau)\bar{\omega}(\tau)}d\tau. \tag{29}$$

If $\|\bar{\Phi}(t,\tau)\|$ is the induced norm of $\bar{\Phi}(t,\tau)$, it follows that $\|\bar{\Phi}(t,\tau)\|$ is bounded for all (t,τ). Let $c_{\bar{\Phi}} = \sup_{t,\tau}\|\bar{\Phi}(t,\tau)\|$, it then follows that the second term in (29) satisfies the inequality $\| - \int_{t_0}^t \bar{\Phi}(t,\tau)\bar{\omega}(\tau)p^T\tilde{\bar{\omega}}(\tau)/1 + \bar{\omega}^T(\tau)\bar{\omega}(\tau)d\tau\| \leq \int_{t_0}^t \|\bar{\Phi}(t,\tau)\|\|\bar{\omega}(\tau)/1+\bar{\omega}^T(\tau)\bar{\omega}(\tau)\|\|p^T\tilde{\bar{\omega}}(\tau)\|d\tau \leq c_{\bar{\Phi}}c/\gamma(1-e^{-\gamma(t-t_0)}) \leq c_{\bar{\Phi}}c/\gamma$. Given any positive constant error bound ϵ_0, Λ can be chosen such that $\gamma \geq c_{\bar{\Phi}}c/\epsilon_0$. Therefore,

$$\tilde{p}_j(t) = \bar{\Phi}(t,t_0)\tilde{p}_j(t_0) + \epsilon(t,t_0) \tag{30}$$

where $\|\epsilon(t,t_0)\| \leq \epsilon_0$ for all $t \geq t_0$. Theorem 1 can now be extended to the case when the plant state variables are not accessible.

Theorem 3: If N identification models described by (26) are initialized with the parameter values $\hat{p}_j(t_0)$, and the latter are adjusted according to the adaptive laws (27), and if the unknown plant parameter vector p can be expressed as $p = \sum_{j=1}^N \alpha_j\hat{p}_j(t_0)$ for nonnegative constants α_j satisfying $\sum_{j=1}^N \alpha_j = 1$, then given a positive constant ϵ_0, which represents the error bound, the parameters of the observer (i.e., Λ) can be chosen such that $p = \sum_{j=1}^N \alpha_j\hat{p}_j(t) + \epsilon(t,t_0)$, where $\|\epsilon(t,t_0)\| \leq \epsilon_0$.

Proof: The proof follows directly from (28), (29), and (30). ■

Comment 3.4.1: By Theorem 1 it is possible to express the best possible estimate of p as $p = \sum_{j=1}^N \alpha_j\hat{p}_j(t)$. With nonzero initial conditions, this equation is modified to $p = \sum_{j=1}^N \alpha_j\hat{p}_j(t) + \epsilon(t,t_0)$. (The additional term $\epsilon(t,t_0)$ represents the deviation of a virtual model located at p and having the same nonzero initial conditions).

Comment 3.4.2: If the plant and the observers are started from rest, the convex hull property is strictly valid since $\bar{\omega}^*(t) = \bar{\omega}(t)$ and $\tilde{\bar{\omega}}(t) \equiv 0$.

Example 4: The following problem was simulated to illustrate the property discussed in Theorem 3. The transfer function of the plant and the reference model are chosen as $W_p(s) = s + 1/s^2 - 5s + 6$, $W_m(s) = 1/s + 1$. Choosing $(\Lambda, l) = (-1, 1)$, it follows that $\omega_1 = 1/s + 1u$, $\omega_2 = 1/s + 1y_p$. In this parameterization, the plant parameter vector p which has to be identified can be computed as $p = (\beta_0^*, \beta_1^*, \alpha_0^*, \alpha_1^*)^T = (1, 0, 7, -12)^T$. For ease of explanation, we assume that it is known that $\beta_0^* = 1$ and $\beta_1^* = 0$. It is further assumed that bounds on α_0^* and α_1^* are known as $5 \leq \alpha_0^* \leq 10$ and $-15 \leq \alpha_1^* \leq -10$. Four models were chosen with initial values $\hat{p}_1(t_0) = (1,0,10,-10)^T$, $\hat{p}_2(t_0) = (1,0,5,-10)^T$, $\hat{p}_3(t_0) = (1,0,5,-15)^T$, $\hat{p}_4(t_0) = (1,0,10,-15)^T$. This permits the trajectories of the parameters of the four models, and the convex hull formed by them to be viewed in $\mathbb{R}^2$. In Fig. 6, p lies in the convex hull of $\hat{p}_i(t_0) (i = 1,2,3,4)$ but soon falls outside it due to the effect of initial conditions discussed earlier. However, as described in the text, the convex hull property will be asymptotically restored. As seen in the figure, the convex hull is almost a straight line which passes through p.

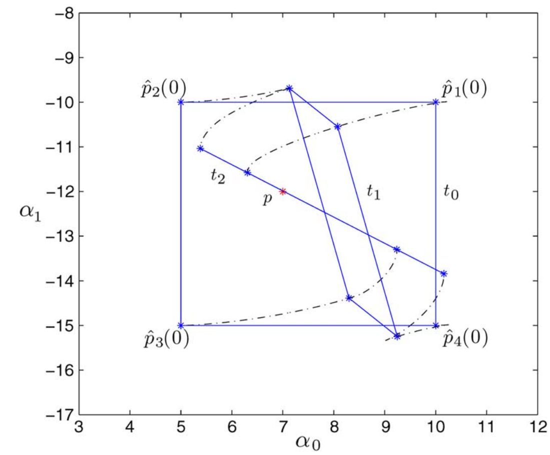


Fig. 6. Convex hull property of multiple adaptive models.

E. Control Using Time-Varying Convex Combination of Multiple Adaptive Models

The results given in Section II-H when a time-varying convex combination of $\hat{p}_j(t)$ is used to determine the virtual model can also be extended to the present case as indicated in this section. In contrast to Section II-H, where all state variables were assumed to be accessible, only inputs and outputs are assumed to be accessible. More specifically, stability and asymptotic convergence of $e_c(t)$ to zero are achieved when control is based on the following time-varying convex combination of $\hat{p}_j(t)$, i.e., $\bar{p}(t) = \sum_{j=1}^{N} \alpha_j(t)\hat{p}_j(t)$ where $\alpha_j(t)$ are bounded piecewise differentiable coefficients satisfying $\sum_{j=1}^{N} \alpha_j(t) = 1$.

If the system is unstable, it follows as in [1] that all signals can grow at most exponentially. Let $\omega_p = (\omega_1^T, y_p, \omega_2^T)^T$ be the "equivalent state" of the system, and let ω_p grow in an unbounded fashion. It follows that ω_1, ω_2, $\underline{\omega}$, ω, and $\bar{\omega}$ are all $\mathcal{O}(\sup_{\tau\leq t}\|\omega_p(\tau)\|)$, and $\dot{\omega}_p(t) = \mathcal{O}(\sup_{\tau\leq t}\|\omega_p(\tau)\|)$. Further, y_p grows at the same rate as ω_p. From Section II-B, it is seen that the error $\underline{\omega} - \underline{\omega}^*$ tends to zero exponentially fast. Therefore, for purposes of stability analysis, it can be assumed without loss of generality that $\underline{\omega} - \underline{\omega}^* \equiv 0$. It then follows that

$$\beta_0^*\tilde{u}_j = -\tilde{p}_j^T\omega + \tilde{\beta}_{0j}(u - u_j). \tag{31}$$

For convenience, if the high frequency gain k_p of the plant is assumed to be known, it follows that u_j, θ_j, and $\hat{p}_j$ are linearly related. This permits the control parameter vector to be computed as $\bar{\theta}$, so that $\bar{u} = \bar{\theta}^T\underline{\omega}$. (31) then becomes $\beta_0^*\tilde{u}_j = -\tilde{p}_j^T\omega$. The control error can be expressed as $e_c = W_m(s)\{-\beta_0^*\tilde{u}\} = W_m(s)\{-\beta_0^*\tilde{\theta}^T\underline{\omega}\} = W_m(s)\{\tilde{p}^T\omega\}$. From standard arguments [9], it follows that $\tilde{p}_j^T(t)\bar{\omega}(t) = o(\sup_{\tau\leq t}\|\omega_p(\tau)\|)$. Since $\dot{\tilde{p}}_j \in \mathcal{L}^2$, using the flipping lemma, that $W_m(s)\{\tilde{p}_j^T\omega\} = o(\sup_{\tau\leq t}\|\omega_p(\tau)\|)$. Further, since we have $\tilde{p}_j^T\omega = (\tilde{\beta}_{1j}^T, \tilde{\alpha}_{0j}, \tilde{\alpha}_{1j}^T)\omega_p$, $\tilde{p}_j^T\omega = o(\sup_{\tau\leq t}\|\omega_p(\tau)\|)$, or $\tilde{u}_j = o(\sup_{\tau\leq t}\|\omega_p(\tau)\|)$. Since $\bar{u} = \sum_{j=1}^{N}\alpha_j u_j$, $\tilde{u} = o(\sup_{\tau\leq t}\|\omega_p(\tau)\|)$. This results in a contradiction and therefore the overall system is stable. Furthermore, using the same argument as in Section II, it can be shown that $e_c(t)$ tends to zero.

IV. Comments on Robustness Issues and Simulation Study

We conclude the paper with a few brief comments concerning the robustness of the proposed scheme, and also include a simulation study to compare the latter with conventional adaptive control when the plant described in Example 4 has an additive bounded disturbance at the output.

The robustness of adaptive systems is a well investigated area with a vast literature, and the principal results have been collected and presented in a convenient form by Ioannou and Sun [3]. Many of the methods proposed by them (for a single model) can be suitably modified for use in second-level adaptation. For example if a bounded additive disturbance $d(t)$ is present at the output $y_p(t)$ of the plant, the adaptive law with σ-modification can be used. The multiple models are then described by the equations $\dot{\hat{p}}_j = -e_j\bar{\omega}/1 + \bar{\omega}^T\bar{\omega} - \sigma\hat{p}_j/1 + \bar{\omega}^T\bar{\omega}$, where σ is a suitably chosen constant parameter. Using the methods described in Section III it can be shown that the parametric error $\tilde{p}_j(t)$ satisfies the differential equation $\tilde{p}_j(t) = \bar{\Phi}(t, t_0)\tilde{p}_j(t_0) + \epsilon(t, t_0)$ where the bounded term $\epsilon(t, t_0)$ depends upon the disturbance. This implies that while the plant may not lie in the convex hull $\mathcal{K}(t)$ of the models, it lies in a neighborhood whose magnitude depends upon the disturbance. Similar results can also be

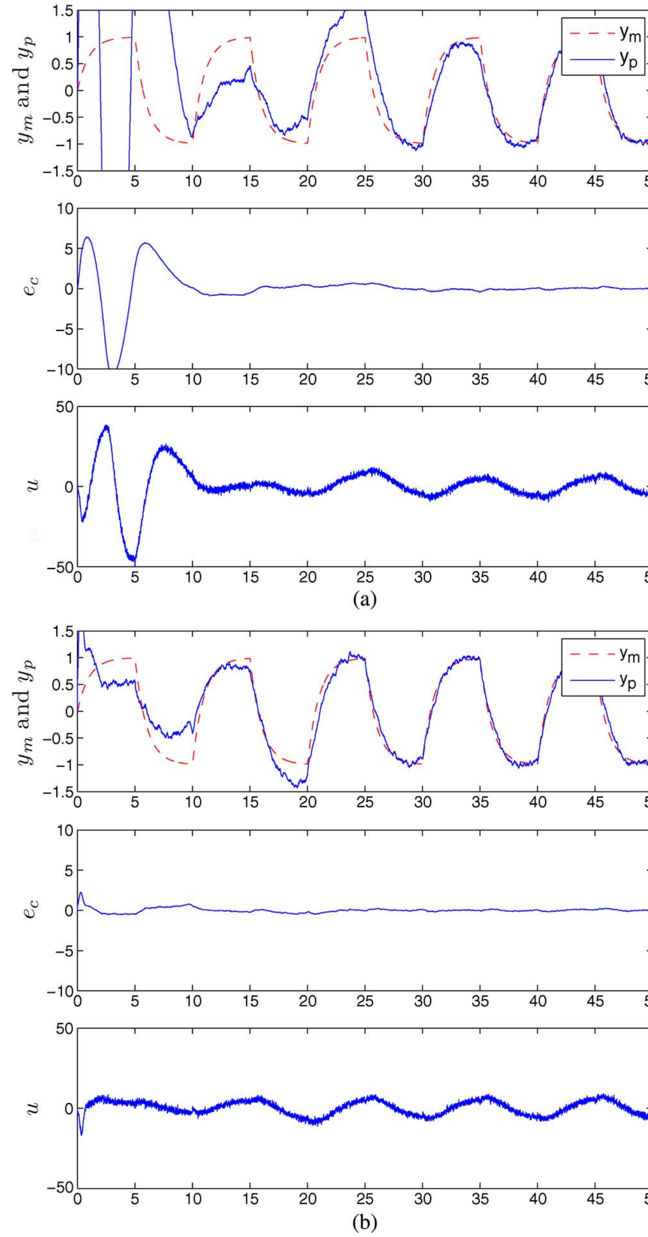


Fig. 7. System responses for example 6 using first- and second-level adaptation. (a) Classical adaptive control with a single model. (b) Second-level adaptation with four adaptive models.

derived using other methods in [3] which were proposed for different perturbations. The following simulation study indicates that the advantages enjoyed by the new method for the noise free case also carry over when perturbations are present.

Example 5: Let the unstable plant considered in Example 4 have additive noise $d(t)$ at the output with standard deviation $\delta = 0.1$. If the plant is parametrized as described earlier (i.e., $p = (\beta_0^*, \beta_1^*, \alpha_0^*, \alpha_1^*)^T = (1, 0, 7, -12)^T$), it is assumed that $\beta_0^* = 1$ is known and that β_1^*, α_0^*, and α_1^* are to be identified. Four models located initially at $(1, -5, 10, 5)^T$, $(1, -5, -10, -15)^T$, $(1, -5, 10, -15)^T$, and $(1, 5, 10, -15)^T$ are used to identify the unknown parameters. The responses obtained using adaptive control based on a single model are indicated in Fig. 7(a). Very large transient output errors are observed as expected, and the control input has a maximum amplitude of 50 during the initial interval of 5 units of time. In contrast to this, second level adaptation results in significantly faster convergence, and consequently substantially smaller transient with a relatively small input as shown in Fig. 7(b).

V. Conclusion

The paper deals with the adaptive control of an LTI plant with unknown parameters using multiple identification models and the following results were derived: 1) If the plant parameter vector θ_p lies in the convex hull of the parameters $\theta_i(t_0)(i \in \Omega)$ of the identification models, it also lies in the convex hull of $\theta_i(t)$ for $t \geq t_0$. 2) Every point in the convex hull of $\{\theta_i(t)\}(i \in \Omega)$ can be considered as a virtual model with the same convergence properties as the regular models. 3) A virtual model can also be adjusted using information derived from regular models. This gives rise to second level adaptation. 4) Second-level adaptation results in much faster and smoother convergence, and consequently improved performance. 5) All the results obtained in 1–4 are for the case when the state variables of the plant are accessible and the state errors $e_i(t_0)$ are zero. When initial conditions are present (when only input and output of the plant are accessible), the same results are found to hold asymptotically. 6) A general stability result was derived concerning second level adaptation with time-varying $\alpha_i(t)$.

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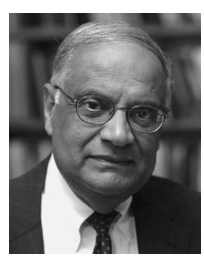

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