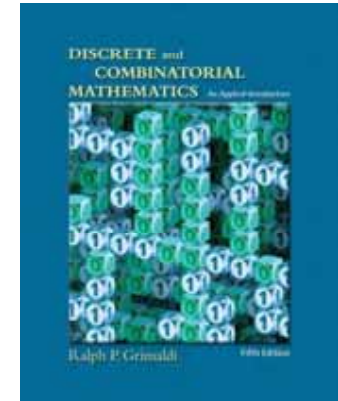
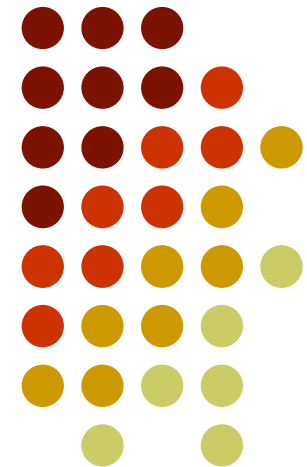


Discrete Mathematics

-- Chapter 8: The Principle of Inclusion and Exclusion



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Outline

- The Principle of Inclusion and Exclusion
- Generalization of the Principle
- Derangements: Nothing Is in Its Right Place
- Rook Polynomials
- Arrangements with Forbidden Positions



8.1 The principle of Inclusion and Exclusion

- For a given finite set S ($|S| = N$) with conditions C_i
- $N(\overline{c_1} \overline{c_2}) = N - [N(c_1) + N(c_2)] + N(c_1 c_2)$ $N(\overline{c_1} \text{ and } \overline{c_2})$

$$= N(\overline{c_1}) - N(\overline{c_1} c_2)$$

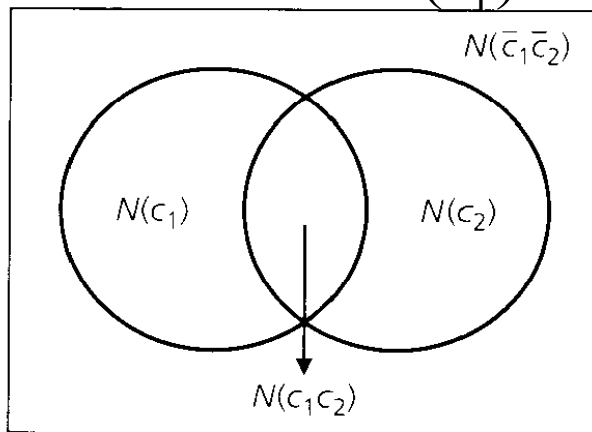


Figure 8.1

$$N(\overline{c_1} \overline{c_2}) = N - N(c_1 c_2)$$

$$\neq N(\overline{c_1} \overline{c_2})$$
 $N(\overline{c_1} \text{ or } \overline{c_2})$

- $N(\overline{c_1} \overline{c_2} \overline{c_3}) = N - [N(c_1) + N(c_2) + N(c_3)]$
 $+ [N(c_1 c_2) + N(c_1 c_3) + N(c_2 c_3)] - N(c_1 c_2 c_3)$

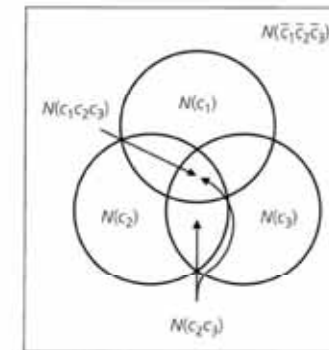


Figure 8.2



Four sets

- Ex 8.3 :

$$\begin{aligned} N(\overline{c_1} \overline{c_2} \overline{c_3} \overline{c_4}) &= N - [N(c_1) + N(c_2) + N(c_3) + N(c_4)] \\ &+ [N(c_1c_2) + N(c_1c_3) + N(c_1c_4) + N(c_2c_3) + N(c_2c_4) + N(c_3c_4)] \\ &- [N(c_1c_2c_3) + N(c_1c_2c_4) + N(c_1c_3c_4) + N(c_2c_3c_4)] + N(c_1c_2c_3c_4) \end{aligned}$$

- For each element $x \in S$, we have five cases:
 - (0) x satisfies none of the four conditions;
 - (1) x satisfies only one of the four conditions;
 - (2) x satisfies exactly two of the four conditions;
 - (3) x satisfies exactly three of the four conditions;
 - (4) x satisfies all the four conditions.

Four sets

$$\begin{aligned}
 N(\overline{c_1} \overline{c_2} \overline{c_3} \overline{c_4}) &= N - [N(c_1) + N(c_2) + N(c_3) + N(c_4)] \\
 &+ [N(c_1 c_2) + N(c_1 c_3) + N(c_1 c_4) + N(c_2 c_3) + N(c_2 c_4) + N(c_3 c_4)] \\
 &- [N(c_1 c_2 c_3) + N(c_1 c_2 c_4) + N(c_1 c_3 c_4) + N(c_2 c_3 c_4)] + N(c_1 c_2 c_3 c_4)
 \end{aligned}$$

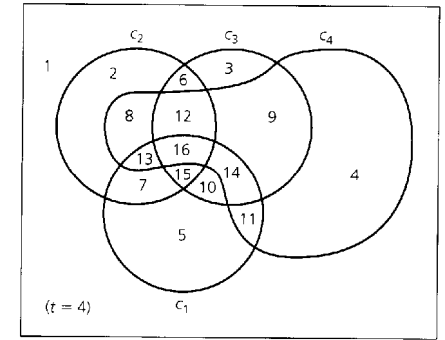


Figure 8.5

1. x satisfies no condition. x is counted once in $N(\overline{c_1} \overline{c_2} \overline{c_3} \overline{c_4})$ and once in N .
[1=1]
2. x satisfies c_1 . It is not counted on the left side. It is counted once in N and once in $N(c_1)$. [0 = 1-1 = 0]
3. x satisfies c_2 and c_4 . It is not counted on the left side. It is counted once in N , $N(c_2)$, $N(c_4)$ and $N(c_2 c_4)$.
[0 = 1 - (1 + 1) + 1 = 1 - $\binom{2}{1}$ + $\binom{2}{2}$ = 0]
4. x satisfies c_1 , c_2 and c_4 . It is not counted on the left side. It is counted once in N , $N(c_1)$, $N(c_2)$, $N(c_4)$, $N(c_1 c_2)$, $N(c_1 c_4)$, $N(c_2 c_4)$ and $N(c_1 c_2 c_4)$.
[0 = 1 - (1 + 1 + 1) + (1 + 1 + 1) - 1 = 1 - $\binom{3}{1}$ + $\binom{3}{2}$ - $\binom{3}{3}$ = 0]
5. x satisfies all conditions. It is not counted on the left side. It is counted once in all the subsets on the right side.
[0 = 1 - $\binom{4}{1}$ + $\binom{4}{2}$ - $\binom{4}{3}$ + $\binom{4}{4}$ = 0]

The Principle of Inclusion and Exclusion



- Theorem 8.1:
 - $|S| = N$, conditions c_i , $1 \leq i \leq t$
 - $\overline{N} = N(\overline{c_1} \overline{c_2} \cdots \overline{c_t})$ denote the number of elements of S that satisfy none of the conditions.

$$\begin{aligned} \overline{N} = N - \sum_{1 \leq i \leq t} N(c_i) + \sum_{1 \leq i < j \leq t} N(c_i c_j) - \sum_{1 \leq i < j < k \leq t} N(c_i c_j c_k) + \cdots \quad (2) \\ + (-1)^t N(c_1 c_2 c_3 \cdots c_t) \end{aligned}$$

The other possibility is that x satisfies *exactly* r of the conditions where $1 \leq r \leq t$. In this case x contributes nothing to \overline{N} . But on the right-hand side of Eq. (2), x is counted

- (1) One time in N .
- (2) r times in $\sum_{1 \leq i \leq t} N(c_i)$. (Once for each of the r conditions.)
- (3) $\binom{r}{2}$ times in $\sum_{1 \leq i < j \leq t} N(c_i c_j)$. (Once for each pair of conditions selected from the r conditions it satisfies.)
- (4) $\binom{r}{3}$ times in $\sum_{1 \leq i < j < k \leq t} N(c_i c_j c_k)$. (Why?)
-
- ($r + 1$) $\binom{r}{r} = 1$ time in $\sum N(c_{i_1} c_{i_2} \cdots c_{i_r})$, where the summation is taken over all selections of size r from the t conditions.

Consequently, on the right-hand side of Eq. (2), x is counted

$$1 - r + \binom{r}{2} - \binom{r}{3} + \cdots + (-1)^r \binom{r}{r} = [1 + (-1)]^r = 0^r = 0 \text{ times,}$$



The Principle of Inclusion and Exclusion

- Corollary 8.1: $N(c_1 \text{ or } c_2 \text{ or } \dots \text{ or } c_t) = N - \overline{N}$.
- Some notation for simplifying Theorem 8.1

$$S_0 = N,$$

$$S_1 = [N(c_1) + N(c_2) + \dots + N(c_t)],$$

$$S_2 = [N(c_1 c_2) + N(c_1 c_3) + \dots + N(c_1 c_t) + N(c_2 c_3) + \dots + N(c_{t-1} c_t)],$$

and, in general,

$$S_k = \sum N(c_{i_1} c_{i_2} \dots c_{i_k}), \quad 1 \leq k \leq t,$$

where the summation is taken over all selections of size k from the collection of t conditions.

Hence S_k has $\binom{t}{k}$ summands in it.

Using this notation we can rewrite the result in Eq. (2) as

$$\overline{N} = S_0 - S_1 + S_2 - S_3 + \dots + (-1)^t S_t.$$



The Principle of Inclusion and Exclusion

- **Ex 8.4** : Determine the number of positive integers n where $1 \leq n \leq 100$ and n is not divisible by 2, 3 or 5.

- Condition c_1 if n is divisible by 2.

- Condition c_2 if n is divisible by 3.

- Condition c_3 if n is divisible by 5.

- Then the answer to this problem is

$$N(\overline{c_1 c_2 c_3}) = S_0 - S_1 + S_2 - S_3 \quad \lfloor 100 / (2 * 3) \rfloor = 16$$

$$= N - [N(c_1) + N(c_2) + N(c_3)] + [N(c_1 c_2) + N(c_1 c_3) + N(c_2 c_3)] - N(c_1 c_2 c_3)$$

$$= 100 - [50 + 33 + 20] + [16 + 10 + 6] - 3 = 26.$$



The Principle of Inclusion and Exclusion

- **Ex 8.5** : Determine the number of nonnegative integer solutions to the equation

$$x_1 + x_2 + x_3 + x_4 = 18 \text{ and } x_i \leq 7 \text{ for all } i.$$

- We say that a solution x_1, x_2, x_3, x_4 satisfies condition c_i if $x_i > 7$ (i.e., $x_i \geq 8$).
- Then the answer to this problem is

$$N(\overline{c_1}\overline{c_2}\overline{c_3}\overline{c_4}) = S_0 - S_1 + S_2 - S_3 + S_4 =$$

$$\binom{21}{18} - \binom{4}{1}\binom{13}{10} + \binom{4}{2}\binom{5}{2} - 0 + 0 = 246.$$

$\binom{4+18-1}{18}$

$\binom{4+10-1}{10}$

$\binom{4+2-1}{2}$



The Principle of Inclusion and Exclusion

- **Ex 8.6** : For finite sets A, B , where $|A| = m \geq n = |B|$, and function $f: A \rightarrow B$, determine the number of onto functions f .
 - Let $A = \{a_1, a_2, \dots, a_m\}$ and $B = \{b_1, b_2, \dots, b_n\}$.
 - Let c_i be the condition that b_i is not in the range of f .
Then $N(c_1)$ is the number of functions in S that have b_i in their range.
 - Then the answer to this problem is $N(\overline{c_1} \overline{c_2} \dots \overline{c_n})$.

$$N(\overline{c_1} \overline{c_2} \overline{c_3} \dots \overline{c_n}) = S_0 - S_1 + S_2 - S_3 + \dots + (-1)^n S_n$$

$$N = S_0 = |S| = n^m$$

$$N(c_i) = (n-1)^m \Rightarrow S_1 = \binom{n}{1} (n-1)^m$$

$$N(c_i c_j) = (n-2)^m \Rightarrow S_2 = \binom{n}{2} (n-2)^m$$

$$= n^m - \binom{n}{1} (n-1)^m + \binom{n}{2} (n-2)^m - \binom{n}{3} (n-3)^m$$

$$+ \dots + (-1)^n (n-n)^m = \sum_{i=0}^n (-1)^i \binom{n}{i} (n-i)^m$$

$$= \sum_{i=0}^n (-1)^i \binom{n}{n-i} (n-i)^m.$$



The Principle of Inclusion and Exclusion

- **Ex 8.10** : In a certain area of the countryside are five villages. An engineer is to devise a system of two-way roads so that after the system is completed, no village will be isolated. In how many ways can he do this?
- Let c_i denote the condition that a system of these roads isolates village a, b, c, d , and e , respectively.

$$N(\bar{c}_1 \bar{c}_2 \bar{c}_3 \bar{c}_4 \bar{c}_5) = 2^{10} - \underset{\substack{\uparrow \\ C(5,2)}}{\binom{5}{1} 2^6} + \underset{\substack{\uparrow \\ C(4,2)}}{\binom{5}{2} 2^3} - \binom{5}{3} 2^1 + \binom{5}{4} 2^0 - \binom{5}{5} 2^0 = 768.$$

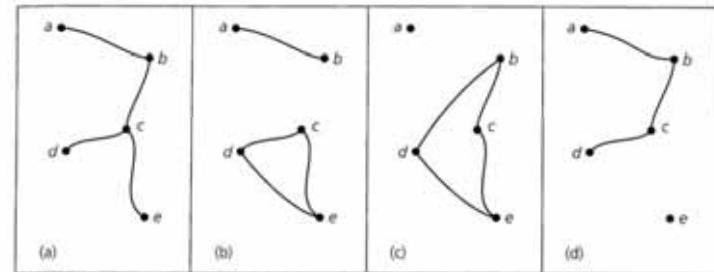


Figure 8.3



8.2 Generalizations of the Principle

- E_m denotes the number of elements in S that satisfy exactly m of the t conditions.
- $E_1 = N(\overline{c_1} \overline{c_2} \overline{c_3} \cdots \overline{c_t}) + N(\overline{c_1} \overline{c_2} \overline{c_3} \cdots \overline{c_t}) + \cdots + N(\overline{c_1} \overline{c_2} \cdots \overline{c_{t-1}} c_t).$
 $E_2 = N(c_1 c_2 \overline{c_3} \cdots \overline{c_t}) + N(c_1 \overline{c_2} c_3 \cdots \overline{c_t}) + \cdots + N(\overline{c_1} c_2 \cdots c_{t-2} c_{t-1} c_t).$
- $E_1 = \mathbf{2+3+4} = N(c_1) + N(c_2) + N(c_3) - 2[N(c_1 c_2) + N(c_1 c_3) + N(c_2 c_3)] + 3N(c_1 c_2 c_3)$
 $= S_1 - 2S_2 + 3S_3 = S_1 - \binom{2}{1} S_2 + \binom{3}{1} S_3$
- $E_2 = \mathbf{5+6+7} = N(c_1 c_2) + N(c_1 c_3) + N(c_2 c_3) - 3N(c_1 c_2 c_3)$
 $= S_2 - 3S_3 = S_2 - \binom{3}{1} S_3$
- $E_3 = \mathbf{8} = N(c_1 c_2 c_3) = S_3$

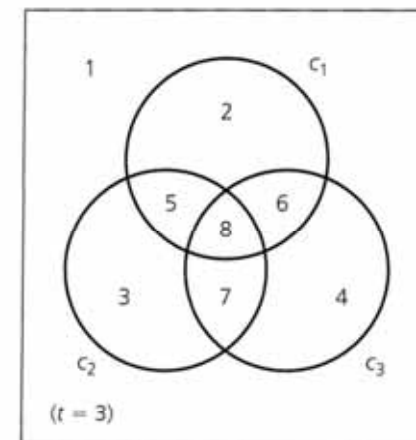


Figure 8.4



Generalizations of the Principle

$$E_1 = S_1 - 2S_2 + 3S_3 - 4S_4$$

$$= S_1 - \binom{2}{1}S_2 + \binom{3}{2}S_3 - \binom{4}{3}S_4$$

$$E_2 = S_2 - 3S_3 + 6S_4$$

$$= S_2 - \binom{3}{1}S_3 + \binom{4}{2}S_4$$

$$E_3 = S_3 - 4S_4 = S_3 + \binom{4}{1}S_4$$

$$E_4 = S_4$$

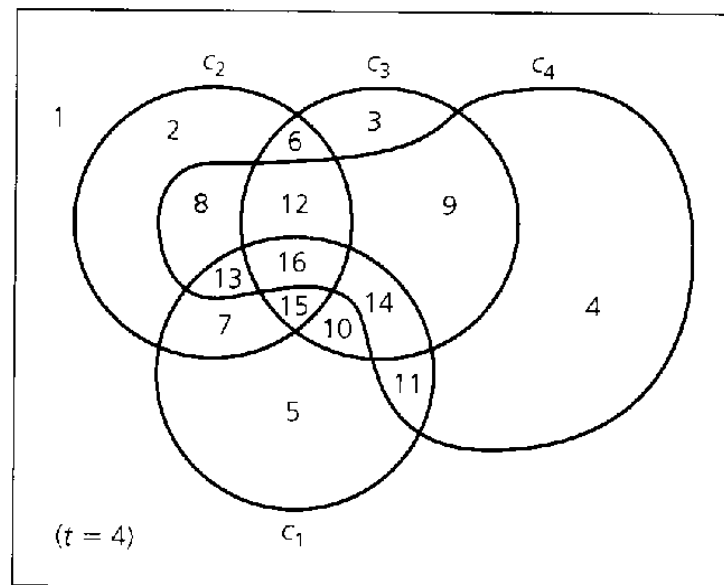


Figure 8.5

Table 8.1

| S_2 | S_3 | S_4 |
|-----------------------------|------------------------|-----------------------|
| $N(c_1c_2): 7, 13, 15, 16$ | $N(c_1c_2c_3): 15, 16$ | $N(c_1c_2c_3c_4): 16$ |
| $N(c_1c_3): 10, 14, 15, 16$ | $N(c_1c_2c_4): 13, 16$ | |
| $N(c_1c_4): 11, 13, 14, 16$ | $N(c_1c_3c_4): 14, 16$ | |
| $N(c_2c_3): 6, 12, 15, 16$ | $N(c_2c_3c_4): 12, 16$ | |
| $N(c_2c_4): 8, 12, 13, 16$ | | |
| $N(c_3c_4): 9, 12, 14, 16$ | | |



Theorem 8.2

$$E_m = S_m - \binom{m+1}{1} S_{m+1} + \binom{m+2}{2} S_{m+2} - \cdots + (-1)^{t-m} \binom{t}{t-m} S_t. \quad (1)$$

(If $m = 0$, we obtain Theorem 8.1.)

Proof: Arguing as in Theorem 8.1, let $x \in S$ and consider the following three cases.

- When x satisfies fewer than m conditions, it contributes a count of 0 to each of the terms $E_m, S_m, S_{m+1}, \dots, S_t$, so it is not counted on either side of the equation.
- When x satisfies exactly m of the conditions, it is counted once in E_m and once in S_m , but not in S_{m+1}, \dots, S_t . Consequently, it is included once in the count for either side of the equation.
- Suppose x satisfies r of the conditions, where $m < r \leq t$. Then x contributes nothing to E_m . Yet it is counted $\binom{r}{m}$ times in S_m , $\binom{r}{m+1}$ times in S_{m+1}, \dots , and $\binom{r}{r}$ times in S_r , but 0 times for any term beyond S_r . So on the right-hand side of the equation, x is counted $\binom{r}{m} - \binom{m+1}{1} \binom{r}{m+1} + \binom{m+2}{2} \binom{r}{m+2} - \cdots + (-1)^{r-m} \binom{r}{r-m} \binom{r}{r}$ times.



For $0 \leq k \leq r - m$,

$$\begin{aligned} \binom{m+k}{k} \binom{r}{m+k} &= \frac{(m+k)!}{k! m!} \cdot \frac{r!}{(m+k)!(r-m-k)!} \\ &= \frac{r!}{m!} \cdot \frac{1}{k!(r-m-k)!} = \frac{r!}{m!(r-m)!} \cdot \frac{(r-m)!}{k!(r-m-k)!} \\ &= \binom{r}{m} \binom{r-m}{k}. \end{aligned}$$

Consequently, on the right-hand side of Eq. (1), x is counted

$$\begin{aligned} &\binom{r}{m} \binom{r-m}{0} - \binom{r}{m} \binom{r-m}{1} + \binom{r}{m} \binom{r-m}{2} - \dots + (-1)^{r-m} \binom{r}{m} \binom{r-m}{r-m} \\ &= \binom{r}{m} \left[\binom{r-m}{0} - \binom{r-m}{1} + \binom{r-m}{2} - \dots + (-1)^{r-m} \binom{r-m}{r-m} \right] \\ &= \binom{r}{m} [1 - 1]^{r-m} = \binom{r}{m} \cdot 0 = 0 \text{ times,} \end{aligned}$$

and the formula is verified.

8.3 Derangements: Nothing Is in Its Right Place



- $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $e^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots$
- $e^{-1} = 0.36788$, $1 - 1 + (1/2!) - (1/3!) + \cdots - (1/7!) \approx 0.36786$
- **Derangement** means that all numbers are in the wrong positions.
- **Ex 8.12** : Determine the number of derangements of $1, 2, \dots, 10$.
Let c_i be the condition that integer i is in the i th place.

$$\begin{aligned}
 d_{10} &= N(\bar{c}_1 \bar{c}_2 \bar{c}_3 \cdots \bar{c}_{10}) = 10! - \binom{10}{1}9! + \binom{10}{2}8! - \binom{10}{3}7! + \cdots + \binom{10}{10}0! \\
 &= 10! \left[1 - \binom{10}{1}(9!/10!) + \binom{10}{2}(8!/10!) - \binom{10}{3}(7!/10!) + \cdots + \binom{10}{10}(0!/10!) \right] \\
 &= 10! \left[1 - 1 + (1/2!) - (1/3!) + \cdots + (1/10!) \right] \doteq (10!)(e^{-1}).
 \end{aligned}$$

Derangements: Nothing Is in Its Right Place



- The general formula:

$$d_n = n!e^{-1} = n!\left[1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \dots - \frac{1}{n!}\right] \quad P = d_n/n!$$

- **Ex 8.14** : Peggy has seven books and hires seven reviewers. She wants two reviewers per book. In how many ways can she make the distributions?
 - The first time: $7!$ ways
 - The second time: $d_7 = 7! * e^{-1}$ Ways (different position)
 - Totally, we have $7! \times d_7$ ways

8.4 Rook Polynomials



- In Fig. 8.6, we want to determine *the number of ways* in which *k rooks* can be placed on the unshaded squares of this chessboard so that no two of them can take each other—that is, *no two of them are in the same row or column of the chessboard C*. This number is denoted as $r_k(C)$.

| | | |
|---|---|---|
| 3 | 2 | 1 |
| 4 | | |
| | 5 | 6 |



Rook Polynomials

- In Fig. 8.6, we have $r_0=1$, $r_1= 6$, $r_2= 8$, $r_3= 2$ and $r_k= 0$ for $k \geq 4$.
- Rook polynomial: $r(C, x) = 1+6x+8x^2+2x^3$. For each $k \geq 0$, the coefficient of x^k is the number of ways we can place k nontaking rooks on chessboard C .

| | | |
|---|---|---|
| 3 | 2 | 1 |
| 4 | | |
| | 5 | 6 |



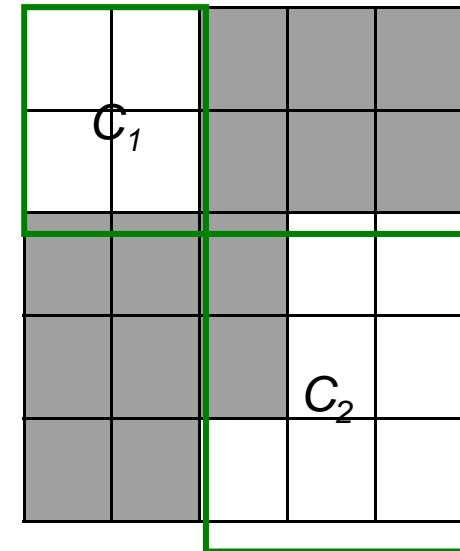
Disjoint Subboards

- Break up a larger board into smaller subboards.
- In Fig. 8.7, the chessboard contains two disjoint subboards that have no squares in the same column or row of C.
- $r(C, x) = r(C_1, x) \cdot r(C_2, x)$

$$r(C_1, x) = 1 + 4x + 2x^2$$

$$r(C_2, x) = 1 + 7x + 10x^2 + 2x^3$$

$$\begin{aligned} r(C, x) &= 1 + 11x + 40x^2 + 56x^3 + 28x^4 + 4x^5 \\ &= r(C_1, x) \cdot r(C_2, x) \end{aligned}$$





Disjoint Subboards

- r_3 for C
 - a) All three rooks are on subboard C_2 (and none is on C_1): $(2)(1) = 2$ ways.
 - b) Two rooks are on subboard C_2 and one is on C_1 : $(10)(4) = 40$ ways.
 - c) One rook is on subboard C_2 and two are on C_1 : $(7)(2) = 14$ ways.
- In general, if C is a chessboard made up of pairwise disjoint subboards C_1, C_2, \dots, C_n , then $r(C, x) = r(C_1, x) \cdot r(C_2, x) \cdot \dots \cdot r(C_n, x)$.



Recursive Formula

- Fig. 8.8 (a), For a given designated square (*), we either (b) place one rook here, or (c) do not use this square.
- $r_k(C) = r_{k-1}(C_s) + r_k(C_e)$
 - C_s : denote the remaining smaller subboard (Fig. 8.8(b))
 - C_e : C with the one designed square eliminated (Fig. 8.8(c))
- $r_k(C)x^k = r_{k-1}(C_s)x^k + r_k(C_e)x^k$ for $1 \leq k \leq n$.

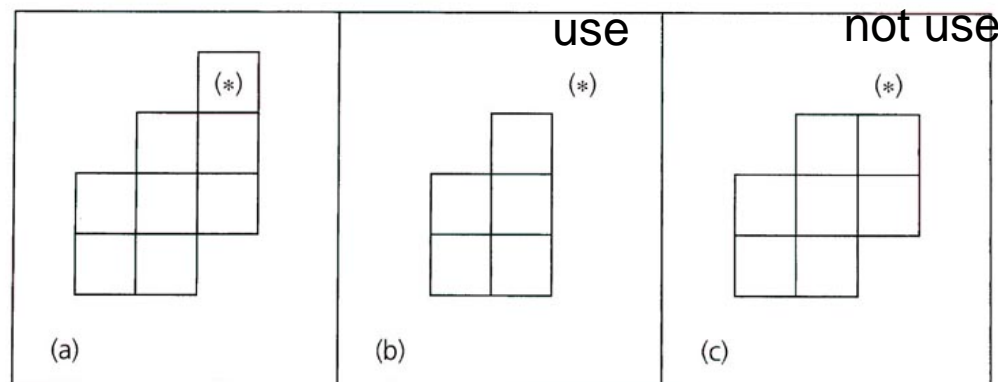


Figure 8.8



Recursive Formula

$$\sum_{k=1}^n r_k(C)x^k = \sum_{k=1}^n r_{k-1}(C_s)x^k + \sum_{k=1}^n r_k(C_e)x^k.$$

$r_0(C_s)x^0$
typo in p405

$$\sum_{k=1}^n r_k(C)x^k = x \sum_{k=1}^n r_{k-1}(C_s)x^{k-1} + \sum_{k=1}^n r_k(C_e)x^k$$

$$1 + \sum_{k=1}^n r_k(C)x^k = x \cdot r(C_s, x) + \sum_{k=1}^n r_k(C_e)x^k + 1,$$

$$r(C, x) = x \cdot r(C_s, x) + r(C_e, x).$$



Apply the Recursive Formula

$$\begin{aligned}
 & \left(\begin{array}{ccc} & & (*) \\ & \square & \square \\ \square & \square & \square \\ \square & & \end{array} \right) = x \left(\begin{array}{cc} & (*) \\ & \square \\ \square & \square \end{array} \right) + \left(\begin{array}{ccc} & & (*) \\ \square & \square & \square \\ \square & & \end{array} \right) \\
 & = x \left[x \left(\begin{array}{c} \square \\ \square \end{array} \right) + \left(\begin{array}{cc} \square & \square \\ \square & \square \end{array} \right) \right] + \left[x \left(\begin{array}{cc} \square & \square \\ \square & \square \end{array} \right) + \left(\begin{array}{ccc} & & (*) \\ \square & \square & \square \end{array} \right) \right] \\
 & \quad \text{use } * \quad \text{not use } * \\
 & = x^2 \left(\begin{array}{c} \square \\ \square \end{array} \right) + 2x \left(\begin{array}{cc} \square & \square \\ \square & \square \end{array} \right) + \left[x \left(\begin{array}{cc} & \square \\ \square & \square \end{array} \right) + \left(\begin{array}{ccc} & & (*) \\ \square & \square & \square \end{array} \right) \right] \\
 & = x^2(1 + 2x) + 2x(1 + 4x + 2x^2) + x(1 + 3x + x^2) \\
 & \quad + \left[x \left(\begin{array}{c} \square \\ \square \end{array} \right) + \left(\begin{array}{cc} \square & \square \\ \square & \square \end{array} \right) \right] \\
 & = 3x + 12x^2 + 7x^3 + \underline{x(1 + 2x) + (1 + 4x + 2x^2)} = 1 + 8x + 16x^2 + 7x^3.
 \end{aligned}$$



8.5 Arrangements with Forbidden Positions

- **Ex 8.15** : In making seating arrangements, the shaded square of the figure means relative R_i will not sit at table T_j .
- Determine the number of ways that we can seat these four relatives at five different tables.
- Let $|S|$ be the total number of ways we can place the four relatives. ($|S| = 5!$)
- Let c_i be the condition that R_i is seated in a forbidden position but at different tables.
 - $N(c_1) = 4! + 4!$ ($R_1 \rightarrow T_1$ or $R_1 \rightarrow T_2$)
 - $N(c_2) = 4!$ ($R_2 \rightarrow T_2$)
 - $N(c_3) = ?$ $4! + 4!$
 - $N(c_4) = ?$ $4! + 4!$
- $S_1 = 7(4!)$

| | T_1 | T_2 | T_3 | T_4 | T_5 |
|-------|-------|-------|-------|-------|-------|
| R_1 | | | | | |
| R_2 | | | | | |
| R_3 | | | | | |
| R_4 | | | | | |

number of shaded squares



Arrangements with Forbidden Positions

- $S_2 = 16(3!)$
 - $N(c_1c_2) = 3! (R_1 \rightarrow T_1 \text{ or } R_2 \rightarrow T_2)$
 - $N(c_1c_3) = 4(3!)$
 - $N(c_1c_4) = ?, N(c_2c_3) = ?, N(c_2c_4) = ?, N(c_3c_4) = ?$
- Observation: 16 is the number of ways two nontaking rooks can be placed on the shaded chessboard.
- Let r_i be the number of ways in which it is possible to place i nontaking rooks on the shaded chessboard.
 - For all $0 \leq i \leq 4$, $S_i = r_i(5 - i)!$
- Decompose C into the disjoint subboards in the upper left and lower right corners.
- $r(C, x) = (1 + 3x + x^2)(1 + 4x + 3x^2) = 1 + 7x + 16x^2 + 13x^3 + 3x^4$

$$\begin{aligned} \therefore \overline{N(c_1c_2c_3c_4)} &= S_0 - S_1 + S_2 - S_3 + S_4 = 1(5!) - 7(4!) + 16(3!) - 13(2!) + 3(1!) \\ &= \sum_{i=0}^4 (-1)^i r_i(5 - i)! = 25 \end{aligned}$$