ساختمان داده و الگوریتم ها

مبحث هفدهم:
درهم سازی

سجاد شیرعلی شهرضا

پاییز 1402

دوشنبه، 20 آذر 1402

Based on Stanford CS161 Slides from Summer 2020 by Karey Shi

اطلاع رسانی

- بخش مرتبط کتاب برای این جلسه: 11
- انتقال امتحانک سوم به شنبه 2 دی 1402 (در ساعت کلاس، از ساعت 8:15)

جدول درهم سازی

چه عملگرهایی برای ما مهم هستند؟

THE TASK

Again, we want to keep track of objects that have keys $\boxed{5}$ (aka, **nodes** with **keys**)

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Sorted Arrays

1	2	3	4	5	7

O(n) INSERT/DELETE: first, find the relevant element (via SEARCH) and move a bunch of elements in the array

O(log n) SEARCH: use binary search to see if an element is in A

Linked Lists

HEAD → 3 → 5 → 1 → 4 → 7 → 2

O(1) INSERT: just insert the element at the head of the linked list

O(n) SEARCH/DELETE: since the list is not necessarily sorted, you need to scan the list (delete by manipulating pointers)

BINARY SEARCH TREE PERFORMANCE

OPERATION	SORTED ARRAY	UNSORTED LINKED LIST	BALANCED BST
SEARCH	O(log(n))	O(n)	O(log(n))
DELETE	O(n)	O(n)	O(log(n))
INSERT	O(n)	O(1)	O(log(n))

HASH TABLE MOTIVATION

OPERATION	SORTED ARRAY	UNSORTED LINKED LIST	BALANCED BST	HASH TABLES (HOPEFULLY)
SEARCH	$O(\log(n))$	$O(n)$	$O(\log(n))$	$O(1)$
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INSERT	O(n)	O(1)	O(log(n))	O(1)

What is a *naive* way to achieve these runtimes?

ATTEMPT 1: DIRECT ADDRESSING

Suppose you're storing numbers from 1 - 1000: 2 4 5 998 999

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Reasonable Attempt: ***Direct Addressing!***

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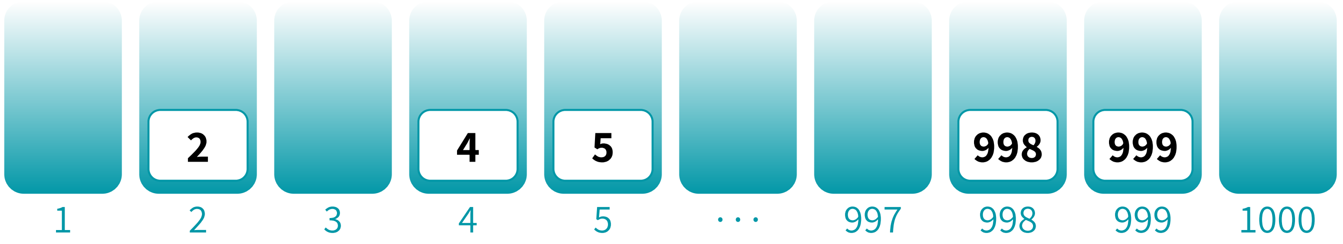

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ATTEMPT 1: DIRECT ADDRESSING

Suppose you're storing numbers from 1 - 1000: **2** **4** **5** **998** **999**

Reasonable Attempt: ***Direct Addressing!***

Not bad!

But what's the issue with this approach?

1 2 3 4 5 · · · 997 998 999 1000

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ATTEMPT 1: DIRECT ADDRESSING

Suppose you're storing numbers from 1 - 10^{10}:

2	**3**	**1000**	**1002**	**10^{10}**

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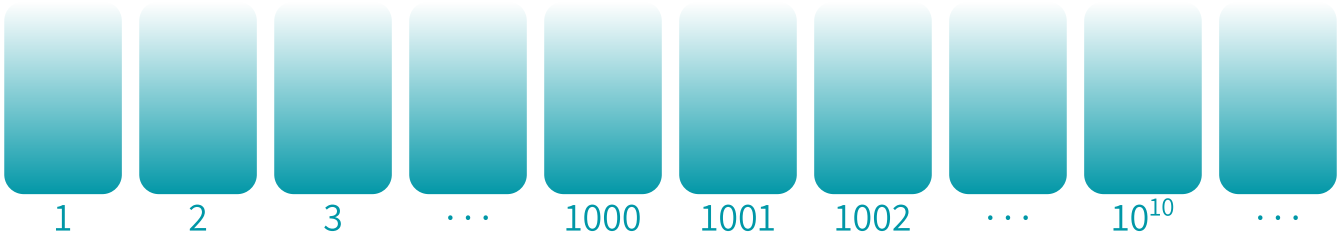

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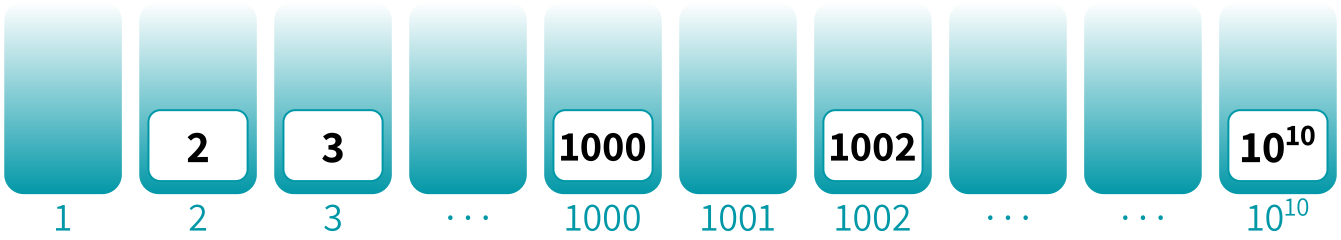

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The space requirement is HUGE...

O(1) INSERT/DELETE/SEARCH: Just index into the bucket!

سوال؟

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On the other extreme, we could save a lot of space by using linked lists!

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Good news: Space is now proportional to the number of objects you deal with

Bad news: Searching for an object is now going to scale with the number of inputs you deal with… not close to our desired O(1)!

The direct-addressing approach still has merit because of it’s fast object search/access

HOW DO WE IMPROVE THIS?

We like the functionality of a direct-addressable array for constant time access (super fast INSERT/DELETE/SEARCH)

But reserving an bucket/array slot for each possible key leads to unreasonable space requirements… (kind of like CountingSort)

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Let's try bucketing **by the least-significant digit**…

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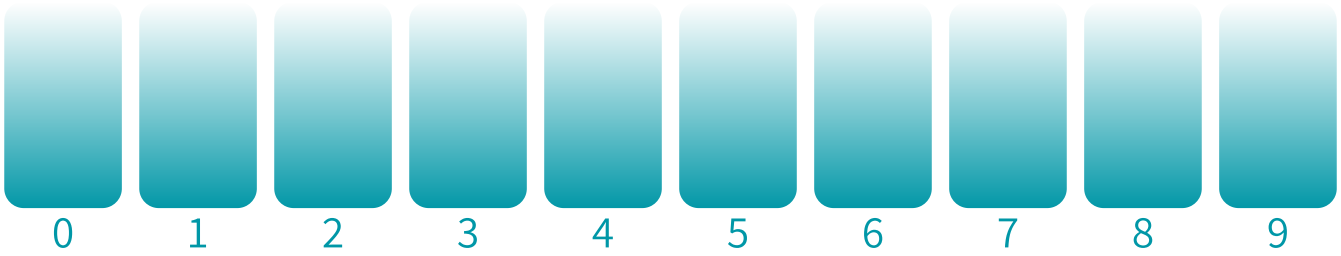
Bucket by last digit?



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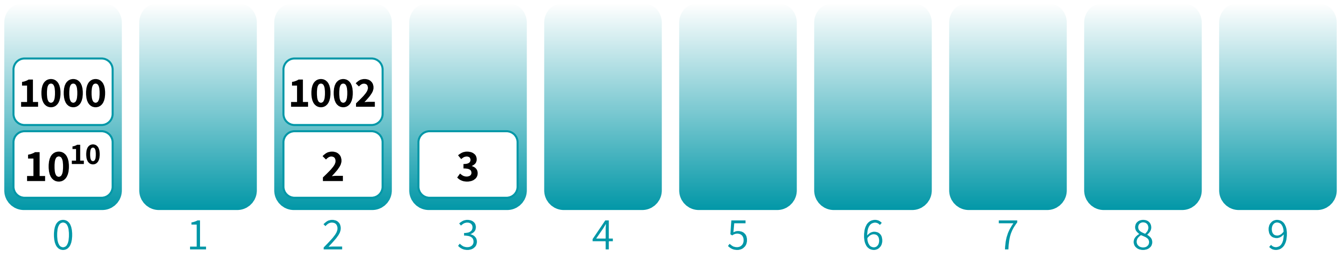

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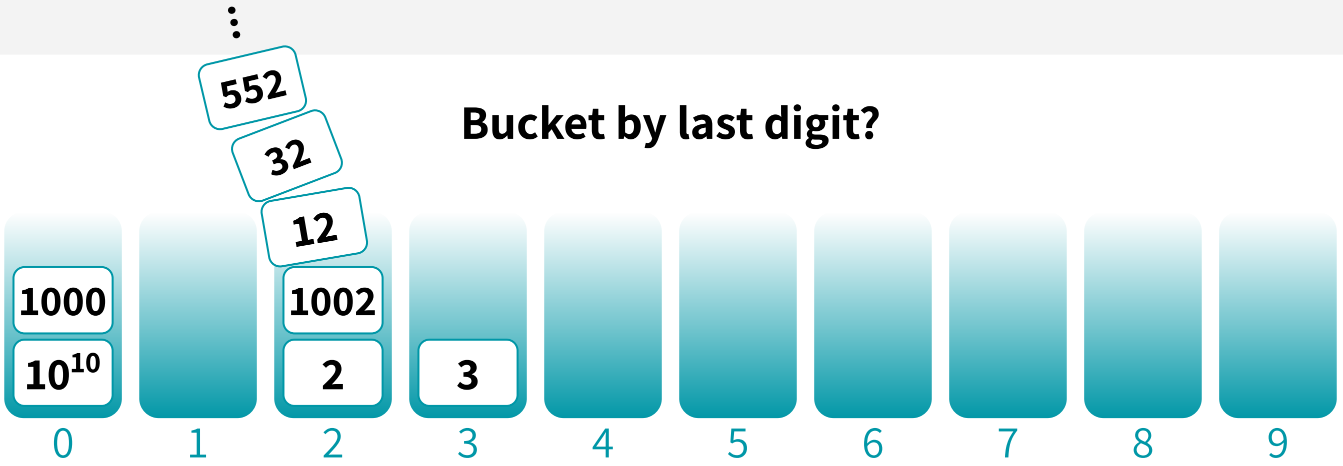

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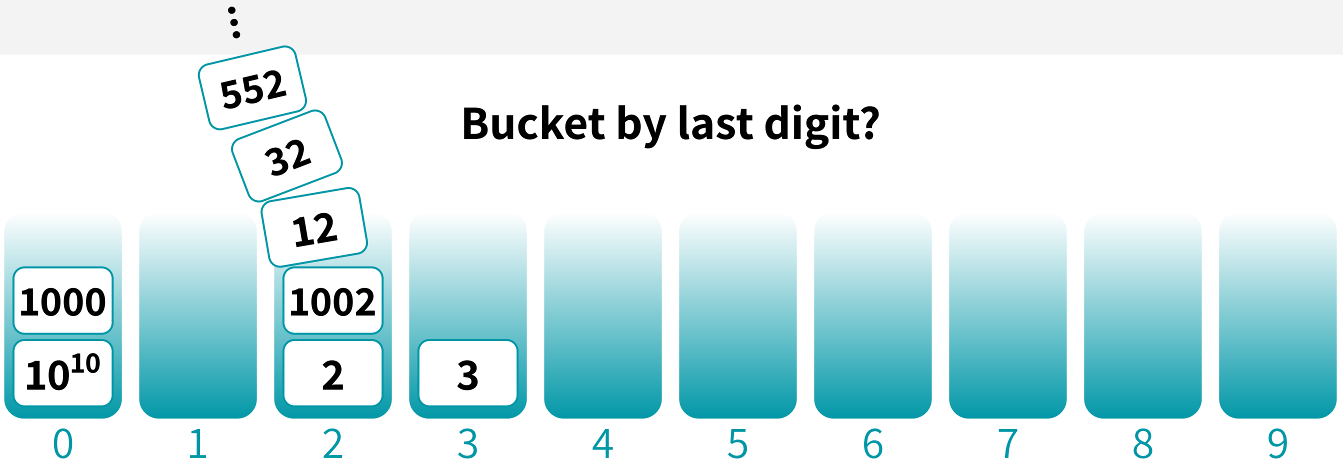

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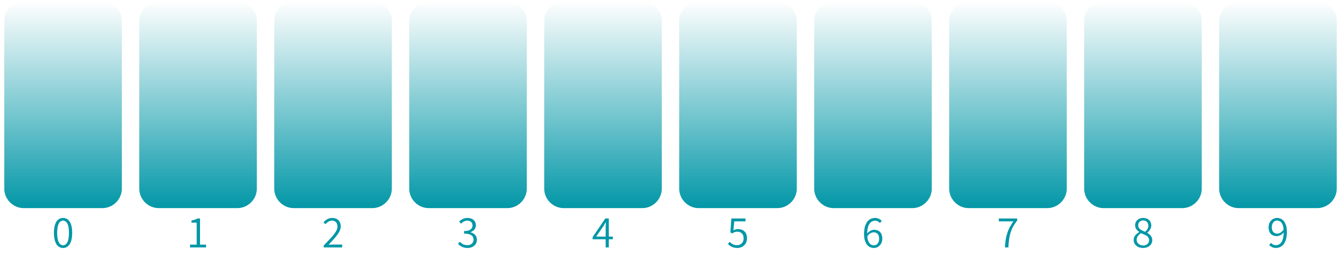
Maybe another bucketing scheme?

BUCKETING ATTEMPT 2:

Suppose you're storing numbers from 1 - 10^{10}: 2 | 3 | 1000 | 1002 | 10^{10}

Bucket by last digit of (number * 7) mod 3



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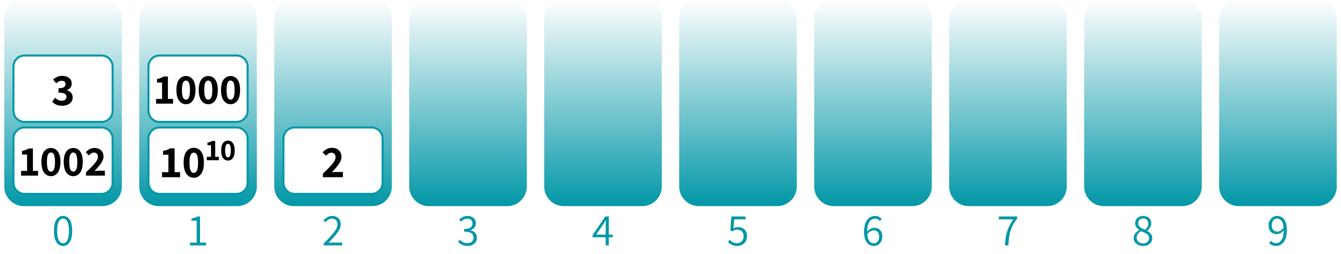

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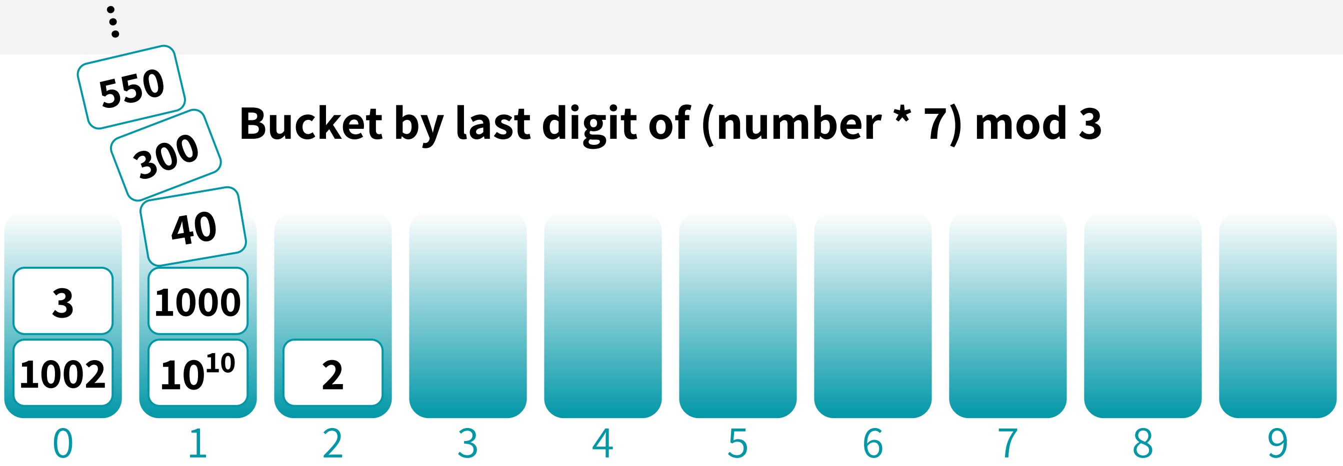

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Bucket by last digit of (number * 7) mod 3

Seems like a bad guy could still thwart us.
There are other bucketing schemes we could use,
so to reason about them more formally,
let's talk about **HASH FUNCTIONS.**

O(1) INSERT: Just index into the bucket (& insert at front of a linked list)!
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سوال؟

تابع درهم ساز

چه تابع درهم سازی خوب است؟

SOME TERMINOLOGY

There exists a universe **U** of keys, with size M.

Generally, M is *really big*. Examples:

- U = the set of all ASCII strings of length 20. M = 26^{20}
- U = the set of all IPv4 addresses. M = 2^{32}
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maps elements of U to buckets 1, ..., n

SOME TERMINOLOGY

Ther   e M.

- U =
- U =
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Our   >> n

Only a few (at mos   now up in advance.

NOTE:

For this lecture, I'm assuming that the # of elements I receive is the same as the # of buckets (both are **n**). This doesn't have to be the case, but we usually aim for
#buckets = O(# elements that show up)
(otherwise, we're using "too much" space)

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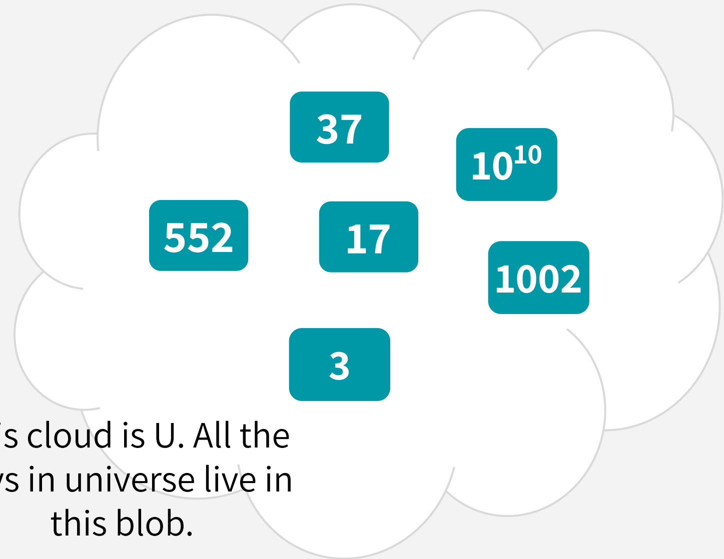
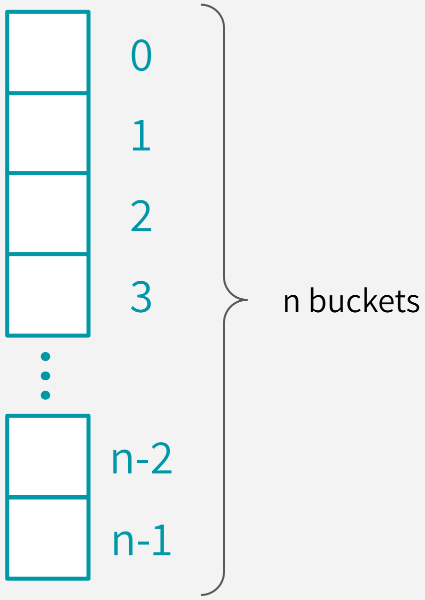


This cloud is U. All the keys in universe live in this blob.



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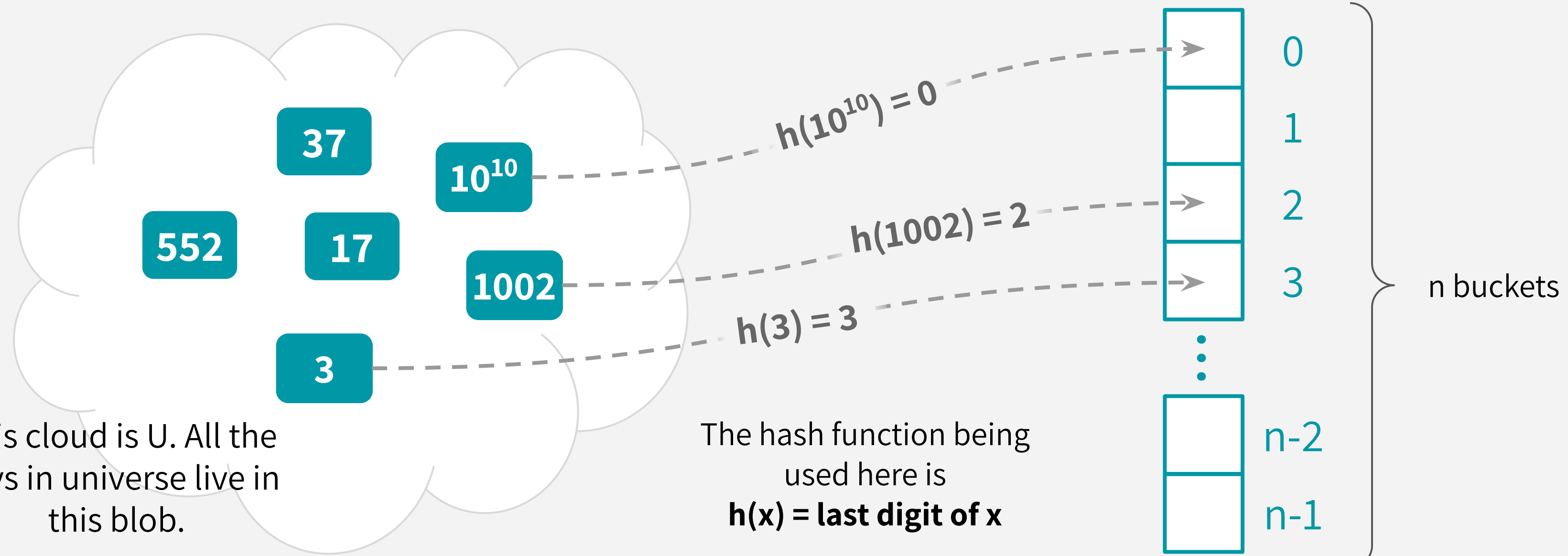

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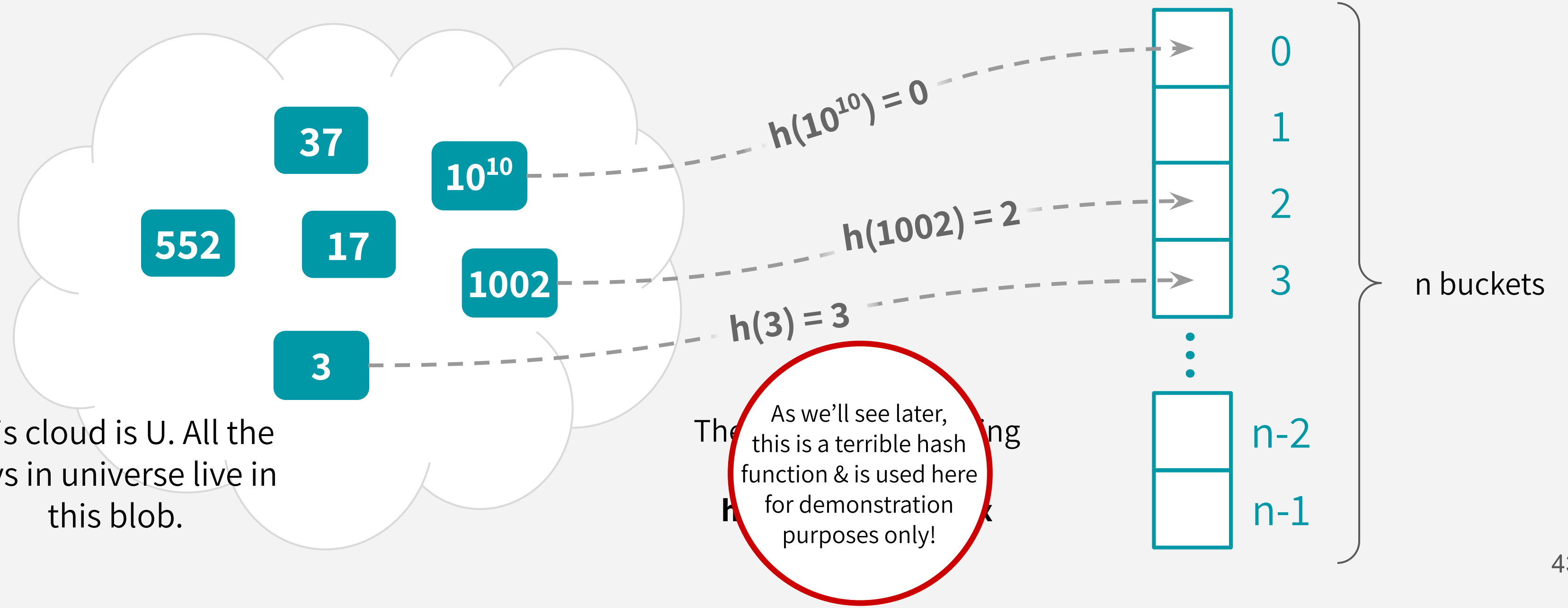

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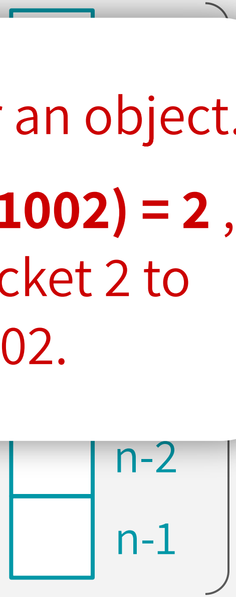
A hash function tells you where to start looking for an object.

For example, if a particular hash function **h** has **h(1002) = 2** , then we say "1002 *hashes* to 2", and we go to bucket 2 to search for 1002, or insert 1002, or delete 1002.



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سوال؟

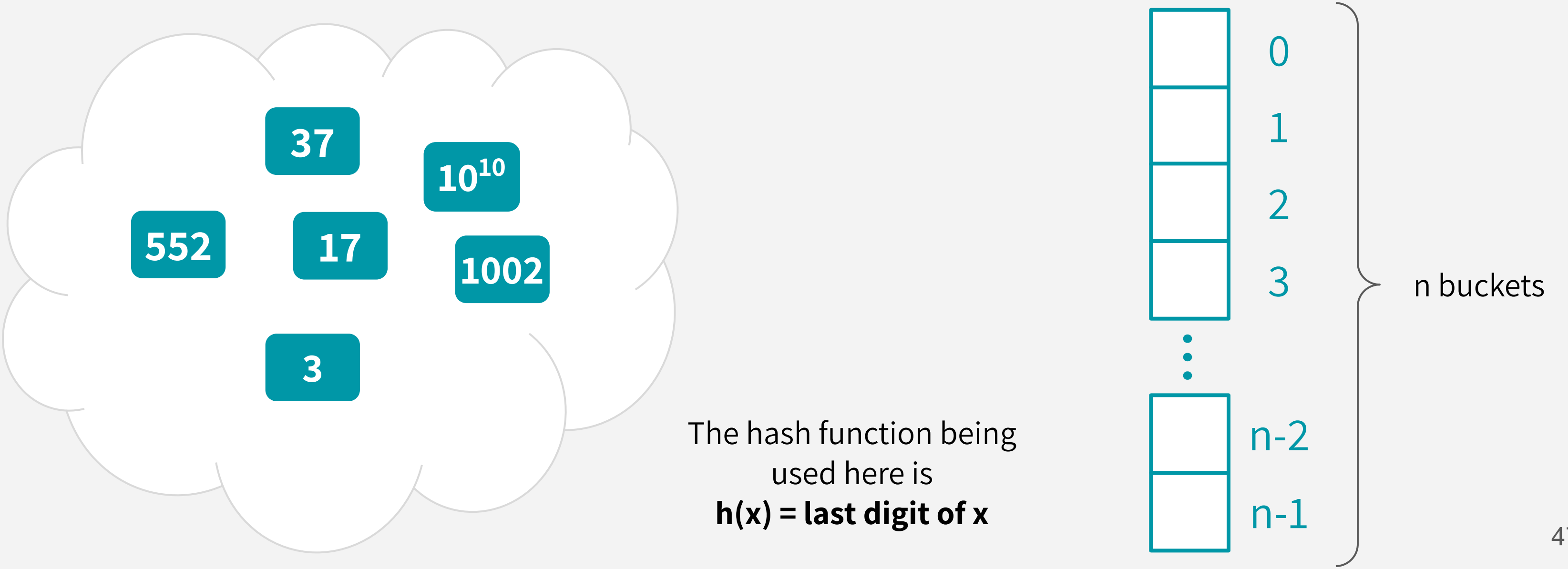
برخورد در درهم ساز!

چه مشکلی ممکن است پیش بیاید؟

COLLISIONS

Collisions (when a hash function would map 2 different keys to the same bucket) **are inevitable!**

This is because of the *Pigeonhole Principle*. Since the size of universe U > # of buckets, every hash function (no matter how clever), suffers from at least one collision.



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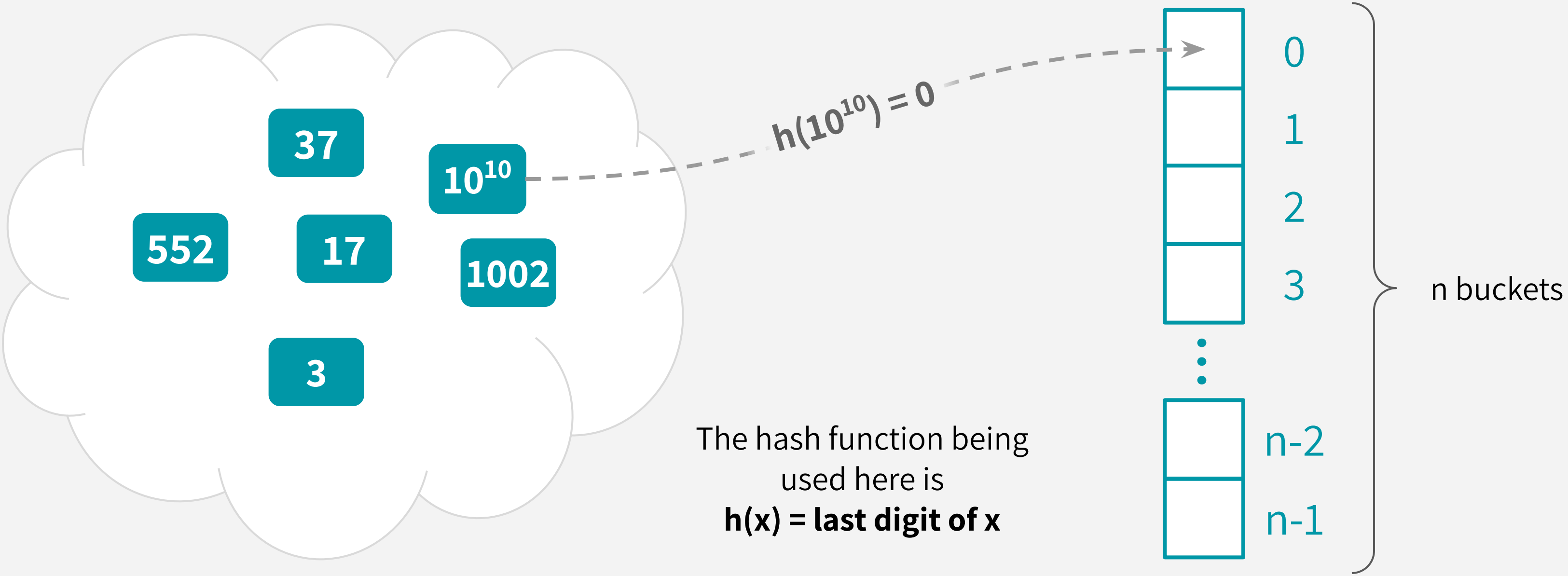

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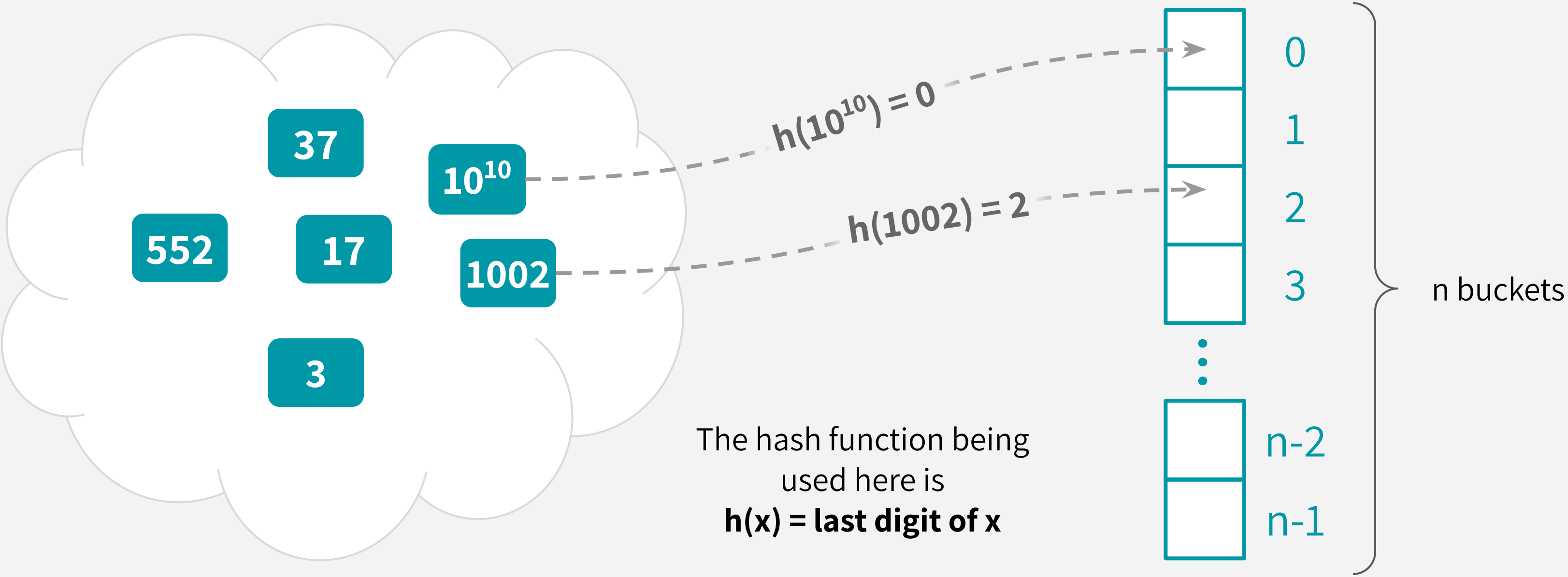

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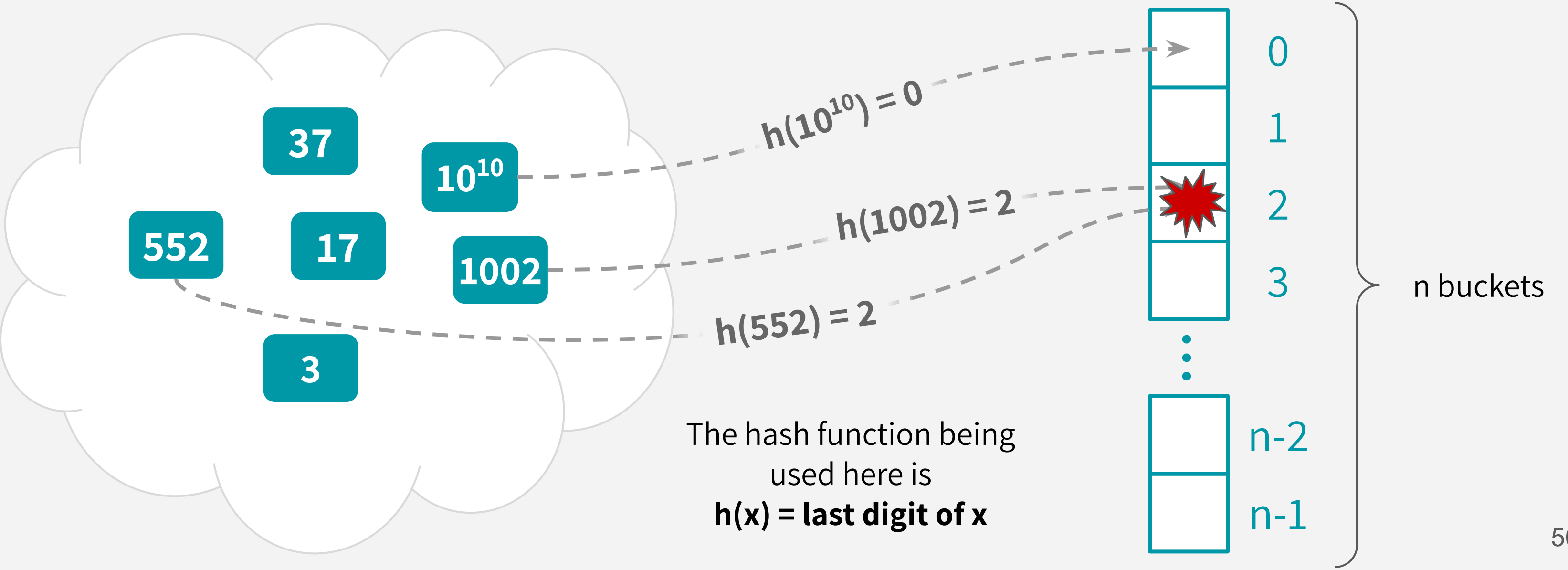

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COLLISION RESOLUTION: CHAINING

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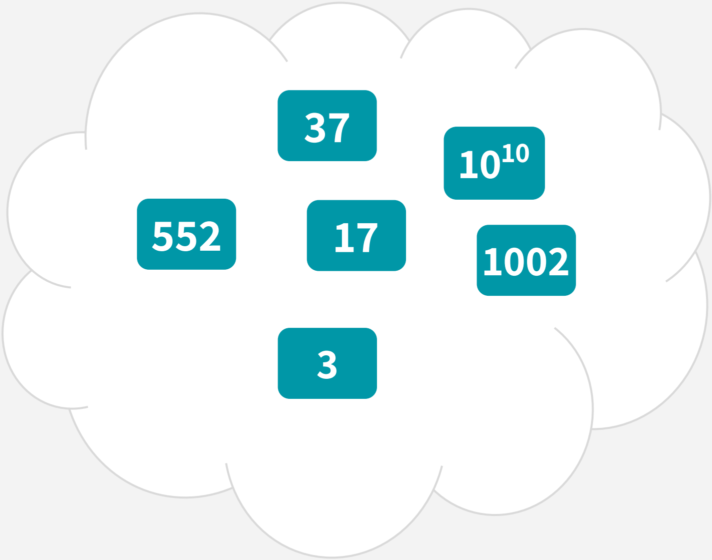

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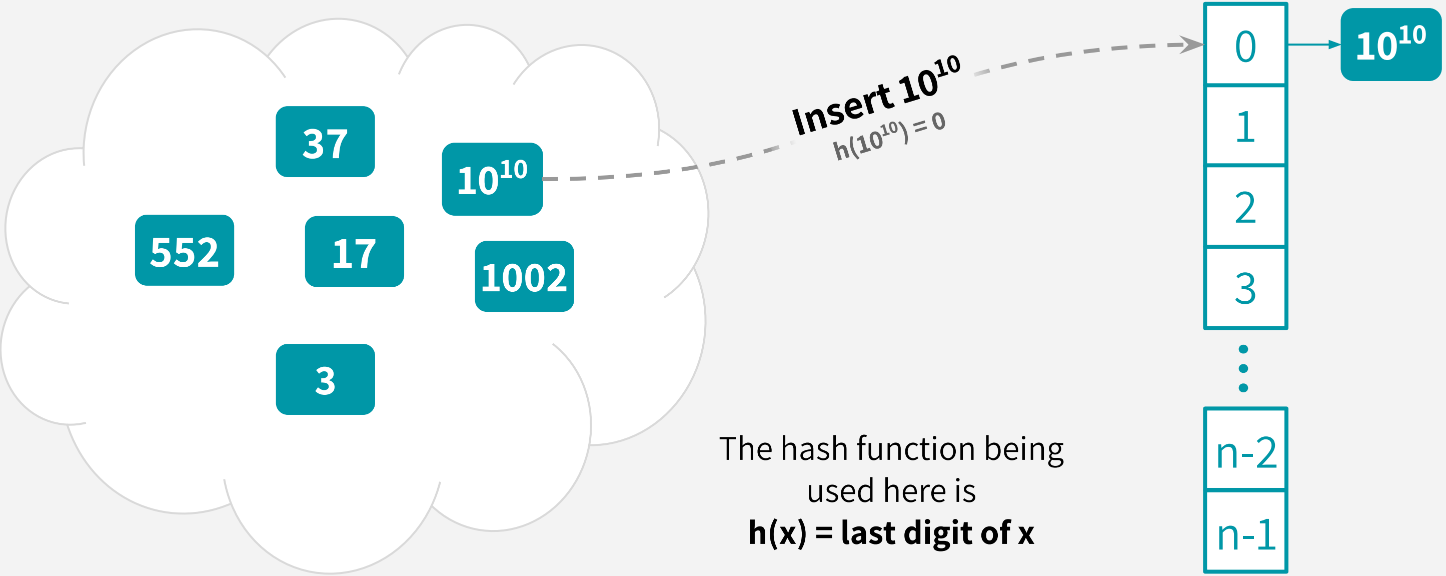

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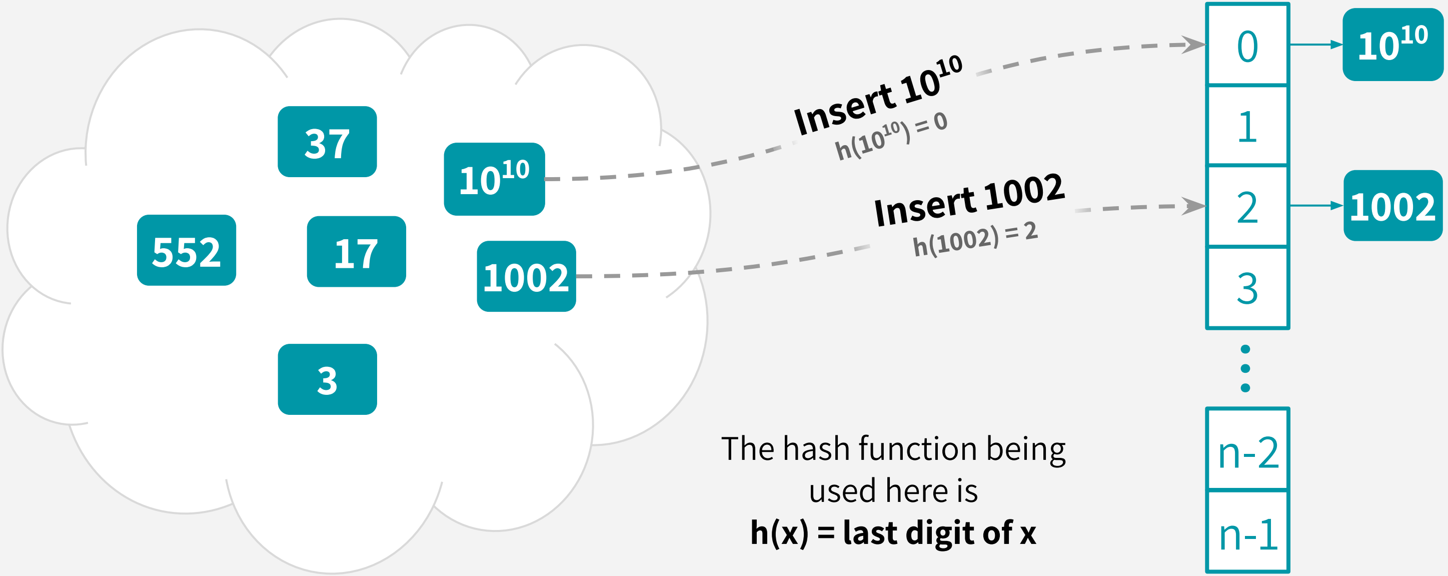

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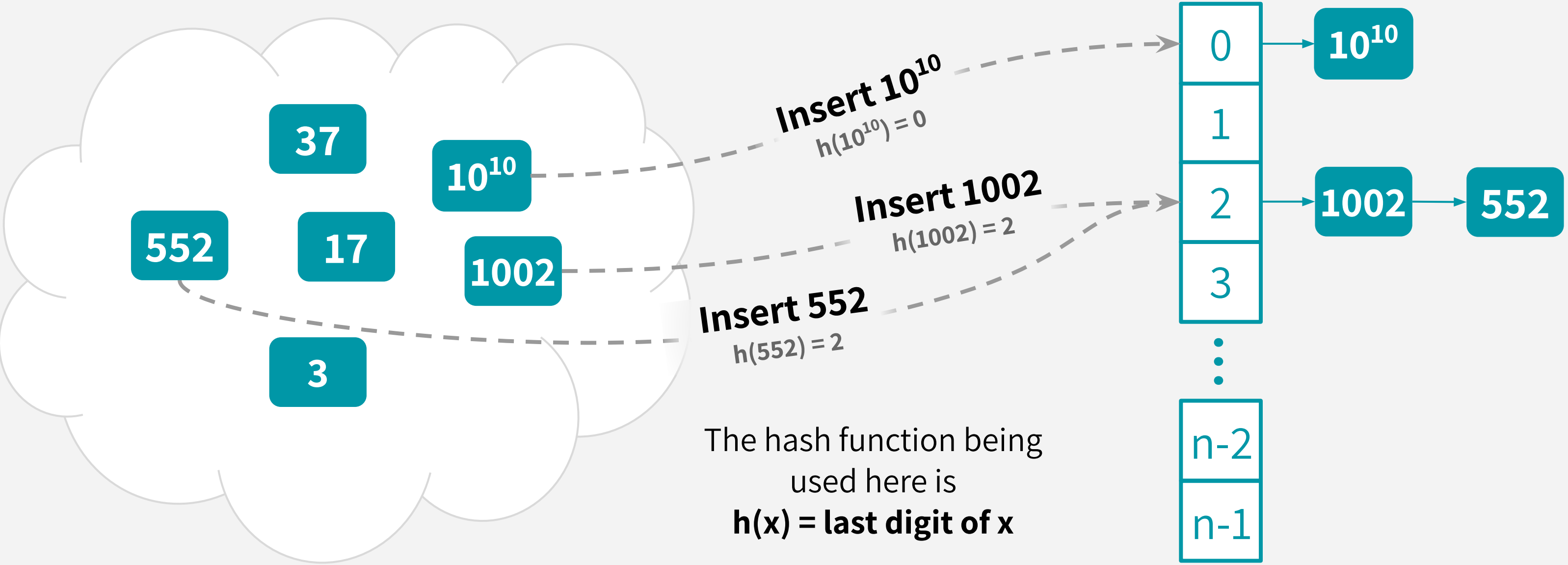

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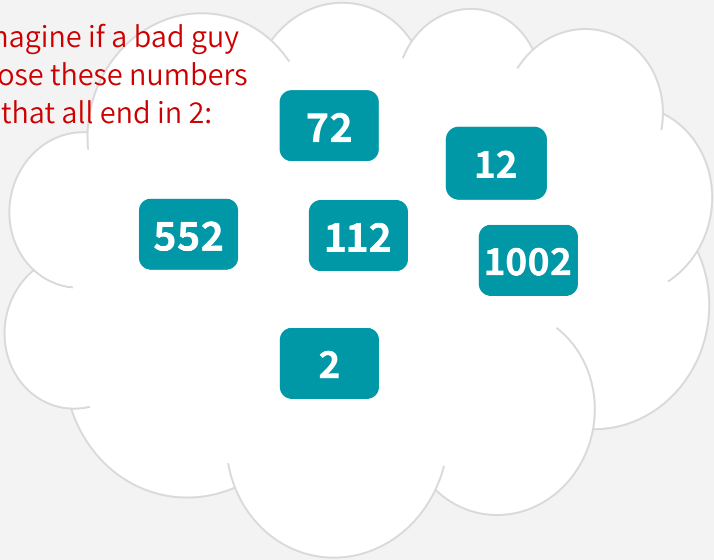

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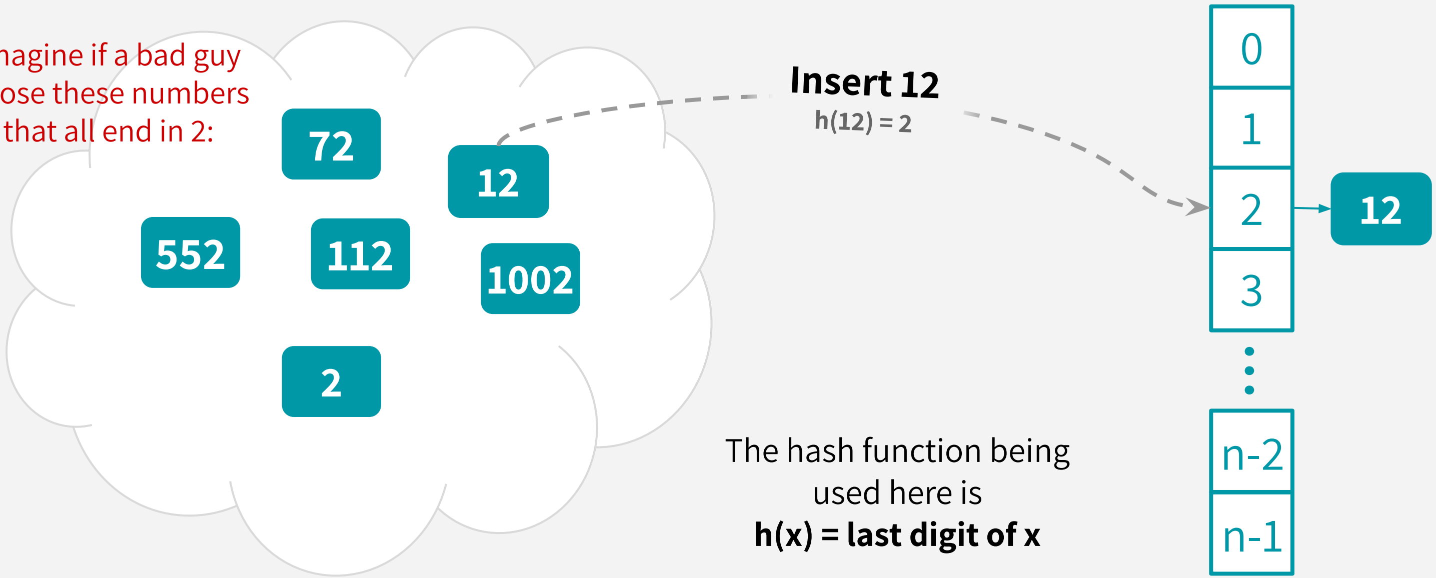

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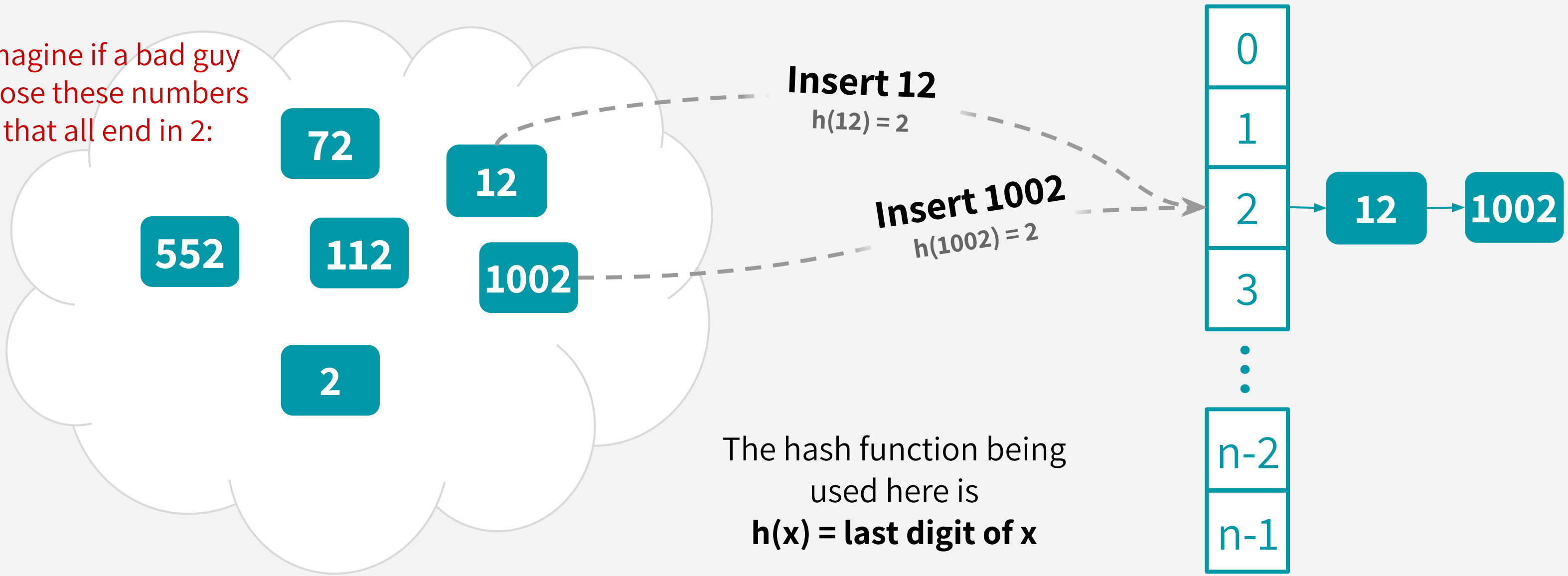

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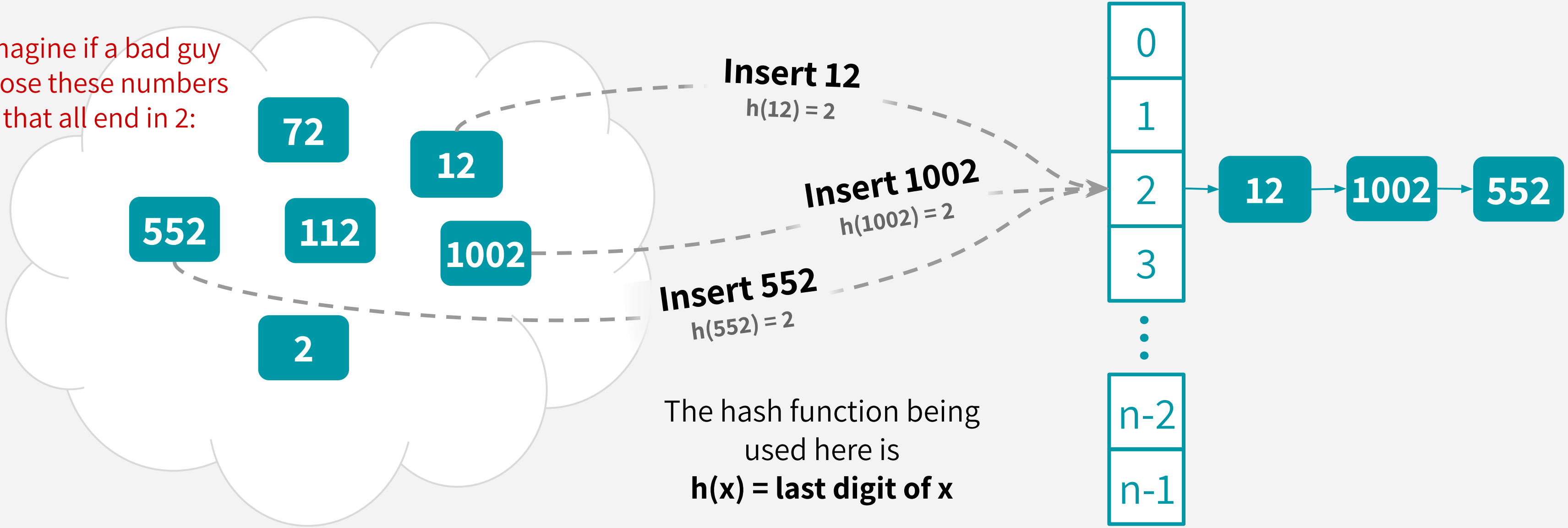

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سوال؟

درهم سازی اعلی!

چه درهم سازی خوب خواهد بود؟

HASH TABLE GOALS

Remember worst-case analysis:

***OUR GOAL*:** Design a function **h: U** $\rightarrow$ **{1, …, n}** so that no matter what **n** items of **U** a bad guy chooses & the operations they choose to perform, the buckets will be balanced.

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Can you come up with such a function?

HASH TABLE GOALS

BAD NEWS!

OUR GOAL: Design a function **h: U → {1, …, n}** so that no matter what **n** items of **U** a bad guy chooses, the buckets will be balanced (have O(1) size).

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RANDOMNESS!

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سوال؟

تصادفی کردن تابع درهم ساز

چگونه میتوان بدخواهان را تضعیف کرد؟

INTUITION

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You can think of it like a game:

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What would make a "good" set of hash functions H?

تابع درهم ساز خوب

معنی خوب بودن چیست؟

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Consider these two goals:

Which goal better represents what we want?

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SUPER IMPORTANT:

The randomness is over the choice of hash function **h** from a set of hash functions **H**.

You should *not* think of it as if you've chosen a fixed hash function and are thinking about randomness over possible items the adversary could choose, or randomness over the n possible buckets in your table, or randomness over the M possible items, or anything like that.

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Then, **E[# items in bucket i] = 1 = O(1)** for all i…

Bucket i has **n** elements with prob. 1/n, and 0 elements with prob. (n-1)/n

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WHAT DOES "GOOD" MEAN?

An analogy to explain the difference between the two:

Suppose a university offers 10 classes. 9 classes have only 1 student in them, and 1 class has 491 students.
Using the reasoning on the left, the university might say "Average class size is 50!" but in reality, it should instead report class sizes experienced by the average student (~482).

Desig… whe… that i… and a…

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WHAT WE WANT

Design a set $\mathbf{H} = \{h_1, h_2, h_{3,} \ldots, h_k\}$ where $h_i : U \rightarrow \{1, \ldots, n\}$, such that if we chose a random **h** in **H** and after an adversary chooses **n** items $\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_n}\}$ to hash,

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خانواده درهم سازی سراسری

مجموعه ای خوب از توابع درهم سازی که خیلی هم بزرگ نیست!

UNIVERSAL HASH FAMILY

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when **h** is chosen uniformly at random from **H**,

$$\text{for all } u_i, u_j \in U \text{ with } u_i \neq u_j,$$

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Then if we randomly choose **h** from a universal hash family **H**, we'll be guaranteed that:

E[# of items in u_i's bucket] ≤ 2 = O(1)

(OVERVIEW OF THE MATH)

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$$\begin{aligned}
\mathbb{E}[\text{\# of items in } u_i \text{ 's bucket}] &= \sum_{j=1}^{n} P[h(u_i) = h(u_j)] \\
&= P[h(u_i) = h(u_i)] + \sum_{j \neq i} P[h(u_i) = h(u_j)] \\
&= 1 + \sum_{j \neq i} P[h(u_i) = h(u_j)] \\
&\leq 1 + \sum_{j \neq i} \frac{1}{n} \\
&= 1 + \frac{n-1}{n} \leq 2
\end{aligned}$$

This inequality is now what a universal hash family guarantees!

O(1)

This is what we wanted!

AN EXAMPLE

Here is one of the more well-studied universal hash families:

Pick a prime $\mathbf{p \geq M}$

Define $\mathbf{h_{a,b}(x) = ((ax + b) \bmod p) \bmod n}$

$$\mathbf{H = \{\, h_{a,b} : \ a \in \{1, \ldots, p - 1\},\ b \in \{0, \ldots, p - 1\} \,\}}$$

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Example: Suppose n = 3, and p = 5. Here's $\mathbf{h_{2,4}}$:

$\mathbf{h_{2,4}(1) =}$ ((2*1 + 4) mod 5) mod 3 = (6 mod 5) mod 3 = 1 mod 3 = **1**

$\mathbf{h_{2,4}(4) =}$ ((2*4 + 4) mod 5) mod 3 = (12 mod 5) mod 3 = 2 mod 3 = **2**

$\mathbf{h_{2,4}(3) =}$ ((2*3 + 4) mod 5) mod 3 = (6 mod 5) mod 3 = 1 mod 3 = **1**

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To draw a hash function **h** from **H**:

Pick a random **a** in {1, …, p - 1}.

&

Pick a random **b** in {0, …, p - 1}.

AN EXAMPLE

To store your $\mathbf{h_{a,b}}$, you just need to store two numbers: **a** and **b**!

Since **a** and **b** are at most p-1, we need **~2·log(p) bits**.

p is a prime that's close-ish to M, so this means the space needed =

O(log M)

Pick a random **a** in {1, …, p - 1}. & Pick a random **b** in {0, …, p - 1}.

AN EXAMPLE

Claim: This **H** is a universal hash family!

The proof is a bit complicated, and relies on number theory. See CLRS (Theorem 11.5) for details if you're curious, but **YOU ARE NOT RESPONSIBLE** for the proof in this class.

What you should know:

There exists a small universal hash family! A hash function from this universal hash family is quick to compute, lightweight to store, and relies on number theory to achieve our expected O(1) operation costs!

سوال؟

جدول درهم سازی

جمع بندی مطالب درهم سازی و استفاده عملی از آن!

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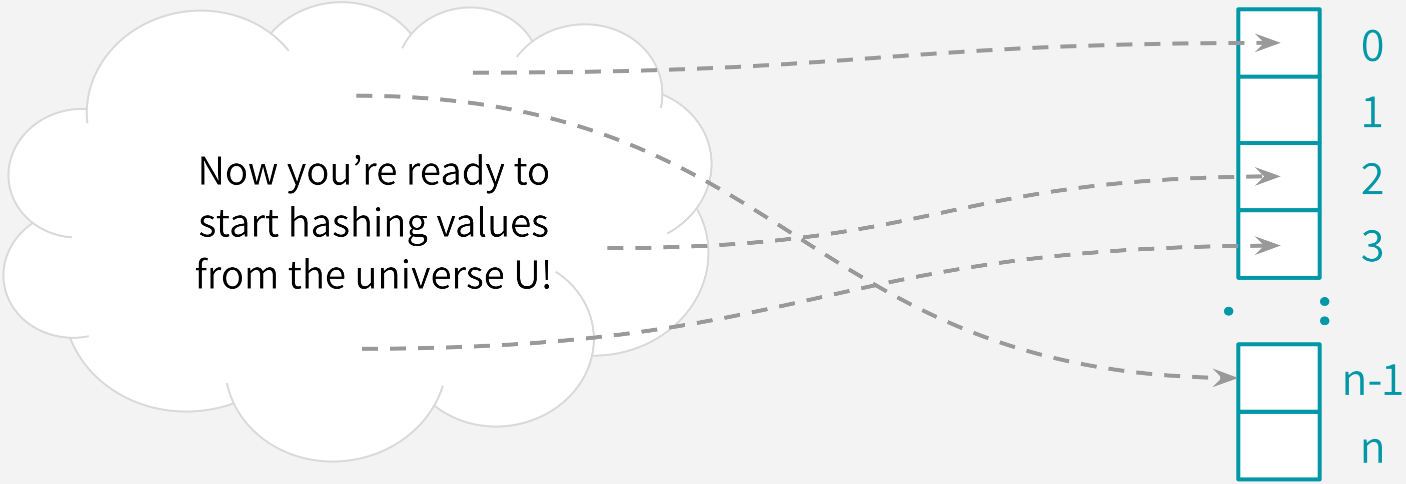

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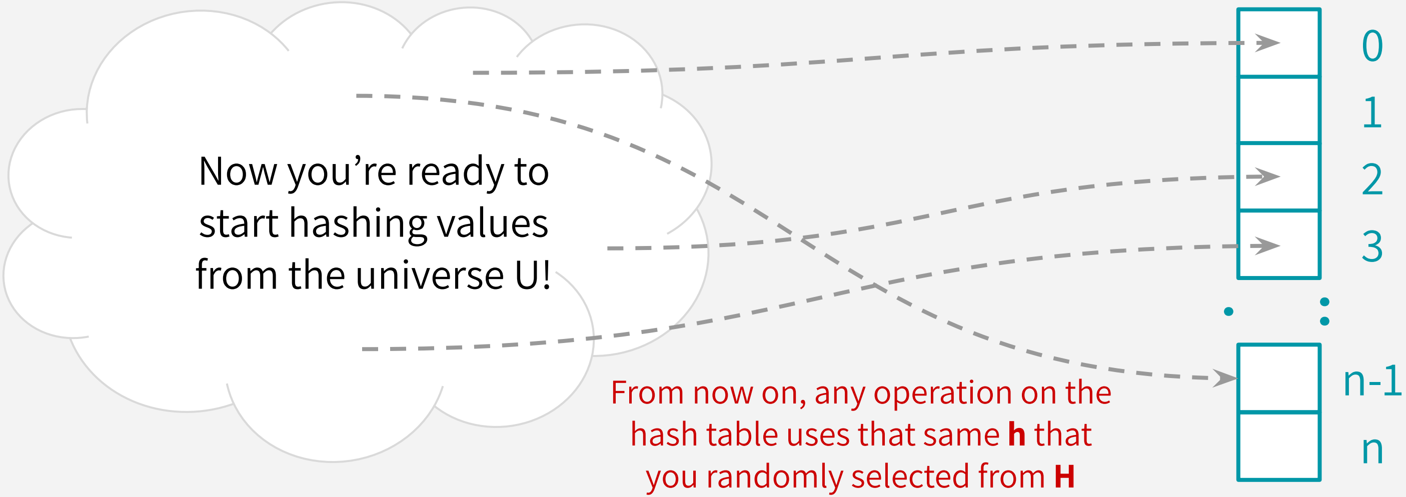

We can now expect that these buckets will be pretty balanced

From now on, any operation on the hash table uses that same **h** that you randomly selected from **H**

HASH TABLE MOTIVATION

OPERATION	SORTED ARRAY	UNSORTED LINKED LIST	HASH TABLES (WORST-CASE)	HASH TABLES (EXPECTED)*
SEARCH	O(log(n))	O(n)	O(n)	O(1)
DELETE	O(n)	O(n)	O(n)	O(1)
INSERT	O(n)	O(1)	O(1)	O(1)

*** Assuming we implement it cleverly with a “good” hash function**

سوال؟