دانشگاه صنعتی امیرکبیر
(پلی تکنیک تهران)

Electrical and Electronic Circuits

chapter 8. Sinusoidal Steady State Analysis

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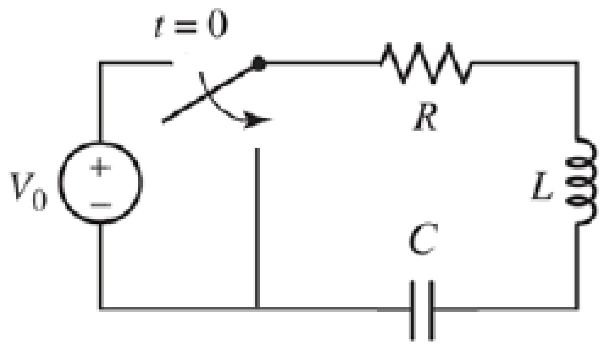
Objectives of the Lecture

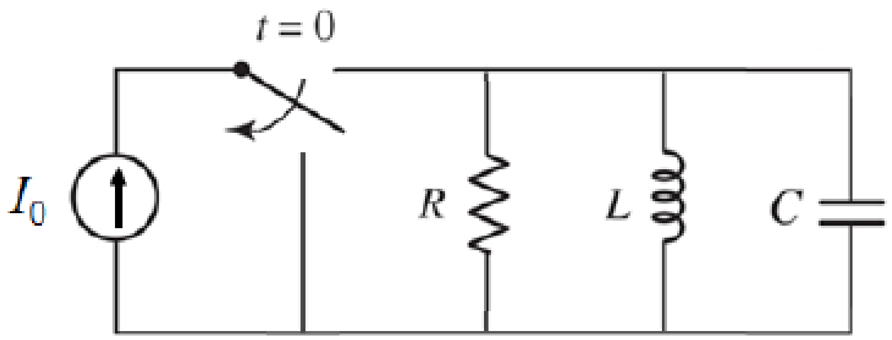
- Introduction: Sinusoidal Waves and Complex Numbers
- Force Response to Sinusoidal Input
- The Concept of the Phasor

The Rl and RC Circuit

If we know that the only independent sources in our circuit are **sinusoidal**, and we know that all transients are gone (steps, switches, pulses)…



$$\alpha = \frac{R}{2L}, \quad \omega_0 = \frac{1}{\ }$$



$$\alpha = \frac{1}{\ }, \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

NOT NECESSARY

$\alpha > \omega_0$	$\alpha = \omega_0$	$\alpha < \omega_0$
OVERDAMPED	**CRITICALLY DAMPED**	**UNDERDAMPED**
$x(t) = X_1 e^{c_1 t} + X_2 e^{c_2 t} + X_3$	$x(t) = e^{-\alpha t}(X_1 t + X_2) + X_3$	$x(t) = e^{-\alpha t}\begin{bmatrix} X_1 \cos(\omega_d t) \\ + X_2 \sin(\omega_d t) \end{bmatrix} + X_3$

…we may instead solve the circuit **algebraically** (e.g. nodal, mesh) without determining initial conditions, final conditions, etc.

Sinusoidal Waves

Amplitude V_m

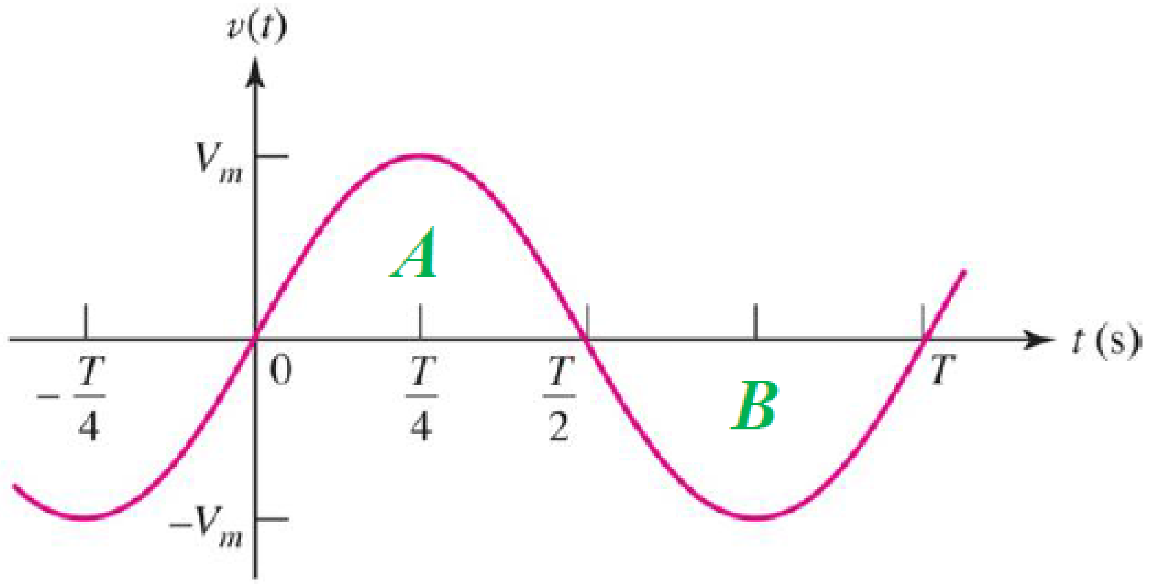
Argument ωt

Angular Frequency ω

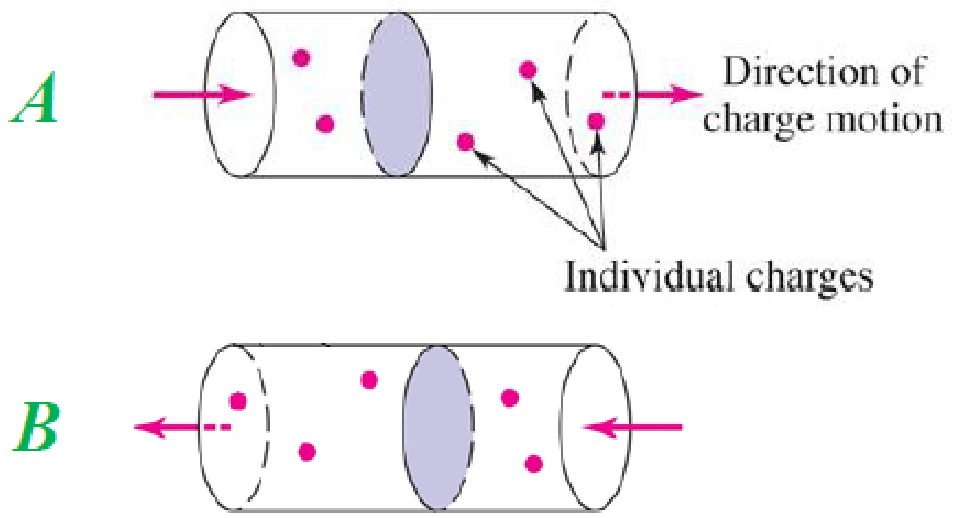
Period T

Frequency f

$$v(t) = V_m \cdot \sin(\omega t + \phi_0) \ , \quad \phi_0 = 0$$



$+ \quad v(t) \quad -$



V_m = amplitude (in Volts), ϕ_0 = phase (in radians)

ω = frequency (in radians/second)

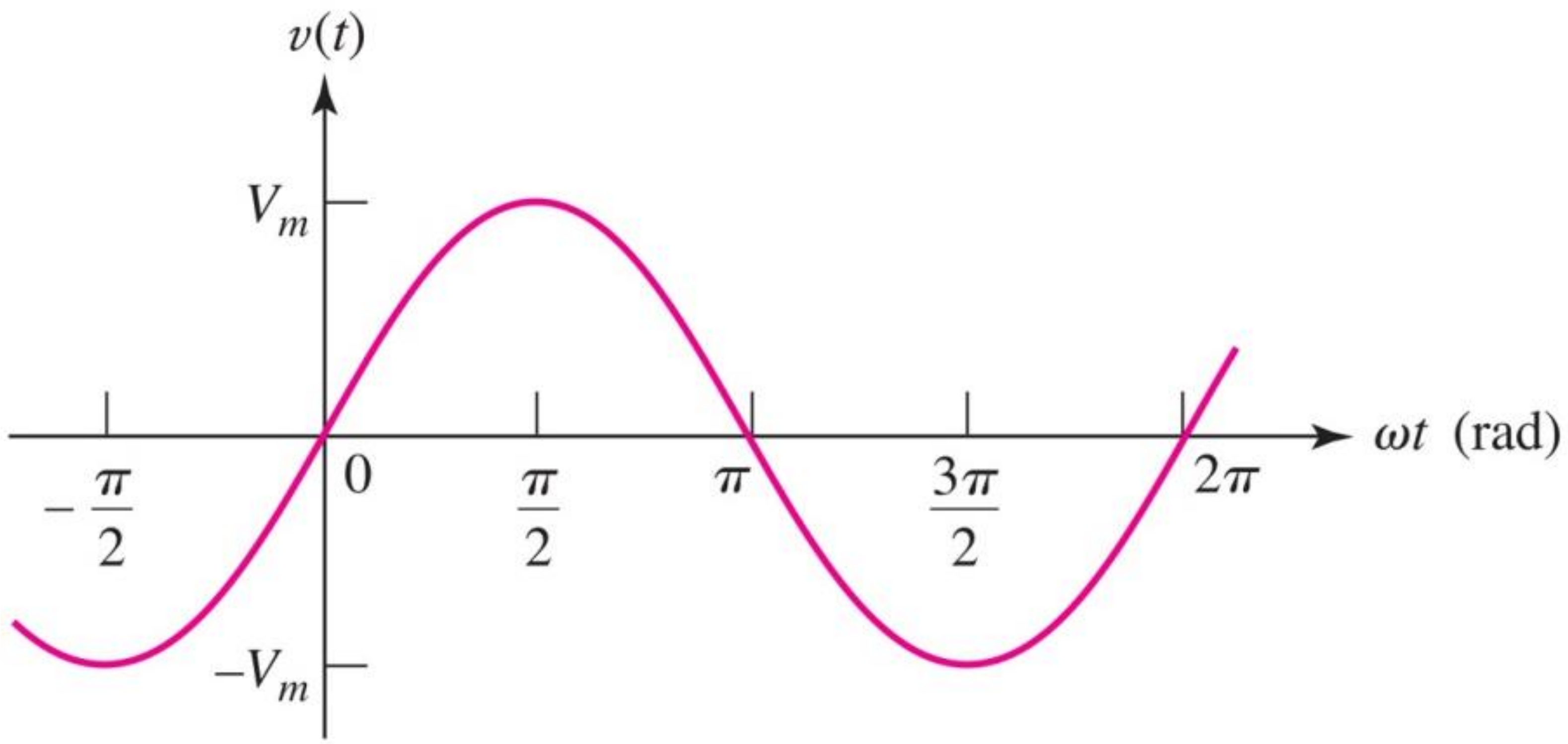
T = period (in seconds)

f = frequency (in cycles/second) = $1/T$ = $\omega / 2\pi$

$$V_m \cdot \sin(\omega t + \phi_0) = V_m \cdot \cos(\omega t + \phi_0 - \pi/2)$$

Sinusoidal Waves

$$v(t) = V_m \sin(\omega t + \phi) \qquad T = \frac{2\pi}{\omega} \qquad f = \frac{1}{T}$$



where

V_m is the amplitude of the sinusoid

w is the angular frequency in radians/s

ϕ is the phase angle in degrees

$wt + \phi$ is the argument of the sinusoid

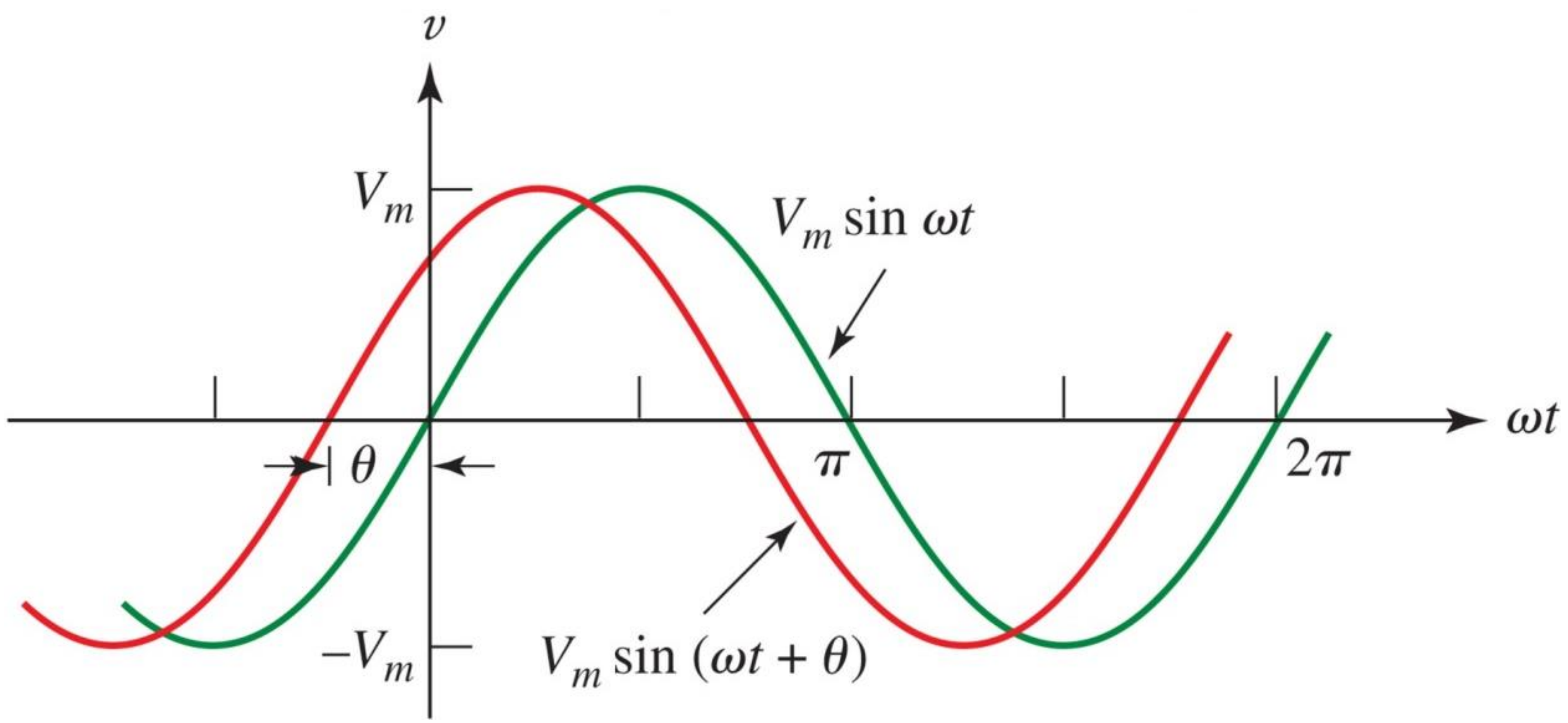
T is the period of a sinusoid in seconds

f is the frequency with units of Hz (cycles per second)

Phase of the Sinusoidal Wave

In the more general case, a sinusoidal wave includes a phase θ.

$$v(t) = V_m \sin(\omega t + \theta)$$

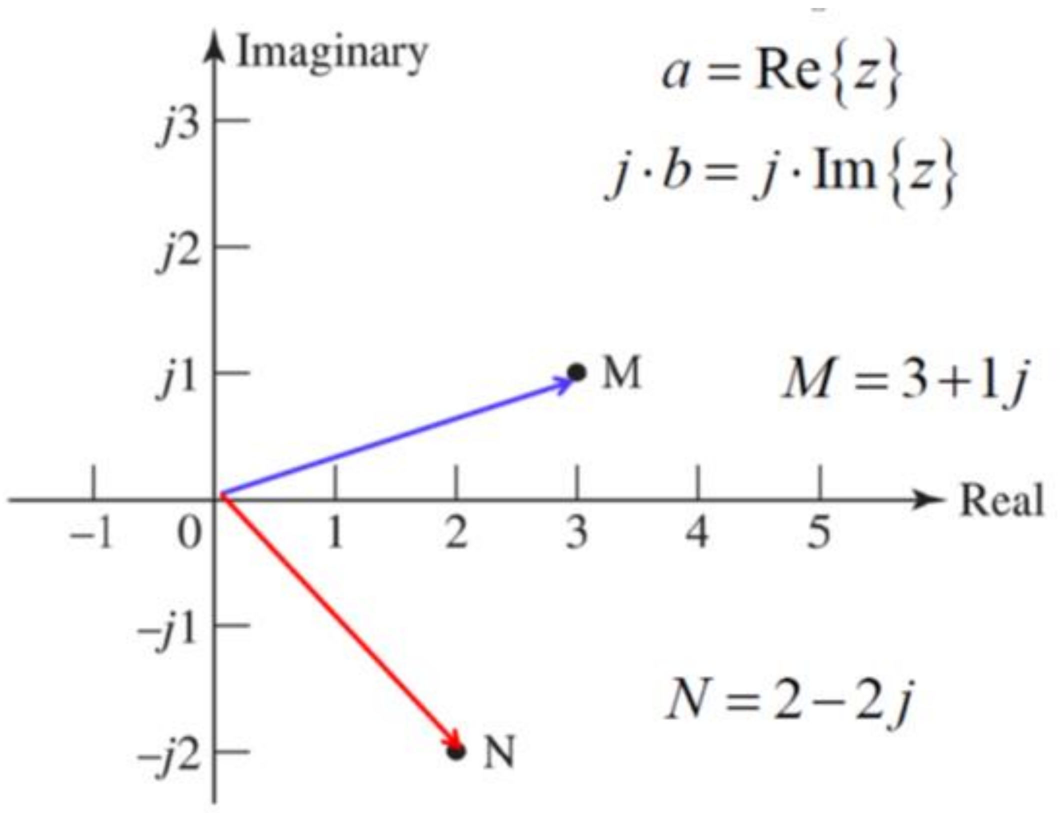


We say that the new wave leads the original wave by a phase of θ.

We say that the original wave lags behind the new wave by a phase of θ.

Complex numbers

- A complex number *z* can be written in rectangular form as



❖ $a + bj \leftrightarrow Ae^{j\theta}$

❖ $A = \sqrt{a^2 + b^2},\ \theta = \tan^{-1}\frac{b}{a}$

❖ $a = A\cos\theta\,,\quad b = A\sin\theta$

- The complex number *z* can also be written in polar or exponential form as $z = A\measuredangle\theta = Ae^{j\theta}$

✓ $2\measuredangle 45° = 2\mathrm{e}^{\mathrm{j}45} = 2\cos 45 + j2\sin 45 = \sqrt{2} + \sqrt{2}j$

✓ $\frac{1}{1+2j} = \frac{1\measuredangle 0}{\sqrt{5}\measuredangle\tan^{-1}2} = \frac{1}{\sqrt{5}}\measuredangle - \tan^{-1}2$

Multiplication and Division

✓ Subtraction with Rectangular Coordinates

If $Z = X + Y$ and $X = a + jb$ and $Y = c + jd$

$Z = (a - c) + j(b - d) \rightarrow Re\,(Z) = (a - c)$ and $Im(Z) = (b - d)$

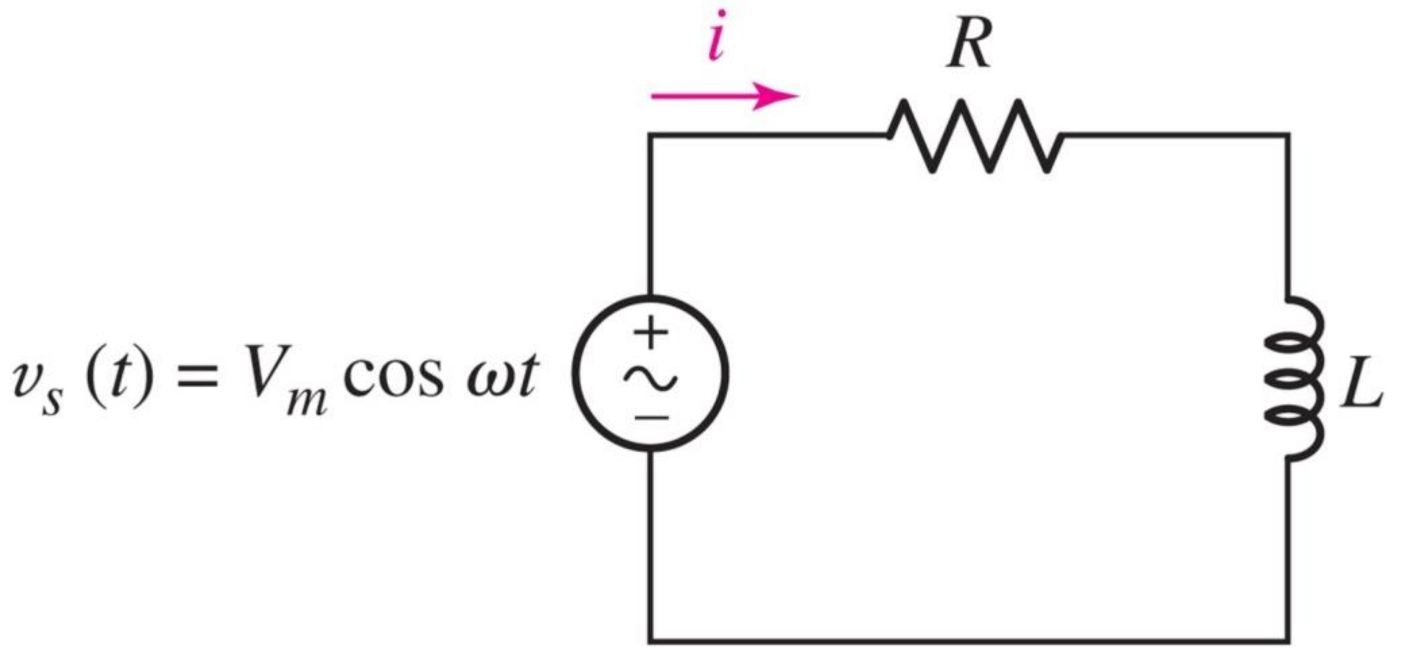
✓ Multiplication and Division

If $V = v\angle f$ and $I = i\angle q$

$P = VI = vi\,\angle(f + q)$ and $Z = V/I = v/i\,\angle(f - q)$

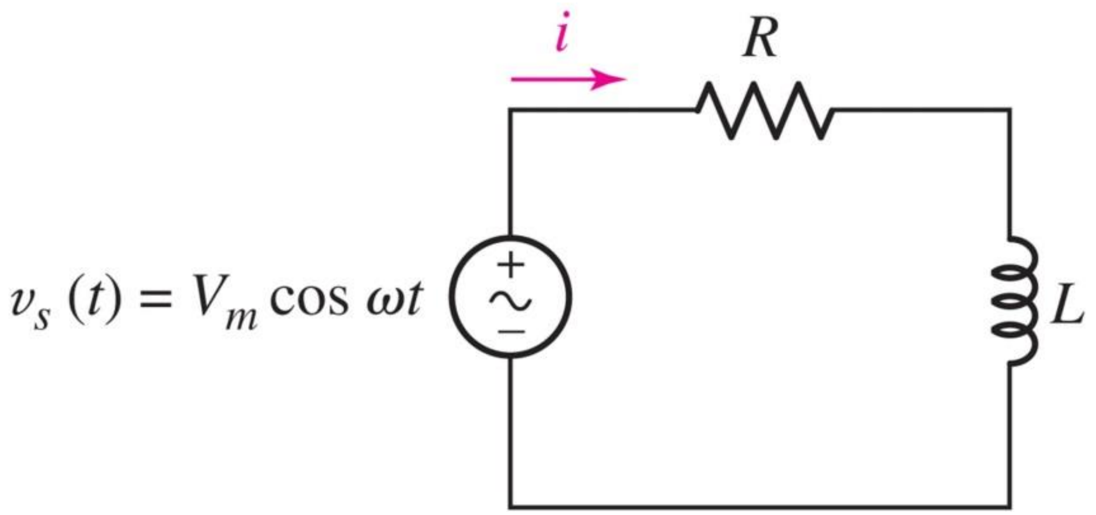
Forced Response to Sinusoidal Input

- In many applications, when the input is sinusoidal, the transient (natural) response is not of interest; instead, we are primarily concerned with finding the steady-state (forced) response.



- Here, we seek a method to simplify obtaining this response.

- ✓ We apply KVL (Kirchhoff's Voltage Law):



$$L\frac{di}{dt} + Ri = V_m \cos(\omega t)$$

- ✓ The forced response is of the same form as the input.

$$i(t) = I_1 \cos \omega t + I_2 \sin \omega t$$

- ✓ By substituting into the differential equation, the coefficients are determined.

$$i(t) = \frac{RV_m}{R^2 + \omega^2 L^2} \cos \omega t + \frac{\omega L V_m}{R^2 + \omega^2 L^2} \sin \omega t$$

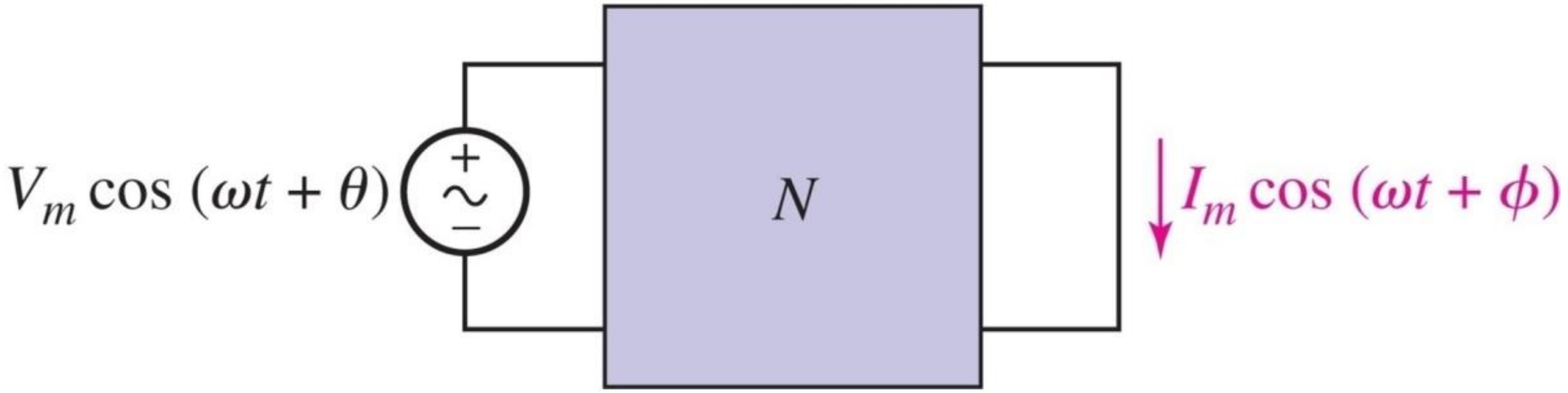
Conclusion?

- ✓ The forced response to a sinusoidal input can be obtained as before by solving the differential equation.

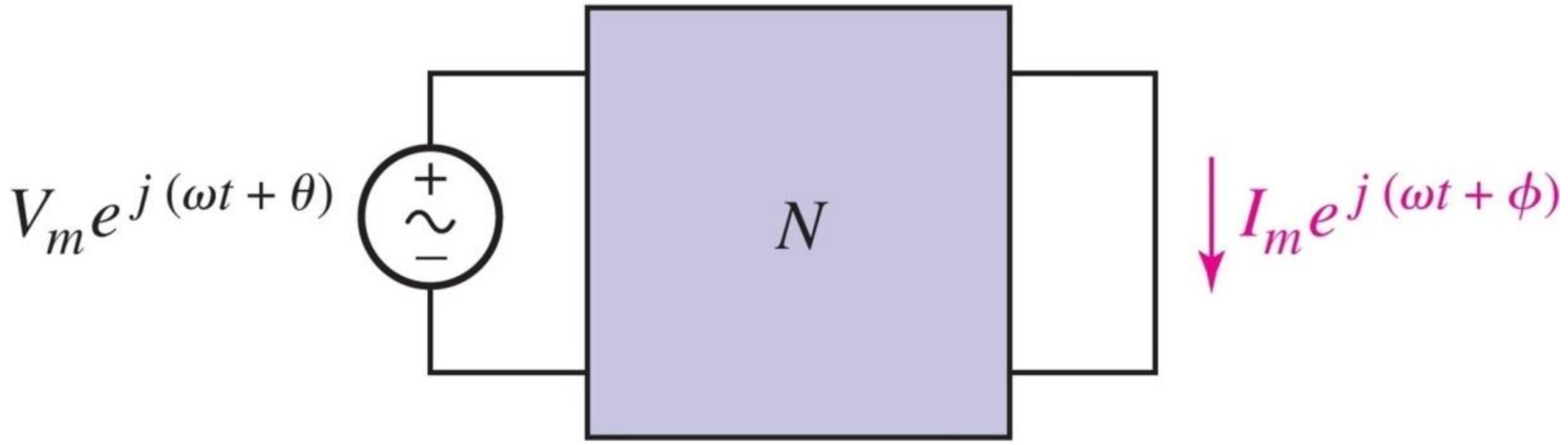
- ✓ Is there a way to avoid differential equations and rely solely on algebraic calculations to find the forced sinusoidal response?

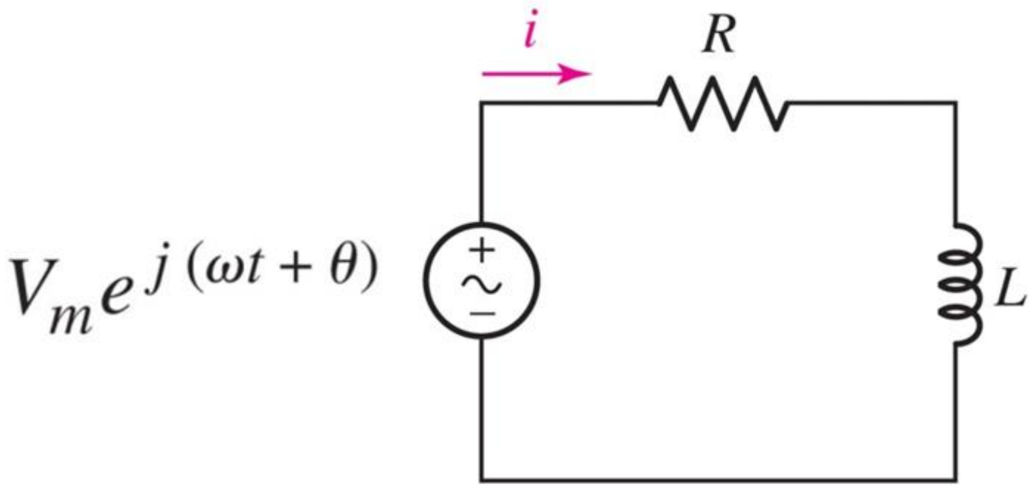
- ✓ Yes, by using the concept of the phasor!

➢ Main problem: finding the steady-state response of circuit N to a sinusoidal input.



➢ Alternative approach: let us determine the response of the circuit to the following exponential input:





1. By applying KVL, we have:

$$L\frac{di}{dt} + Ri = v_s$$

2. The forced response is of the same form as the output.

$$i(t) = I_m e^{j(\omega t+\phi)}$$

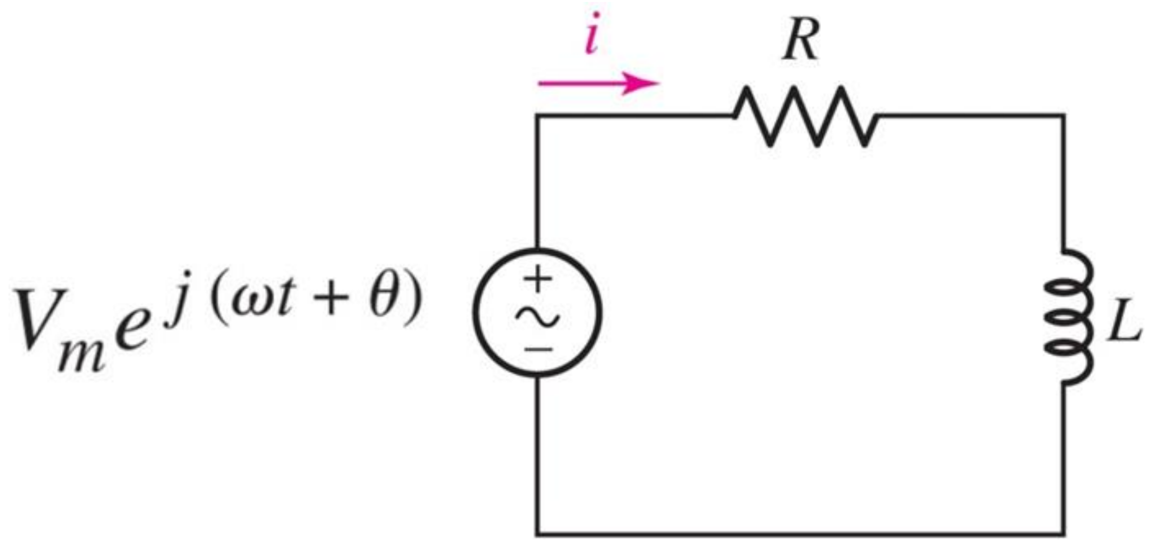
3. Substituting into the differential equation:

- $j\omega L I_m e^{j(\omega t+\phi)} + R I_m e^{j(\omega t+\phi)} = V_m e^{j(\omega t+\theta)}$
- $j\omega L I_m e^{j\phi} + R I_m e^{j\phi} = V_m e^{j\theta}$
- $I_m e^{j\phi} = \frac{V_m e^{j\theta}}{R+j\omega L} \rightarrow \; I_m = \frac{V_m}{\sqrt{R^2+\omega^2 L^2}}, \quad \phi = \theta - \tan^{-1}\frac{\omega L}{R}$
- $i(t) = Re\left[I_m e^{j(\omega t+\phi)}\right] = \frac{V_m}{\sqrt{R^2+\omega^2 L^2}}\cos(\omega t + \theta - \tan^{-1}\frac{\omega L}{R})$

The concept of phasor

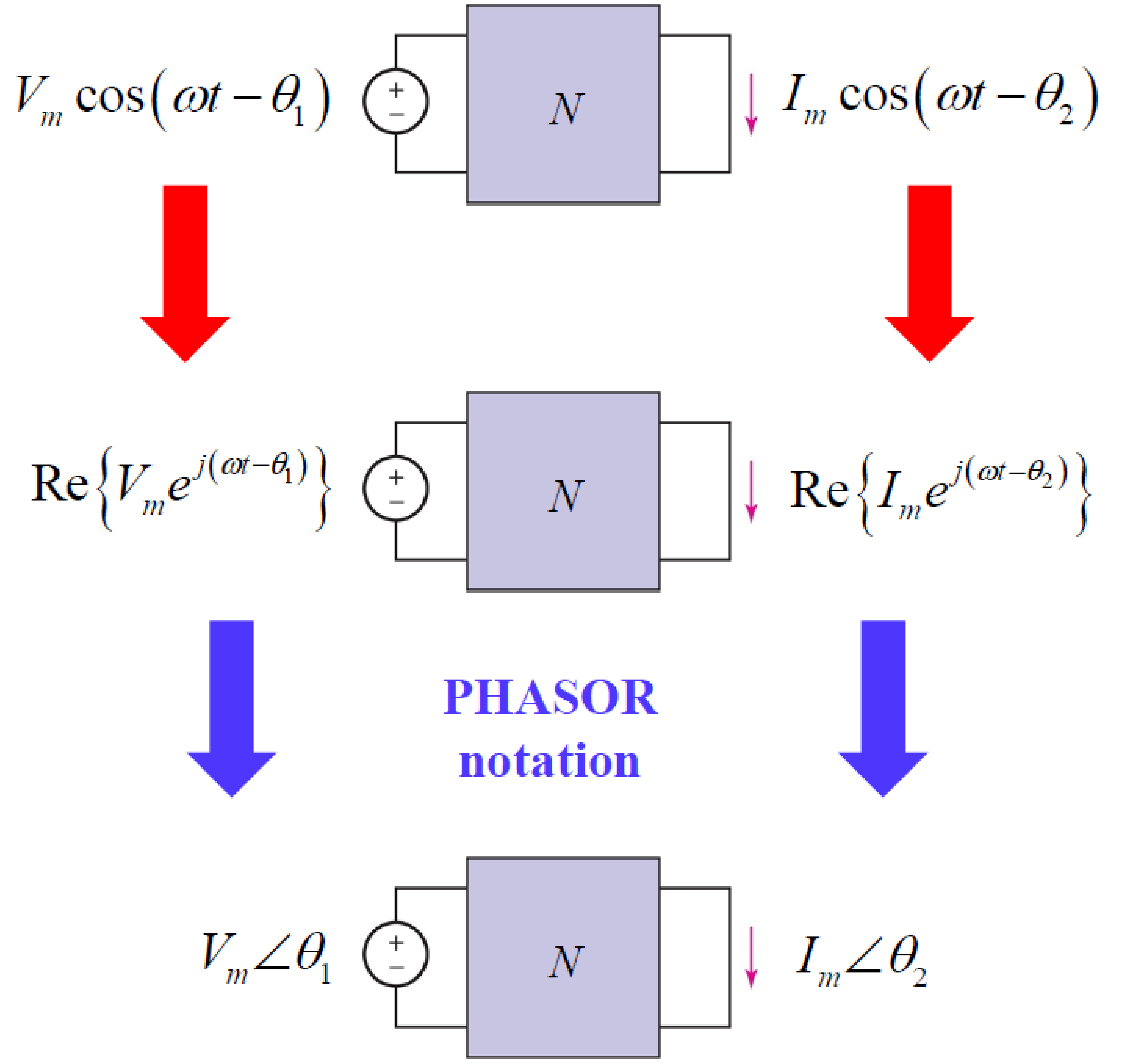
- ✓ In the sample circuit shown, we arrive at the following relationship:

$$I_m e^{j\phi} = \frac{V_m e^{j\theta}}{R + j\omega L}$$



- ✓ A complex number with a magnitude of V_m and an angle θ, represented as $V_m e^{j\theta}$, is referred to as a phasor.
- ✓ Similarly, the phasor for the current is $I_m e^{j\phi}$.
- ✓ The relationship between the phasors in the circuit is an algebraic relationship, not a differential one!

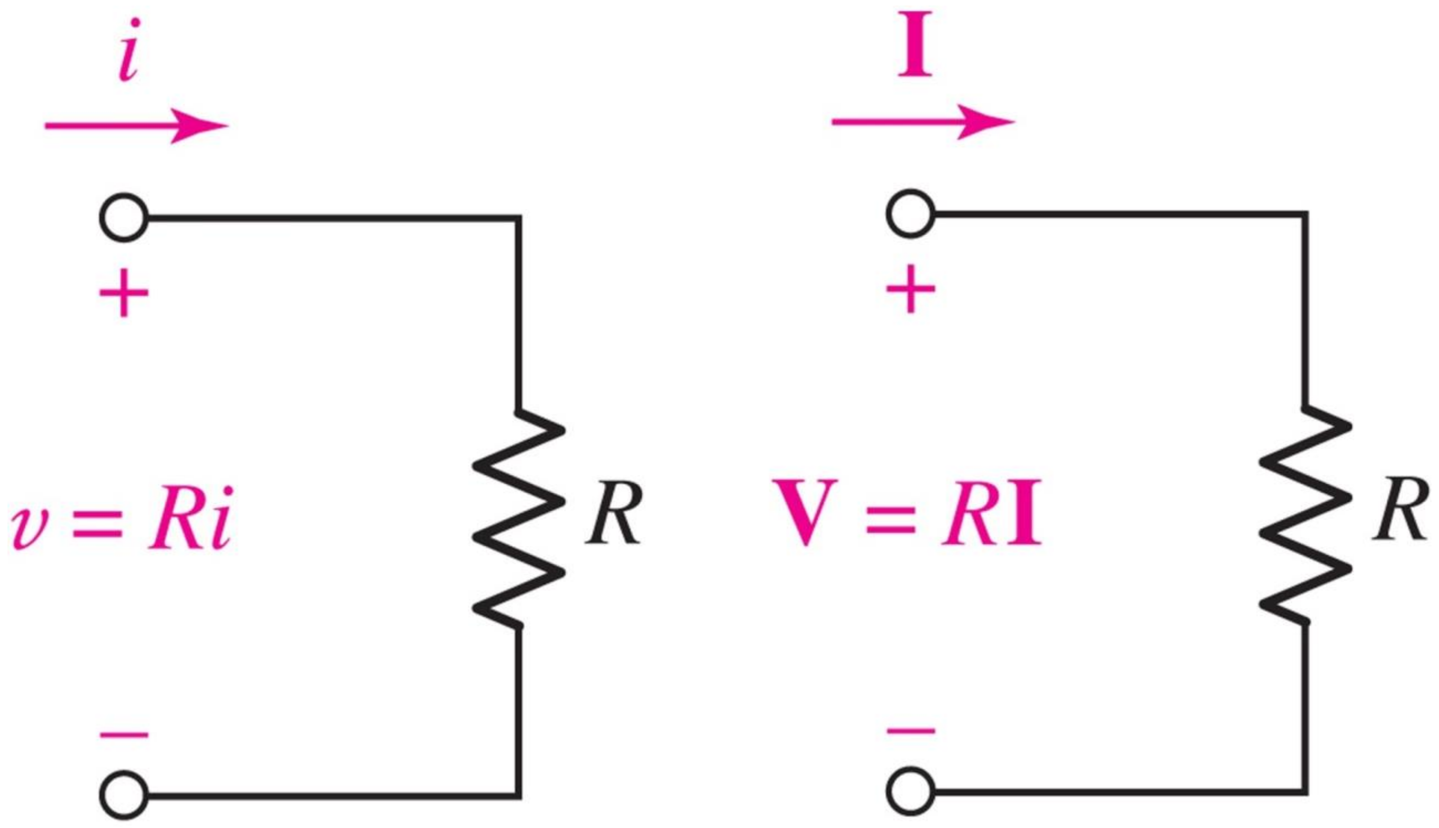
Sinusoidal vs. Complex Representation



The phasor of the resistor

The voltage phasor and current phasor of a resistor also follow Ohm's Law.

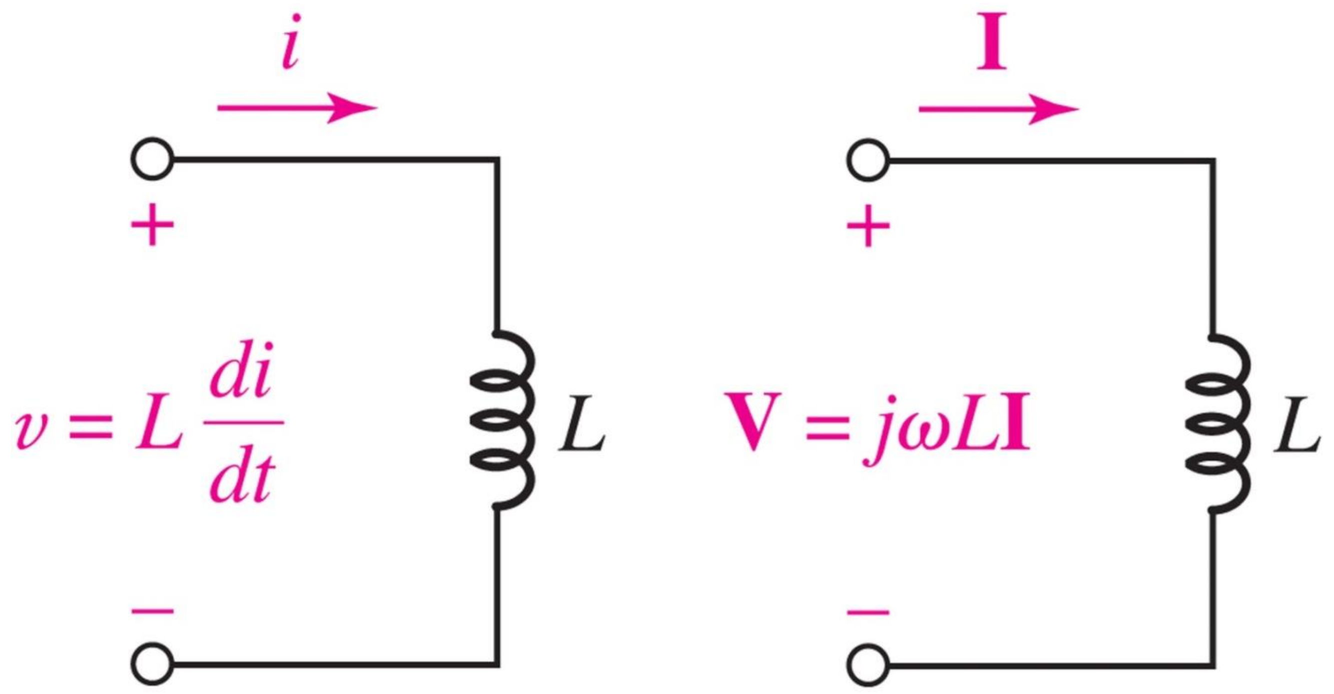
Thus, the phasor of the resistor is simply R.



The phasor of the inductor

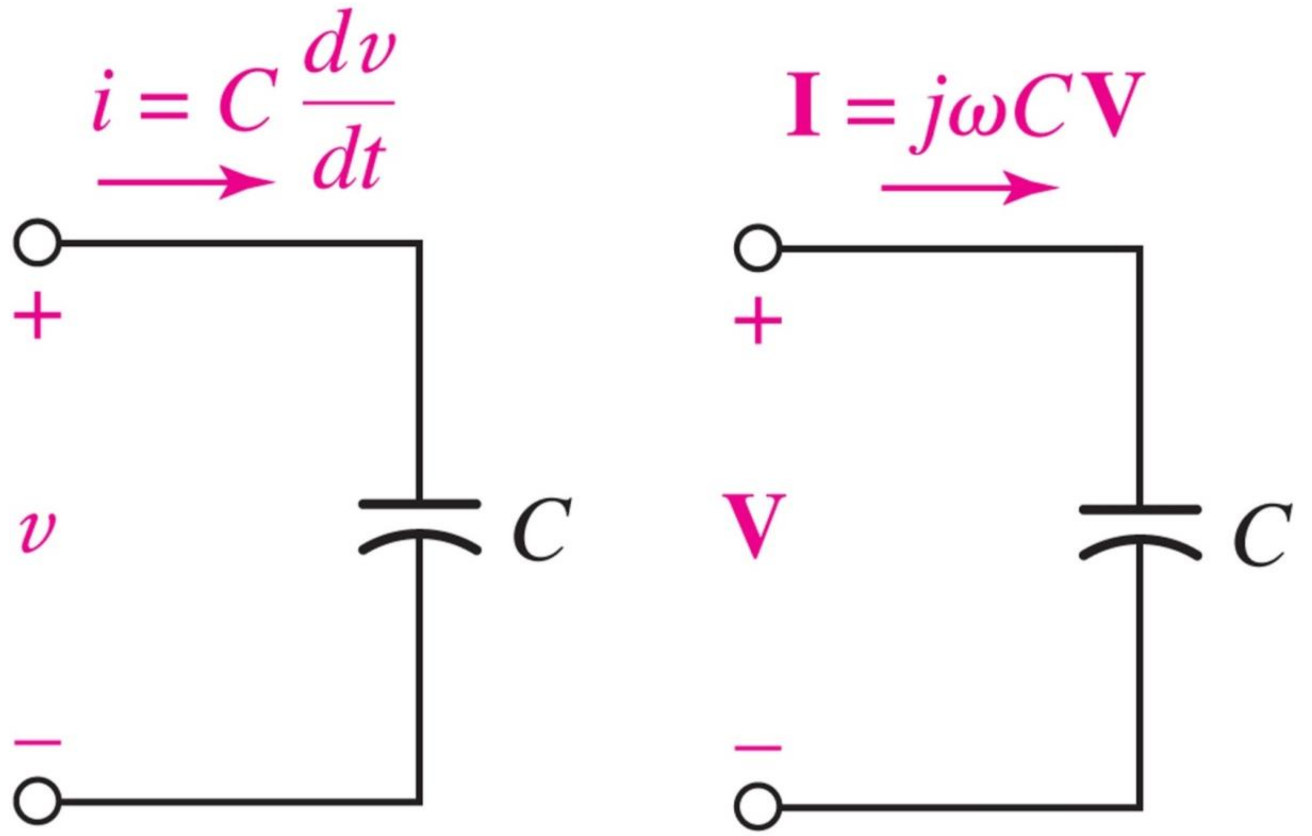
The differential relationship between the current and voltage of an inductor in the time domain transforms into an algebraic relationship in the phasor domain.

The phasor of the inductor is $j\omega L$, and its voltage and current phasors satisfy Ohm's law.



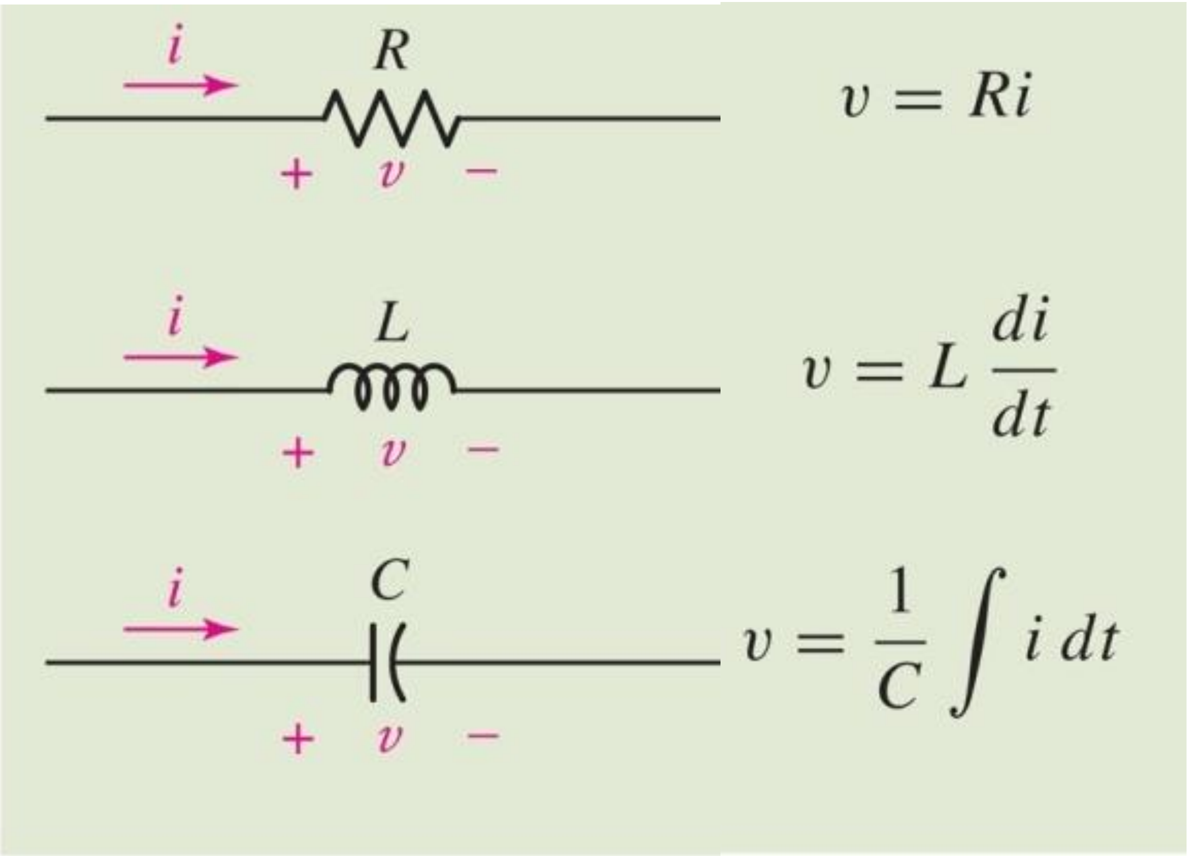
The phasor of the capacitor

The phasor of the capacitor is $\frac{1}{j\omega C}$, and the relationship between its voltage and current in the phasor domain, like that of the inductor and resistor, follows Ohm's law.

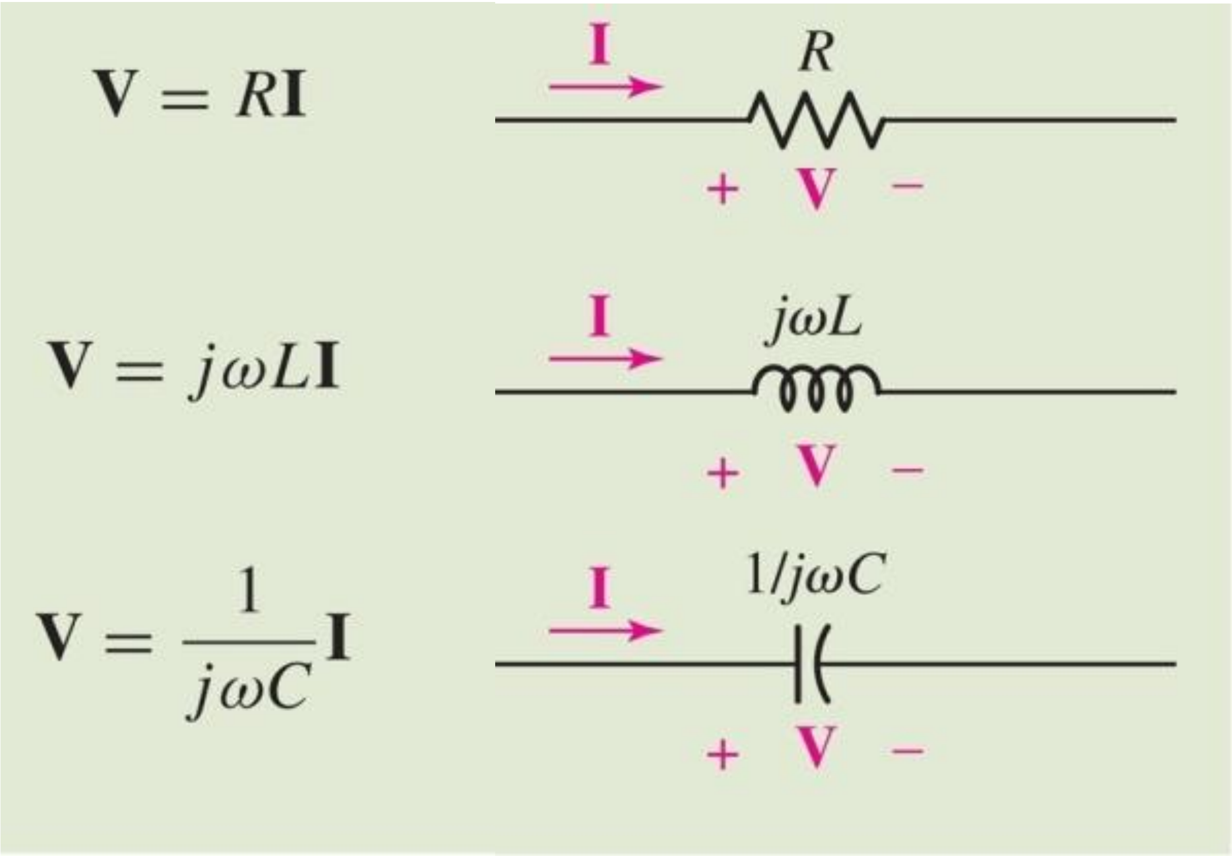


Summery

Time Domain



Phasor Domain



Differential calculations with real numbers

Algebraic calculations with complex numbers

Kirchhoff's laws for phasors

- The KVL relationship also holds for voltage phasors in a loop.

$$\mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_N = 0$$

- Similarly, the KCL relationship holds for current phasors at a node.

$$\mathbf{I}_1 + \mathbf{I}_2 + \cdots + \mathbf{I}_N = 0$$

Impedance

- ✓ The ratio of the voltage phasor to the current phasor is called impedance.

$$Z_R = R \qquad Z_L = j\omega L \qquad Z_C = 1/j\omega C$$

- ✓ Impedance is equivalent to resistance in the phasor domain.

- ✓ Impedance is a complex number and is measured in ohms.

- ✓ The real part of impedance is called resistance, and the imaginary part is called reactance.

  Series and parallel impedances can be combined in the same way as resistances.

Admittance

- The inverse of impedance is called admittance.

$$Y_R = 1/R \qquad Y_L = 1/j\omega L \qquad Y_C = j\omega C$$

- Admittance is equivalent to conductance.

- Admittance is a complex number, and its unit is Siemens.

- The real part of admittance is called conductance, and the imaginary part is called susceptance.

Summary of the Phasor Method

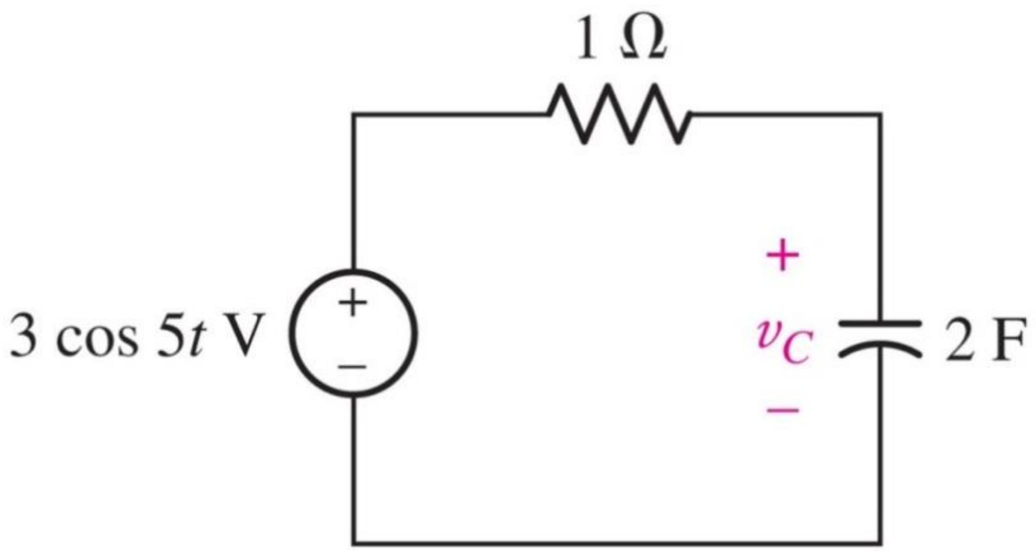
1. Determine the impedance of all elements based on the source frequency.

$$Z_R = R \qquad Z_L = j\omega L \qquad Z_C = 1/j\omega C$$

1. Replace the source values with their corresponding phasors.
2. Analyze the circuit in the phasor domain as if it were a resistive circuit, and find the phasors for all currents and voltages in the circuit.
3. Converting phasors back to the time domain: Multiply the desired phasor by $e^{j\omega t}$ and take the real part as the final response.

Example 1: Using Phasors

Find the steady-state response of the voltage across the capacitor.



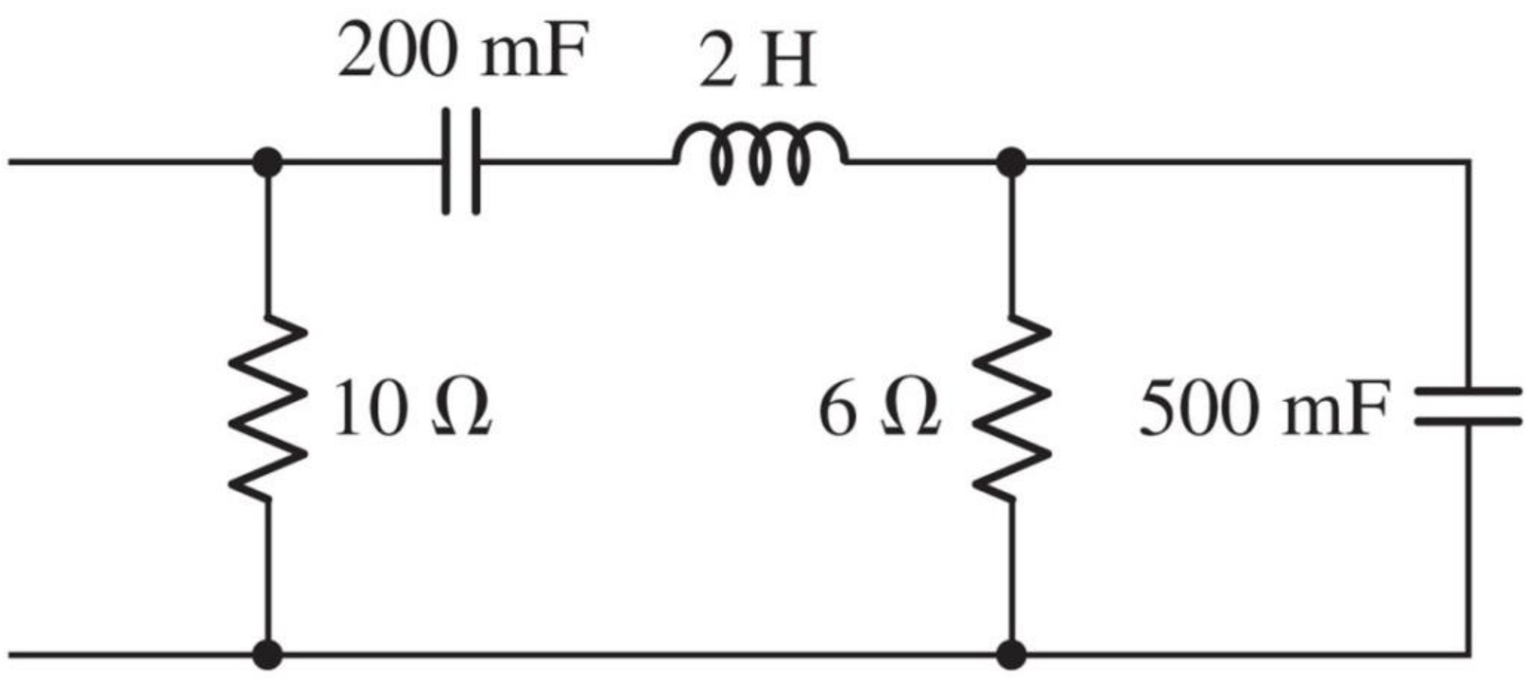
$$V_s = 3\measuredangle 0, \qquad Z_R = 1, \qquad Z_C = \frac{1}{j\omega C} = \frac{1}{10j} = -0.1j$$

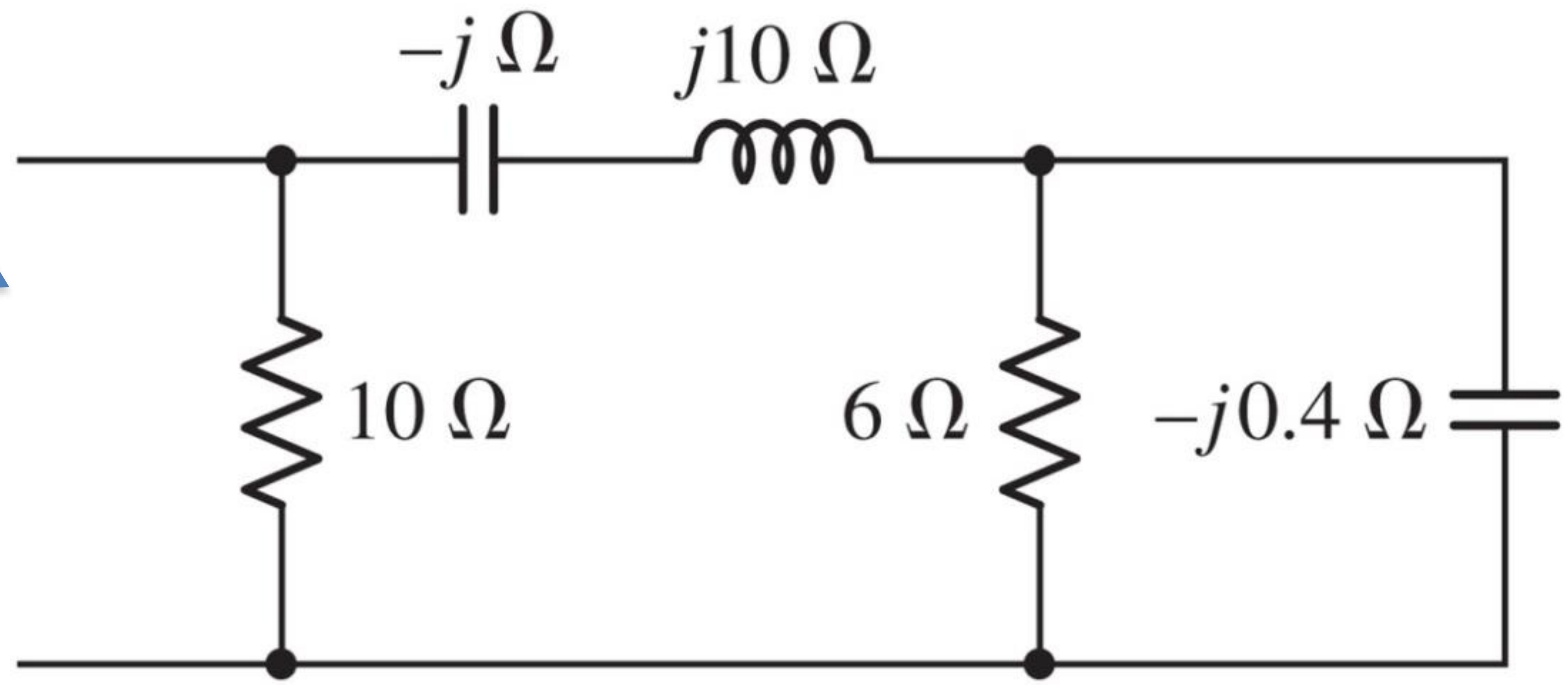
$$V_c = \frac{-0.1j}{1-0.1j} \times 3 = \frac{3}{1+10j} = \frac{3\measuredangle 0}{\sqrt{101}\measuredangle \tan^{-1} 10} = 0.298\measuredangle -84.3$$

$$v_C(t) = Re\left[0.298e^{j(5t-84.3)}\right] = 0.298\cos(5t-84.3)$$

Example 2: Calculation of Equivalent Impedance

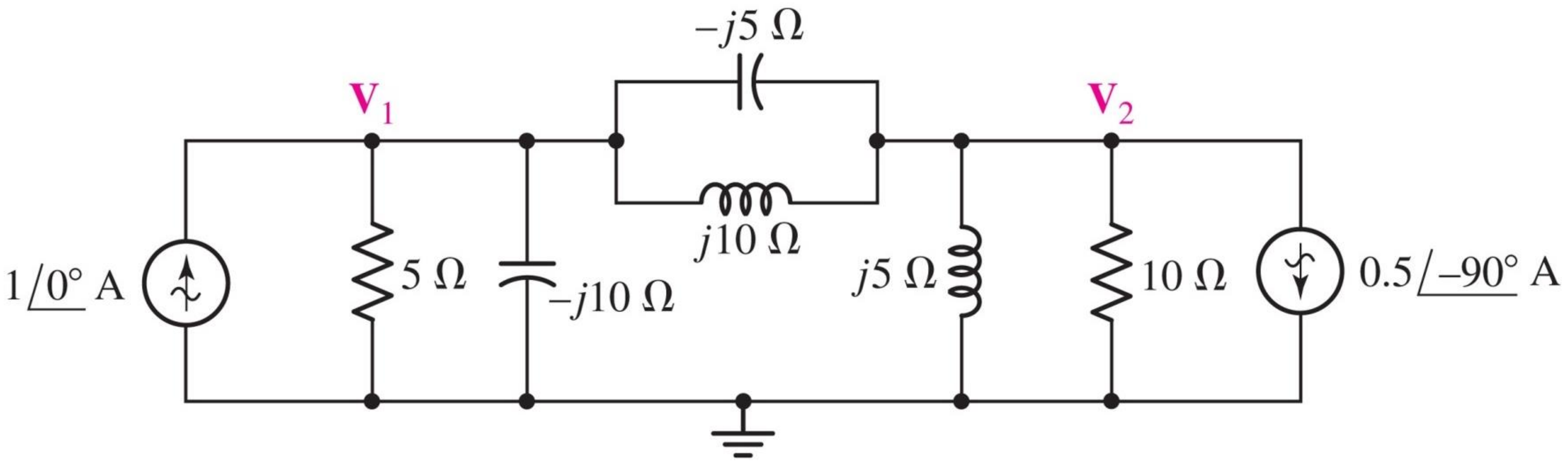
Determine the equivalent impedance of the circuit below at a frequency of 5 rad/s.





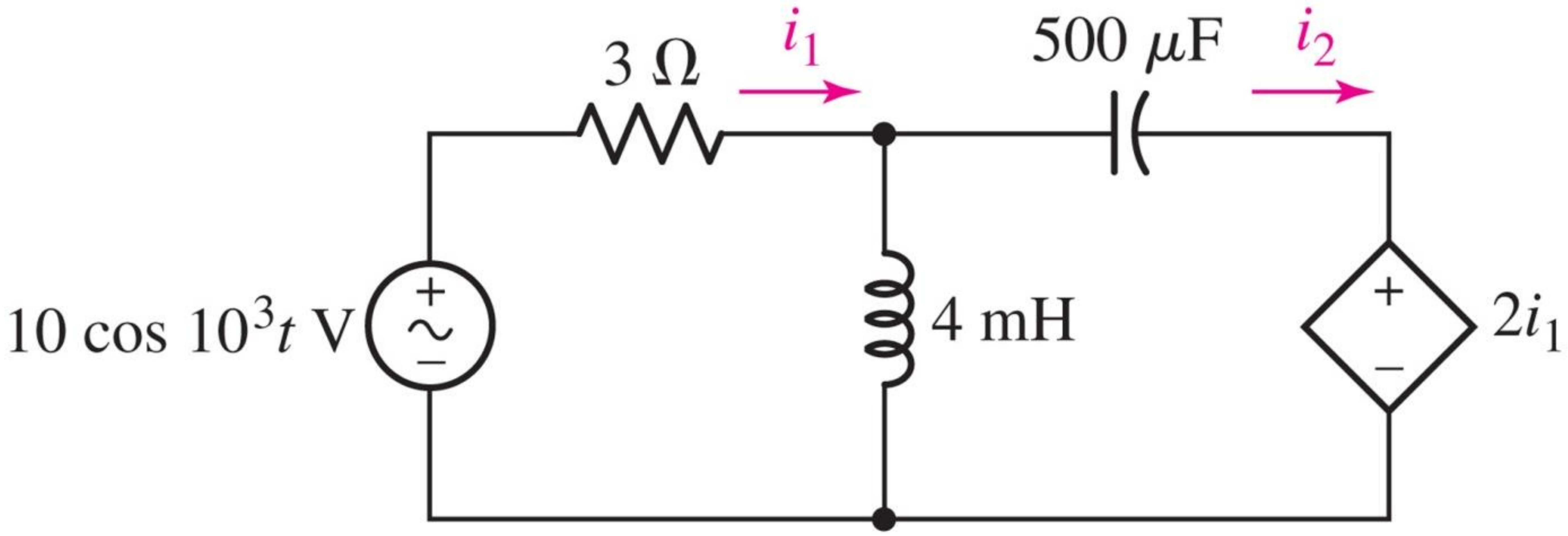
Answer: 4.255 + j4.929 Ω

Determine the phasors for the voltages V1 and V2.



Example 4: Mesh Analysis

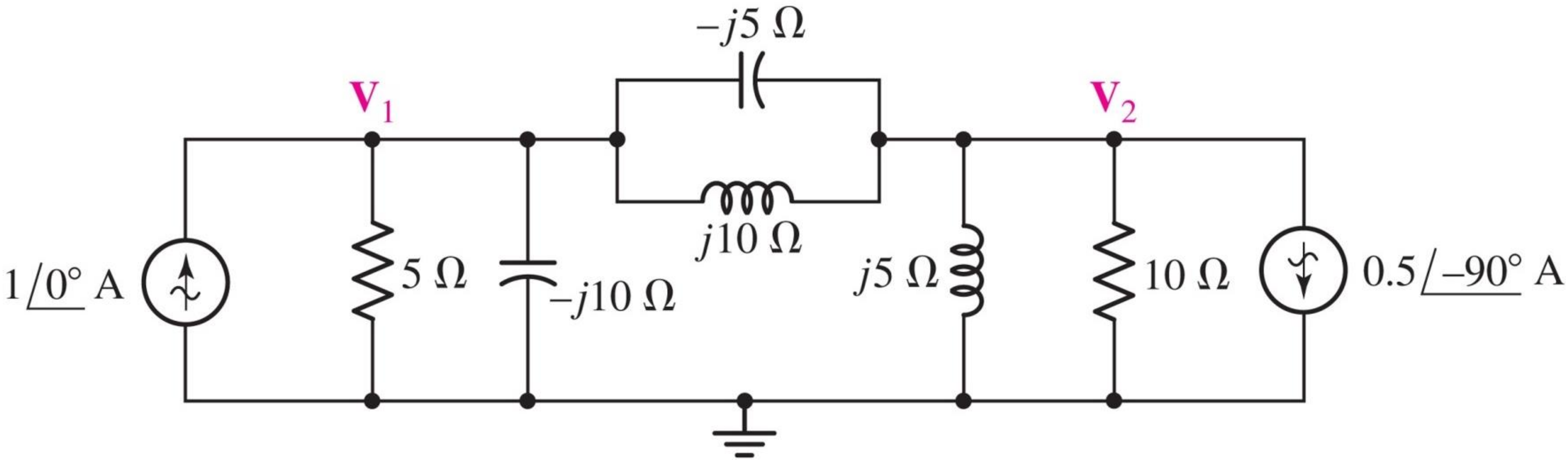
Determine the currents i1(t)i_1(t)i1(t) and i2(t)i_2(t)i2(t).



Answer: $i_3(t) = 1.333 \sin t$ V

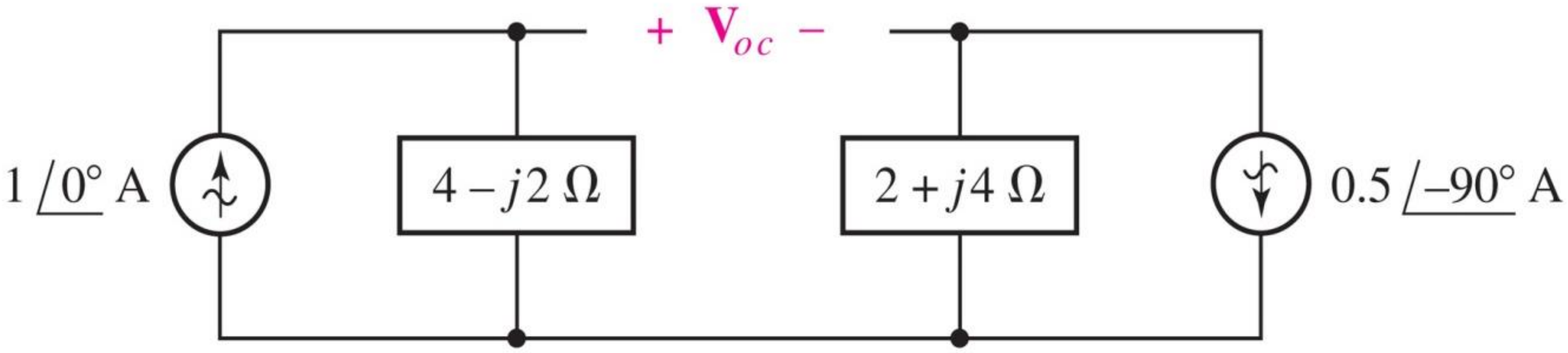
Example 5: Superposition Theorem

The principle of superposition is also valid for phasors. Use this principle to determine the voltage V1



Example 6: Thevenin Equivalent Circuit

Theorems of Thevenin and Norton are also applicable to phasors. By calculating the Thevenin equivalent circuit of the given diagram, first determine the current flowing between nodes V1 and V2, and then find the voltage V1.



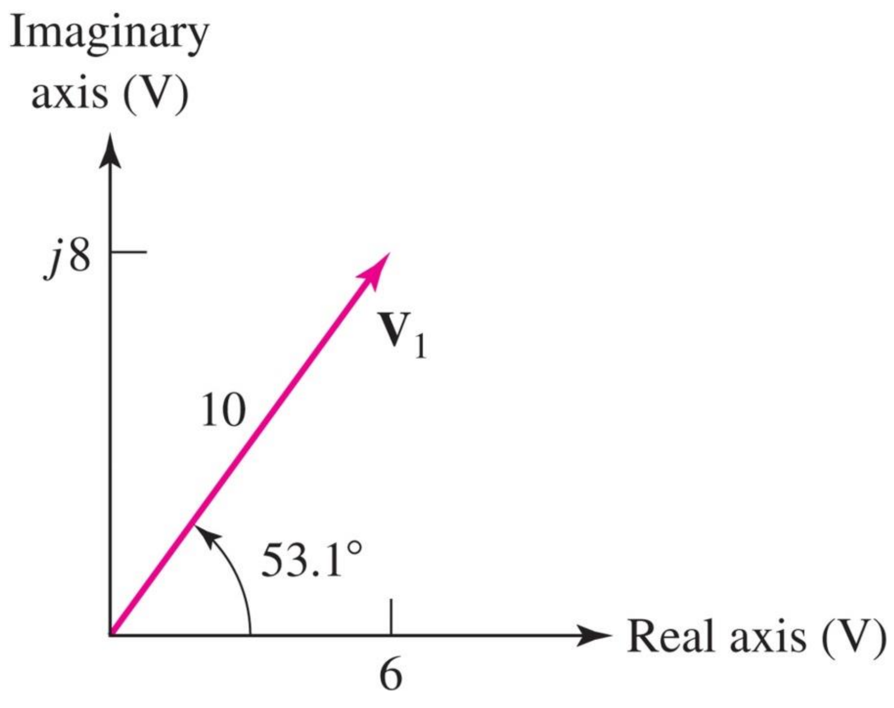
Phasor Vector Diagram

- Assume we have a phasor vector in the complex plane that rotates around the origin with an angular speed ω.
- The figure below illustrates the position of this vector at the moment t =0. This represents the phasor voltage V1 with a magnitude of 10 and an angle of 53.1 degrees.

$$V_1 = 10e^{j53.1}$$

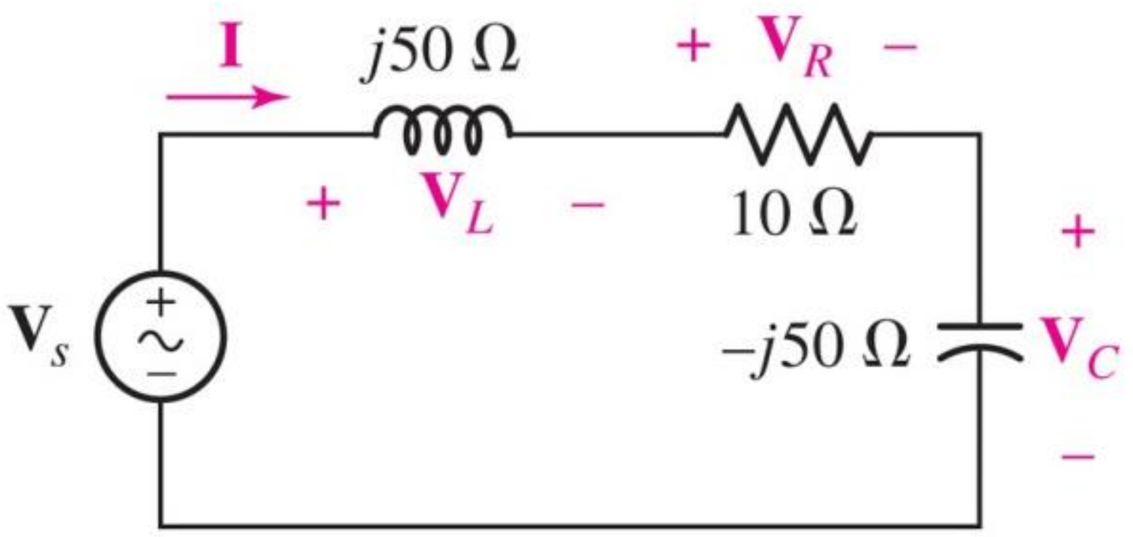
- The projection of this vector onto the real axis at any moment indicates the instantaneous voltage v1(t):

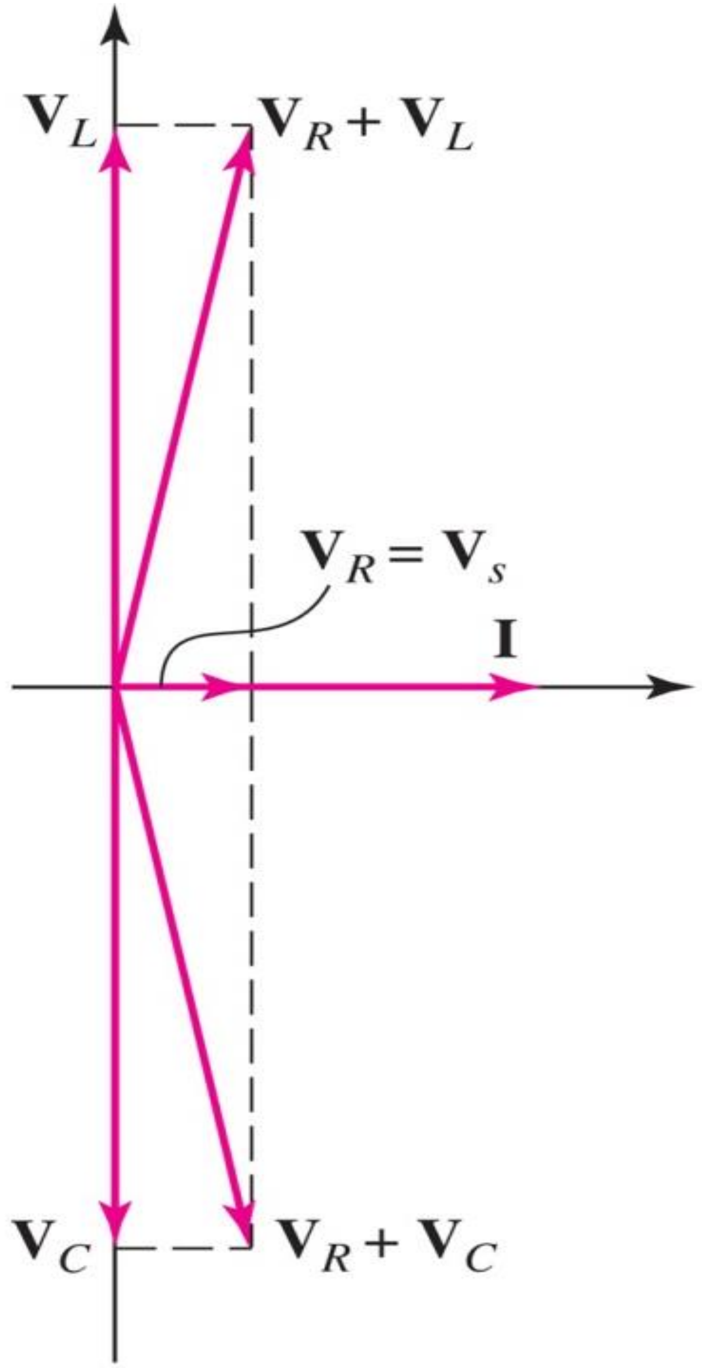
$$v_1(t) = 10\cos(\omega t + 53.1)$$



Phasor Vector Diagram: Example 1

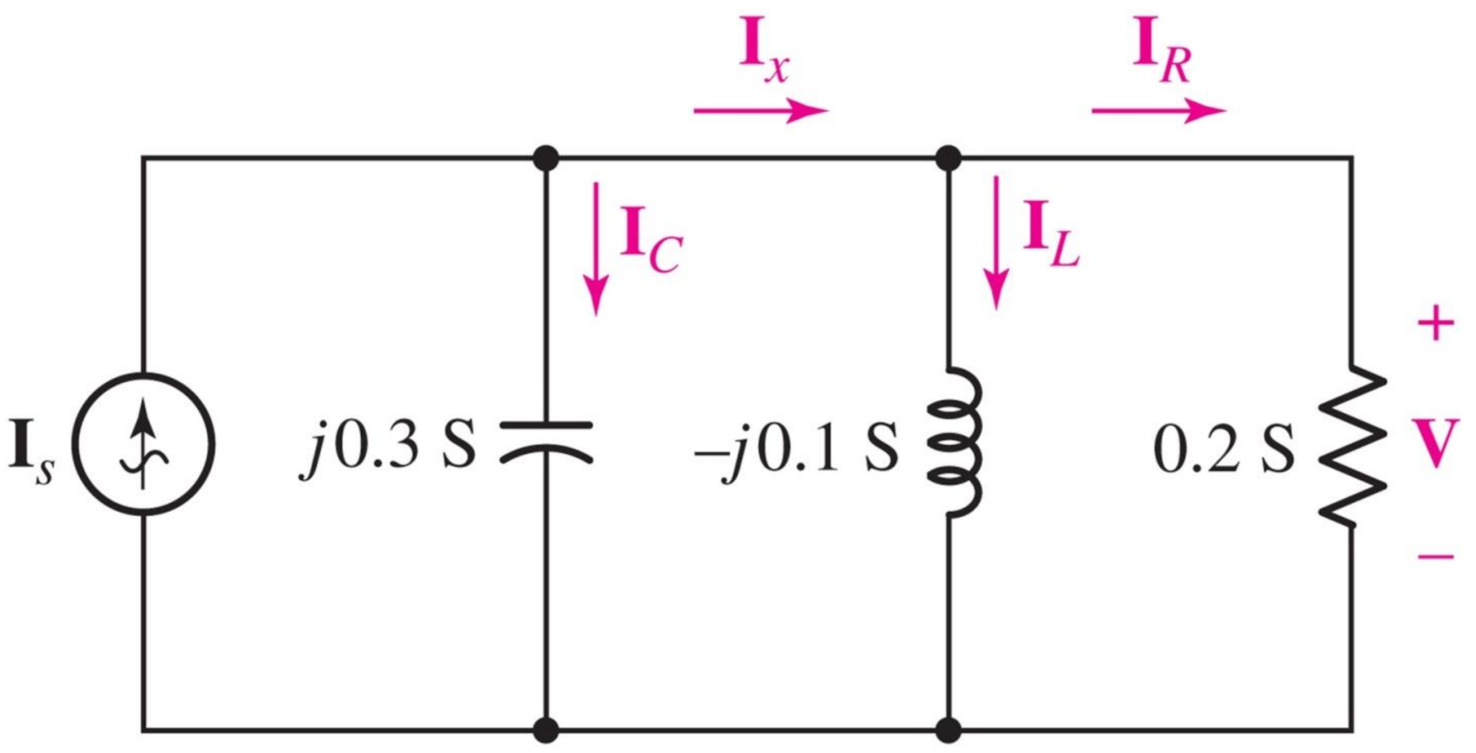
Assuming $I = 1\angle 0$:

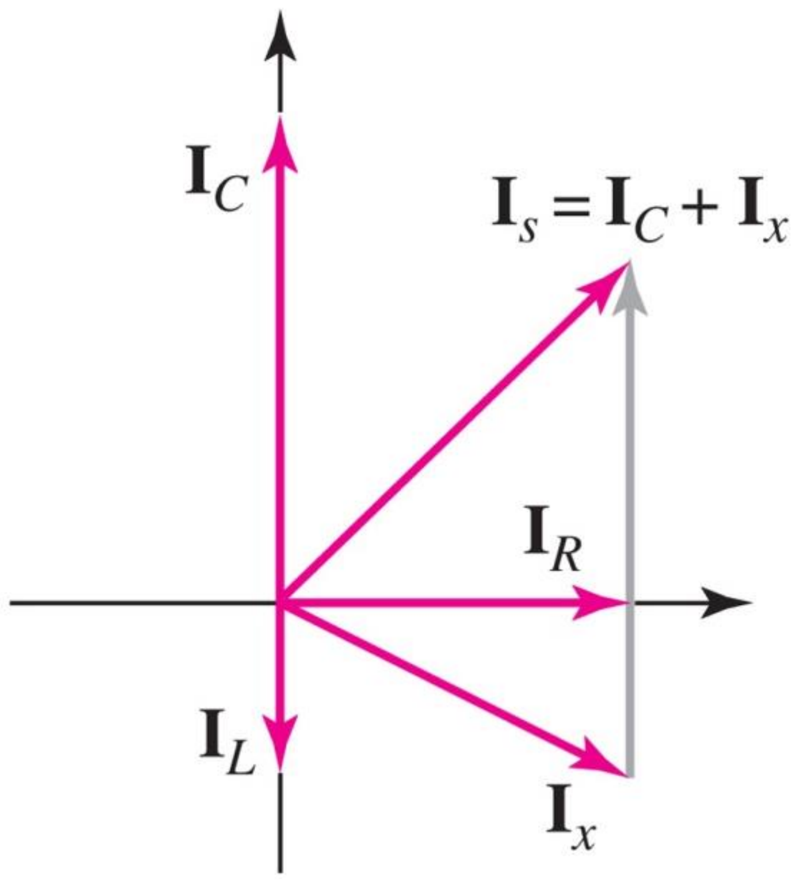




In this context, the behaviour of inductors and capacitors can be understood in terms of their phase relationships with voltage.

Assuming $V = 1\measuredangle 0$:





Question: What should be done if a circuit contains sources with different frequencies?

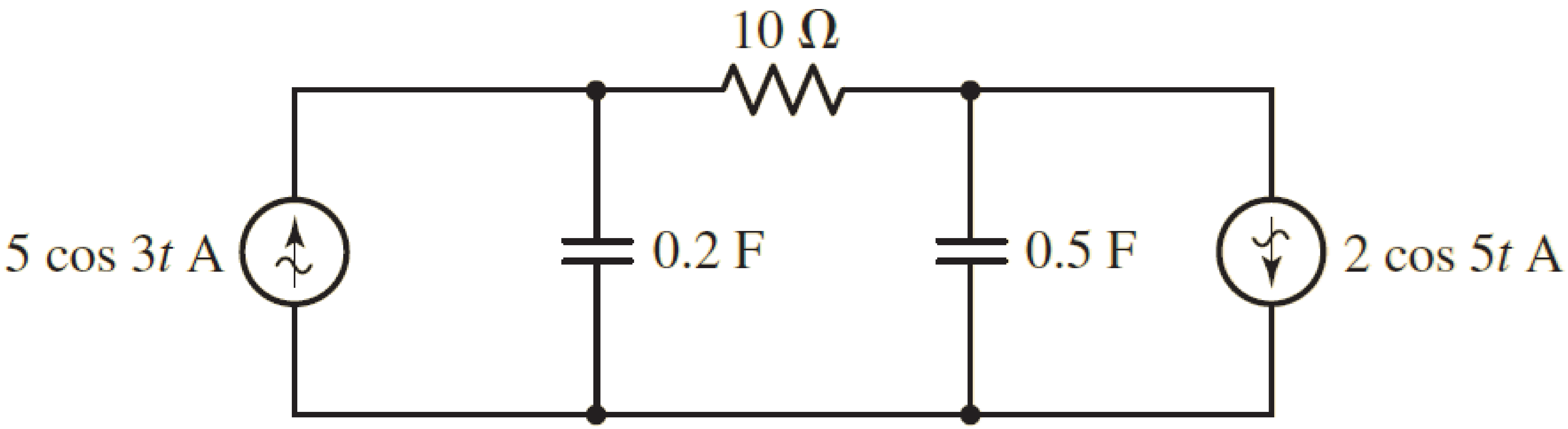
When calculating $j\omega L$ and $\frac{1}{j\omega C}$, which ω should be used?

Answer: In this case, the effect of each source must be determined separately, and the responses should be combined in the time domain (using the principle of superposition).

Important: Note that phasors corresponding to two different frequencies cannot be directly summed! They must first be transformed to the time domain and then added together.

Practice 2

Determine the power consumed by the 10-ohm resistor in the circuit shown below under steady-state conditions.



Thanks