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Adaptive control using multiple models, switching and tuning

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SUMMARY

The past decade has witnessed a great deal of interest in both the theory and practice of adaptive control using multiple models, switching, and tuning. The general approach was introduced in the early 1990s to cope with large and rapidly varying parameters in control systems. During the following years, detailed mathematical analyses of special classes of systems were carried out. Considerable empirical evidence was also collected to demonstrate the practical viability of the methods proposed. This paper attempts to review critically the stability questions that arise in the study of such systems, describes recent extensions of the approach to non-linear adaptive control, and discuss briefly promising new areas of research, particularly related to the location of models. Copyright © 2003 John Wiley & Sons, Ltd.

KEY WORDS: multiple models; adaptive control; stability; learning

1. INTRODUCTION

Adaptive control systems have been investigated for over four decades. Since the beginning, for the sake of mathematical tractability, adaptive control theorists confined their attention to time-invariant systems with unknown parameters [1, 2]. The accepted philosophy was that if an adaptive system was fast and accurate when the plant parameters were constant but unknown, they would also prove satisfactory when the parameters varied with time, provided the latter occurred on a relatively slower time-scale. Based on these general principles, adaptive control was extensively studied and numerous globally stable and robust adaptive control algorithms were derived [3]. Extensive computer simulations of adaptive control algorithms have revealed that when there are large errors in the initial parameter estimates, the tracking error is quite often oscillatory with unacceptably large amplitudes during the transient phase. In the highly competitive industrial world, new classes of problems are arising where such variations in parameters are quite common. These are due to large variations in load in mechanical systems

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[4], failure of actuators in flight control systems [5, 6], and transition control in chemical systems [7]. It was to cope with such situations that the multiple model switching and tuning approach (MMST) was introduced in the early 1990s [8].

This paper is written with three objectives. The first is to review the stability results when the MMST approach is used to adaptively control an unknown deterministic linear plant, or a linear plant with a stochastic disturbance. The second is to discuss recent efforts to extend the methodology to nonlinear systems. Since the effectiveness of MMST depends critically on the choice of the models, the third objective is to examine how learning can be used to accomplish the latter efficiently.

2. GENERAL METHODOLOGY

In this section the motivation for the MMST approach, the basic concepts used throughout the paper, the structure of the overall system, and some of the choices that face the designer are briefly described.

2.1. Biological motivation

Every biological system is faced with a multiplicity of choices at any instant of time. Having a wide repertoire of behaviours suited to different situations, possessing the ability to recognize the specific situation that has arisen, and taking the appropriate action, result in rapid and efficient adaptation in time-varying environments. These features of biological system behaviour, involving learning and adaptation, are what we attempt to capture in the MMST method.

2.2. Multiple-model based adaptive control

The architecture for MMST based adaptive control originally proposed in Reference [8] and later developed in References [9, 10] is described in this section and is used throughout the paper. The structure of the control system is shown in Figure 1. The plant P to be controlled has an input u and an output y. A reference model provides a desired output y^*, and the objective is to make the control error $e_c = (y - y^*)$ tend to zero (or lie within specified bounds), for large values of time. N identification models $M_1, M_2, \ldots, M_N$ are used in parallel to estimate the parameters of the plant, and the outputs $\hat{y}_i$ $(i \in \Omega = \{1, 2, \ldots, N\})$ of the models are N estimates of the output y. The estimation error of the model M_j is $e_j = \hat{y}_j - y$. Corresponding to each model M_j there exists a parameterized controller C_j such that M_j together with C_j in the feedback path would behave like the reference model (or alternatively, achieve the desired objective). At every instant, based on a switching criterion, one of the model/controller (M_j, C_j) pairs is chosen. The output u_j of the controller is used to control the plant. Given prior information about the plant (i.e. linear, non-linear, stochastic, slowly time-varying, etc.), the design problem is to choose the models M_j and the controllers C_j together with the rules for switching between the controllers, and demonstrate that the overall system will be stable and improved performance will be obtained. For mathematical convenience, as well as for a precise definition of the control problem, we shall assume that the plant and all the identification models (unless otherwise stated) can be parameterized in the same fashion. If the unknown plant parameter is a vector p and the estimates of p given by the models are $\hat{p}_i$, we assume that p and $\hat{p}_i \in S \subset R^r$ $(i \in \Omega)$ where S is a compact set. For ease of exposition, we shall refer to p as the

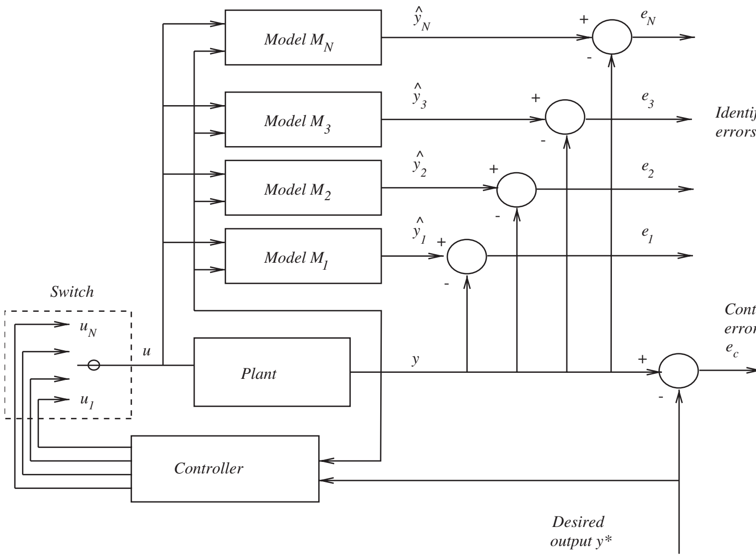


Figure 1. Adaptive control based on MMST.

plant and $\hat{p}_i$ as the model M_i. The general problem of MMST can also be considered as one of choosing $\hat{p}_i$ $(i \in \Omega)$ so that for any $p \in S$ the control objectives can be achieved, while the overall system is stable.

2.3. Models

The primary reason for using multiple models, as seen from Section 2.2, is to detect changes in the environment and initiate appropriate action. The models chosen can be either continuous-time or discrete-time, linear or non-linear, and deterministic or stochastic. Models with fixed parameters are referred to as fixed models, while those in which the parameters are continuously adjusted are referred to as adaptive models. Fixed models, in contrast to adaptive models, require very little computational overhead and are used (as seen in the following sections) mainly to provide better initial conditions for parametric adaptation. After extensive computation studies, it was decided [9, 10] that $(N-2)$ fixed models and two adaptive models should be used as a compromise between computational complexity and performance.

Many other reasons can also be given for using multiple models, besides detecting changes in the environment. For example, when all the relevant information needed to design a controller is not available (e.g. delays, bound on disturbance, etc.) multiple models can be used to estimate them. Another important reason is to combine the advantages of different controllers, to achieve both stability and improved performance [11]. Due to space limitations, this problem is not considered in this paper.

2.4. Tuning, switching, and switching and tuning

Tuning: Direct and indirect adaptive control are two distinct methods of controlling a plant with unknown parameters. In classical adaptive control theory, estimation of parameters (indirect control) and adjustment of control parameters (direct and indirect control) are carried out incrementally. Such adjustments are generally referred to as tuning.

Switching: When the plant parameter vector switches rapidly (or discontinuously) between two values p_1 and p_2, the adaptive method must be able to detect the change and switch to the appropriate controller to avoid catastrophic failure. The use of multiple (fixed) models permits the rapid selection of a fixed controller from a set of controllers, which can also be considered to be adaptive. Thus the MMST approach freed the adaptive control theorist from the traditional view that all adaptation has to be incremental.

Switching and Tuning: From the above discussion it is clear that switching is rapid but generally not sufficiently accurate. Tuning, on the other hand, is relatively slow, but is desirable for improving the performance of the system.

2.5. *The switching criterion*

When multiple models are used for adaptive control, the following are some of the important questions that have to be addressed:

(i) When should we switch from one model to another? To which model should we switch?
(ii) When is a switching scheme stable? Will switching stop after a finite time?
(iii) Will the switching scheme improve performance?

The switching criterion (or switching rule) which provides the answer to question (i) plays a crucial role in the design of MMST systems. Different performance indices can be chosen, based on the estimation error e_i, to determine which of the models best fits the plant at any instant and consequently should be used to control it. These may assume the following forms for continuous time systems and their discrete-time counterparts:

$$\begin{array}{lll}
\text{(i)} & J_i(t) = e_i^2(t), & J_i(k) = e_i^2(k) \\
\text{(ii)} & J_i(t) = \int_0^t e_i^2(\tau)\,\mathrm{d}\tau, & J_i(k) = \sum_0^{\tau=k} e_i^2(\tau) \\
\text{(iii)} & J_i(t) = \alpha e_i^2(t) + \beta \int_0^t e_i^2(\tau)\,\mathrm{d}\tau, & J_i(k) = \alpha e_i^2(k) + \beta \sum_0^{\tau=k} e_i^2(\tau) \\
\text{(iv)} & J_i(t) = \alpha e_i^2(t) + \beta \int_0^t e^{-\lambda(t-\tau)} e_i^2(\tau)\,\mathrm{d}\tau, & J_i(k) = \alpha e_i^2(k) + \beta \sum_0^{\tau=k} \rho^{k-\tau} e_i^2(\tau) \;\; \rho \in [0,1]
\end{array} \qquad (I)$$

Based on one of the criteria (i) – (iv) the model/controller pair (M_j, C_j) is used at any instant to control the plant. If $J_j(t) = \min_i\{J_i(t)\}$, the jth controller is used at that instant. The switching criterion depends upon the prior information assumed about the plant, and is chosen to ensure stability as well as to improve performance.

3. SWITCHING AND TUNING: EXAMPLE

To illustrate some of the concepts introduced in Section 2, we consider a simple example in which switching, tuning, as well as switching and tuning are used. A discrete-time plant is described by the scalar difference equation

$$y(k+1) = a(k)y(k) + b(k)u(k), \quad p(k) = [a(k), b(k)]^{\mathrm{T}}$$

where the plant parameter vector is piecewise constant and unknown. $p(k)$ is periodic with period 400 and has constant values shown in Table I over intervals of 100 units. The plant is stable in the first and third intervals and unstable in the second and fourth. The objective of the control is for the plant output $y(k)$ to track the output of a reference model $y_m(k+1) = r(k)$

Table I. Plant parameter values.

Plant	θ_1	θ_2
P_1	0.8	0.6
P_2	1.8	−0.4
P_3	−0.4	0.8
P_4	−1.1	−1.2

with a small error in the presence of discontinuous changes in plant parameters, when $r(k) = 0.5[\cos(2\pi k/150) + \sin(2\pi k/50)]$.

The responses of the system, when different control strategies are used, are shown in Figure 2(a)–2(e). In all cases, solid lines correspond to the desired response and dotted lines to the output of the plant. In the first simulation study, a single classical adaptive controller is used to control the system. The response shown in Figure 2(a) is clearly unstable. In the second simulation, the different values of plant parameters are assumed to be known, but the instants at which the switching takes place are unknown. Four fixed models with parameters identical to the known plant parameter values were used in the MMST approach, but only switching between fixed models was attempted. To ensure fast response, the instantaneous switching criterion $I(i)$, i.e. $J_i(t) = e_i^2(t)$ was used. Except for errors at one instant of time following switching, exact model following is achieved as seen in Figure 2(b). However, it is evident that exact asymptotic tracking cannot be achieved in the time-varying situation even with complete knowledge of discontinuous plant parameters.

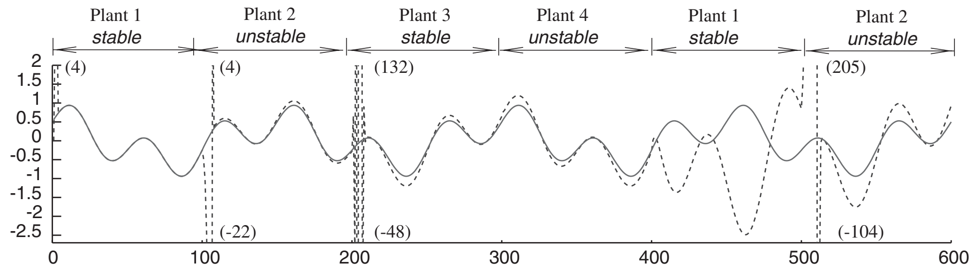
Figure 2(c) shows the response obtained by switching when the model parameters do not coincide with those of the plant but lie in their neighbourhood (i.e. plant parameters are not known exactly). Obviously, the performance deteriorates as the mismatch between plant and model increases.

In Figure 2(d), the switching criterion is changed to I(iv). As expected, the performance deteriorates rapidly as $\lambda \to 0$, since the model (and controller) switching lags increasingly behind that of the plant. In contrast to the above, when the plant is stable over all the intervals, the response using the switching criterion I(iv) was found to be satisfactory for appropriate choice of parameters α, β and λ. In Figure 2(e) the fixed models used in case c were replaced by adaptive models. The control parameters obtained prior to switching were retained and used in the following cycle to initiate adaptation. Only the response over the interval $3600 \leqslant k \leqslant 4000$ is shown to indicate the improvement in performance as compared to 2c (refer to Section 7).

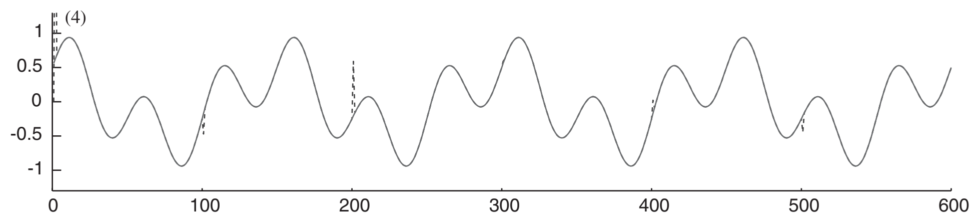
While the example is simple and contrived, and avoids many of the technical difficulties encountered in more complex systems, it nevertheless serves to underscore the close relationship between switching, tuning, stability, performance, and the role played by prior information in the choice of the switching criterion. Qualitatively, the same ideas also carry over to the stochastic and nonlinear systems considered later in this paper.

4. ADAPTIVE CONTROL OF LINEAR TIME SYSTEMS USING MMST

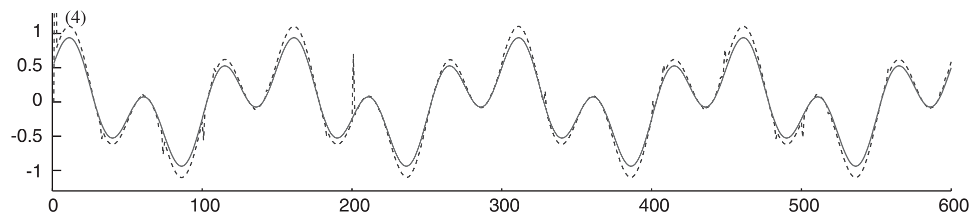
From the qualitative comments in Section 2, and the simulation results discussed in Section 3, it is evident that the MMST approach is a promising one for adaptively controlling dynamical



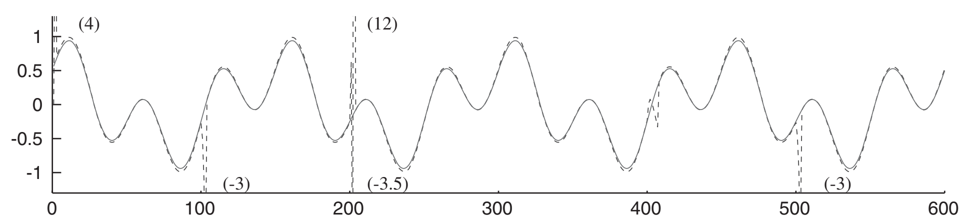
(a) Classical adaptive control using a single model - Unstable



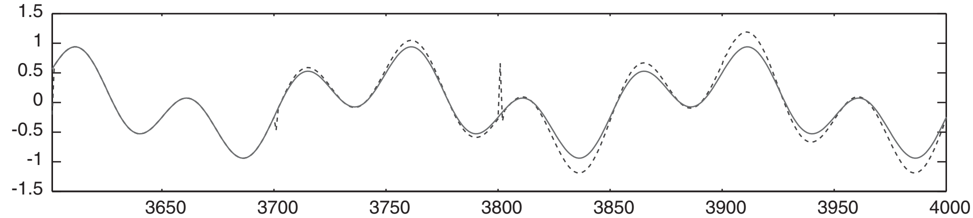
(b) Perfect model matching - Multiple fixed models



(c) Multiple fixed models in the neighborhood of the plant



(d) Switching based on criterion I(iv)



(e) Adaptive identification and control

Figure 2. Switching and tuning—Example.

systems with large variations in parameters. However, it is only after their stability properties are well understood that they can be designed with confidence. The stability analysis of MMST adaptive control for continuous-time and discrete-time systems in the noise free case, and discrete-time systems in the presence of stochastic disturbances have been investigated extensively in the past [10, 14, 15]. In this section, we attempt to review critically the stability proofs that have been given for the above cases. The following points are worth noting:

(i) MMST, by its very definition, is an indirect approach, (ii) Plant parameter identification plays a central role in it. The models used for identification may either have the same structure with different initial conditions, or have different structures. (iii) $(N-2)$ fixed models and 2 adaptive models are used in the adaptive process. Among the adaptive models, one is a free running model with arbitrary initial conditions, while the second is a re-initialized model with the same initial conditions as a fixed model. (iv) Control at any instant is based on certainty equivalence. (v) Switching among controllers is based on one of the switching criteria discussed in Section 2. (vi) The proof of stability of the overall system is closely related to the stability of adaptive control systems using a single model.

Since the stability theory of deterministic and stochastic linear-time invariant systems using a single model is well known [1, 2, 12] we shall attempt to focus only on those aspects of MMST based control which are significantly different from the former.

4.1. Stability of linear discrete-time systems (One model)

A plant P is described by the deterministic equation

$$P\colon\ y(k+d) = \sum_{i=0}^{n-1} a_i y(k-i) + \sum_{j=0}^{n-1} b_j u(k-j) = p^{\mathrm{T}}\phi(k) \tag{1}$$

where the constant parameter vector $p^{\mathrm{T}} = [a_0, a_1, \ldots, a_{n-1}, b_0, b_1, \ldots, b_{n-1}]$ is unknown and $\phi^{\mathrm{T}}(k) = [y(k), y(k-1), \ldots, y(k-n+1), u(k), u(k-1), \ldots, u(k-n+1)]$. The plant is assumed to be minimum phase and its relative degree d (delay) is assumed to be known. The objective of the control is to determine a bounded input u such $\lim_{k\to\infty}|y(k) - y^*(k)| = 0$, where $y^*(k)$ is a reference output. It is assumed that $y^*(k+d)$ is known at time k (or alternatively if $y^*(k+d) = r(k)$, that $r(k)$ is specified).

An estimation model is used to estimate the unknown parameters of P, described by

$$M\colon\ \hat{y}(k+d) = \sum_{i=0}^{n-1} \hat{a}_i(k) y(k-i) + \sum_{j=0}^{n-1} \hat{b}_j(k) u(k-j) = \hat{p}^{\mathrm{T}}(k)\phi(k) \tag{2}$$

where $\hat{p}^{\mathrm{T}}(k) = [\hat{a}_0(k), \hat{a}_1(k), \ldots, \hat{a}_{n-1}(k), \hat{b}_0(k), \hat{b}_1(k), \ldots, \hat{b}_{n-1}(k)]$ is the estimate of p.

The estimation error $e(k)$ is defined by $e(k) = [\hat{p}(k-d) - p]^{\mathrm{T}}\phi(k-d) = \tilde{p}^{\mathrm{T}}(k-d)\phi(k-d)$. The algorithm (one of many) for estimating $\hat{p}(k)$ has the form $\hat{p}(k) = \hat{p}(k-1) + P(k-d)\phi(k-d)e(k)$, where $P(k-d)$ is a suitably chosen matrix. The consequences of using the above algorithm can be summarized as follows [12]:

(i) $\|\hat{p}(k) - p\|^2 \leqslant k_1\|\hat{p}(0) - p\|^2, \quad k_1 > 0$

(ii) $\lim_{N\to\infty} \sum_{k=1}^{N} \frac{e^2(k)}{1 + k_2\phi^{\mathrm{T}}(k-d)\phi(k-d)} < \infty, \quad k_2 > 0 \quad \text{or} \quad (E)$

$\lim_{k\to\infty} \frac{e(k)}{[1 + k_2\phi^{\mathrm{T}}(k-d)\phi(k-d)]^{\frac{1}{2}}} = 0, \quad k_2 > 0$

(iii) $\lim_{k\to\infty} \|\hat{p}(k) - \hat{p}(k-m)\|^2 = 0$ for any finite m.

From E(i) it follows that the parameter estimates are bounded. From E(iii) it follows that the changes in $\hat{p}(k)$ over a finite interval tend to zero. Finally, from E(ii), the estimation error $e(k)$, even when it is unbounded, grows at a slower rate than the norm of the regression vector.

4.2. *Proof of stability*

At every instant k, after $\hat{p}(k)$ has been estimated, the control input $u(k)$ is chosen so that $y^*(k+d) = \hat{p}^{\mathrm{T}}(k)\phi(k)$ based on the certainty equivalence principle (i.e. using $\hat{p}(k)$ as the true parameter p). From the above choice of $u(k)$, it follows that

$$e_c(k) = e(k) + [\hat{p}(k-1) - \hat{p}(k-d)]^{\mathrm{T}}\phi(k-d) \tag{3}$$

and consequently from E(ii), $\lim_{k\to\infty} e_c(k)/[1 + k_2\phi^{\mathrm{T}}(k-d)\phi(k-d)]^{\frac{1}{2}} = 0$. If the plant is minimum phase, this results in a contradiction, if we assume that $e(k)$ and $\|\phi(k)\|$ grow in an unbounded fashion. Hence, it follows that (i) *the regression vector* $\phi(k)$ *is bounded*, and (ii) $\lim_{k\to\infty} e_c(k) = 0$.

4.3. *Stability results in MMST (Deterministic case)*

The results derived in Section 4.1 find direct application in MMST. Since in the latter, multiple models are used in place of the single model in Section 4.1, denoting the N estimates of the parameter vector by $\hat{p}_i$ $(i \in \Omega)$, and the corresponding estimation errors by $e_i(k)$, it follows that all of them satisfy the inequalities (E). If p_f is a constant vector, the estimation error e_f is given by $e_f(k) = \tilde{p}_f^{\mathrm{T}}\phi(k)$ where $\tilde{p}_f = p_f - p$ is a constant. We now consider several different ways in which multiple models can be used to control a linear plant with unknown constant parameters.

4.3.1. *Case* (i) : *All adaptive models*

If N models are used to estimate the plant parameters, and at time k the ith model $\hat{p}_i$ is chosen, the control input $u(k)$ is computed from the equation $\hat{p}_i^{\mathrm{T}}(k)\phi(k) = y^*(k+d)$. The control error at time k is given by $e_c(k) = e_i(k) + [\hat{p}_i(k-1) - \hat{p}_i(k-d)]^{\mathrm{T}}\phi(k-d)$. If the model at the next instant is chosen randomly as $\hat{p}_j$, e_c satisfies an equation similar to the latter with i replaced by j. The important fact to note is that the regression vector is the same for both of them. This process can be repeated at every instant with a model chosen randomly from $\hat{p}_i$, $(i \in \Omega)$. Hence, using the same arguments as in the single model case, it follows that $e_c(k)$ grows at a slower rate than $\|\phi(k)\|$, $\|\phi(k)\|$ is bounded, and that $\lim_{k\to\infty} e_c(k) = 0$.

Comment 1: Even when the models are chosen randomly, the tracking error tends to zero asymptotically. Hence, choosing multiple adaptive models for control decouples stability and performance, and the switching procedure can be based entirely on the latter.

Comment 2: Conventional adaptive control is not satisfactory when $\tilde{p}(0)$ is large. By initializing several models with different initial conditions we hope that one model is close to the plant and will result in a smaller e_i, and hence a smaller e_c. However, if no adaptive model is close to the plant, there may be no improvement in performance.

4.3.2. Case (ii) : *One adaptive model and one fixed model*

All adaptive models can be used only for a time invariant problem but not for a time-varying case. This is because all adaptive models may converge to the neighbourhood of the same point if a plant has a constant parameter vector for an extended period of time. To avoid such a possibility, fixed models can be used. These can be thought of as providing convenient initial conditions for adaptation. In this section we consider the case of two models, one adaptive and one fixed.

The two models result in error equations $e(k) = \tilde{p}(k)^{\mathrm{T}}\phi(k)$ and $e_f(k) = \tilde{p}_f^{\mathrm{T}}\phi(k)$ where, in general only $e(k)$ tends to zero. If $|e_f(0)| < |e(0)|$, using the criterion I(iii), the system will initially start with the fixed model, and the controller corresponding to it will be used. However, since $J_i(k)$ is bounded while $J_f(k)$ (of the fixed model) grows monotonically with k, the system will switch to the adaptive model in a finite time. The above arguments are still valid if an arbitrary number N of fixed models are used. The importance of using the criterion I(iv) rather than I(i) to assure convergence is clear from the above discussion.

Comment 3: When all the models used are adaptive and the plant is known to be time-invariant, the performance index used for the switching criterion is $J_i(k) = \sum_{\tau=1}^{k} e_i^2(\tau)$. Performance in a time-varying environment can be improved by using a finite window (i.e. $\sum_{k-T}^{k} e^2(\tau)$) or a forgetting factor as given by $J_i(k) = \sum_{\tau=1}^{k} \rho^{k-\tau} e_i^2(\tau)$, $0 < \rho \leqslant 1$. In this case $J_i(k) \to 0$ for all adaptive models, while $J_f(k) > 0$ for the fixed model.

Comment 4: No advantage is gained if all fixed models are far from the plant, relative to the adaptive model.

4.3.3. Case (iii) : $(N-2)$ *fixed models and 2 adaptive models*

In Section 4.3.2 if the fixed model is close to the plant compared to the adaptive model (as determined by the performance criterion J), the system switches to it and switches back to the adaptive model as described earlier. However, faster tracking can be obtained if a new adaptive model is initialized at the same value as the fixed model.

Since the plant is unknown, assuming that the fixed model lies close to the plant begs the question. However, what can be concluded is that if a large number of fixed models are used and one of them lies close to the plant, convergence may be rapid and performance satisfactory if a reinitialized adaptive model is used.

5. THE STOCHASTIC ADAPTIVE CONTROL PROBLEM

In recent years the results obtained for deterministic adaptive control using multiple models have been extended to the stochastic case [14, 15]. The linear system with unknown parameters has a stochastic input $v(k)$ (in addition to the control input $u(k)$), which is modeled as a realization of a random process driven by a white noise sequence $\{\omega(k)\}$ with zero conditional mean $(E(\omega(k)|k-1) = 0)$, finite variance $(E(\omega^2(k)|k-1) = \sigma^2)$, and which is mean square bounded $(\lim_{N\to\infty} \sup 1/N \sum_{k=1}^{N} \omega^2(k) < \infty \ a.s)$.

The plant is usually described by the ARMAX model representation

$$A(q^{-1})y(k) = B(q^{-1})u(k) + C(q^{-1})\omega(k) = p^{\mathrm{T}}(k)\psi(k) \tag{4}$$

where q^{-1} is a unit delay and A, B and C are polynomials in q^{-1} with unknown parameters, $p^{\mathrm{T}} = [a_0, a_1, \ldots, a_{n-1}, b_0, b_1, \ldots, b_{n-1}, c_0, c_1, \ldots, c_{n-1}]$ is the extended plant parameter vector, and $\psi^{\mathrm{T}}(k) = [y(k), y(k-1), \ldots, y(k-n+1), u(k), u(k-1), \ldots, u(k-n+1), \omega(k), \omega(k-1), \ldots, \omega(k-n+1)]$ is the plant regression vector defined in terms of the past plant input and output values, as well as of the past values of the stochastic sequence $\omega(k)$. The objective is to choose an appropriate model (or models) for the plant, estimate its (their) parameters, and use the estimate to choose a control input $u(k)$ which asymptotically converges to the minimum variance control. To achieve such a goal, the parameters corresponding to both plant and disturbance must be simultaneously identified, in such a way that the prediction error $e(k)$ becomes asymptotically white. The output of a predictor model $\hat{y}(k+1)$ having such properties will have the form

$$\hat{y}(k+1) = -\hat{A}^*(q^{-1})y(k) + \hat{B}^*(q^{-1})u(k-d) + \hat{C}^*(q^{-1})e(k) = \hat{p}^{\mathrm{T}}(k)\phi(k) \tag{5}$$

where the time-varying coefficients of $\hat{A}^*, \hat{B}^*$, and $\hat{C}^*$ are the estimates of suitably defined polynomials A^*, B^*, and C^*, and $\phi(k)$, the model regression vector, is defined in terms of the past a posteriori estimates of $y(k)$ as $\phi^{\mathrm{T}}(k) = [y(k), y(k-1), \ldots, y(k-n+1), u(k), u(k-1), \ldots, u(k-n+1), \hat{y}(k), \hat{y}(k-1), \ldots, \hat{y}(k-n+1)]$. Primarily due to space limitations, and also because much of the procedure follows along the same lines as in the deterministic case, we state below those features of the problem that are relevant to the stochastic case when the MMST approach is used:

(i) Even in the non-adaptive case where the parameters of the system are known, if the performance criterion $E[y(k+d) - y^*(k+d)]^2$ is to be minimized, it can be shown that the best that the controller can do is to minimize $E[y^0(k+d) - y^*(k+d)]^2$ where $y^0(k+d)$ is the optimal d-step ahead prediction of $y(k)$. This implies that a suitable parameterization must be found such that $y^0(k+d)$ can be expressed in terms of the quantities that are defined at or before k. If such a representation can be expressed as $y^0(k+d) = p_0^{\mathrm{T}}\phi(k)$, where p_0 is the unknown parameter vector, a procedure similar to that in the deterministic case can be used. In [12] it was shown that this leads to the solution of the minimum variance control problem. (ii) The regression vector $\phi(k)$ in Equation (5) contains in addition to the past values of the input and output, the past estimates $\hat{y}(k), \hat{y}(k-1), \ldots$. While this poses no problem in analysing the convergence of the stochastic adaptive controller when a single model is used, it makes the multiple model case more complex. This is because the regression vector is no longer the same for all the models, and hence arguments similar to those in the deterministic case cannot be used. The principal contribution of Reference [14] was the demonstration that when multiple models with the same estimation structure are used, all regression vectors are equivalent in a certain sense, and that the overall system exhibits a behaviour similar to that of the single model case, even when switching among controllers is allowed.

(iii) There is no single estimation procedure which can be considered to be the best for stochastic systems. Numerous methods such as the extended least squares (ELS), the output error with extended prediction model (OEEPM) and the recursive maximum likelihood (RML) have been proposed for the identification of linear systems in the presence of different noise characterizations. Quite often, it is not known a priori what the structure of the noise is, and what estimation model is best suited for a given situation. The advantages of using multiple models with different estimation structure (based on the above schemes) for adaptively controlling a linear stochastic system are discussed in Reference [15]. Simulation studies clearly demonstrate the improvement in performance that can be achieved in cases where uncertainty about the noise structure is present. In References [14, 15], for the different cases mentioned in (i) – (iii), it is shown that a minimum variance control can be achieved using the MMST approach. As in the deterministic case, the principal advantage of using multiple models is that the transient performance is significantly improved.

6. MMST OF NONLINEAR SYSTEMS

The adaptive control of the systems considered in Sections 4 and 5 is simplified by the fact that the unknown parameters occur linearly. Hence, if the methodology is to be extended to the non-linear domain it is natural to consider the class of systems where the latter condition is satisfied. Such systems have been studied in the literature, and globally stable controllers have been derived [16, 17]. Simulations studies and industrial applications [20] have also revealed that the transient error of these adaptive systems are significantly larger than in the linear case, so that the MMST approach is even more relevant for them.

There are two factors that have to be taken into account while extending the methodology to such problems. The first arises only in continuous-time adaptive systems, in which there is a possibility of infinite switching frequency. In such cases, the accepted convention is to assume a finite dwell time (i.e. if the controller is chosen at an instant, it is used over a minimum interval of time, called the dwell time). This poses stability questions in nonlinear systems that need to be addressed. The second concerns direct and indirect methods for adaptive control. The adaptive methods given in References [16, 17] are direct methods, while the MMST methodology is based on the indirect approach. This consequently implies that existing methods have to be suitably modified before the MMST approach can be used. Recently, the adaptive control of a limited class of nonlinear plants using MMST was proposed in Reference [18], and later extended to more general classes in [19, 20]. Due to space limitations we indicate briefly in this section only the principal questions that arise and discuss the high points of the proofs contained in Reference [19].

Statement of the problem:

A non-linear plant P is described by the differential equation

$$\begin{aligned} \dot{x}_1 &= x_2 + p^{\mathrm{T}}\phi_1(x_1) \\ \dot{x}_2 &= x_3 + p^{\mathrm{T}}\phi_2(x_1, x_2) \\ &\vdots \\ \dot{x}_{n-1} &= x_n + p^{\mathrm{T}}\phi_{n-1}(x_1, x_2, \ldots, x_{n-1}) \\ \dot{x}_n &= p^{\mathrm{T}}\phi_n(x_1, x_2, \ldots, x_n) + u \\ y &= x_1 \end{aligned} \tag{6}$$

The functions $\phi_i(.)$ are assumed smooth and to be known, while the parameter vector $p \in R^r$ is unknown. A bounded reference signal $y_r(t)$ and its $n-1$ derivatives are specified, and the objective is to determine a bounded control input $u(t)$ using MMST, such that $\lim_{t\to\infty}|y(t)-y_r(t)|=0$.

The above problem can be attempted in several stages of increasing complexity. In case (i) it is assumed that $\phi_1,\phi_2,\ldots,\phi_{n-1}$ are identically zero, and that the entire state $x(t)=[x_1(t),x_2(t),\ldots,x_n(t)]^{\mathrm{T}}$ of the plant is accessible. This is treated in [18]. In case (ii), $\phi_i(i\in[1,2,\ldots,n]=\bar{\Omega})$ are non-zero, but $x(t)$ is assumed to be accessible. Finally, in case (iii), ϕ_i are non-zero, but only the input and the output of the plant are known. Cases (ii) and (iii) are treated in [19].

Case (i): Since the state x of the system is accessible, u can be chosen as $u=k^{\mathrm{T}}x+v$, so that system (6) can be expressed as $\dot{x}=A_mx+b[p^{\mathrm{T}}\phi_n(x)+v]$, where A_m is asymptotically stable, A_m, b is in companion form and p is unknown. The reference model can be chosen as $\dot{x}_m=A_mx_m+b_r$ with $r(t)$ specified and $y_r(t)$ the first element x_{m1} of x_m. N models are set up, where the ith model is described by $\dot{\hat{x}}_i=A_m\hat{x}_i+b[\hat{p}_i^{\mathrm{T}}\phi_n(x)+v]$, $i\in\bar{\Omega}$. Defining the state estimation error as $e_i=\hat{x}_i-x$, we have the error equation $\dot{e}_i=A_me_i+b\tilde{p}_i^{\mathrm{T}}\phi_n(x)$, where $\tilde{p}_i=\hat{p}_i(t)-p$. Using standard adaptive control methods [1], if $\hat{p}$ is adjusted according to the adaptive law $\dot{\tilde{p}}_i=-e_i^{\mathrm{T}}Pb\phi_n(x)$ (where P is the solution of the Lyapunov equation $A_mP+PA_m=-Q<0$) it follows that $e_i\in\mathscr{L}_2\cap\mathscr{L}_\infty$, and $\tilde{p}_i\in\mathscr{L}_\infty$ for all $i\in\Omega$.

If at any instant the jth model is selected based on the switching criterion, the input $v(t)$ is chosen as $v(t)=-\hat{p}_j^{\mathrm{T}}\phi_n(x)+r(t)$. This results in $\dot{\hat{x}}_j=A_m\hat{x}_j+br$ describing $\hat{x}_j$ over the interval in which M_j is used. $\hat{x}_i(t)$, $i\neq j$ is described by $\dot{\hat{x}}_i=A_m\hat{x}_i+b[r+(\hat{p}_i-\hat{p}_j)^{\mathrm{T}}\phi_n(x)]$ over the same interval. Since $\hat{x}_j$ and x_m satisfy the same differential equation (when the jth model is used), $\mathrm{d}/\mathrm{d}t(\hat{x}_j-x_m)=A_m(\hat{x}_j-x_m)$ on that interval. This also can be expressed in terms of the estimation error e_j and the control error e_c (since $\hat{x}_j-x=\hat{x}_j-x+x_m-x_m=e_j-e_c$) as $\mathrm{d}/\mathrm{d}t$ $(e_j-e_c)=A_m(e_j-e_c)$. This together with the fact that $e_j\in\mathscr{L}_2\cap\mathscr{L}_\infty$, $j\in\Omega$ is used in [18] to show that the state x is bounded for all $t\in\Re$, and hence $\lim_{t\to\infty}e_c(t)=0$.

Case (ii): Since $\phi_i(i\in\bar{\Omega})$ are assumed to be nonzero, this problem is considerably more complex than case (i), However the principal ideas involved in proving the stability while using MMST are the same as in Case (i). Since the existence of a Lyapunov function is central to the latter, back stepping is used. As in case (i) the problem can be conveniently divided into two parts: an estimation part involving multiple models for estimating p, and a control part for generating control inputs corresponding to the different estimates.

Estimation: Assuming that the first equation in (6) is used to estimate p, N models are set up described by $\dot{\hat{x}}_{j1}=-\hat{x}_{j1}+x_1+x_2+\hat{p}_j(t)\phi_1(x_1)$ $j\in\Omega$, where $\hat{p}_j(t)$ is the jth estimate of p. Defining the error estimates as $e_{j1}=\hat{x}_{j1}-x_1$ and $\tilde{p}_j=\hat{p}_j-p$ we obtain the error equation $\dot{e}_{j1}=-e_{j1}+\tilde{p}_j\phi(x_1)$, which in turn leads to the adaptive law $\dot{\hat{p}}_j=-\dot{\tilde{p}}_j=e_{j1}\phi(x_1)$ for the parameter estimate. As in case (i) it follows that $e_{j1}\in\mathscr{L}_2\cap\mathscr{L}_\infty$ and $\tilde{p}_j\in\mathscr{L}_\infty$.

Control: While in principle the choice of the controller C_j is based on the model M_j over an interval as in case (*i*), the computation of the control input is substantially more involved and is based on backstepping. The structure of a typical controller C_j has the form [19] $\dot{z}_j=Az_j+bv_j$ $u_j=f[z_j,\hat{p}_j]$ $V_j=g[x,\hat{x}_{j1},\hat{p}_j]$, which consists of a linear dynamical part with static non-linear input and output maps. If this controller is used for all $t\in\Re$, it can be shown that the system has a quadratic Lyapunov function $V_j(z_j,\tilde{p}_j)$ with a time derivative $\dot{V}_j=-z_j^{\mathrm{T}}Q_jz_j\leqslant 0$, and hence results in asymptotic tracking of the desired output. When multiple models are used to control the system, a different model/controller pair is used over each interval, and the condition $V_j>0$, $\dot{V}_j\leqslant 0$ is satisfied over each interval on which the jth model is used. In Reference [19] it is shown that this fact along with $\tilde{x}_{j1}\in\mathscr{L}_2\cap\mathscr{L}_\infty$ $(j\in\Omega)$ results in all signals being bounded and the tracking error tending to zero.

Case (iii): If the state variables of the plant are not accessible, the problem is significantly more complex and an adaptive observer has to be used to estimate them. However, it is shown in Reference [19], that arguments very similar to those in case (ii) can be used to demonstrate that all signals will remain bounded and that the tracking error tends to zero asymptotically.

Comment 5: Due to the problem of finite escape time in nonlinear systems, fixed controllers are not used in MMST. Fixed models merely provide initial conditions for adaptation over every interval.

7. LEARNING AND ADAPTATION

In the preceding sections it was shown that MMST is a theoretically stable approach for adaptively controlling linear deterministic and stochastic systems, as well as a class of non-linear systems. However, its success in improving transient response in time-varying contexts depends upon one fundamental assumption, i.e. at least one model $\hat{p}_j$ lies in the neighbourhood of the plant $p(t)$ at any instant of time. In practice, time variations in the plant may occur in infinitely many ways, and it is not generally evident as to where the models should be located. A proper choice of the location of the models requires prior knowledge based on the past behaviour of the plant, i.e. it involves learning.

Work has been initiated by the first and third authors to determine a procedure for locating the adaptive models of a system operating in a random environment. In focusing on this problem they were influenced by earlier work by several authors, and in particular by Kawato *et al*. [21] on motor learning and control. The problem may be posed as follows: The unknown parameter vector $p(t)$ of the plant belongs to a finite set $P = [p_1, p_2, \dots, p_N]$, where p_i are unknown constant vectors, and assumes one of these values at random instants of time. If $p(t) = p_i$ at a given instant, we shall refer to it as the ith environment. If k_l and the ith environment, which exists over the interval $[k_l, k_{l+1}]$ are known (note that i and not p_i is assumed to be known), the identification problem becomes a relatively straightforward one. In such a case, N models can be used to identify the N values p_i, $i \in \Omega$ with only one model M_i being updated at any instant of time (knowing that the ith environment exists at that instant). The problem becomes truly difficult when neither k_l nor the class i to which the plant belongs are known at any instant. It is then not evident as to which of the N models is to be updated. Our objective, following Reference [21], is to adapt N models simultaneously so that every model converges asymptotically to a different element of the set P, and the controlled system is stable.

Statement of the problem:
Let a linear time-varying system be given by $y(k+1) = p^{\mathrm{T}}(k)\phi(k)$, where $p(k), \phi(k) \in \Re^r$ and $\phi(k)$ is the regression vector defined in Section 4. $p(k) \in P$ is an unknown time-varying vector. N estimation models are defined as

$$\hat{p}_j(k+1) = \hat{p}_j(k) + \eta_j(k)\phi(k)e_j(k+1), \quad e_j(k) = \hat{y}_j(k) - y(k)$$

and $\eta_j(k)$ is a function of all the estimation errors $e_i(k)$ $(i \in \Omega)$. The important point to note is that all the models are updated at every instant but with different step sizes. The N models compete, and those that have smaller estimation errors have larger step sizes. The convergence of the overall scheme is found to be very sensitive to the choice of $\eta_j(k)$. For some choices, all the models converge to the same value of the plant while for others none of the models converge. A choice of $\eta_j(k)$ which has proved successful in simulation studies is

$$\eta_j(k) = \eta_0 \frac{1/J_j(k)}{\sum_{i=1}^{N} 1/J_i(k)} \tag{8}$$

where $\eta_0 \in (0, 1]$ and $J_j(k)$ is a performance index corresponding to the jth model, as described in Section 2.

For static systems, theoretical results have been obtained when $J_i(k)$ is chosen to be $e_i^2(k)$. Assuming that over a finite interval T, $p(t)$ assumes all values in P, the objective is to choose $\eta_j(k)$ so that asymptotically each vector $\hat{p}_j$ converges to an element of P and that no two vectors converge to the same element.

Both static and dynamic systems have been studied, and considerable empirical data has been collected. As mentioned above, some theoretical results have also been obtained in the static case. Due to space limitations, these results that are contained in Reference [22] are not included here. However some qualitative statements concerning the convergence observed in simulation studies are listed below:

(i) If the number of models used is N (equal to the number of distinct values that the plant $p(t)$ can assume), each model M_i converges to one element of P, and no two models converge to the same element.

(ii) If the number of models used is $L<N$, the models do not converge. If $N<L$, N models converge to the set P and $L-N$ evolve to arbitrary values in parameter space.

(iii) For convergence, the plant must assume each of the values $p_i \in P$ at least once in a finite amount of time. Periodic switching between the elements is a typical example. We refer to this as a *persistently exciting environment*.

From the foregoing comments it is clear that both learning and adaptation are involved in the above procedure. We refer to it as Simultaneous Identification and Control (SIC). We note that the performance of a system based on SIC improves asymptotically. While the MMST method (with $N-2$ fixed models) performs satisfactorily when the fixed model values are close to p_i, it performs poorly if the initial parameter errors are large. In contrast to this, SIC results in accurate tracking after a brief transient period.

8. COMMENTS AND CONCLUSIONS

Adaptive control theorists are increasingly being called upon to design controllers that can perform satisfactorily in rapidly time-varying environments in which parameter variations can be significant. The MMST methodology presented in this paper is particularly suited to cope with such problems.

As in classical adaptive control theory, stability and improvement in performance of the overall system are the two features that are of principal interest to the control theorist. While stability depends upon the controller structure, the prior information assumed concerning the plant, and the switching criterion, overall performance to a large extent is determined by the proximity of one of the models to the plant, even as the latter varies with time. The paper reviewed some of the theoretical results that are available and examined the stability issues involved for linear deterministic and stochastic adaptive control assuming that the plant is time-invariant but unknown.

Recently the above results have been extended to a class of non-linear plants, and in Section 6 the highlights of the proofs of this important problem were presented. The MMST methodology for adaptively controlling a plant with large and/or rapidly varying parameters is still in its initial stages and much work remains to be done, both to increase its scope of applications as well as to obtain better performance in practical applications. Some of these areas are listed below:

(i) How the switching criterion should be modified to assure robustness when multiple models are used, needs to be investigated.

(ii) In Sections 4–6 it was tacitly assumed that the plant (whether linear or nonlinear) is minimum phase (or has asymptotically stable zero dynamics). However, if the objective is merely set point regulation, the problem can be formulated so that this is no longer required. A comparison of the two situations would prove useful.

(iii) In all the problems treated in Sections 4–6 the choice of the control input $u(k)$ was chosen to minimize the error $|y^* - \hat{y}(k)|$ at one instant of time. Empirical studies have shown that adaptive control results in much better performance if optimization is carried out over a finite-time horizon. However, the stability of even conventional adaptive controllers based on such a strategy has not been established thus far. Proving the latter takes on a greater significance in the MMST context.

(iv) The number of models N proposed in Sections 4 can be arbitrary large. However, practical considerations dictate that this number be kept as small as possible, compatible with the performance that is desired. In this context, the process of relocation and/or deletion of models, even as the system is in operation, without affecting either stability or performance needs further investigation.

(v) A problem of great importance and interest was posed in Section 6, in which learning and adaptation have to be combined in a stable fashion. This is currently an active area of research [21, 22] in which numerous open problems exist. Results obtained in recent months by the authors will appear elsewhere in the adaptive control literature.

In conclusion, the many directions in which the MMST methodology can be extended, and the variety of potential practical applications, make this an exciting field of research.

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