Adaptive Control Using Multiple Models

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***Abstract*— Intelligent control may be viewed as the ability of a controller to operate in multiple environments by recognizing which environment is currently in existence and servicing it appropriately. An important prerequisite for an intelligent controller is the ability to adapt rapidly to any unknown but constant operating environment. This paper presents a general methodology for such adaptive control using multiple models, switching, and tuning. The approach was first introduced in [1] and [2] for improving the transient response of adaptive systems in a stable fashion. This paper proposes different switching and tuning schemes for adaptive control which combine fixed and adaptive models in novel ways. The principal mathematical results are the proofs of stability when these different schemes are used in the context of model reference control of an unknown linear time-invariant system. A variety of simulation results are presented to demonstrate the efficacy of the proposed methods.**

***Index Terms*— Adaptive control, multiple models, stability, switching, transient response.**

I. Introduction

THE requirements of any good control system are speed, accuracy, and stability. Achieving these in complex systems, in the presence of large uncertainty concerning the process to be controlled, is the challenge for the control theorist today. The realization that conventional controllers do not possess all the attributes necessary to achieve such control has, in recent years, given rise to several definitions for intelligent control. We adopt the perspective that intelligent control is merely the ability of the control system to operate successfully in a wide variety of situations by detecting the specific situation that exists at any instant and servicing it appropriately. External disturbances, changes in subsystem dynamics, parameter variations, etc., are examples of different unknown environments in which the system has to operate. Research has been in progress at Yale University, New Haven, CT, since 1991, toward the development of a general methodology for the design of stable, fast, and accurate controllers to cope with such time varying situations, using multiple models, switching, and tuning. A qualitative description of the approach may be found in [3]. The work described in this paper is the analytical part of the research activity, dealing with linear systems.

It is well known that efficient design methods for various classes of control systems can be developed only when their stability properties are well understood. This is the case with linear time-invariant (LTI) systems as well as adaptive systems, the stability properties of which have been investigated extensively. We adopt a similar philosophy for the design of controllers which involve multiple models, switching, and tuning. One of the objectives of this paper is to take a first step in this direction by addressing the question of stability for a fairly broad class of switching and tuning systems.

In [1] and [2] the problem of model reference adaptive control was considered, where multiple adaptive identification models were used to identify an LTI plant, and the adaptive controller corresponding to the one yielding the minimum of a performance index was chosen at every instant. The method resulted in improved transient response in the presence of large parametric uncertainties. In this paper, the basic idea contained in [1] and [2] is extended into a general methodology. The paper introduces different classes of switching and tuning schemes which combine fixed and adaptive models in novel ways. The aim is to determine flexible architectures for stable intelligent control that can yield a fast and accurate response, yet remain computationally efficient. This constitutes the conceptual contribution of the paper. The theoretical contribution of the paper is the proof of stability of the overall system for these different kinds of switching and tuning schemes, in the context of model reference control of unknown LTI systems. A series of simulation results is presented which demonstrates that the proposed methods indeed result in substantial improvement in performance.

The individual concepts of multiple models, switching, or tuning are not new in control theory. Multiple Kalman filter-based models were studied in the 1970's to improve the accuracy of the state estimate in estimation and control problems, by Magill, Lainiotis, and others [4]–[6]. This was followed in later years by several practical applications [7]–[10]. In all these cases, no switching was involved, and only a convex combination of the control determined by different models was used. Further, no stability results were reported.

In the context of adaptive control, switching was first introduced by Mårtennson [11]. Two kinds of switching schemes have been proposed in the literature. In direct switching [11]–[15], the choice of when to switch to the next controller, in a predetermined sequence, is based directly on the output of the plant. Such schemes have little practical utility. Indirect switching methods, in which multiple models are used to determine both when and to which controller one should switch, are more attractive for applications. This approach was first proposed by Middleton *et al*. [16] and later adapted

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in [17] and [18]. The objective in all the above efforts is to achieve stability in adaptive control with minimal prior information. In recent years, Morse [19] has been studying the use of multiple fixed models and optimization for robust set point control. In contrast to that, the work reported here focuses on improving performance in multiple environments while retaining stability.

The use of switching and tuning using multiple models containing neural networks as components has been proposed in [20] to improve the performance of nonlinear systems. However, the stability characteristics of such systems are substantially more difficult to analyze. The stability of linear switching and tuning systems of the type analyzed in this paper, together with the implicit function theorem, are currently being used in the investigation of the stability of nonlinear switching and tuning systems. The methods presented in this paper have found practical applications in robotic manipulator control [3] and are being investigated in the context of biological gait control and control of fermentation processes. All these different practical applications reveal that the methodology of control based on multiple models, switching, and tuning, while being theoretically attractive, is also practically viable.

II. General Methodology

Control in Multiple Environments: Control system design has traditionally been based on a single fixed or slowly adapting model of the system. This implicitly assumes that the operating environment is either time invariant, or varies slowly with time. As control theory is extended to more complex systems, many types of changes other than slow parameter variations are encountered, e.g., faults in the system, changes in subsystem dynamics, sensor and actuator failures, external disturbances, and changes in system parameters. In general, complex systems operate in multiple environments which may change abruptly from one context to another.

The speed and accuracy with which a controller responds to sudden and large changes may be considered as a measure of its "intelligence." From this point of view, "intelligent control" is the efficient control of dynamical systems operating in rapidly time-varying environments. While conventional robust control is restricted to sufficiently small ranges of variations, conventional adaptive control reacts too slowly to abrupt changes, resulting in large transient errors before convergence. Hence alternate control methods are needed.

Multiple Models: When the environment of a system changes abruptly, the original model (and hence controller) is no longer valid. If models are available for different environments, controllers corresponding to them can be designed *a priori*. During system operation, one has to identify the existing environment to determine the correct controller. Such identification can again be achieved if a model for each environment is known in advance. Based on these two ideas, the control strategy proposed is to determine the best model for the current environment at every instant and activate the corresponding controller.

Switching and Tuning: Since the number of available models is finite, whereas the number of possible environments can be uncountable, identification of the environment takes place in two stages. Assuming that the models and environments are parameterized suitably, the model with the smallest error, according to some criterion, is selected rapidly (switching) and then its parameters are adjusted over a slower time scale to improve accuracy (tuning). In switching, the problem is to determine when the current parameter value is unsatisfactory (i.e., when to switch) and which one to replace it with (i.e., what to switch to). In tuning, the problem is to determine the rule by which the parameter value is to be adjusted at each instant.

A. Architecture of the Control System

The proposed architecture for intelligent control is shown in Fig. 1. The system to be controlled has input u and output y. The objective is to make the control error $e_c = y^* - y$ tend to zero, where y^* is the desired output. The control system contains N identification models, denoted by $\{I_j\}_{j=1}^N$, operating in parallel. The parameter vector $\hat{p}_j$ of each I_j may either be fixed or may be tuned from an initially chosen value. The identification error between the output y_j of I_j and that of the plant is denoted as $e_j = \hat{y}_j - y$. Corresponding to each I_j is a parameterized controller C_j, whose parameter vector θ_j is chosen such that C_j achieves the control objective for I_j. The output of C_j is denoted by u_j. At every instant, one of the models I_j is selected by a switching rule, and the corresponding control input u_j is used to control the plant. This architecture was introduced in [1], where all the models and controllers were adaptive with identical structures.

Given prior knowledge of the different possible environments, the design problem is to choose the number and structure of the models and controllers as well as their parameter vectors. The control problem is to determine suitable rules for switching and tuning these parameters to yield the best performance for the given objective while assuring stability.

The architecture described above is quite general and applies to both linear and nonlinear systems. However, in this paper it is assumed that the system to be controlled is linear, primarily because the stability of the resulting switching and tuning systems can be proved. Qualitative treatments of the nonlinear case may be found in [3] and [20].

B. Choice of the Switching Rule

A natural way to decide when, and to which controller, one should switch, is to determine performance cost indexes $J_j(t)$ for each controller C_j and switch to the one with the minimum index at every instant. However, since only one control input can be used at any instant, the performance of any candidate controller can be evaluated only after it used. On the other hand, the performance of all the identification models can be evaluated in parallel at every instant. Hence the indexes $\{J_j(t)\}$ must be based on the performance of the models rather than the controllers, i.e., using identification errors $\{e_j\}$ rather than the control error e_c. From an adaptive control point of view, this rationale extends the principle of certainty equivalence from tuning to switching.

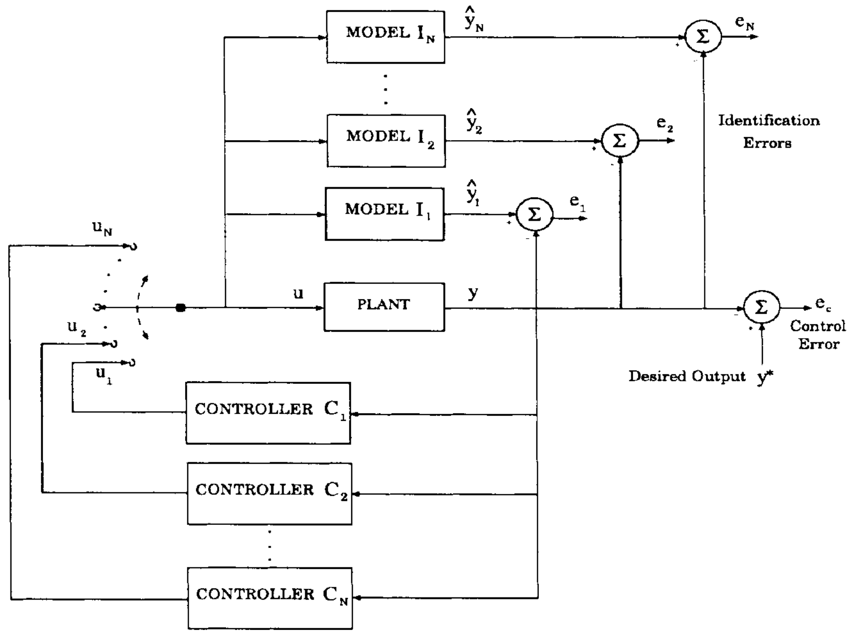

Fig. 1. A general architecture for control using N identification models and controllers.

The specific performance index proposed has the form

$$J_j(t) = \alpha e_j^2(t) + \beta \int_0^t e^{-\lambda(t-\tau)} e_j^2(\tau)\, d\tau,$$
$$\alpha \geq 0,\ \beta,\ \lambda > 0 \tag{1}$$

where α and β can be chosen to yield a desired combination of instantaneous and long-term accuracy measures. The forgetting factor λ determines the memory of the index in rapidly switching environments and ensures boundedness of $J_j(t)$ for bounded e_j. The reader is referred to [2] and [21] for more detailed discussions on the choice of this index.

The switching scheme (first proposed in [1] for the case $\lambda = 0$) consists of monitoring the performance indexes $\{J_j(t)\}$ at every instant. After every switch, a waiting period of length $T_{\min} > 0$ is allowed to elapse, and then the controller corresponding to the model with the minimum index is chosen (switched) to control the plant. The waiting period is introduced to prevent arbitrarily fast switching.

III. Model Reference Control

The effectiveness of the proposed methodology is illustrated in this paper in the context of model reference adaptive control (MRAC) of a LTI system. The structure of the models and controllers are described in this section. The derivation of specific switching and tuning schemes and their stability analysis are the subject of the next section.

The MRAC Problem [22]: The unknown plant to be controlled is LTI, single-input/single-output (SISO) with control input u and output y_p. Its transfer function is $W_p(s) = k_p[Z_p(s)/R_p(s)]$, where k_p and the $2n-1$ coefficients of the monic polynomials $Z_p(s)$ and $R_p(s)$ constitute the unknown plant parameter vector p. It is assumed that p belongs to a known compact set $\mathcal{S} \subset \mathbb{R}^{2n}$. Each plant in $\mathcal{S}$ is further assumed to satisfy the standard assumptions of MRAC [22], i.e., the order n, relative degree n^*, and sign of k_p are known and constant, and $W_p(s)$ is minimum phase. An LTI, SISO, asymptotically stable, minimum phase reference model is given with a bounded, piecewise differentiable input r and output y_m. Its transfer function $W_m(s) = 1/R_m(s)$ has the same relative degree as $W_p(s)$. The MRAC objective is to determine a differentiator-free control input u such that all signals in the overall system remain bounded, and the control error $e_c(t) \triangleq y_p(t) - y_m(t)$ tends to zero. □

A. Structure of the Models and Controllers

Parameterization of the Plant: The plant is parameterized as in conventional adaptive control. Define "sensitivity vectors" ω_1, ω_2: $\mathbb{R}_+ \to \mathbb{R}^{n-1}$ as

$$\begin{aligned}\dot{\omega}_1 &= \Lambda\omega_1 + lu\\ \dot{\omega}_2 &= \Lambda\omega_2 + ly_p\end{aligned} \tag{2}$$

where (Λ, l) is an asymptotically stable, controllable system with $\det(sI - \Lambda) = \lambda_1(s)Z_m(s)$, where $\lambda_1(s)$ is Hurwitz. Regression vectors $\underline{\omega}$, ω, $\overline{\omega}$: $\mathbb{R}_+ \to \mathbb{R}^{2n}$ are defined as

$$\begin{aligned}\underline{\omega} &\triangleq (r,\, \omega_1^T,\, y_p,\, \omega_2^T)^T\\ \omega &\triangleq (u,\, \omega_1^T,\, y_p,\, \omega_2^T)^T\\ \overline{\omega} &\triangleq W_m(s)I_{2n\times 2n}\{\omega\}.\end{aligned} \tag{3}$$

It is known [22], [23] that unique constants $\beta_0^* \triangleq k_p/k_m \in \mathbb{R}$, $\beta_1^* \in \mathbb{R}^{n-1}$, $\alpha_0^* \in \mathbb{R}$, $\alpha_1^* \in \mathbb{R}^{n-1}$ exist such that the output of the plant can be expressed as

$$\begin{aligned}y_p &= W_m(s)\{\beta_0^* u + \beta_1^{*T}\omega_1 + \alpha_0^* y_p + \alpha_1^{*T}\omega_2\}\\ &\triangleq W_m(s)\{p^{*T}\omega\}\\ &= p^{*T}\overline{\omega}\end{aligned} \tag{4}$$

where $p^* \triangleq (\beta_0^*,\, \beta_1^{*T},\, \alpha_0^*,\, \alpha_1^{*T})^T \in \mathbb{R}^{2n}$ represents the parameter vector of the plant.

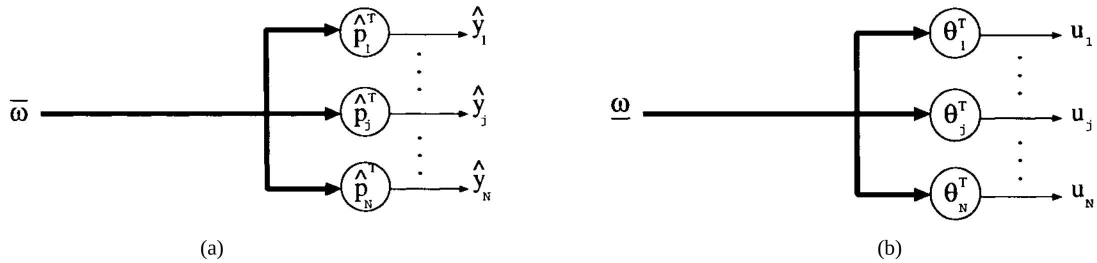

Fig. 2. Structure of the identification models and controllers. The output of model I_j is $\hat{y}_j = \hat{p}_j^T\overline{\omega}$, and the output of controller C_j is $u_j = \theta_j^T\underline{\omega}$. $\hat{p}_j$ and θ_j are constant vectors if the pair (I_j, C_j) is fixed and are functions of time if (I_j, C_j) is adaptive. (a) Identification models and (b) controllers.

Structure of the Identification Models and Controllers: The structure of the models and controllers is shown in Fig. 2. The identification error of I_j is defined as $e_j = \hat{y}_j - y_p$ and has the expression

$$\begin{aligned} e_j &= (\hat{p}_j - p^*)^T\overline{\omega} \\ &\triangleq \tilde{p}_j^T\overline{\omega} \end{aligned} \tag{5}$$

where $\tilde{p}_j$ is the parameter error vector $\hat{p}_j - p^*$. The corresponding control input error for C_j is defined as $\tilde{u}_j(t) \triangleq u_j(t) - u^*(t)$, where $u^* \triangleq \theta^{*T}\underline{\omega}$ is the ideal control input to the plant which results in $y_p(t) \equiv y_m(t)$, and the ideal control parameter vector θ^* is computed from p^* as

$$\begin{aligned} k^* &\triangleq \frac{1}{\beta_0^*} \\ \theta_1^* &\triangleq -\frac{\beta_1^*}{\beta_0^*} \\ \theta_0^* &\triangleq -\frac{\alpha_0^*}{\beta_0^*} \\ \theta_2^* &\triangleq -\frac{\alpha_1^*}{\beta_0^*} \\ \theta^* &\triangleq (k^*, \theta_1^{*T}, \theta_0^*, \theta_2^{*T})^T. \end{aligned} \tag{6}$$

For fixed (I_j, C_j), θ_j is the ideal control parameter vector [determined using (6)] corresponding to $\hat{p}_j$. If (I_j, C_j) is adaptive, $\hat{p}_j(t)$ is tuned using a suitable tuning algorithm (e.g., gradient or least squares) and $\theta_j(t)$ is computed at every instant using (6) assuming that $\hat{p}_j(t) = p^*$. Alternately, $\theta_j(t)$ may be dynamically adjusted at every instant as proposed in [23].

The input to the plant $u(t)$ is chosen by the switching rule to be one of the elements of the set $\{u_1(t), u_2(t), \cdots, u_N(t)\}$. It can also be expressed as $u(t) = \theta^T(t)\underline{\omega}(t)$, where the control parameter vector $\theta(t) \in \{\theta_1(t), \theta_2(t), \cdots, \theta_N(t)\}$.

Error Equation: The following error equation is used in the stability analysis of the system:

$$\beta_0^*\tilde{u}_j = \epsilon_{\theta_j}^T\underline{\omega} - \tilde{p}_j^T\omega + \tilde{\beta}_{0_j}(u - u_j). \tag{7}$$

While the relation was first given in [2], its derivation is included in Appendix A for the sake of completeness. The errors $\epsilon_{\theta_j}(t)$, which are defined in (27) in Appendix A, represent the deviation of the parameters $\hat{p}_j$ and θ_j from the certainty equivalence relation (6) [2], [23]. Hence $\epsilon_{\theta_j}(t) \equiv 0$ for a fixed model and for an adaptive model with a certainty equivalence controller. For an adaptive model using dynamical adjustment of control parameters, tuning algorithms can be designed so that $\epsilon_{\theta_j}(t) \to 0$ and belongs to $\mathcal{L}^2$ [23]. Hence with no loss of generality we can assume that $\epsilon_{\theta_j}(t) \equiv 0$ in (7) in all the discussions that follow.

Comment 1: In view of the importance of (7) in the proof of stability, a qualitative description of its physical implication would be helpful to the reader. $u(t)$ denotes the actual input to the plant at instant t, while $u_j(t)$ denotes the control computed by the controller C_j. $\tilde{p}_j$ is the identification parameter error vector of I_j, and $\tilde{u}_j$ is the corresponding control input error of C_j. Assuming that $\epsilon_{\theta_j}(t) \equiv 0$ (ref. the previous paragraph), (7) relates at every instant $\tilde{u}_j(t)$, $\tilde{p}_j(t)$, and $u(t)$. If at any instant the control input $u(t)$ is chosen to be $u_j(t)$, the last term in (7) is identically zero. In such a case, the equation directly relates parametric uncertainty (i.e., $\tilde{p}_j$) to control input error ($\tilde{u}_j$) when the jth controller is being used. The last fact reflects one's intuitive feeling that a small identification parametric error should result in a small control input error. □

Comment 2: The problem stated above, and the results in Section IV, can be readily extended to cases where a finite number of distinct model structures are used, i.e., when the sign of k_p is unknown and only upper bounds on the order n and relative degree n^* are known. □

IV. Fixed and Adaptive Models

The rationale for using multiple models as proposed in this paper is to ensure that there is at least one model with parameters sufficiently close to those of the unknown plant. The parameters of the models thus chosen may be either fixed or adjustable. For the designer to decide which strategy to adopt, the rationale for using fixed models or adaptive models must be clearly understood.

From a computational point of view, adaptive models are inefficient because of the need to update their parameter vectors $\{\hat{p}_j(t)\}$ at every instant (see Fig. 2). A more significant drawback arises when the environment of the plant changes with time. If the environment is initially constant over a long interval, the adaptive parameters $\{\hat{p}_j(t)\}$ will converge to a neighborhood of the parameter vector p^*, representing that environment. When the environment changes at this stage, i.e., p^* changes discontinuously, the change has to be detected, and the parameters $\{\hat{p}_j(t)\}$ must be reset to their original starting locations to identify the new p^* rapidly. Fixed models do not have these drawbacks, implying that the same strategy can be used in stationary and time-varying environments. However, since fixed models can represent exactly only a finite number of environments, adaptive models are needed to improve accuracy asymptotically, as explained in Section II. Thus, in the triad—speed, accuracy, and stability—we see that fixed models together with switching yield the desired speed, while adaptive models and tuning provide the desired accuracy. The fundamental problem of stability consequently remains.

In this section, we consider in succession four different combinations of fixed and adaptive models. These correspond to i) all adaptive models, ii) all fixed models, iii) fixed models and one free-running adaptive model, and iv) fixed models with one free-running adaptive model and one adaptive model with reinitializable parameters. Our objective is to determine the one which provides the best tradeoff between performance and computational efficiency. As emphasized in the introduction, for any one of these methods to be practically applicable, their stability properties must be well understood. For the first configuration, the proof of stability has already appeared in [2]. The stability of the overall system when the other three configurations are used is proved in this section. The approach used here is distinctly different from that adopted in [2].

A. All Adaptive Models

In this configuration, which can be viewed as an extension of conventional indirect adaptive control, multiple adaptive identification models are used with initial parameter values distributed over the set $\mathcal{S}$. The method was considered in detail in [1] and [2].

Tuning Scheme: The identification parameters $\hat{p}_j$ of I_j and the control parameters θ_j of the corresponding C_j are tuned simultaneously. In particular, θ_j can be determined from $\hat{p}_j$ using either certainty equivalence or dynamical adjustment [23]. Any stable tuning scheme can be used, provided the following conditions are satisfied [21].

Identification Conditions:

1) $$\hat{p}_j,\, \theta_j \in \mathcal{L}^\infty$$

2) $$\dot{\hat{p}}_j,\, \dot{\theta}_j \in \mathcal{L}^\infty \cap \mathcal{L}^2$$

3) $$\frac{e_j}{\sqrt{1+\overline{\omega}^T\overline{\omega}}} \in \mathcal{L}^2$$

4) $$\epsilon_{\theta_j}(t) \to 0. \tag{8}$$

Theorem 1 states the stability properties of the above configuration. Its proof may be found in [2].

Theorem 1: Consider the switching and tuning system defined in Section III-A, where the N models are all adaptive. Assume that the tuning scheme is as given above. Let any arbitrary switching scheme be used such that there is some minimum interval $T_{\min}$ between successive switches. Then for any $T_{\min} > 0$ it follows that all signals in the overall system are uniformly bounded. Further, all the identification errors $\{e_j(t)\}$ belong to $\mathcal{L}^2$ and tend to zero asymptotically. Finally, the input errors $\{\tilde{u}_j(t)\}$, the overall input error $\tilde{u}(t)$, and control error $e_c(t)$ also tend to zero. □

The specific class of switching schemes described in Section II-B (with $\lambda = 0$) was recommended in [2] for performance improvement. Simulation results demonstrated a dramatic improvement in transient response when multiple adaptive models are used together with this switching scheme. The number of models required to obtain a desired level of transient performance depends upon the region of uncertainty $\mathcal{S}$ as well as the sensitivity of the transient response to mismatches in parameters [21].

B. All-Fixed Models

The advantages of fixed models over adaptive models were discussed earlier in this section. Hence, only the stability aspects are discussed here.

The problem differs from the all-adaptive case in three ways. In the first place, to ensure stability with switching, each plant in $\mathcal{S}$ must be stabilized by at least one of the fixed controllers. This implies a sufficient density of fixed models in $\mathcal{S}$. Second, stability is not assured for arbitrary switching schemes, since there may be controllers which destabilize the given plant. Ideally, we would like to determine the class of all stable switching schemes and then optimize from this set for the best performance. Since this is analytically intractable, we seek a parameterized class of stable switching schemes which provides sufficient flexibility for performance improvement. The scheme described in Section II-B was shown in [1] and [2] to offer such flexibility; in this subsection, we prove that the same class results in stability with fixed models. Finally, while a model activated at time t according to this scheme has the minimum index at t, it may deviate during the waiting period $(t,\, t+T_{\min})$, resulting in inaccurate control inputs. Hence $T_{\min}$ must be sufficiently small to limit such deviations. The above stability considerations are summarized in Theorem 2.

Theorem 2: Consider the switching and tuning system described in Section III-A, where the N models are all fixed and the proposed switching scheme is used with $\beta,\, \lambda,\, T_{\min} > 0$, and $\alpha \geq 0$. Then, for each plant with parameter vector $p \in \mathcal{S}$, there is a positive number T_S and a function $\mu_S(p,\, T_{\min}) > 0$, such that if:

- the waiting time $T_{\min} \in (0,\, T_S)$;
- there is at least one model I_k with parameter error $\|\hat{p}_k - p\| < \mu_S(p,\, T_{\min})$;

then all the signals in the overall system, as well as the performance indexes $\{J_j(t)\}$, are uniformly bounded. Here T_S depends only upon $\mathcal{S}$, and μ_S also depends upon α, β, λ, and $\mathcal{S}$. □

The proof is by contradiction and proceeds in three stages. In the first stage, it is assumed that the state grows in an unbounded fashion. It is then shown that if the overall input error $\tilde{u}$ is sufficiently small compared to the state, the latter must be bounded, which is a contradiction. In the second stage, it is shown that if the waiting time $T_{\min}$ is sufficiently small and the state is large enough, the above condition on $\tilde{u}$ is in turn assured by a corresponding sufficient condition on the identification errors. The latter condition states that if u_j is used as the control input over an interval $[t_1,\, t_2)$, the corresponding identification error e_j must be sufficiently small compared to the state over an interval $[t_1 - T^*,\, t_2)$. The final stage in the proof establishes that the proposed switching scheme, together with the fact that μ_S in Theorem 2 is chosen to be sufficiently small, assures that the condition on e_j is satisfied if the state is large enough.

The proof makes use of two lemmas, Lemmas 1 and 2, which are stated and proved[1] in Appendix B.

Proof: From the results of [22], the overall system can be represented as

$$\dot{x} = Ax + b\beta_0^* \tilde{u} + br, \quad y_p = c^T x \tag{9}$$

where A is an asymptotically stable matrix, $\tilde{u}(t) = \tilde{\theta}^T(t)\underline{\omega}(t)$ is the input error at time t, r is the bounded reference input, and $c^T(sI - A)^{-1}b = W_m(s)$. Since there is a minimum interval $T_{\min}$ between switches and the individual control parameters θ_j are bounded, (9) is a linear time-varying system with bounded, piecewise continuous coefficients and inputs. Hence a unique solution exists on $[0, \infty)$.

Let the supremum function of the state be defined by

$$M(t) \triangleq \sup_{\tau \le t} \|x(\tau)\|, \qquad t \ge 0.$$

We assume that $\lim_{t\to\infty} M(t) = \infty$. With no loss of generality we can assume that $M(t) > 1$, $t \ge 0$. Then from (9), since $M(t)$ can grow at most exponentially, there exist constants $\overline{c} \ge 1$ and $\overline{\lambda} > 0$ such that

$$M(t) < \overline{c}e^{\overline{\lambda}(t-\tau)}M(\tau), \qquad t \ge \tau \ge 0. \tag{10}$$

Let $\eta^* \triangleq 1/\|e^{At}b\|_1$. From Lemma 1 in Appendix B, it follows that if the input error satisfies the inequality

$$|\beta_0^* \tilde{u}(t)| < \eta^* M(t), \qquad t \ge t_0 \tag{11}$$

for some finite time t_0, then (9) will have bounded solutions, resulting in an immediate contradiction. The proof is completed in the next two stages by establishing this condition.

Since $\tilde{u}(t)$ belongs to the set $\{\tilde{u}_1(t), \tilde{u}_2(t), \cdots, \tilde{u}_N(t)\}$ for every t, (11) is clearly equivalent to the following:

$$|\beta_0^* \tilde{u}_j(t)| < \eta^* M(t), \qquad \text{whenever} \quad u(t) \equiv u_j(t),\, t \ge t_0. \tag{12}$$

The above condition states that the input error of the active controller must be small at every instant. A sufficient condition for (12) to hold is derived in the following paragraph.

Consider a typical switching interval $[t_1, t_2)$, $t_2 \ge t_1 + T_{\min}$, such that C_j is activated at t_1 and deactivated at t_2. Here $0 < T_{\min} < T_S$ where the latter is to be chosen. For establishing (12), $[t_1, t_2)$ is divided into three parts—the instant t_1, the waiting period $(t_1, t_1 + T_{\min})$, and the remaining interval $[t_1 + T_{\min}, t_2)$ (which could be empty if C_j is deactivated at $t_1 + T_{\min}$). Let

$$\mathcal{T} = \{t_1\} \cup [t_1 + T_{\min}, t_2)$$

denote the set of instants where the choice of C_j to control the plant is made by minimizing the performance indexes. Assume that over the set $\mathcal{T}$, the input error satisfies

$$|\beta_0^* \tilde{u}_j(t)| < \frac{\eta^*}{2} M(t), \qquad t \in \mathcal{T}. \tag{13}$$

[1] J. Hespanha at Yale pointed out an error in an earlier version of the proof of Lemma 2.

Since switching is forbidden over $(t_1, t_1 + T_{\min})$, it is not certain that the input error will remain small over this interval. However, it can be imagined that if $T_{\min}$ is sufficiently small, $\tilde{u}_j(t)$ must remain close to $\tilde{u}_j(t_1)$ over the interval $[t_1, t_1 + T_{\min})$. It is straightforward to show [21], using (13), the mean value theorem, and the assumed unboundedness of $M(t)$, that there exists a maximum time T_S, proportional to η^*, such that if $T_{\min} < T_S$, then

$$|\beta_0^* \tilde{u}_j(t)| < \eta^* M(t), \qquad t \in (t_1, t_1 + T_{\min}). \tag{14}$$

Inequalities (13) and (14) together imply (12). It thus remains to establish (13). The latter is stated in terms of the input error of the active controller. On the other hand, the switching scheme—which determines which controller is active—is expressed in terms of the identification error. In what follows, a sufficient condition for (13) to hold is given in terms of the identification error of the active model.

Lemma 2 in Appendix B is used for this purpose. The lemma relates the identification error $e_j = \tilde{p}_j^T \overline{\omega} = W_m(s)\{\tilde{p}_j^T \omega\}$ to the corresponding signal $\tilde{p}_j^T \omega$. To apply the lemma, we use the error (6), and the fact that $\epsilon_{\theta_j}(t) \equiv 0$ for a fixed model, to equate (13) to the following:

$$|\tilde{p}_j^T \omega(t)| < \frac{\eta^*}{2} M(t), \qquad t \in \mathcal{T}. \tag{15}$$

The conditions required for Lemma 2 to be applicable are satisfied, since the fixed models I_j satisfy the tuning conditions (30), there is a minimum interval of $T_{\min} > 0$ between switches, and the assumed unboundedness of $M(t)$ assures that it will be greater than the required lower bound in Lemma 2 after some time T_0. Applying Lemma 2, it is seen that for (15) to hold, it is sufficient if e_j satisfies the following condition for some $t_0 > T_0$:

$$|e_j(\tau)| < \epsilon^* M(\tau), \qquad \tau \in [t - T^*, t], \quad \forall t \in \mathcal{T},\, t_1 \ge t_0 \tag{16}$$

where $\epsilon^* > 0$ is chosen to satisfy

$$\max_j \{g_j(\epsilon^*, T_{\min})\} < \frac{\eta^*}{2}$$

the maximum being taken over all the fixed models, and T^* is defined as

$$T^* \triangleq \max\{T_{\min}, \Delta_1(\epsilon^*), \Delta_2(\epsilon^*), \cdots, \Delta_N(\epsilon^*)\} > 0.$$

Here the functions $g_j(\cdot, \cdot)$ and $\{\Delta_j(\cdot)\}$, defined in the statement of Lemma 2, are continuous and $\lim_{\epsilon\to 0} g_j(\epsilon, T_{\min}) = 0$.

Equation (16) implies that a controller C_j is activated at instant t_1 only if the corresponding identification error e_j is sufficiently small compared to $M(t)$, over a sufficiently long interval prior to t_1. Further, C_j is retained over an interval $[t_1 + T_{\min}, t_2)$, after the end of the waiting period, only if e_j remains small over the interval $[t_1 + T_{\min} - T^*, t_2)$. This completes the second stage of the proof in which (11), for contradicting the unboundedness of $M(t)$, has been converted into (16) on the active identification error. According to the sequence of arguments used thus far, if e_j is small according to

(16), then $\hat{p}_j^T\omega$ is small according to (15), which in turn implies that $\tilde{u}_j$ is small in the sense of (12), which finally implies the boundedness of $M(t)$. The final stage of the proof is thus to establish (16) by invoking the properties of the switching scheme.

Let I_k be the fixed model with parameters closest to those of the plant. Let its parameter error $\|\hat{p}_k - p^*\| \triangleq \mu_S$, where μ_S is to be chosen. Its identification error satisfies $|e_k(t)| \leq \|\hat{p}_k - p\| \, \|\overline{\omega}(t)\| < \mu_S c_0 M(t)$, where $c_0 > 0$ is a constant such that $\|\overline{\omega}(t)\| < c_0 M(t)$. Letting $\mu^* \triangleq \mu_S c_0$, we obtain that

$$|e_k(t)| < \mu^* M(t), \qquad t \geq 0. \tag{17}$$

Choosing $\mu^* < \epsilon^*$ it is seen that (16) holds for the model I_k for all $t \geq 0$. We will show that (16) holds for any arbitrary model I_j, over the intervals in which it is actually used. The intuitive basis for this is that the switching scheme ensures that any model I_j will be chosen in preference to I_k at any instant $t \in \mathcal{T}$ only if $J_j(t) \leq J_k(t)$. The presence of the integral term in $J_j(t)$ ensures this will be true only if $|e_j|$ is smaller than $|e_k|$ over a sufficiently long interval prior to t. By choosing μ_S sufficiently small, the above interval can be made larger than T^*, ensuring that (16) is satisfied.

From the definition of the switching scheme and the set $\mathcal{T}$, it follows that

$$J_j(t) \leq J_k(t), \qquad t \in \mathcal{T}. \tag{18}$$

To show that condition (16) is satisfied by model I_j, we assume the contrary, i.e., there is a $\tau_1 \in \mathcal{T}$ and a $\tau_0 \in [\tau_1 - T^*, \tau_1]$ such that

$$|e_j(\tau_0)| \geq \epsilon^* M(\tau_0). \tag{19}$$

In Appendix C, this is shown to imply that

$$J_j(\tau_1) > \epsilon_0^2 M^2(\tau_1) \tag{20}$$

where $\epsilon_0^2 \triangleq [\beta/(\lambda+2\overline{\lambda})](\epsilon^{*^2}/4\overline{c}^2)[1-e^{-(\lambda+2\overline{\lambda})\overline{T}}]e^{-(\lambda+2\overline{\lambda})T^*}$. Here $\overline{T}$ is a positive constant not depending on μ^*. For model I_k, it follows from (1) and (17) that

$$J_k(t) < \left(\alpha + \frac{\beta}{\lambda}\right)\mu^{*^2} M^2(t), \qquad t \geq 0. \tag{21}$$

Choosing μ^* as

$$\mu^* \triangleq \frac{\epsilon_0}{2\left(\alpha + \dfrac{\beta}{\lambda}\right)^{1/2}} \tag{22}$$

it follows from inequalities (20) and (21) that $J_j(\tau_1) > J_k(\tau_1)$, which contradicts (18), and thus establishes (16). Recall that the second stage in the proof demonstrated that the latter assured that (11) holds, which in turn was shown in the first stage to ensure the boundedness of $M(t)$. Since the latter was assumed to be unbounded to begin with, a contradiction arises, and the theorem is proved. ■

In light of the proof given above, Theorem 2 has the following interpretation which is intuitively appealing. Given N fixed models with parameters $\{\hat{p}_j\}_{j=1}^N$, and a switching scheme with parameters α, β, and λ, there is a waiting time $T_S(\hat{p}_j)$ associated with each I_j such that for every waiting time $T_{\min} < \min_j \{T_S(\hat{p}_j)\}$, there exist uncertainty radii $\{\mu_S(\hat{p}_j, T_{\min})\}$ which assure stability for every plant in the set $\mathcal{S} \triangleq \bigcup_{j=1}^N \{p| \, \|p - \hat{p}_j\| < \mu_S(\hat{p}_j, T_{\min})\}$. Alternately, given α, β, and λ, and a compact set $\mathcal{S}$ to which the plant parameters p belong, the number N of fixed models, to be uniformly distributed in $\mathcal{S}$, can be chosen such that Theorem 2 holds.

Comment 3: In this section as well as in the following sections, a nonzero value of $T_{\min}$ is chosen to prevent switching with infinite frequency. The proof of Theorem 2 is by contradiction and follows from the fact that the input $\tilde{u}$ is small. This in turn is assured if the control error $\tilde{u}_j$ of each controller C_j chosen, or alternately the corresponding identification error, is small. This is precisely what necessitates the use of a sufficiently small $T_{\min}$ in the proof.

The authors believe that any finite $T_{\min} > 0$ would result in stability, but a large value of $T_{\min}$ would result in slow switching and hence poor performance. Hence in the above proof, our objective is only to show that an arbitrarily small $T_{\min} > 0$ can be used. As in the case of other design parameters in adaptive control theory (e.g., σ-modification, adaptive parameter μ for unmodeled dynamics), only the existence of $T_{\min}$ can be assured. The choice of design parameters such as $T_{\min}$ and N is problem dependent and must be determined on the basis of the desired performance and the sensitivity of the plant response over the specified parameter set $\mathcal{S}$ (see simulation results in Section V). □

Comment 4: Fixed models cannot yield zero steady-state error for all reference inputs (except with exact parameter matching). The magnitude of this error will decrease with the parameter error of the closest model and increase with the input magnitude. Hence a large number of models may be needed for a small error. The simulation results in Section V verify this. This in turn motivates the inclusion of a single free-running adaptive model in the following section. □

C. Fixed Models and One Adaptive Model

The previous section reveals that a large number of fixed models may be needed to assure stability and good steady-state performance. This shortcoming may be overcome by including a single free-running adaptive model I_A which operates in parallel with the fixed models.

Switching and Tuning Scheme: The switching scheme is as described in Section II-B. The parameters of I_A are tuned exactly as in the case of all adaptive models, i.e., (8) is satisfied.

In the case of all fixed models, the requirement that the parameter error of at least one model be small enough is used merely to ensure that its identification error is sufficiently small compared to the state (17). This condition is automatically satisfied by an adaptive model after finite time since its identification error grows at a slower rate than the state of the system [2]. Hence there is no need to have a minimum number of fixed models to ensure stability. Further, stability is independent of the parameters of the switching scheme. Finally, the magnitude of the steady-state error is independent of the number of fixed models, and hence the latter need be chosen solely to improve the transient response.

The stability properties of this configuration are summarized in the following theorem.

Theorem 3: Consider the switching and tuning system described in Section III-A with N_1 fixed models and $N_2 \geq 1$ free-running adaptive models, where the latter are assumed to satisfy the identification conditions (8). Let the switching scheme described above be used with β, λ, and $T_{\min} > 0$. There exists a $T_S > 0$ such that if $T_{\min} \in (0, T_S)$, then all the signals in the overall system, as well as the performance indexes $\{J_j(t)\}$, are uniformly bounded. Here T_S depends only upon the set $\mathcal{S}$. □

Proof: The proof is a simple application of that given for Theorem 2. Let T_S be the maximum waiting time computed in the same manner as in Theorem 2, and let $T_{\min} \in (0, T_S)$. Let $\mu_S(T_{\min}) > 0$ be the corresponding minimum parameter error required for stability, and $\mu^* = \mu_S c_0$ as before. Since I_A is a free-running adaptive model, it follows from the results of [2] that $e_A(t) = o[M(t)]$, i.e., it grows at a slower rate than $M(t)$. Hence there exists a T_0 large enough such that

$$|e_A(t)| < \mu^* M(t), \qquad t \geq T_0 \tag{23}$$

implying that I_A satisfies (17). Further, from [2], $\tilde{p}_A^T(t)\overline{\omega}(t)$ is also $o[M(t)]$, which, from the argument used in the second stage of the proof of Theorem 2, further implies that

$$|\beta_0^* \tilde{u}_A(t)| < \eta^* M(t), \qquad \text{whenever } u(t) \equiv u_A(t)$$
$$t \in [t_1, t_2), \; t_1 \geq T_0 \tag{24}$$

where η^* is the same as in (11). Hence, by replacing the role of the fixed model I_k in the proof of Theorem 2 with I_A, and resetting the time origin to T_0, it follows that (24) also holds for the fixed models, and boundedness follows as in Theorem 2. ■

Even though a free-running adaptive model is present, Theorem 3 does not guarantee that the control error will tend to zero asymptotically, unlike conventional adaptive control. This is because it cannot be guaranteed that switching will settle down at the adaptive model I_A, which in turn is due to the exponentially decaying integral term in the performance index. Even though boundedness of all signals implies that the index $J_A(t)$ of I_A will tend to zero, it is still possible that switching may alternate between two fixed models I_1 and I_2 without encountering the adaptive model. This in turn implies that their indexes $J_1(t)$ and $J_2(t)$ will be smaller than $J_A(t)$ during the switching instants; however, any increase in e_1 or e_2 during the waiting periods after switching will cause $J_1(t)$ or $J_2(t)$ to increase correspondingly. This in turn will cause consequent "bursts" in the control error e_c. However, it is easily seen that the magnitude of these bursts may be made arbitrarily small by choosing $T_{\min}$ to be sufficiently small. Based on this discussion, the following corollary is stated without proof. Its proof may be obtained by adapting the proof of Theorem 3.

Corollary 1: In Theorem 3, for every $\epsilon > 0$, there exists a $T_P > 0$ such that if $T_{\min} \in (0, T_P)$, then all signals in the overall system are bounded, and further

$$\limsup_{t \to \infty} |e_c(t)| < \epsilon. \qquad □$$

D. Fixed Models with One Free-Running and One Reinitialized Adaptive Model

It is commonly accepted that the convergence time of an adaptive model will be large for large initial parametric errors. Hence in the configuration described above, a large number of fixed models may be needed to keep the transient response under control until the adaptive model has converged. If the fixed model which is closest to the given plant is assumed to be known, faster convergence can be obtained by initiating a new adaptive model from the location of the former. The same objective can be achieved on-line by starting adaptation from the location of each different fixed model that is successively chosen by the switching scheme. Either new adaptive models may be started from these locations, or a single adaptive model may be reused by resetting its parameters to each new location (i.e., fixed model) that is successively chosen. The second method is obviously preferable since only one extra adaptive model is needed, and this is the one that we recommend. The free-running adaptive model is retained for the same advantages as before.

Switching and Tuning Scheme: The switching scheme is the same as that described in earlier sections, with the reinitialized adaptive model I_R included in the switching process. The parameter vector $\hat{p}_R$ of I_R is determined as follows: if a fixed model I_j is activated by the switching scheme at any instant t, then $\hat{p}_R(t)$ is reinitialized to the value of $\hat{p}_j$. Further, the initial condition of the integral in the performance index of I_R is also reset to the corresponding value of I_j [ensuring that $J_R(t) = J_j(t)$]. Thereafter, I_R is left to adapt until the next reinitialization. The tuning algorithm used to adjust $\hat{p}_R(t)$ between reinitializations is such that (8) will hold if I_R is left to run free, and further, the following tuning conditions are always satisfied.

Tuning Conditions:

1) $$\hat{p}_R, \, \dot{\hat{p}}_R, \, \theta_R, \, \dot{\hat{\theta}}_R \in \mathcal{L}^\infty$$

2) $$\|\dot{\hat{p}}_R(t)\| \leq \gamma_R \frac{|e_R(t)|}{\sqrt{1 + \xi^T(t)\xi(t)}}, \qquad t \geq 0$$

3) $$\epsilon_{\theta_R}(t) \equiv 0 \tag{25}$$

where ξ is an equivalent state of the system [21], [22] comprising the output of the plant and the state of the controller.

The inclusion of the reinitialized adaptive model does not alter the stability and convergence properties of the overall system, as asserted in Theorem 4.

Theorem 4: Let an additional reinitialized adaptive model I_R be added to the configuration considered in Section IV-C and the switching and tuning scheme described above be used. Then the conclusions of Theorem 3 hold without change. □

The proof of the theorem is given in Appendix D. It is based on the proofs of Theorems 2 and 3, with additional arguments included to take into account the discontinuities arising from reinitialization. A qualitative discussion of the proof is given below.

If there are no reinitializations after some finite time, I_R becomes a free-running adaptive model, and hence the proof of Theorem 3 applies. Further, if I_R is never chosen by the switching scheme, stability is once again assured by Theorem 3. The remaining situation to be considered is when I_R is reinitialized infinitely often and becomes active between reinitializations. We show that even in this case, the input error $\tilde{u}_R$ of the corresponding controller C_R remains small whenever C_R is active, thus assuring stability.

As in the case of Theorems 2 and 3, this is shown by demonstrating that the identification error e_R satisfies (17). If the input u_R is used over $[t_1, t_2)$, and there is no reinitialization over the interval $[t_1 - T^*, t_1)$, then the argument used in Theorem 2 verifies (17) in this case. On the other hand, if I_R was reinitialized from a fixed model I_j which was activated at some instant $t_R \in [t_1 - T^*, t_1)$, then the fact that the error e_j of I_j satisfies (16) over the interval $[t_R - T^*, t_R]$ is used to assure the same for I_R.

Comment 5: It can be shown in a straightforward manner that Corollary 1 will also carry over exactly. Further, the proof of the theorem only requires the presence of a model with sufficiently small identification error. Hence, Theorem 2 also holds for the case when a reinitialized adaptive model is added. □

Comment 6: The procedure described in this section implements the two-stage identification process outlined in Section II for the general problem of control in time-varying environments. □

E. Robustness

It is well known that adaptive control methods which assure convergence of the control error to zero in the ideal case tend to become unstable when external disturbances are present. During the 1980's numerous methods were consequently developed to assure the robustness of adaptive controllers in the presence of bounded disturbances, time-varying parameters, and unmodeled dynamics. These methods involve modifications in the standard adaptive scheme such as the inclusion of a dead zone in the adaptive algorithm, projection of estimated parameters onto a compact convex set in parameter space, and the use of σ-modification and ϵ-modification in the adaptive algorithms. Since all the models used in the procedure outlined in this paper are either fixed or adaptive, one would expect the overall system to be robust under perturbations, if each model-controller pair is individually robust. This indeed turns out to be the case. As in the single model case, the proofs for the multiple model case can be provided for each class of perturbations separately and do not differ significantly from the former. Due to space limitations, we merely outline the principal steps involved in one specific case, where all the models used are adaptive. The reader familiar with the robust adaptive control literature [22, Appendix D] can recognize the principal arguments involved and can use the same procedure to conclude robustness in the other cases as well.

We consider the case of bounded external disturbances and demonstrate that the parameter projection method assures robustness when the given compact set $\mathcal{S}$ to which the unknown plant parameters belong is convex.

If $\xi(t)$ is the equivalent state of the system [21], [22], the proof of boundedness for the case $N = 1$ proceeds by contradiction [22]. It is assumed that $\xi(t)$ is unbounded, and hence there exist unbounded sequences $\{t_i\}$ and $\{a_i\}$ such that $\|\xi(t_i)\| = a_i$ and $\|\xi(t)\| \geq a_i$, for $t \in I_i \triangleq [t_i, t_i + a_i]$. Let e_I and ϵ_θ be the identification error and closed-loop estimation error, respectively. The set T_{i2} is defined as that subset of I_i (i.e., $I_i = T_{i1} \bigcup T_{i2}$ with $T_{i1} = T_{i2}^c$) over which

$$|e_I(t)|/\sqrt{1 + \xi^T(t)\xi(t)} + \|\epsilon_\theta(t)\| \leq \epsilon.$$

This in turn assures that $|\epsilon_\theta^T(t)\underline{\omega}(t) - \tilde{p}^T(t)\omega(t)| < c\epsilon\|\xi(t)\|$, for $t \in T_{i2}$ and some $c > 0$, and hence the input error satisfies

$$|\tilde{u}(t)| < c\epsilon\|\xi(t)\|, \qquad t \in T_{i2}. \tag{26}$$

It is then shown that ϵ can be chosen to be sufficiently small, and hence that $\xi(t)$ decays exponentially with a minimum rate $-\lambda_1 < 0$ over the set T_{i2}. Over the set T_{i1} the state cannot grow faster than a maximum rate λ_2. By choosing a_i large enough, it is shown that the measure $\mu\{T_{i2}\}$ can be made much larger than $\mu\{T_{i1}\}$, implying that $\|\xi(t)\|$ is smaller than a_i at the end of the interval I_i. Since this is a contradiction, boundedness of signals follows.

When multiple adaptive models are present, by identical arguments it follows that $|\epsilon_{\theta_j}^T(t)\underline{\omega}(t) - \tilde{p}_j^T(t)\omega(t)| < c\epsilon\|\xi(t)\|$, $t \in T_{i2}$, for *each* j. However, for the same reason as in the case when no perturbations are present, it does not follow that each input error $\tilde{u}_j$ satisfies the same condition; (26) is true only for the actual switched input error $\tilde{u}(t)$ along any permissible switching sequence of controllers that is used. In fact, using (7), it follows (see ref. Comment 1 and the derivation of (15) in the proof of Theorem 2) that if u_j is used as the input over any subinterval of T_{i2}, then $|\beta_0^*\tilde{u}_j(t)| < c\epsilon\|\xi(t)\|$ over that subinterval. Hence, considering the switching sequence of inputs chosen by the switching scheme over T_{i2}, it follows that $|\beta_0^*\tilde{u}(t)| < c\epsilon\|\xi(t)\|$ over T_{i2}. Hence, boundedness of signals follows as in the case $N = 1$ treated above.

In the case of a single adaptive model, if unmodeled dynamics are present and sufficiently small, it is well known that modifying the normalization of the adaptive algorithm to include a dynamic normalization term [24] will assure robustness. When multiple models are present, the same procedure described above can be used to assure robustness by considering the actual switched input $u(t)$ to the plant rather than the individual control inputs $\{u_j(t)\}$.

In summary, the comments made in this section show that if each model-controller pair is individually robust to bounded external disturbances or unmodeled dynamics, then the overall multiple adaptive model system is also robust for any arbitrary switching scheme with an arbitrary waiting period $T_{\min} > 0$. Simulation results presented in [21] demonstrate that the switching schemes proposed in this paper improve the transient response of conventional adaptive systems even when such perturbations are present.

V. Simulation Results

In this section we show how the different strategies proposed for adaptive control using multiple models may be used to improve the performance of conventional adaptive control systems. From a mathematical perspective, there is little agreement on the term "performance," and a variety of indexes have proposed to measure performance. In the context of the proposed adaptive systems using multiple models, a rigorous demonstration that performance is improved relative to conventional adaptive control systems involves choosing a particular performance index and proving that for all possible reference inputs, initial conditions for states and parameters, etc., the value of the index for the former is less than that for the latter. Since this is mathematically intractable at present, we resort to computer simulations to demonstrate performance improvement.

The methods proposed have been simulated extensively, over a wide variety of scenarios, and reported in [21]. Due to space limitations, only three sets of simulations are presented in this section. The first two sets pertain to transient response improvement for a time-invariant plant, while continuous performance improvement for a rapidly switching plant is considered in the third.

In all simulations presented below, the plant to be controlled has the transfer function $W_p(s) = k_p/(s^2 + a_1 s + a_0)$, where the parameters k_p, a_1, and a_0 are unknown. The control objective is to track the output $y_m(t)$ of a reference model $W_m(s) = 1/(s^2 + 1.4s + 1)$ to a square wave reference input $r(t)$ with unit amplitude and period 10 units of time. The least-squares method [25] was used to tune the parameters of the adaptive models.

Example 1: This example demonstrates the importance of switching in the proposed methodology. Let I_F be a fixed model which is close to the plant to be controlled and I_A be an adaptive model which starts far from the plant. The transient response using conventional adaptive control with I_A alone will be unsatisfactory. If I_F is included, the system will quickly switch to it because of its smaller initial identification error, resulting in improved transient response. I_A will eventually take over and bring the control error to zero. If a second reinitialized adaptive model I_R is allowed to start from I_F, the system will switch to it immediately after it switches to I_F, and fast error convergence is obtained.

In the simulation, the unknown plant has the unstable transfer function $0.5/(s^2 + 2s - 1.5)$. I_F has the transfer function $0.65/(s^2 + 1.8s - 1.7)$, and I_A is started from $1.25/(s^2 + 1.4s)$. The response obtained using I_A alone is seen from Fig. 3(a) to be unsatisfactory. When I_F is added, the system quickly switches to it at time $t = 0.15$ [Fig. 3(c)] and remains there until $t = 13.2$, at which point I_A takes over. Fig. 3(b) shows that the resulting tracking error is small over the entire interval. When a reinitialized adaptive model I_R is added, the system switches to it at $t = 0.5$ [Fig. 3(e)], and the error converges rapidly to zero [Fig. 3(d)]. □

Example 2: In this example, three increasingly larger regions of uncertainty concerning the plant parameters (k_p, a_1, a_0) are considered. These are denoted by $\mathcal{S}_1$, $\mathcal{S}_2$, and $\mathcal{S}_3$, with $\mathcal{S}_1 \subset \mathcal{S}_2 \subset \mathcal{S}_3$, where

$$\begin{aligned}\mathcal{S}_1 &\triangleq \{k_p, a_1, a_0\,|0.5 \le k_p \le 0.8,\\ &\qquad 0.5 \le a_1 \le 0.8, -2 \le a_0 \le -1.7\}\\ \mathcal{S}_2 &\triangleq \{k_p, a_1, a_0|\,0.5 \le k_p \le 1.25,\\ &\qquad 0.5 \le a_1 \le 1.25, -2 \le a_0 \le -1.25\}\\ \mathcal{S}_3 &\triangleq \{k_p, a_1, a_0|\,0.5 \le k_p \le 2,\\ &\qquad 0.5 \le a_1 \le 2, -2 \le a_0 \le 1\}.\end{aligned}$$

For each set $\mathcal{S}_i$, a comparison is made between conventional adaptive control with a single adaptive model and the four multiple model-based control strategies described in Section IV. The results are shown in Fig. 4, where each column shows the responses obtained for a given level of uncertainty. In all cases, the unknown plant has the unstable transfer function $0.5/(s^2 + 0.5s - 2)$, with parameters belonging to $\mathcal{S}_1$. The initial conditions of the plant are zero.

For the smallest region of uncertainty $\mathcal{S}_1$, a single adaptive model initiated at the center of $\mathcal{S}_1$ provides satisfactory response [Fig. 4(a)]. Control based on a fixed model at the same location does not even stabilize the plant. Even with $N = 64$ fixed models, there is a significant steady-state error [Fig. 4(f)].

When the region of uncertainty is increased to $\mathcal{S}_2$, one adaptive model yields an unacceptable response [Fig. 4(b)], and 27 adaptive models are needed [Fig. 4(d)] to obtain a response comparable to Fig. 4(a). With only fixed models, even with $N = 216$ the response is unsatisfactory [Fig. 4(g)]. With 48 fixed models and one adaptive model, the response improves markedly [Fig. 4(i)] and is significantly better than with one adaptive model, but still not as good as that with 27 adaptive models. By merely including a second reinitialized adaptive model [Fig. 4(k)], a response comparable to the latter is obtained.

The same trend was observed for the region $\mathcal{S}_3$, except that a larger number of models was needed in each case to cope with the increased uncertainty, as seen from the third column of Fig. 4. □

The basis for the proposed use of multiple models in intelligent control is that improved performance in rapidly switching environments can be achieved by ensuring fast and accurate transient response in each of the individual environments. This is demonstrated in the next scenario, in which the same approach used in the previous example is applied to a rapidly switching plant.

Example 3: The problem is the same as in Example 2, except that the parameters of the plant switch periodically between constant values in the set $\mathcal{S}_3$ at the end of every interval of 50 units. The transfer functions of the plant during six successive intervals are: $2/(s^2 + 2s + 1)$, $1/(s^2 + 2s - 1)$, $0.75/(s^2 + s - 2)$, $0.5/(s^2 + 0.75s + 0.25)$, $0.5/(s^2 + 1.5s - 0.5)$, and $1.5/(s^2 + 1s + 1)$, respectively. The response obtained using a single adaptive model is compared with that obtained with a combination of 320 fixed models and two adaptive models as used in the previous example. No attempt was made to detect plant parameter changes. The substantial

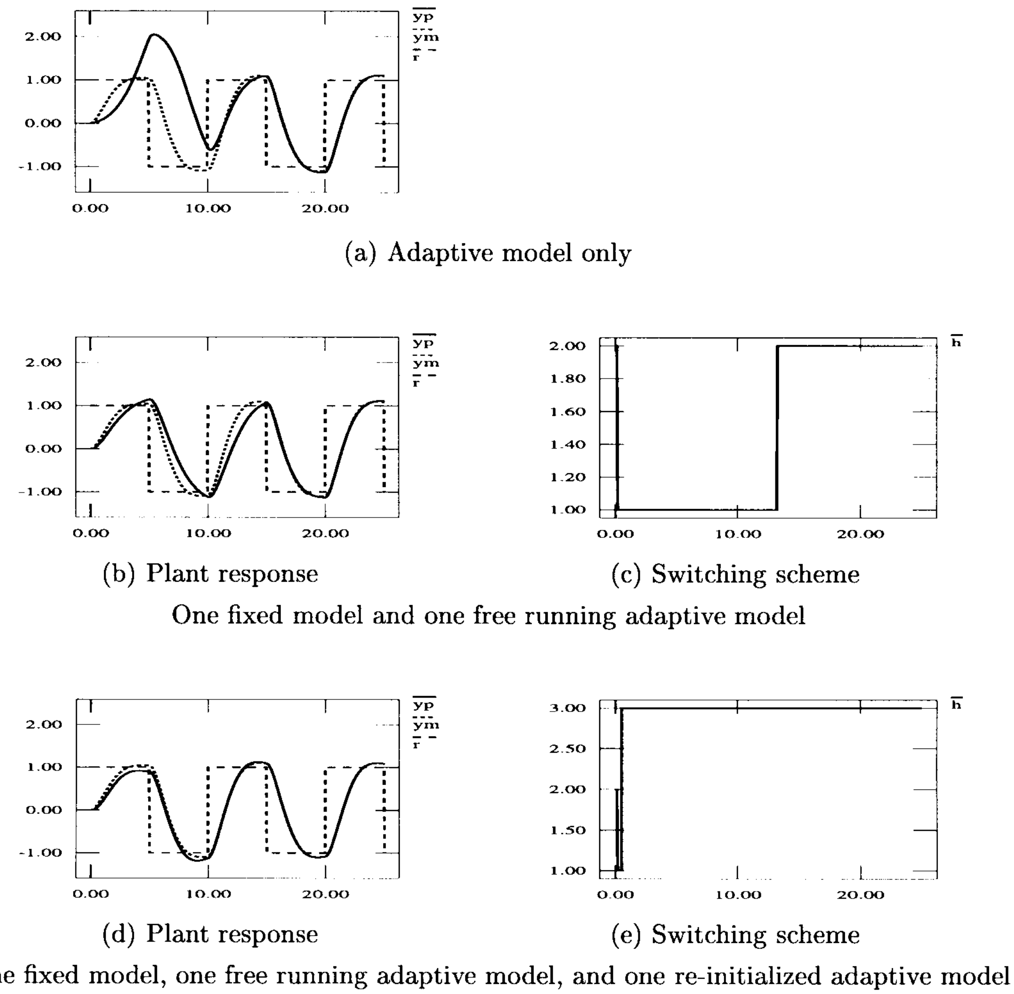

Fig. 3. Illustrative scenario of transient response improvement by switching between an adaptive model (Model 1) started far from the plant, a fixed model (Model 2) located close to the plant, and a second adaptive model (Model 3) initiated from the fixed model. The plant is unstable with transfer function $W_p(s) = 0.5/(s^2 + 2s - 1)$. The performance index used is $J_j(t) = e_j^2(t) + 5\int_0^t e^{-0.001(t-\tau)} e_j^2(\tau)\,d\tau$. (a) Adaptive model only. One fixed model, and one free-running adaptive model: (b) plant response, and (c) switching scheme, respectively. One fixed model, one free-running adaptive model, and one reinitialized adaptive model: (d) plant response, and (e) switching scheme, respectively.

improvement in performance using multiple models is evident from Fig. 5. □

Comment 7: The stability of the overall system, when the plant parameter vector switches at intervals between values in the finite set $\{p_j\}_{j=1}^N$ and control is effected using N fixed models with the same parameter vectors $\{p_j\}$, has been proved in [21]. □

VI. Conclusion

As stated in the introduction, the perspective adopted by the authors is that intelligent control is merely the ability of the control system to operate successfully in a wide variety of situations. The main idea introduced in this paper is the use of multiple models, each one of which corresponds to a different environment in which the plant may have to operate. The assumption made is that if the plant is sufficiently close to a model in some sense, the latter can be used to choose the controller. The objective, consequently, is to determine the most appropriate model at any instant, using a suitable performance criterion based on the identification error, and to activate the corresponding controller.

It is the opinion of the authors that the above methodology has applications in widely different contexts. The same ideas have been applied successfully in robotic systems and have also been used to control nonlinear systems using neural networks. The theoretical basis for the design of such systems is presented in this paper in the context of linear systems. In particular, stability proofs were provided for different situations, in which either switching alone, or both switching and tuning, are called for to improve performance.

Appendix A
Derivation of (7)

The closed-loop estimation errors ϵ_{θ_j} are defined as follows [23]:

$$\begin{aligned}
\epsilon_{k_j} &\triangleq \hat{\beta}_{0_j} k_j - 1 \\
\epsilon_{\theta_{1_j}} &\triangleq \hat{\beta}_{0_j} \theta_{1_j} + \hat{\beta}_{1_j} \\
\epsilon_{\theta_{0_j}} &\triangleq \hat{\beta}_{0_j} \theta_{0_j} + \hat{\alpha}_{0_j} \\
\epsilon_{\theta_{2_j}} &\triangleq \hat{\beta}_{0_j} \theta_{2_j} + \hat{\alpha}_{1_j}.
\end{aligned} \tag{27}$$

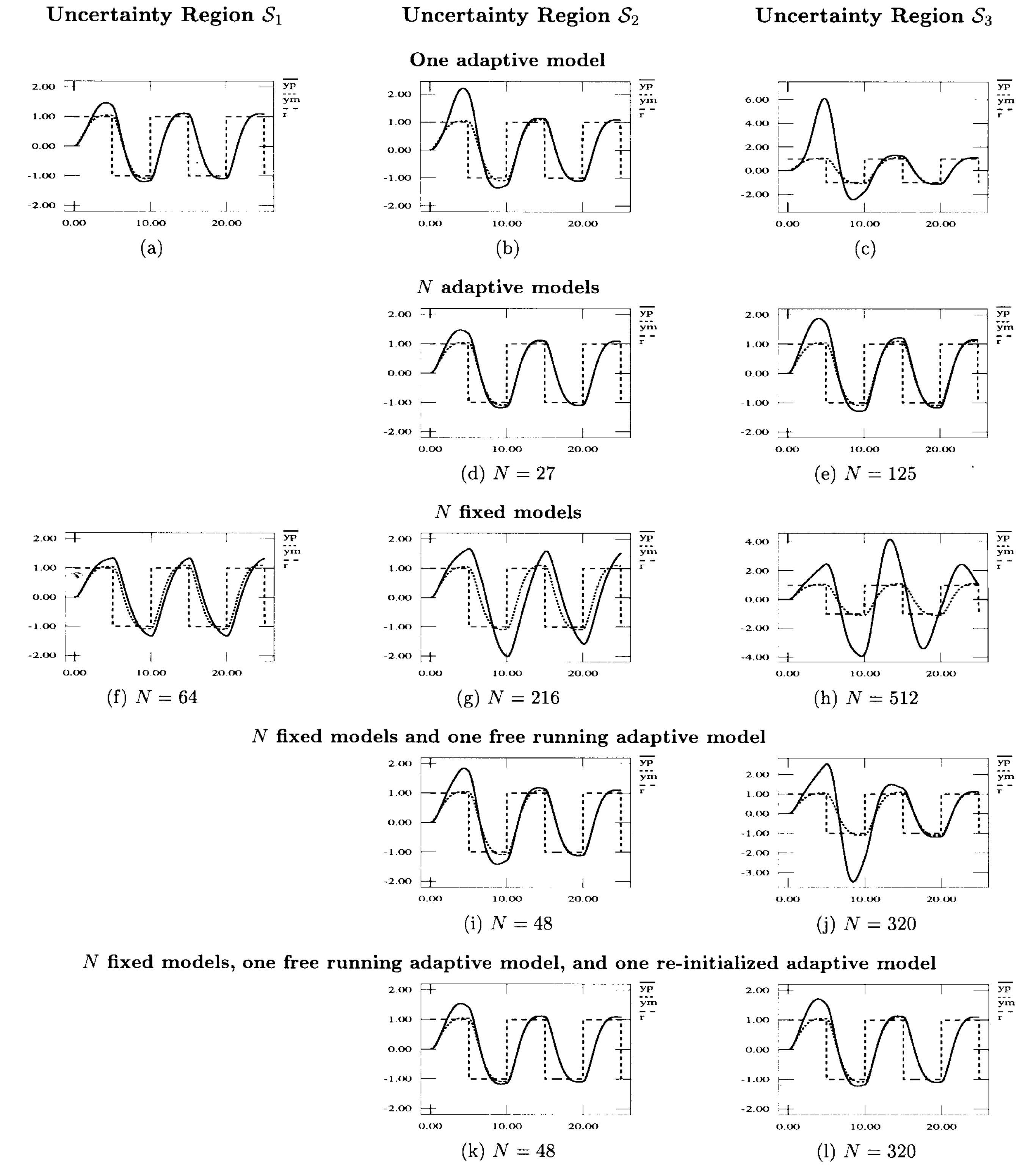

Fig. 4. Response of an unstable plant $W_p(s) = 0.5/(s^2 + 0.5s - 2)$ with different control architectures and different levels of parametric uncertainty. The performance index used is $J_j(t) = e_j^2(t) + 3\int_0^t e^{-0.001(t-\tau)} e_j^2(\tau)\, d\tau$. One adaptive model: (a)–(c). N adaptive models: (d) $N = 27$ and (e) $N = 125$, respectively; (f) $N = 64$, (g) $N = 216$, and (h) $N = 512$, respectively. N fixed models and one free-running adaptive model: (i) $N = 48$ and (j) $N = 320$, respectively. N fixed models, one free-running adaptive model, and one reinitialized adaptive model: (k) $N = 48$ and (l) $N = 320$, respectively.

From this and the definition of the parameter errors $\tilde{p}_j$ and $\tilde{\theta}_j$ and (6), it follows that (27) can be rewritten as

$$
\begin{aligned}
\epsilon_{k_j} &= \beta_0^* \tilde{k}_j + \tilde{\beta}_{0_j} k_j \\
\epsilon_{\theta_{1_j}} &= \beta_0^* \tilde{\theta}_{1_j} + \tilde{\beta}_{0_j} \theta_{1_j} + \tilde{\beta}_{1_j} \\
\epsilon_{\theta_{0_j}} &= \beta_0^* \tilde{\theta}_{0_j} + \tilde{\alpha}_{0_j} \theta_{0_j} + \tilde{\alpha}_{0_j} \\
\epsilon_{\theta_{2_j}} &= \beta_0^* \tilde{\theta}_{2_j} + \tilde{\alpha}_{0_j} \theta_{2_j} + \tilde{\alpha}_{1_j}.
\end{aligned} \tag{28}
$$

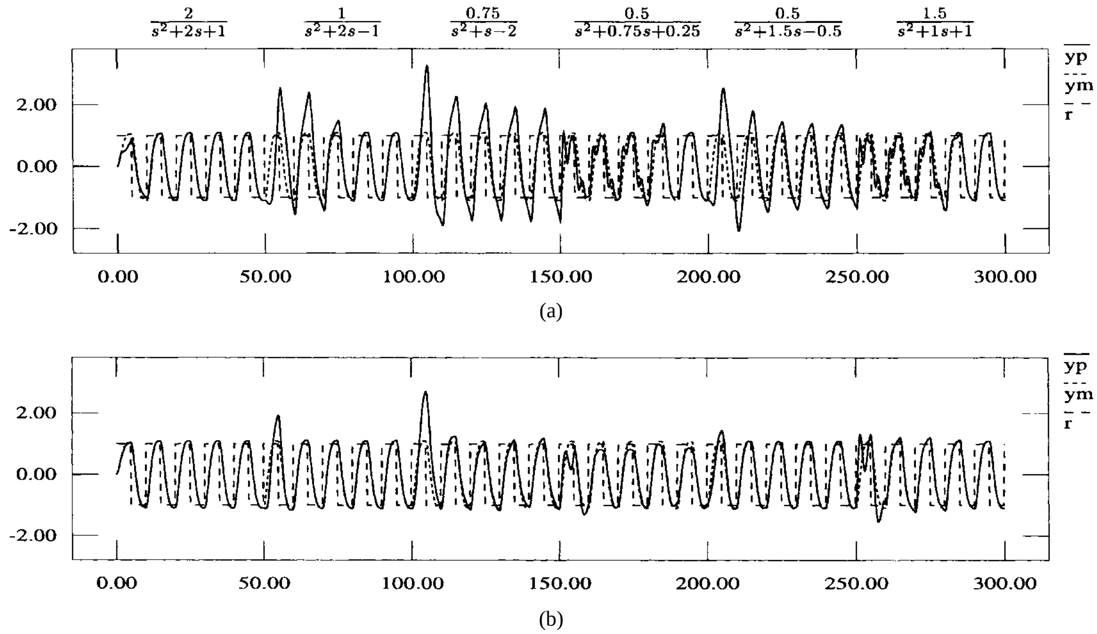

Fig. 5. Response of a plant with parameters switching between values in $\mathcal{S}_3$ every 50 units of time. The performance index used in (b) is $J_j(t) = e_j^2(t) + 3\int_0^t e^{-0.75(t-\tau)}\, e_j^2(\tau)\, d\tau$. (a) One adaptive model and (b) 320 fixed models, one free-running adaptive model, and one reinitialized adaptive model.

Using (28) and the definitions of ω, $\underline{\omega}$ (3), u_j, and θ_j, the input error $\tilde{u}_j$ can be expressed as

$$\begin{aligned}
\beta_0^*\tilde{u}_j &= \beta_0^*(\theta_j - \theta^*)^T\underline{\omega} \\
&= \beta_0^*\tilde{k}_j\, r + \beta_0^*\tilde{\theta}_{1_j}^T\omega_1 + \beta_0^*\tilde{\theta}_{0_j}y_p + \beta_0^*\tilde{\theta}_{2_j}^T\omega_2 \\
&= \epsilon_{k_j}\, r + \epsilon_{\theta_{1_j}}^T\omega_1 + \epsilon_{\theta_{0_j}}y_p + \epsilon_{\theta_{2_j}}^T\omega_2 \\
&\quad - \tilde{\beta}_{0_j}(k_j\, r + \theta_{1_j}^T\omega_1 + \theta_{0_j}y_p + \theta_{2_j}^T\omega_2) \\
&\quad - (\tilde{\beta}_{1_j}^T\omega_1 + \tilde{\alpha}_{0_j}y_p + \tilde{\alpha}_{1_j}^T\omega_2) \\
&= \epsilon_{\theta_j}^T\underline{\omega} - \tilde{\beta}_{0_j}u_j - (\tilde{\beta}_{1_j}^T\omega_1 + \tilde{\alpha}_{0_j}y_p + \tilde{\alpha}_{1_j}^T\omega_2) \\
&= \epsilon_{\theta_j}^T\underline{\omega} - \tilde{p}_j^T\omega + \tilde{\beta}_{0_j}(u - u_j)
\end{aligned} \tag{29}$$

which is (7).

APPENDIX B
STATEMENT AND PROOF OF LEMMAS 1 AND 2

This Appendix contains the statement and proof of Lemmas 1 and 2 which are used in the proofs of Theorems 2–4. Lemma 1 gives a sufficient condition on the inputs for the boundedness of the state of an asymptotically stable linear system.

Lemma 1: Consider a linear system described by the equation

$$\dot{x} = Ax + b(u + v)$$

where A is an asymptotically stable matrix and $v \in \mathcal{L}^\infty$. If u is bounded over the interval $[0, t_0)$ and $|u(t)| < \eta M(t)$ for $t \geq t_0$, where $M(t) \triangleq \sup_{\tau\leq t}\|x(\tau)\|$ and

$$\eta < \frac{1}{\|e^{At}b\|_1}$$

with $\|e^{At}b\|_1$ denoting the $\mathcal{L}^1$ norm of $e^{At}b$, then the state x is uniformly bounded. □

Proof: Since $u + v$ is bounded over the interval $[0, t_0)$, the solution $x(t)$ exists over this interval and $M(t_0) = \sup_{\tau\leq t_0}\|x(\tau)\|$ is finite. Let $\delta \triangleq \eta\|e^{At}b\|_1 < 1$. By the variation of constants formula, the solution for $t \geq t_0$ satisfies

$$\begin{aligned}
\|x(t)\| &< \|e^{A(t-t_0)}\|\,\|x(t_0)\| + \delta M(t) + \|e^{At}b\|_1\,\|v\|_\infty \\
&\leq \|e^{A(t-t_0)}\| M(t_0) + \delta M(t) + \|e^{At}b\|_1\,\|v\|_\infty.
\end{aligned}$$

Since $\|e^{A(t-t_0)}\|$ is uniformly bounded, taking supremum on both sides, we obtain that for $t \geq t_0$

$$\begin{aligned}
&M(t) \\
&\quad = \sup_{\tau\leq t}\|x(\tau)\| \\
&\quad = \max\left\{\sup_{\tau\leq t_0},\ \|x(\tau)\|,\ \sup_{t_0\leq\tau\leq t}\|x(\tau)\|\right\} \\
&\quad \leq \max\{M(t_0),\ \|e^{A(t-t_0)}\|_\infty M(t_0) \\
&\qquad + \delta\, M(t) + \|e^{At}b\|_1\,\|v\|_\infty\} \\
&\quad \leq \max\left\{M(t_0),\ \frac{\|e^{A(t-t_0)}\|_\infty M(t_0) + \|e^{At}b\|_1\,\|v\|_\infty}{1-\delta}\right\} \\
&\quad < \infty
\end{aligned}$$

showing that $x(t)$ is uniformly bounded. ■

Lemma 2 assures that if the identification error $e_j = \tilde{p}_j^T\overline{\omega}$ of a typical model I_j is small compared to the state over a sufficiently long interval, then the associated signal $\tilde{p}_j^T\omega$ is correspondingly small compared to the state over that interval.

The lemma assumes that the tuning scheme for model I_j satisfies (25), repeated here for convenience:

$$\begin{aligned}
&1) \quad \hat{p}_j,\, \dot{\hat{p}}_j,\, \theta_j,\, \dot{\theta}_j \in \mathcal{L}^\infty\\
&2) \quad \|\dot{\hat{p}}_j(t)\| \le \gamma_j \frac{|e_j(t)|}{\sqrt{1+\xi^T(t)\xi(t)}}, \qquad t \ge 0\\
&3) \quad \epsilon_{\theta_j}(t) \equiv 0.
\end{aligned} \tag{30}$$

Choosing $\gamma_j = 0$, it is seen that these conditions are automatically satisfied by fixed models.

Definition: $\mathcal{PD}[T]$ is the class of right continuous, piecewise continuously differentiable functions with a minimum interval of T between instants of nondifferentiability. □

Lemma 2: Let I_j denote a typical fixed or adaptive model in the MRAC system described in Section III-A. Assume that I_j satisfies the tuning conditions (30), and there is an interval of $T_{\min}$ between switches. Then, there exist nonnegative continuous functions $g_j(\cdot, \cdot)$ and $\Delta_j(\cdot)$, with $\lim_{\epsilon\to 0} g_j(\epsilon, T_0) = 0$ for every fixed $T_0 > 0$ and $\lim_{\epsilon \to 0} \Delta_j(\epsilon) = 0$, such that if:

- $|e_j(t)| < \epsilon M(t),\ t \in [t_1, t_1 + T)$, for any $t_1 \ge 0$, $\epsilon > 0$, and $T \ge \max\{T_{\min}, \Delta_j(\epsilon)\}$;
- there are no switches in the interval $[t_1 + T - T_{\min},\, t_1 + T)$;
- $M(t_1) > \|\tilde{\beta}_{0_j}(t)k(t)r(t)\|_\infty/\epsilon$, where $\tilde{\beta}_{0_j}(t)k(t)$ is defined in the proof of the lemma, it follows that

$$|\tilde{p}_j^T(t)\omega(t)| < g_j(\epsilon, T_{\min})M(t), \qquad t \in [t_1, t_1 + T).$$

Here, for fixed $\epsilon > 0$, $g_j(\epsilon, T_{\min}) \to \infty$ as $T_{\min} \to 0$. □

Proof: Let $R_m(s) = s^{n^*} + \sum_{n=0}^{n^*-1} a_n s^n$, where $W_m(s) = 1/R_m(s)$. Then from (3), $\tilde{p}_j^T(t)\omega(t) = \tilde{p}_j^T(t)\overline{\omega}^{(n^*)}(t) + \sum_{n=0}^{n^*-1} a_n \tilde{p}_j^T(t)\overline{\omega}^{(n)}(t)$, where $\overline{\omega}^{(n)}$ denotes the nth derivative of $\overline{\omega}$.

In the first part of the proof, we prove by induction that there are positive constants $\{\epsilon_n\}_{n=0}^{n^*-1}$, depending continuously on ϵ and tending to zero as $\epsilon \to 0$, and a continuous function $\Delta_j(\epsilon) \triangleq \max\{\sqrt{\epsilon_0}, \sqrt{\epsilon_1}, \cdots, \sqrt{\epsilon_{n^*-2}}\}$, such that if $T \ge \Delta_j(\epsilon)$, then

$$|\tilde{p}_j^T(t)\overline{\omega}^{(n)}(t)| < \epsilon_n M(t), \qquad n = 0, 1, \cdots, n^* - 1,\quad t \in [t_1, t_1 + T). \tag{31}$$

In the second part of the proof, the above inequality is used to prove the lemma.

Choosing $\epsilon_0 \triangleq \epsilon$, inequality (31) is seen to be true for $n = 0$ from the hypothesis of the lemma. We assume that it is true for some $n \in \{0, 1, \cdots, n^* - 2\}$. For $n + 1$, we choose $\Delta_n = \sqrt{\epsilon_n} \le \Delta_j(\epsilon)$ and let $t \in [t_1, t_1 + T - \Delta_n]$. Using the induction hypothesis and the second condition in (30), we obtain by partial integration that

$$\begin{aligned}
&\left|\int_t^{t+\Delta_n} \tilde{p}_j^T(\tau)\overline{\omega}^{(n+1)}(\tau)\, d\tau\right|\\
&\quad= \left|\tilde{p}_j^T(t+\Delta_n)\overline{\omega}^{(n)}(t+\Delta_n) - \tilde{p}_j^T(t)\overline{\omega}^{(n)}(t) - \int_t^{t+\Delta_n} \dot{\tilde{p}}_j^T(\tau)\overline{\omega}^{(n)}(\tau)\, d\tau\right|\\
&\quad\le \epsilon_n[M(t) + M(t+\Delta_n)] + \gamma_j \int_t^{t+\Delta_n} \frac{|e_j(\tau)|\|\overline{\omega}^{(n)}(\tau)\|}{\sqrt{1+\xi^T(\tau)\xi(\tau)}}\, d\tau.
\end{aligned} \tag{32}$$

From the hypothesis of the lemma and the monotonicity of $M(t)$, we then obtain that

$$\left|\int_t^{t+\Delta_n} \tilde{p}_j^T(\tau)\overline{\omega}^{(n+1)}(\tau)\, d\tau\right| < (2\epsilon_n + \epsilon\gamma_j c_n \Delta_n)M(t+\Delta_n) \tag{33}$$

where we have used the fact [21] that

$$\begin{aligned}
&\|\overline{\omega}^{(n)}(t)\| < c_n\|\xi(t)\|,\\
&\|\xi(t)\| < c_\xi M(t), \qquad n = 0, 1, \cdots, n^*.
\end{aligned} \tag{34}$$

From the mean value theorem, there exists a $t_2 \in [t, t + \Delta_n)$ such that

$$|\tilde{p}_j^T(t_2)\overline{\omega}^{(n+1)}(t_2)|\Delta_n = \left|\int_t^{t+\Delta_n} \tilde{p}_j^T(\tau)\overline{\omega}^{(n+1)}(\tau)\, d\tau\right|. \tag{35}$$

Again, by the mean value theorem, if $\tau \in [t, t + \Delta_n]$, there is a τ_0 between t_2 and τ such that

$$|\tilde{p}_j^T(\tau)\overline{\omega}^{(n+1)}(\tau)| < |\tilde{p}_j^T(t_2)\overline{\omega}^{(n+1)}(t_2)| + \Delta_n\left|\frac{d}{dt}\left[\tilde{p}_j^T(\tau_0)\overline{\omega}^{(n+1)}(\tau_0)\right]\right|. \tag{36}$$

From inequality (34) and the second tuning condition in (30), we obtain that

$$\begin{aligned}
&\left|\frac{d}{dt}\left[\tilde{p}_j^T(\tau_0)\overline{\omega}^{(n+1)}(\tau_0)\right]\right|\\
&\quad= |\dot{\tilde{p}}_j^T(\tau_0)\overline{\omega}^{(n+1)}(\tau_0) + \tilde{p}_j^T(\tau_0)\overline{\omega}^{(n+2)}(\tau_0)|\\
&\quad\le (\epsilon\gamma_j c_{n+1} + \|\tilde{p}_j\|_\infty c_{n+2}\, c_\xi)M(t+\Delta_n).
\end{aligned}$$

Using this, the definition of Δ_n, and equations (10), (33), (35), and (36), it follows that

$$\begin{aligned}
|\tilde{p}_j^T(\tau)\overline{\omega}^{(n+1)}(\tau)| &< \left[\frac{2\epsilon_n}{\Delta_n} + (\epsilon\gamma_j c_{n+1} + \|\tilde{p}_j\|_\infty c_{n+2}\, c_\xi)\Delta_n + \epsilon\gamma_j c_n\right]\overline{c}e^{\overline{\lambda}\Delta_n}M(\tau)\\
&< [(2 + \epsilon\gamma_j c_{n+1} + \|\tilde{p}_j\|_\infty c_{n+2}\, c_\xi)\sqrt{\epsilon_n} + \epsilon\gamma_j c_n]\overline{c}e^{\overline{\lambda}\sqrt{\epsilon_n}}M(\tau)\\
&\triangleq \epsilon_{n+1}M(\tau), \qquad \tau \in [t, t+\Delta_n).
\end{aligned}$$

The induction step follows since $t \in [t_1, t_1 + T - \Delta_n)$ was arbitrary. This proves inequality (31).

For the second part of the proof, let $\Delta > 0$ and $[t, t+\Delta) \subset [t_1, t_1 + T)$. Using inequality (31) and the same arguments used to prove inequality (33), it follows that

$$\left|\int_t^{t+\Delta} \tilde{p}_j^T(\tau)\omega(\tau)\, d\tau\right| = \left|\int_t^{t+\Delta} \tilde{p}_j^T(\tau)\overline{\omega}^{(n^*)}(\tau)\, d\tau\right|$$

$$+\sum_{n=0}^{n^*-1} a_k \int_t^{t+\Delta} \tilde{p}_j^T(\tau)\overline{\omega}^{(n)}(\tau)\,d\tau\Bigg|$$
$$\leq \left[2\epsilon_{n^*-1} + \Delta\left(\epsilon\gamma_j c_{n^*-1} + \sum_{n=0}^{n^*-1} a_k\epsilon_k\right)\right] \cdot M(t+\Delta). \tag{37}$$

From the proof given in [2], the signal $\tilde{p}_j^T\omega$ can be decomposed as

$$\tilde{p}_j^T(t)\omega(t) = \tilde{\beta}_{0_j}(t)k(t)r(t) + v_j(t). \tag{38}$$

Here $k(t)$ is the first element of the control parameter vector $\theta(t)$, and $\tilde{\beta}_{0_j}(t)$ is the first element of $\tilde{p}_j(t)$. Hence the first term is uniformly bounded. Further [2], the second term $v_j(t) \in \mathcal{PD}[T_{\min}]$, with

$$|v_j(t)| < k_1 M(t), \qquad t \geq 0$$

and

$$|\dot{v}_j(t)| < k_2 M(t) \quad \text{between discontinuities.} \tag{39}$$

From (38) and the inequality (37), it follows that

$$\left|\int_t^{t+\Delta} v_j(\tau)\,d\tau\right| \leq \left|\int_t^{t+\Delta} \tilde{p}_j^T(\tau)\omega(\tau)\,d\tau\right| + \Delta\|\tilde{\beta}_{0_j}(t)k(t)r(t)\|_\infty$$
$$< \left[2\epsilon_{n^*-1} + \Delta\left(\epsilon\gamma_j c_{n^*-1} + \sum_{n=0}^{n^*-1} a_k\epsilon_k + \epsilon\right)\right] M(t+\Delta)$$
$$\triangleq (\overline{\epsilon}_1 + \Delta\overline{\epsilon}_2)M(t+\Delta) \tag{40}$$

if $M(t_1) > \|\tilde{\beta}_{0_j}(t)k(t)r(t)\|_\infty/\epsilon$. Note from the properties of $\{\epsilon_n\}$ that both $\overline{\epsilon}_1$ and $\overline{\epsilon}_2$ are continuous functions of ϵ and tend to zero as $\epsilon \to 0$. The above holds for arbitrary Δ. We now prove that if Δ is chosen as

$$\Delta \triangleq \min\left\{\frac{T_{\min}}{2},\, f^{-1}\left(\frac{\overline{\lambda}\overline{\epsilon}_1}{2k_2}\right)\right\} > 0 \tag{41}$$

where f^{-1} is the inverse of the monotonic function $f(\Delta) \triangleq \Delta(1 - e^{\overline{\lambda}\Delta})$, then v_j satisfies

$$|v_j(t)| < \frac{2\overline{c}^2\overline{\lambda}e^{\overline{\lambda}\Delta}}{1 - e^{-\overline{\lambda}\Delta}}(\overline{\epsilon}_1 + \Delta\overline{\epsilon}_2)M(t), \qquad t \in [t_1,\, t_1 + T). \tag{42}$$

Note that $f^{-1}(\overline{\lambda}\overline{\epsilon}_1/2k_2) \to 0$ as $\overline{\epsilon}_1 \to 0$. To prove inequality (42), we assume the contrary, i.e.,

$$|v_j(t_0)| > \frac{2\overline{c}^2\overline{\lambda}e^{\overline{\lambda}\Delta}}{1 - e^{-\overline{\lambda}\Delta}}(\overline{\epsilon}_1 + \Delta\overline{\epsilon}_2)M(t_0) \tag{43}$$

for some $t_0 \in [t_1,\, t_1 + T)$. Since $v_j \in \mathcal{PD}[T_{\min}]$, successive discontinuities in v_j are $T_{\min}$ apart. Further, from the hypothesis of the lemma, $T > T_{\min}$ and there are no switchings in $[t_1 + T - T_{\min},\, t_1 + T)$. Consequently, there exists an interval I of the form

$$I = \left[t_0,\, t_0 + \frac{T_{\min}}{2}\right)$$

or

$$I = \left[t_0 - \frac{T_{\min}}{2},\, t_0\right)$$

such that v_j is continuously differentiable over I. Without loss of generality, we take $I = [t_0,\, t_0 + T_{\min}/2)$. Using the mean value theorem, the right continuity of v_j, and inequalities (10), (39), (41), and (43), it follows that for every $t \in [t_0,\, t_0 + \Delta) \subset I$

$$|v_j(t)| > \frac{\overline{c}\overline{\lambda}(\overline{\epsilon}_1 + \Delta\overline{\epsilon}_2)}{1 - e^{-\overline{\lambda}\Delta}} M(t). \tag{44}$$

The above, together with the continuity of v_j over I, further implies that the sign of v_j is constant over $[t_0,\, t_0 + \Delta)$. Hence, the integral of v_j satisfies

$$\left|\int_{t_0}^{t_0+\Delta} v_j(\tau)\,d\tau\right| = \int_{t_0}^{t_0+\Delta} |v_j(\tau)|\,d\tau$$
$$> \frac{\overline{\lambda}(\overline{\epsilon}_1 + \Delta\overline{\epsilon}_2)}{1 - e^{-\overline{\lambda}\Delta}} \int_{t_0}^{t_0+\Delta} \cdot e^{-\overline{\lambda}(t_0+\Delta-\tau)} M(t_0+\Delta)\,d\tau$$
$$> (\overline{\epsilon}_1 + \overline{\epsilon}_2\Delta)M(t_0 + \Delta)$$

which contradicts (40), thereby proving (42). Using (41) and the fact $\overline{\lambda}\Delta/(1 - e^{-\overline{\lambda}\Delta}) \leq \overline{\lambda}\Delta + 1$, inequality (42) reduces to

$$|v_j(t)| < 2\overline{c}^2 e^{\overline{\lambda}f^{-1}(\overline{\lambda}\overline{\epsilon}_1/2k_2)} \cdot \left\{f_1(\overline{\epsilon}_1,\, T_{\min}) + \left[\overline{\lambda}f^{-1}\left(\frac{\overline{\lambda}\overline{\epsilon}_1}{2k_2}\right) + 1\right]\overline{\epsilon}_2\right\} M(t) \tag{45}$$

where the function $f_1(\cdot,\, \cdot)$ is defined as

$$f_1(\overline{\epsilon}_1,\, T_{\min}) \triangleq \begin{cases} \dfrac{\overline{\epsilon}_1}{1 - e^{-\overline{\lambda}T_{\min}/2}}, & \dfrac{T_{\min}}{2} < f^{-1}\left(\dfrac{\overline{\lambda}\overline{\epsilon}_1}{2k_2}\right) \\ 2k_2\, f^{-1}\left(\dfrac{\lambda\overline{\epsilon}_1}{2k_2}\right), & \text{else.} \end{cases}$$

f_1 is seen to be a continuous function of $(\overline{\epsilon}_1,\, T_{\min})$ for $T_{\min} > 0$, $\overline{\epsilon}_1 \geq 0$. Further, for any fixed $T_{\min} > 0$, $f_1(\overline{\epsilon}_1,\, T_{\min}) \to 0$ as $\overline{\epsilon}_1 \to 0$. From (38), (45), and the choice of $M(t_1) > \|\tilde{\beta}_{0_j}(t)k(t)r(t)\|_\infty/\epsilon$, it follows that for any $t \in [t_1,\, t_1 + T)$

$$|\tilde{p}_j^T(t)\omega(t)| < \left(2\overline{c}e^{\overline{\lambda}f^{-1}(\lambda\overline{\epsilon}_1/2k_2)}\left\{f_1(\overline{\epsilon}_1,\, T_{\min}) + \left[\overline{\lambda}f^{-1}\left(\frac{\lambda\overline{\epsilon}_1}{2k_2}\right) + 1\right]\overline{\epsilon}_2\right\} + \epsilon\right) M(t)$$
$$\triangleq g_j(\epsilon,\, T_{\min})\, M(t) \tag{46}$$

which proves the lemma. ■

APPENDIX C
PROOF OF INEQUALITY (20)

The objective is to prove inequality (20), given (19). It is first shown that the latter implies that there exists a $T_j > 0$ such that

$$|e_j(t)| > \frac{\epsilon^*}{2} M(t), \qquad t \in [\tau_0 - T_j, \tau_0]. \tag{47}$$

From the tuning conditions (25) and (34), it can be checked that $|(d/dt)e_j(t)| < d_j M(t)$, for some $d_j > 0$. Using the mean value theorem it follows from (19) that if $T_j \triangleq \epsilon^*/2d_j$, then for any $t \in [\tau_0 - T_j, \tau_0]$

$$\begin{aligned} |e_j(t)| &\geq |e_j(\tau_0)| - |t - \tau_0| \left| \frac{d}{dt} e_j(\tau) \right| \\ &\qquad \text{for some } \tau \in (t, \tau_0) \\ &\geq \epsilon^* M(\tau_0) - T_j d_j M(\tau) \\ &> \frac{\epsilon^*}{2} M(t) \end{aligned}$$

which proves (47).

It is given that $\tau_0 \in [\tau_1 - T^*, \tau_1]$. Using (1), (10), and (47), it follows that

$$\begin{aligned} J_j(\tau_1) &\geq \beta \int_0^{\tau_1} e^{-\lambda(\tau_1 - \tau)} e_j^2(\tau)\, d\tau \\ &> \beta \int_{\tau_0 - T_j}^{\tau_0} e^{-\lambda(\tau_1 - \tau)} e_j^2(\tau)\, d\tau \\ &\geq \frac{\beta \epsilon^{*^2}}{4} \int_{\tau_0 - T_j}^{\tau_0} e^{-\lambda(\tau_1 - \tau)} M^2(\tau)\, d\tau \\ &> \frac{\beta \epsilon^{*^2}}{4\overline{c}^2} \int_{\tau_0 - T_j}^{\tau_0} e^{-(\lambda + 2\overline{\lambda})(\tau_1 - \tau)} M^2(\tau_1)\, d\tau \\ &> \frac{\beta}{\lambda + 2\overline{\lambda}} \frac{\epsilon^{*^2}}{4\overline{c}^2} [1 - e^{-(\lambda + 2\overline{\lambda})T_j}]\, e^{-(\lambda + 2\overline{\lambda})T^*} M^2(\tau_1). \end{aligned}$$

Choosing $\overline{T}$ to be the minimum of the values $\{T_j\}$ over all the fixed models I_j, $j \neq k$, verifies (20). ■

APPENDIX D
PROOF OF THEOREM 4

Proof: The proof consists of demonstrating that the input error $\tilde{u}_R$ of the reinitialized adaptive model I_R satisfies

$$\begin{aligned} |\beta_0^* \tilde{u}_R(t)| < \eta^* M(t), \qquad &\text{whenever } u(t) \equiv u_R(t), \\ &t \in [t_1, t_2), \quad t_1 \geq T_r \end{aligned} \tag{48}$$

for some T_r large enough, where η^* is as defined in the proof of Theorem 2. Since the proofs of Theorems 2 and 3 ensure that the same is true for the rest of the fixed and free-running adaptive models, boundedness follows as in the previous cases.

As in the case of Theorem 2, this is shown via (16). Let $[t_1, t_2)$ be any arbitrary switching interval over which I_R is active. From Lemma 2 (Appendix B) applied to I_R, it follows that for the given η^* in (48), there is an $\epsilon_r > 0$ and an interval $\Delta_r \triangleq \max\{T_{\min}, \Delta_R(\epsilon_r)\} > 0$ such that if $e_R(t) = \tilde{p}_R^T(t)\overline{\omega}(t)$ is continuous over the interval $[t_1 - \Delta_r, t_2)$ and satisfies

$$|e_R(t)| < \epsilon_r M(t), \qquad t \in [t_1 - \Delta_r, t_2) \tag{49}$$

then the corresponding signal $\tilde{p}_R^T \overline{\omega}$ satisfies $|\tilde{p}_R^T(t)\omega(t)| < \eta^* M(t)$ over the same interval. From the error (7) and the fact that $\epsilon_{\theta_R}(t) \equiv 0$ for model I_R, it follows that condition (48) is satisfied if (49) is true.

For proving condition (48), we consider two possibilities: either I_R was not reinitialized over the interval $[t_1 - \Delta_r, t_1)$ prior to its activation, or it was. In the former case, both $e_R(t)$ and $J_R(t)$ are continuous over the entire interval $[t_1 - \Delta_r, t_2)$. Then the argument used in the second part of the proof of Theorem 2 applies, and it follows that (49) will be satisfied if μ^* is chosen small enough and $M(t)$ large enough. Hence, by the argument in the previous paragraph, (48) is verified in this case.

Considering the second possibility, assume that $t_r \in [t_1 - \Delta_r, t_1)$ was the last instant at which I_R was reinitialized. Let I_j be the fixed model from which it was reinitialized. This implies that I_j was activated at instant t_r, and hence, by the proof of Theorem 2, it satisfies condition (16), i.e.,

$$|e_j(t)| < \epsilon^* M(t), \qquad t \in [t_r - T^*, t_r]. \tag{50}$$

Note that we may assume that $\epsilon^* < \epsilon_r$ and $T^* > \Delta_r$ by choosing μ^* sufficiently small in the proof of Theorem 2. Over the remaining interval (t_r, t_2), $e_R(t)$, and $J_R(t)$ are continuous and the earlier arguments can be repeated to obtain that

$$|e_R(t)| < \epsilon_r M(t), \qquad t \in (t_r, t_2). \tag{51}$$

To combine the above two facts, we define a new model I_n, operative over the interval $[t_r - T^*, t_2)$, as the concatenation of model I_j over the interval $[t_r - T^*, t_r]$ and model I_R over the interval (t_r, t_2). The parameter vector $\hat{p}_n$ of I_n is defined as

$$\hat{p}_n(t) \triangleq \begin{cases} \hat{p}_j, & t \in [t_r - T^*, t_r] \\ \hat{p}_R(t), & t \in (t_r, t_2) \end{cases}$$

and its identification error is $e_n(t) \triangleq \tilde{p}_n^T(t)\overline{\omega}(t)$. Hence, $e_n(t)$ is continuous and satisfies $|e_n(t)| < \epsilon_r M(t)$ over the interval $[t_r - T^*, t_2)$ which contains the interval $[t_1 - \Delta_r, t_2)$. Moreover, I_n satisfies (25) over this interval. Hence, applying Lemma 2 (Appendix B) to $\tilde{p}_n^T \overline{\omega}$ and $\tilde{p}_n^T \omega$ and using the definition of $\hat{p}_n$, it follows that $|\tilde{p}_R^T(t)\omega(t)| < \eta^* M(t)$, $t \in [t_1, t_2)$. From this, as in the proof of Theorem 2, we conclude that (48) holds for this case also. ■

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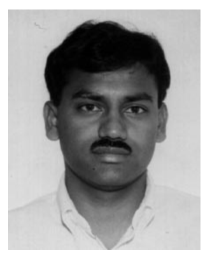

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