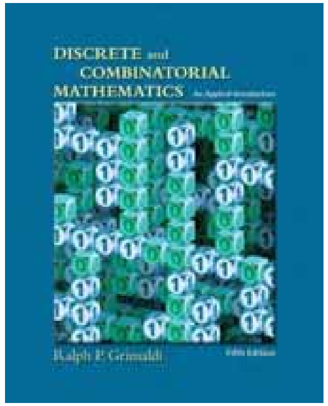
Discrete Mathematics
-- Chapter 1: Fundamental Principles of Counting



Hung-Yu Kao (高宏宇)
Department of Computer Science and Information Engineering,
National Cheng Kung University

Outline

- Preface & Introduction
- Sum & Product
- Permutations
- Combinations: The Binomial Theorem (二項式定理)
- Combinations with Repetition
- The Catalan Numbers
- Summary

Counting

- Capable of solving difficult problems.
  - Coding theory, probability and statistics

- Help the analysis and design of efficient algorithms.
- E.g.,

$$1+2+3=?$$

$$\frac{3\times 4}{2}=?$$

1.1 The Rules of Sum and Product

- **The Rule of Sum**
  - If a first task can be performed in m ways, while a second task can be performed in n ways, and the two tasks cannot be performed simultaneously, then performing either task can be accomplished in any one of $m+n$ ways.

- **The Rule of Product**
  - If a procedure can be broken down into first and second stages, and if there are m possible outcomes for the first stage and if, for each of these outcomes, there are n possible outcomes for the second stage, then the total procedure can be carried out, in the designated order, in mn ways.

    - **Ex 1.6**: If a license plate consists of two letters followed by four digits, how many different plates are there? $26\times26\times10\times10\times10\times10$

Rule combination

1.2 Permutations

- Permutation: counting linear arrangements of **distinct** objects
- If there are n distinct objects and r is an integer, with $1 \leq r \leq n$, then **by the rule of product**,
- The number of permutations of size r for the n objects is

$$P(n,r) = n(n-1)(n-2)\cdots(n-r+1)$$

$$= n(n-1)(n-2)\cdots(n-r+1) \times \frac{(n-r)(n-r-1)\cdots(3)(2)(1)}{(n-r)(n-r-1)\cdots(3)(2)(1)}$$

$$= \frac{n!}{(n-r)!} \quad \longleftarrow \textit{n factorial}$$

- **Ex 1.9**: Given 10 students, *three* are to be chosen and seated in a row.

How many such linear arrangements are possible?

Permutations with Repeated Objects

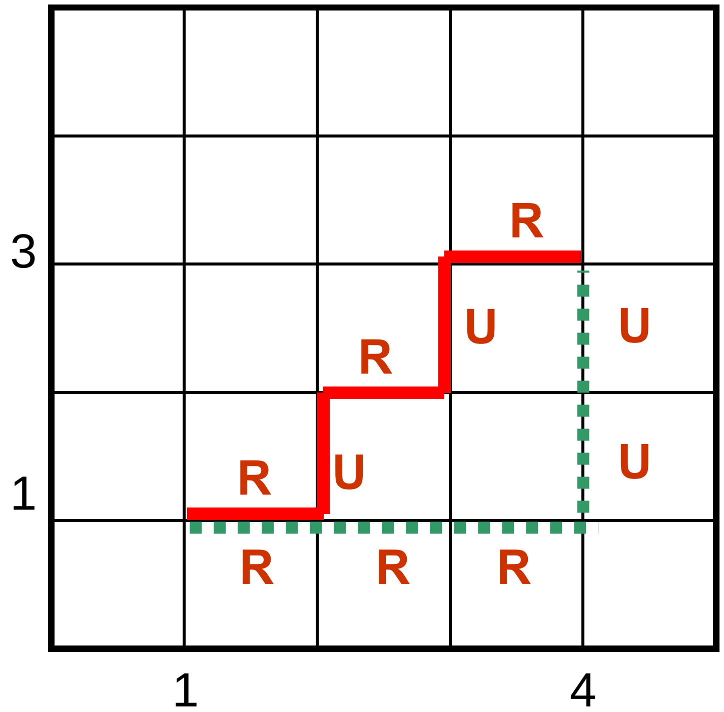
- If there are n objects with n_1 indistinguishable objects of a first type, n_2 indistinguishable objects of a second type, …, and n_r indistinguishable objects of an rth type, where $n_1 + n_2 + \ldots + n_r = n$.
  - → the number of (linear) arrangements of the given n objects

    $$= \frac{n!}{n_1!n_2!\ldots n_r!}$$

- **Ex 1.13**: Arranging all of the letters in MASSASAUGA, we find there are $\frac{10!}{4!\,3!\,1!\,1!\,1!}$ possible arrangements, $\frac{7!}{3!\,1!\,1!\,1!}$ arrangements while all four A's are together.

Permutations with Repeated Objects

- **Ex 1.14**: Determine the number of (staircase) paths in the xy-plane from (1, 1) to (4, 3), where each such path is made up of individual steps going one unit to the right or one unit upward.
  - As for xyz-space, from (1,1,1) to (4, 3, 2)?



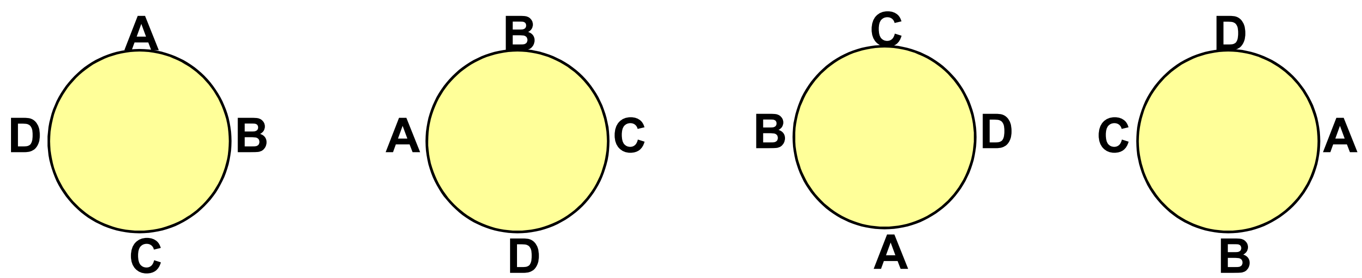
Combinatorial Proofs

- Prove that $\frac{(2k)!}{2^k}$ is an integer.
  - Consider $2k$ symbols $x_1, x_1, x_2, x_2, \ldots, x_k, x_k$.
  - The number of ways they can be arranged is
    $$\frac{(2k)!}{2^k} = \frac{(2k)!}{2!2!\cdots 2!}$$
  - It must be an integer.

- Prove that $\frac{(mk)!}{(m!)^k}$ is also an integer.

Arrangement around a Circle

- Consider n distinct objects
- Two arrangements are considered the same when one can be obtained from the other by rotation.
- How many different circular arrangements?
  - Thinking distinct linear arrangements for 4 objects, e.g., ABCD, BCDA, CDAB, and …
  - So, the number of circular arrangements is? **4!/4 = 3!**



Arrangement around a Circle

- Ex 1.17: arrange six people (3 males, 3 females) around a tableso that males/females alternate.

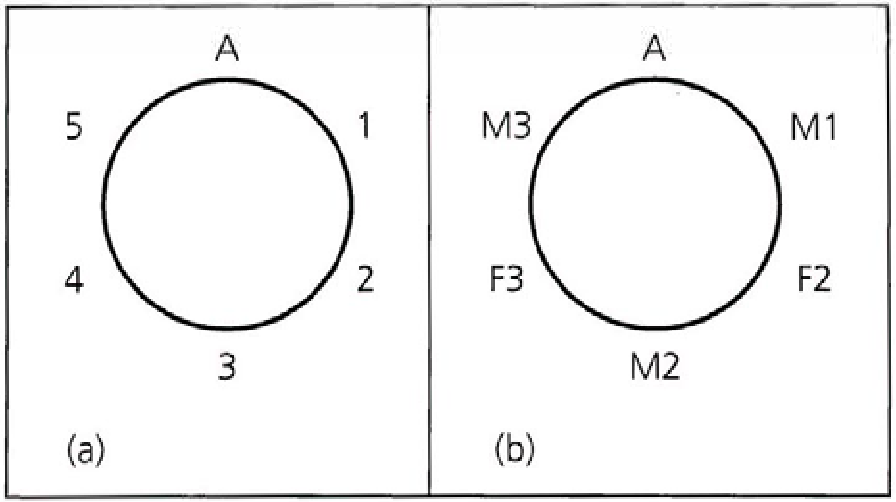


Figure 1.3

No constraint 6!/6 = 5! = 120

m/f alternate
3 x 2 x 2 x 1 x 1 = 12

1.3 Combinations: The Binomial Theorem

- If there are n distinct objects and r is an integer, with $1 \le r \le n$
- The number of combinations (***selections without reference to order***) of size r for the n objects is

$$C(n,r) = \boxed{\binom{n}{r}} = \frac{P(n,r)}{r!} = \frac{n(n-1)\cdots(n-r+1)}{r(r-1)\cdots 1} = \frac{n!}{r!(n-r)!}$$

  - are sometimes read "n choose r".

- $C(n, 0) = C(n, n) = 1$, for all $n \ge 0$.
- $C(n, r) = C(n, n\text{-}r)$, for all $n \ge 0$.

Combinations

- **Ex 1.18**
  - A hostess is having a dinner party for some members of her charity committee. Because of the size of her home, she can invite only 11 of the 20 committee members.
  - So, how many different ways can she invite "the lucky 11"?

$$C(20,11) = \binom{20}{11} = \frac{20!}{11! \cdot 9!} = 167,960$$

  - Once the 11 arrive, how to arranges them around her rectangular dining table is an arrangement problem.

Combinations

- <u>**Ex 1.20**</u>
  - A student taking a history examination is directed to answer any <u>seven of 10</u> essay questions. There is no concern about order here.
    - She can answer the examination in $\binom{10}{7}=120$ ways
  - If the student must answer three questions from the first five and four questions from the last five.

    $$\binom{5}{3}\binom{5}{4}=5\times 10=50$$
  - If the student must answer **at least** three questions from the first five.

    $$\binom{5}{3}\binom{5}{4}+\binom{5}{4}\binom{5}{3}+\binom{5}{5}\binom{5}{2}=50+50+10=110$$

    $$also, \sum_{i=3}^{5}\binom{5}{i}\binom{5}{7-i}$$

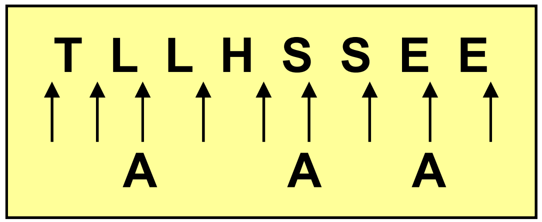
Combinations

- **Ex 1.23**
  - The number of arrangements of the letters in TALLAHASSEE is?

$$\frac{11!}{3!\,2!\,2!\,2!\,1!\,1!} = 831{,}600 \quad \textit{Permutations with Repeated Objects}$$

  - How many of these arrangements have no adjacent A's?

$$\left(\frac{8!}{2!\,2!\,2!\,1!\,1!}\right)\binom{9}{3} = 5040 \times 84 = 423{,}360$$



Theorem 1.1: The Binomial Theorem

- $$(x+y)^n = \binom{n}{0}x^0y^n + \binom{n}{1}x^1y^{n-1} + \ldots + \binom{n}{n}x^ny^0 = \sum_{k=0}^{n}\binom{n}{k}x^ky^{n-k}$$

- There are $C(n, k)$ ways to choose k x's and n-k y's.
- $C(n, k)$ is often referred to as a binomial coefficient.

- In case $(x+y)^2$

$$\begin{array}{r}
x + y \\
x + y \\
\hline
xy + y^2 \\
x^2 + xy \\
\hline
x^2y^0 + 2x^1y^1 + x^0y^2
\end{array}$$

The Binomial Theorem

- $$(x+y)^n = \binom{n}{0}x^0y^n + \binom{n}{1}x^1y^{n-1} + \ldots + \binom{n}{n}x^ny^0 = \sum_{k=0}^{n}\binom{n}{k}x^ky^{n-k}$$

- **Ex 1.26**
  - a) What is the coefficient of x^5y^2 in the expansion of $(x + y)^7$?
  - b) What is the coefficient of a^5b^2 in the expansion of $(2a - 3b)^7$?

a) the coefficient of x^5y^2 in $(x+y)^7$ is $\binom{7}{5}$

$b)$ *Set* $x = 2a,\ y = -3b$

$$\binom{7}{5}x^5y^2 = \binom{7}{5}(2a)^5(-3b)^2 = \binom{7}{5}(2)^5(-3)^2a^5b^2 = 6048a^5b^2$$

Corollaries of The Binomial Theorem

- Corollary 1.1: a) $\binom{n}{0}+\binom{n}{1}+\ldots+\binom{n}{n}=?$ **2^n**

  b) $\binom{n}{0}-\binom{n}{1}+\ldots+(-1)^n\binom{n}{n}=?$ **0**

  - Proof
    - Part (a) set $x=y=1$
    - Part (b) set x=-1 and y=1

$$(x+y)^n=\binom{n}{0}x^0y^n+\binom{n}{1}x^1y^{n-1}+\ldots+\binom{n}{n}x^ny^0=\sum_{k=0}^{n}\binom{n}{k}x^ky^{n-k}$$

how about $\binom{n}{0}+2\binom{n}{1}+2^2\binom{n}{2}+\ldots+2^k\binom{n}{k}+\ldots+2^n\binom{n}{n}=?$

Theorem 1.2 : The Multinomial Theorem

- The coefficient of $x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t}$ in the expansion of

  $(x_1 + x_2 + \cdots + x_t)^n$ is $\dfrac{n!}{n_1!n_2!\ldots n_t!} = \dbinom{n}{n_1, n_2, \cdots, n_t}$

  where $0 \le n_i \le n$, and $n_1 + n_2 + \ldots + n_t = n$.

- Proof
  - The number of ways we can select ① x_1 from n_1 of the n factors, ② x_2 from n_2 of the $n - n_1$ remaining factors, ③ …

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-n_1-n_2-\ldots-n_{t-1}}{n_t} = \underline{\frac{n!}{n_1!n_2!\cdots n_t!} = \binom{n}{n_1, n_2, \cdots, n_t}}$$

Multinomial coefficient

$t=2$ → binomal coefficient

The Multinomial Theorem

- **Ex 1.27**
  - What is the coefficient of x^5y^2 in the expansion of $(x+y+z)^7$?

    $$\binom{7}{5\;2\;0} = \frac{7!}{5!\,2!\,0!} = 21$$

  - What is the coefficient of $a^2b^3c^2d^5$ in the expansion of $(a+2b-3c+2d+5)^{16}$?

$$\text{Set } \mathrm{v} = a,\ w = 2b,\ x = -3c,\ y = 2d,\ z = 5$$

$$\text{the coefficient of } v^2w^3x^2y^5z^4 \text{ in } (v+w+x+y+z)^{16} \text{ is } \binom{16}{2,3,2,5,4}$$

$$\binom{16}{2,3,2,5,4}(a)^2(2b)^3(\text{-}3\mathrm{c})^2(2\mathrm{d})^5(5)^4 = \binom{16}{2,3,2,5,4}(1)^2(2)^3(\text{-}3)^2(2)^5(5)^4a^2b^3\mathrm{c}^2\mathrm{d}^5$$

$$= 435{,}891{,}456{,}000{,}00a^2b^3\mathrm{c}^2\mathrm{d}^5$$

1.4 Combination with Repetition

- **Ex 1.28**
  - How many different purchases are possible for **seven** students each having one of the following, a cheeseburger, a hot dog, a taco, or a fish sandwich?

Possible way	**Another way**
c,c,h,h,t,t,f	xx|xx|xx|x
c,c,c,c,h,t,f	xxxx|x|x|x
c,c,c,c,c,c,f	xxxxxx|||x
h,t,t,f,f,f,f	|x|xx|xxxx

7 x's + 3 |'s $\quad \binom{10}{7} = \frac{10!}{7!3!} = \frac{(4+7-1)!}{7!(4-1)!}$

- The number of combinations of n objects taken r at a time, with repetition, is

(foods) (students)

$$C(n+r-1,r) = \frac{(n+r-1)!}{r!(n-1)!} = \binom{n+r-1}{r}$$

1.4 Combination with Repetition

- **Ex 1.31**
  - In how many ways can we distribute seven bananas and six oranges among four children so that each child receives at least one banana?
  - Remaining bananas: 7-4=3
  - 3 bananas was distributed 4 children: (n=4, r=3) C(4+3-1, 3) = C(6, 3) =20
  - 6 oranges was distributed 4 children: (n=4, r=6) C(4+6-1, 6)= C(9, 6)=84
  - Thus, 20×84=1680

Distribute 3 bananas to 4 children	
c_1, c_2, c_3	b|b|b|
c_1, c_3, c_3	b||bb|
c_3, c_4, c_4	||b|bb
c_4, c_4, c_4	|||bbb

3 b's + 3 |'s

1.4 Combination with Repetition

- Ex 1.33
  - Determine all integer solutions to the equation $x_1+x_2+x_3+x_4=7$, where $x_i \geq 0$ for all $1 \leq i \leq 4$.
  - ***n*=4, *r*=7 → *C*(4+7-1, 7)**
- Equivalence: ***C*(*n* + *r* - 1, *r*)**
  - The number of integer solutions of the equation $x_1+x_2+_{\dots}+x_n=r$, where $x_i \geq 0$ for all $1 \leq i \leq n$.
  - The number of selections, with repetition, of size r from a collection of size n.
  - The number of ways r identical objects can be distributed among n distinct containers.

=the number of ways r distinct objects be distributed among n identical containers **?**

- Difference
  - The arrangement of size r from n distinct objects can be obtained in n^r ways.

1.4 Combination with Repetition

*the number of ways **r** objects be distributed among **n** containers*

	r distinct	*r identical*
n distinct	n^r	$C(n+r-1, r)$
n identical	$n^r/n!$ See Chapter 5	See Chapter 9

$$\sum_{i=1}^{n} S(r,i)$$

1.4 Combination with Repetition

- **Ex 1.35**
  - How many nonnegative integer solutions to the inequality $x_1 + x_2 + \dots + x_6 < 10$?
  - Transform the problem to $x_1 + x_2 + \dots + x_6 + x_7 = 10$, $x_i \geq 0$ for all $1 \leq i \leq 6$, but $x_7 > 0$.
  - $y_1 + y_2 + \dots + y_6 + y_7 = 9$, where $y_1 = x_i$ for all $1 \leq i \leq 6$, and $y_7 = x_7 - 1$.
  - $C(7+9-1, 9) = 5050$.

1.4 Combination with Repetition

- **Ex 1.39**

Consider the following program segment, where i, j, and k are integer variables.

```
for i := 1 to 20 do
  for j := 1 to i do
    for k := 1 to j do
      print (i * j + k)
```

How many times is the **print** statement executed in this program segment?

k j i

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

(i, j, k) = (13, 8, 6)

(I, j, k)=(8, 4, 4)

(i, j, k) = (13, 8, 11) X

|||xx||||x|||||||||||, 3 x's 19|'s combinations = C(22, 3)=C(22, 9)=1540

1.4 Combination with Repetition

- **Ex 1.39**
  - Summation formula
  - *counter* = C(n+2-1, 2)=C(n+1, 2)
  - Also, *counter* = $\sum_{i=1}^{n} i = 1+2+3+\cdots+n = \binom{n+1}{2} = \frac{n(n+1)}{2}$

```
counter := 0
for i := 1 to n do
  for j := 1 to i do
    counter := counter + 1
```

1.5 Catalan Number

- Count paths from (0,0)→(5,5) but never rise over the line y=x
- No constraint C(10,5)
- With constraint
  - C(10,5)-C(10,4)
- Exchange R,U after the first "crossing" U

  RUUURRRUUR

→ RUURUUURRU

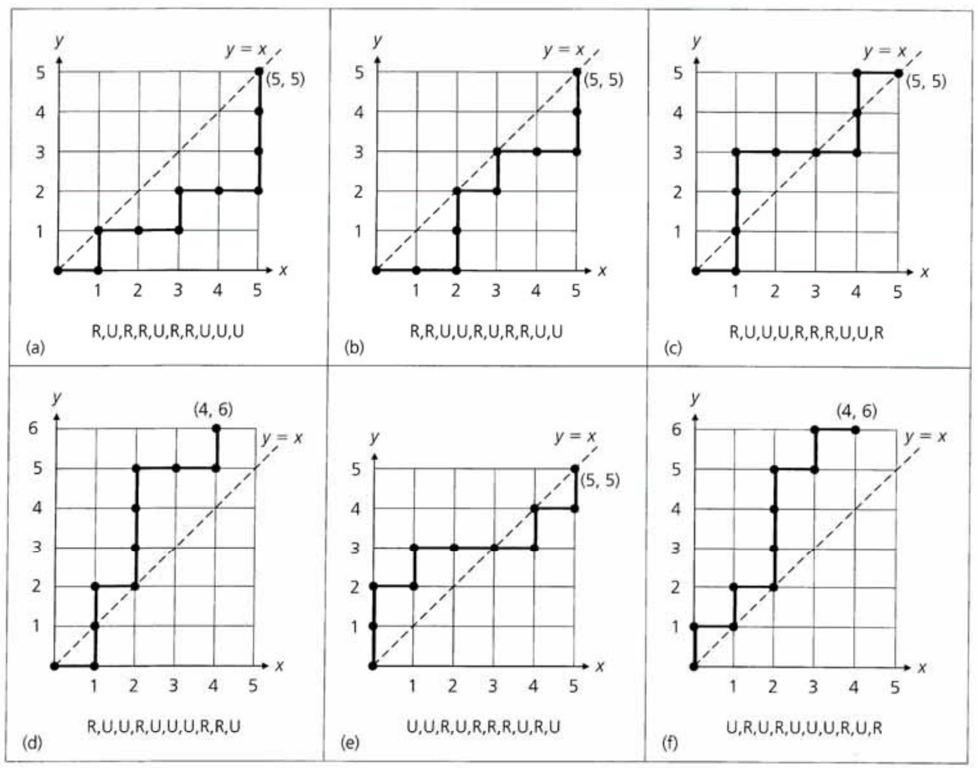


Figure 1.9

1.5 Catalan Number

- From (0,0)→(n,n)

$$\# path = b_n = \binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n}$$

- which can determine the number of ways to parenthesize the product $x_1 x_2 x_3 x_4 \cdots x_n$.
  - E.g., n=4, #parenthesis = b3 = ¼*C(6,3) = 5

$$(((x_1 x_2)x_3)x_4), ((x_1(x_2 x_3)x_4)), ((x_1 x_2)(x_3 x_4)),$$
$$(x_1((x_2 x_3)x_4)), (x_1(x_2(x_3 x_4)))$$

1.5 Catalan Number

- The number of valid parenthesis expressions that consist of *n* right parentheses and *n* left parentheses is equal to the n^{th} Catalan number
- For example, $C_3 = 5$ and there are 5 ways to create valid expressions with 3 sets of parenthesis:
  - ( ) ( ) ( )
  - (( )) ( )
  - ( ) (( ))
  - ((( )))
  - (( ) ( ))

1.6 Summary

- Fundamental techniques in counting:
  - ***Top-down approach***: Divide the problems into subproblems suitable for discrete and combinatorial mathematics.

Order Is Relevant	Repetitions Are Allowed	Type of Result	Formula	Location in Text
Yes	Yes	Arrangement	n^{r}	Page 7
Yes	No	Permutation	$P(n,r)=\dfrac{n!}{(n-r)!}$	Page 7
No	No	Combination	$C(n,r)=\dbinom{n}{r}=\dfrac{n!}{r!(n-r)!}$	Page 15
No	Yes	Combination with repetition	$C(n+r-1,r)=\dbinom{n+r-1}{r}=\dfrac{(n+r-1)!}{r!(n-1)!}$	Page 27