دانشگاه صنعتی امیرکبیر
(پلی تکنیک تهران)

Electrical and Electronic Circuits

chapter 9. Frequency Response

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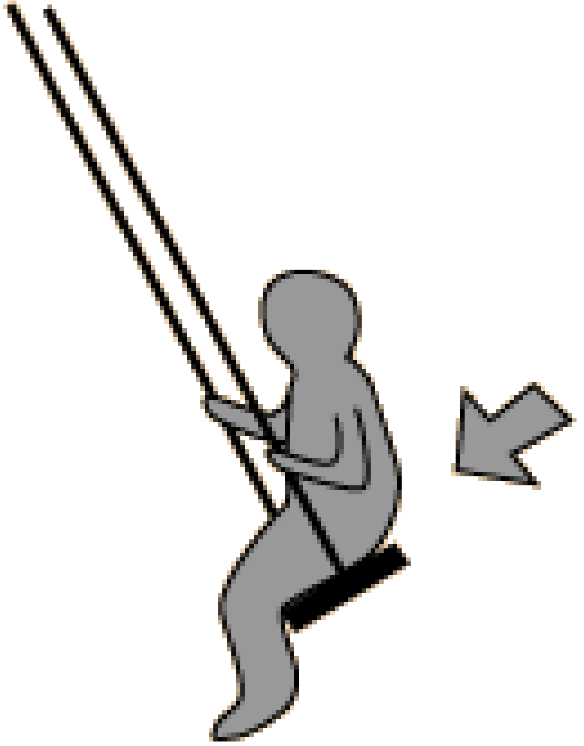
Objectives of the Lecture

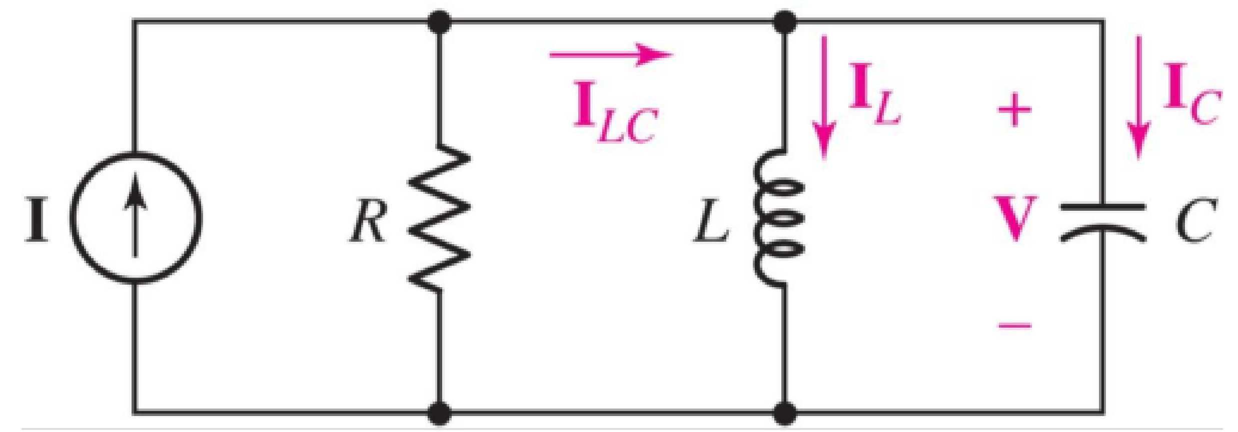
- Resonance
- Frequency filter
- Describe the sinusoidal steady-state behavior of a circuit as a function of frequency.

- Resonance is a phenomenon in which an external force causes a system to oscillate with a greater amplitude.
- The frequency at which resonance occurs is called the resonance frequency.

Tacoma bridge, 1940, US



In the circuit below, what is the frequency of the sinusoidal source so that the ratio V/I is maximized (Resonance occurs)?



$$\frac{V}{I} = Z_{eq} = \frac{R}{1 + jR\left(c\omega - \frac{1}{L\omega}\right)} \rightarrow Z_{max} = R$$

The resonance frequency is $\omega_0 = \frac{1}{\sqrt{LC}}$.

Inductor and capacitor begin to exchange energy between themselves and no longer receive energy from the source.

Electrical Resonance

- In the previous example, you observed that when resonance occurs, the imaginary part of the impedance becomes zero.

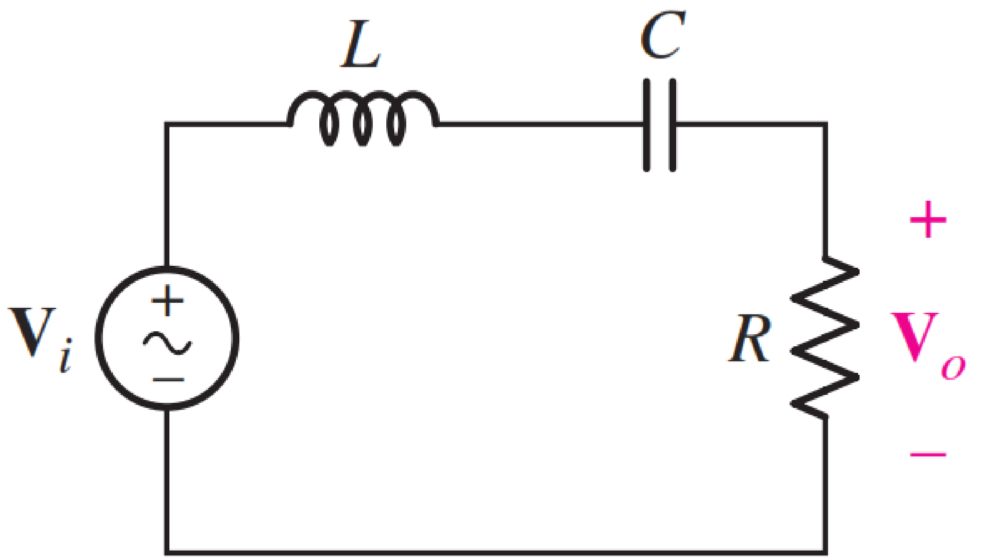
$$\boldsymbol{Z_{eq}} = \frac{\boldsymbol{R}}{\boldsymbol{1 + jR\left(c\omega - \frac{1}{L\omega}\right)}} \rightarrow \boldsymbol{Z_{max} = R}$$

- This is true for all RLC circuits, meaning resonance occurs when the imaginary part of the impedance or admittance becomes zero.

- In this state, the current and voltage of the circuit become in phase (since the equivalent impedance of the circuit is a real number and behaves like a resistor).

Electrical Resonance

What is the resonance frequency in a series RLC circuit?



- $Z_{eq} = R + j\omega L + \dfrac{1}{j\omega C}$
- $Img(Z_{eq}) = 0 \rightarrow \omega_0 = \dfrac{1}{\sqrt{LC}}$

Frequency Response

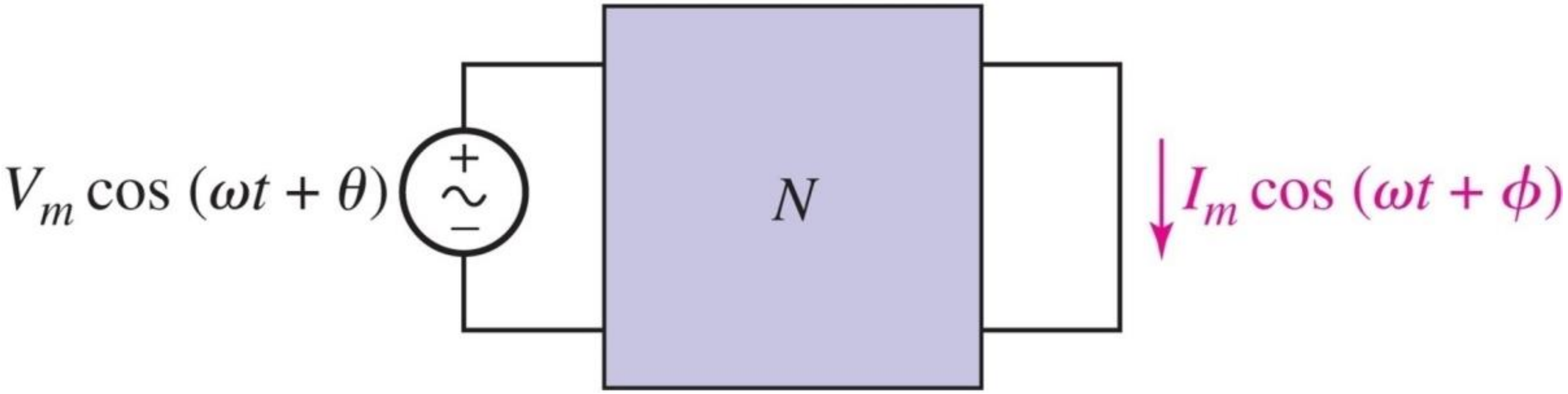
- If we let the amplitude of the sinusoidal source remain constant and vary the frequency,
  - we obtain the circuit’s frequency response.
- The frequency response of a circuit is the variation in its behavior with change in signal frequency.
- The frequency response of a circuit may also be considered as the variation of the gain and phase with frequency.

Frequency Response

In an n-order circuit with a sinusoidal input, frequency response analysis involves finding:

•The ratio of the output amplitude to the input amplitude ($\frac{I_m}{V_m}$), known as the gain A.

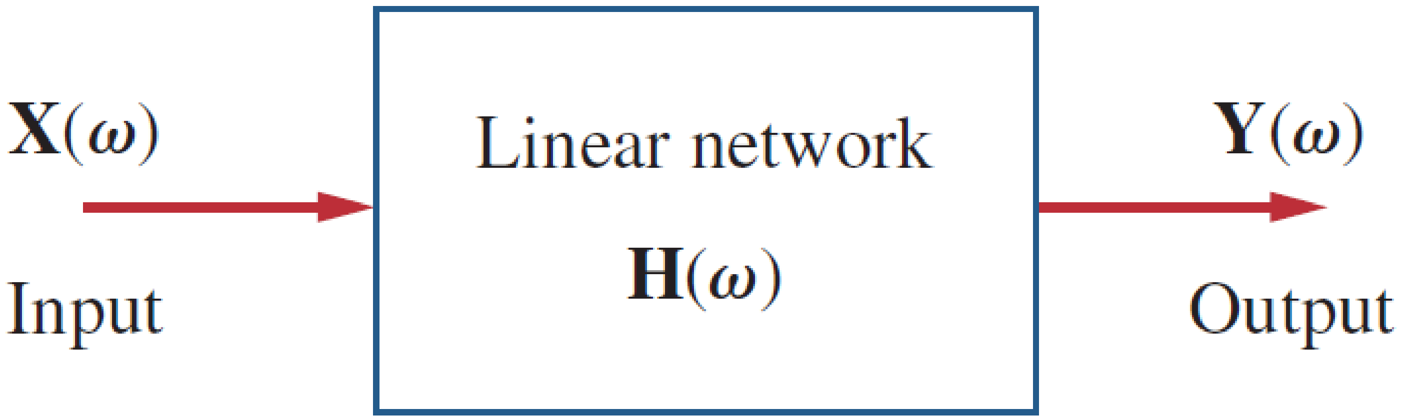
•The phase difference between them $(\phi - \theta)$ at various frequencies.



To perform this analysis, we use the transfer function.

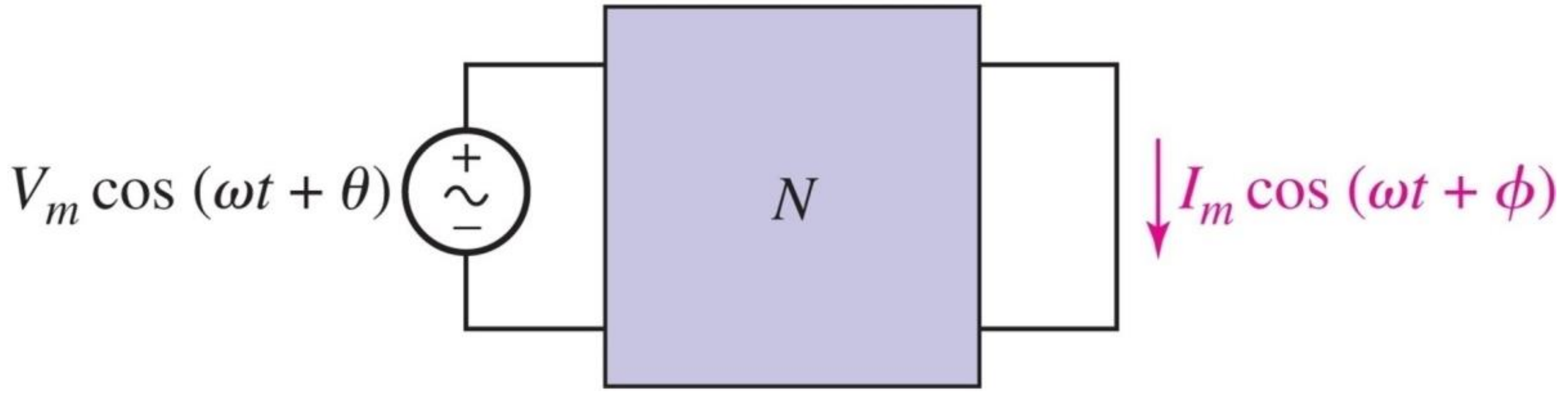
Transfer Function

- The transfer function $\mathbf{H}(\omega)$ of a circuit is the frequency-dependent ratio of a phasor output $\mathbf{Y}(\omega)$ (an element voltage or current) to a phasor input $\mathbf{X}(\omega)$ (source voltage or current).



$$\boxed{\mathbf{H}(\omega) = \frac{\mathbf{Y}(\omega)}{\mathbf{X}(\omega)}}$$

The ratio of the output phasor to the input phasor is called the transfer function, $\mathbf{H}(j\omega)$.



- $\mathbf{H}(j\omega) = \frac{I_m e^{j\phi}}{V_m e^{j\theta}}$
- $A = |\mathbf{H}(j\omega)| = \frac{I_m}{V_m}$
- $Phase = \measuredangle \mathbf{H}(j\omega) = \phi - \theta$

All three are functions of frequency.

Transfer Function

- Since the input and output can be either voltage or current at any place in the circuit,
  - there are four possible transfer functions:

$$\mathbf{H}(\omega) = \text{Voltage gain} = \frac{\mathbf{V}_o(\omega)}{\mathbf{V}_i(\omega)}$$

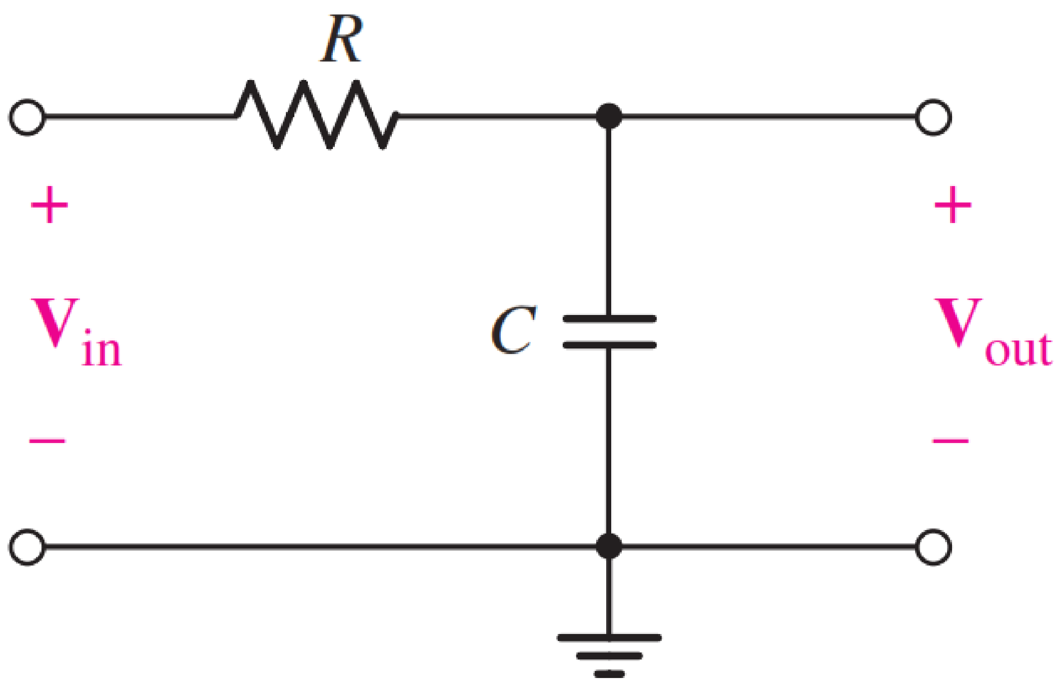
$$\mathbf{H}(\omega) = \text{Current gain} = \frac{\mathbf{I}_o(\omega)}{\mathbf{I}_i(\omega)}$$

$$\mathbf{H}(\omega) = \text{Transfer Impedance} = \frac{\mathbf{V}_o(\omega)}{\mathbf{I}_i(\omega)}$$

$$\mathbf{H}(\omega) = \text{Transfer Admittance} = \frac{\mathbf{I}_o(\omega)}{\mathbf{V}_i(\omega)}$$

A low-pass RC filter

$$V_{out} = \frac{\frac{1}{j\omega C}}{\frac{1}{j\omega C} + R} V_{in} = \frac{1}{1 + j\omega RC} V_{in}$$



Assuming the capacitor voltage as the circuit output, the transfer function is given by:

$$\mathbf{H}(j\omega) = \frac{V_{out}}{V_{in}} = \frac{1}{1 + j\omega RC}$$

Bode Diagram

A Bode diagram is a plot that shows the magnitude and phase of the transfer function as a function of frequency on a logarithmic scale.

- ✓ The horizontal axis represents frequency, plotted on a logarithmic scale.
- ✓ The vertical axis of the magnitude plot represents the gain in decibels (dB).
- ✓ The vertical axis of the phase plot represents the phase angle on a linear scale.

This diagram is useful for analyzing the frequency response of a system, showing how both the amplitude and phase of the output signal vary with changes in input frequency.

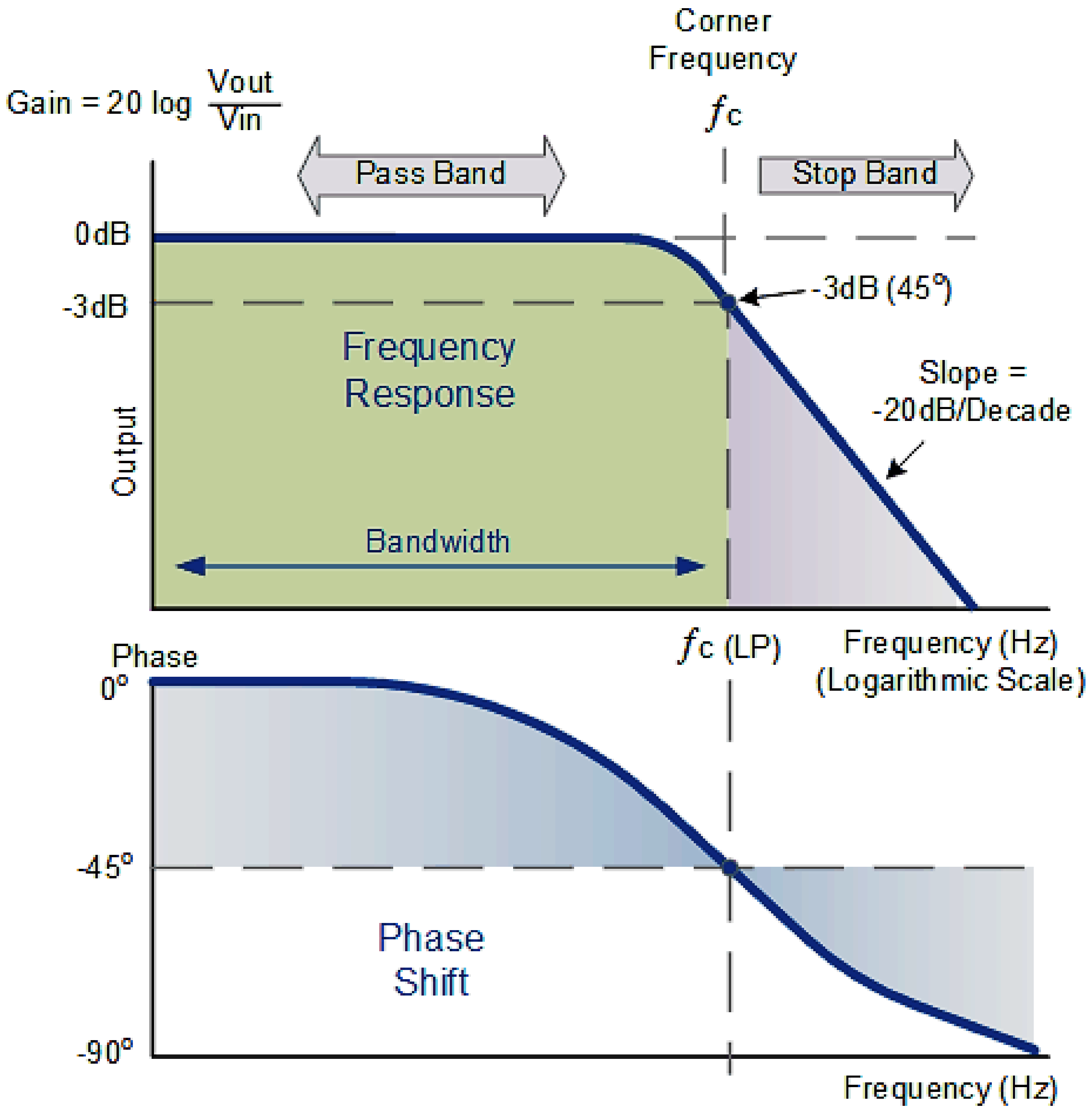
$$H_{dB} = 20\log\left|\mathbf{H}(j\omega)\right|$$

The Bode diagram for an RC low-pass filter

- $\mathbf{H}(j\omega) = \frac{V_{out}}{V_{in}} = \frac{1}{1+j\omega RC}$
- $|\mathbf{H}(j\omega)| = \frac{1}{\sqrt{1+\omega^2R^2C^2}}$
- $\measuredangle\mathbf{H}(j\omega) = -\tan^{-1}\omega RC$

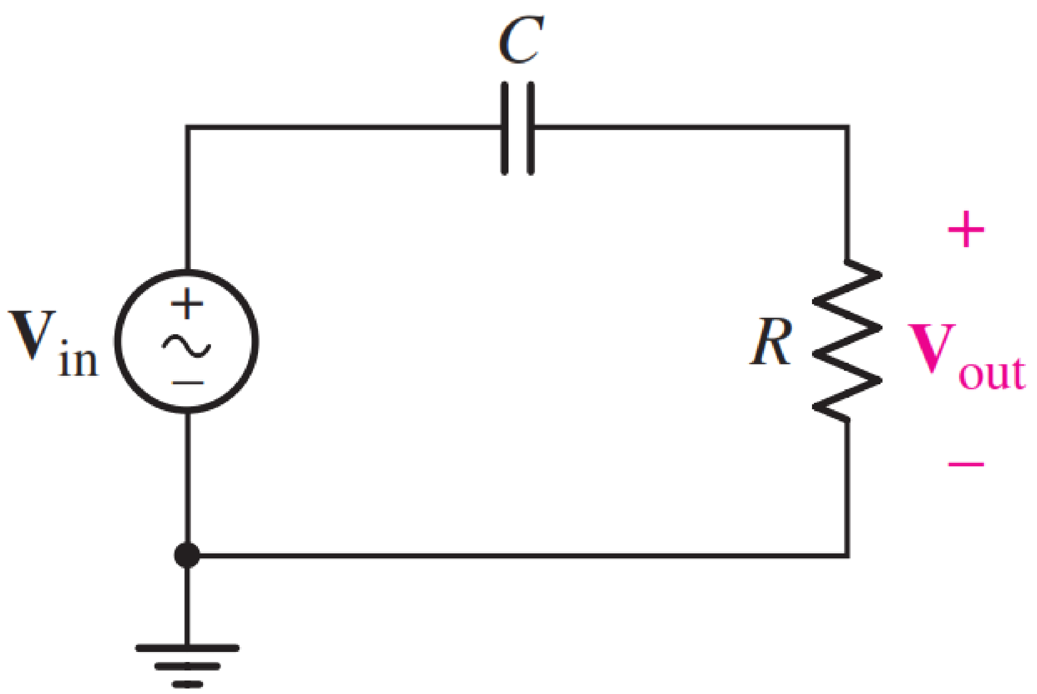
The **cutoff frequency** is the frequency at which the magnitude of the transfer function drops to $\frac{1}{\sqrt{2}}$ of its maximum value. This corresponds to a -3 dB, point on the magnitude plot.

- $f_c = \frac{1}{2\pi RC}$



RC high-pass filter

$$V_{out} = \frac{R}{\frac{1}{j\omega C} + R} V_{in} = \frac{j\omega RC}{1 + j\omega RC} V_{in}$$

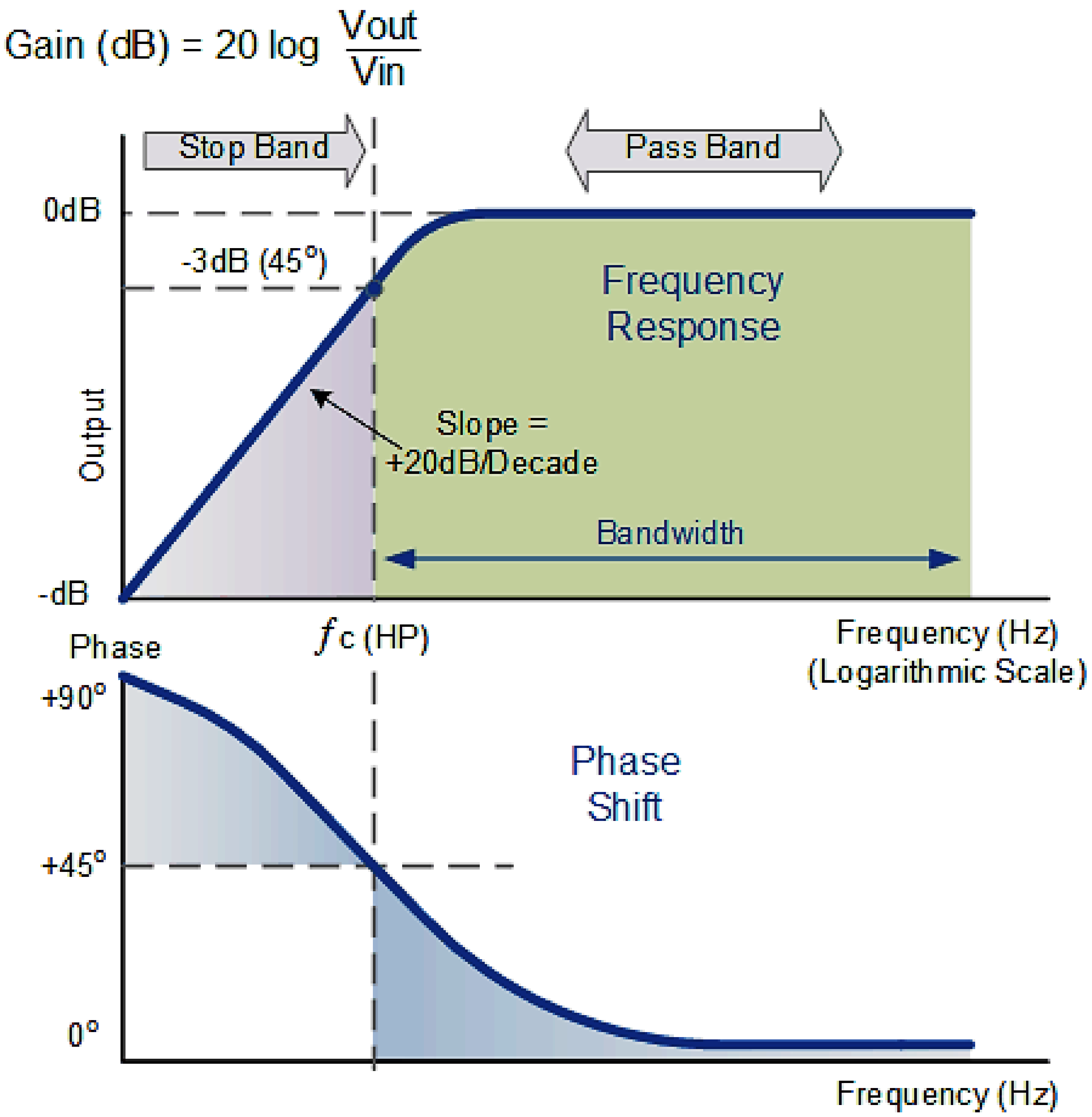


- Transfer function

$$\mathbf{H}(j\omega) = \frac{V_{out}}{V_{in}} = \frac{j\omega RC}{1 + j\omega RC}$$

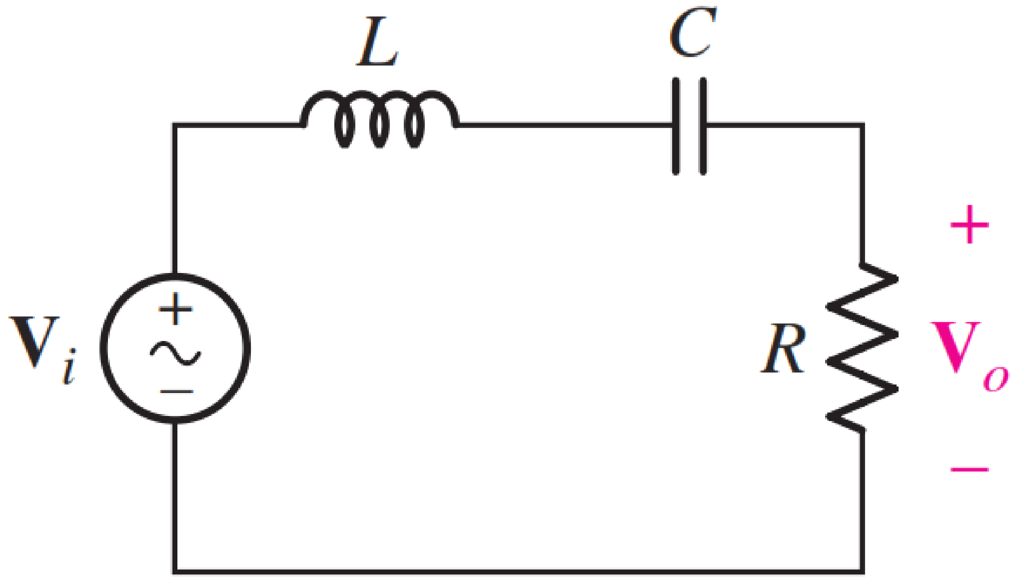
The Bode diagram for an RC high-pass filter

- $\mathbf{H}(j\omega) = \frac{V_{out}}{V_{in}} = \frac{j\omega RC}{1+j\omega RC}$
- $|\mathbf{H}(j\omega)| = \frac{\omega RC}{\sqrt{1+\omega^2 R^2 C^2}}$
- $\measuredangle \mathbf{H}(j\omega) = \frac{\pi}{2} - \tan^{-1} \omega RC$
- $f_c = \frac{1}{2\pi RC}$



Band-Pass RLC Filter

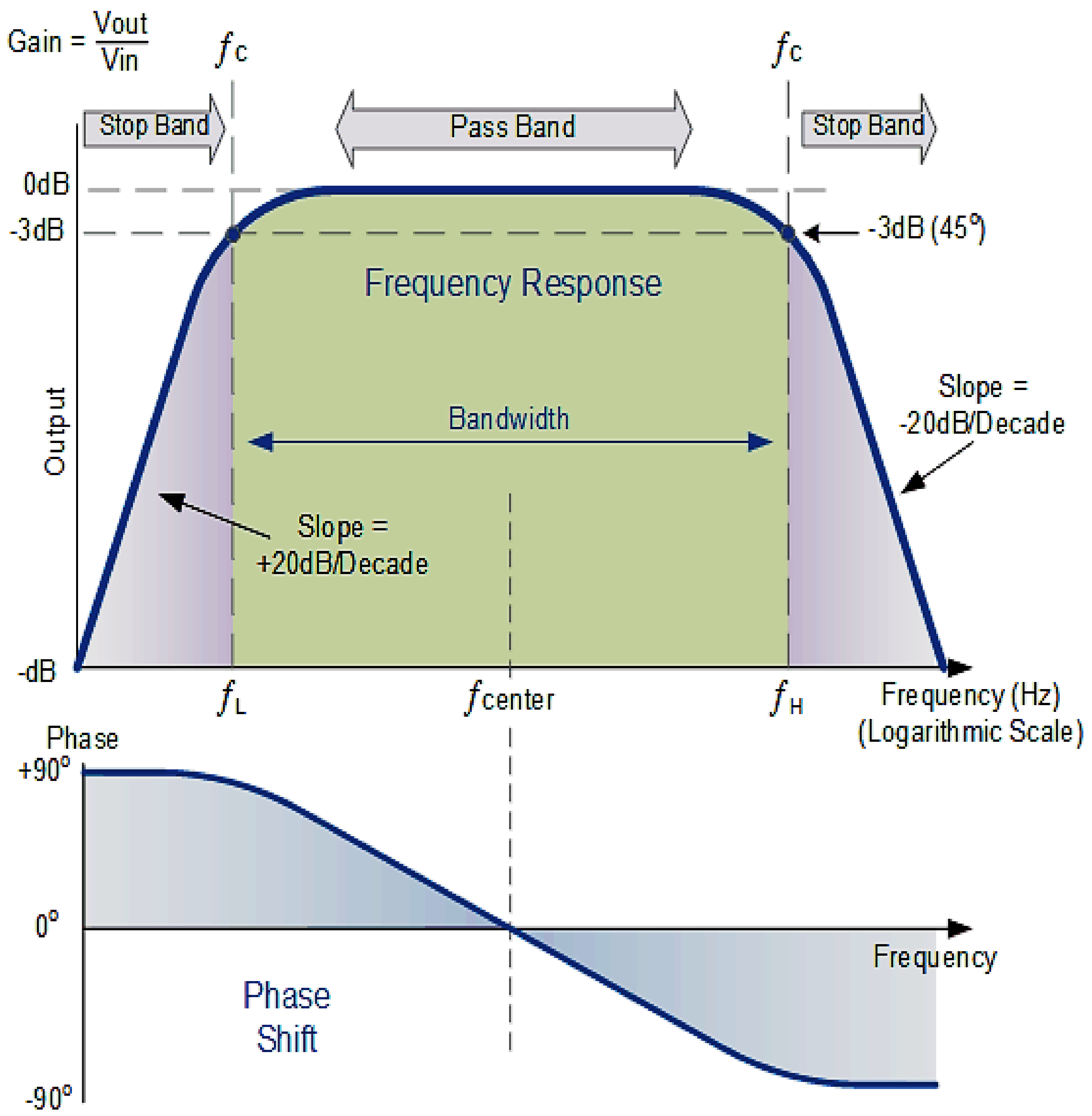
$$V_{out} = \frac{R}{j\omega L + \dfrac{1}{j\omega C} + R} V_{in}$$



- Transfer function

$$\mathbf{H}(j\omega) = \frac{V_{out}}{V_{in}} = \frac{j\omega RC}{1 + j\omega RC - \omega^2 LC}$$

- $\mathbf{H}(j\omega) = \frac{V_{out}}{V_{in}} = \frac{j\omega RC}{1+j\omega RC-\omega^2 LC}$
- $|\mathbf{H}(j\omega)| = \frac{\omega RC}{\sqrt{(1-\omega^2 LC)^2+\omega^2 R^2 C^2}}$
- $\measuredangle \mathbf{H}(j\omega) = \frac{\pi}{2} - \tan^{-1}\frac{\omega RC}{1-\omega^2 LC}$



- Find the cutoff frequencies f_L and f_H.
- Bandwidth: the distance between the two frequencies.

- $BW = 2\pi(f_H - f_L)$

Thanks