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Feedback error learning and nonlinear adaptive control

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Abstract

In this paper, we present our theoretical investigations of the technique of feedback error learning (FEL) from the viewpoint of adaptive control. We first discuss the relationship between FEL and nonlinear adaptive control with adaptive feedback linearization, and show that FEL can be interpreted as a form of nonlinear adaptive control. Second, we present a Lyapunov analysis suggesting that the condition of strictly positive realness (SPR) associated with the tracking error dynamics is a sufficient condition for asymptotic stability of the closed-loop dynamics. Specifically, for a class of second order SISO systems, we show that this condition reduces to $K_D^2 > K_P$, where K_P and K_D are positive position and velocity feedback gains, respectively. Moreover, we provide a 'passivity'-based stability analysis which suggests that SPR of the tracking error dynamics is a *necessary and sufficient* condition for asymptotic hyperstability. Thus, the condition $K_D^2 > K_P$ mentioned above is not only a sufficient but also necessary condition to guarantee asymptotic hyperstability of FEL, i.e. the tracking error is bounded and asymptotically converges to zero. As a further point, we explore the adaptive control and FEL framework for feedforward control formulations, and derive an additional sufficient condition for asymptotic stability in the sense of Lyapunov. Finally, we present numerical simulations to illustrate the stability properties of FEL obtained from our mathematical analysis.

Keywords: Feedback error learning; Adaptive control; Feedback and feedforward control; Strictly positive realness; Lyapunov stability; Passivity

1. Introduction

This paper presents a reformulation and formal stability analysis of the *feedback error learning* (FEL) scheme (Gomi & Kawato, 1993; Kawato, 1987, 1990) for a class of nonlinear systems from a viewpoint of the adaptive control theory. Originally, FEL was proposed from a biological perspective to establish a computational model of the cerebellum for learning motor control with internal models in the central nervous system (CNS) (Kawato, 1987). The research presented here is inspired by our insight into the close relationship between FEL and adaptive control algorithms which we gained during our recent development of a new adaptive control framework with advanced statistical learning (Nakanishi, Farrell, & Schaal, 2002, 2004).

From a control theoretic viewpoint, FEL can be conceived of as an adaptive control technique (Doya, Kimura, & Miyamura, 2001; Miyamura & Kimura, 2002). Stability analyses of FEL for a class of *linear* systems and a two-link planar robot arm in a horizontal plane are presented by Miyamura and Kimura (2002) and Ushida and Kimura (2002), respectively. However, the plant dynamics considered in Miyamura and Kimura (2002) are confined to a restricted class of linear systems (stable and stably invertible),[2] and these studies do not address practical issues, e.g. as to how to select feedback gains to ensure the stability in FEL. In this paper, we present a more general treatment of the formulation and stability properties of FEL for a class of nonlinear systems. Fig. 1 depicts the block diagram of the FEL scheme which was

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[2] An extension to a non-invertible case is also discussed by Miyamura and Kimura (2002).

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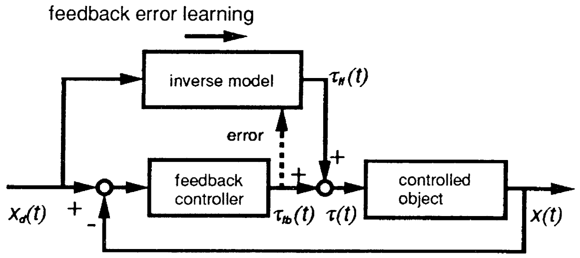

Fig. 1. Feedback error learning scheme originally proposed by Kawato (1990). (This diagram is taken from Kawato, 1990).

originally proposed for inverse model learning with an adaptive feedforward component. This paper considers two different formulations—one with an *adaptive state feedback* controller (see Fig. 2) and one with an *adaptive feedforward* controller (see Fig. 3). In the adaptive feedback formulation, the actual state is used to compute basis functions of a function approximator for parameter update and cancellation of nonlinearities. On the other hand, in the adaptive feedforward formulation, the desired state is used instead of the actual state. We first consider the formulation with the adaptive state feedback controller in Fig. 2 since it allows us to directly apply several theoretical results of the adaptive control framework (Choi & Farrell, 2000; Nakanishi et al., 2002, 2004; Sanner & Slotine, 1992). Subsequently, we investigate the formulation with the adaptive feedforward component in Fig. 3, which requires some additional treatment in the stability analysis. The adaptive feedforward controller, as developed in this paper, can be conceived of as a special case of the original FEL inverse model learning formulation, where the inverse model has a simplified parametrization. Note that a particular formulation of FEL was discussed by Gomi and Kawato (1992) as a computational model of the cerebellum for adaptive feedback control such as the vestibulo-ocular reflex (VOR) and opto-kinetic response (OKR). It can be considered as a combination of the feedback and feedforward formulations, where the desired state, its time derivative and the actual state are used as an input to the adaptive controller at the same time, and the adaptive controller will acquire both the inverse dynamics of the plant and a nonlinear PD feedback controller. The FEL formulations discussed in this paper slightly differ from this original work (Gomi & Kawato, 1992) in that the adaptive part will only learn unknown nonlinearities in the plant dynamics, and we treat the adaptive feedback and feedforward formulations separately.

This paper is organized as follows: Section 2 presents the structure of the control system and function approximation of unknown nonlinearities in the plant dynamics as considered in this paper. In Section 3, we discuss the adaptive feedback formulation. In Section 3.1, we review the nonlinear adaptive control framework (Choi & Farrell, 2000; Sanner & Slotine, 1992; Slotine & Li, 1991). In Sections 3.2 and 3.3, we discuss the relationship between nonlinear adaptive control and FEL in the adaptive state feedback formulation. Section 3.4 provides a Lyapunov stability analysis of adaptive control and FEL. In Section 3.5, we provide a sufficient condition on the choice of feedback gains to guarantee the stability of learning based on the Lyapunov stability analysis, which is associated with strictly positive realness (SPR) of the tracking error dynamics. Section 3.6 presents a 'passivity'-based stability analysis, which suggests that SPR of the tracking error dynamics is not only a sufficient but also a necessary condition for asymptotic hyperstability. To the best of our knowledge, no other previous work has explicitly addressed *necessary* conditions for the stability of nonlinear adaptive control. This missing treatment is largely due to the application of Lyapunov theory for stability analyses which can generally show only sufficiency at best. Section 4 considers the adaptive feedforward formulation. We derive an additional sufficient condition to guarantee the stability of learning, which is associated with the characteristics of nonlinearities of the plant dynamics. Finally, in Section 5, we present numerical examples to illustrate the theoretical stability properties of FEL.

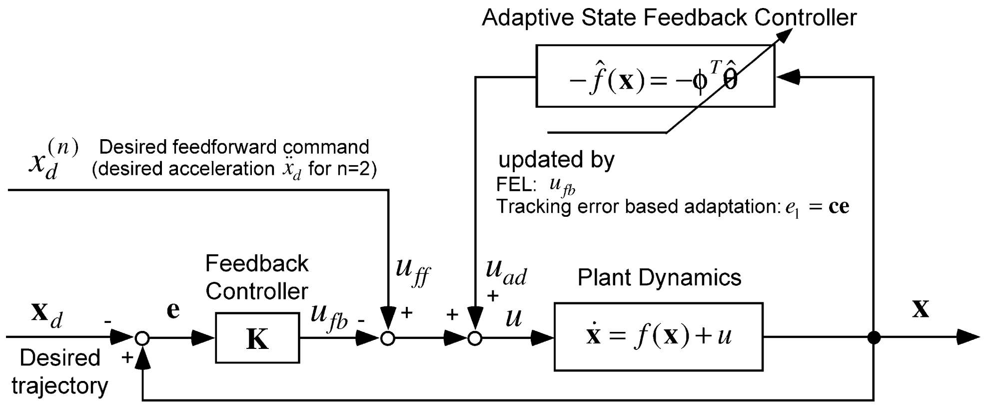

Fig. 2. Nonlinear adaptive control and FEL with an *adaptive state feedback* controller for a class of nth order nonlinear SISO systems.

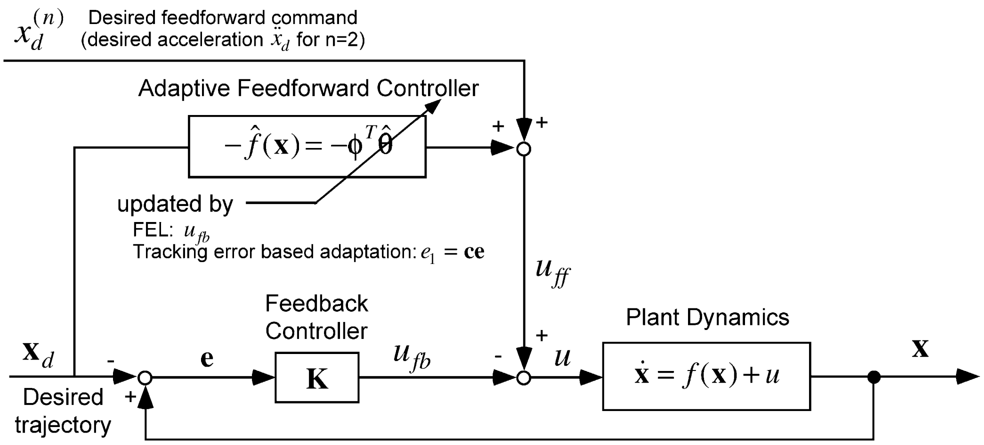

Fig. 3. Nonlinear adaptive control and FEL with an *adaptive feedforward* controller for a class of nth order nonlinear SISO systems.

2. Problem setup

2.1. Plant dynamics and function approximation

The general structure of the control system of interest is a class of nonlinear MIMO systems of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{G}(\mathbf{x})\mathbf{u} \tag{1}$$

$$\mathbf{z} = \mathbf{h}(\mathbf{x}) \tag{2}$$

where $\mathbf{x} \in \Re^n$ is a state, $\mathbf{z} \in \Re^p$ is an output, $\mathbf{u} \in \Re^m$ is an input, $\mathbf{f} : \Re^n \to \Re^n$, $\mathbf{G} : \Re^n \to \Re^{n \times m}$ are nonlinear functions, and $\mathbf{h} : \Re^n \to \Re^p$ denotes a mapping from the state to the output.

For our initial mathematical development in this paper, we consider a simplified nth order SISO system of the form

$$\begin{aligned} \dot{x}_1 &= x_2 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= f(\mathbf{x}) + u \end{aligned} \tag{3}$$

where $x = x_1$, $\mathbf{x} = [x_1, \dots, x_n]^{\mathrm{T}} \in \Re^n$, and $u \in \Re$. Note that a more general form of the plant dynamics including the $g(\mathbf{x})$ term such as $\dot{x}_n = f(\mathbf{x}) + g(\mathbf{x})u$ can be treated by introducing the parameter projection algorithm as discussed by Nakanishi et al. (2004).

Suppose that $f(\mathbf{x})$ can be represented in a linearly parametrized form as

$$f(\mathbf{x}) = \boldsymbol{\phi}^{\mathrm{T}}(\mathbf{x})\boldsymbol{\theta} + \Delta(\mathbf{x}) \tag{4}$$

where $\boldsymbol{\phi}$ is the vector of nonlinear basis functions defined by $\boldsymbol{\phi}(\mathbf{x}) = [\boldsymbol{\phi}_1^{\mathrm{T}}, \dots, \boldsymbol{\phi}_N^{\mathrm{T}}]^{\mathrm{T}}$, $\boldsymbol{\theta}$ is the parameter vector defined by $\boldsymbol{\theta} = [\boldsymbol{\theta}_1^{\mathrm{T}}, \dots, \boldsymbol{\theta}_N^{\mathrm{T}}]^{\mathrm{T}}$, and $\Delta(\mathbf{x})$ is the approximation error. If the structure of f is known, i.e. we know all the correct basis functions, Δ will be zero. If the structure of f is unknown, we can use a linearly parametrized function approximator such as locally linear models (Choi & Farrell; 2000; Nakanishi et al., 2002, 2004; Schaal & Atkeson, 1998) or RBF neural networks (Sanner & Slotine, 1992). In this paper, we assume a perfect approximation where $\Delta = 0$; the case where $\Delta \neq 0$ can be treated as in Nakanishi et al. (2002, 2004) by introducing an adaptation deadzone for parameter update.

3. Adaptive feedback formulation

This section considers the adaptive feedback formulation of nonlinear adaptive control and FEL as depicted in Fig. 2. We discuss the relationship between nonlinear adaptive control and FEL, and the stability properties of FEL with respect to the choice of feedback gains.

3.1. Nonlinear adaptive control

In order to facilitate a coherent development of our research results, in this section we review a general formulation of nonlinear adaptive control (Choi & Farrell, 2000; Sanner & Slotine, 1992; Slotine & Li, 1991) with the adaptive feedback controller as depicted in Fig. 2. The goal of adaptive control is to achieve asymptotic tracking to the desired trajectory under the presence of unknown parameters in the plant dynamics by adjusting them during operation from input–output data while guaranteeing stability of the closed-loop system.

Consider a control law

$$u = u_{ad} + u_{ff} - u_{fb} := -\hat{f}(\mathbf{x}) + x_d^{(n)} - \mathbf{K}\mathbf{e} \tag{5}$$

where

$$u_{ad} = -\hat{f}(\mathbf{x}), \qquad u_{ff} = x_d^{(n)}, \qquad \text{and } u_{fb} = \mathbf{K}\mathbf{e}. \tag{6}$$

$x^{(n)}$ denotes the nth time derivative of x, $\mathbf{K} = [K_1, K_2, \dots, K_n]$ is the feedback gain row vector chosen such that the polynomial $s^n + \sum_{i=1}^{n} K_i s^{i-1} = 0$ has all roots in the left half of the complex number plane, and $\mathbf{e} = [e, \dot{e}, \dots, e^{(n)}]^{\mathrm{T}}$ is the tracking error vector with $e = x - x_d$ and $x_d(t)$ denotes a desired trajectory that is smooth and also has a smooth nth order derivatives. $\hat{f}$ is the estimate of f defined by

$$\hat{f}(\mathbf{x}) = \boldsymbol{\phi}^{\mathrm{T}}(\mathbf{x})\hat{\boldsymbol{\theta}} \tag{7}$$

where $\hat{\boldsymbol{\theta}}$ is an estimate of $\boldsymbol{\theta}$. The tracking error dynamics with the estimate of f can be expressed in the controllable canonical form of the state space representation as:

$$\dot{\mathbf{e}} = \mathbf{A}\mathbf{e} + \mathbf{b}(f - \hat{f}) = \mathbf{A}\mathbf{e} + \mathbf{b}(-\boldsymbol{\phi}^{\mathrm{T}}\tilde{\boldsymbol{\theta}}) \tag{8}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & 1 \\ -K_1 & -K_2 & \cdots & -K_n \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{9}$$

where $\tilde{\boldsymbol{\theta}}$ is the parameter error vector defined as $\tilde{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}} - \boldsymbol{\theta}$. Note that $\mathbf{A}$ is Hurwitz. Define the sliding surface

$$e_1 = \mathbf{c}\mathbf{e} \tag{10}$$

where $\mathbf{c} = [\Lambda_1, \ldots, \Lambda_n]$ and $\Lambda_i > 0$ is chosen such that $(\mathbf{A}, \mathbf{b}, \mathbf{c})$ is minimal (controllable and observable) and $H(s) = \mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}$ is strictly positive real (SPR) (see Appendix A for details). This filtered tracking error will be used in the tracking error-based parameter update (Choi & Farrell, 2000) and the strictly positive real assumption will be necessary in the Lyapunov stability analysis.

If we select the tracking error-based parameter adaptation law

$$\dot{\hat{\boldsymbol{\theta}}} = \boldsymbol{\Gamma}\boldsymbol{\phi}(\mathbf{x})e_1 \tag{11}$$

where $\boldsymbol{\Gamma}$ is a positive definite adaptation gain matrix, it is possible to prove the desirable stability properties of the adaptive controller as discussed in Section 3.4. Note that the actual state $\mathbf{x}$ is used to compute $\hat{f}(\mathbf{x})$ in the control law (5) and $\boldsymbol{\phi}(\mathbf{x})$ in the update law (11). As we will discuss in Section 4, in the adaptive feedforward formulation, the desired state $\mathbf{x}_d$ is used instead of the actual state $\mathbf{x}$.

3.2. Feedback error learning

By reformulating the results of Gomi and Kawato (1993) and Kawato (1990) in our notation, the parameter update law in FEL can be expressed as

$$\dot{\hat{\boldsymbol{\theta}}} = \boldsymbol{\Gamma}\left(\frac{\partial h}{\partial \boldsymbol{\theta}}\right)(-u_{fb}) \tag{12}$$

when the inverse plant dynamics is given in form of

$$u = h(\mathbf{x}, \dot{\mathbf{x}}, \boldsymbol{\theta}) = -f(\mathbf{x}, \boldsymbol{\theta}) + \dot{x} \tag{13}$$

for (3). In case of a linear parametrization (4), the parameter update law in the adaptive feedback formulation can be derived as

$$\dot{\hat{\boldsymbol{\theta}}} = \boldsymbol{\Gamma}\left(-\frac{\partial f}{\partial \boldsymbol{\theta}}\right)(-u_{fb}) = \boldsymbol{\Gamma}\boldsymbol{\phi}(\mathbf{x})u_{fb}. \tag{14}$$

Notice the close relationship between the parameter update law of tracking error-based adaptive control (11) and that of FEL (14). In Section 3.3, we compare the properties of these two parameter update laws.

3.3. Comparison of parameter update laws

In summary, the parameter update law for the tracking error-based adaptive control (11) and FEL (14) are given by

Tracking error-based adaptation

$$\dot{\hat{\boldsymbol{\theta}}} = \boldsymbol{\Gamma}\boldsymbol{\phi}(\mathbf{x})\mathbf{c}\mathbf{e} = \boldsymbol{\Gamma}\boldsymbol{\phi}(\mathbf{x})e_1 \tag{15}$$

where $e_1 = \mathbf{c}\mathbf{e} = \Lambda_1 e + \Lambda_2 \dot{e} + \cdots + \Lambda_n e^{(n)}$.

Feedback error earning

$$\dot{\hat{\boldsymbol{\theta}}} = \boldsymbol{\Gamma}\boldsymbol{\phi}(\mathbf{x})\mathbf{K}\mathbf{e} = \boldsymbol{\Gamma}\boldsymbol{\phi}(\mathbf{x})u_{fb} \tag{16}$$

where $u_{fb} = \mathbf{K}\mathbf{e} = K_1 e + K_2 \dot{e} + \cdots + K_n e^{(n)}$.

We can see that these two update laws are identical except for the difference in the driving signal to update the parameters. In the tracking error-based adaptive control the filtered error e_1 is used while in FEL the output of the feedback controller u_{fb} is used. For example, for second order systems, these driving signals become $e_1 = \Lambda_1 e + \Lambda_2 \dot{e}$ and $u_{fb} = K_P e + K_D \dot{e}$, where $K_P = K_1$, $K_D = K_2$. In FEL, the driving signal to update the parameters depends on the choice of feedback gains. In contrast, in adaptive control, we have the additional freedom in selecting the parameters in $\mathbf{c} = [\Lambda_1, \ldots, \Lambda_n]$ to scale the tracking error and its derivatives.

3.4. Lyapunov stability analysis

We present Lyapunov stability analyses for the nonlinear adaptive controller and FEL. Consider the Lyapunov function

$$V = \frac{1}{2}\mathbf{e}^{\mathrm{T}}\mathbf{S}\mathbf{e} + \frac{1}{2}\tilde{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{\Gamma}^{-1}\tilde{\boldsymbol{\theta}}. \tag{17}$$

By the Lefschetz–Kalman–Yakubovich lemma (Tao & Ioannou, 1990), with the positive real assumption of $(\mathbf{A}, \mathbf{b}, \mathbf{c})$, there exist real symmetric positive definite matrices $\mathbf{S}$ and $\mathbf{L}$, a real vector $\mathbf{q}$, and $\mu > 0$ such that

$$\mathbf{A}^{\mathrm{T}}\mathbf{S} + \mathbf{S}\mathbf{A} = -\mathbf{q}\mathbf{q}^{\mathrm{T}} - \mu\mathbf{L} \tag{18}$$

$$\mathbf{S}\mathbf{b} = \mathbf{c}^{\mathrm{T}}. \tag{19}$$

By inserting the error dynamics (8), the adaptation law (11) and the properties (18) and (19), the time derivative of V

with can be calculated as

$$\dot{V} = \frac{1}{2}(\dot{\mathbf{e}}^{\mathrm{T}}\mathbf{S}\mathbf{e} + \mathbf{e}^{\mathrm{T}}\mathbf{S}\dot{\mathbf{e}}) + \tilde{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{\Gamma}^{-1}\dot{\tilde{\boldsymbol{\theta}}}$$

$$= \frac{1}{2}\mathbf{e}^{\mathrm{T}}(\mathbf{A}^{\mathrm{T}}\mathbf{S} + \mathbf{S}\mathbf{A})\mathbf{e} + \tilde{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{\Gamma}^{-1}(\dot{\tilde{\boldsymbol{\theta}}} - \boldsymbol{\Gamma}\boldsymbol{\phi}e_1)$$

$$= -\frac{1}{2}\mathbf{e}^{\mathrm{T}}(\mathbf{q}\mathbf{q}^{\mathrm{T}} + \mu\mathbf{L})\mathbf{e} \le -\beta_1\mathbf{e}^{\mathrm{T}}\mathbf{e} \le 0 \tag{20}$$

where

$$\beta_1 = \frac{1}{2}\lambda_{\min}(\mathbf{q}\mathbf{q}^{\mathrm{T}} + \mu\mathbf{L}) > 0 \tag{21}$$

and $\lambda_{\min}(\cdot)$ denotes the minimum eigenvalue of a matrix. Note that the term $\tilde{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{\Gamma}^{-1}(\dot{\tilde{\boldsymbol{\theta}}} - \boldsymbol{\Gamma}\boldsymbol{\phi}e_1)$ in (20) is cancelled out with the adaptation law (11) since

$$\dot{\tilde{\boldsymbol{\theta}}} = \dot{\hat{\boldsymbol{\theta}}} = \boldsymbol{\Gamma}\boldsymbol{\phi}e_1. \tag{22}$$

This Lyapunov analysis implies that the tracking error, e, converges to zero with $e \in L_2$ by Barbalat's lemma, and the parameter error, $\tilde{\boldsymbol{\theta}}$, is bounded with $\tilde{\boldsymbol{\theta}} \in L_\infty$ (Slotine & Li, 1991). For asymptotic parameter error convergence to zero, $\boldsymbol{\phi}(\mathbf{x})$ needs to satisfy the so-called persistent excitation (PE) condition (Slotine & Li, 1991).

3.5. Choice of feedback gains in FEL

As discussed in Section 3.1 the parameter $\mathbf{c} = [\Lambda_1, \ldots, \Lambda_n]$ in (10) must be chosen to satisfy the strictly positive real (SPR) condition (see Appendix A). Given that FEL is equivalent to the tracking error-based adaptive controller for the plant dynamics (3), in order to ensure the stability of FEL, we need to choose the feedback gains K_i in FEL so that the SPR condition holds for the pair $(\mathbf{A}, \mathbf{b}, \mathbf{c})$:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & 1 \\ -K_1 & -K_2 & \cdots & -K_n \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{23}$$

and $\mathbf{c} = [K_1, \ldots, K_n]$.

For the general nth order systems, the SPR condition is somewhat abstract and the criterion as to how we can choose the feedback gains K_i to satisfy the SPR condition for $(\mathbf{A}, \mathbf{b}, \mathbf{c})$ is not obvious. However, for the case of second order SISO systems with

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -K_P & -K_D \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{24}$$

and $\mathbf{c} = [K_P, K_D]$,

it is possible to derive a condition for the choice of K_i to guarantee the stability of FEL in a very simple form using Theorem 1 in Appendix A. The conditions in this theorem are:

1. *All eigenvalues of A have negative real parts*: This condition holds since we choose K_P and K_D such that $\mathbf{A}$ is Hurwitz ($K_P > 0$ and $K_D > 0$).
2. $\mathbf{CB} = (\mathbf{CB})^{\mathrm{T}} > 0$: This condition also holds since $\mathbf{cb} = (\mathbf{cb})^{\mathrm{T}} = K_D > 0$.
3. $\mathbf{CAB} + (\mathbf{CAB})^{\mathrm{T}} < 0$: A simple calculation yields $\mathbf{cAb} + (\mathbf{cAb})^{\mathrm{T}} = 2(K_P - K_D^2) < 0$.

Thus, for second order SISO systems, the feedback gains must satisfy the condition

$$K_D^2 > K_P, \quad \text{where } K_P > 0 \text{ and } K_D > 0 \tag{25}$$

to guarantee the stability of FEL. Interestingly, this condition does not depend on the nonlinearity $f(\mathbf{x})$. Table 1 lists several choices of feedback gains with the stability condition (25). It is important to note that there are cases in which certain choices of feedback gains do not guarantee the stability of FEL, typically when K_D is small. Thus, care must be taken for selecting feedback gains in FEL to achieve stable learning as well as good tracking performance. In contrast, in the case of tracking error-based adaptive control, the corresponding stability condition is given by

$$\Lambda_1 - \Lambda_2 K_D < 0, \qquad K_P > 0 \text{ and } K_D > 0. \tag{26}$$

In this case, there is a large amount of freedom in scaling the tracking error and its time derivative by choosing $\mathbf{c}$ in (10). For example, with $K_P = 5.0$ and $K_D = 1.0$ (case 1 in Table 1), we can still choose $\Lambda_1 = 0.1$ and $\Lambda_2 = 1.0$ to guarantee stability.

3.6. Passivity-based stability analysis

The notion of passivity and output-dissipativity has been used as a means of advanced stability analysis of nonlinear systems motivated by the concept of energy conservation and dissipation in network theory (Arimoto, 1996). In brief, a passive system does not generate energy, and a dissipative system loses energy. Mathematical definitions of passivity and output-dissipativity for dynamical systems are given in Appendix B. For linear systems, equivalence between passivity and positive realness (PR), and output-dissipativity and SPR are demonstrated by Arimoto and Naniwa (2000, 2001). Applications of passivity-based stability

Table 1
Choice of PD gains and FEL stability

Gains	K_P	K_D	$K_D^2 > K_P$ holds?	Stability
Case 1	5.0	1.0	No	Not guaranteed
Case 2	5.0	3.0	Yes	Guaranteed
Case 3	10.0	2.0	No	Not guaranteed
Case 4	10.0	4.0	Yes	Guaranteed
Case 5	100.0	20.0	Yes	Guaranteed

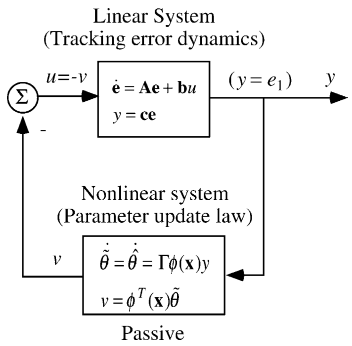

Fig. 4. Closed-loop dynamics of the learning adaptive control system in terms of a negative feedback connection of a passive nonlinear dynamical system (parameter update law) and a linear system (tracking error dynamics) to apply Theorem 3 in Appendix B (Arimoto, 1996) (see Fig. B1).

analysis to nonlinear mechanical systems such as robot arms are presented by Arimoto (1996).

In Section 3.4, we presented a Lyapunov analysis which suggests that SPR of the tracking error dynamics (8) and (10) is a sufficient condition for asymptotic stability of the learning adaptive controller and FEL. In the following, we show that SPR of the tracking error dynamics is a *necessary and sufficient* condition for asymptotic hyperstability, i.e. the tracking error $\mathbf{e}$ is bounded and $\mathbf{e} \to 0$ as $t \to \infty$, by applying Theorem 3 described in Appendix B (Arimoto, 1996) (see Appendix B for details of the definition of asymptotic hyperstability).

The closed-loop dynamics of the learning adaptive control system considered in this paper can be formulated as a negative feedback connection of a passive nonlinear system (parameter update law) and a linear system (tracking error dynamics) as depicted in Fig. 4 where

Linear system: Tracking error dynamics defined by (8) and (10):

$$\dot{\mathbf{e}} = \mathbf{Ae} + \mathbf{b}u \tag{27}$$

$$y = \mathbf{ce} \tag{28}$$

where $y = e_1$, $u = -\boldsymbol{\phi}^{\mathrm{T}}(\mathbf{x})\tilde{\boldsymbol{\theta}} = -v$. Note that in FEL, $\mathbf{c} = \mathbf{K}$.

Nonlinear system: Parameter update law in (11):

$$\dot{\tilde{\boldsymbol{\theta}}} = \dot{\hat{\boldsymbol{\theta}}} = \boldsymbol{\Gamma}\boldsymbol{\phi}(\mathbf{x})y \tag{29}$$

$$v = \boldsymbol{\phi}^{\mathrm{T}}(\mathbf{x})\tilde{\boldsymbol{\theta}} \tag{30}$$

where $y = e_1$.

Note that the input–output pair $\{y, v\}$ of the nonlinear system defined in (29) and (30) satisfies passivity since

$$\int_{t_0}^{t} v^{\mathrm{T}} y \, \mathrm{d}\tau = \frac{1}{2}\tilde{\boldsymbol{\theta}}^{\mathrm{T}}(t)\boldsymbol{\Gamma}^{-1}\tilde{\boldsymbol{\theta}}(t) - \frac{1}{2}\tilde{\boldsymbol{\theta}}^{\mathrm{T}}(t_0)\boldsymbol{\Gamma}^{-1}\tilde{\boldsymbol{\theta}}(t_0) \geq -\frac{1}{2}\tilde{\boldsymbol{\theta}}^{\mathrm{T}}(t_0)\boldsymbol{\Gamma}^{-1}\tilde{\boldsymbol{\theta}}(t_0) = -\gamma_0^2 \tag{31}$$

where the following identity was exploited

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\tilde{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{\Gamma}^{-1}\tilde{\boldsymbol{\theta}}) = \tilde{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{\Gamma}^{-1}\dot{\tilde{\boldsymbol{\theta}}} = \tilde{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{\phi}(\mathbf{x})e_1 = v^{\mathrm{T}}y(= vy). \tag{32}$$

Now we can apply Theorem 3 in Appendix B (Arimoto, 1996) to conclude that SPR of the tracking error dynamics is a *necessary and sufficient* condition for asymptotic hyperstability of learning adaptive control and FEL, i.e. $\|\mathbf{e}\|$ is bounded and $\mathbf{e} \to 0$ as $t \to \infty$, Therefore, the conditions on the choice of $\mathbf{c}$ and $\mathbf{K}$ (25) for FEL

$$K_D^2 > K_P, \quad \text{where } K_P > 0 \text{ and } K_D > 0$$

and (26) for the tracking error-based adaptive controller

$$\Lambda_1 - \Lambda_2 K_D < 0, \quad K_P > 0 \text{ and } K_D > 0,$$

which are derived from SPR of the tracking error dynamics for a class of second order SISO systems, are necessary as well as sufficient for the learning system to be asymptotically hyperstable.

4. Adaptive feedforward formulation

Recall that in the adaptive feedback formulation in Fig. 2 discussed above, the actual state $\mathbf{x}$ is used to compute $\hat{f}(\mathbf{x})$ in the control law (5) and $\boldsymbol{\phi}(\mathbf{x})$ in the update laws (15) and (16). We explore the theoretical analysis of the adaptive feedforward formulation in which the desired state $\mathbf{x}_d$ is used instead such as $\hat{f}(\mathbf{x}_d)$ and $\boldsymbol{\phi}(\mathbf{x}_d)$ as illustrated in Fig. 3. This formulation is identical to the original FEL model for our simplified plant in (3). We derive a condition to ensure the stability of learning for the adaptive feedforward formulation, which is more conservative than that of the adaptive feedback case.

4.1. Control law

The control law in the adaptive feedforward formulation is given by

$$u = u_{ff} - u_{fb} := -\hat{f}(\mathbf{x}_d) + x_d^{(n)} - \mathbf{Ke} \tag{33}$$

where

$$u_{ff} = x_d^{(n)} - \hat{f}(\mathbf{x}_d) \quad \text{and} \quad u_{fb} = \mathbf{Ke}. \tag{34}$$

Note that $\mathbf{K}$ is chosen such that $\mathbf{A}$ in (23) is Hurwitz. As mentioned above, the desired state $\mathbf{x}_d$ is used for $\hat{f}$ instead of the actual state $\mathbf{x}$.

4.2. Parameter update law

The parameter update laws for the tracking error-based adaptive control and FEL in the adaptive feedforward formulation are given by

Tracking error-based adaptation

$$\dot{\hat{\boldsymbol{\theta}}} = \boldsymbol{\Gamma\phi}(\mathbf{x}_d)\mathbf{ce} = \boldsymbol{\Gamma\phi}(\mathbf{x}_d)e_1 \quad (35)$$

where $e_1 = \mathbf{ce} = \Lambda_1 e + \Lambda_2 \dot{e} + \cdots + \Lambda_n e^{(n)}$.

Feedback error earning

$$\dot{\hat{\boldsymbol{\theta}}} = \boldsymbol{\Gamma\phi}(\mathbf{x}_d)\mathbf{Ke} = \boldsymbol{\Gamma\phi}(\mathbf{x}_d)u_{fb} \quad (36)$$

where $u_{fb} = \mathbf{Ke} = K_1 e + K_2 \dot{e} + \cdots + K_n e^{(n)}$.

4.3. Lyapunov stability analysis

In the adaptive feedforward formulation, the error dynamics can be written as

$$\begin{aligned}\dot{\mathbf{e}} &= \mathbf{Ae} + \mathbf{b}(f(\mathbf{x}) - \hat{f}(\mathbf{x}_d)) \\ &= \mathbf{Ae} + \mathbf{b}(f(\mathbf{x}_d) - \hat{f}(\mathbf{x}_d) + f(\mathbf{x}) - f(\mathbf{x}_d)) \\ &= \mathbf{Ae} + \mathbf{b}(-\boldsymbol{\phi}^{\mathrm{T}}(\mathbf{x}_d)\tilde{\boldsymbol{\theta}} + f(\mathbf{x}) - f(\mathbf{x}_d)) \quad (37)\end{aligned}$$

where $e_1 = \mathbf{ce}$ and $(\mathbf{A}, \mathbf{b}, \mathbf{c})$ are defined in (9) and (10). $\mathbf{K}$ is chosen such that $\mathbf{A}$ is Hurwitz, $\mathbf{c}$ is chosen such that $(\mathbf{A}, \mathbf{b}, \mathbf{c})$ is minimal (controllable and observable), and $H(s) = \mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}$ is SPR. Note that in (37) there is, in comparison to (8), an additional term $f(\mathbf{x}) - f(\mathbf{x}_d)$ which drives the tracking error.

Consider the same Lyapunov function as in (17)

$$V = \frac{1}{2}\mathbf{e}^{\mathrm{T}}\mathbf{Se} + \frac{1}{2}\tilde{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{\Gamma}^{-1}\tilde{\boldsymbol{\theta}}. \quad (38)$$

By the Lefschetz–Kalman–Yakubovich lemma (Tao & Ioannou, 1990), with the positive real assumption of $(\mathbf{A}, \mathbf{b}, \mathbf{c})$, the time derivative of V with error dynamics (37) can be calculated as

$$\begin{aligned}\dot{V} &= \frac{1}{2}(\dot{\mathbf{e}}^{\mathrm{T}}\mathbf{Se} + \mathbf{e}^{\mathrm{T}}\mathbf{S}\dot{\mathbf{e}}) + \tilde{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{\Gamma}^{-1}\dot{\tilde{\boldsymbol{\theta}}} \\ &= -\frac{1}{2}\mathbf{e}^{\mathrm{T}}(\mathbf{qq}^{\mathrm{T}} + \mu\mathbf{L})\mathbf{e} + (f(\mathbf{x}) - f(\mathbf{x}_d))\mathbf{ce} \\ &= -\frac{1}{2}\mathbf{e}^{\mathrm{T}}(\mathbf{qq}^{\mathrm{T}} + \mu\mathbf{L})\mathbf{e} + \mathbf{e}^{\mathrm{T}}(\nabla f(\mathbf{z})\mathbf{c})\mathbf{e} \\ &\le -(\beta_1 - \beta_2)\mathbf{e}^{\mathrm{T}}\mathbf{e} \le -(\beta_1 - \overline{\beta_2})\mathbf{e}^{\mathrm{T}}\mathbf{e} \quad (39)\end{aligned}$$

where

$$\beta_1 = \frac{1}{2}\lambda_{\min}(\mathbf{qq}^{\mathrm{T}} + \mu\mathbf{L}) > 0 \quad (40)$$

$$\begin{aligned}\beta_2 &= \lambda_{\max}(\nabla f(\mathbf{z})\mathbf{c}) = \mathbf{c}\nabla f(\mathbf{z}) \\ &= \Lambda_1 \nabla f(\mathbf{z})_1 + \cdots + \Lambda_n \nabla f(\mathbf{z})_n \quad (41)\end{aligned}$$

and $\overline{\beta_2}$ is the upper bound of β_2

$$\overline{\beta_2} = \Lambda_1\overline{(\nabla f)_1} + \cdots + \Lambda_n\overline{(\nabla f)_n} \quad (42)$$

assuming that the upper bound of $(\nabla f)_i$, denoted by $\overline{(\nabla f)_i}$, is known. $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ denote the minimum and maximum eigenvalues of a matrix, respectively. In the derivation above, the mean value theorem (Khalil, 1996)

$$f(\mathbf{x}) - f(\mathbf{x}_d) = (\nabla f(\mathbf{z}))^{\mathrm{T}}(\mathbf{x} - \mathbf{x}_d) = (\nabla f(\mathbf{z}))^{\mathrm{T}}\mathbf{e} = \mathbf{e}^{\mathrm{T}}(\nabla f(\mathbf{z})) \quad (43)$$

is used, where $\mathbf{z}$ is the vector on the line segment $L(\mathbf{x}, \mathbf{x}_d)$ joining $\mathbf{x}$ and $\mathbf{x}_d$ (when $\mathbf{x}$ and $\mathbf{x}_d$ are distinct) defined by

$$L(\mathbf{x}, \mathbf{x}_d) = \{\mathbf{z} | \mathbf{z} = \alpha\mathbf{x} + (1-\alpha)\mathbf{x}_d,\ \ 0 < \alpha < 1\}. \quad (44)$$

The Lyapunov analysis (39) implies that if $\beta_1 - \overline{\beta_2} > 0$, then $\dot{V} \le 0$. Thus, in the adaptive feedforward formulation, we need to choose $\mathbf{K}$ and $\mathbf{c}$ such that $\beta_1 - \overline{\beta_2} > 0$ in addition to the SPR condition for $(\mathbf{A}, \mathbf{b}, \mathbf{c})$. To illustrate this more closely, let us examine this condition for first and second order systems:

First order systems: The Lyapunov function (38) reduces to

$$V = \frac{1}{2}\Lambda e^2 + \frac{1}{2}\tilde{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{\Gamma}^{-1}\tilde{\boldsymbol{\theta}} \quad (45)$$

where $\mathbf{A} = -K(K > 0)$, $\mathbf{b} = 1$, $\mathbf{c} = \Lambda > 0$ and $\mathbf{S} = \Lambda$. The time derivative of V becomes

$$\begin{aligned}\dot{V} &= -K\Lambda e^2 + (f(x) - f(x_d))\Lambda e \\ &= -\Lambda(K - f'(z))e^2 \le -\Lambda(K - \overline{f'})e^2 \quad (46)\end{aligned}$$

Thus, if K is selected such $K > \overline{f'}$ where $K > 0$, then stability is guaranteed where $\overline{f'}$ denotes the upper bound of the gradient of f assuming that it is known.

Second order systems: β_1 and β_2 in (40) and (42) can be calculated as

$$\beta_1 = \min\{\Lambda_1 K_P, -\Lambda_1 + \Lambda_2 K_D\} \quad (47)$$

$$\overline{\beta_2} = \Lambda_1\overline{(\nabla f)_1} + \Lambda_2\overline{(\nabla f)_2} \quad (48)$$

Thus, in tracking error-based adaptive control, $\mathbf{K}$ and $\mathbf{c}$ should be selected such that $\beta_1 - \overline{\beta_2} > 0$ in addition to the SPR condition (26). For FEL with $\Lambda_1 = K_P$ and $\Lambda_2 = K_D$, this condition becomes

$$\beta_1 = \min\{K_P^2, -K_P + K_D^2\} \quad (49)$$

$$\overline{\beta_2} = K_P\overline{(\nabla f)_1} + K_D\overline{(\nabla f)_2} \quad (50)$$

in addition to the SPR condition (25).

In summary, in the adaptive feedforward formulation, an additional condition

$$\beta_1 - \overline{\beta_2} > 0 \quad (51)$$

needs to be satisfied to guarantee the stability, where β_1 and $\overline{\beta_2}$ are defined in (40) and (42), respectively, in addition to the SPR condition with respect to the tracking error dynamics characterized by $(\mathbf{A}, \mathbf{b}, \mathbf{c})$. In contrast to the adaptive feedback formulation, the stability condition for the adaptive feedforward case depends not only the choice of the parameters $\mathbf{K}$ and $\mathbf{c}$ but also on the intrinsic characteristics of the plant dynamics which appears as β_2 in the Lyapunov analysis in (39). However, in practice such

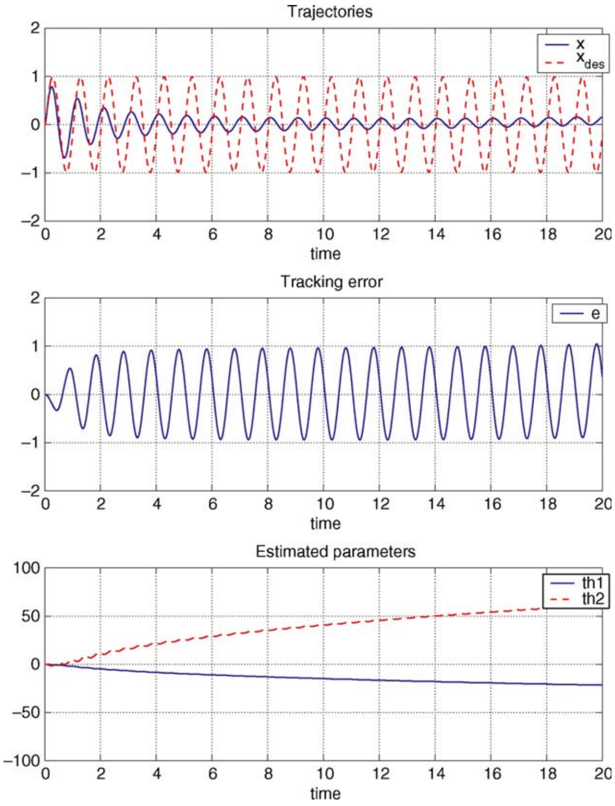

Fig. 5. FEL simulation in the adaptive feedback formulation for the case 1 where $K_P = 5.0$ and $K_D = 1.0$. The result shows that the tracking error and the estimated parameters do not converge as the choice of feedback gains do not guarantee the stability of FEL.

prior information on the upper bound of the gradient of f is not available when f is unknown. Thus, when f is unknown, feedback gains should be carefully chosen based on an initial guess/estimate of the characteristics of the nonlinearities in the plant dynamics. Since β_1 increases if $\mathbf{K}$ increases, choosing feedback gains sufficiently large would suffice for first and second order plant dynamics.[3] As we gain information about f through learning, we may be able to fine tune the feedback gains. The difference between the adaptive feedback formulation and the adaptive feedforward formulation becomes significant when the plant dynamics is unstable, as the numerical simulations in Section 5 will demonstrate.

5. Numerical simulations

We illustrate the stability properties of FEL in numerical simulations. In Section 5.1, we first present simulation results on the adaptive feedback formulation. Then, in Section 5.2, we consider the case of the adaptive feedforward formulation. As an example, we use a mass-damper-spring system for plant dynamics

$$m\ddot{x} + d\dot{x} + kx = u \tag{52}$$

where $m = 1$ is assumed to be known, but we suppose that d and k are unknown.

5.1. Adaptive feedback formulation

The FEL and tracking error-based adaptive control algorithms in the adaptive feedback formulation depicted in Fig. 2 are implemented as follows. First, we rewrite the system dynamics as

$$\ddot{x} = f(x, \dot{x}) + u = -d\dot{x} - kx + u \tag{53}$$

and use the linear model for $f(x, \dot{x})$ as

$$f(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}\boldsymbol{\theta} \tag{54}$$

where $\mathbf{x} = [x, \dot{x}]^{\mathrm{T}}$. Then, we design a feedback linearizing controller with an estimate of f

$$u = -\hat{f}(\mathbf{x}) + \ddot{x}_d - \mathbf{Ke} = -\hat{f}(\mathbf{x}) + \ddot{x}_d - (K_P e + K_D \dot{e}). \tag{55}$$

In the simulations, the parameters $d = k = 2$ and $\boldsymbol{\Gamma} = 2.0\mathbf{I}$. are chosen, and we use $x_d(t) = \sin(2\pi t)$ as the desired trajectory. Initial conditions are set to $\mathbf{x}(0) = 0$ and

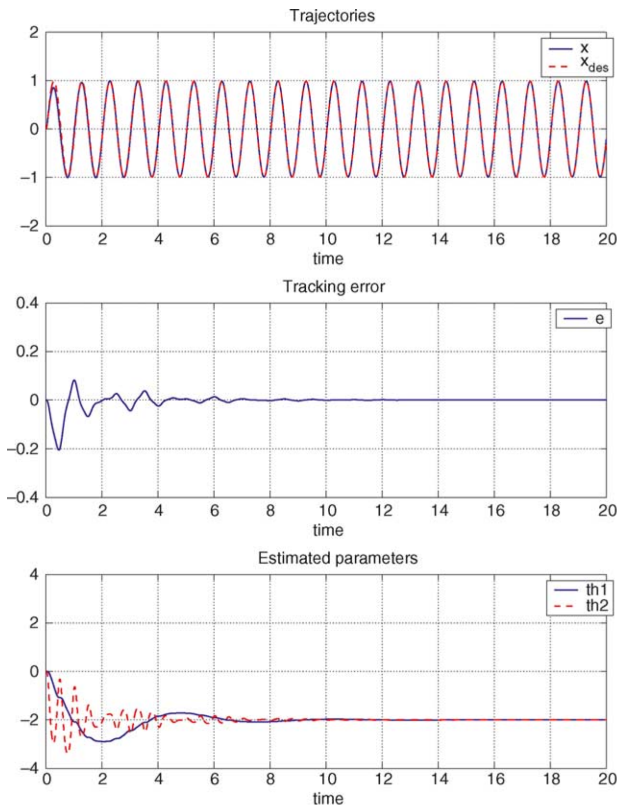

Fig. 6. FEL simulation in the adaptive feedback formulation for the case 2 where $K_P = 5.0$ and $K_D = 3.0$. The result shows the convergence of tracking error to zero as well as the estimated parameters to $\boldsymbol{\theta} = [-k, -d]^{\mathrm{T}} = [-2, -2]^{\mathrm{T}}$ as the choice of feedback gains ensures the stability of FEL.

[3] Note that this strategy only applies to the first and second order plant dynamics. For $n > 2$, simply choosing large feedback gains may not satisfy the Hurwitz condition for $\mathbf{A}$.

$\hat{\boldsymbol{\theta}}(0) = 0$. The dynamics and parameter update laws are integrated using the adaptive step-size Runge-Kutta algorithm in Matlab. In the following figures, we plot the actual trajectory x as a solid line and the desired trajectory x_d as a dotted line in the top of each figure, the tracking error e in the middle of each figure, and the estimated parameters $\hat{\boldsymbol{\theta}}$ at the bottom of each figure.

Feedback error learning. Fig. 5 shows the simulation result for the case 1 in Table 1 with FEL using $K_P = 5.0$ and $K_D = 1.0$. This choice of feedback gains does not satisfy the condition $K_D^2 > K_P$ (25), which implies that the stability of FEL is not guaranteed. The simulation result illustrates that the tracking error and the parameter error do not converge. In contrast, Fig. 6 shows the result for the case 2 in Table 1 where $K_P = 5.0$ and $K_D = 3.0$. This choice satisfies the condition (25) ensuring the stability of FEL. The result in Fig. 6 demonstrates the convergence of the tracking error to zero as well as the convergence of the parameters to the desired values $\boldsymbol{\theta} = [-k, -d]^{\mathrm{T}} = [-2, -2]^{\mathrm{T}}$.

Adaptive control. Consider the case 1 presented above where $K_P = 5.0$ and $K_D = 1.0$. This choice does not guarantee the stability for FEL since the condition (25) does not hold. However, in the tracking error-based adaptive control, we have the freedom to choose the parameters of the filtered error (10) to guarantee stability independent of the feedback gains. Fig. 7 shows the result of the tracking error-based adaptation with the additional choice of $\Lambda_1 = 0.1$ and $\Lambda_2 = 1.0$. This choice satisfies the condition to guarantee stability, $\Lambda_1 - \Lambda_2 K_D < 0$ in (26). The simulation result illustrate that the tracking error converges to zero and the estimated parameter converges to the desired values $\boldsymbol{\theta} = [-k, -d]^{\mathrm{T}} = [-2, -2]^{\mathrm{T}}$.

Unstable plant dynamics learning with FEL. Consider the case of learning unstable plant dynamics where $d = k = -2$ using FEL with the adaptive feedback controller. The feedback gains $K_P = 5.0$ and $K_D = 3.0$ are used (case 2 in Table 1), which satisfy the condition (25) to ensure stability. The simulation result shown in Fig. 8 suggests that stable learning and asymptotic tracking can be achieved in the adaptive feedback formulation as long as the condition (25) is satisfied even if the original plant dynamics are unstable. However, simulations of unstable plant dynamics learning with FEL in the adaptive feedforward formulation presented in Section 5.2 will illustrate that the same choice $K_P = 5.0$ and $K_D = 3.0$ yields instability since this choice not satisfy the condition (51) for the case where $d = k = -2$.

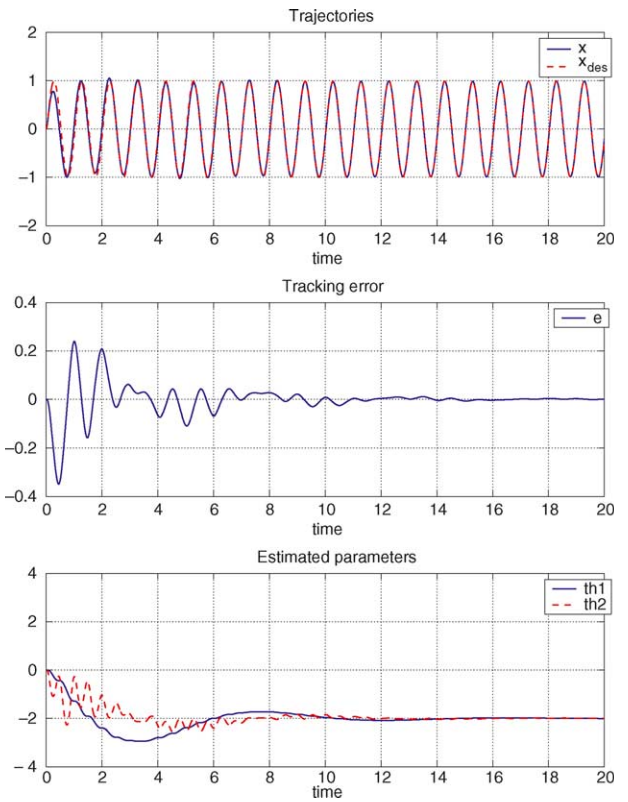

Fig. 7. Adaptive control simulation in the adaptive feedback formulation for the case 2 where $K_P = 5.0$ and $K_D = 1.0$. Although this choice of PD gains does not guarantee the stability for FEL, an additional choice of $\Lambda_1 = 0.1$ and $\Lambda_2 = 1.0$ ensures stability. The result demonstrates the convergence of the tracking error to zero and that of the estimated parameters to $\boldsymbol{\theta} = [-k, -d]^{\mathrm{T}} = [-2, -2]^{\mathrm{T}}$.

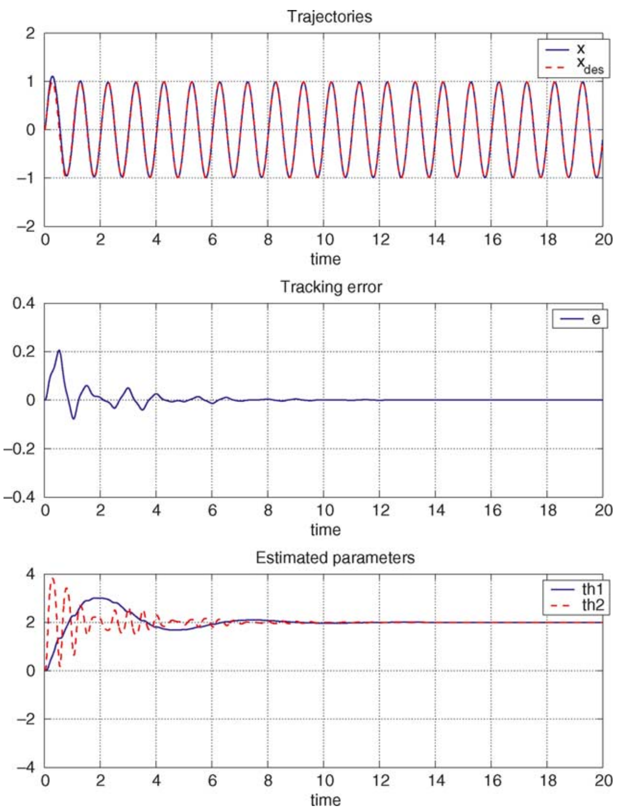

Fig. 8. FEL simulation of learning unstable dynamics in the adaptive feedback formulation for the case 2 where $K_P = 5.0$ and $K_D = 3.0$. The result demonstrates the convergence of the tracking error to zero as well as the estimated parameters to $\boldsymbol{\theta} = [-k, -d]^{\mathrm{T}} = [2, 2]^{\mathrm{T}}$.

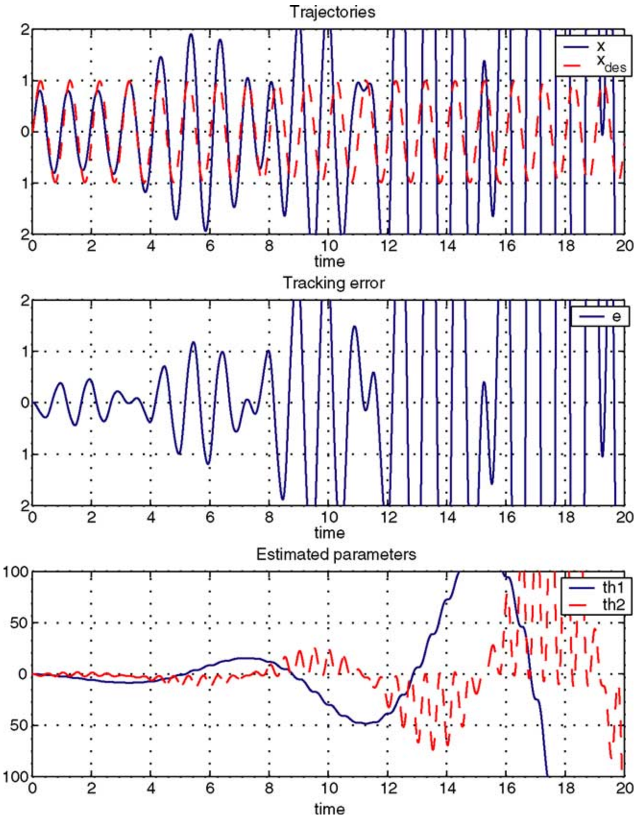

Fig. 9. FEL simulation in the adaptive feedforward formulation for the case 1 where $K_P = 5.0$ and $K_D = 1.0$. The result shows that the tracking error and the estimated parameters do not converge as the choice of feedback gains do not guarantee the stability of FEL.

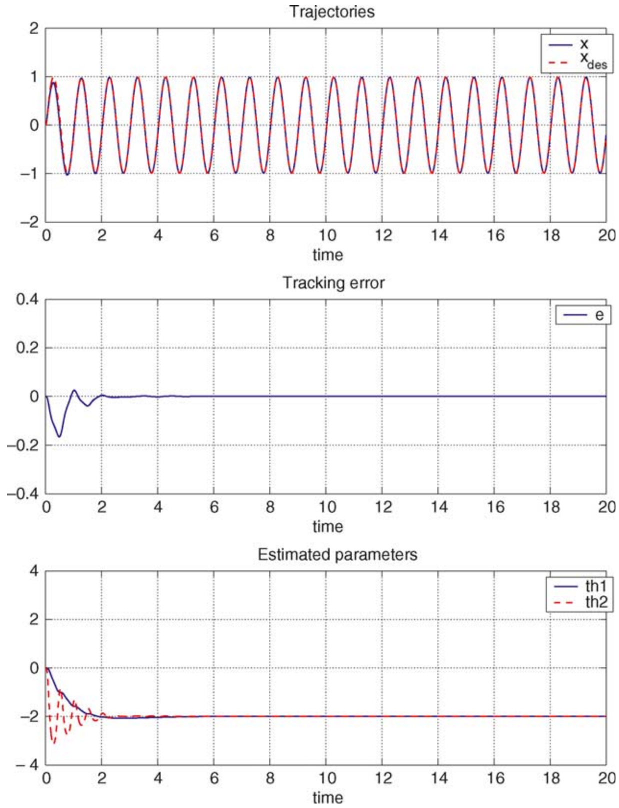

Fig. 10. FEL simulation in the adaptive feedforward formulation for the case 2 where $K_P = 5.0$ and $K_D = 3.0$. The result demonstrates the convergence of tracking error to zero as well as the estimated parameters to $\boldsymbol{\theta} = [-k, -d]^{\mathrm{T}} = [-2, -2]^{\mathrm{T}}$.

5.2. Adaptive feedforward formulation

In the adaptive feedforward formulation, we use the desired state $\mathbf{x}_d$ to compute the basis functions of the function approximator. The control law is formulated as

$$u = -\hat{f}(\mathbf{x}_d) + \ddot{x}_d - \mathbf{K}\mathbf{e} = -\hat{f}(\mathbf{x}_d) + \ddot{x}_d - (K_P e + K_D \dot{e}). \quad (56)$$

We use the same parameters and desired trajectory which were used in the adaptive feedback case above.

Feedback error learning. Fig. 9 shows the simulation result for the case 1 in Table 1 with FEL where $K_P = 5.0$ and $K_D = 1.0$. This choice of feedback gains does not satisfy the SPR condition $K_D^2 > K_P$ (25), which implies that the stability of FEL is not guaranteed. The simulation result illustrates that the tracking error and the parameter error do not converge. In contrast, Fig. 10 shows the result for the case 2 in Table 1 where $K_P = 5.0$ and $K_D = 3.0$. This choice satisfies both the SPR condition (25) and the additional condition (51) for the adaptive feedforward formulation to ensure stability. The result in Fig. 10 demonstrates the convergence of the tracking error to zero as well as the convergence of the parameter to the desired values where $\boldsymbol{\theta} = [-k, -d]^{\mathrm{T}} = [-2, -2]^{\mathrm{T}}$.

Unstable plant dynamics learning with FEL. Consider the case of learning unstable plant dynamics where $d = k = -2$ using FEL with the adaptive feedforward controller. Fig. 11 exhibits the case of instability in learning with $K_P = 5.0$ and $K_D = 3.0$ (case 2 in Table 1) since this choice does not satisfy condition (51). However, it can be stabilized with the choice of different feedback gains $K_P = 20.0$ and $K_D = 10.0$ which satisfy the condition (51) as well as the SPR condition as the simulation result in Fig. 12 demonstrates.

6. Conclusions

This paper presented a theoretical treatment of feedback error learning from a nonlinear adaptive control viewpoint for a class of nth order nonlinear systems. First, we considered the adaptive control and FEL algorithms in an adaptive feedback formulation. We showed that FEL can be viewed as a form of tracking error-based adaptive control. A Lyapunov analysis suggests that SPR is a sufficient condition to guarantee asymptotic stability of adaptive control and FEL. Specifically, for a class of second order SISO systems, we derived that this condition simplifies to $K_D^2 > K_P$ to guarantee asymptotic stability of FEL, where K_P and K_D are positive constants. A passivity-based stability analysis showed that SPR of the error dynamics is

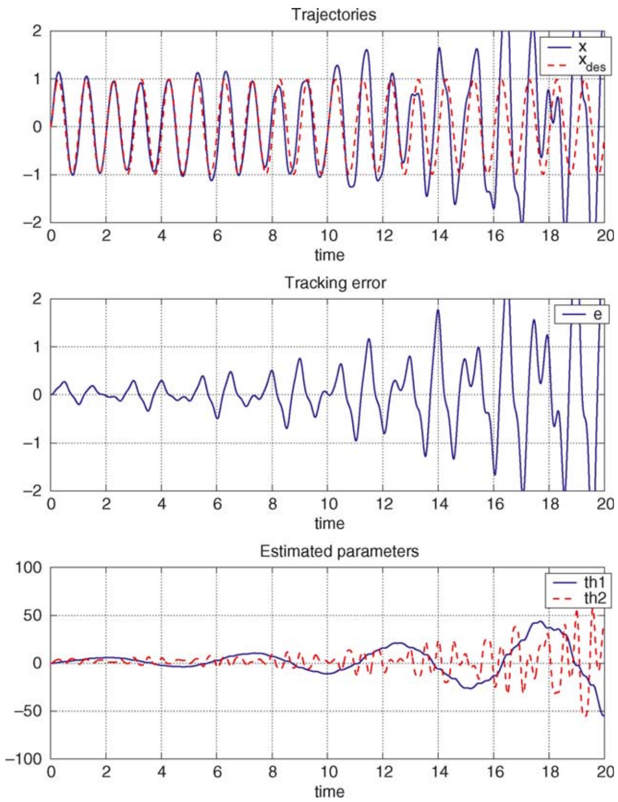

Fig. 11. FEL simulation of unstable dynamics learning in the adaptive feedforward formulation for the case 2 where $K_P = 5.0$ and $K_D = 3.0$. The simulation result shows that learning goes unstable learning as K_P and K_D do not satisfy the condition (51).

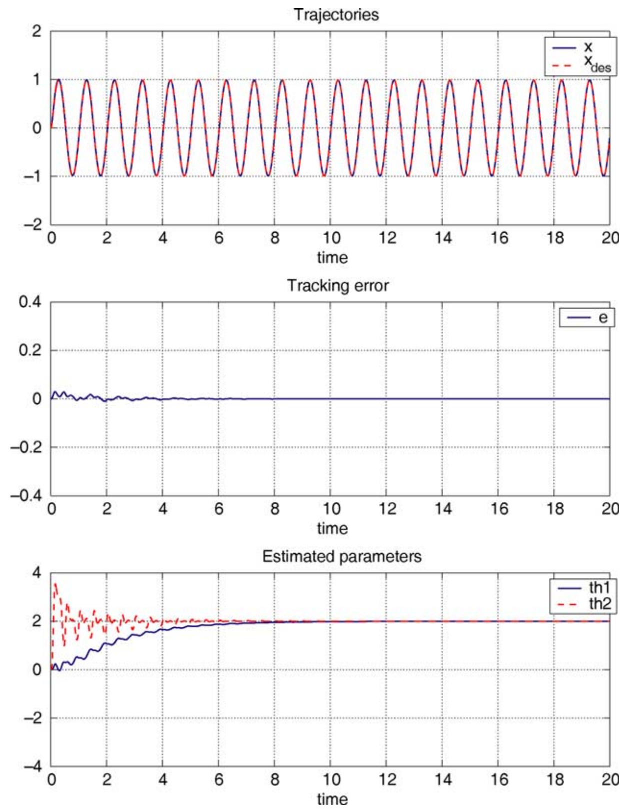

Fig. 12. FEL simulation of unstable dynamics learning in the adaptive feedforward formulation with a large choice of feedback gains where $K_P = 20.0$ and $K_D = 10.0$. This choice satisfies the condition (51) in addition to the SPR condition to guarantee the stability. The simulation result demonstrates the convergence of tracking error to zero as well as the estimated parameters to $\boldsymbol{\theta} = [-k, -d]^{\mathrm{T}} = [2, 2]^{\mathrm{T}}$.

a *necessary and sufficient* condition for asymptotic hyperstability, i.e. the tracking error is bounded and asymptotically converges to zero. This result indicates that the condition in FEL, $K_D^2 > K_P$, is not only a sufficient but also necessary condition for asymptotic convergence of the tracking error. To the best of our knowledge, this is the first time that SPR of the tracking error dynamics is explicitly shown to be a *necessary* condition for the stability in a nonlinear adaptive control framework in terms of asymptotic hyperstability.

Second, we considered the adaptive feedforward formulation of the adaptive control and FEL. This is closer to the original formulation of FEL which only employs desired states in the feedforward model, e.g. similar to a computed torque controller. It turns out that the adaptive feedforward formulation imposes a much tighter condition for the choice of the feedback gains associated with the property of the plant dynamics to guarantee the stability of learning. Thus, care must be taken in the choice of feedback gains for FEL, especially in the adaptive feedforward formulation. In Ushida and Kimura (2002), similar theoretical studies of FEL including time delay are presented for a specific class of plant dynamics such as a robot arm. One of the differences between our work and Ushida and Kimura's (2002) is that we explicitly address the problem of how to select feedback gains to ensure stability of FEL. In contrast, the study by Ushida and Kimura (2002) only mentions that feedback gains need to be selected such that the tracking error dynamics with respect to the non-adaptive control system using all the correct parameters in the plant dynamics are stable. In practice, analysis of the tracking error dynamics by Ushida and Kimura (2002) is not likely to be straightforward since the tracking error dynamics are highly nonlinear and rather complicated even without time delay. In contrast, in this paper, the tracking error dynamics of the non-adaptive system are linear, and we provide more explicit condition for the choice of feedback gains to guarantee stability.

Numerical results demonstrate the significant difference in the stability property between the adaptive feedback formulation and the adaptive feedforward formulation, particularly when learning of unstable plant dynamics is considered. As a straightforward extension to more general plant dynamics with $g(\mathbf{x}) \neq 1$ in (3) for the adaptive feedback formulation, the Lyapunov stability results remain the same by introducing the parameter projection method as discussed by Nakanishi et al. (2004). For the adaptive feedforward formulation with $g(\mathbf{x}) \neq 1$, it can be shown that the upper bound of the control input u needs to be considered to guarantee stability. In our future work, we will address a generalization of our work to a class of nonlinear MIMO systems and an inverse dynamics representation.

We believe that the theoretical results presented in this paper can be beneficial for the growing number of studies on motor learning and control in humans and humanoid robots in interdisciplinary fields such as robotics and computational neuroscience. In robotics, the ideas presented in this paper can be applied to design theoretically principled algorithms for motor learning such as biomimetic oculomotor control in humanoid robots (Shibata & Schaal, 2001). In computational neuroscience, we hope that the mathematical developments in this paper could help gaining further insights into the theoretical principles of human motor learning such as impedance learning in unstable force fields (Burdet, Osu, Franklin, Milner, & Kawato, 2001).

Acknowledgements

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Appendix A. Lefschetz–Kalman–Yakubovich lemma and positive real transfer function

Lemma 1. *Lefschetz–Kalman–Yakubovich lemma (Tao & Ioannou, 1990). Given $\mu > 0$, a matrix $\mathbf{A}$ such that $\det(s\mathbf{I} - \mathbf{A})$ has only zeros in the open left half plane, a real vector $\mathbf{b}$ such that $(\mathbf{A}, \mathbf{b})$ is completely controllable, a real vector $\mathbf{c}$, a scalar d, and an arbitrary real symmetric positive definite matrix $\mathbf{L}$; then a real vector $\mathbf{q}$ and a real matrix $\mathbf{P} = \mathbf{P}^{\mathrm{T}} > 0$ satisfying*

$$\mathbf{A}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{A} = -\mathbf{q}\mathbf{q}^{\mathrm{T}} - \mu\mathbf{L} \tag{A1}$$

$$\mathbf{P}\mathbf{b} - \mathbf{c}^{\mathrm{T}} = \sqrt{(2d)}\mathbf{q} \tag{A2}$$

exist if and only if $h(s) = \mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} + d$ is a strictly positive real matrix and μ is sufficiently small.

Definition 1. Positive real transfer function (page 509 in Krstić, Kanellakopoulos, & Kokotović, 1995). A rational transfer function $G(s)$ is said to be positive real if $G(s)$ is real for all real s, and $\mathrm{Re}\{G(s)\} \geq 0$ for all $\mathrm{Re}\{s\} \geq 0$. If, in addition, $G(s - \epsilon)$ is positive real for some $\epsilon > 0$, then $G(s)$ is said to be strictly positive real.

Note that the scalar positive real transfer function $G(s)$ is stable, minimum-phase and of relative degree not exceeding one (Kaufman, Bar-Kana & Sobel, 1993). In addition, any scalar transfer function of a relative degree higher than one is not positive real (Khalil, 1996).

Consider the following linear time-invariant system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \qquad \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \tag{A3}$$

where $\mathbf{x} \in \Re^n$, $\mathbf{u} \in \Re^m$ and $\mathbf{y} \in \Re^m$. $\mathbf{H}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$ is the transfer matrix of the system above.

The following theorem states the necessary and sufficient condition of strictly positive real system matrices for a special case of the dynamical system given by (A3) (Tao & Ioannou, 1990).

Theorem 1. (*Tao & Ioannou, 1990*). *In (A3), let $n = 1$ and $m = 1$ or $n = 2$ and $m = 1$ and let $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ be minimal, $\mathbf{D} = 0$ and $\mathbf{B} \neq 0$, then $\mathbf{H}(s)$ is a strictly positive and real matrix if and only if the following conditions hold*

1. *All eigenvalues of A have negative real parts*
2. $\mathbf{CB} = (\mathbf{CB})^{\mathrm{T}} > 0$
3. $\mathbf{CAB} + (\mathbf{CAB})^{\mathrm{T}} < 0$

Appendix B. Passivity, output-dissipativity, and asymptotic hyperstability

Definition 2. Passivity (Arimoto & Naniwa, 2000). If for any initial state $x(0)$ and any $t > 0$ the input–output pair $\{u, y\}$ of the objective system satisfies

$$\int_0^t \mathbf{y}^{\mathrm{T}}(\tau)\mathbf{u}(\tau)\mathrm{d}\tau \geq -\gamma_0^2 \tag{B1}$$

with $\gamma_0^2 > 0$ dependent on only the initial state $\mathbf{x}(0)$ and vanishing at $\mathbf{x}(0) = 0$, then the pair $\{\mathbf{u}, \mathbf{y}\}$, concerning the system is said to satisfy passivity.

Definition 3. Output-dissipativity (Arimoto & Naniwa, 2000). If the input–output pair $\{\mathbf{u}, \mathbf{y}\}$, of the objective system satisfies

$$\int_0^t \mathbf{y}^{\mathrm{T}}(\tau)\mathbf{u}(\tau)\mathrm{d}\tau \geq -\gamma_0^2 + \frac{\gamma^2}{2}\int_0^t \|\mathbf{y}(\tau)\|^2\,\mathrm{d}\tau \tag{B2}$$

with some positive constant γ^2 that does not depend on $\mathbf{x}(0)$, and a constant $\gamma_0^2 > 0$ that depends on only the initial state $\mathbf{x}(0)$ and vanishes when $\mathbf{x}(0) = 0$, then the pair of the system satisfies output-dissipativity.

Consider the negative feedback connection of the following linear dynamical system (B3) with a passive

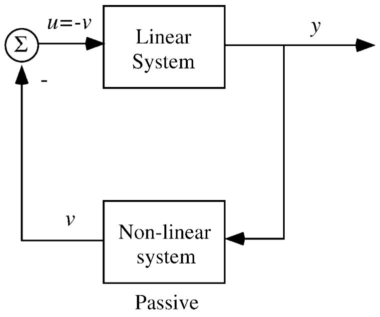

Fig. B1. A negative feedback connection of a linear system and a passive (hyperstable) nonlinear system. (This diagram is a revised version of Fig. B2 in Arimoto (1996) through personal communication with Prof. Arimoto.)

nonlinear system as depicted in Fig. B1:

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu} \qquad \mathbf{y} = \mathbf{Cx} + \mathbf{Du} \tag{B3}$$

where $\dim(\mathbf{u}) = \dim(\mathbf{y}) = m$. $\mathbf{F}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$ is the transfer matrix of the system above.

Definition 4. Hyperstablity and asymptotic hyperstablity (Arimoto, 1996). If the linear part can be expressed by (B3) and, for any output $\mathbf{v}(t)$ of the nonlinear part satisfying passivity such that

$$\int_{t_0}^{t} \mathbf{v}^{\mathrm{T}}(\tau)\mathbf{y}(\tau)\mathrm{d}\tau \geq -\gamma_0^2 \tag{B4}$$

where γ_0 is a constant dependent only on the state of the nonlinear part at $t = t_0$, the solution to the first equation (B3) satisfies

$$\|\mathbf{x}(t)\| \leq K\{\|\mathbf{x}(t_0)\| + \gamma_0\} \tag{B5}$$

with some constant $K > 0$, then the overall system of Fig. B1 is said to be hyperstable. Further, if the overall system is hyperstable and, for any bounded output $\mathbf{v}(t)$ satisfying (B4), $\mathbf{x}(t) \to 0$ as $t \to \infty$, then it is said to be asymptotically hyperstable.

Theorem 2. *Theorem B4 in Arimoto (1996). Assume that* $\{\mathbf{A}, \mathbf{B}\}$, *and* $\{\mathbf{C}, \mathbf{A}\}$, *of the linear system* (B3) *are controllable and observable, respectively, and both* $\mathbf{B}$ *and* $\mathbf{C}$ *are of full rank. Then a necessary and sufficient condition for the overall system of Fig. B1 to be hyperstable is that the transfer function* $\mathbf{F}(s)$ *of the linear part is positive real.*

Theorem 3. *Theorem B5 in Arimoto (1996). Under that same assumptions on the linear part, a necessary and sufficient condition for the overall system of Fig. B1 to be asymptotically hyperstable is that* $\mathbf{F}(s)$ *is strictly positive real.*

Proofs of the sufficiency of Theorems 2 and 3 are presented by Arimoto (1996). The necessity part was proven by Arimoto, but it is unpublished (S. Arimoto, personal communication, 2003).

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