ساختمان داده و الگوریتم ها

مبحث سوم:
تحلیل زمانی الگوریتم ها

سجاد شیرعلی شهرضا

پائیز 1402

سه شنبه، 10 مهر 1402

Based on Stanford CS161 Slides from Summer 2020 by Karey Shi

بخشهای مرتبط در کتاب

- جلسه قبل (ضرب و تقسیم و حل): 2.3 و 4.4
- این جلسه (تحلیل زمانی): 3
- واژه نامه ی انگلیسی به فارسی و فارسی به انگلیسی (پیوستهای 3 و 4 کتاب دکتر قدسی):
http://sharif.edu/~ghodsi/books/ds-algf-dics-both.pdf

FROM LAST WEEK

THE POINT OF ASYMPTOTIC NOTATION

suppress $\underbrace{\text{constant factors}}_{\textit{too system dependent}}$ and $\underbrace{\text{lower-order terms}}_{\textit{irrelevant for large inputs}}$

- **Some guiding principles:** we care about how the running time/number of operations *scales* with the size of the input (i.e. the runtime's *rate of growth*), and we want some measure of runtime that's independent of hardware, programming language, memory layout, etc.
    - We want to reason about high-level algorithmic approaches rather than lower-level details

A NOTE ON RUNTIME ANALYSIS

There are a few different ways to analyze the runtime of an algorithm:

Worst-case analysis:
What is the runtime of the algorithm on the *worst* possible input?

Best-case analysis:
What is the runtime of the algorithm on the *best* possible input?

Average-case analysis:
What is the runtime of the algorithm on the *average* input?

A NOTE ON RUNTIME ANALYSIS

There are a few different ways to analyze the runtime of an algorithm:

We'll mainly focus on worst case analysis since it tells us how fast the algorithm is on *any* kind of input

Worst-case analysis:
What is the runtime of the algorithm on the *worst* possible input?

Best-case analysis:
What is the runtime of the algorithm on the *best* possible input?

Average-case analysis:
What is the runtime of the algorithm on the *average* input?

We'll also work on this in some cases.

BIG-O NOTATION

Let $T(n)$ & $f(n)$ be functions defined on the positive integers.

(In this class, we'll typically write $T(n)$ to denote the worst case runtime of an algorithm)

What do we mean when we say "$T(n)$ is $O(f(n))$"?

English Definition

Pictorial Definition

Mathematical Definition

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What do we mean when we say "T(n) is O(f(n))"?

In English

$T(n) = O(f(n))$ if and only if $T(n)$ is *eventually* **upper bounded** by a constant multiple of $f(n)$

Pictorial Definition

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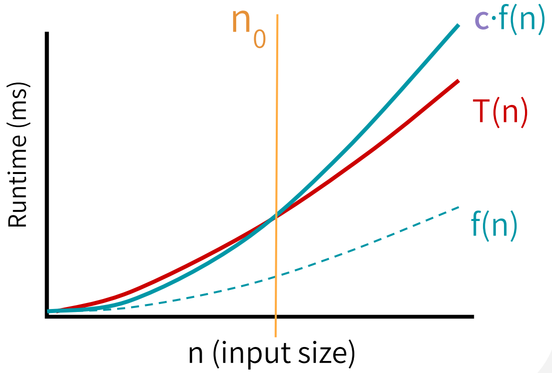

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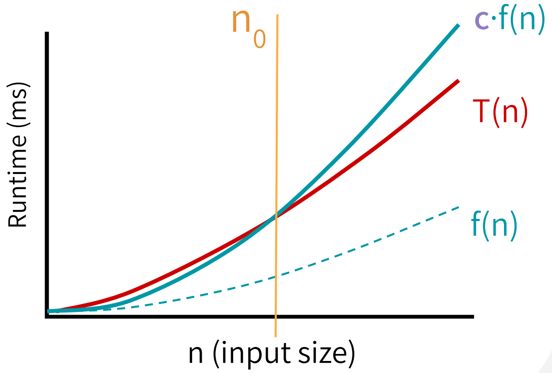

In Math

$T(n) = O(f(n))$ if and only if there exists positive **constants** **c** and $\mathbf{n_0}$ such that *for all* $n \geq n_0$

$$T(n) \leq c \cdot f(n)$$

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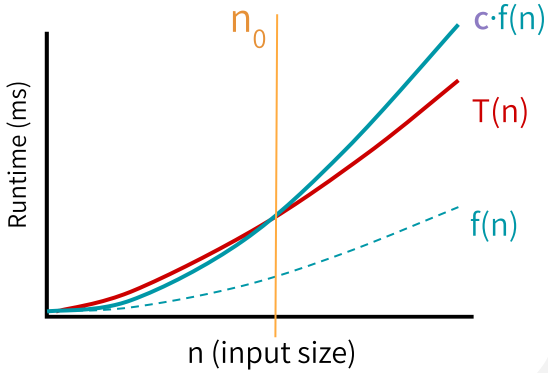

In Math

$$T(n) = O(f(n))$$
$$\Leftrightarrow$$
$$\exists\ c, n_0 > 0 \text{ s.t. } \forall\ n \geq n_0,$$
$$\mathbf{T(n) \leq c \cdot f(n)}$$

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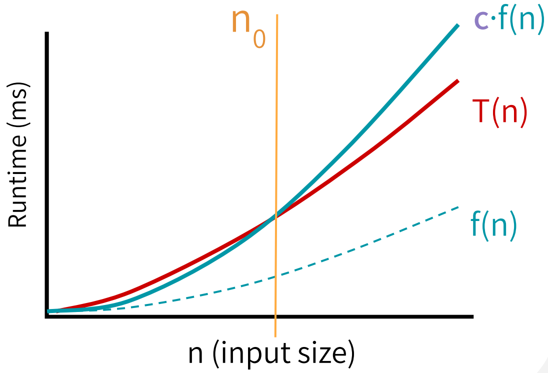

In Math

$$T(n) = O(f(n))$$

"if and only if" $\Leftrightarrow$

"for all"

$$\exists\ c, n_0 > 0 \text{ s.t. } \forall\ n \geq n_0,$$

"there exists" "such that"

$$T(n) \leq c \cdot f(n)$$

PROVING BIG-O BOUNDS

If you're ever asked to formally prove that T(n) is O(f(n)), use the *MATH* definition:

$$T(n) = O(f(n))$$
$$\Leftrightarrow$$
$$\exists\ c, n_0 > 0 \text{ s.t. } \forall\ n \geq n_0,$$
$$\mathbf{T(n) \leq c \cdot f(n)}$$

must be constants!
i.e. c & n_0 cannot
depend on n!

- To **prove** T(n) = O(f(n)), you need to announce your c & n_0 up front!
    - Play around with the expressions to find appropriate choices of c & n_0 (positive constants)
    - Then you can write the proof! Here how to structure the start of the proof:

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    - Play around with the expressions to find appropriate choices of c & n_0 (positive constants)
    - Then you can write the proof! Here how to structure the start of the proof:

"Let c = __ and n_0 = __. We will show that $T(n) \leq c \cdot f(n)$ for all $n \geq n_0$."

PROVING BIG-O BOUNDS: EXAMPLE

$$T(n) = O(f(n))$$

$$\Leftrightarrow$$

$$\exists\ c, n_0 > 0 \text{ s.t. } \forall\ n \geq n_0,$$

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Prove that $3n^2 + 5n = O(n^2)$.

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Let $c = 4$ and $n_0 = 5$. We will now show that $3n^2 + 5n \leq c \cdot n^2$ for all $n \geq n_0$.

■

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Prove that $3n^2 + 5n = O(n^2)$.

Let $c = 4$ and $n_0 = 5$. We will now show that $3n^2 + 5n \leq c \cdot n^2$ for all $n \geq n_0$.
We know that for any $n \geq n_0 = 5$, we have:

$$5 \leq n$$
$$5n \leq n^2$$
$$3n^2 + 5n \leq 4n^2$$

Using our choice of c and n_0, we have successfully shown that $3n^2 + 5n \leq c \cdot n^2$ for all $n \geq n_0$. From the definition of Big-O, this proves that $3n^2 + 5n = O(n^2)$. ■

DISPROVING BIG-O BOUNDS

If you're ever asked to formally disprove that T(n) is O(f(n)), use **proof by contradiction!**

DISPROVING BIG-O BOUNDS

If you're ever asked to formally disprove that $T(n)$ is $O(f(n))$, use **proof by contradiction!**

For sake of contradiction, assume that $T(n)$ is $O(f(n))$. In other words, assume there does indeed exist a choice of c & n_0 s.t. $\forall\ n \geq n_0$, $\mathbf{T(n) \leq c \cdot f(n)}$

pretend you have a friend that comes up and says "I have a c & n_0 that will prove $T(n) = O(f(n))$!!!", and you say "ok fine, let's assume your c & n_0 does prove $T(n) = O(f(n))$"

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Treating c & n_0 as variables, derive a contradiction!

although you are skeptical, you'll entertain your friend by saying: "let's see what happens. [some math work... and then...] AHA! regardless of what your constants c & n_0, trusting you has led me to something *impossible!!!*"

DISPROVING BIG-O BOUNDS

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↓

Conclude that the original assumption must be false, so **T(n) is *not* O(f(n))**.

you have triumphantly proven your silly (or lying) friend wrong.

DISPROVING BIG-O: EXAMPLE

$T(n) = O(f(n))$
$\Leftrightarrow$
$\exists\ c, n_0 > 0$ s.t. $\forall\ n \geq n_0$,
$T(n) \leq c \cdot f(n)$

Prove that $3n^2 + 5n$ is *not* $O(n)$.

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Prove that $3n^2 + 5n$ is *not* $O(n)$.

For sake of contradiction, assume that $3n^2 + 5n$ is $O(n)$. This means that there exists positive constants c & n_0 such that $3n^2 + 5n \leq c \cdot n$ for all $n \geq n_0$.

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For sake of contradiction, assume that $3n^2 + 5n$ is $O(n)$. This means that there exists positive constants c & n_0 such that $3n^2 + 5n \leq c \cdot n$ for all $n \geq n_0$.
Then, we would have the following:

$$3n^2 + 5n \leq c \cdot n$$
$$3n + 5 \leq c$$
$$n \leq (c - 5)/3$$

■

DISPROVING BIG-O: EXAMPLE

$T(n) = O(f(n))$

$\Leftrightarrow$

$\exists\ c, n_0 > 0$ s.t. $\forall\ n \geq n_0$,

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Prove that $3n^2 + 5n$ is *not* $O(n)$.

For sake of contradiction, assume that $3n^2 + 5n$ is $O(n)$. This means that there exists positive constants c & n_0 such that $3n^2 + 5n \leq c \cdot n$ for all $n \geq n_0$.
Then, we would have the following:

$$3n^2 + 5n \leq c \cdot n$$

$$3n + 5 \leq c$$

$$n \leq (c - 5)/3$$

However, since $(c - 5)/3$ is a constant, we've arrived at a contradiction since n cannot be bounded above by a constant for all $n \geq n_0$. For instance, consider $n = n_0 + c$: we see that $n \geq n_0$, but $n > (c - 5)/3$. Thus, our original assumption was incorrect, which means that $3n^2 + 5n$ is not $O(n)$. ■

BIG-O EXAMPLES

$$\log_2 n + 15 = O(\log_2 n)$$

$$3^n = O(4^n)$$

Polynomials

Say $p(n) = a_k n^k + a_{k-1} n^{k-1} + \cdots + a_1 n + a_0$ is a polynomial of degree $k \geq 1$.

Then:

i. $p(n) = O(n^k)$
ii. $p(n)$ is **not** $O(n^{k-1})$

$$6n^3 + n \log_2 n = O(n^3)$$

$$25 = O(1)$$

[any constant] = $O(1)$

BIG-O EXAMPLES

lower order terms don't matter!

$$\log_2 n + 15 = O(\log_2 n)$$

remember, big-O is upper bound!

$$3^n = O(4^n)$$

Polynomials

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Then:

i. $p(n) = O(n^k)$
ii. $p(n)$ is **not** $O(n^{k-1})$

constant multipliers & lower order terms don't matter

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$$25 = O(1)$$

$$[\text{any constant}] = O(1)$$

سوال؟

BIG-Ω NOTATION

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What do we mean when we say "$T(n)$ is $\Omega(f(n))$"?

English Definition

Pictorial Definition

Mathematical Definition

BIG-Ω NOTATION

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In English

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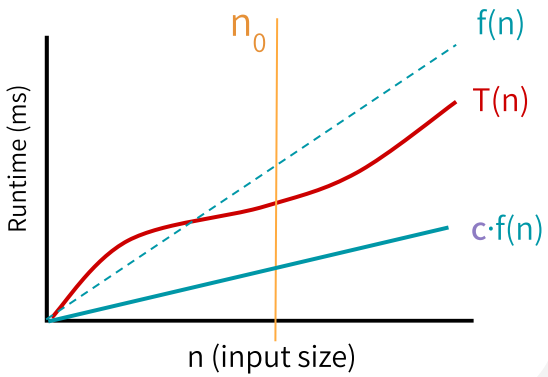

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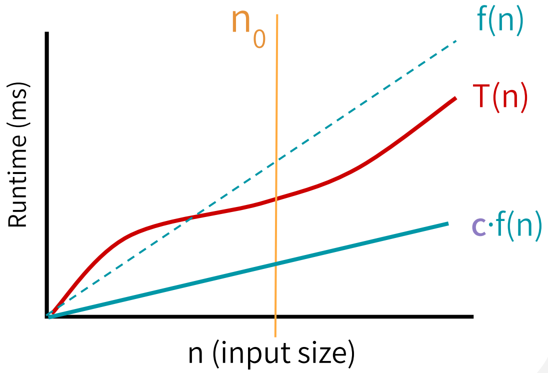

In Math

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$$\Leftrightarrow$$

$$\exists \ c, n_0 > 0 \ \text{s.t.} \ \forall \ n \geq n_0,$$

$$\mathbf{T(n) \geq c \cdot f(n)}$$

inequality switched directions!

BIG-Θ NOTATION

We say **"$T(n)$ is $\Theta(f(n))$"** if and only if both

$$\mathbf{T(n) = O(f(n))}$$

and

$$\mathbf{T(n) = \Omega(f(n))}$$

$$T(n) = \Theta(f(n))$$

$$\Leftrightarrow$$

$$\exists\ c_1, c_2, n_0 > 0 \text{ s.t. } \forall\ n \geq n_0,$$

$$\mathbf{c_1 \cdot f(n) \leq T(n) \leq c_2 \cdot f(n)}$$

ASYMPTOTIC NOTATION CHEAT SHEET

BOUND	DEFINITION (HOW TO PROVE)	WHAT IT REPRESENTS
$T(n) = O(f(n))$	$\exists\ c > 0,\ \exists\ n_0 > 0$ s.t. $\forall\ n \geq n_0,\ T(n) \leq c \cdot f(n)$	upper bound
$T(n) = \Omega(f(n))$	$\exists\ c > 0,\ \exists\ n_0 > 0$ s.t. $\forall\ n \geq n_0,\ T(n) \geq c \cdot f(n)$	lower bound
$T(n) = \Theta(f(n))$	$T(n) = O(f(n))$ and $T(n) = \Omega(f(n))$	tight bound

سوال؟