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Stability of feedback error learning scheme

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Abstract

This paper aims to establish control theoretical validity of the feedback error learning scheme proposed as an architecture of brain motor control with deep physiological root in computational neuroscience. The feedback error learning method is formulated as a two-degree-of-freedom adaptive control. The stability of the adaptive control law is proved based on the strict positive realness, under the assumption that the plant is stable and stably invertible. Extension to non-invertible cases is also discussed. Some simulation results are given to illustrate the effectiveness of the method. © 2002 Elsevier Science B.V. All rights reserved.

Keywords: Motor control; Adaptive control; Feedforward; Learning

1. Introduction

Each voluntary movement of human being is controlled by central nervous system, which is usually referred to as *motor control* in neurophysiology. Though the human muscular-skeleton system is not as powerful as hydraulic actuators and peripheral sensors of human body is not so accurate with slow rate of information transmission, no man-made robot can achieve such quick, smooth and precise movements carried out by human motor control. It is natural to ascribe such excellent performances of human motor control to the superiority of motor control algorithm built in the human neural network. It is a central theme of motor control to devise architectures and algorithms which explain high performances of motor control based on physiological functions and anatomical structure of brain neural system. The recent progress of motor control verifies the importance of models of the outer world that are considered to be built in human neural network. The existence of internal models gives great flexibility in architecture of motor control, and the method of learning models becomes a new issue of motor control. The architecture must incorporate learning algorithm in its control architecture. It is remarkable that brain motor control community now uses the common language with control community, so that the interplay between control theory and motor control is now important and to be facilitated to solve various issues of motor control [4,8,9].

Recently, Kawato and his group proposed a novel architecture of motor control called *feedback error learning* (FEL) method which combines learning and control efficiently [5,6] (1). It is essentially an adaptive two-degree-of-freedom (TDOF) control with inverse model in feedforward path. In some sense, the method

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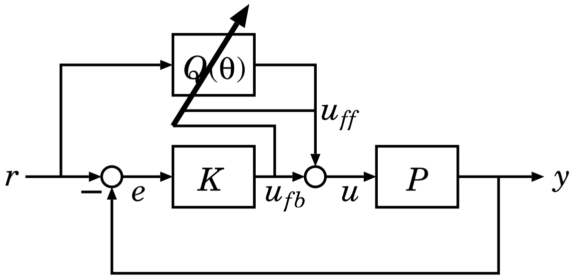

Fig. 1. Feedback error learning scheme.

is closely related to the adaptive internal model control that has recently been exploited by Datta et al. [2,3]. The novelty of FEL method lies in its use of feedback error as a teaching signal for learning the inverse model, which is essentially new in control literature. Kawato et al. [7] claims that the reality of FEL method in cerebellum has been experimentally validated through recent research in neurophysiology. This paper aims to establish a control theoretical ground of the FEL method. We formulate the problem in the framework of adaptive control and prove its stability based on the strict positive realness (2).

In Section 2, the FEL method is briefly reviewed. In Section 3, the problem is formulated and stability proof for the case of invertible plants is given. Extension to non-invertible cases is discussed in Section 4. Section 5 gives simulation results to illustrate the effectiveness of FEL:

(1) This method was identical to an adaptive control scheme of robot manipulators developed independently by Slotine and Li [13].
(2) The technical results here are much better than those in [12].

With a slight abuse of notation, we write

$$y(t) = L(s)u(t)$$

to denote the input/output relation of the system having the transfer function $L(s)$.

2. Feedback error learning (FEL) method for brain motor control

Fig. 1 illustrates the feedback error learning architecture. The objective of control is to minimize the error e between the command signal r and the plant output y. The input u to the plant P is composed of the output u_{ff} of feedforward controller $Q(\theta)$ which contains tunable parameters θ, and u_{fb} of feedback controller K_1. If we disregard the learning part of the architecture, this is a typical TDOF control system. If P is known and P^{-1} exists and stable, choosing $Q = P^{-1}$ makes the tracking perfect. Indeed, from the relations $u_{\mathrm{ff}} = P^{-1}r$, $u_{\mathrm{fb}} = K_1(r - y)$ and $y = P(u_{\mathrm{ff}} + u_{\mathrm{fb}})$, we easily see that $y = r$.

The novelty of feedback error learning method lies in its way to learn the inverse model of P. In Kawato's original work [5], the feedforward controller $Q(\theta)$ is implemented by a neural network. In this paper, we formulate the method in the usual framework of adaptive control, instead of neural network, assuming that the plant P is linear time-invariant. It is assumed that a true parameter θ_0 exists such that

$$P^{-1} = Q(\theta_0).$$

The learning law is derived from the gradient method with error function

$$V = \tfrac{1}{2}(u_{\mathrm{ff}} - u_0)^{\mathrm{T}}(u_{\mathrm{ff}} - u_0), \tag{1}$$

where u_0 denotes the correct input $u_0 = P^{-1}r$. Then, the gradient method yields

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\frac{\partial V}{\partial \theta}, \tag{2}$$

which results in the tuning rule

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\frac{\partial u_{\mathrm{ff}}}{\partial \theta}(u_{\mathrm{ff}} - u_0). \tag{3}$$

The tuning rule (3) cannot be implemented because u_0 is unknown. If we assume that the input u to the plant is correct, i.e.,

$$u = u_0 = P^{-1}r, \tag{4}$$

then, the tuning rule becomes

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\partial u_{\mathrm{ff}}}{\partial \theta} u_{\mathrm{fb}}, \tag{5}$$

which can be implemented. The above tuning rule is the core of the FEL method which depends on the approximation (4) crucially. The main purpose of this paper is to justify the approximation (4) by proving the stability of the algorithm (5).

In what follows, we set up the problem of FEL method in the simplest framework of linear time-invariant systems and give a stability proof of the feedback error learning algorithm (5). Another important point which was not dealt with in Kawato's work is the problem of non-invertibility of plant, which is the big nuissance for adaptive control. We shall prove stability for non-invertible cases.

3. Analysis of FEL for invertible plants

3.1. Parameterization of feedforward controller

Now, we describe a method of adaptive construction of a desired feedforward controller Q. Throughout this paper, we use the following parameterization of the unknown system Q:

$$\frac{\mathrm{d}\eta_1(t)}{\mathrm{d}t} = F\eta_1(t) + gr(t), \tag{6}$$

$$\frac{\mathrm{d}\eta_2(t)}{\mathrm{d}t} = F\eta_2(t) + gu_0(t), \tag{7}$$

$$u_0(t) = c_0^{\mathrm{T}}\eta_1(t) + d_0^{\mathrm{T}}\eta_2(t) + k_0 r(t), \tag{8}$$

where F is any stable matrix and g is any vector with $\{F, g\}$ being controllable. Actually, the parameterization (6)–(8) is used in adaptive observer [12]. In (6)–(8), c_0, d_0 and k_0 are unknown parameters to be estimated. It is easy to see that (6)–(8) can yield an arbitrary transfer function from $r(t)$ to $u_0(t)$. To see this, let the matrix F and vector g be in a controllable canonical form

$$F = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ -f_1 & -f_2 & -f_3 & \cdots & -f_n \end{bmatrix}, \quad g = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \tag{9}$$

The transfer function $T_{u_0 r}$ from $r(t)$ to $u_0(t)$ is given by

$$T_{u_0 r} = \frac{k_0 + c_0^{\mathrm{T}}(sI - F)^{-1}g}{1 - d_0^{\mathrm{T}}(sI - F)^{-1}g} = \frac{k_0 s^n + (f_n k_0 + c_n)s^{n-1} + \cdots + (f_1 k_0 + c_1)}{s^n + (f_n - d_n)s^{n-1} + \cdots + (f_1 - d_1)}, \tag{10}$$

$$c_0 = [c_1 \quad c_2 \quad \cdots \quad c_n]^{\mathrm{T}},$$

$$d_0 = [d_1 \quad d_2 \quad \cdots \quad d_n]^{\mathrm{T}}.$$

Therefore, we can construct any transfer function of degree less than or equal to n by selecting parameters c_0, d_0 and k_0 appropriately. The advantage of the parameterization (6)–(8) is that the unknown parameters enter linearly in the system description.

3.2. Adaptation law

Now we are ready to discuss the FEL method from the viewpoint of adaptive control. As was discussed in Section 2, the feedforward controller Q in Fig. 1 is chosen to be identical to the inverse P^{-1} of P if P is known. Since P is unknown, we must employ some adaptive scheme for estimating Q so that Q converges to P^{-1}.

We make the following assumptions:

(A1) The plant P is stable and has stable inverse P^{-1}.
(A2) The upper bound of the order of P is known.
(A3) The high frequency gain $K_0 = \lim_{s\to\infty} P(s)$ is assumed to be positive.

The assumption (A1) is rather restrictive in the context of control system design. In the context of motor control, this assumption is not restrictive because the plant is always a neuro-muscular system with low order so that the computed torque method, which is essentially equivalent to constructing an inverse model, works. Nevertheless, we relax this assumption in Section 4, where the infinite zero is allowed. If K_0 is negative in (A3), the subsequent results are valid by taking $-P(s)$ instead of $P(s)$. Hence, (A3) is relaxed to the assumption that *the sign of the high frequency gain is known*. For the sake of the simplicity of exposition, however, we retain (A3).

We take (6)–(8) for the parameterization of the feedforward controller Q. Instead of $u_0(t)$ in (7), we use the plant input $u(t)$ and $u_0(t)$ in (8) corresponds to the output $u_{\mathrm{ff}}(t)$ of Q:

$$\frac{\mathrm{d}\xi_1(t)}{\mathrm{d}t} = F\xi_1(t) + gr(t), \tag{11}$$

$$\frac{\mathrm{d}\xi_2(t)}{\mathrm{d}t} = F\xi_2(t) + gu(t), \tag{12}$$

$$u_{\mathrm{ff}}(t) = c^{\mathrm{T}}(t)\xi_1(t) + d^{\mathrm{T}}(t)\xi_2(t) + k(t)r(t), \tag{13}$$

$$u(t) = u_{\mathrm{ff}}(t) + K_1 e(t), \tag{14}$$

where F is stable and $\{F, g\}$ is controllable. Assume that the *true* inverse system is described as (11)–(13) with $u(t) = u_0(t) = u_{\mathrm{ff}}(t)$. Since the ideal system generates $u_0(t) = P^{-1}(s)r(t)$, the true values c_0, d_0 and k_0

of $c(t)$, $d(t)$ and $k(t)$, respectively, satisfy

$$\frac{k_0 + c_0^{\mathrm{T}}(sI-F)^{-1}g}{1-d_0^{\mathrm{T}}(sI-F)^{-1}g} = P^{-1}(s) \tag{15}$$

as was given in (10). The learning rule (5) becomes equivalent to

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \alpha \frac{\partial u_{\mathrm{ff}}(t)}{\partial \theta} K_1(s)e(t), \tag{16}$$

where we introduce a parameter α to adjust adaptation speed, and $\theta(t)$ is given by

$$\theta(t) := [c(t)^{\mathrm{T}} \quad d(t)^{\mathrm{T}} \quad k(t)]^{\mathrm{T}}.$$

From (13), it follows that

$$u_{\mathrm{ff}}(t) = \theta(t)^{\mathrm{T}}\xi(t).$$

where $\xi(t)$ is given by

$$\xi(t) := [\xi_1(t)^{\mathrm{T}} \quad \xi_2(t)^{\mathrm{T}} \quad r(t)]^{\mathrm{T}}.$$

Then, (16) can be written as

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \alpha K_1(s)e(t)\xi(t). \tag{17}$$

The problem is to prove stability of adaptive scheme (11)–(14), (17) for the plant $P(s)$.

3.3. Stability of the algorithm

Since $e(t) = r(t) - P(s)u(t)$, we have

$$u_0(t) - u(t) = P^{-1}(s)e(t),$$

From (8) and (13), it follows that

$$u_{\mathrm{ff}}(t) - u_0(t) = \theta(t)^{\mathrm{T}}\xi(t) - \theta_0^{\mathrm{T}}\eta(t),$$

$$\theta_0 := [c_0^{\mathrm{T}} \quad d_0^{\mathrm{T}} \quad k_0^{\mathrm{T}}]^{\mathrm{T}}, \quad \eta(t) = [\eta_1(t) \quad \eta_2(t) \quad r(t)]^{\mathrm{T}}. \tag{18}$$

Since F is stable, the comparison of (6)–(8) and (11)–(13) yields asymptotic relations

$$\xi_1(t) \to \eta_1(t), \quad \xi_2(t) \to \eta_2(t) - d_0^{\mathrm{T}}(sI-F)^{-1}gP^{-1}(s)e(t).$$

Thus, the relation (18) yields the asymptotic relations

$$u_{\mathrm{ff}}(t) - u_0(t) = \psi(t)^{\mathrm{T}}\xi(t) - d_0^{\mathrm{T}}(sI-F)^{-1}gP^{-1}(s)e(t),$$

$$\psi(t) := \theta(t) - \theta_0. \tag{19}$$

From the relation $u(t) = u_{\mathrm{ff}}(t) + K_1(s)e(t)$, we have

$$-P^{-1}(s)e(t) - K_1(s)e(t) = \psi(t)^{\mathrm{T}}\xi(t) - d_0^{\mathrm{T}}(sI-F)^{-1}gP^{-1}(s)e(t),$$

which results in

$$(G(s)+K_1(s))e(t) = -\xi(t)^{\mathrm{T}}\psi(t), \tag{20}$$

$$G(s) := (1-d_0^{\mathrm{T}}(sI-F)^{-1}g)P^{-1}(s).$$

On the other hand, from (17), we have

$$\frac{\mathrm{d}\psi(t)}{\mathrm{d}t} = \frac{\mathrm{d}\theta(t)}{\mathrm{d}t} = \alpha e(t)K_1(s)\xi(t). \tag{21}$$

It should be noted that the relation (15) implies that

$$G(s) = k_0 + c_0^{\mathrm{T}}(sI-F)^{-1}g. \tag{22}$$

Combining (20) and (21), we have

$$\frac{\mathrm{d}\psi(t)}{\mathrm{d}t} = -\alpha_1\xi(t)K_1(s)(G(s)+K_1(s))^{-1}\xi(t)^{\mathrm{T}}\psi(t). \tag{23}$$

Now, the problem is reduced to the stability of the above differential equations, where the well-known notion of strict positive realness plays a crucial role.

A transfer function $L(s)$ is said to be *positive real* iff $\mathrm{Re}[L(s)] \geqslant 0$, for each $\mathrm{Re}[s] \geqslant 0$, where $\mathrm{Re}[\cdot]$ denotes the real part. A transfer function $L(s)$ is said to be *strictly positive real* iff there exists a positive number ε such that $L(s-\varepsilon)$ is positive real [12]. Actually, the positive realness has been one of the major tools of the stability proof in the literature of adaptive control. Here, this notion is used in a different context. The following result which plays a crucial role in the sequel was found in [1].

Lemma 1. *Let $L(s)$ be a strongly positive real transfer function and $\xi(t)$ be an arbitrary time-varying vector. Then, the solution $z(t)$ of the differential equation*

$$\frac{\mathrm{d}z(t)}{\mathrm{d}t} = -\xi(t)L(s)\xi(t)^{\mathrm{T}}z(t) \tag{24}$$

tends to a constant vector z_0 such that $\xi(t)z_0 \to 0$. If $\xi(t)$ satisfies the so-called persistent excitation (*PE*) *condition* [12], *the above z_0 is equal to* 0.

The proof of this lemma is given in [1]. We show an alternative simpler proof in the appendix.

A special case of Lemma 1 where $L(s)$ is strictly proper was given in [10,11] in a different context. Also, the stability of (24) for $L(s)=1$ actually corresponds to the PE condition for $\xi(t)$.

We notice that (23) is the same form as (24), where $L(s)$ is equal to

$$L_0(s) := \alpha K_1(s)(G(s)+K_1(s))^{-1}. \tag{25}$$

According to Lemma 1, the differential equation (23) is asymptotically stable, if $L_0(s)$ given by (25) is strongly positive real. Since $G(s)$ given by (22) is stable, we can always choose positive constant K_1 sufficiently large such that $G(s)+K_1$ is strongly positive real. If $G(s)+K_1$ is strongly positive real, so is $L_0(s)$. Thus, we have established the following fundamental result:

Theorem 1. *Under the assumptions* (A1)–(A3), *the FEL algorithm* (11)–(14) *and* (17) *is stable and the error $e(t)$ tends to* 0, *by choosing sufficiently large positive constant K_1. If $\xi(t)$ satisfies the PE condition, then $Q(s)$ tends to $P^{-1}(s)$.*

Remark. Though the FEL is totally different from conventional adaptive control schemes, it is remarkable that the notion of strictly positive realness, which plays a crucial role in conventional adaptive control, is also fundamental in dealing with FEL. It should be noted that the fundamental equation (23) also appears in the MRAC in a simpler form [12].

Obviously, the tuning rule (17) can be generalized to

$$\frac{\mathrm{d}\theta(t)}{\mathrm{d}t} = \Gamma \xi(t) e(t),$$

where Γ is a positive definite matrix. We can obtain extra degree of freedom to select the tuning speed through the selection of Γ.

4. The non-invertible case

4.1. Motivation

In Section 3, we assumed that $P^{-1}(s)$ exists and is stable. This implies that the relative degree of $P(s)$ is zero and the zeros of $P(s)$ are all stable. Now, we relax the first condition and introduce an approximated inverse $\hat{P}^{-1}$ as

$$\hat{P}^{-1}(s) = P^{-1}(s)W(s),$$

where $W(s)$ is a filter with relative degree identical to that of $P(s)$. Using this approximation, the relative degree of $P(s)$ which is the cause of non-invertibility, is compensated by the relative degree of $W(s)$.

We aim to construct $\hat{P}^{-1}(s) = W(s)P^{-1}(s)$ as a feedforward controller Q by the scheme of the feedback error learning method (Fig. 2).

In this section we make, instead of (A1), the assumption (A1′) and an additional assumption (A4):

(A1′) All the finite zeros of $P(s)$ are stable.
(A4) The upper bound of the relative degree of P is known.

4.2. Parameterization of the feedforward controller $K_2(s)$

We parametrize the feedforward controller $Q(\theta)$ in the same way as Section 3 ((11)–(13)). We aim to construct $P^{-1}(s)W(s)$, which can be realized by the parameterization discussed in Section 3.1, as a feedforward controller. In this case, since $W(s)$ is known, the number of unknown parameters in $Q(\theta)$ can be reduced.

Assume that the plant $P(s)$ has relative degree k ($k \leqslant n$), so we can write $P(s)$ generally as

$$P(s) = \frac{b_k s^{n-k} + b_{k+1} s^{n-k-1} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n}. \tag{26}$$

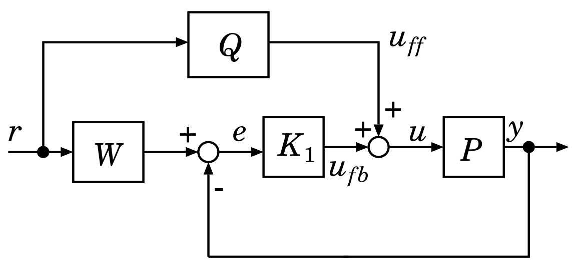

Fig. 2. FEL method non-invertible case.

We select a prefilter $W(s)$ with relative degree k as

$$W(s) = \frac{w_{k+1}}{s^k + w_1 s^{k-1} + \cdots + w_k} \quad (w_i,\ i = 1, 2, \ldots, k+1\text{: known}).$$

Assume that $P^{-1}(s)W(s)$ is represented as

$$\frac{\mathrm{d}\xi_1(t)}{\mathrm{d}t} = F\xi_1(t) + gr(t), \tag{27}$$

$$\frac{\mathrm{d}\xi_2(t)}{\mathrm{d}t} = F\xi_2(t) + gu_0(t), \tag{28}$$

$$u_0(t) = c_w^{\mathrm{T}}\xi_1(t) + d_w^{\mathrm{T}}\xi_2(t) + k_w r(t), \tag{29}$$

where F is given as before. Then, $P^{-1}(s)W(s)$ is written as

$$P^{-1}(s)W(s) = \frac{k_w + c_w^{\mathrm{T}}(sI - F)^{-1}g}{1 - d_w^{\mathrm{T}}(sI - F)^{-1}g} = \frac{k_w s^n + (f_n k_w + c_{w,n})s^{n-1} + \cdots + (f_1 k_w + c_{w,1})}{s^n + (f_n - d_{w,n})s^{n-1} + \cdots + (f_1 - d_{w,1})}, \tag{30}$$

where

$$c_w = [c_{w,1} \quad c_{w,2} \quad \ldots \quad c_{w,n}]^{\mathrm{T}}, \qquad d_w = [d_{w,1} \quad d_{w,2} \quad \ldots \quad d_{w,n}]^{\mathrm{T}}.$$

Hence, c_w, d_w and k_w must satisfy the identity

$$\frac{k_w s^n + (f_n k_w + c_{w,n})s^{n-1} + \cdots + (f_1 k_w + c_{w,1})}{s^n + (f_n - d_{w,n})s^{n-1} + \cdots + (f_1 - d_{w,1})}$$

$$= \frac{s^n + a_1 s^{n-1} + \cdots + a_n}{b_1 s^{n-k} + b_2 s^{n-k-1} + \cdots + b_{n-k+1}} \frac{w_{k+1}}{s^k + w_1 s^{k-1} + \cdots + w_k}.$$

Let $f := [\, f_1 \;\; f_2 \;\; \ldots \;\; f_n \,]^{\mathrm{T}}$. The above identity yields the relation

$$f - d_w = \begin{bmatrix} w_k & 0 & \ldots & 0 \\ w_{k-1} & w_k & \ldots & 0 \\ \vdots & \vdots & \ldots & 0 \\ w_1 & w_2 & \ldots & 0 \\ 1 & w_1 & \ldots & w_k \\ 0 & 1 & \ldots & w_{k-1} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} b_{n-k+1}/b_1 \\ b_{n-k}/b_1 \\ \vdots \\ b_2/b_1 \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ w_k \\ w_{k-1} \\ \vdots \\ w_2 \\ w_1 \end{bmatrix}. \tag{31}$$

Let h_i, $i = 0, \ldots, n-1$ be a sequence of solutions of a difference equation

$$h_i + h_{i-1}w_1 + h_{i-2}w_2 + \cdots + h_{i-k}w_k = 0. \tag{32}$$

Using (32), we have

$$[h_0 \quad h_1 \quad h_2 \quad \ldots \quad h_{n-1}] \cdot [f - d_w] = w_1 h_{n-1} + w_2 h_{n-2} + \cdots + w_k h_{n-k}.$$

The difference equation (32) has k independent solutions $h_i^{(j)}$, $j=1,\dots,k$; $i=0,\dots,n-k$ as

$$h_i^{(j)} = -w_1 h_{i-1}^{(j)} - w_2 h_{i-2}^{(j)} - \cdots - w_k h_{i-k}^{(j)}, \quad i \geqslant k,$$

$$h_i^{(j)} = \begin{cases} 0, & i \neq j-1, \quad i \leqslant k-1, \\ 1, & i = j-1. \end{cases}$$

Thus, we obtain

$$\begin{bmatrix} h_0^{(1)} & h_1^{(1)} & \dots & h_{n-1}^{(1)} \\ h_0^{(2)} & h_1^{(2)} & \dots & h_{n-1}^{(2)} \\ \dots & \dots & \dots & \dots \\ h_0^{(k)} & h_1^{(k)} & \dots & h_{n-1}^{(k)} \end{bmatrix} [f - d_w] = \begin{bmatrix} h_{n-k+1}^{(1)} & h_{n-k+2}^{(1)} & \dots & h_{n-1}^{(1)} \\ h_{n-k+1}^{(2)} & h_{n-k+2}^{(2)} & \dots & h_{n-1}^{(2)} \\ \dots & \dots & \dots & \dots \\ h_{n-k+1}^{(k)} & h_{n-k+2}^{(k)} & \dots & h_{n-1}^{(k)} \end{bmatrix} \begin{bmatrix} w_k \\ w_{k-1} \\ \vdots \\ 1 \end{bmatrix}. \tag{33}$$

Actually, from the selection of $h_i^{(j)}$, the relation (33) implies

$$\begin{bmatrix} 1 & 0 & \dots & 0 & h_k^{(1)} & h_{k+1}^{(1)} & \dots & h_{n-1}^{(1)} \\ 0 & 1 & \ddots & 0 & h_k^{(2)} & h_{k+1}^{(2)} & \dots & h_{n-1}^{(2)} \\ \dots & \ddots & \ddots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 1 & h_k^{(k)} & h_{k+1}^{(k)} & \dots & h_{n-1}^{(k)} \end{bmatrix} [f - d_w] = \begin{bmatrix} h_{n-k+1}^{(1)} & h_{n-k+2}^{(1)} & \dots & h_{n-1}^{(1)} \\ h_{n-k+1}^{(2)} & h_{n-k+2}^{(2)} & \dots & h_{n-1}^{(2)} \\ \dots & \dots & \dots & \dots \\ h_{n-k+1}^{(k)} & h_{n-k+2}^{(k)} & \dots & h_{n-1}^{(k)} \end{bmatrix} \begin{bmatrix} w_k \\ w_{k-1} \\ \vdots \\ 1 \end{bmatrix}. \tag{34}$$

Using this relation, we can represent $d_1, d_2, \dots, d_k$ as affine functions of the rest $(n-k)$ parameters $d_{k+1}, \dots, d_n$. More precisely, we have affine relations

$$\bar{d}(t) = M\hat{d}(t) + m,$$

where M is a known matrix, m is a known vector and

$$\bar{d} = [d_1 \quad d_2 \quad \cdots \quad d_k]^{\mathrm{T}}, \qquad \hat{d} = [d_{k+1} \quad d_{k+2} \quad \cdots \quad d_n]^{\mathrm{T}}.$$

The parameters $d_1, d_2, \dots, d_k$ are determined once $d_{k+1}, \dots, d_n$ are given. Hence, it is sufficient to estimate $(n-k)$ unknowns $\hat{d}$ for estimating d.

4.3. Adaptation law

Using the result of the previous section, we construct an adaptation law. From Fig. 2, the error signal $e(t)$ is defined as

$$e(t) = W(s)r(t) - y(t).$$

The unknown parameters $c(t), d(t), k(t)$ must be updated so that the error signal $e(t)$ decreases.

Let

$$\bar{d}(t) = [d_1 \quad d_2 \quad \ldots \quad d_k]^{\mathrm{T}} = M\hat{d}(t) + m, \tag{35}$$

$$\hat{d}_w(t) = [d_{w,k+1} \quad d_{w,k+2} \quad \ldots \quad d_{w,n}]^{\mathrm{T}},$$

$$\xi_2(t) = [\xi_{21} \quad \xi_{22} \quad \ldots \quad \xi_{2n}]^{\mathrm{T}}, \quad \bar{\xi}_2(t) = [\xi_{21} \quad \xi_{22} \quad \ldots \quad \xi_{2k}]^{\mathrm{T}},$$

$$\hat{\xi}_2(t) = [\xi_{2(k+1)} \quad \xi_{2(k+2)} \quad \ldots \quad \xi_{2n}]^{\mathrm{T}},$$

$$\hat{\theta}(t) = [c(t)^{\mathrm{T}} \quad \hat{d}(t)^{\mathrm{T}} \quad k(t)]^{\mathrm{T}}, \quad \hat{\theta}_w(t) = [c_w^{\mathrm{T}} \quad \hat{d}_w^{\mathrm{T}} \quad k_w]^{\mathrm{T}}.$$

Note that the dimension of the unknown vector $\hat{\theta}(t)$ is now $2n-k$ instead of $2n$ in the previous section. The output of $Q(\hat{\theta})$ given by (13) is written as

$$\begin{aligned} u_{\mathrm{ff}}(t) &= c(t)^{\mathrm{T}}\xi_1(t) + \bar{d}(t)^{\mathrm{T}}\bar{\xi}_2(t) + \hat{d}(t)^{\mathrm{T}}\hat{\xi}_2(t) + k(t)r(t) \\ &= c(t)^{\mathrm{T}}\xi_1(t) + \hat{d}(t)^{\mathrm{T}}(M^{\mathrm{T}}\xi_2(t) + \hat{\xi}_2(t)) + m^{\mathrm{T}}\bar{\xi}_2(t) + k(t)r(t). \end{aligned}$$

As in the invertible case, we use the same adaptation law (16), which can be written as

$$\frac{\mathrm{d}\hat{\theta}(t)}{\mathrm{d}t} = \alpha \begin{bmatrix} \xi_1(t) \\ M^{\mathrm{T}}\bar{\xi}_2(t) + \hat{\xi}_2(t) \\ r(t) \end{bmatrix} K_1(s)e(t). \tag{36}$$

4.4. Stability proof

As in the previous case, let

$$\hat{\psi}(t) := \hat{\theta}(t) - \hat{\theta}_w. \tag{37}$$

be a vector of parameter errors. Differentiation with respect to t results in

$$\frac{\mathrm{d}\hat{\psi}(t)}{\mathrm{d}t} = \frac{\mathrm{d}\hat{\theta}(t)}{\mathrm{d}t} = -\alpha\hat{\xi}(t)K_1(s)e(t), \tag{38}$$

where

$$\hat{\xi}(t) := \begin{bmatrix} \xi_1(t) \\ M^{\mathrm{T}}\bar{\xi}_2(t) + \hat{\xi}_2(t) \\ r(t) \end{bmatrix}.$$

Eq. (20) is written in this case as

$$(G_1(s) + K_1(s))e(t) = -\hat{\xi}(t)^{\mathrm{T}}\hat{\psi}(t), \tag{39}$$

where

$$G_1(s) = (1 - c_w^{\mathrm{T}}(sI - F)^{-1}g_w)P^{-1}(s).$$

Due to (30), we have

$$G_1(s) = (k_w + c_w^{\mathrm{T}}(sI - F)^{-1}g)W(s)^{-1}.$$

Combining (39) with (38) yields

$$\frac{\mathrm{d}\hat{\psi}(t)}{\mathrm{d}t} = -\alpha\hat{\xi}(t)L_1(s)\hat{\xi}(t)^{\mathrm{T}}\hat{\psi}(t), \tag{40}$$

where

$$\begin{aligned} L_1(s) &:= K_1(s)((k_w + c_w^{\mathrm{T}}(sI - F)^{-1}g)W(s)^{-1} + K_1(s))^{-1} \\ &= K_1(s)W(s)(k_w + c_w^{\mathrm{T}}(sI - F)^{-1}g + K_1(s)W(s))^{-1}. \end{aligned} \tag{41}$$

Eq. (40) is of the same form as (23), and we can use the same reasoning as in the previous section.

Theorem 2. *Under the assumptions* ($A1'$), ($A2$)–($A4$), *the FEL scheme* (35) *and* (36) *is stable and* $e(t)$ *tends to* 0, *if* $K_1(s)$ *is chosen such that* $L_1(s)$ *given by* (41) *is strictly positive real.*

Remark. In order that $L_1(s)$ in (41) is strictly positive real, $K_1(s)$ must contain higher derivatives so that the relative degree of $K_1(s)W(s)$ is not greater than two. This seems to be a drawback of Theorem 2 which depends on the notion of the strong positive realness. However, since the error signal $e(s)$ in Fig. 2 is given by $e(s) = W(s)r(s) - P(s)u(s)$, and $W(s)^{-1}P(s)$ is proper, $W(s)^{-1}e(s) = r(s) - W(s)^{-1}P(s)u(s)$ is a proper function. This implies that $W(s)^{-1}e(s)$ can be generated if the state of the plant is available for feedback. Thus, for any proper $U(s)$, $u(s) = U(s)W(s)^{-1}e(s)$ can be constructed. Then, $K_1(s) = U(s)W(s)^{-1}$ satisfies the requirement. Hence, it is not difficult to implement $K_1(s)$ such that the relative degree of $K_1(s)W(s) = U(s)$ is not greater than two.

5. Simulation results

In this section, a couple of simulation results are illustrated to demonstrate the effectiveness of the FEL.

First, we show a simulation results for a plant:

$$P(s) = \frac{s^2 + 2s + 1}{s^2 + 7s + 12}. \tag{42}$$

This plant has a stable inverse. In Fig. 3, the upper figure shows the convergence of error signal $e(t)$ and the lower one shows the comparison of tracking performances before adaptation from step 0 to step 50, with after adaptation from step 50 to step 100. We observe that the tracking is almost perfect.

Next, in Fig. 4 we show four simulation result for a plant:

$$P(s) = \frac{s + 1}{s^2 + 7s + 12}. \tag{43}$$

This plant does not have a inverse with relative degree one. In this case, the feedforward controller Q is parameterized as

$$Q(\theta) = \frac{ks^2 + (5k + c_2)s + (2k + c_1)}{s^2 + (5 - d_2)s + (5 - d_1)}. \tag{44}$$

On the other hand, we write the unknown plant $P(s)$ as

$$P(s) = \frac{b_1 s + b_2}{a_0 s^2 + a_1 s + a_2}. \tag{45}$$

Choosing

$$W(s) = \frac{10}{s + 10},$$

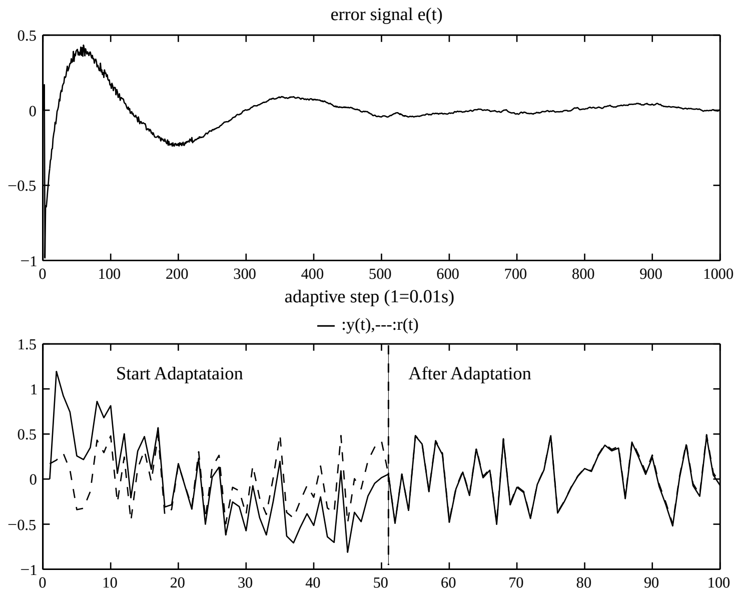

Fig. 3. Simulation result for an invertible case.

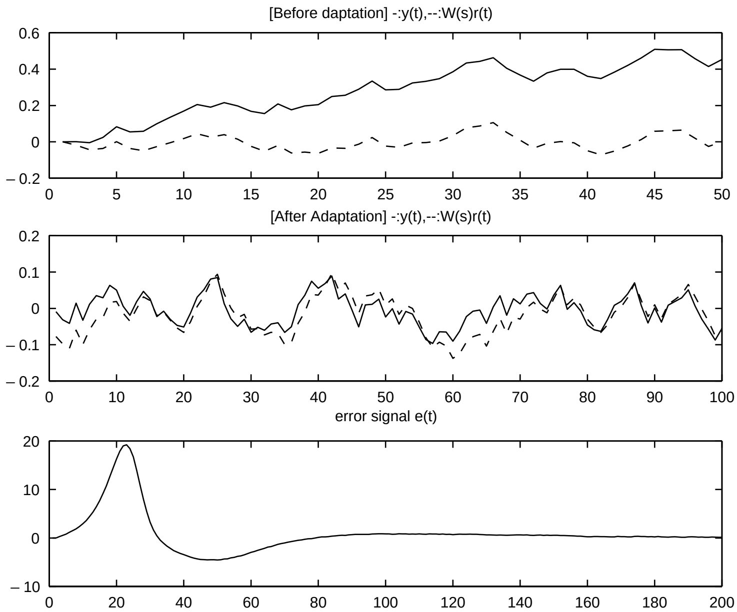

Fig. 4. Simulation results for a non-invertible case.

the target of adaptation is written as

$$P^{-1}(s)W(s) = \frac{10a_0s^2 + 10a_1s + 10a_2}{b_1s^2 + (10b_1 + b_2)s + 10b_2}. \tag{46}$$

From (44) and (46) we obtain the following constraint on parameter d_1.

$$d_1(t) = 55 + 10d_2(t). \tag{47}$$

So $d_1(t)$ is determined by (47), and $d_2(t)$ is adaptively estimated.

In this case, the FEL method gives a relatively poor tracking performance than Fig. 3 where the stable inverse exists.

6. Conclusion

In this paper, FEL proposed as an architecture of brain motor control has been investigated from the viewpoint of adaptive control theory. FEL is formulated as a TDOF control architecture equipped with adaptive capability in the feedforward controller. The novelty of learning algorithm of FEL to acquire an inverse model of the plant has been exploited extensively in the framework of conventional adaptive control. Stability of the algorithm has been established based on the notion of strictly positive realness.

Mathematical essence of FEL is represented by the differential equation

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -\alpha\xi(t)L(s)\xi(t)^{\mathrm{T}}x(t),$$

where x denotes the parameter error. This type of equations have not been discussed in control literature, though their simpler versions were discussed in detail in different context [10,11]. It may be interesting to exploit physiological ground of these equations, as well as to investigate mathematical properties of these equations in depth to obtain less conservative stability conditions of these equations.

FEL can be regarded as an adaptive version of the two-degree-of-freedom control where we are aware of pioneering work of Datta et al. [2,3]. Our convergence proof is much simpler than [2], probably because, in our architecture, adaptive controller is outside the loop.

Appendix A. KYP Lemma (Kalman–Yakubovich–Popov)

A transfer function $L(s) = c^{\mathrm{T}}(sI - A)^{-1}b + d$ is strictly positive real, if and only if $d > 0$ and for each $Q > 0$ there exist positive definite matrix P, a vector q and a number $\varepsilon > 0$ such that

$$A^{\mathrm{T}}P + PA = -qq^{\mathrm{T}} - \varepsilon Q, \tag{A.1}$$

$$Pb = c - \sqrt{2d}q. \tag{A.2}$$

Appendix B. Proof of Lemma 1

Let $L(s) = d + c^{\mathrm{T}}(sI - A)^{-1}b$. Eq. (24) is written as

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = Ax(t) + b\xi(t)^{\mathrm{T}}z(t), \tag{B.1}$$

$$y(t) = c^{\mathrm{T}}x(t) + d\xi(t)^{\mathrm{T}}z(t), \tag{B.2}$$

$$\frac{\mathrm{d}z(t)}{\mathrm{d}t} = -\xi(t)y(t). \tag{B.3}$$

Since $L(s)$ is strictly positive real, KYP Lemma implies the existence of positive definite matrices P and Q, a vector q and $\varepsilon > 0$ such that (A.1) and (A.2) hold. Let us consider a Lyapunov function

$$V(t) = \frac{1}{2}\|z(t)\|^2 + \frac{1}{2}x(t)^{\mathrm{T}}Px(t). \tag{B.4}$$

From (A.1) and (B.1), (B.2),

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t} = -z(t)^{\mathrm{T}}\xi(t)y(t) + \frac{1}{2}x(t)^{\mathrm{T}}(-qq^{\mathrm{T}} - \varepsilon Q)x(t) + \frac{1}{2}(z(t)^{\mathrm{T}}\xi(t)b^{\mathrm{T}}Px(t) + x(t)^{\mathrm{T}}Pb\xi(t)^{\mathrm{T}}z(t)).$$

Using (A.2), we have

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t} = -\frac{1}{2}|q^{\mathrm{T}}x(t) + \sqrt{2d}\,\xi(t)^{\mathrm{T}}z|^2 - \frac{1}{2}\varepsilon x(t)^{\mathrm{T}}Qx(t) \leqslant 0. \tag{B.5}$$

From (B.4), $x(t)$ and $\xi(t)^{\mathrm{T}}z(t)$ converges to 0. Therefore, $\mathrm{d}z(t)/\mathrm{d}t \to 0$. Hence, the first part of Lemma 1 has been established. Since $x(t) \to 0$, (B.3) implies that

$$\frac{\mathrm{d}z(t)}{\mathrm{d}t} = -\mathrm{d}\xi(t)\xi(t)^{\mathrm{T}}z(t), \tag{B.6}$$

for sufficiently large t. Since $L(s)$ is s.p.r., $d > 0$. From the assumption, $\xi(t)$ satisfies the PE condition, $z(t) \to 0$ due to a well-known result established in [10]. Hence, Lemma 1 has been proved. □

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