ساختمان داده و الگوریتم ها

مبحث چهاردهم:
درخت قرمز-سیاه

سجاد شیرعلی شهرضا

پاییز 1402

شنبه، 4 آذر 1402

Based on Stanford CS161 Slides from Summer 2020 by Karey Shi

اطلاع رسانی

- امتحان میان ترم
  - روز دوشنبه، در ساعت کلاس
- بخش مرتبط کتاب برای این جلسه: 13

سوال؟

درخت جستجوی متوازن

چگونه درخت جستجو را متوازن نگه داریم؟

SELF-BALANCING SEARCH TREES

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ROTATIONS

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Note: going forward, we're going to focus on rotations for BINARY search trees (BSTs).

ROTATIONS

IDEA: locally rebalance a node's subtree in O(1) time while maintaining BST property

LEFT ROTATION

RIGHT ROTATION

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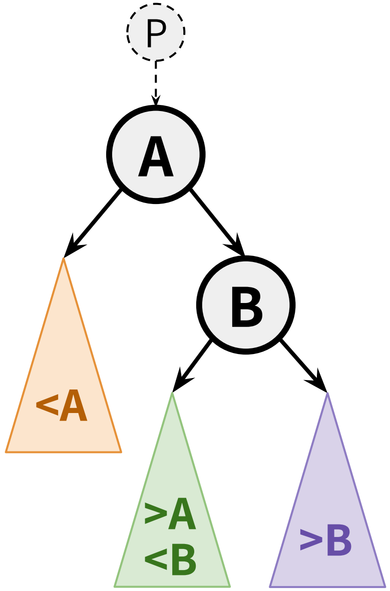

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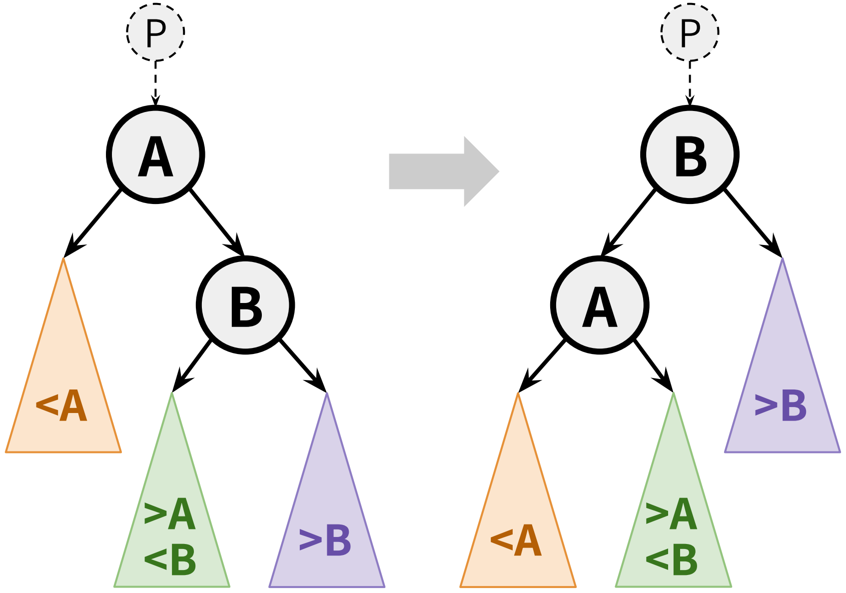

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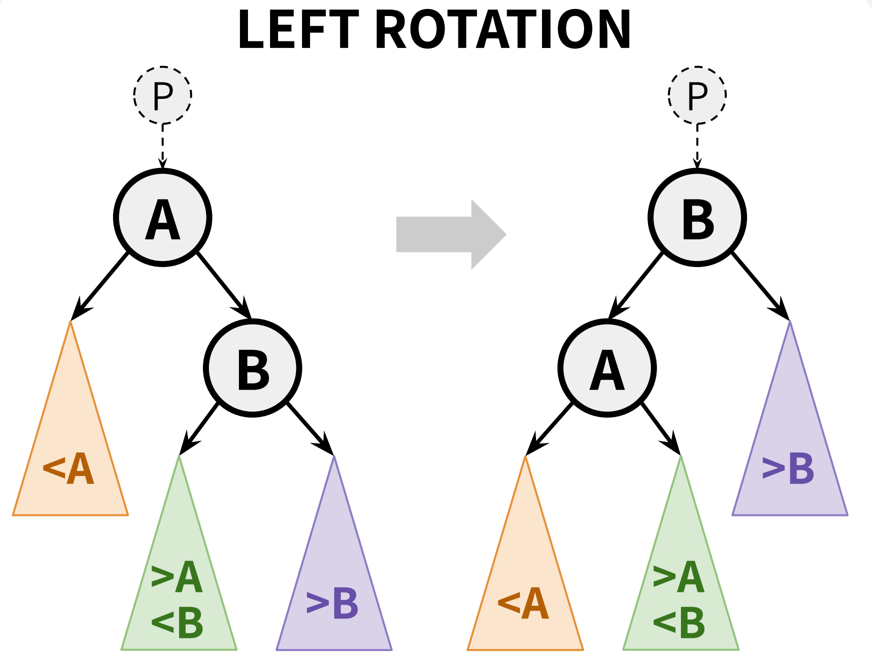

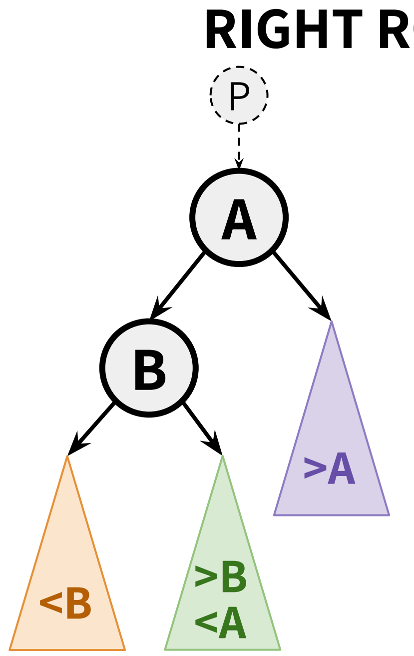

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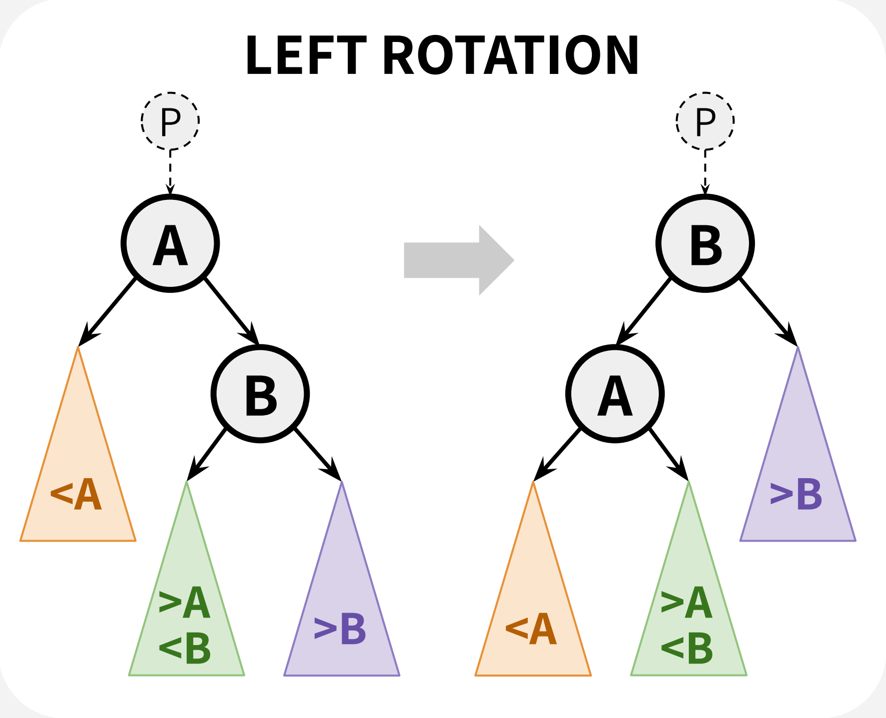

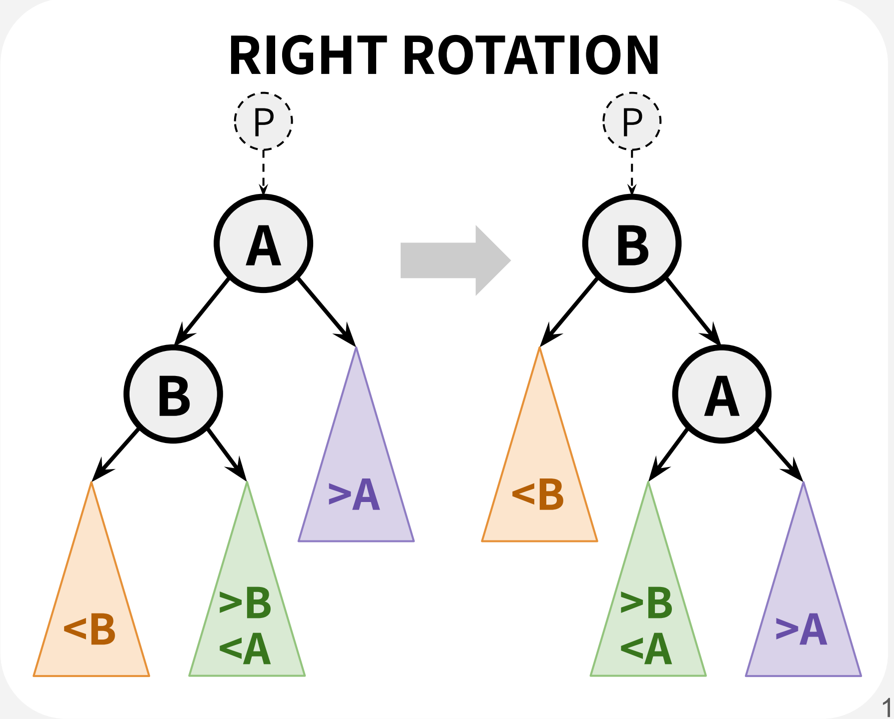

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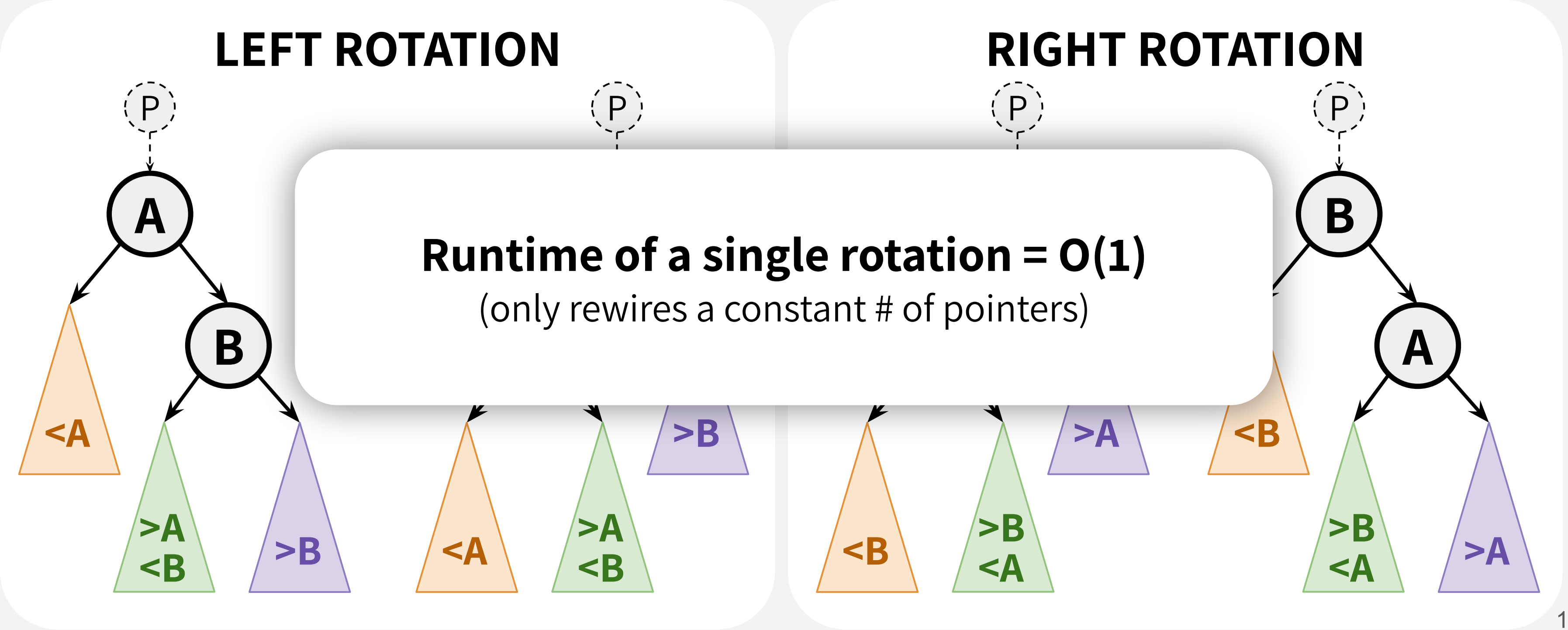

سوال؟

درخت قرمز-سیاه

یک نمونه معروف از درخت‌های جستجوی متوازن

ROTATIONS

When and how do we apply these rotations?

Let’s explore one type of self-balancing BST:

RED-BLACK TREES!

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Let's look at some examples & non-examples!

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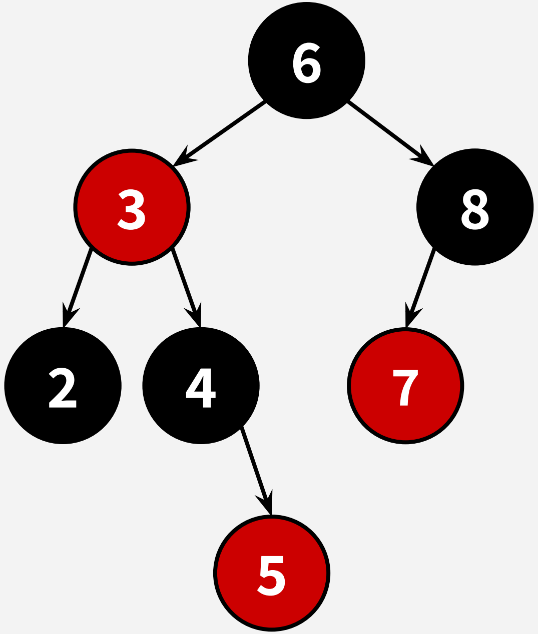

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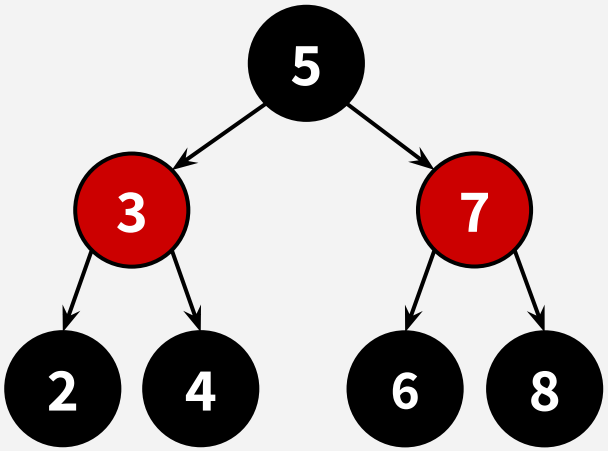

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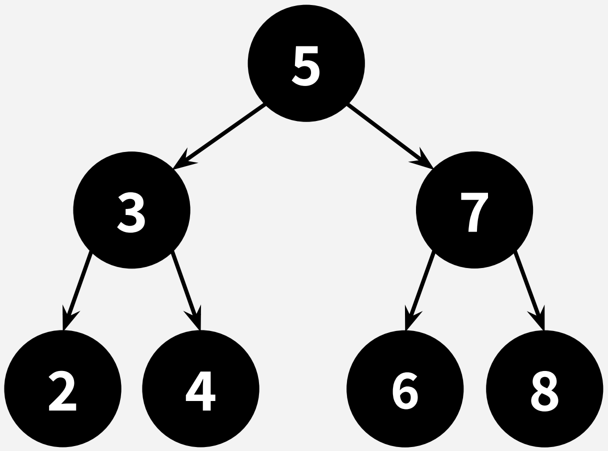

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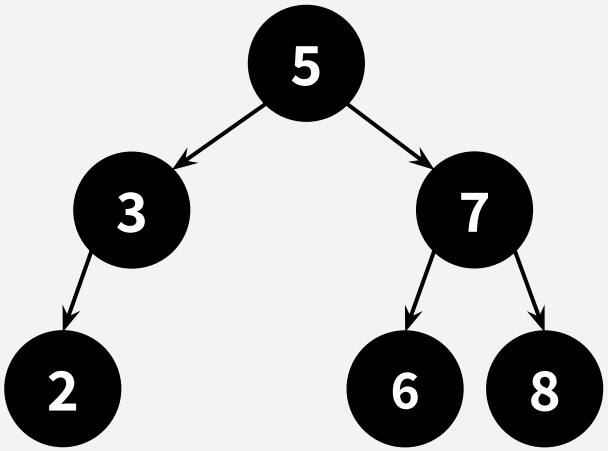

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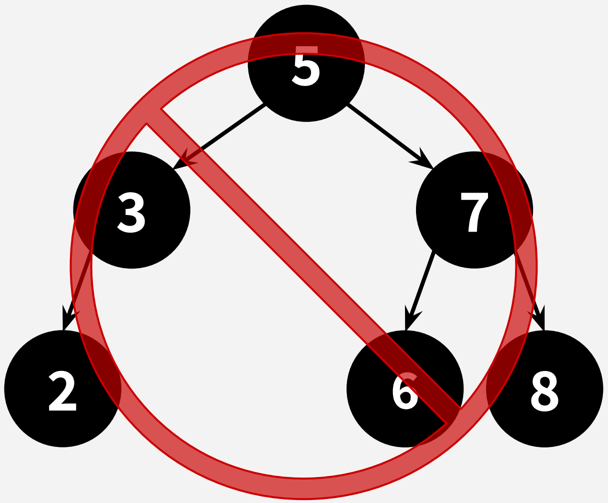

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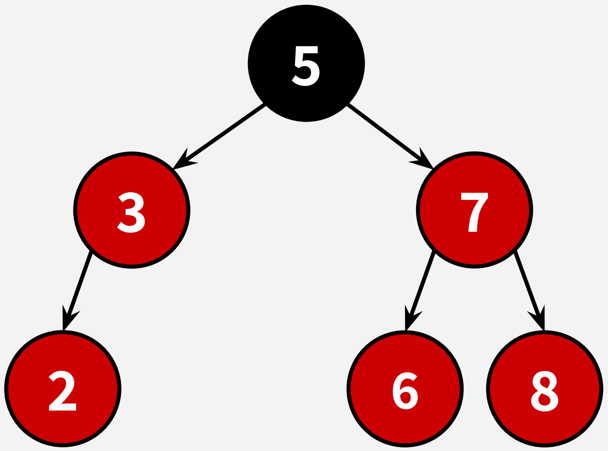

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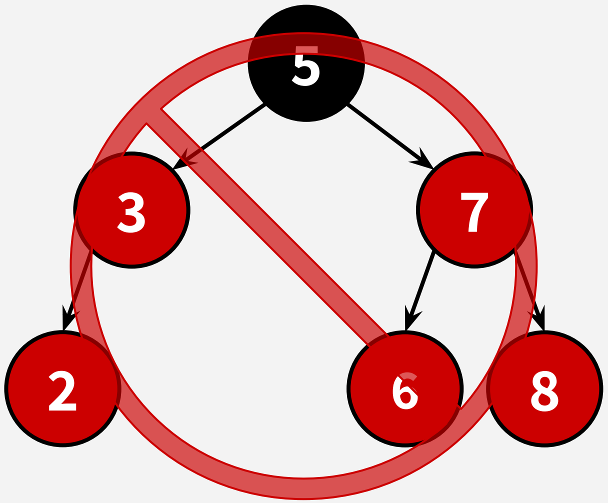

سوال؟

ارتفاع درخت قرمز-سیاه

مزیت ویژگی‌های بیان شده برای درخت قرمز-سیاه چیست؟

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The **black** nodes are ~balanced across the tree.

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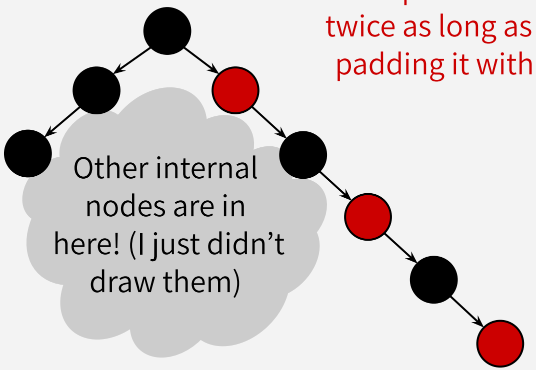

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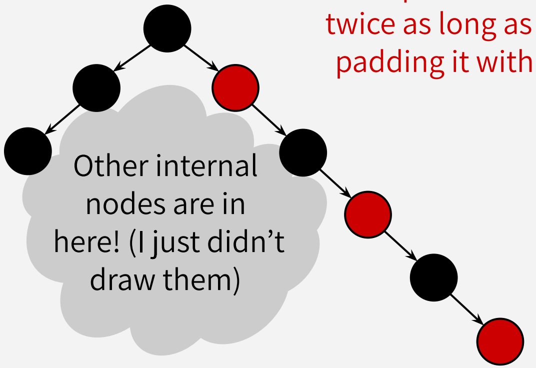

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More formally…

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PROOF IDEA: We can show that any RB tree with **n** nodes has height $\leq 2 \cdot \log_2(n+1)$

O(log n) HEIGHT GUARANTEE (PROOF)

(RB TREE PROPERTIES)

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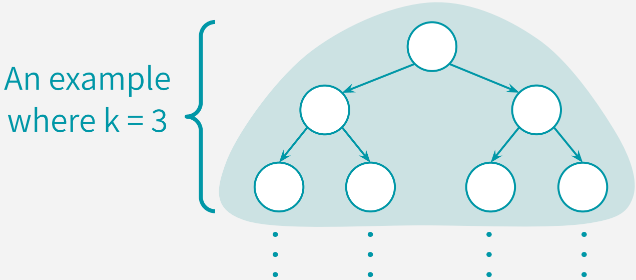

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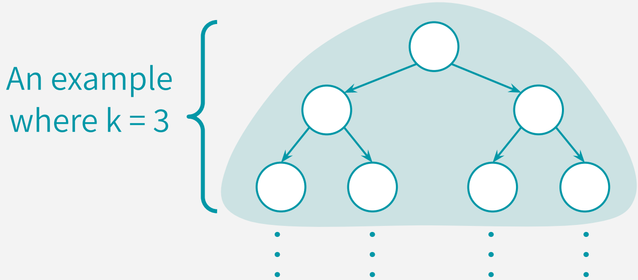

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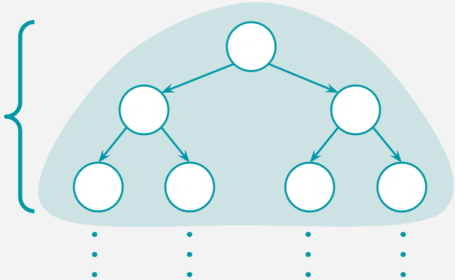

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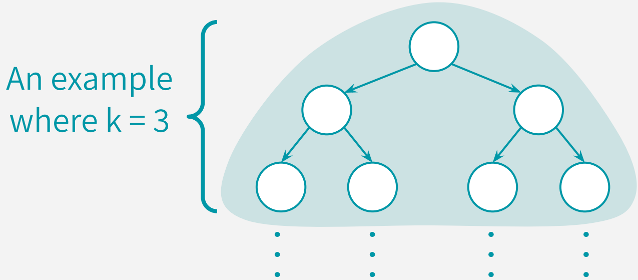

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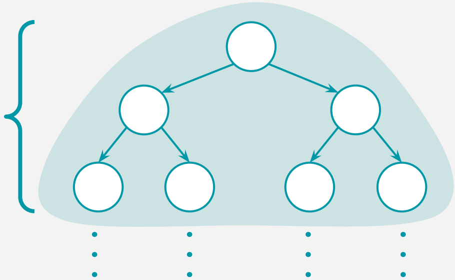

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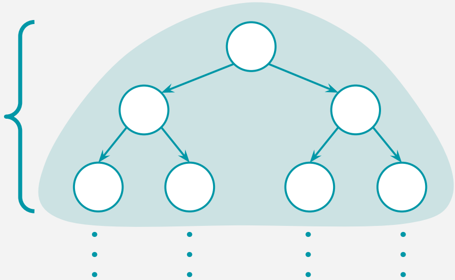

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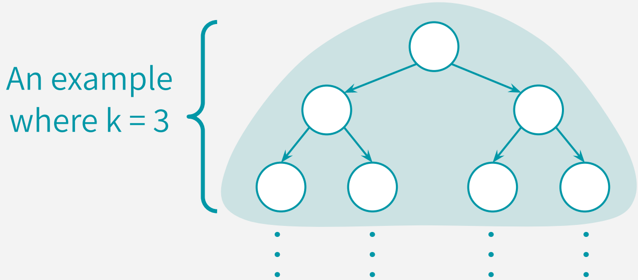

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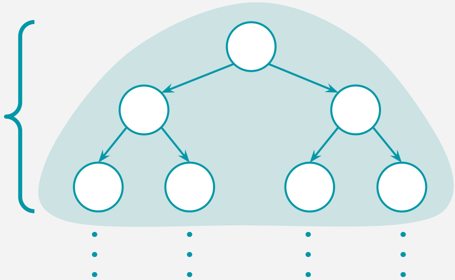

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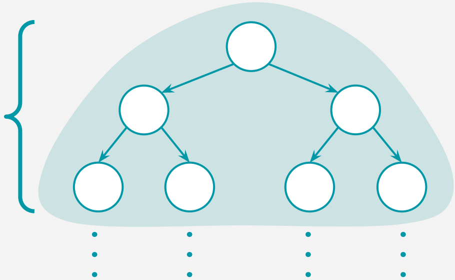

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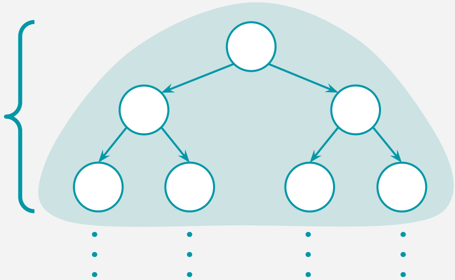

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By PROPERTY 4: *every* root-NIL path has $\leq \log_2(n+1)$ **black** nodes on it.

By PROPERTY 3: *every* root-NIL path has $\leq 2 \cdot \log_2(n+1)$ total nodes on it.

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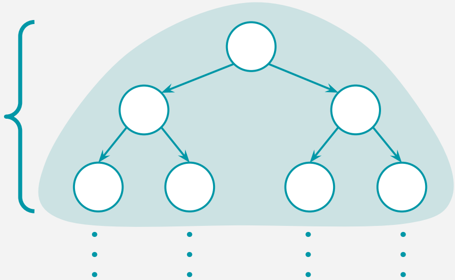

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i.e. $\mathbf{k \leq \log_2(n+1)}$

(RB TREE PROPERTIES)

1. Every node is either **red** or **black**
2. The root is a **black** node
3. No **red** node has a **red** child
4. Every root-NIL path has the same number of **black** nodes on them

This means there must exist some root-NIL path that has $\leq \log_2(n+1)$ nodes on it.

if all root-NIL paths had > $\log_2(n+1)$ nodes, then k wouldn't be upper bounded by $\log_2(n+1)$

Consequently, this path must have $\leq \log_2(n+1)$ **black** nodes on it.

By PROPERTY 4: *every* root-NIL path has $\leq \log_2(n+1)$ **black** nodes on it.

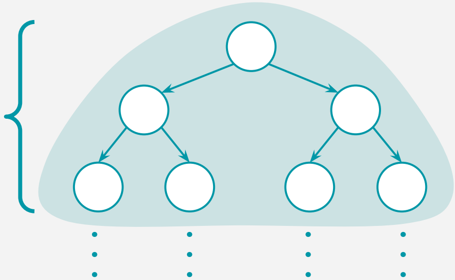
By PROPERTY 3: *every* root-NIL path has $\leq 2 \cdot \log_2(n+1)$ total nodes on it.

Red and black nodes must alternate, so the # of red nodes on the path is at most the # of black nodes

O(log n) HEIGHT GUARANTEE (PROOF)

First, suppose every root-NIL path has ≥ **k** nodes. Then the top part of the RB tree must contain a perfectly balanced BST of height **k - 1**.

An example where k = 3



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O(log n) HEIGHT GUARANTEE (PROOF)

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the RB t

(RB TREE PROPERTIES)

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There's a lot going on, so here's how you should assess your understanding:

Properties 3 and 4 are the non-trivial rules.
Their purpose should ~intuitively make sense.

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سوال؟

تغییر درخت قرمز-سیاه

چگونگی اضافه و حذف کردن در درخت قرمز-سیاه

WHAT HAVE WE LEARNED?

Runtime of **SEARCH** in an BST Tree = **O(height)**

WHAT HAVE WE LEARNED?

The height of an RB Tree is O(log n).

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= O(log n)

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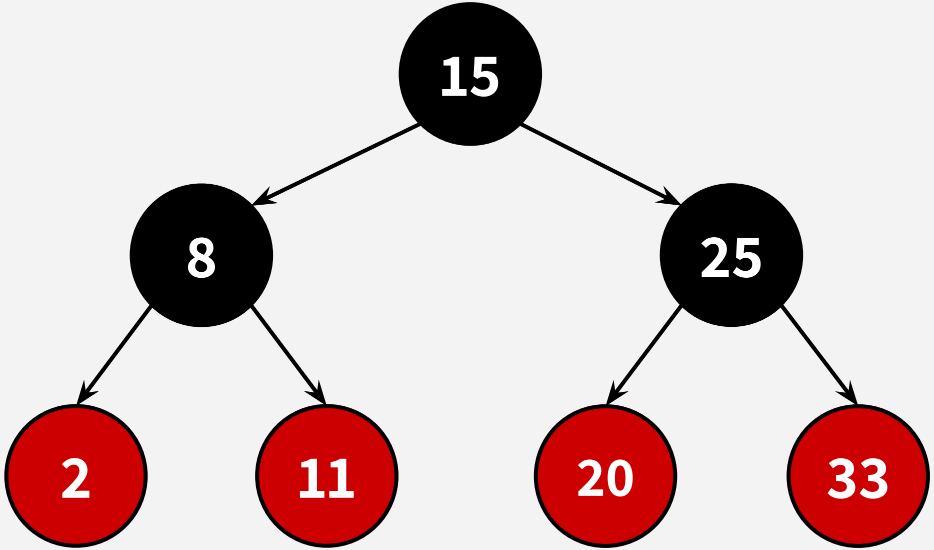
Runtime of **SEARCH** in an RB Tree = **O(height)**

= **O(log n)**

What about INSERT/DELETE?

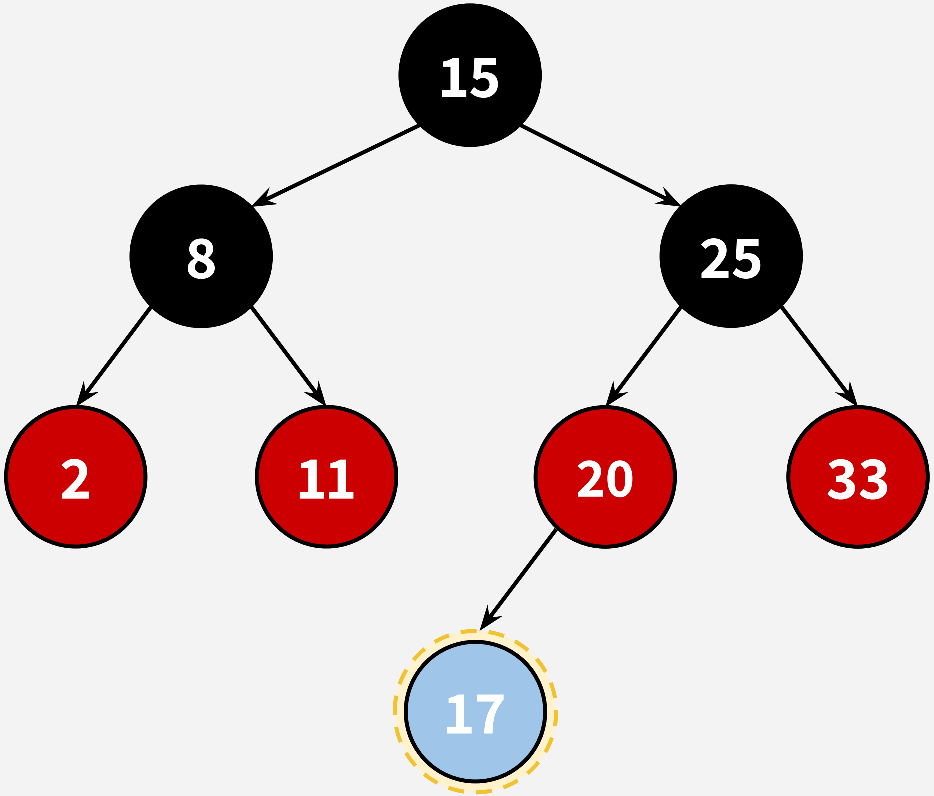
These are the two operations that actually modify the RB Tree, so we need to make sure that we insert & delete without violating our precious RB Tree properties...

INSERTING IN AN RB TREE



EXAMPLE: Insert 17.

INSERTING IN AN RB TREE



EXAMPLE: Insert 17.

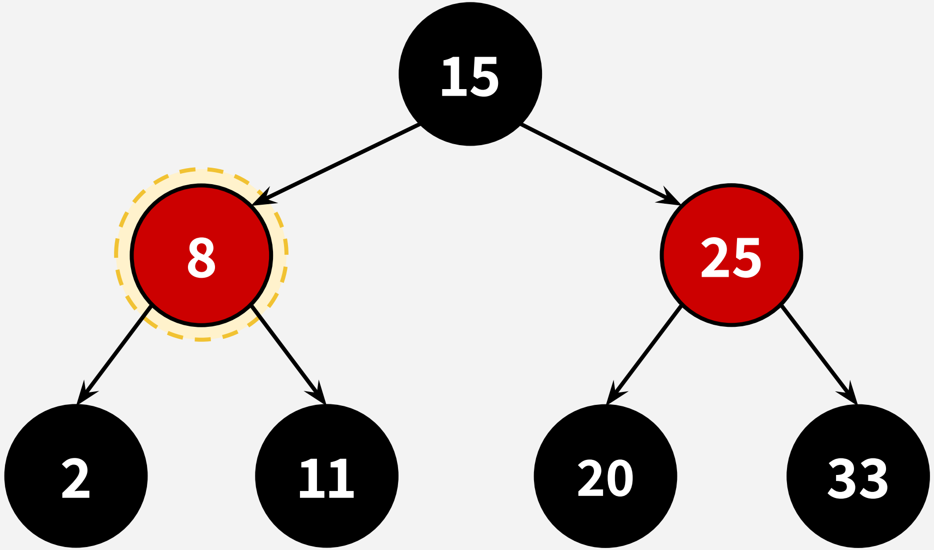
What do we do with 17?

Do we color it **red**?
Do we color it **black**?

Do we need to change the color of other nodes?

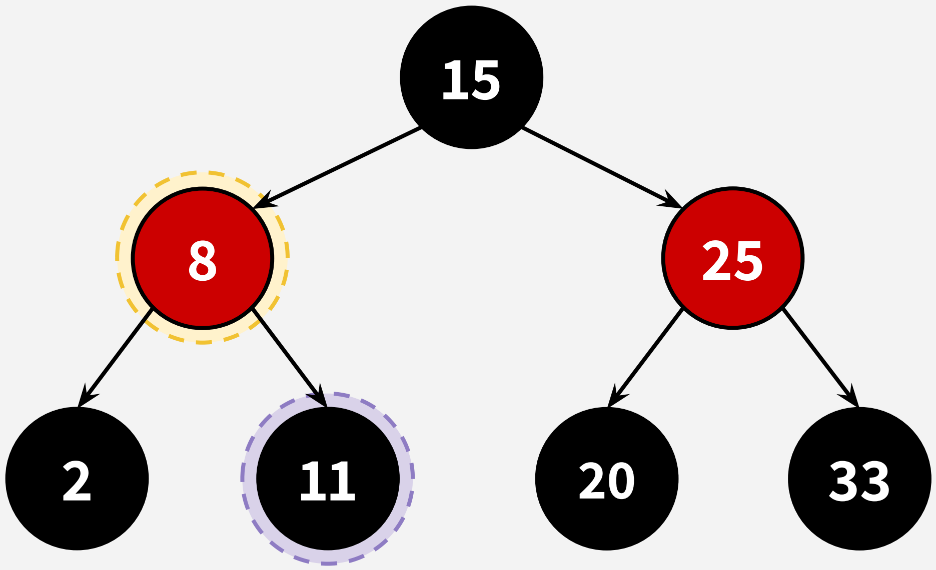
What if we insert 16 next?

DELETING FROM AN RB TREE



EXAMPLE: Delete 8.

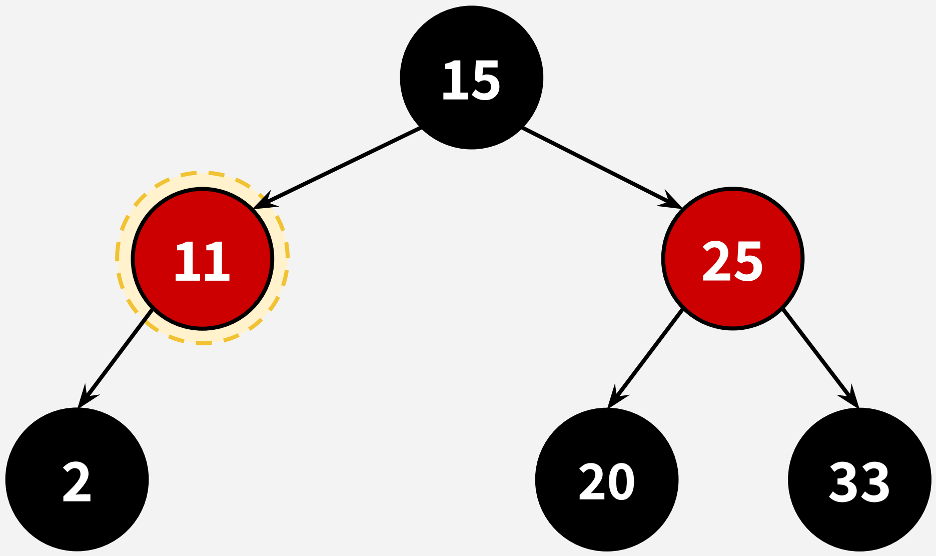
DELETING FROM AN RB TREE



EXAMPLE: Delete 8.

(replace with immediate successor)

DELETING FROM AN RB TREE



EXAMPLE: Delete 8.
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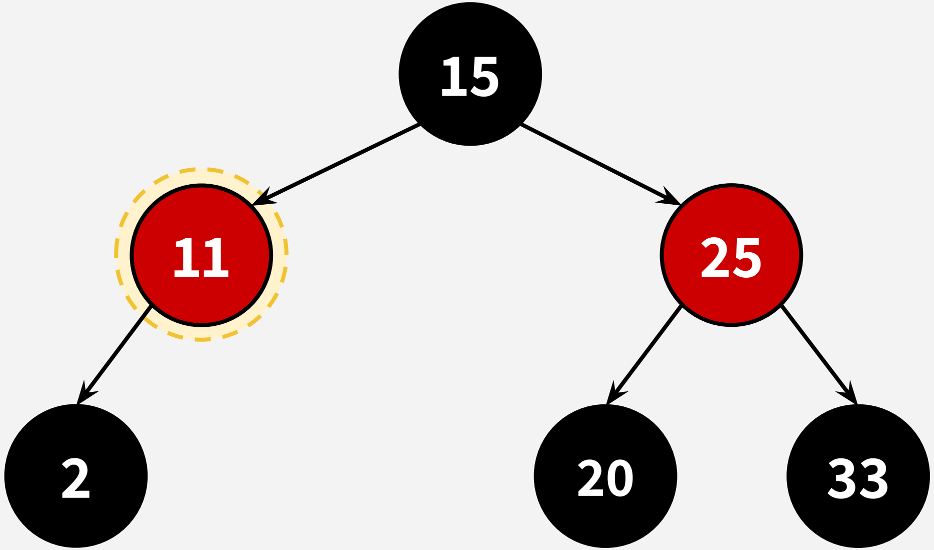
Now we've violated Property 4!
(all root-NIL paths must have the same # of black nodes)

How do we fix this up?

DELETING FROM AN RB TREE



EXAMPLE: Delete 8.
(replace with immediate successor)

Now we've violated Property 4!
(all root-NIL paths must have the same # of black nodes)

How do we fix this up?

Fixing up deletions is *complicated*. See CLRS Section 13.4 if you're curious!

سوال؟

درج در درخت قرمز-سیاه

چگونگی متوازن نگه داشتن درخت هنگام درج

INSERT IN RB TREES

High-level plan

Insert as normal (same insert as BST), and then fix.

Fix = recolor and/or apply rotations until RB Tree properties are met.

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INSERT(x):

- Insert **x** normally (**x** becomes a leaf)
- Color **x** **red**

INSERT IN RB TREES

High-level plan

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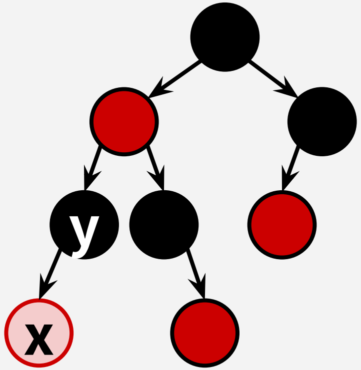
INSERT(x):

- Insert **x** normally (**x** becomes a leaf)
- Color **x** **red**
- If **x**'s parent **y** is **black**, then we're done!



All good :)

INSERT IN RB TREES

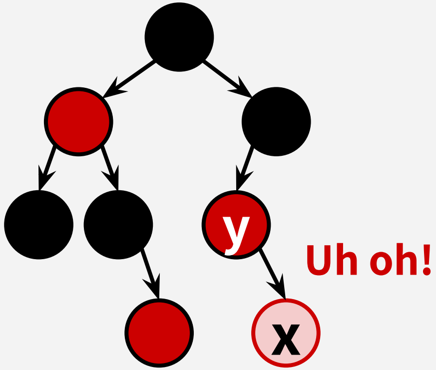
High-level plan

Insert as normal (same insert as BST), and then fix.

Fix = recolor and/or apply rotations until RB Tree properties are met.

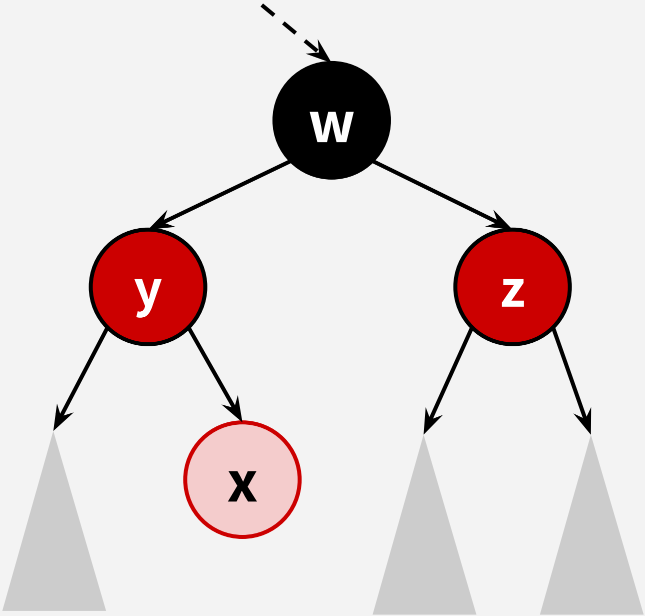
INSERT(x):

- Insert **x** normally (**x** becomes a leaf)
- Color **x** **red**
- If **x**'s parent **y** is **black**, then we're done!
- Otherwise, **y** is **red**, so we have two red nodes in a row and need to do some fixing!



FIXING THINGS UP: CASE 1

CASE 1: parent **y** is **red**, and “uncle” **z** is **red** too!



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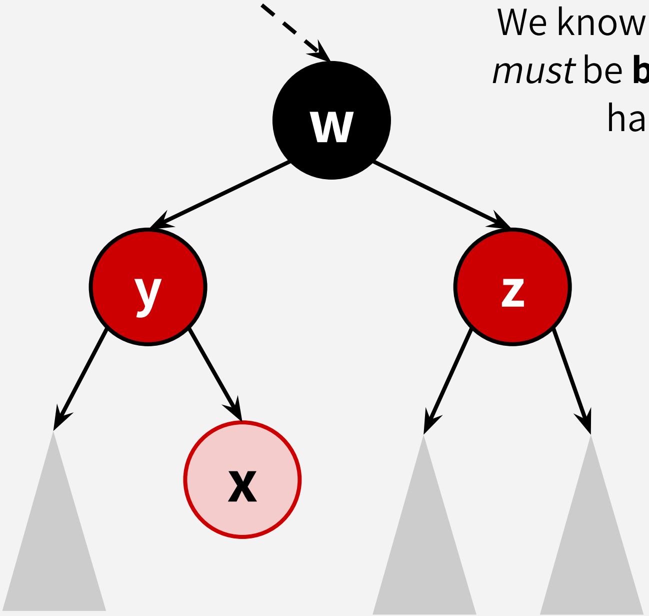


We know **w** (x’s grandparent & y’s parent) *must* be **black** because we could not have had two **red** nodes in a row.

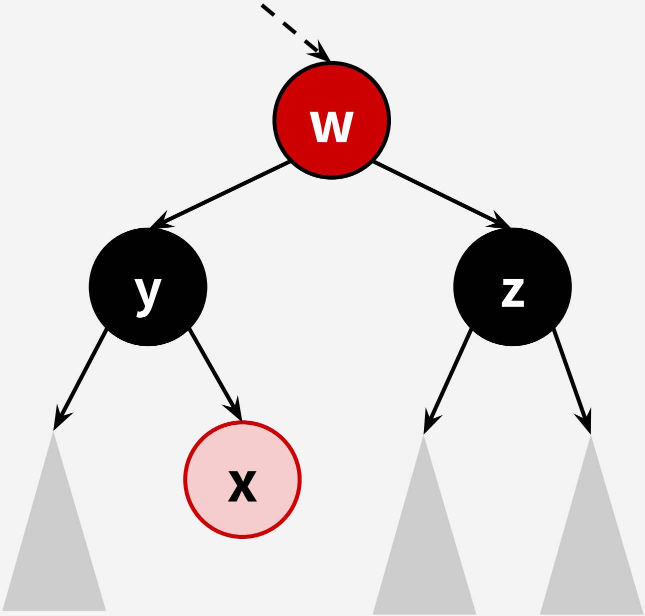
FIXING THINGS UP: CASE 1

CASE 1: parent **y** is **red**, and “uncle” **z** is **red** too!

Recolor!

Change **w** to **red** &
change **y** and **z** both to **black**

One recolor = O(1) time



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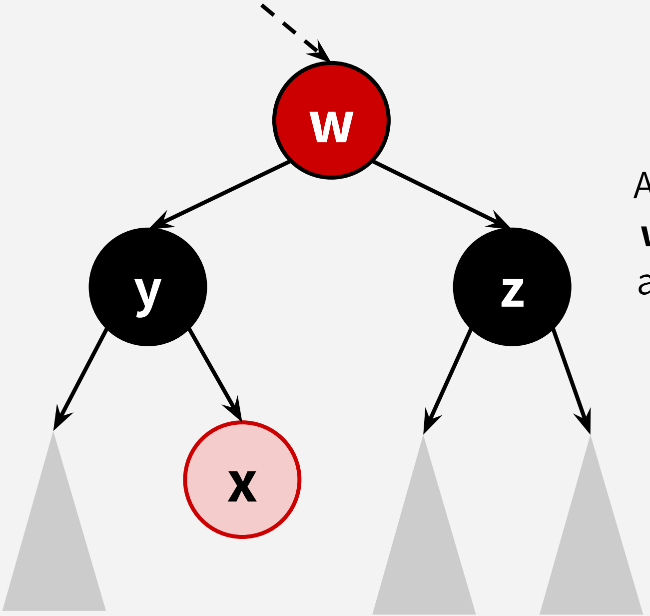

This doesn't hurt Property 4!

All root-NIL paths that interact with **w**, **y**, or **z**, all have to go through **w** and would hit exactly one of **y** or **z**.

Before, **w** contributed one **black** node to each of those paths, and now, **y** (or **z**) still contributes one **black** node (instead of **w**).

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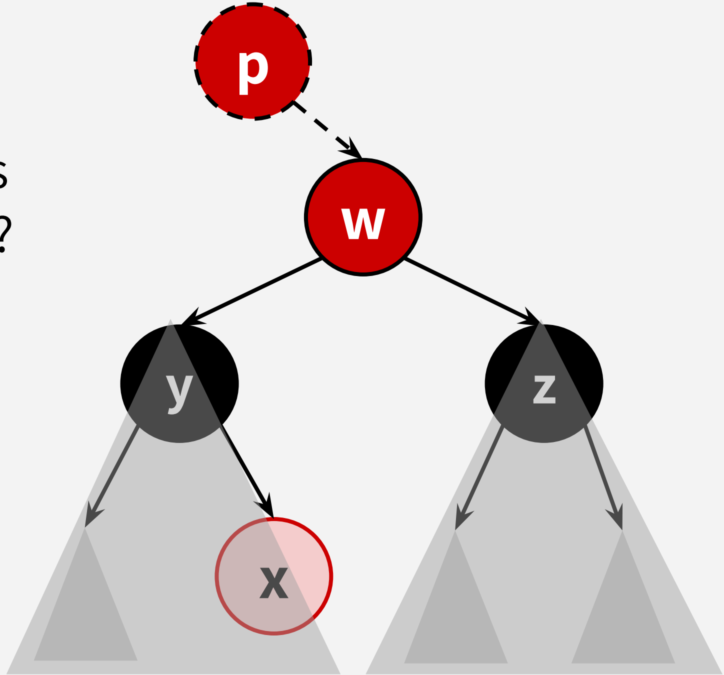
FIXING THINGS UP: CASE 1

CASE 1: parent **y** is **red**, and “uncle” **z** is **red** too!

But wait! What if **w**’s parent was also **red**?



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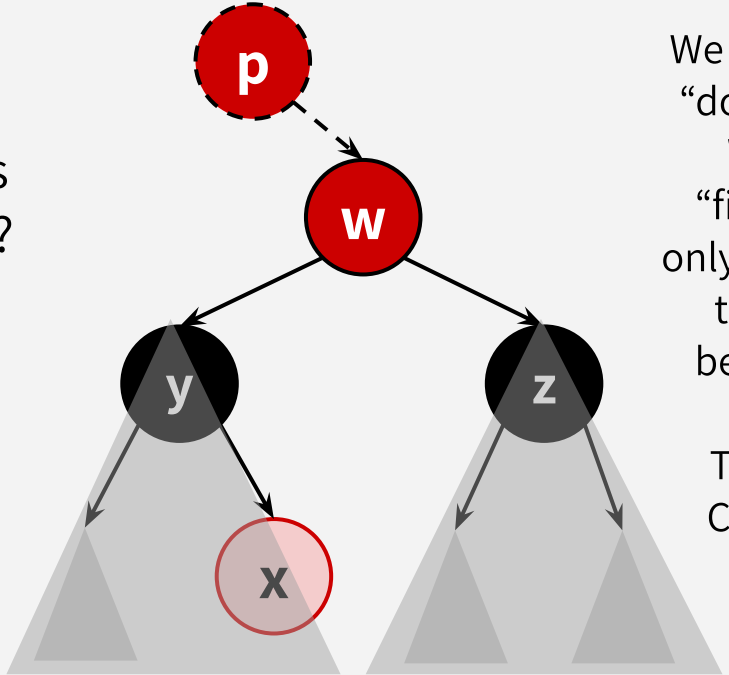


We basically just propagated the "double-red" violation upward! We can recursively do this "fix-up". This propagation can only happen O(log n) times, since the tree was a valid RB Tree before this INSERT operation!

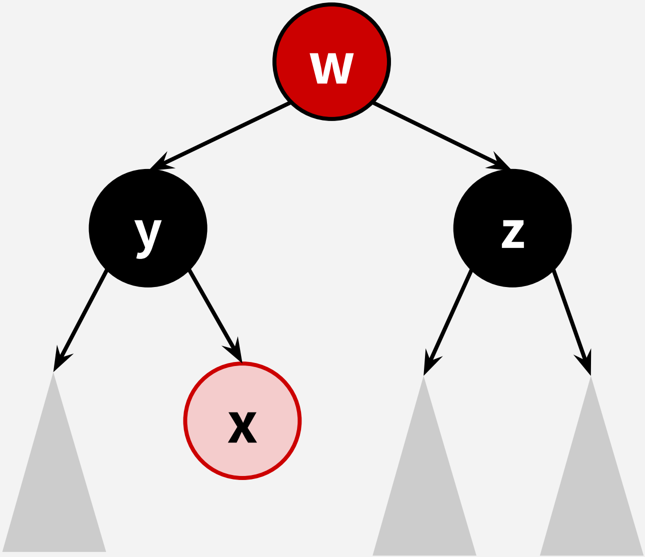
Thus, overall, INSERT in this CASE would be O(log n) still.

سوال؟

FIXING THINGS UP: CASE 1

CASE 1: parent **y** is **red**, and “uncle” **z** is **red** too!

But wait again!!
What if **w** is the root?
The root can’t be **red**!



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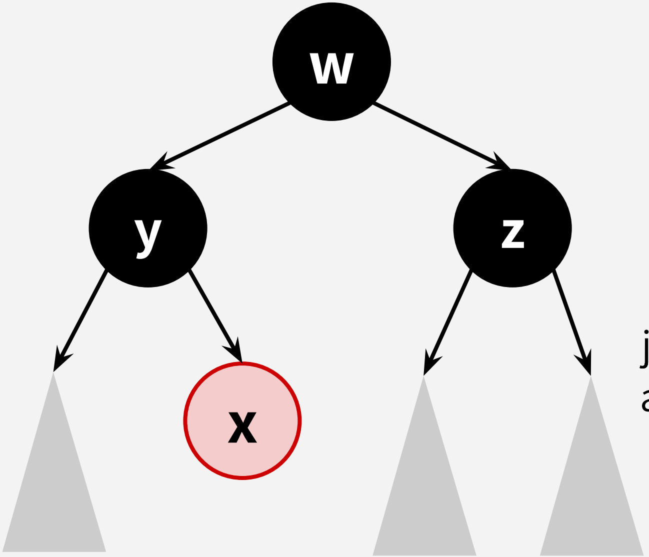


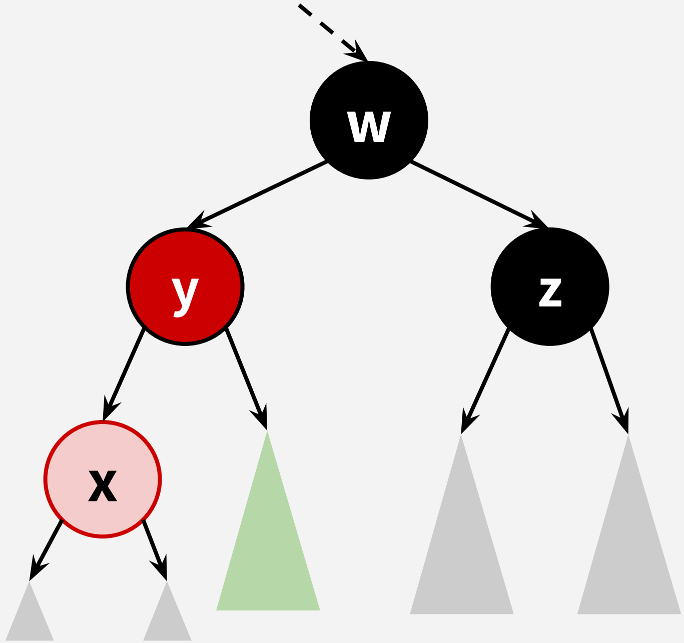
No stress at all,
just color it **black**!

If **w** is the root, then **w** appears once on *every* root-NIL path. Thus, changing **w** to **black** will just add 1 to every root-NIL path and Property 4 is still preserved!

سوال؟

FIXING THINGS UP: CASE 2

CASE 2: parent **y** is **red**, and “uncle” **z** is **black** (or NIL)!



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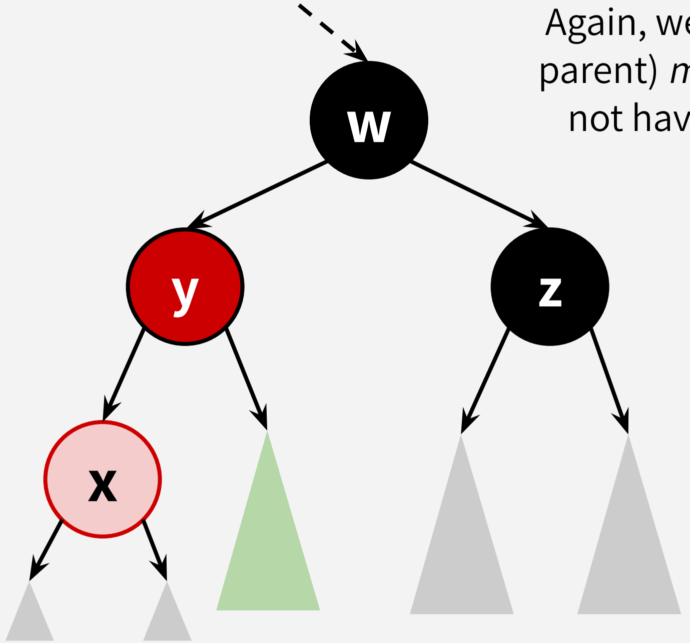

Again, we know **w** (x’s grandparent & y’s parent) *must* be **black** because we could not have had two **red** nodes in a row.

FIXING THINGS UP: CASE 2

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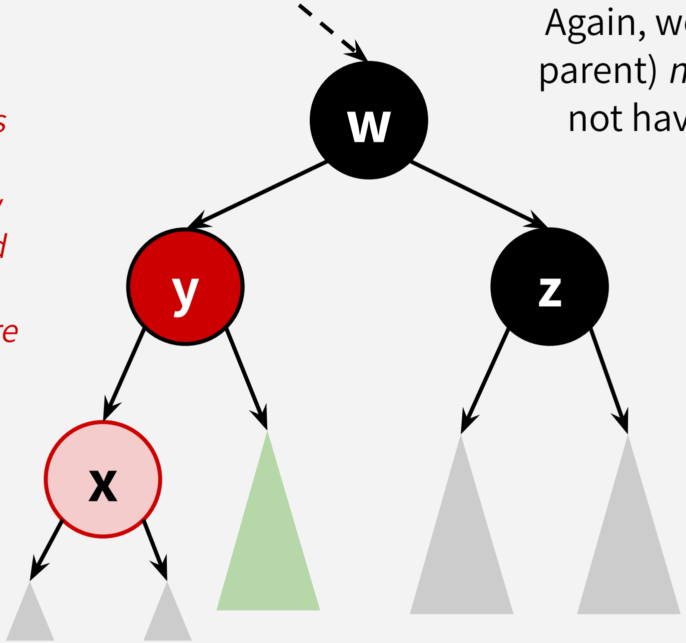
DISCLAIMER:
*This is just one of several sub-cases that fall under this Case 2. To understand all cases and why/how they work **intuitively**, you can read about 2-3-4 trees, which are an **isometry** of RB Trees. 2-3-4 trees are much easier to understand, and operations on 2-3-4 trees map to rotation routines for RB Trees!*



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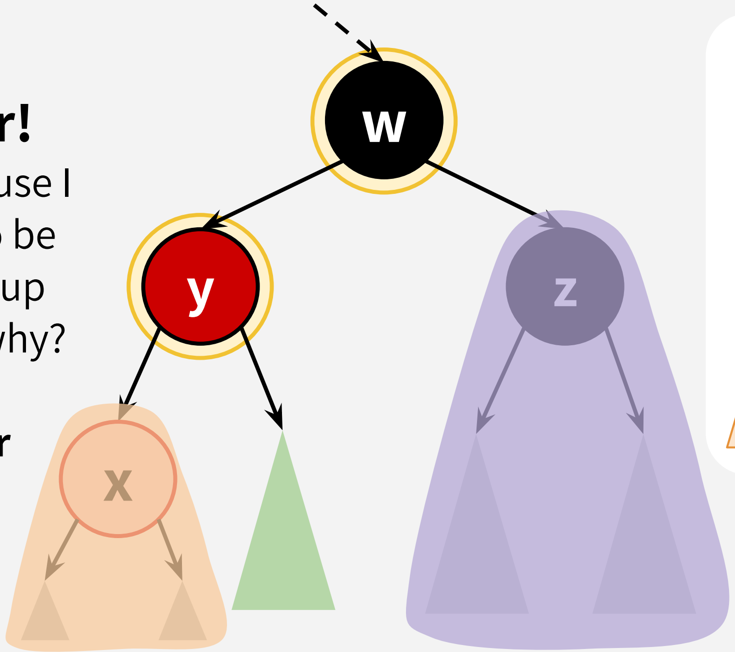
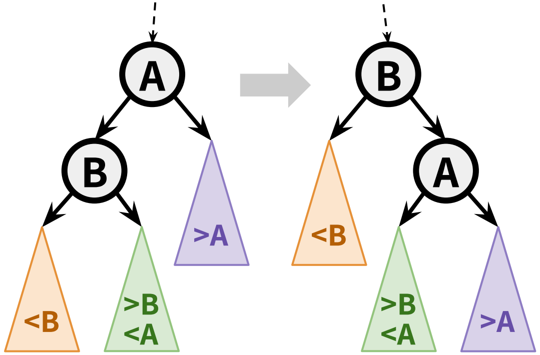


ROTATE & Recolor!

I need to rotate first, because I can’t just recolor **x** or **y** to be **black**! That would mess up Property 4 - can you see why?

One rotation + recolor = O(1) time

RIGHT ROTATION



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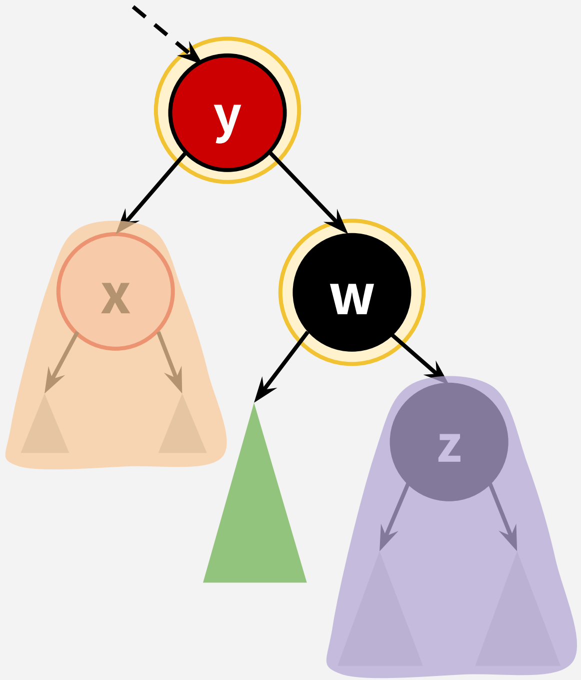

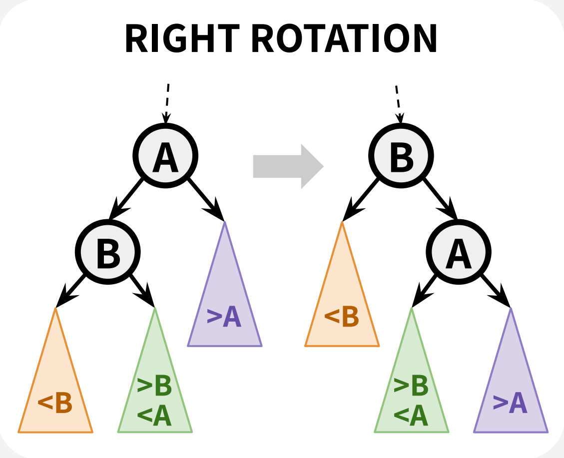

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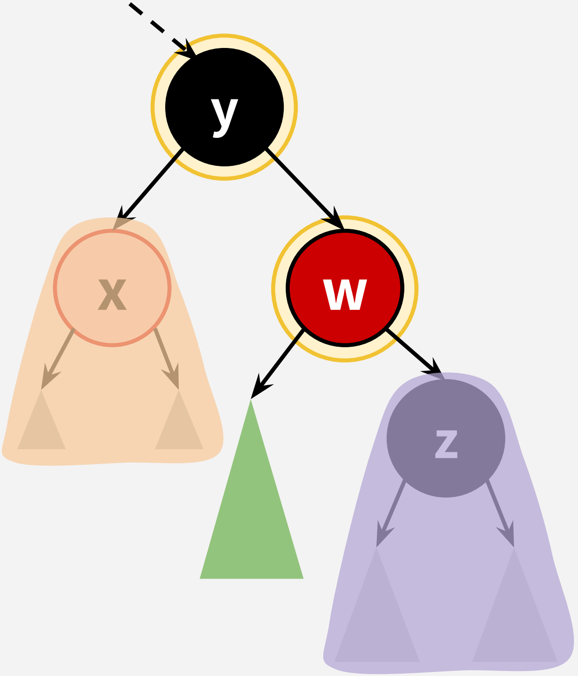

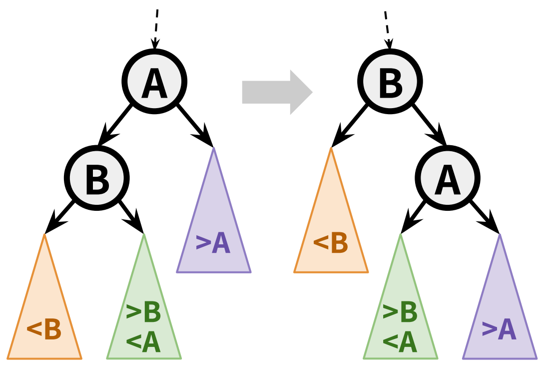

FIXING THINGS UP: CASE 2

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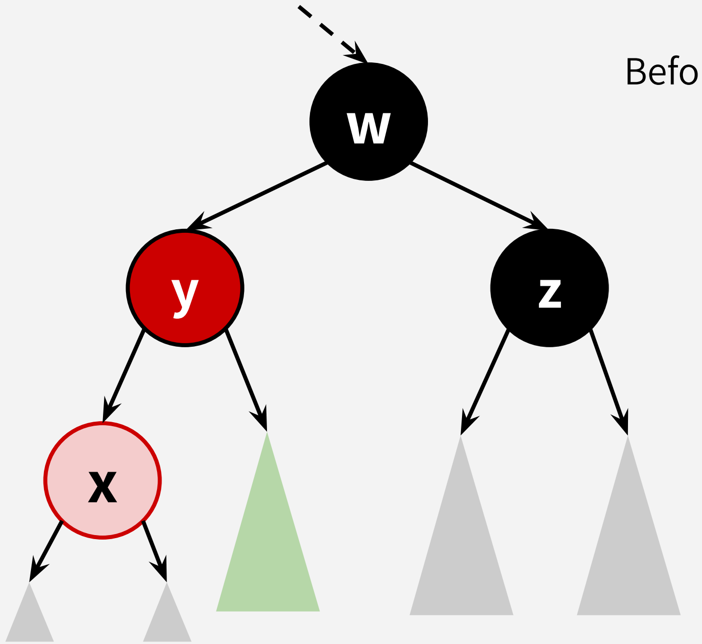
Before rotating and recoloring



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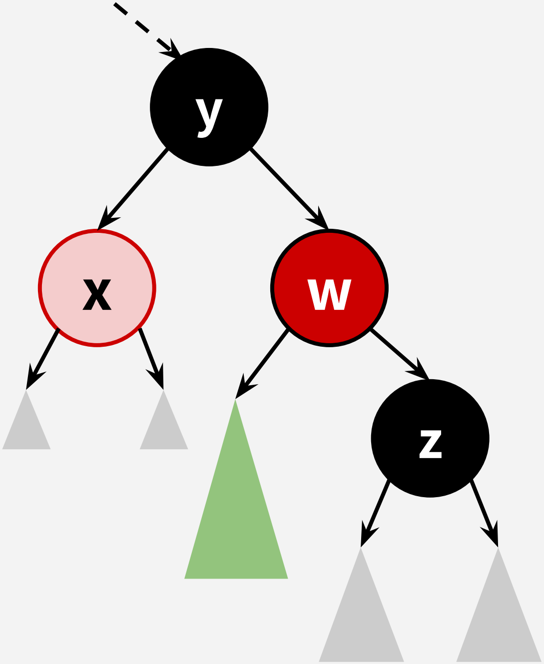

After recoloring, Property 4 is maintained, and we have not propagated the “double-red” further up! We’re done!

FIXING THINGS UP: CASE 2

CASE 2: parent **y** is **red**, and “uncle” **z** is **black** (or NIL)!

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Why we rotated this way should feel like magic to you right now. We needed to recolor in a way that maintained the # of black nodes on any root-NIL path, while getting rid of the “double-red”.

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سوال؟

INSERT IN RB TREES

High-level plan

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You are **not responsible for** the nitty-gritty details of how to insert or delete from RB Trees. And generally, I don't recommend memorizing these rotation/recolor rules.

(if you need to code up a RB Tree, just look up the pseudocode in CLRS)

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(if you need to code up a RB Tree, just look up the pseudocode in CLRS)

You **should know:**

- The properties of a Red-Black tree
- Why do these properties guarantee that they are balanced?

RED-BLACK TREE HIGHLIGHTS

RED-BLACK TREES SUPPORT **SEARCH**, **INSERT**, & **DELETE**
in **O(log n)** time

The key is that RB Trees always have height at most 2·log(n+1).

RED-BLACK TREE HIGHLIGHTS

RED-BLACK TREES SUPPORT **SEARCH**, **INSERT**, & **DELETE**

in **O(log n)** time

The key is that RB Trees always have height at most 2·log(n+1).

Generally, if you need to use a BST to solve a problem, you should think of using a self-balancing BST like Red-Black Trees! Unbalanced BSTs could have worst case O(n) operations.

RED-BLACK TREE HIGHLIGHTS

OPERATION	SORTED ARRAY	UNSORTED LINKED LIST	BST (WORST CASE)	BST (BALANCED)
SEARCH	O(log(n))	O(n)	O(n)	O(log(n))
DELETE	O(n)	O(n)	O(n)	O(log(n))
INSERT	O(n)	O(1)	O(n)	O(log(n))

(Balanced) Binary Search Trees can give us the best of both worlds!