



دانشگاه صنعتی امیر کبیر  
( پلی تکنیک تهران )

# Electrical and Electronic Circuits

## chapter 6. Basic RL and RC Circuits

Afarghadan@aut.ac.ir

عظیم فرقدان 

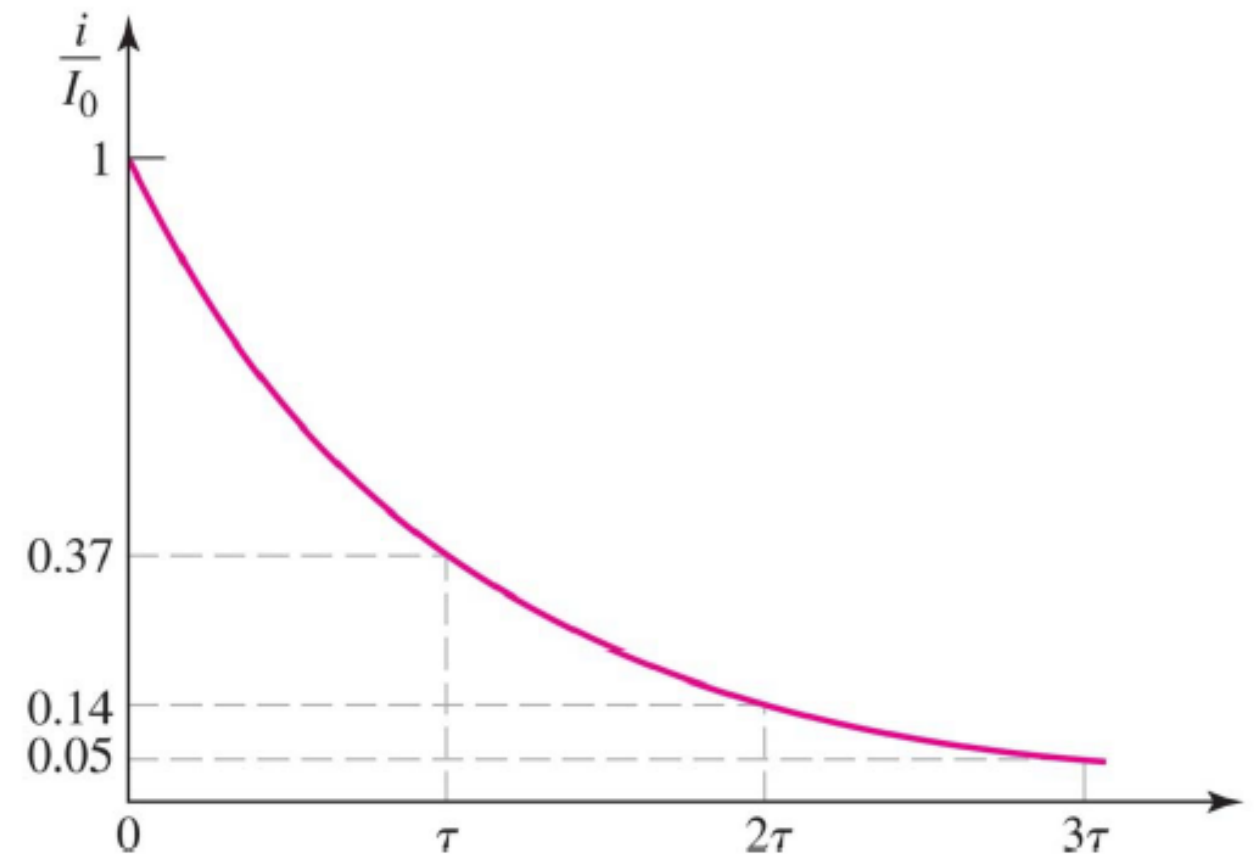
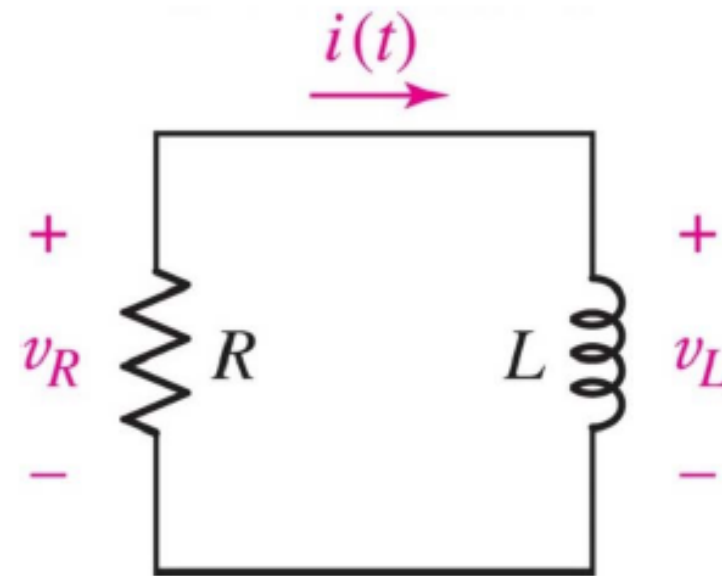
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# Objectives of the Lecture

- Determining the Time Response of First-Order Circuits
  - The Source-Free RL Circuit
  - The Source-Free RC Circuit
  - Driven RL Circuits
  - Driven RC Circuits

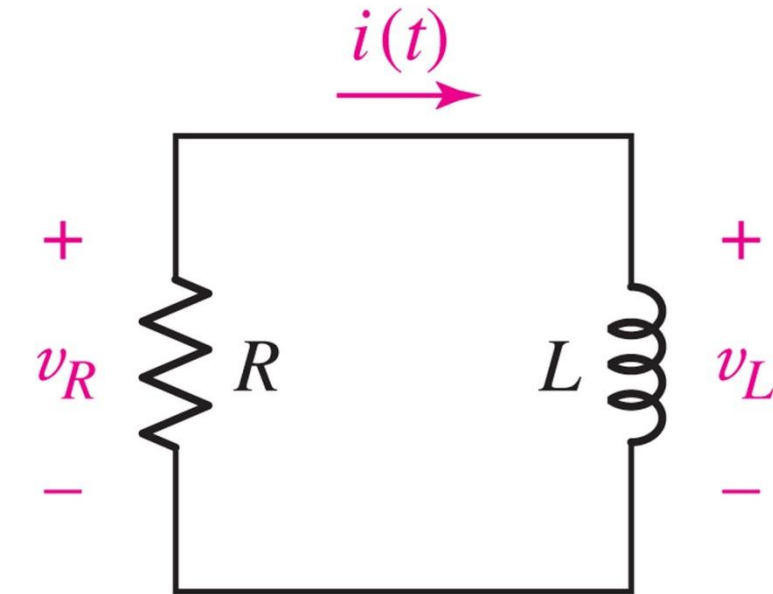
# Objective

- ❖ Time Response Analysis of First-Order (RC or RL) Circuits
- ❖ Examining and analysing the **charging** or **discharging** behaviour of inductors and capacitors over time, and deriving a mathematical expression for this behaviour.



- Applying KVL:

$$Ri + v_L = Ri + L \frac{di}{dt} = 0 \qquad \frac{di}{dt} + \frac{R}{L}i = 0$$



- We can solve for the *natural response* if we know the *initial condition*  $i(0)=I_0$ :

$$i(t) = I_0 e^{\frac{-Rt}{L}} \text{ for } t > 0$$

# Natural response of the first order equation

➤ First solution:

$$\frac{di}{dt} + \frac{R}{L}i = 0 \rightarrow \frac{di}{i} = -\frac{R}{L} dt$$

$$\int_{I_0}^{i(t)} \frac{di'}{i'} = \int_0^t -\frac{R}{L} dt'$$

$$\ln i - \ln I_0 = -\frac{R}{L}(t - 0)$$

$$\ln i = -\frac{Rt}{L} + k \rightarrow i = Ae^{-\frac{Rt}{L}}$$

$$i(0) = I_0 \rightarrow i(t) = I_0 e^{-\frac{Rt}{L}}$$

# Natural response of the first order equation

- The second solution: forming the **characteristic equation**
- Many of the differential equations encountered in circuit analysis have a solution which may be represented by the exponential function or by the sum of several exponential functions.

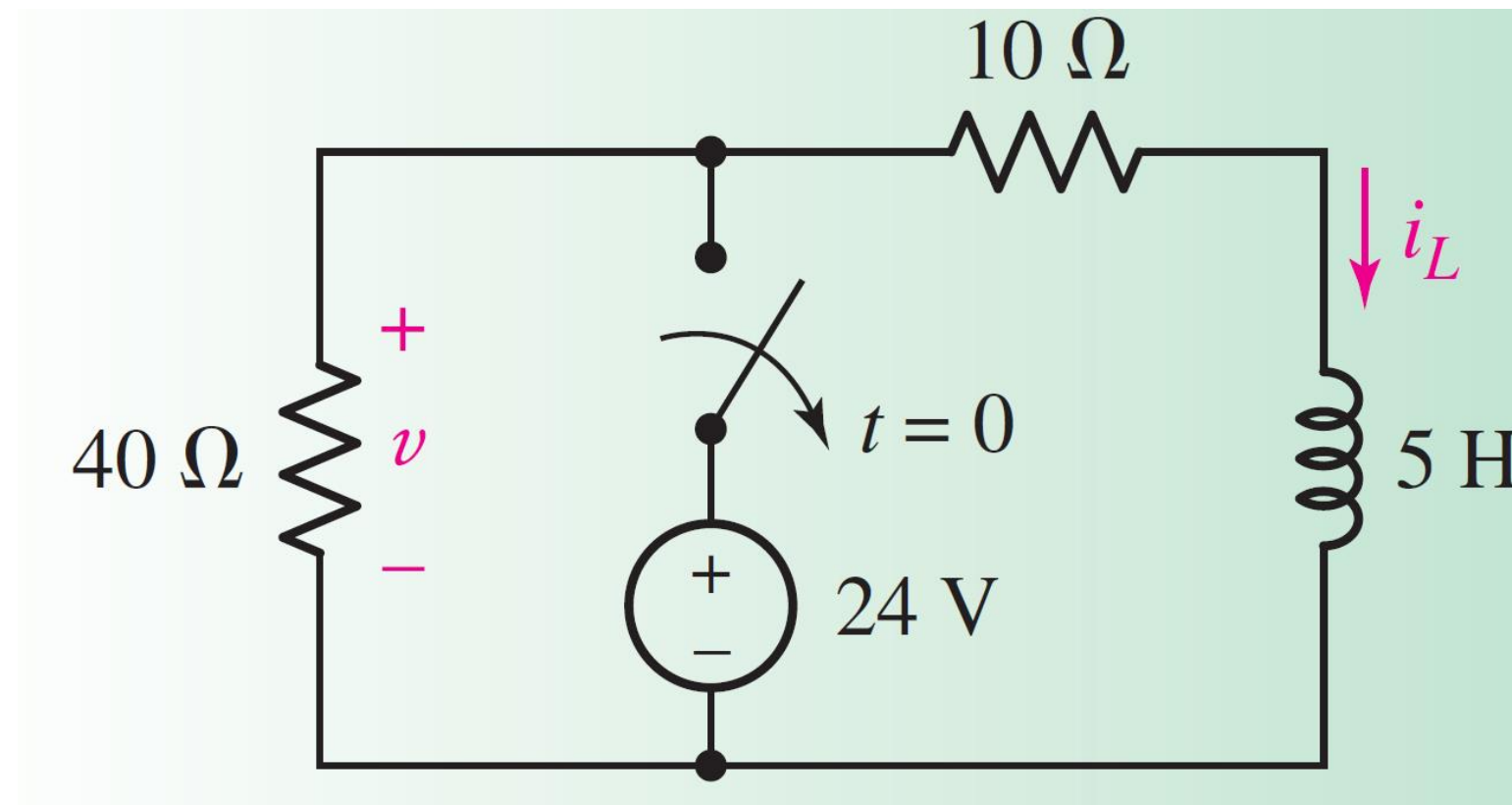
$$s + \frac{R}{L} = 0 \rightarrow s = -\frac{R}{L}$$

$$i = Ae^{st} \rightarrow i = Ae^{-\frac{Rt}{L}}$$

$$i(0) = I_0 \rightarrow i(t) = I_0 e^{-\frac{Rt}{L}}$$

## Example: RL with a Switch

Show that the voltage  $v(t)$  will be -12.99 volts at  $t=200$  ms.

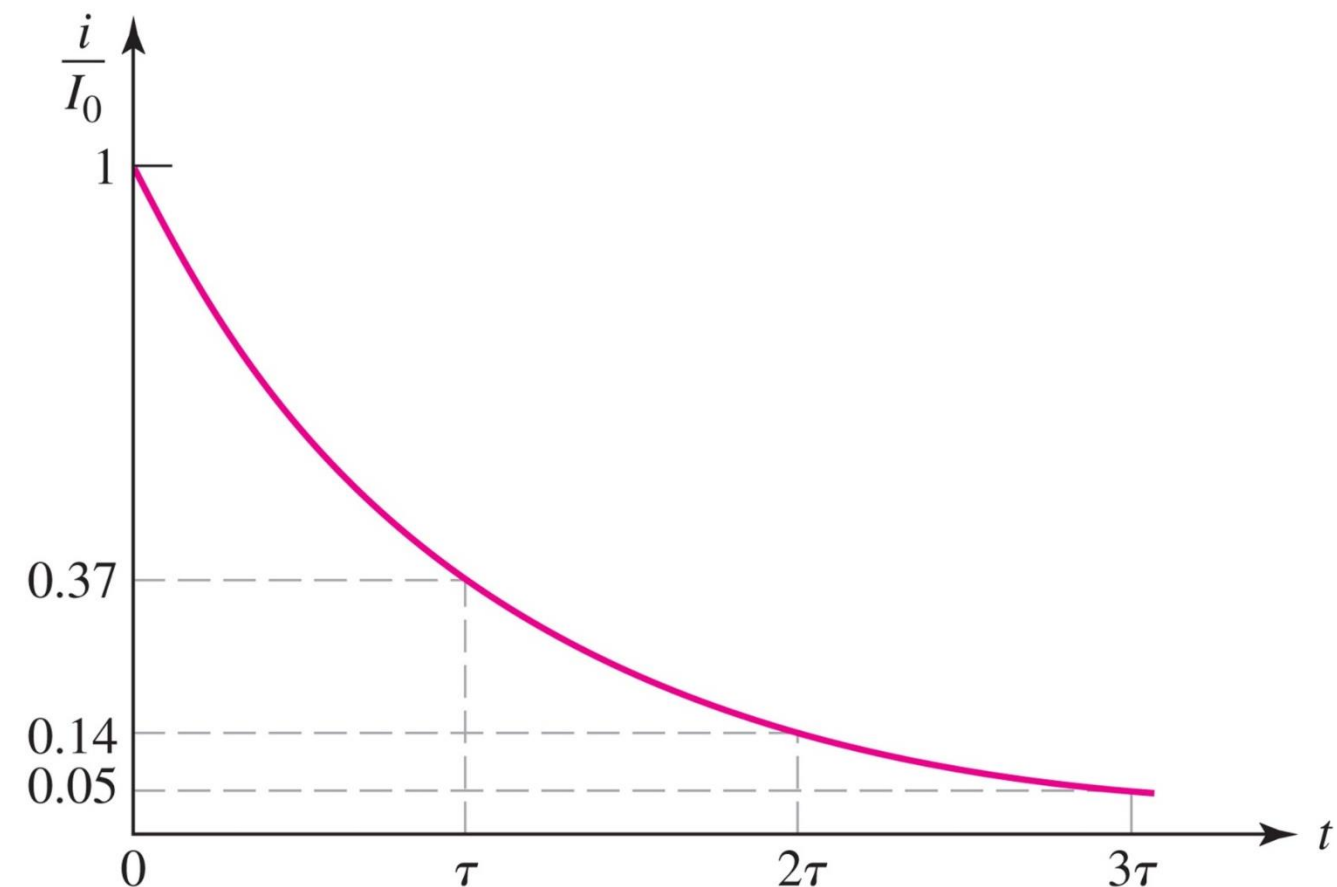
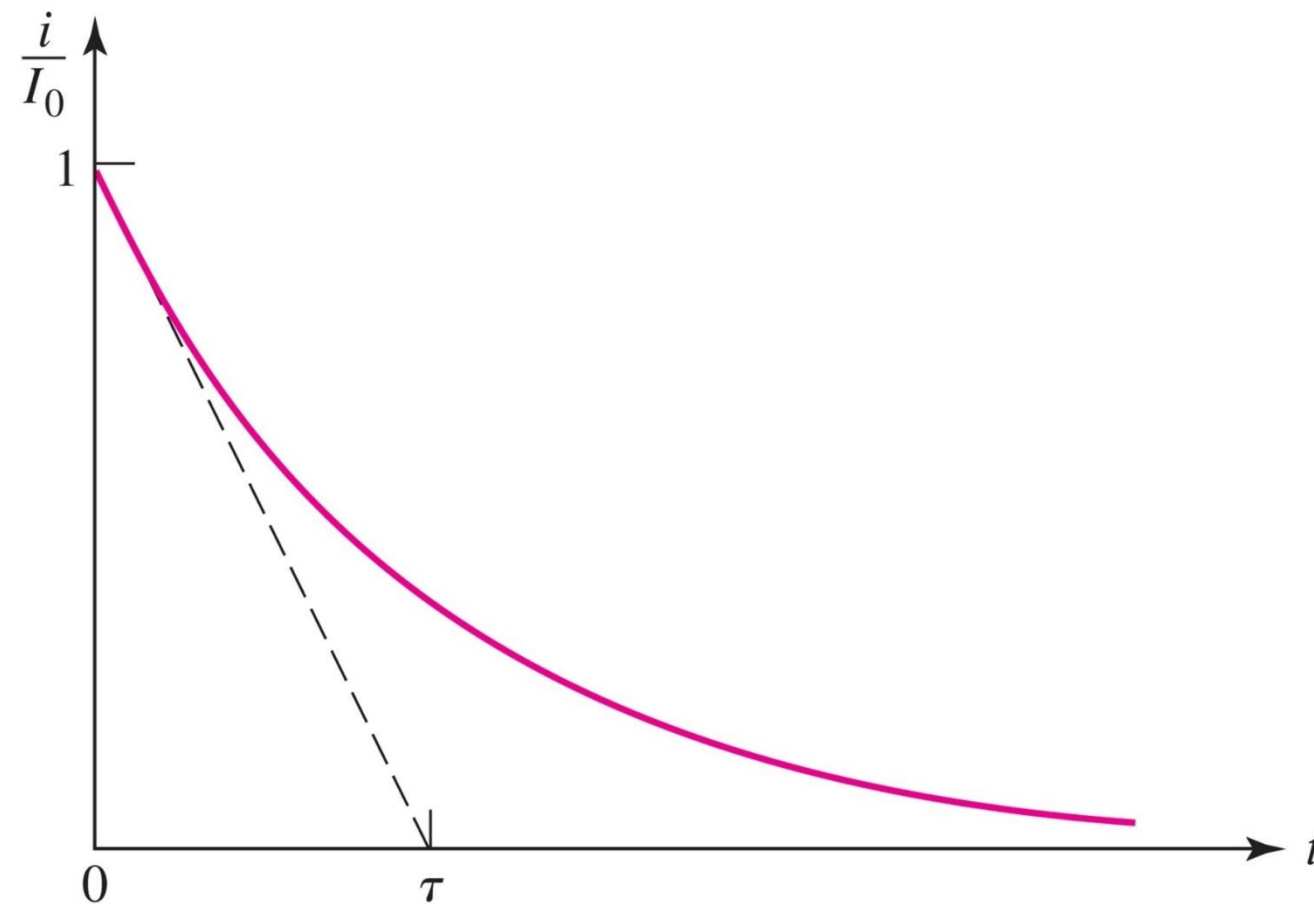


# The Exponential Response

- ✓ The ratio  $L/R$  has the units of seconds, since the exponent  $-Rt/L$  must be dimensionless.
- ✓ The time constant  $\tau = L/R$  determines the rate of decay.
- ✓ The larger the value, the slower the function is damped.

$$i(t) = I_0 e^{-t/\tau}$$

$$\frac{i(\tau)}{I_0} = e^{-1} = 0.3679 \quad \text{or} \quad i(\tau) = 0.3679 I_0$$



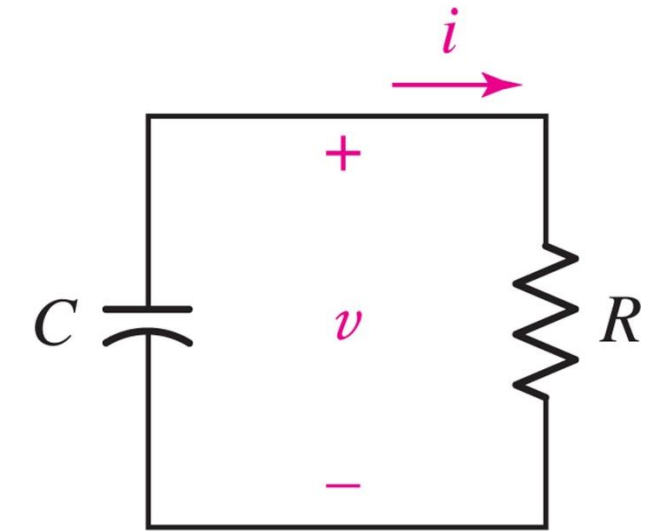


# The Source-Free RC Circuit

Applying KCL:

$$C \frac{dv}{dt} + \frac{v}{R} = 0$$

$$\frac{dv}{dt} + \frac{1}{RC} v = 0$$

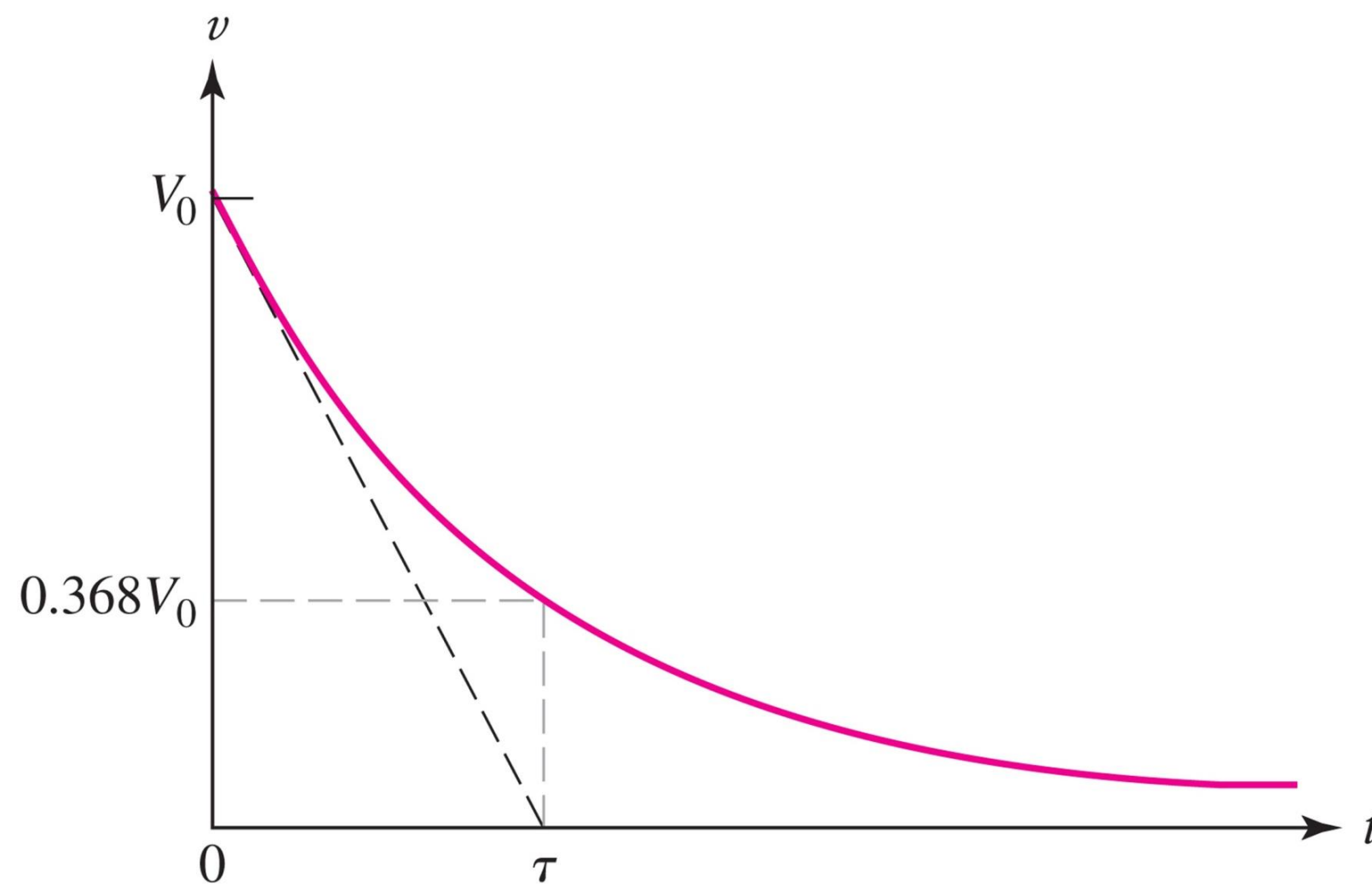


We can solve for the *natural response* if we know the *initial condition*  $v(0) = V_0$

$$v(t) = v_0 e^{\frac{-t}{RC}} \text{ for } t > 0$$

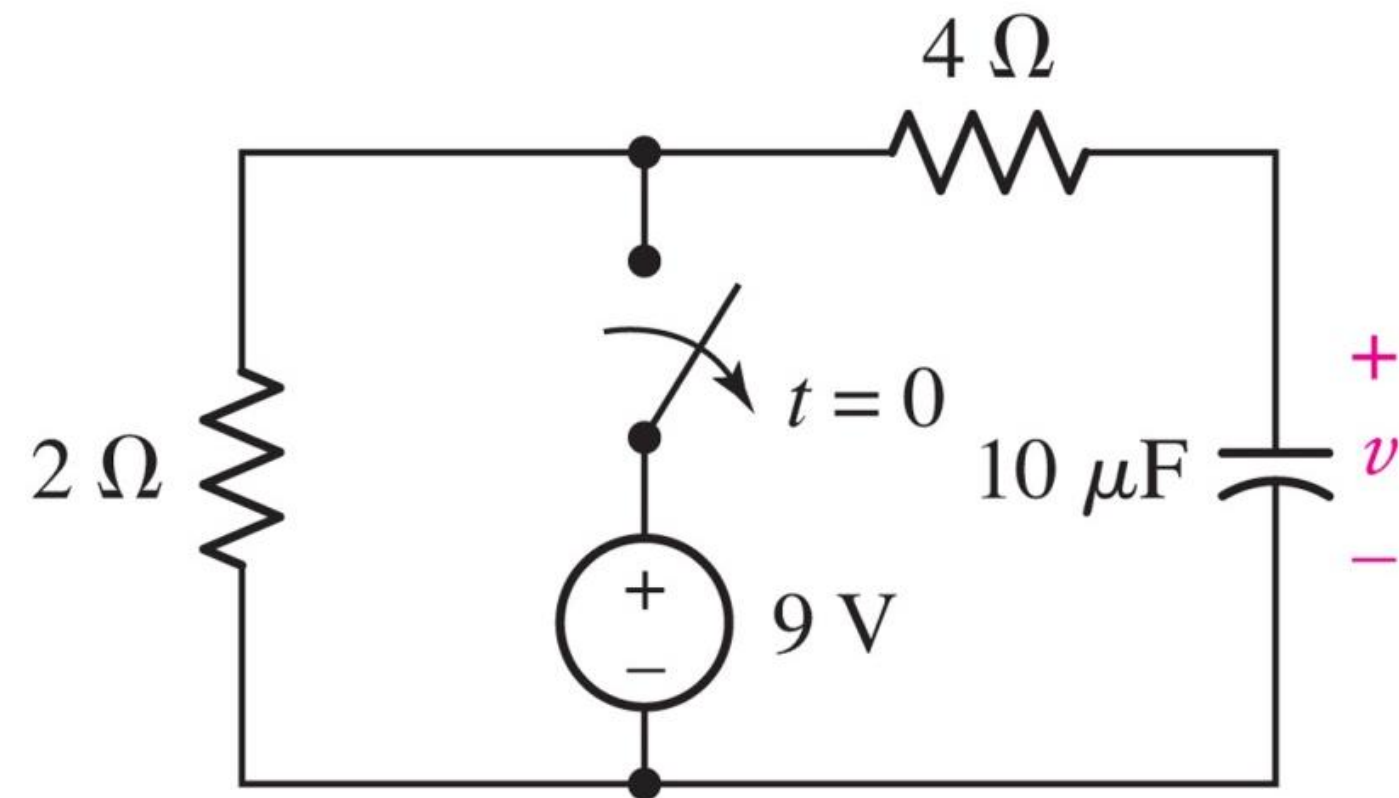
# RC Natural Response

- The time constant is  $\tau=RC$
- The time at which the response has dropped to 37 percent of its initial value

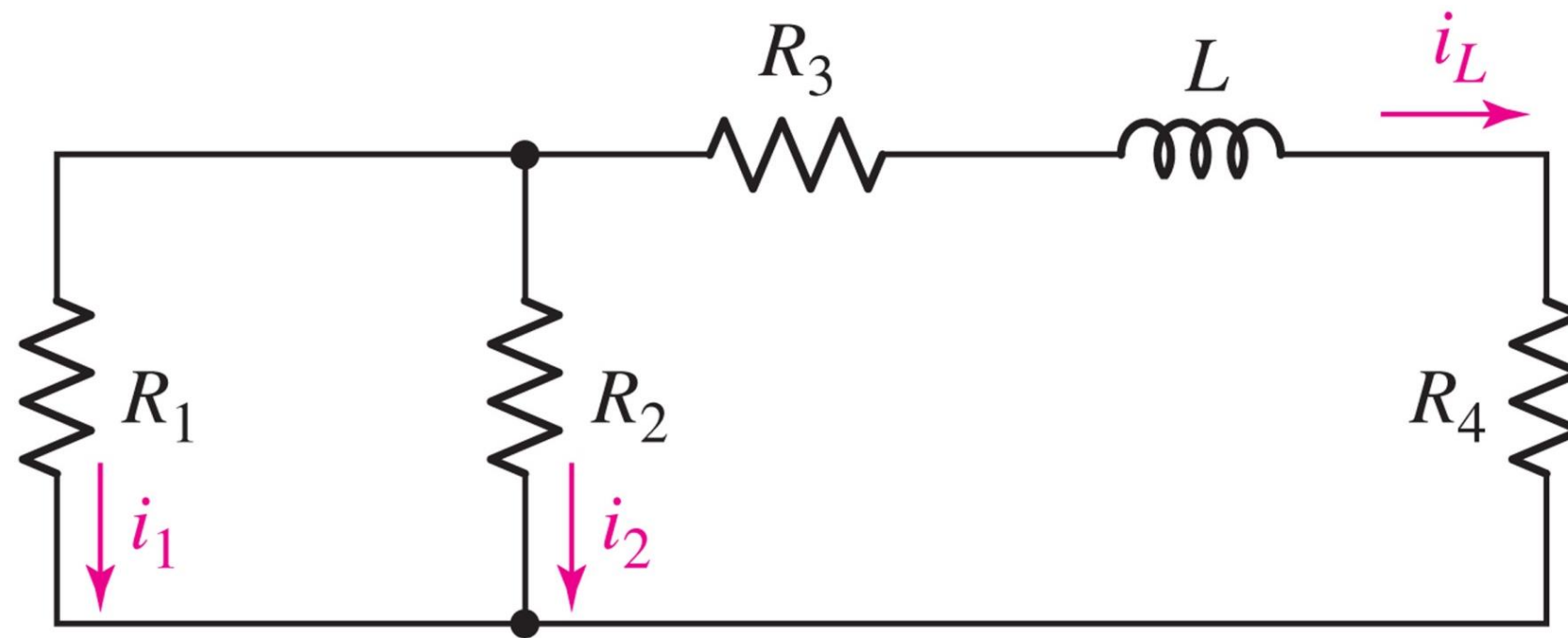


# The Source Free RC Circuit

Show that the voltage  $v(t)$  is 321 mV at  $t=200\ \mu\text{s}$ .



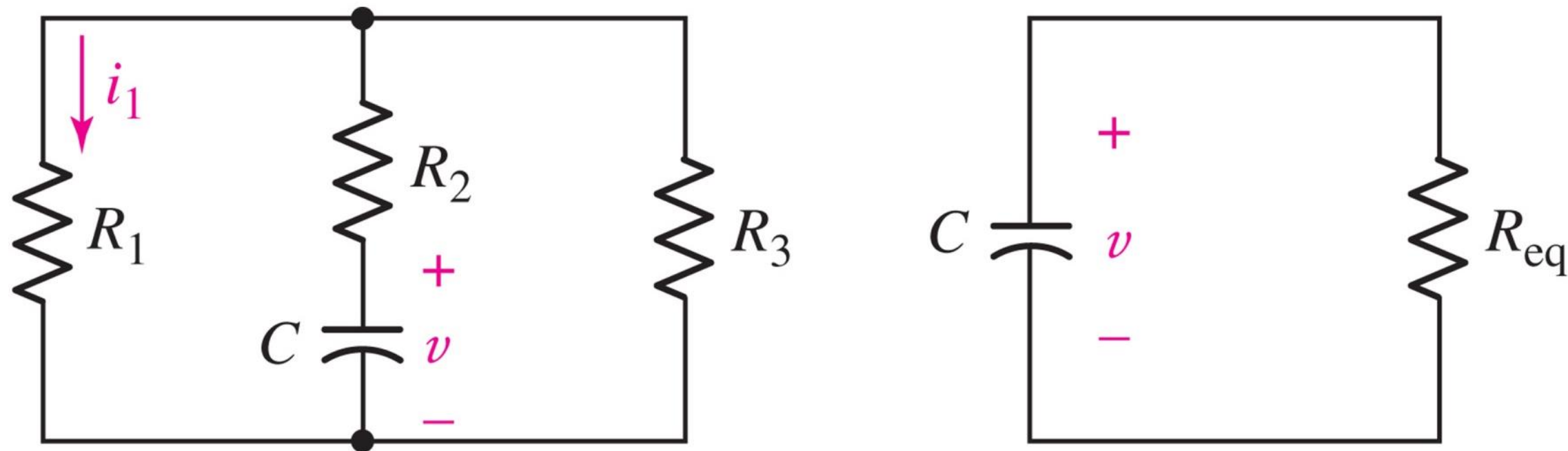
The time constant of a single-inductor circuit will be  $\tau=L/R_{eq}$  where  $R_{eq}$  is the resistance seen by the inductor.



Example:  $R_{eq}=R_3+R_4+\frac{R_1R_2}{(R_1+R_2)}$

# General RC Circuits

The time constant of a single-capacitor circuit will be  $\tau = R_{eq}C$  where  $R_{eq}$  is the resistance seen by the capacitor.



$$\text{Example: } R_{eq} = R_3 + R_4 + \frac{R_1 R_2}{(R_1 + R_2)}$$

# 1<sup>st</sup> Order Response Observations

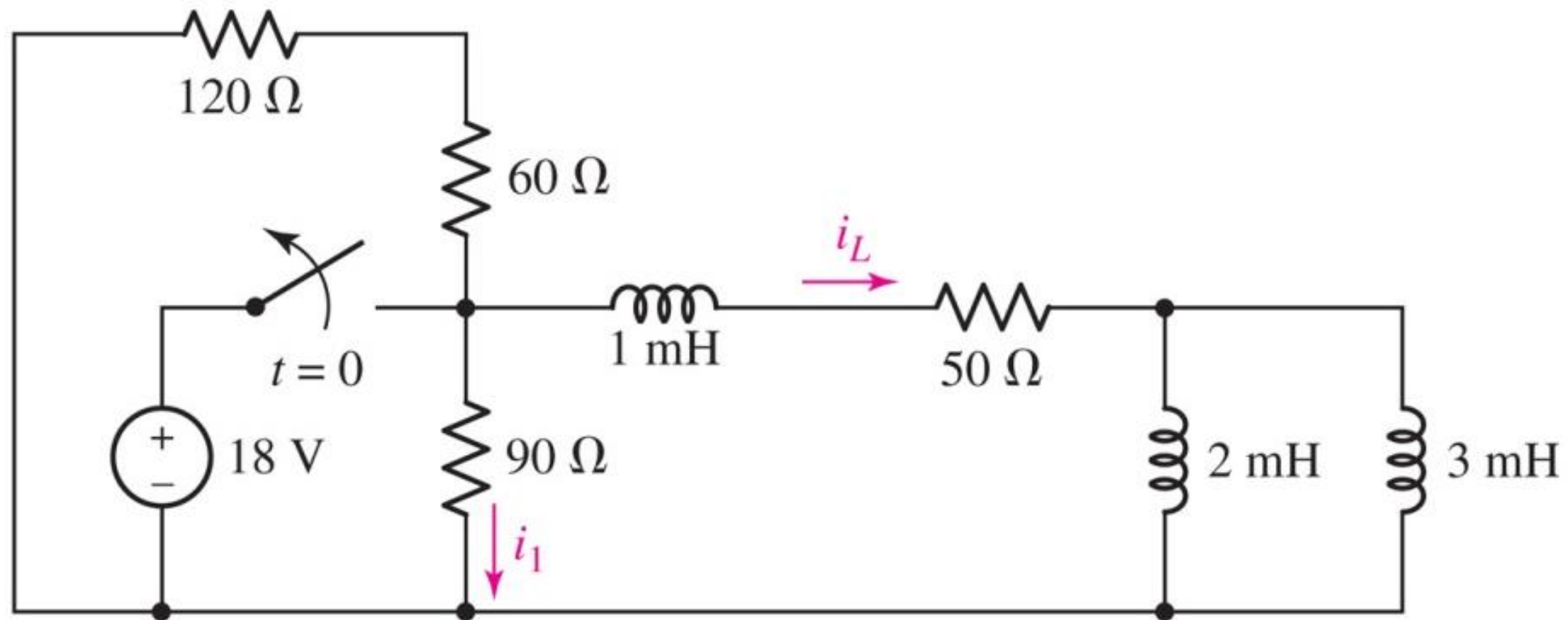
- ✓ The voltage on a capacitor or the current through an inductor is the same *prior to* and *after* a switch at  $t=0$ .

$$v_C(0^+) = v_C(0^-), \quad i_L(0^+) = i_L(0^-)$$

- ✓ Resistor voltage (or current) prior to the switch  $v(0^-)$  can be different from the voltage after the switch  $v(0^+)$ .
- ✓ All voltages and currents in an RC or RL circuit follow the same natural response  $e^{-t/\tau}$ .

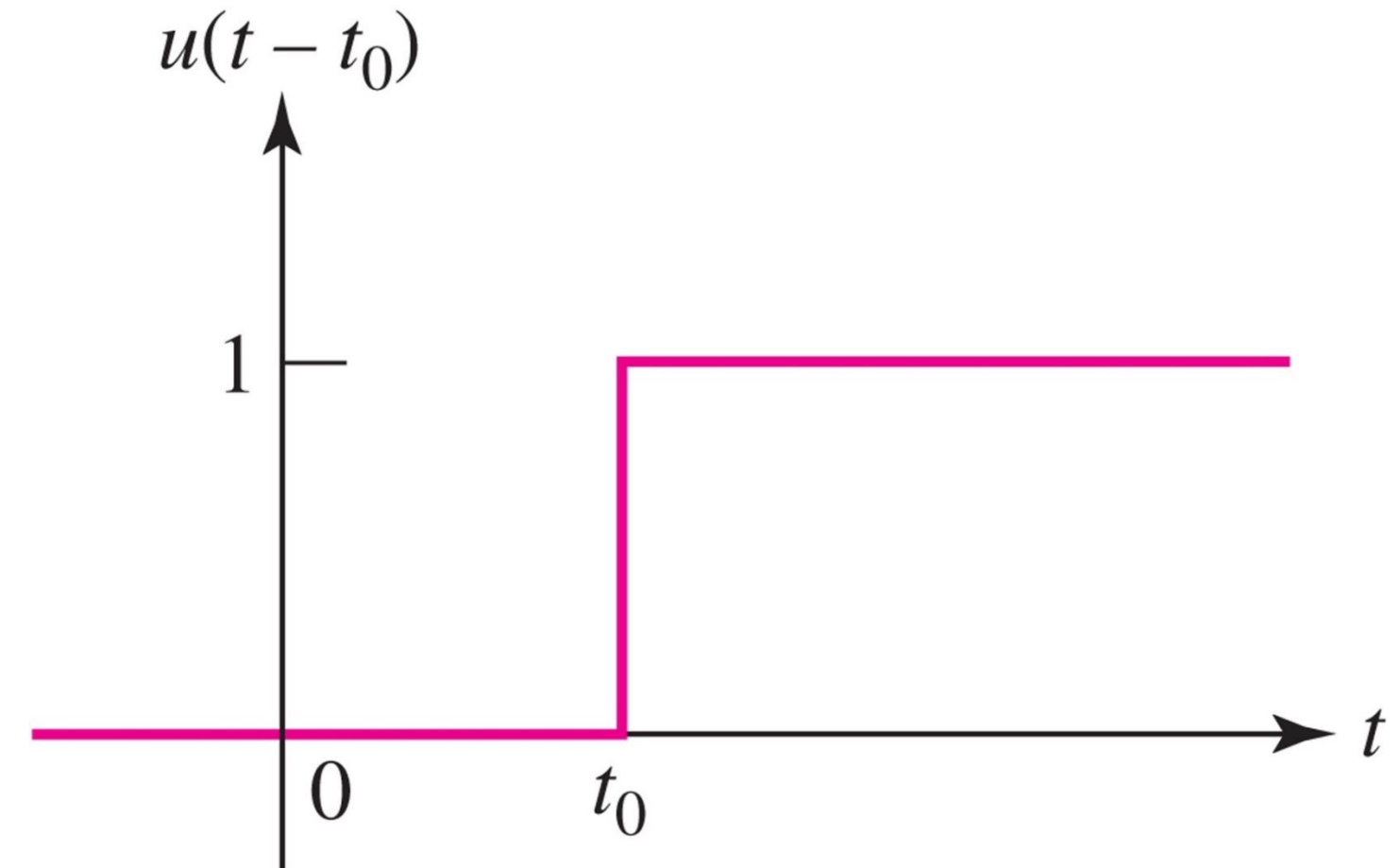
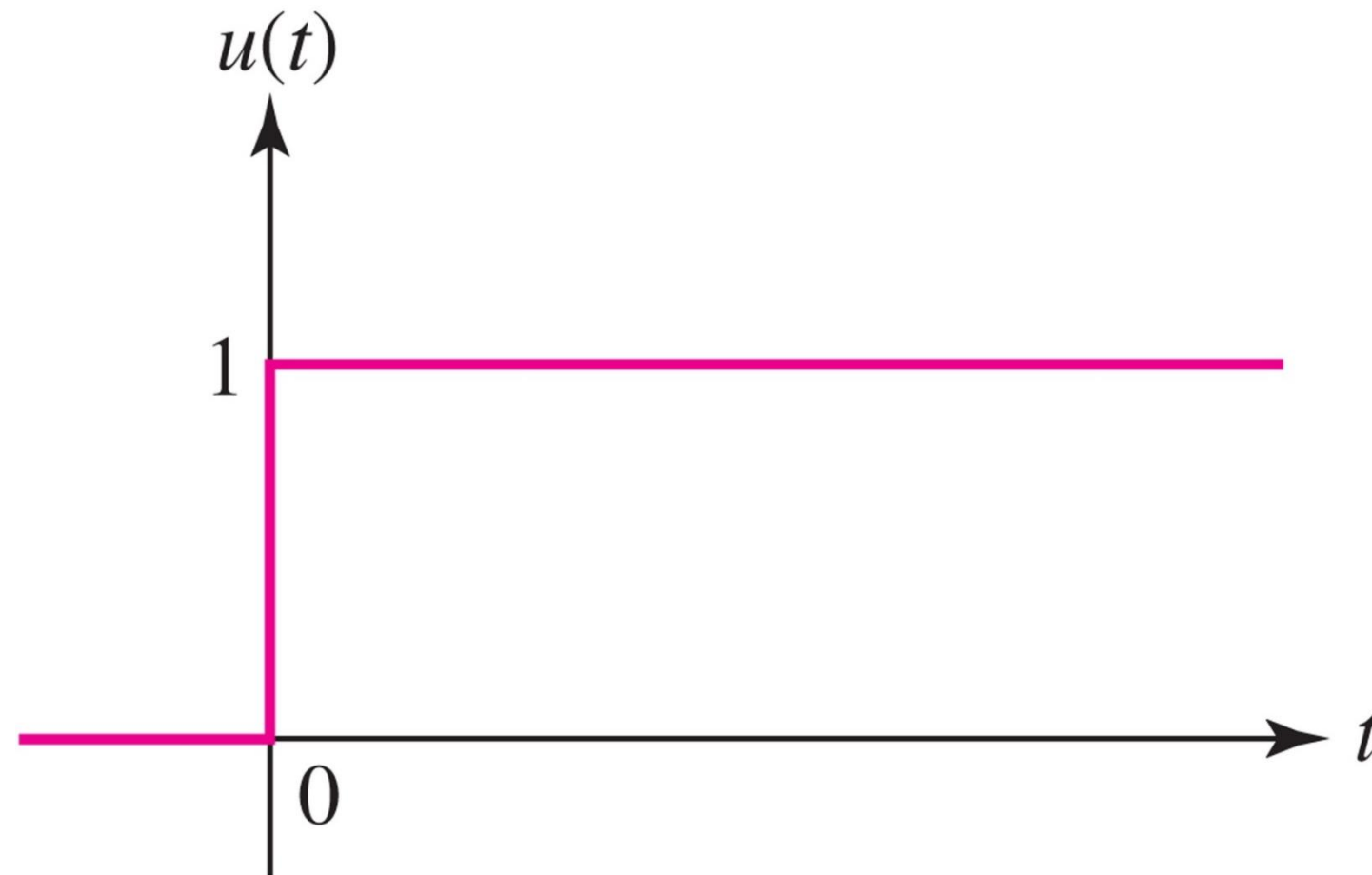
## Example: L and R Current

Find  $i_1(t)$  and  $i_L(t)$  for  $t > 0$ .



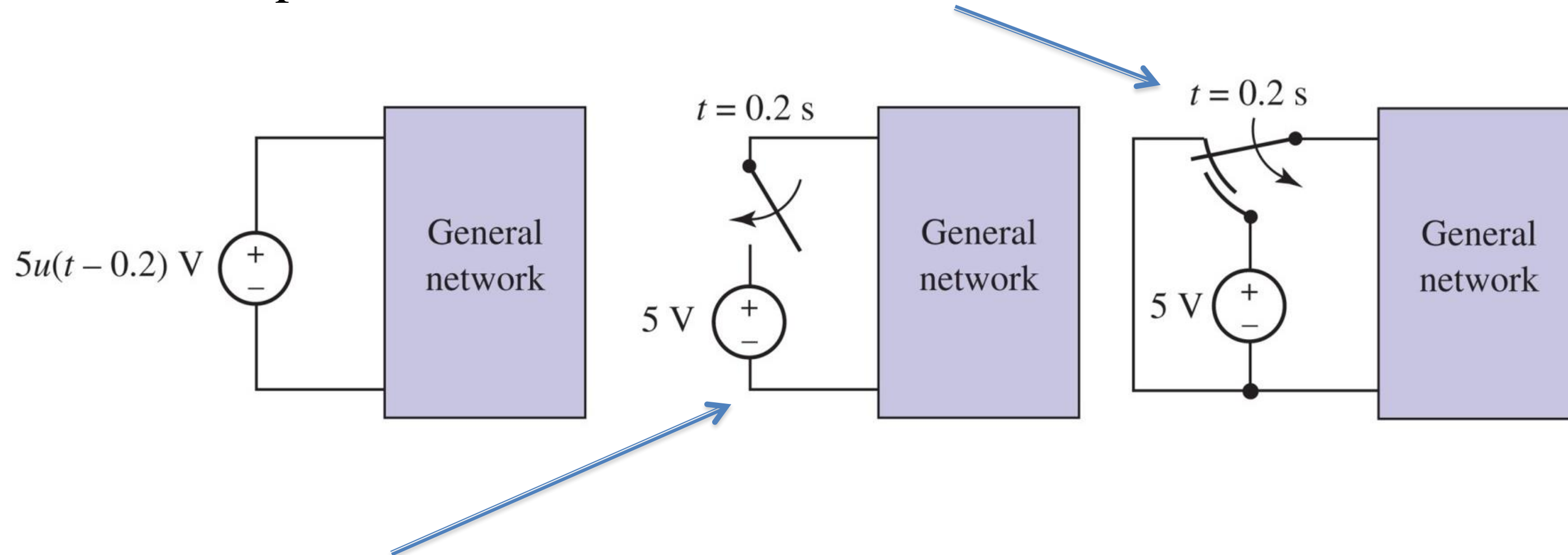
# The Unit Step Function

The unit-step function  $u(t)$  is a convenient notation to represent change at  $t=0$ :



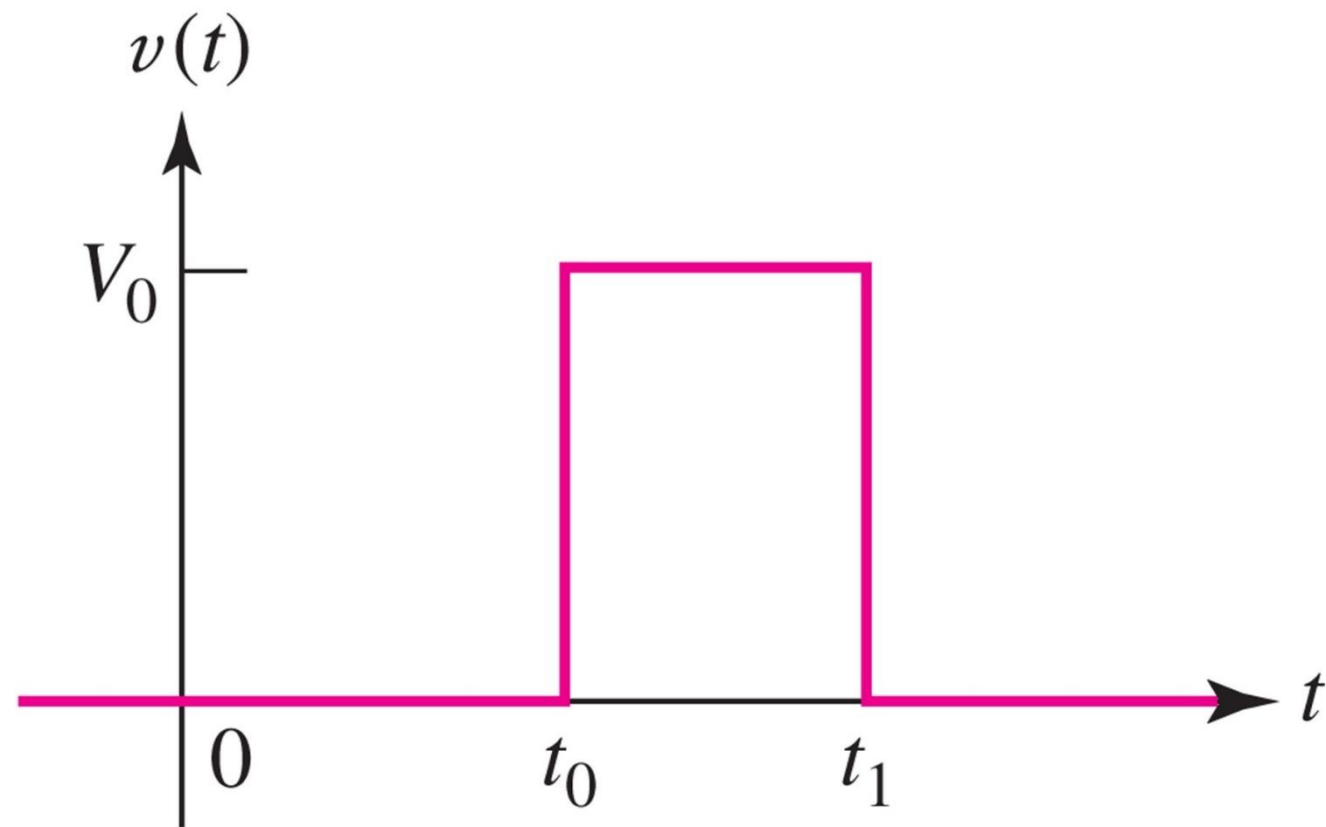


- The unit step models a double-throw switch.



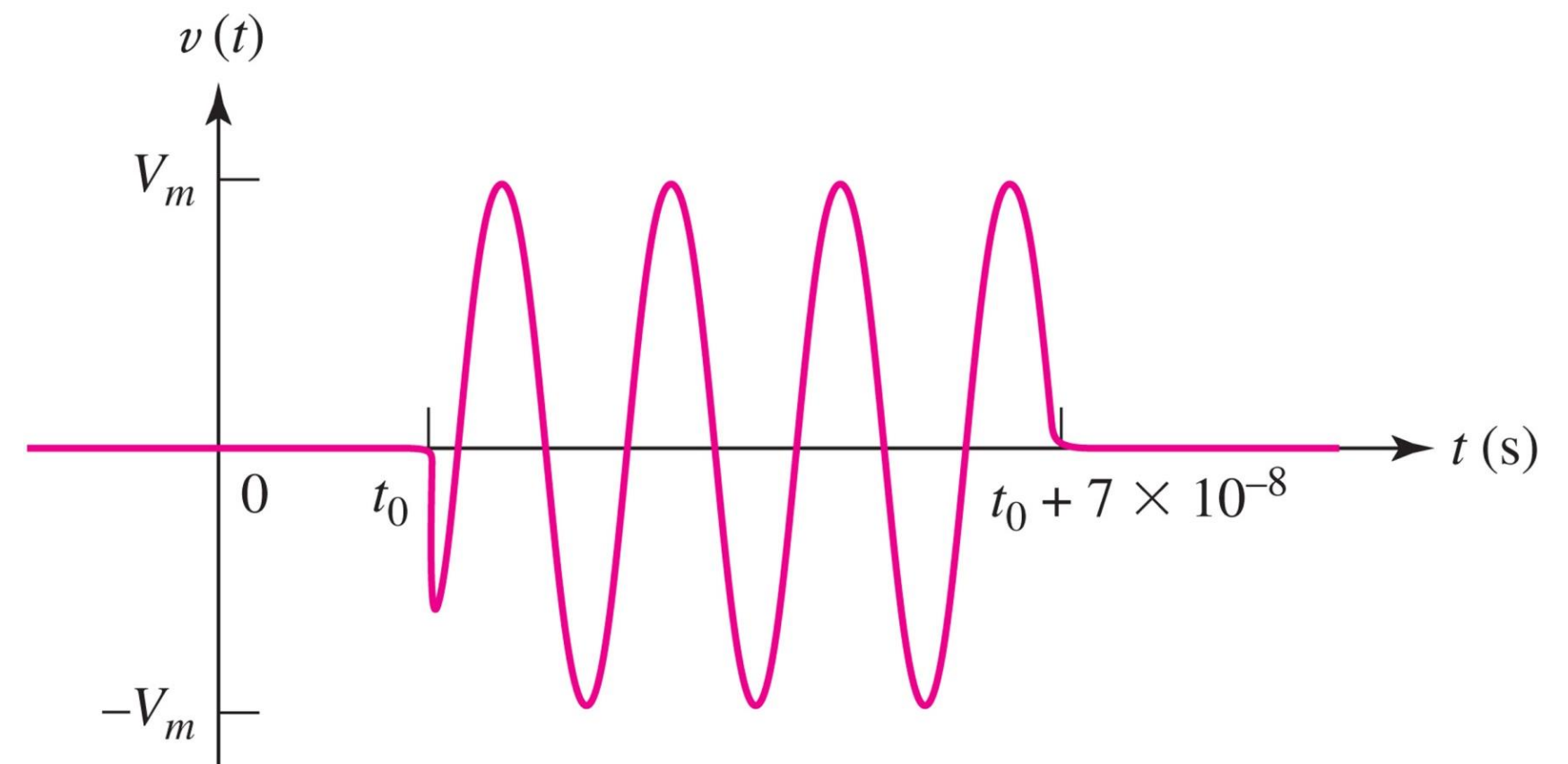
- A single-throw switch is open circuit for  $t < 0$ , not short circuit.

# Modeling Pulses using $u(t)$



Rectangular pulse

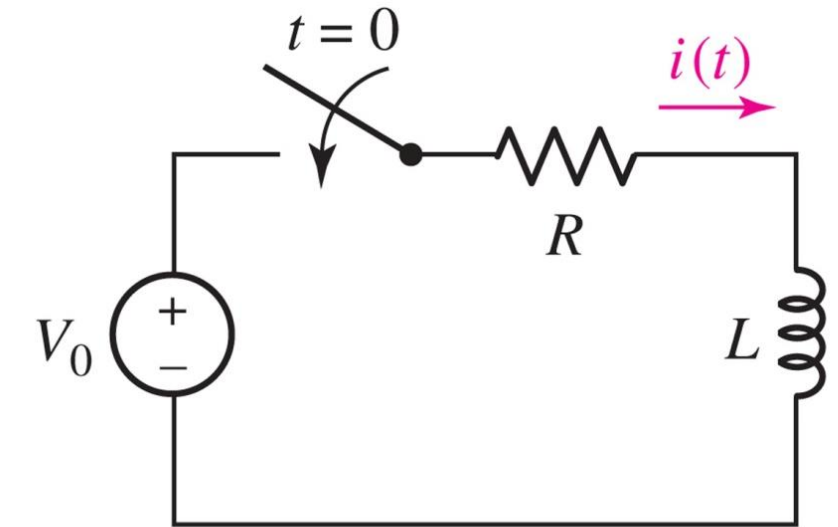
Pulsed sinewave:



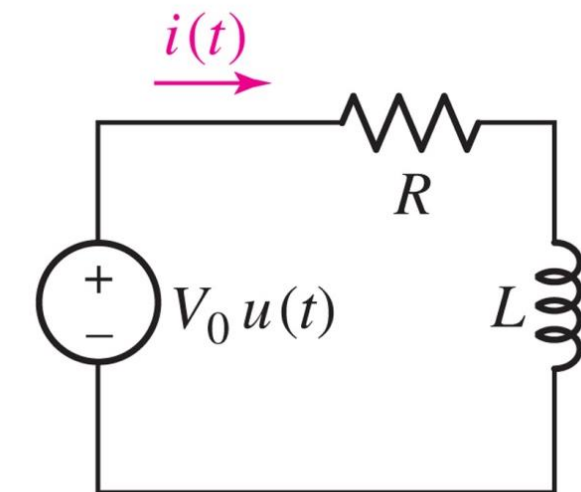
# Driven RL Circuits

- The two circuits shown both have  $i(t)=0$  for  $t<0$  and are also the same for  $t>0$ .
- We now have to find both the *natural response* and the *forced response* due to the source  $V_0$

$$Ri + L \frac{di}{dt} = V_0 u(t)$$



(a)



(b)

$$Ri + L \frac{di}{dt} = V_0 u(t) \quad i(0^+) = 0$$

## 1. Natural response:

$$Ls + R = 0 \rightarrow s = -\frac{R}{L} \quad i_n = Ae^{st} = Ae^{-\frac{Rt}{L}}$$

## 2. Forced response (of the same nature as the source)

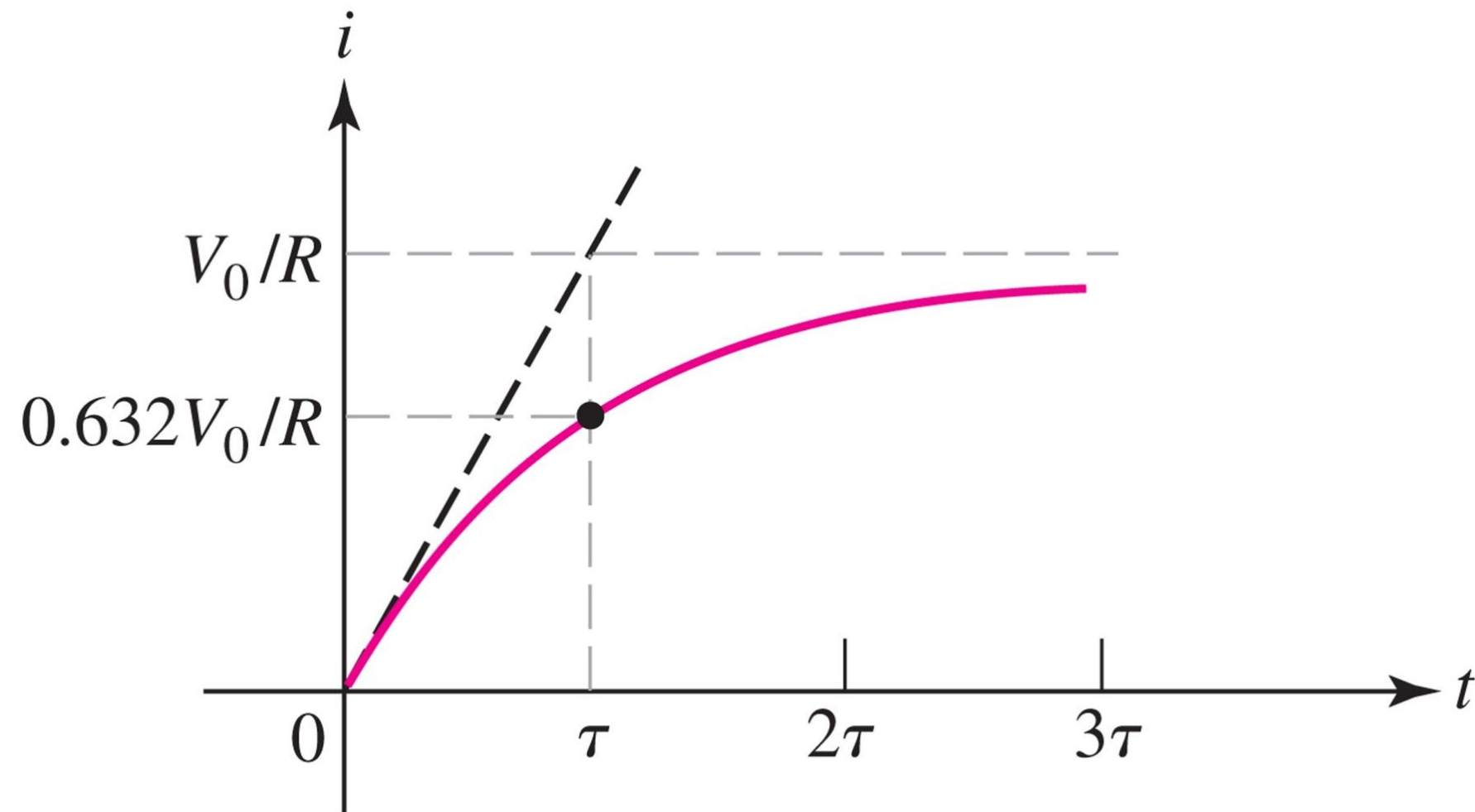
$$i_f = K \xrightarrow[\text{صدق دادن در معادله}]{Ri + L \frac{di}{dt} = V_0} i_f = \frac{V_0}{R}$$

## 3. The complete response = natural response + forced response

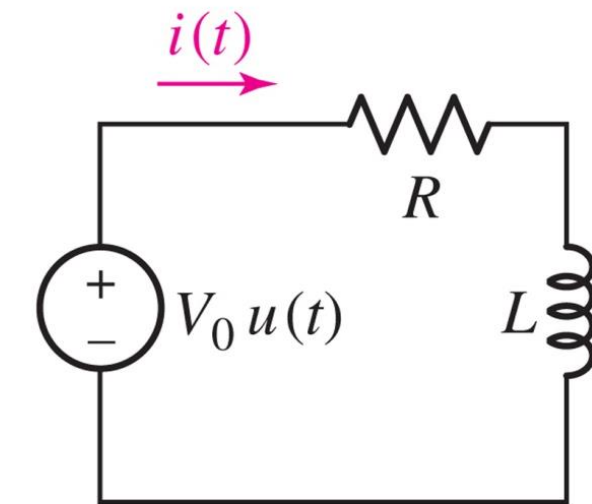
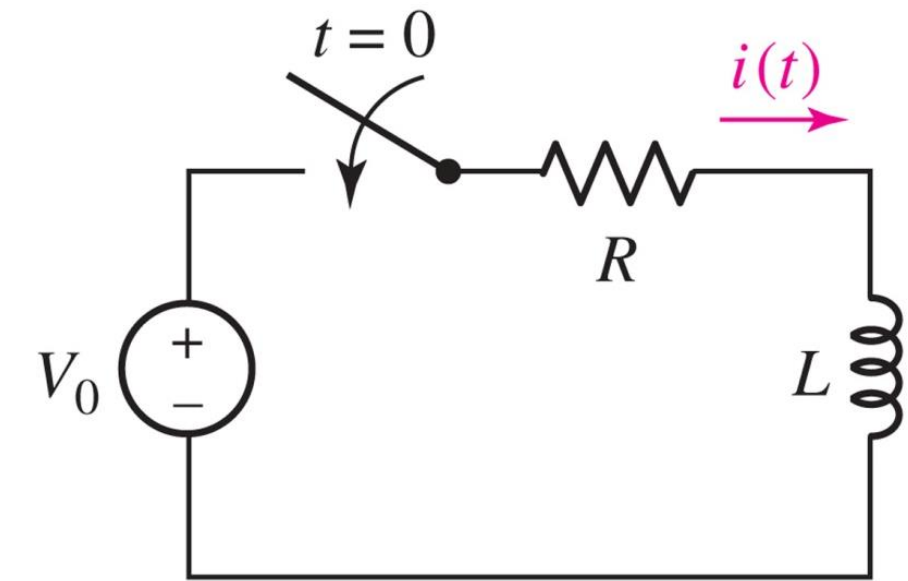
$$i(t) = Ae^{-\frac{Rt}{L}} + \frac{V_0}{R} \xrightarrow[\text{صدق دادن شرایط اولیه}]{} i(t) = \frac{V_0}{R} (1 - e^{-\frac{Rt}{L}}) u(t)$$

# Driven RL Circuits

In this circuit, the inductor current is charged exponentially to the final value  $\frac{V_0}{R}$ .

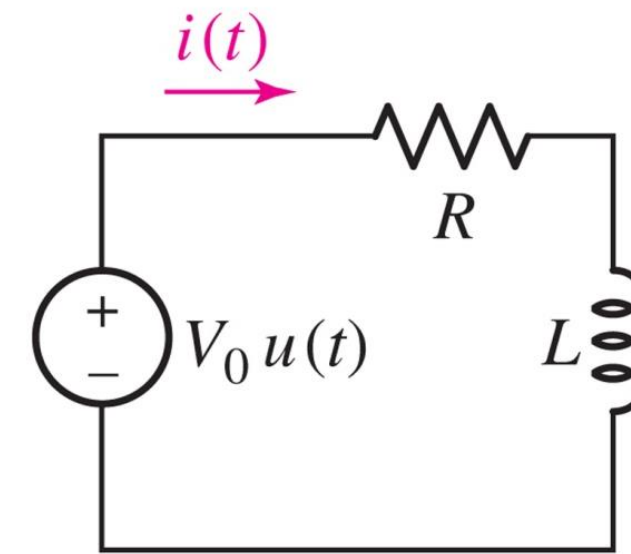
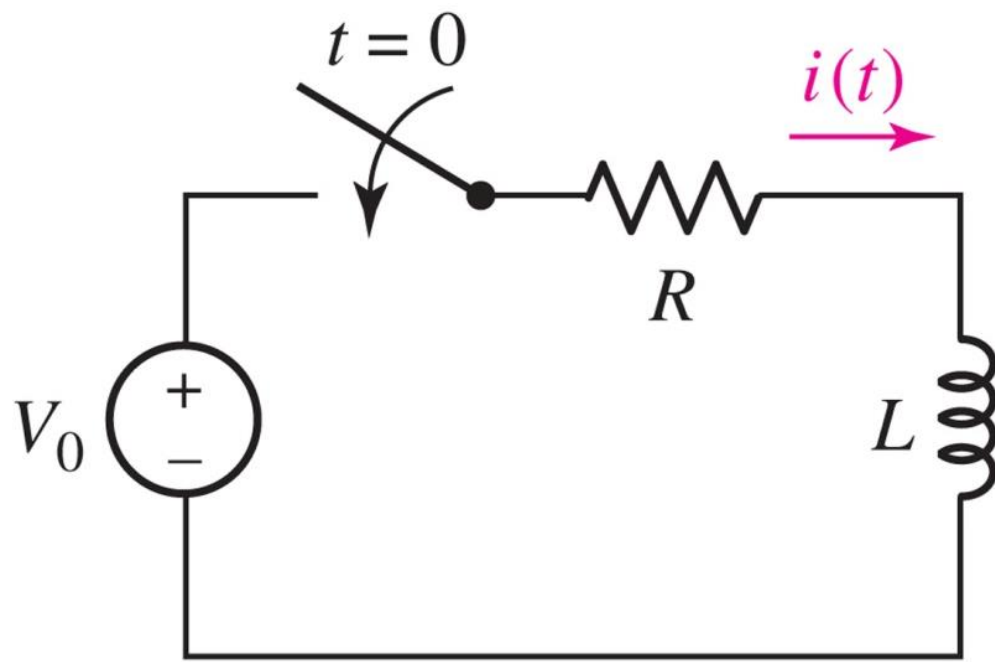


$$i(t) = \frac{V_0}{R} \left(1 - e^{-\frac{Rt}{L}}\right) u(t)$$



The total response is the combination of the transient/natural response and the forced response:

$$i(t) = \frac{V_0}{R} \left( 1 - e^{-\frac{Rt}{L}} \right) u(t)$$



What if the circuit has both a source and an initial condition

- **Solve the differential equation with the given initial condition:**

- Complete response = Natural response + Forced response

- **Use the principle of superposition:**

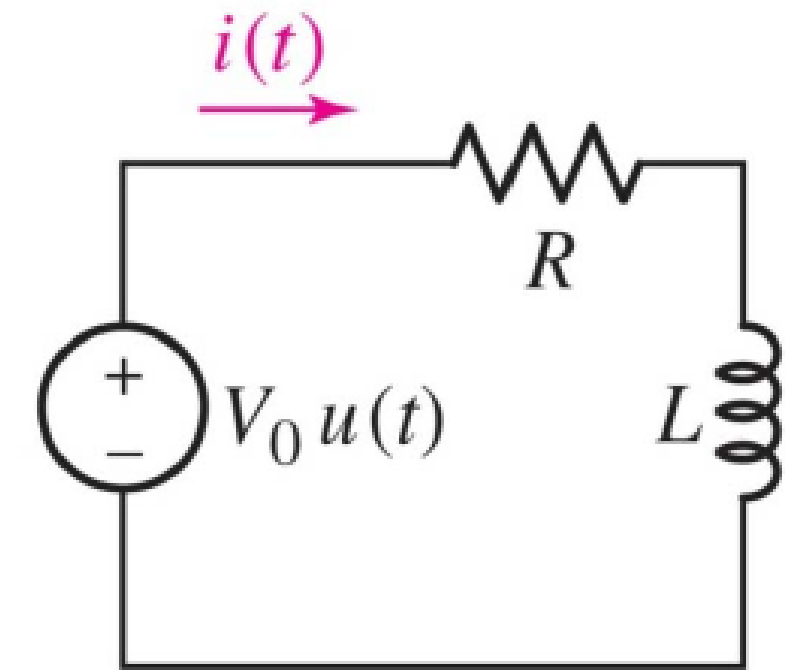
- Complete response = Response of the circuit with the source (without the initial condition) + Response of the circuit without the source (with the initial condition)

Keep the sources but set the initial conditions to zero.

Remove the sources but keep the initial conditions.

Find  $i(t)$  if  $i(0^-) = I_0$ :

**Superposition:**



•Response without the source:

$$i_{sf} = I_0 e^{-Rt/L}$$

•Response with the source:

$$i_d = \frac{V_0}{R} (1 - e^{-Rt/L})$$

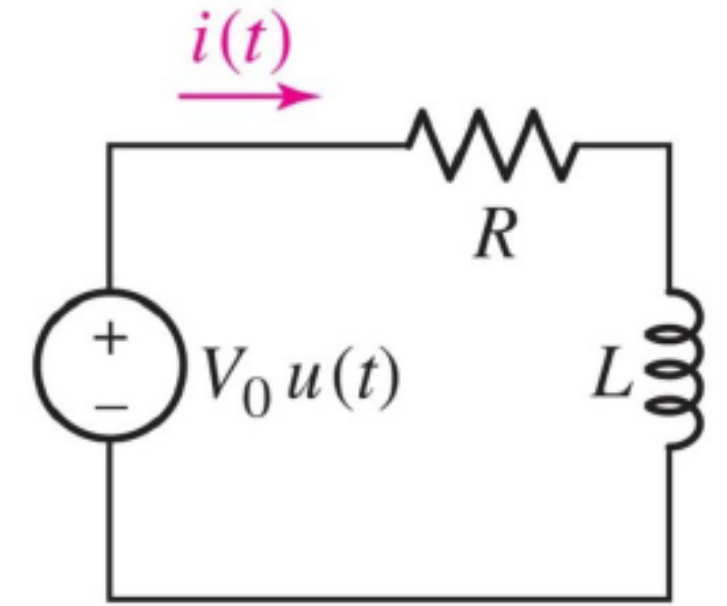
•Complete response:

$$i(t) = I_0 e^{-Rt/L} + \frac{V_0}{R} (1 - e^{-Rt/L})$$



Find  $i(t)$  if  $i(0^-) = I_0$ :

$$i' + \frac{R}{L}i = \frac{V_0}{L}, \quad i(0^+) = I_0$$



**Solving the differential equation with the initial condition:**

Natural response:

$$i_n = K e^{-Rt/L}$$

Forced response:

$$i_f = \frac{V_0}{R}$$

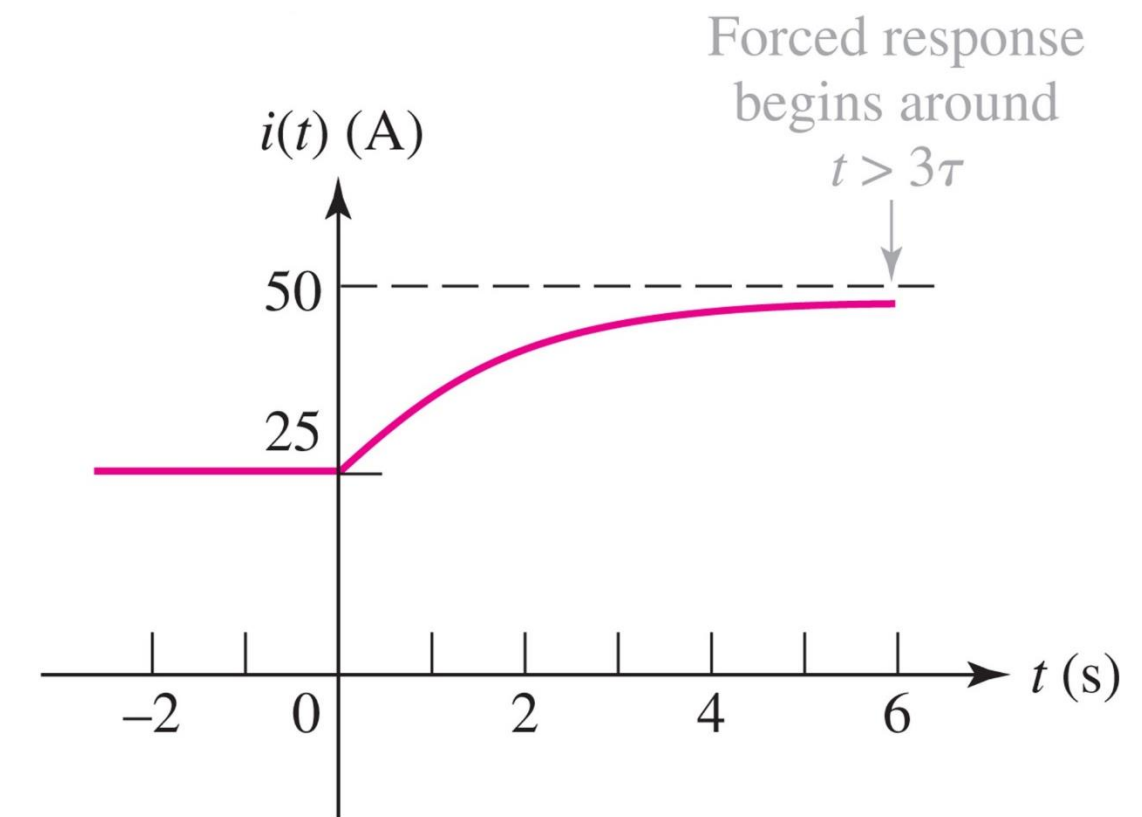
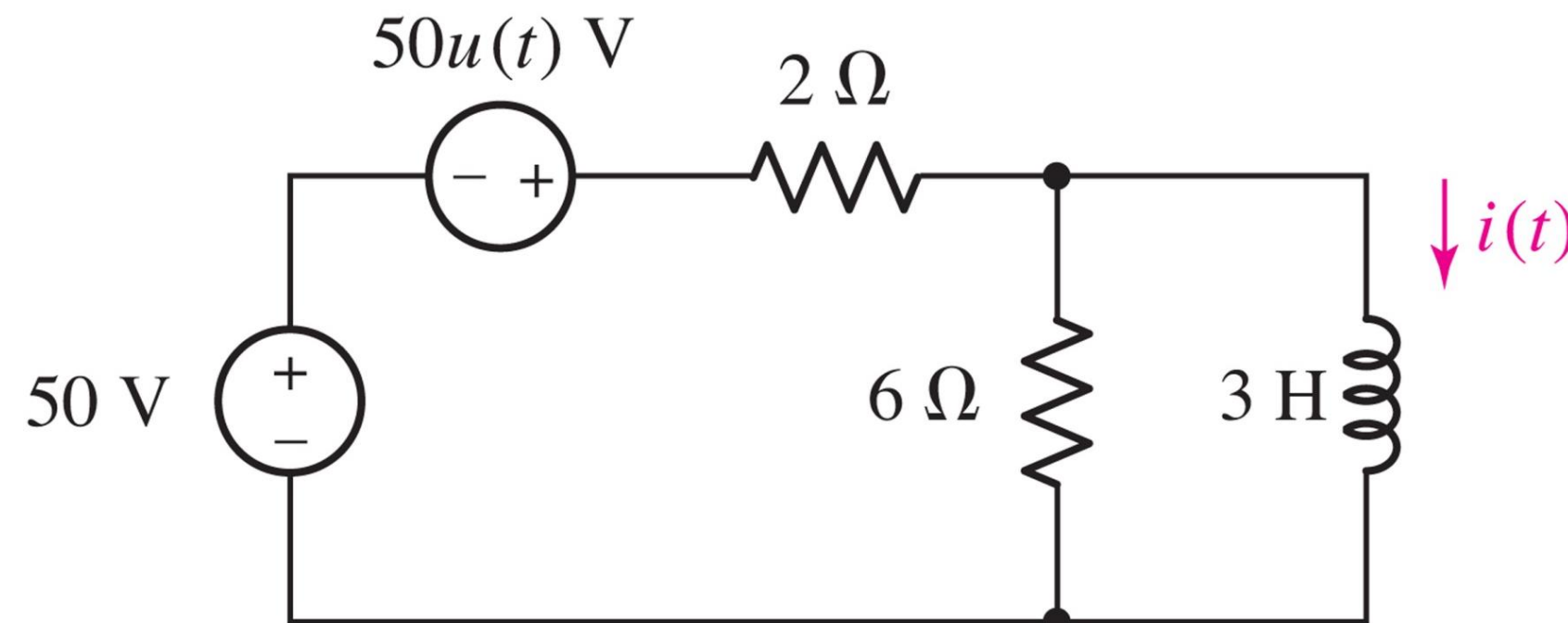
Complete response:

$$i(t) = K e^{-Rt/L} + \frac{V_0}{R}$$

$$i(0) = I_0 \rightarrow K = I_0 - \frac{V_0}{R}, \quad i(t) = \frac{V_0}{R} + \left(I_0 - \frac{V_0}{R}\right)e^{-Rt/L}$$

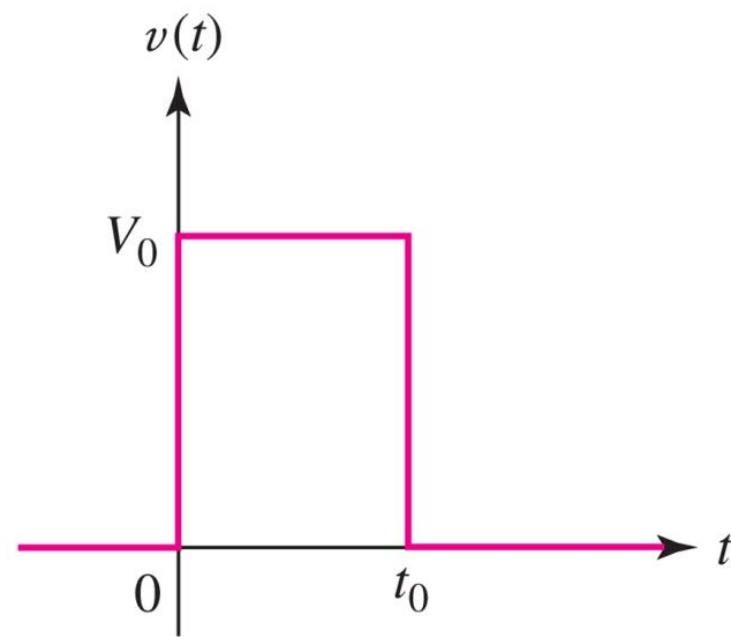
## Example: RL Circuit with Step

Show that  $i(t) = 25 + 25(1 - e^{-t/2})u(t)$  A

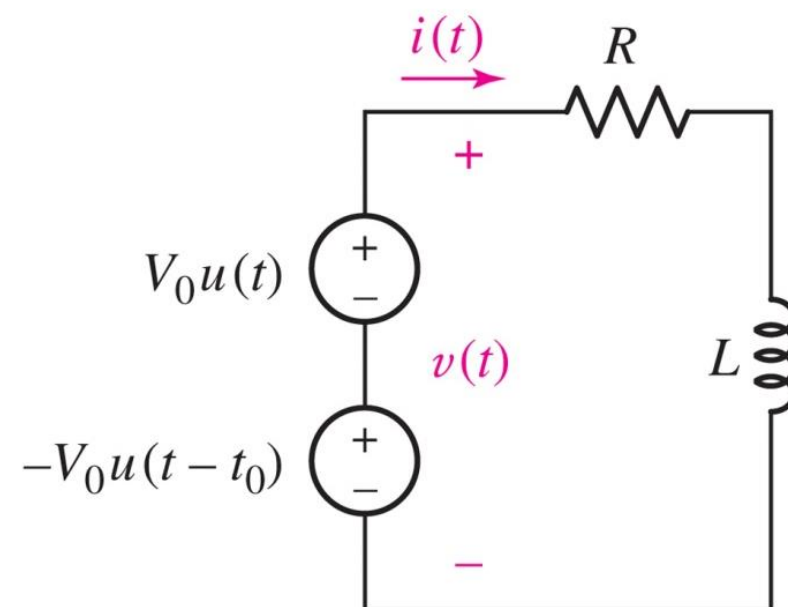
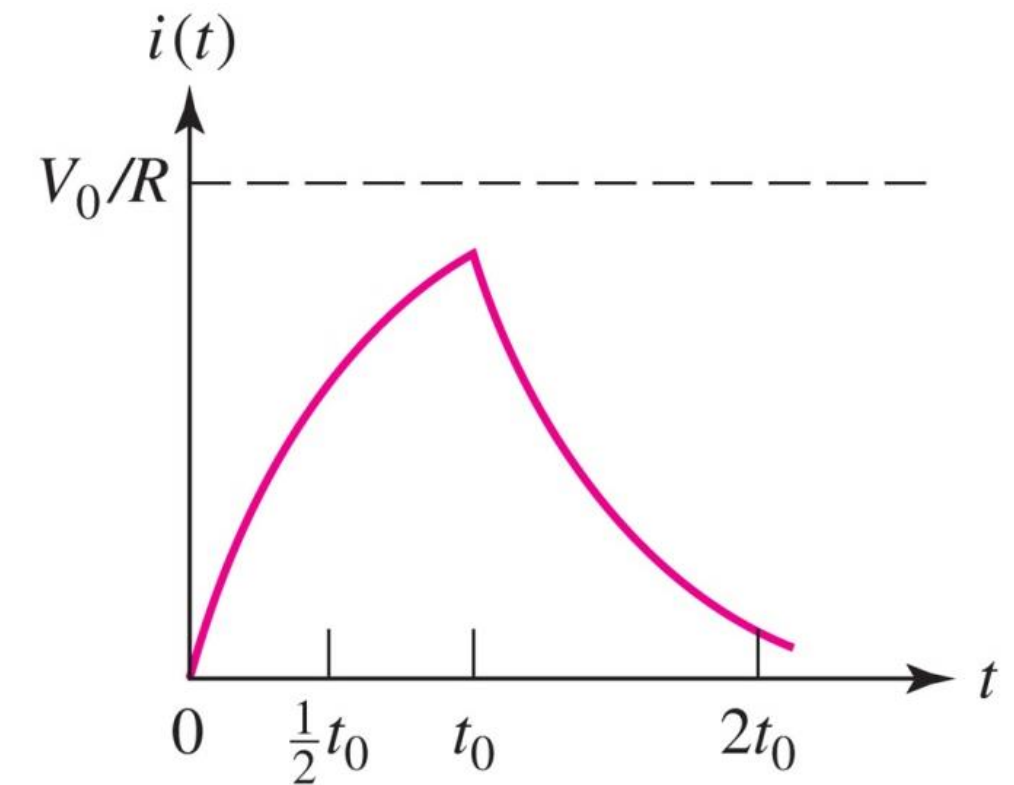


# Example: Voltage Pulse

Assuming the given input voltage, find the current  $i(t)$ .

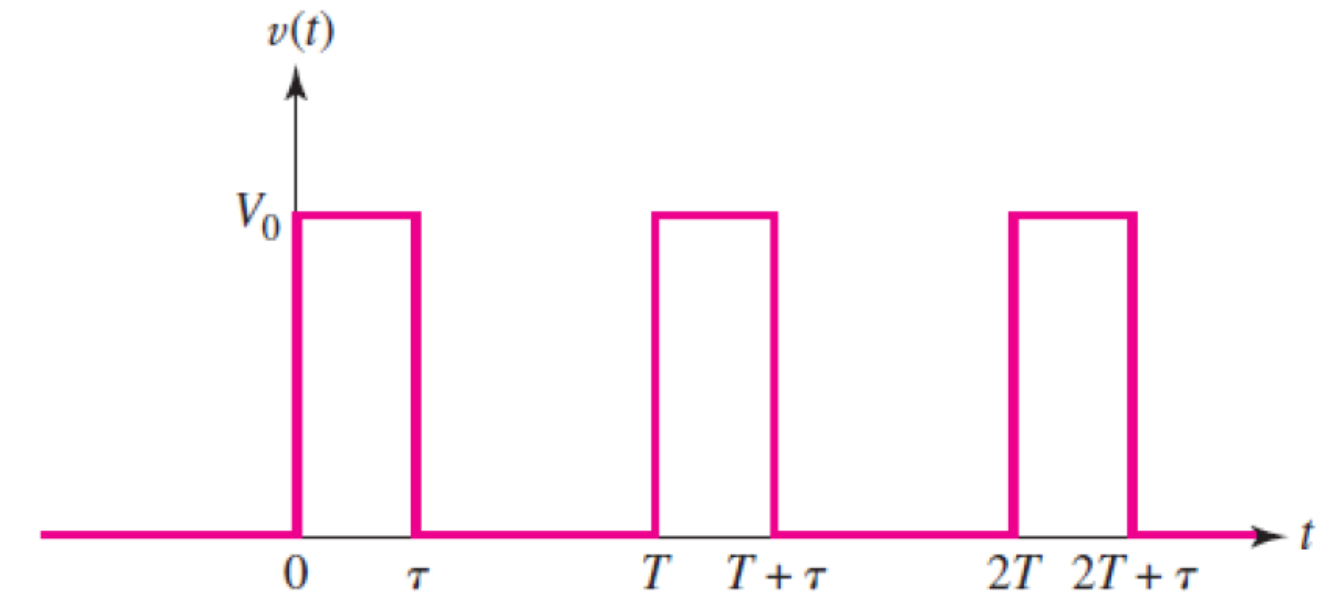


(a)

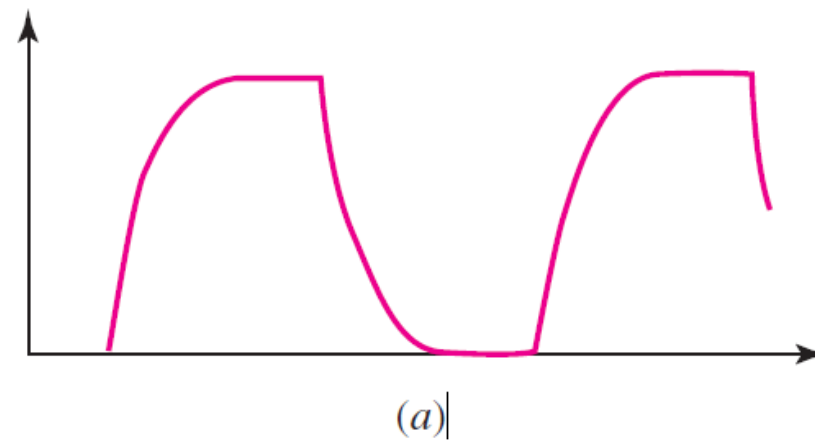


# Response of RL or RC circuit to a pulse train input

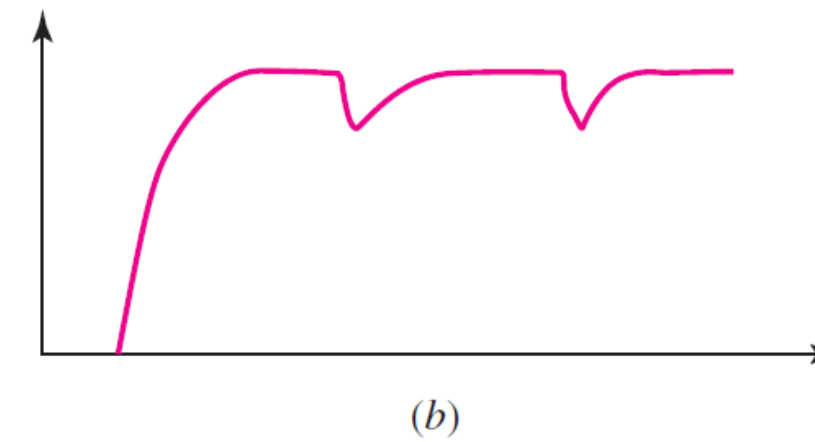
Depending on the values of pulse width ( $\tau$ ), pulse period ( $T$ ), and the circuit time constant ( $RC$ ), four different cases arise:



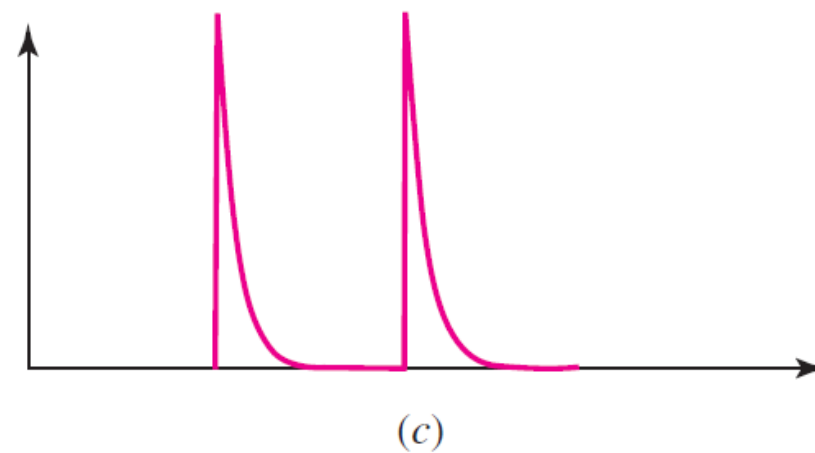
$\tau \gg RC$   
 $T - \tau \gg RC$



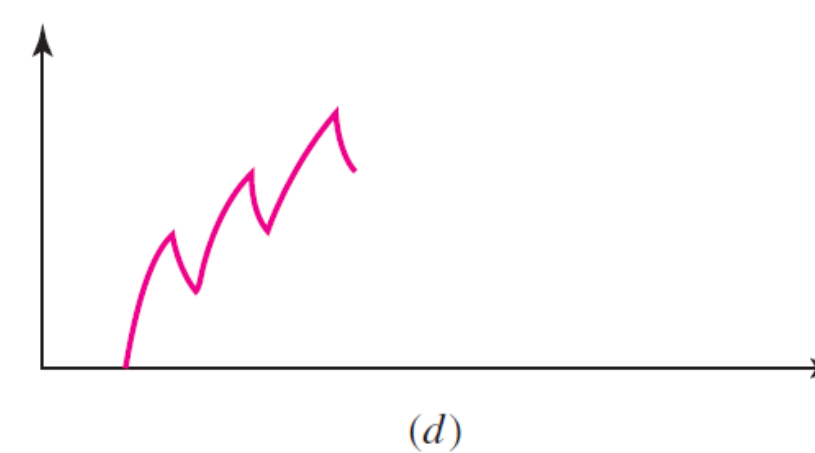
$\tau \gg RC$   
 $T - \tau \ll RC$



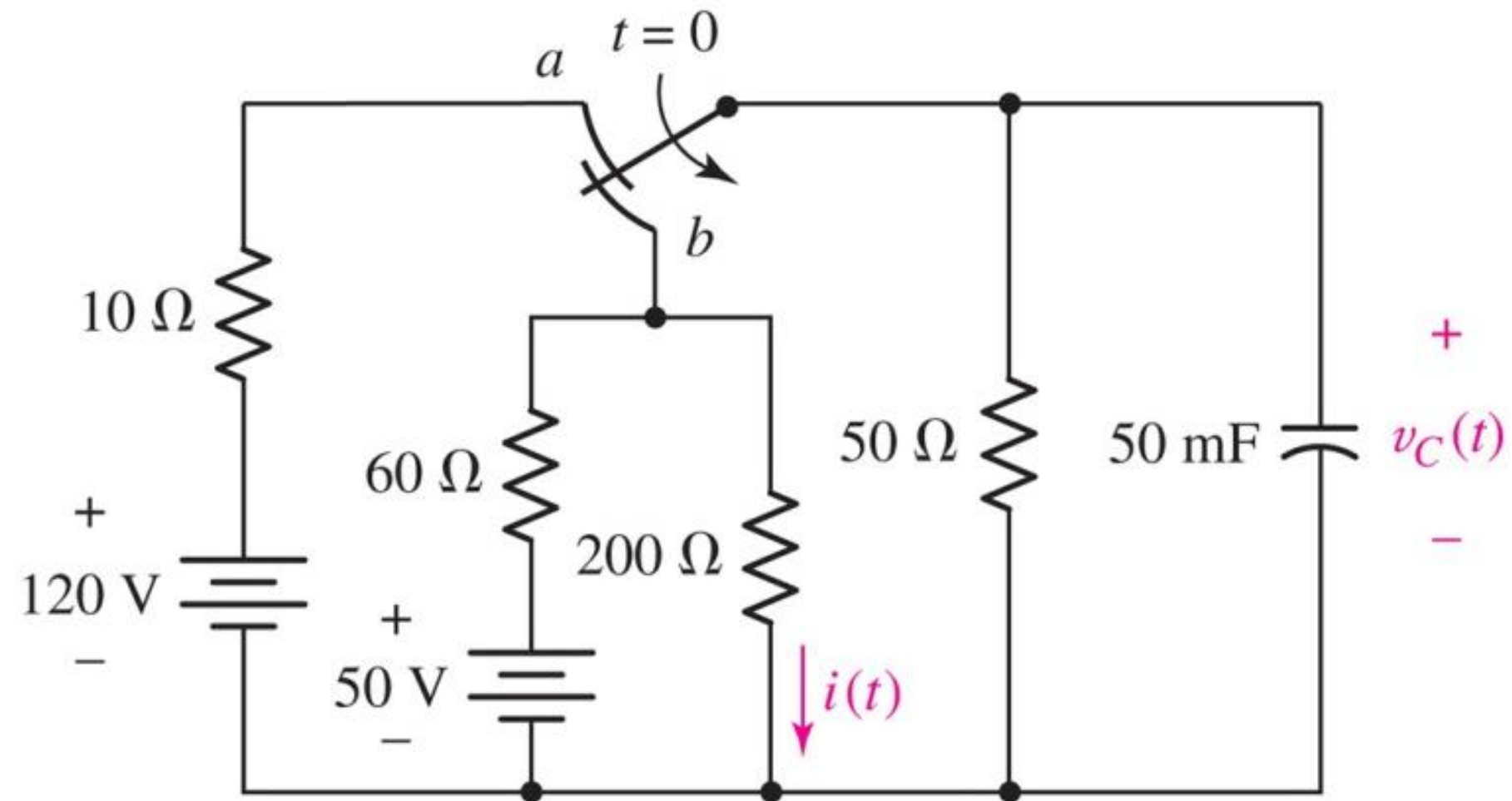
$\tau \ll RC$   
 $T - \tau \gg RC$

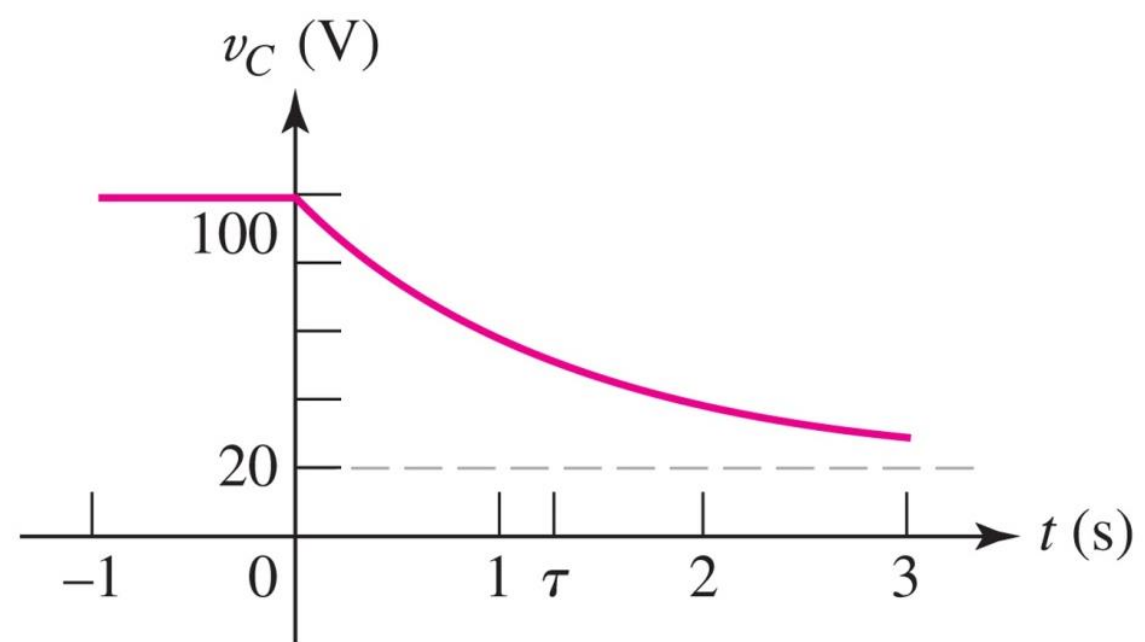


$\tau \ll RC$   
 $T - \tau \ll RC$



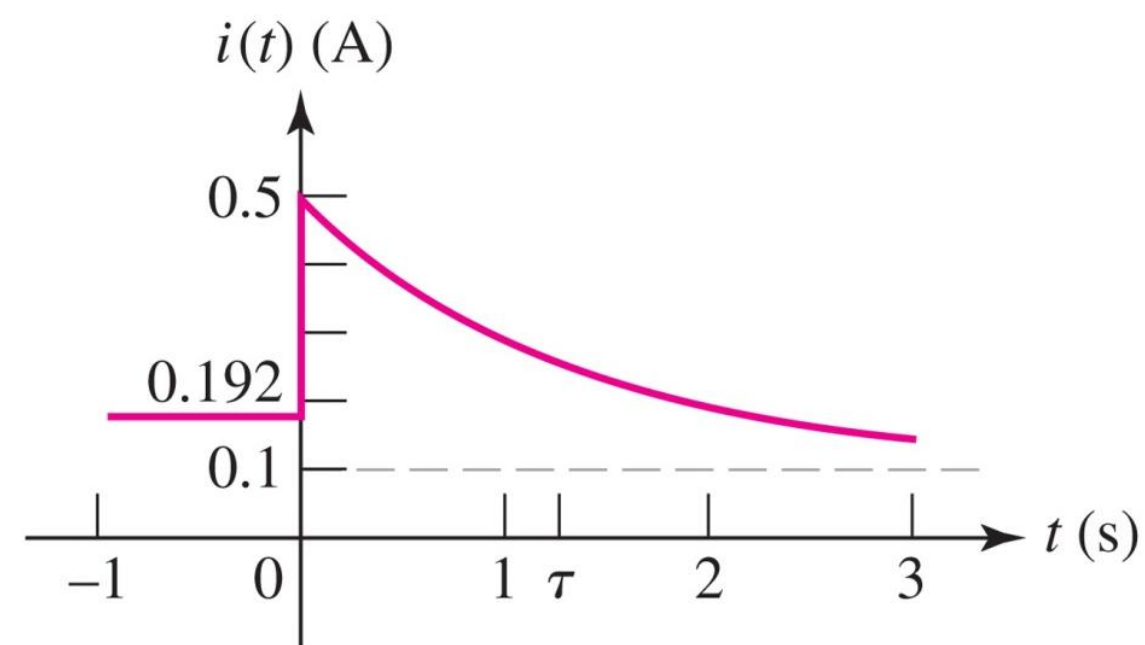
# Driven RC Circuits (part 1 of 2)





(a)

$$v_C = 20 + 80e^{-t/1.2} \text{ V}$$

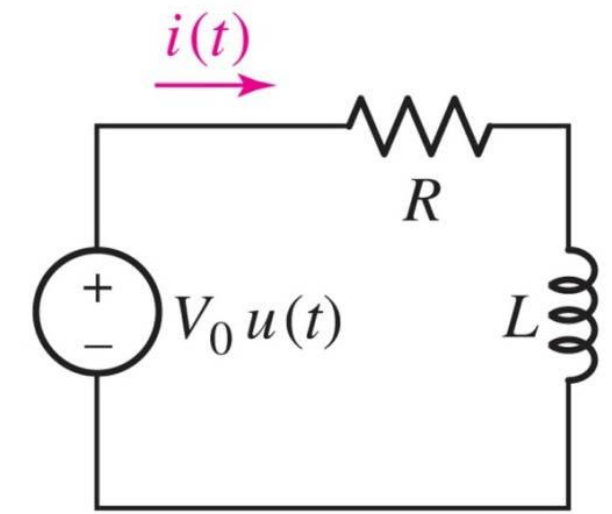


(b)

$$i = 0.1 + 0.4e^{-t/1.2} \text{ A}$$

- ✓ We have previously calculated the current response for an RL circuit as follows:

$$i(t) = \frac{V_0}{R} + (I_0 - \frac{V_0}{R})e^{-Rt/L}$$



- ✓ From the following relation, it can generally be shown that the step response of an RC or RL circuit is also obtainable:

$$x(t) = x(\infty) + [x(0^+) - x(\infty)]e^{-t/\tau}$$

# Response of First-Order Circuits to AC Sources

- What if there are no DC sources?
- The natural response, which is independent of the source, will have the same form ( $Ke^{st}$ ).
- The forced response has the same form as the source but with a different coefficient.

منبع	پاسخ اجباری
$K$	$K'$
$K_1t + K_2$	$K't + k''$
$Ke^{bt} (b \neq s)$	$K'e^{bt}$
$Ke^{bt} (b = s)$	$K'te^{bt}$
$K\cos(\omega t + p)$	$K'\cos(\omega t + p')$

The roots of the characteristic equation of the circuit are  $s$ .



# Complete response

Another method for solving the following first-order differential equation:

$$x'(t) + ax(t) = Q(t)$$

We multiply both sides of the equation by  $e^{at}$ :

$$e^{at}x'(t) + ae^{at}x(t) = e^{at}Q(t)$$

$$\rightarrow (e^{at}x(t))' = e^{at}Q(t)$$

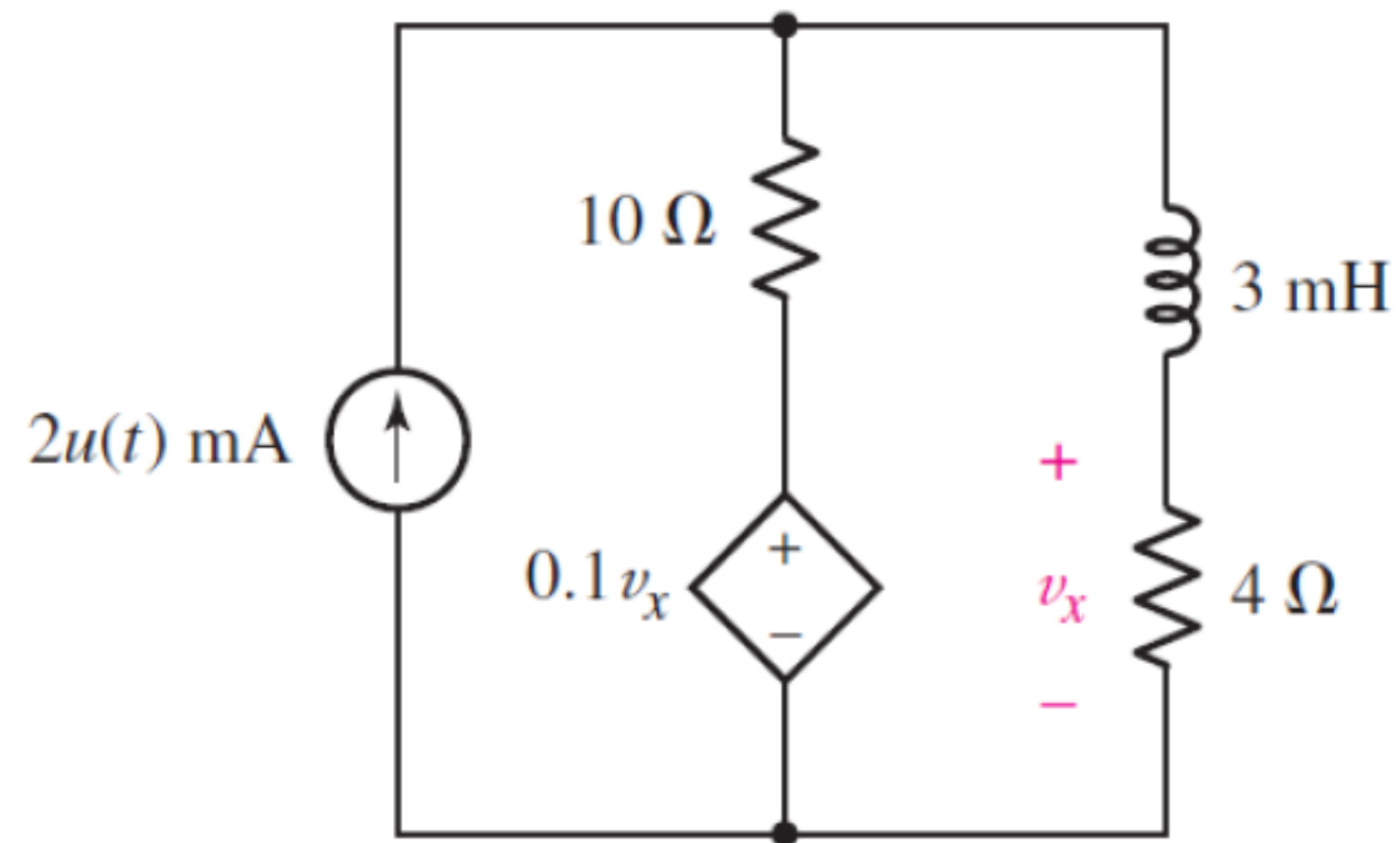
$$\rightarrow x(t) = e^{-at} \int e^{at}Q(t) + c e^{-at}$$

$\longleftrightarrow$   
*forced response*

$\longleftrightarrow$   
*natural response*

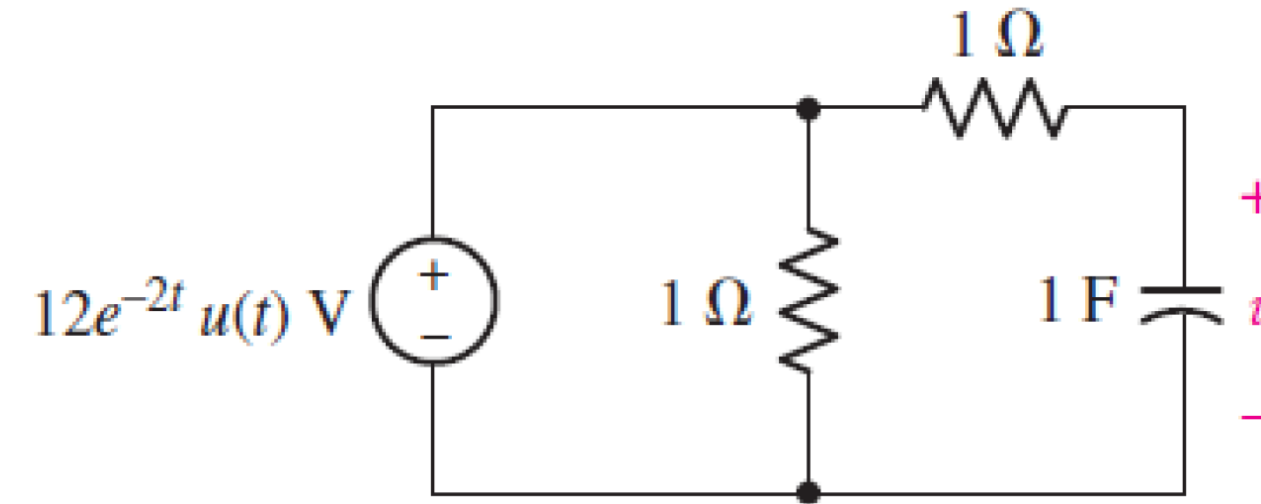
# Practice 1

Find  $v_x(t)$ .



## Practice 2

Find  $v(t)$ .



Soloution:

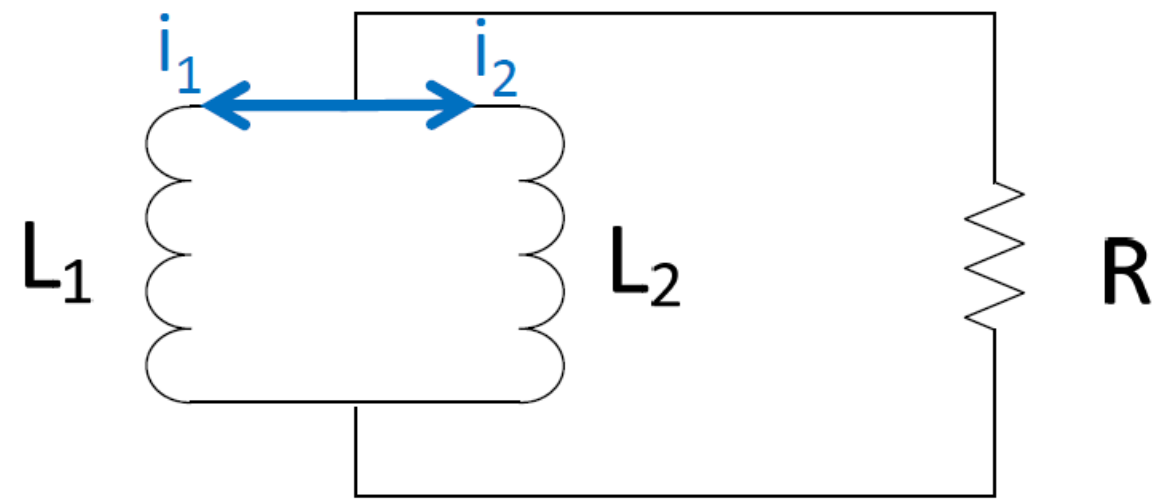
$$v(t) = 12(e^{-t} - e^{-2t})u(t)$$

What if the source value was equal to  $12e^{-t}u(t)$ ?

$$v(t) = 12te^{-t}u(t)$$

## Practice 3

Find  $i_1(t)$  and  $i_2(t)$ .



$$L_1 = 6H, i_1(0) = 2A \quad \square$$

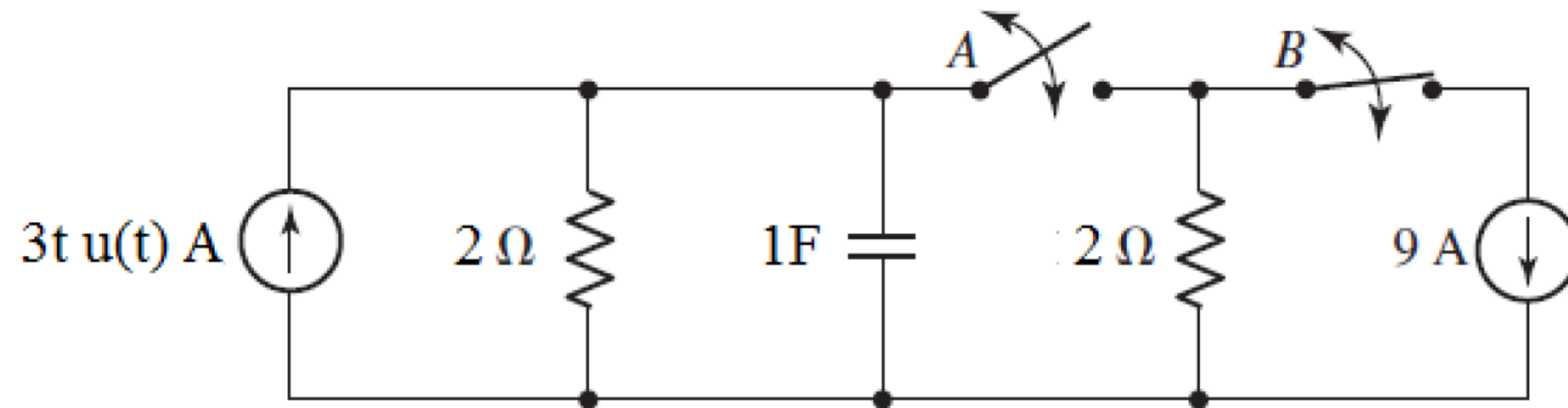
$$L_2 = 3H, i_2(0) = 1A \quad \square$$

$$R = 6\Omega \quad \square$$

Hint:  $V_{L_1} = V_{L_2} \rightarrow L_1 i_1' = L_2 i_2' \rightarrow L_1 i_1 = L_2 i_2 + K$

## Practice 4

Find  $v_c(t)$ . Suppose keys A and B are initially closed and key A is opened in second 2 and key B is opened in second 1.

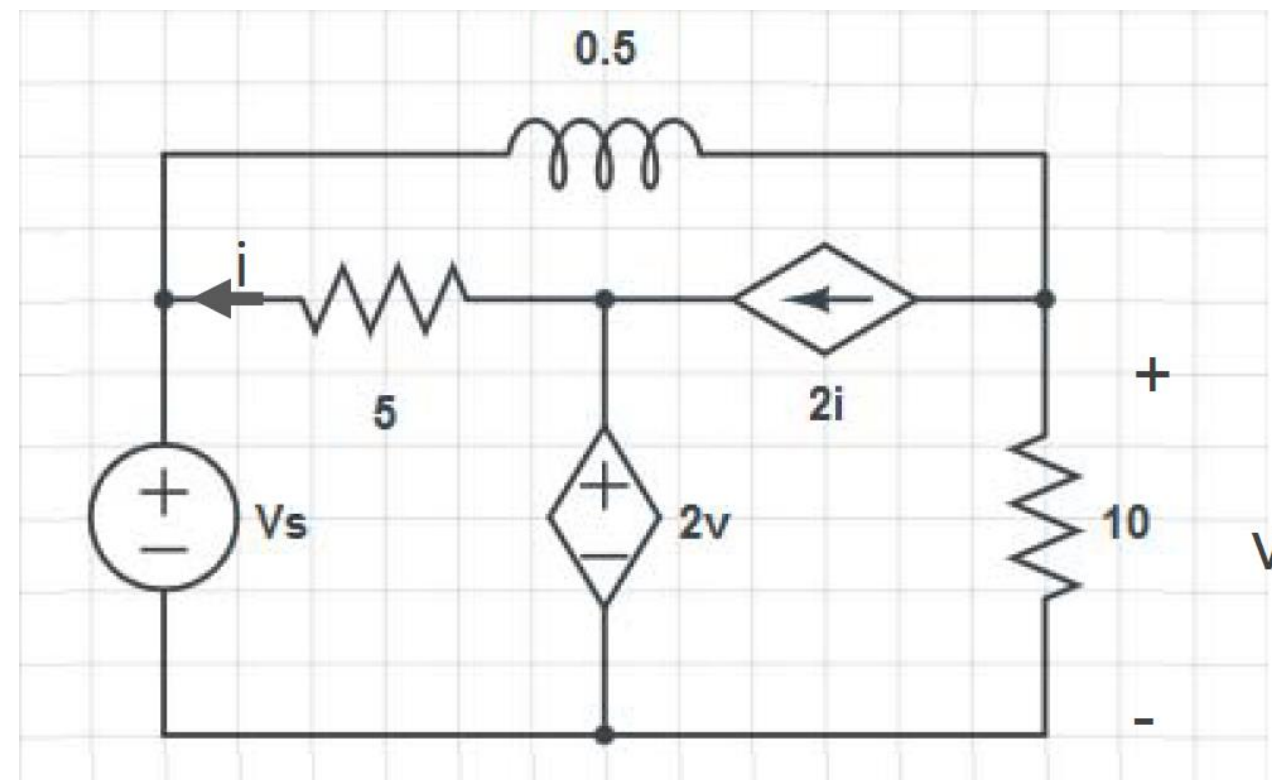


## Practice 5

Find  $v(t)$  if  $v_s = u(t)$ :

$$\left(1 - \frac{5}{9}e^{-\frac{20t}{9}}\right)u(t)$$

And if  $v_s = \cos t \ u(t)$





# Thanks

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