ساختمان داده و الگوریتم ها

مبحث دوازدهم:
درخت

سجاد شیرعلی شهرضا

پاییز 1402

دوشنبه، 22 آبان 1402

Based on Washington University CSE 373 (Autumn 2020 by Hunter Schafer)

اطلاع رسانی

- بخش مرتبط کتاب برای این جلسه: 10.4

درخت

Data Structures

- Data structure
  - Organization or format for storing or managing data
  - Concrete realization of an abstract data type
- Operations
  - Always a tradeoff: some operations more efficient, some less, for any data structure
  - Choose efficient data structure for operations of concern

Example Data Structures

Data Structure	add(val v)	get(int i)
Array [2 | 1 | 3 | 0]		
Linked List 2 → 1 → 3 → 0		

```
add(v): append v
get(i): return element at position i
```

Example Data Structures

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Linked List 2 → 1 → 3 → 0		

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Linked List $2 \to 1 \to 3 \to 0$	$O(1)$	

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Linked List (2) → (1) → (3) → (0)	$O(1)$	

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Example Data Structures

Data Structure	add(val v)	get(int i)	contains(val v)
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Linked List 2 → 1 → 3 → 0	$O(1)$	$O(n)$	

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add(v): append v
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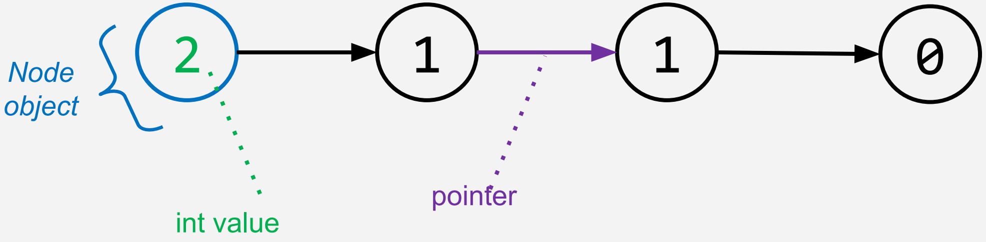
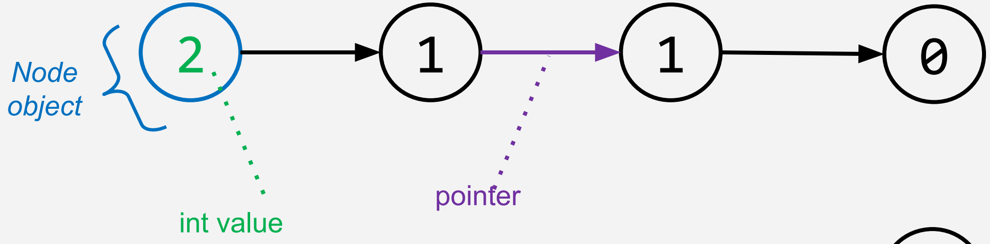
Linked List

Singly linked list:

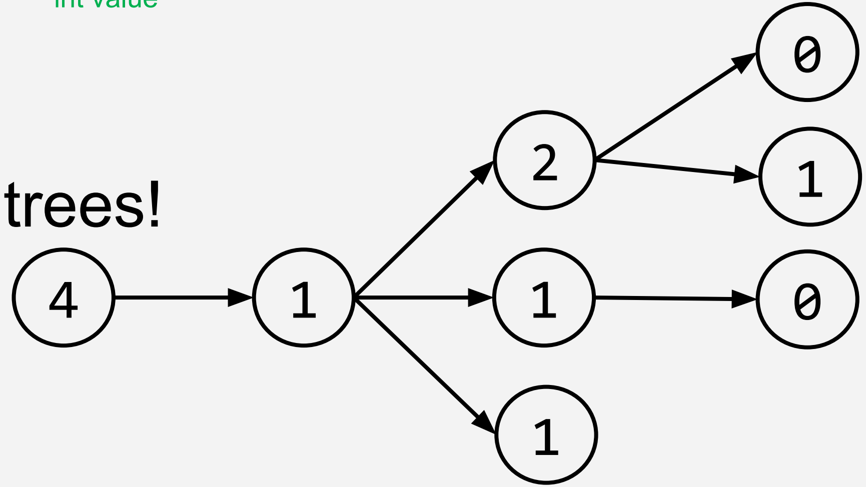


Generalized Linked List (i.e., Tree)

Singly linked list:

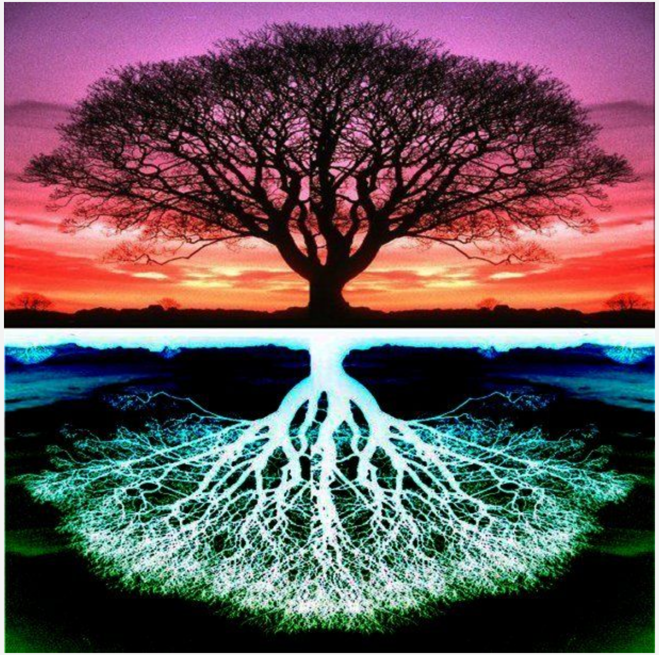


Today: trees!



Tree

- In CS, we draw trees “upside down”



Tree Overview

- Tree:
  - Data structure with nodes
  - Similar to linked list

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- All nodes are reachable from root

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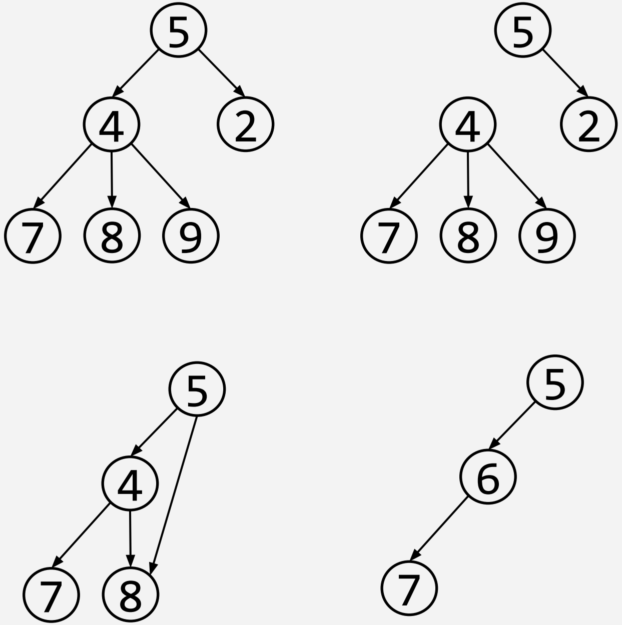
A tree or not a tree?



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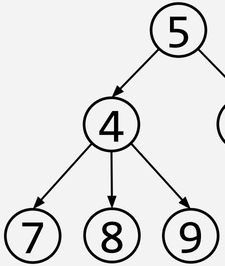
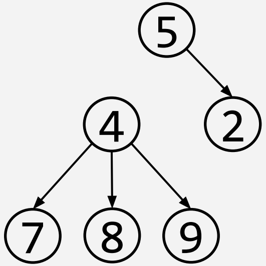
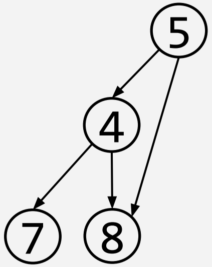
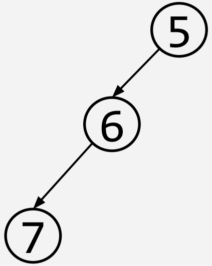




✅A tree





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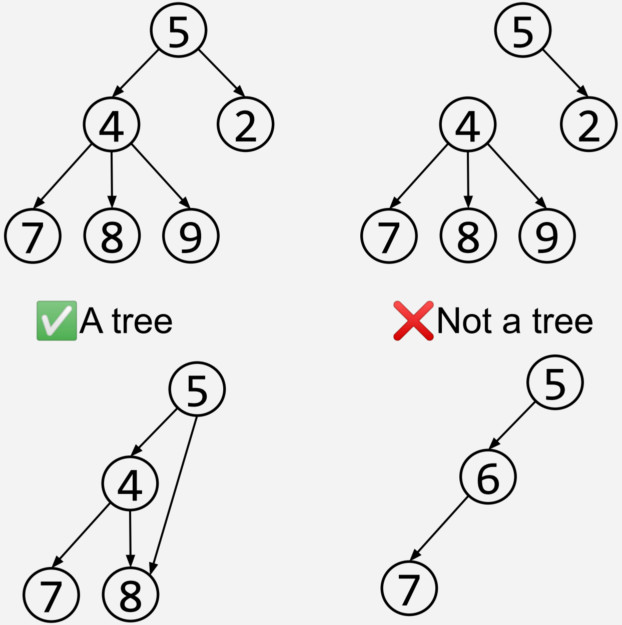

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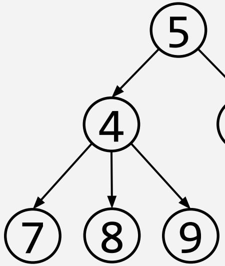

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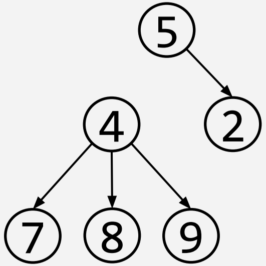

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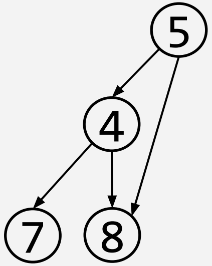

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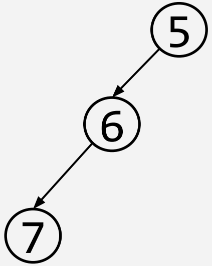

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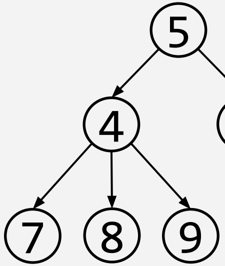

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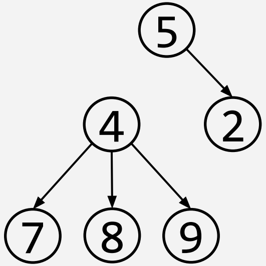

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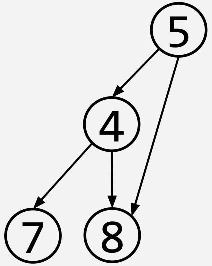

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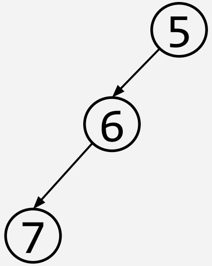


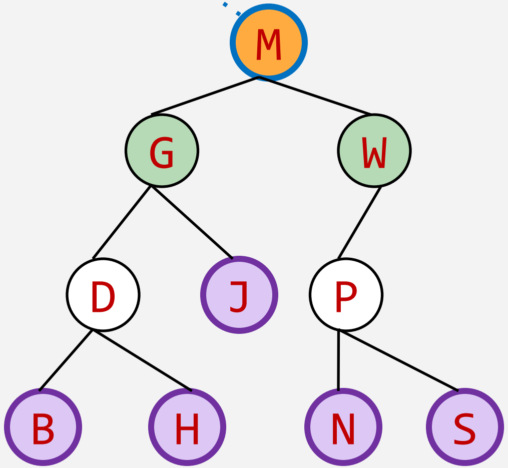
✅A tree

سوال؟

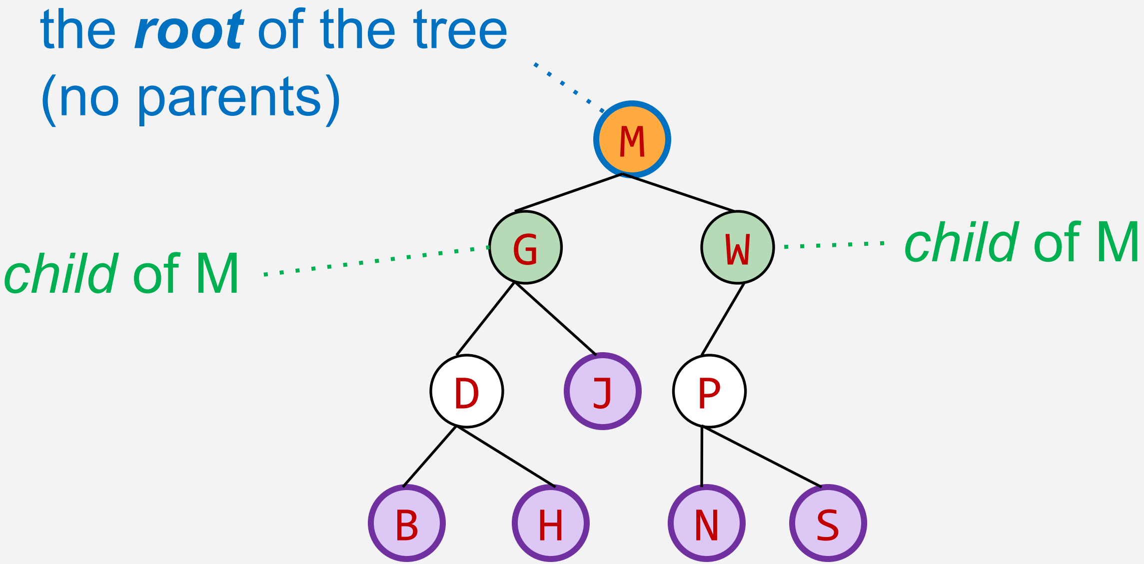
اصطلاحات درخت

Parent, Child, Leaves, Root

the ***root*** of the tree
(no parents)



Parent, Child, Leaves, Root



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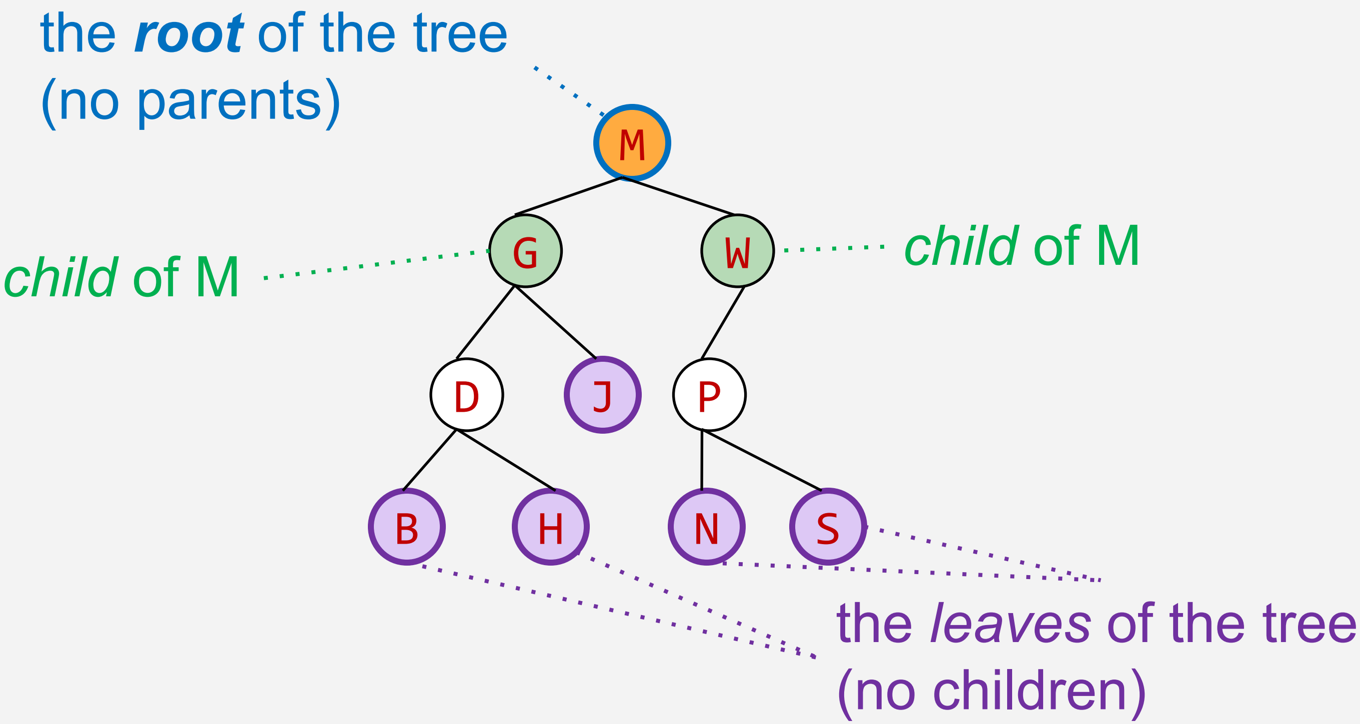
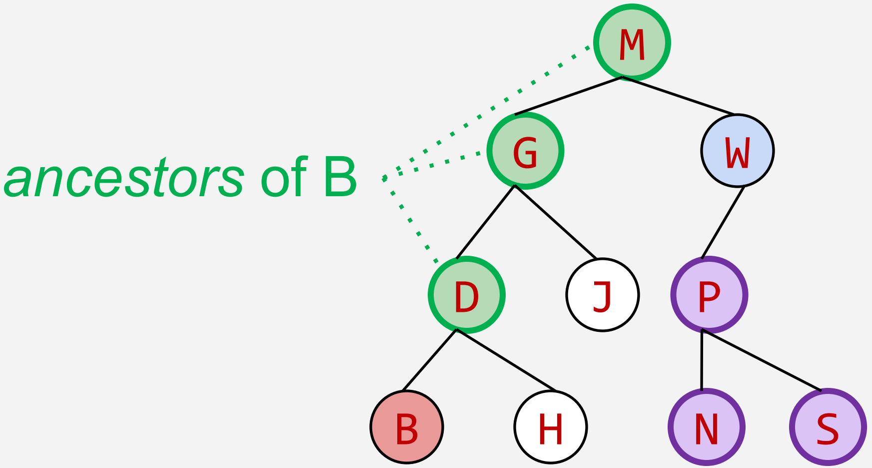


Ancestors and Descendants



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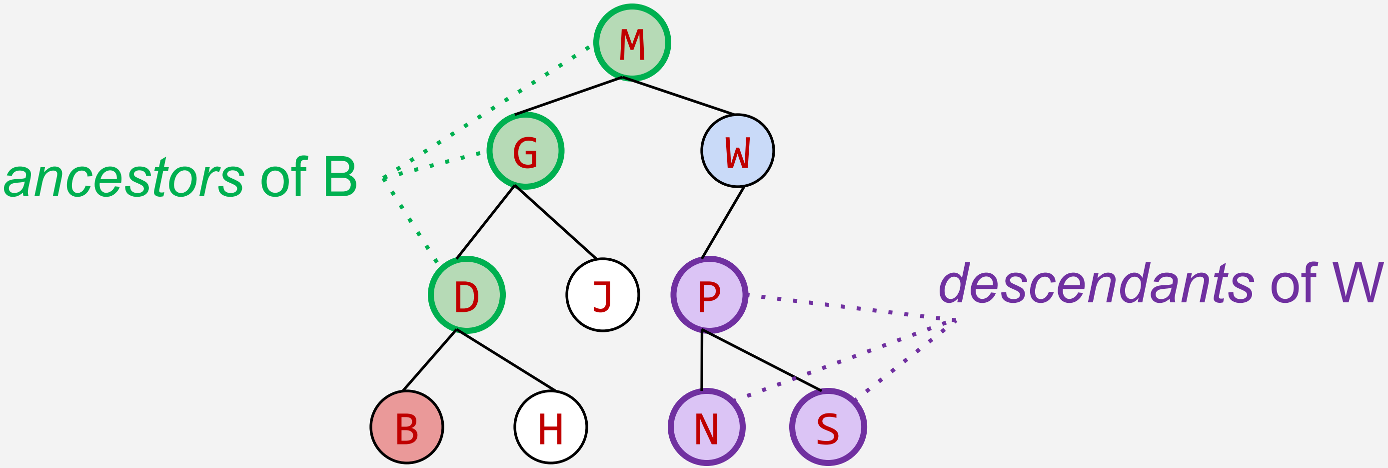
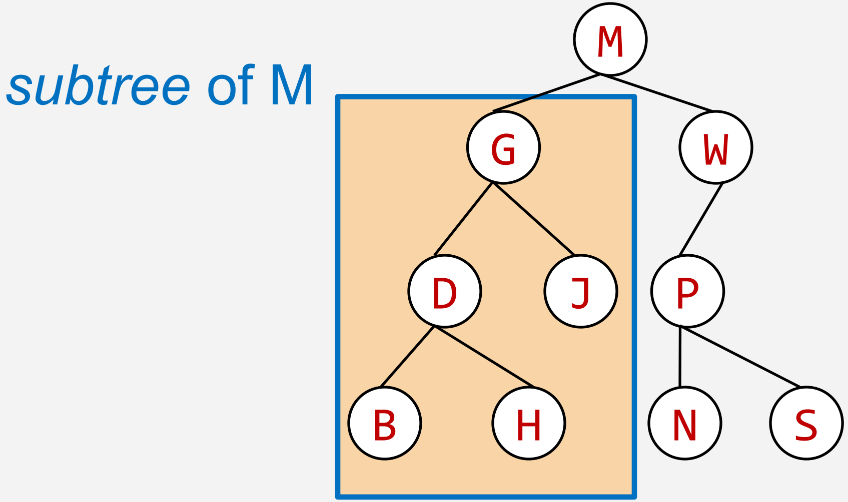
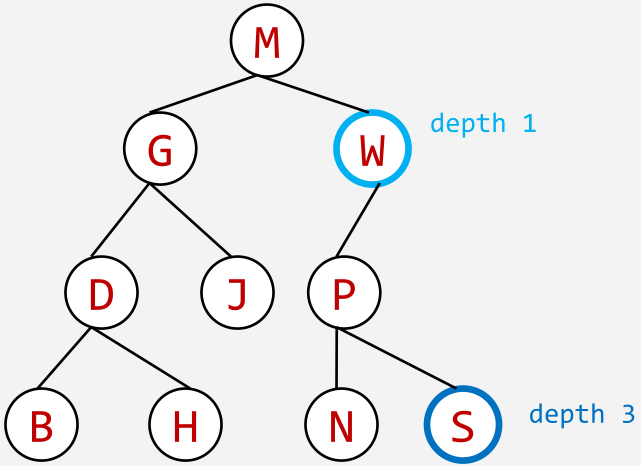


Subtree



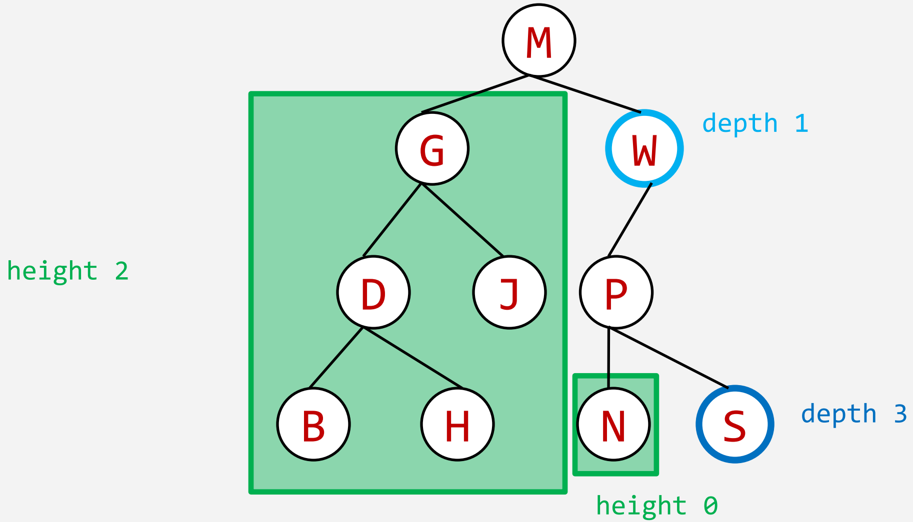
Depth & Height

- **Node depth**: the length of the path to the root



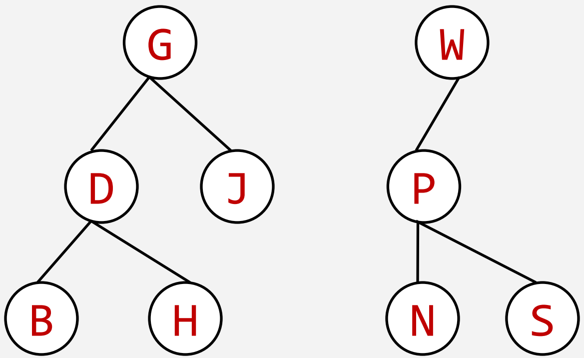
Depth & Height

- **Node depth**: the length of the path to the root
- **Tree (or subtree) height**: the length of the longest path from the root to a leaf



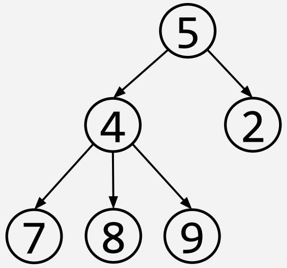
Forest

- Multiple trees!



General vs. Binary Trees

- **General tree**: every node can have an arbitrary number of children



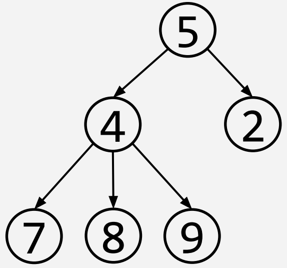
General tree

General vs. Binary Trees

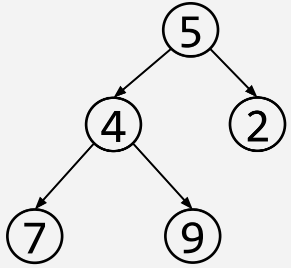
- **General tree**: every node can have an arbitrary number of children

- **Binary tree**: at most two children, called left and right

  ...often "tree" means binary tree



General tree



Binary tree

Nodes at each level in binary tree

- Maximum # of nodes at depth d: 2^d

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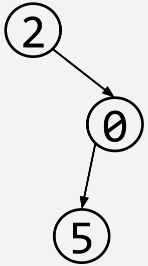
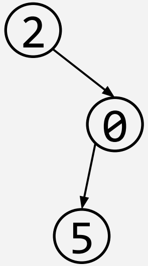


Height 2,
minimum number of nodes

Nodes at each level in binary tree

- Maximum # of nodes at depth d: 2^d
- If height of tree is h:
  - Minimum # of nodes: h + 1
  - Maximum # of nodes:
    - $2^0 + \ldots + 2^h = 2^{h+1} - 1$
    - Known as **Perfect tree**



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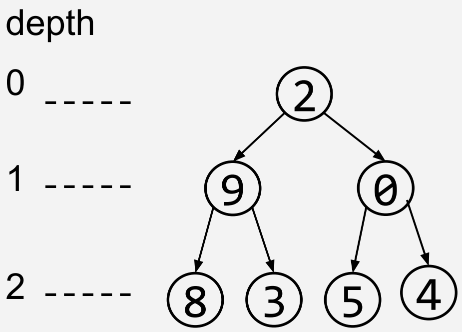
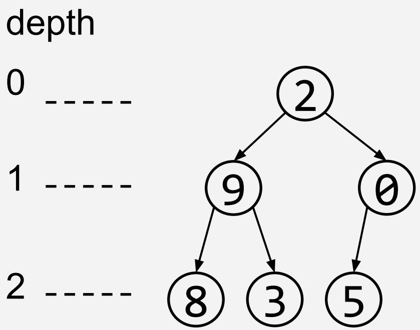


Height 2,
maximum number of nodes

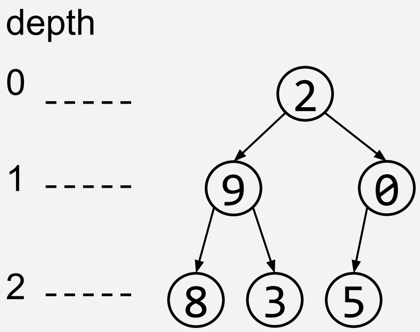
Complete binary tree

- Every level, except last, is completely filled
- Nodes on bottom level as far left as possible
    - I.e., no holes



Complete binary tree

- Every level, except last, is completely filled
- Nodes on bottom level as far left as possible
    - I.e., no holes
- We saw it before in priority queue (heap)!

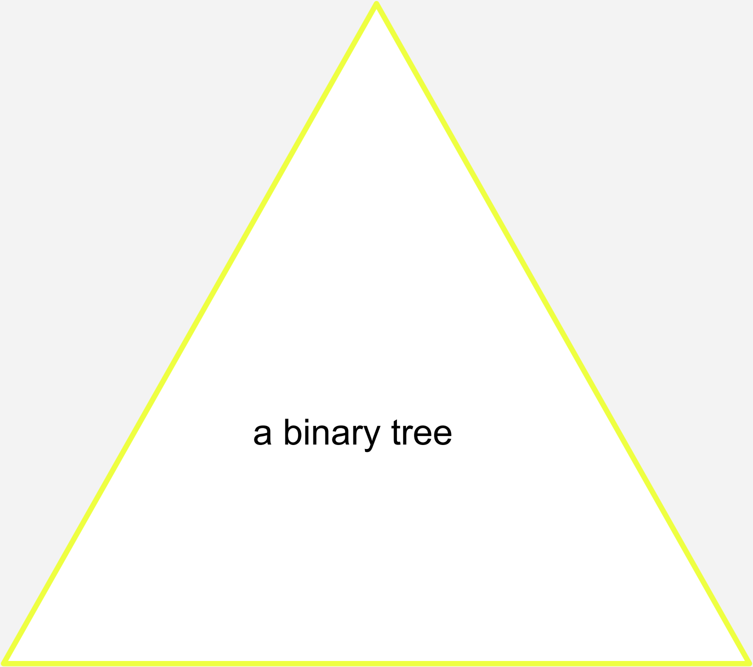


سوال؟

پردازش درخت

انجام عملیات بر روی درخت

Recursive Definition



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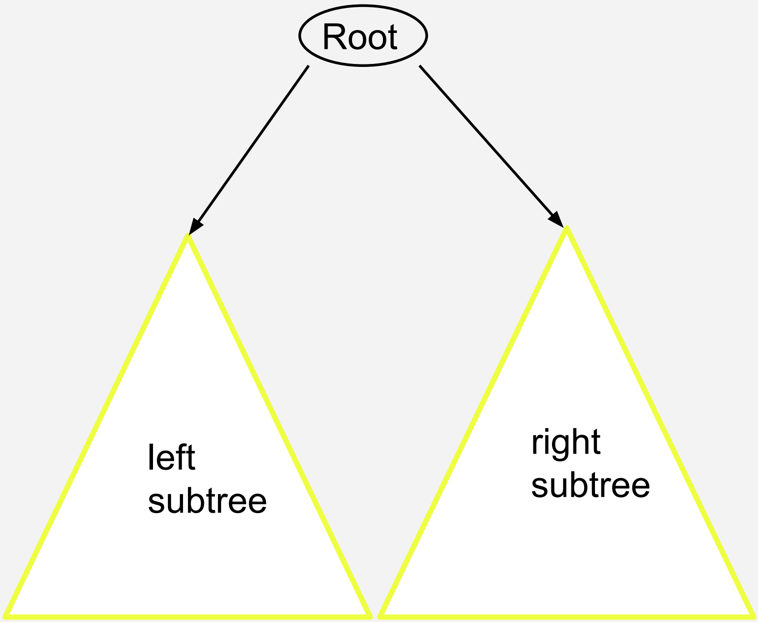

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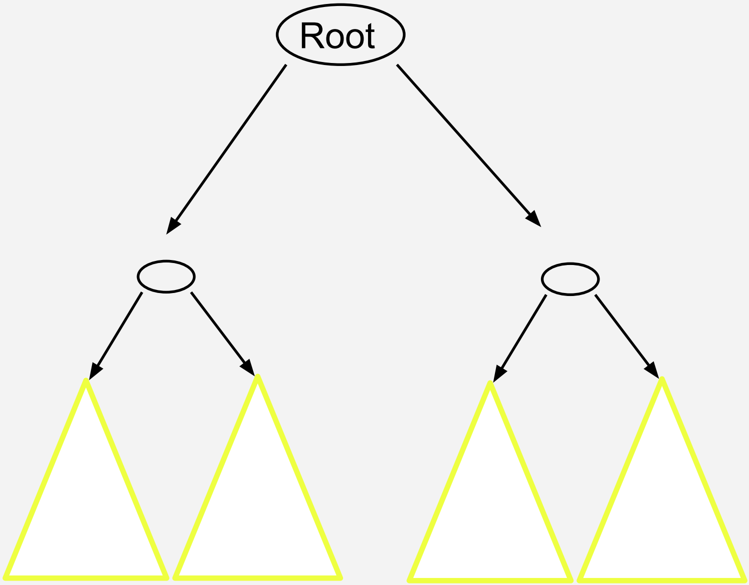

Binary Tree

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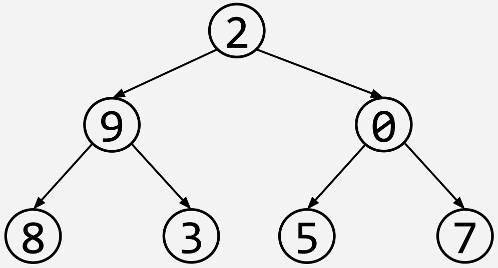

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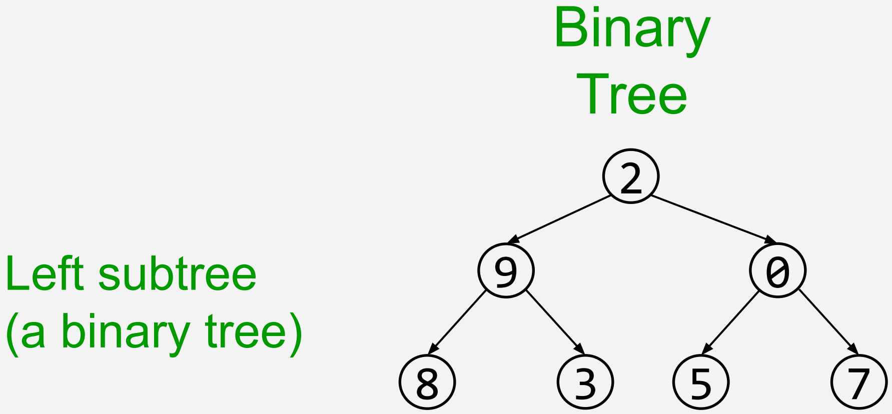

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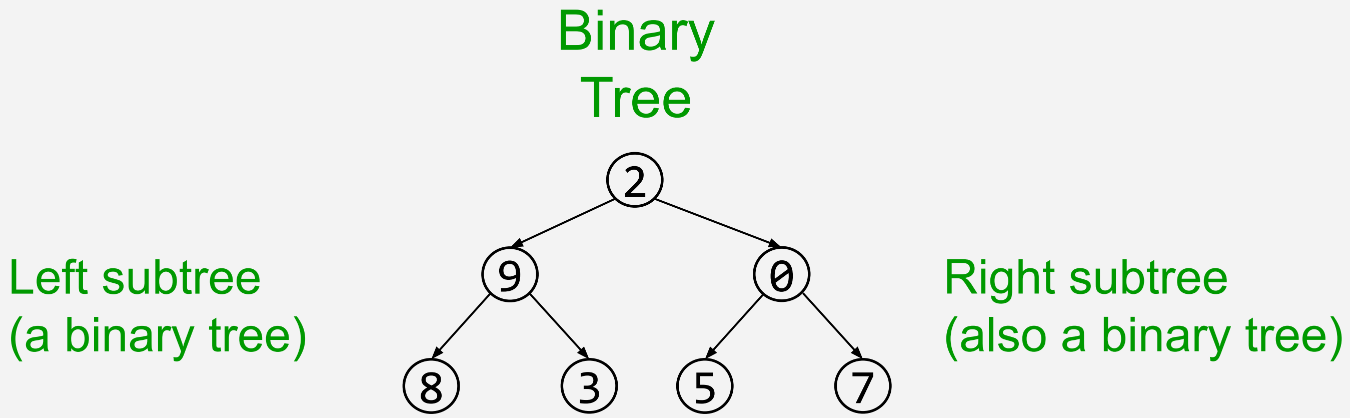
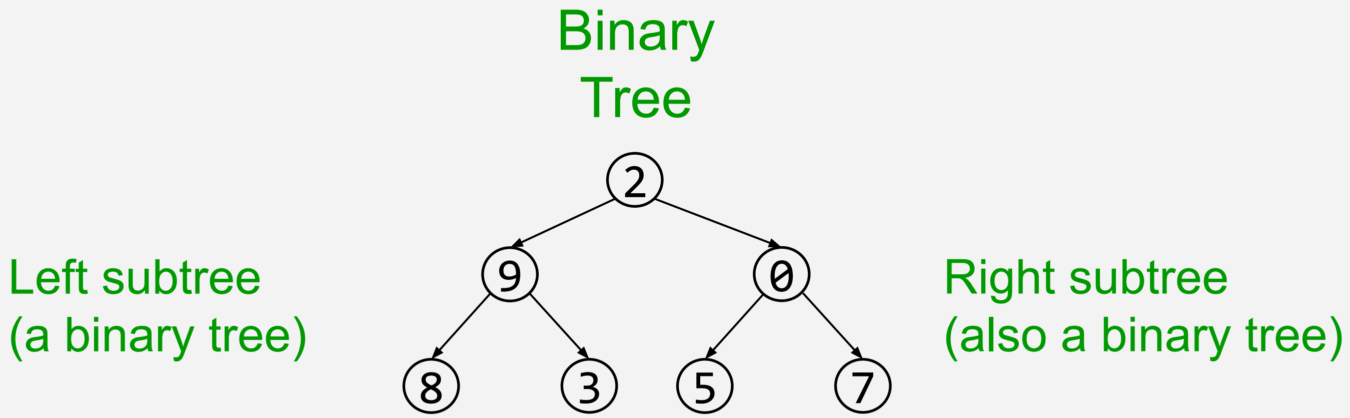


Binary
Tree
2
9
0
8
3
5
7
Left subtree
(a binary tree)
Right subtree
(also a binary tree)

Binary Tree

- A binary tree is either
  - Null
  - An object consisting of a value, a left binary tree, and a right binary tree



Recipe for Recursive Functions

- Base case:
  - If the input is “easy,” just solve the problem directly.

- Recursive case:
  - Get a smaller part of the input (or several parts).
  - Call the function on the smaller value(s).
  - Use the recursive result to build a solution for the full input.

Recipe for Recursive Functions on Binary Trees

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    an empty tree (null), or a leaf

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Recipe for Recursive Functions on Binary Trees

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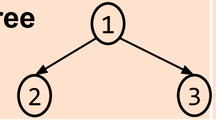
  - If the input is ~~"easy,"~~ just solve the problem directly.

    an empty tree (null), or a leaf

- Recursive case:
  - ~~Get a smaller part of the input (or several parts).~~
  - Call the function on ~~the smaller value(s).~~ each subtree.
  - Use the recursive result to build a solution for the full input.

سوال؟

Search in a Tree

Data Structure	add(val v)	get(int i)	contains(val v)
Array 2 1 3 0	$O(n)$	$O(1)$	$O(n)$
Linked List 2 → 1 → 3 → 0	$O(1)$	$O(n)$	$O(n)$
Binary Tree 			

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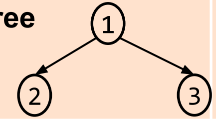
Binary Tree 			$O(n)$

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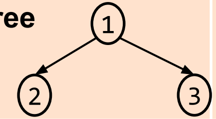
Binary Tree 			$O(n)$

Node could be *anywhere* in tree

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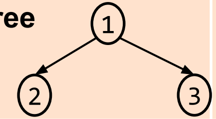
Binary Tree 			$O(n)$

Node could be *anywhere* in tree

Binary search on arrays: O(log n)
Requires invariant: array sorted
…analogue for trees?
TO BE CONTINUED!
(in a future lecture)

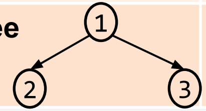
سوال؟

پیمایش درخت

پیمایش و ذخیره یک درخت

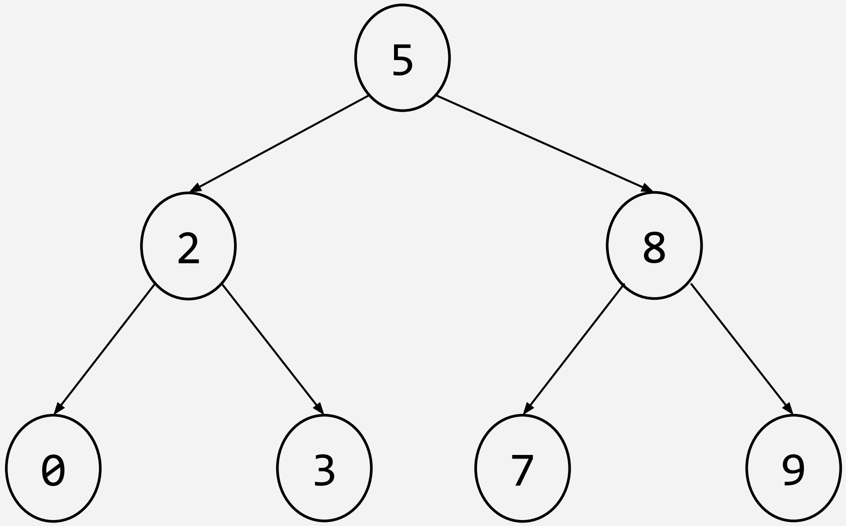
Iterate through data structure

- Iterate: process elements of data structure
  - Sum all elements
  - Print each element

Data Structure	**Order to iterate**
Array 2 1 3 0	Forwards: 2, 1, 3, 0 Backwards: 0, 3, 1, 2
Linked List 2 → 1 → 3 → 0	Forwards: 2, 1, 3, 0
Binary Tree 	**???**

Iterate through a tree

- What would a reasonable order be?



Tree traversals

- Iterating through tree is also known as tree traversal

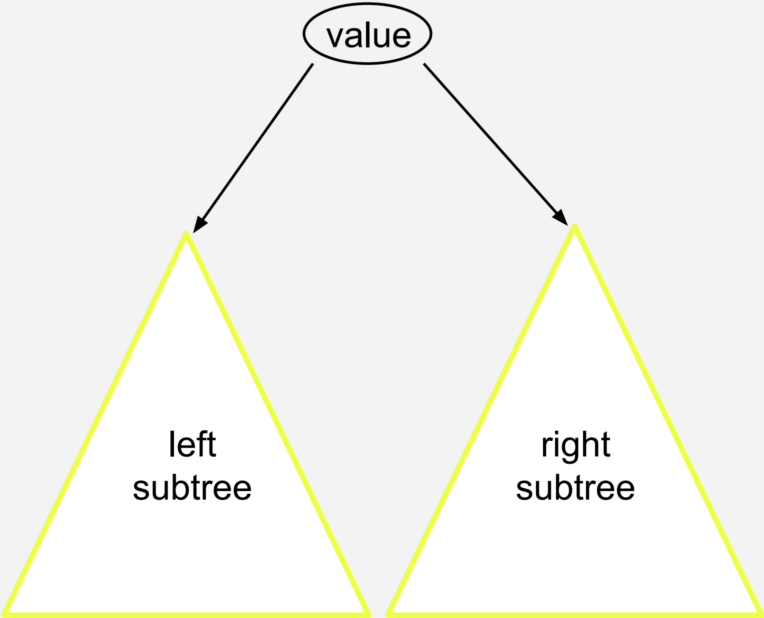
- Well-known recursive tree traversal algorithms:
  - Preorder
  - Inorder
  - Postorder

- Another, non-recursive: level order (BFS!)

پیمایش پیش ترتیب

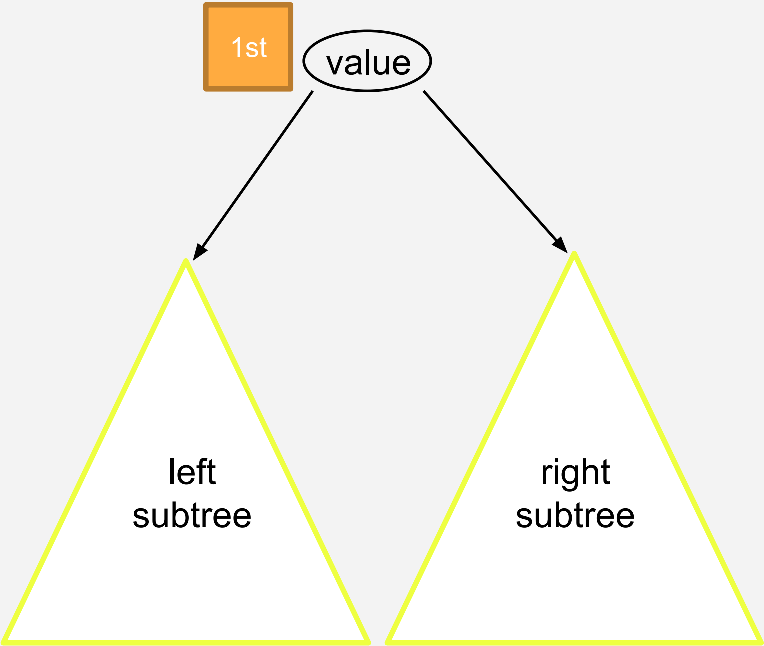
Preorder

- "Pre:" process root before subtrees



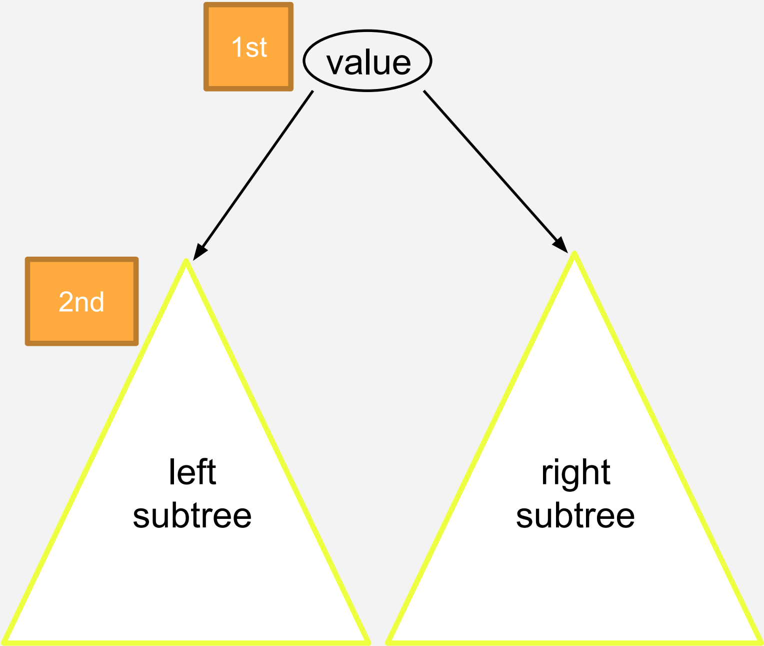
Preorder

- “Pre:” process root before subtrees



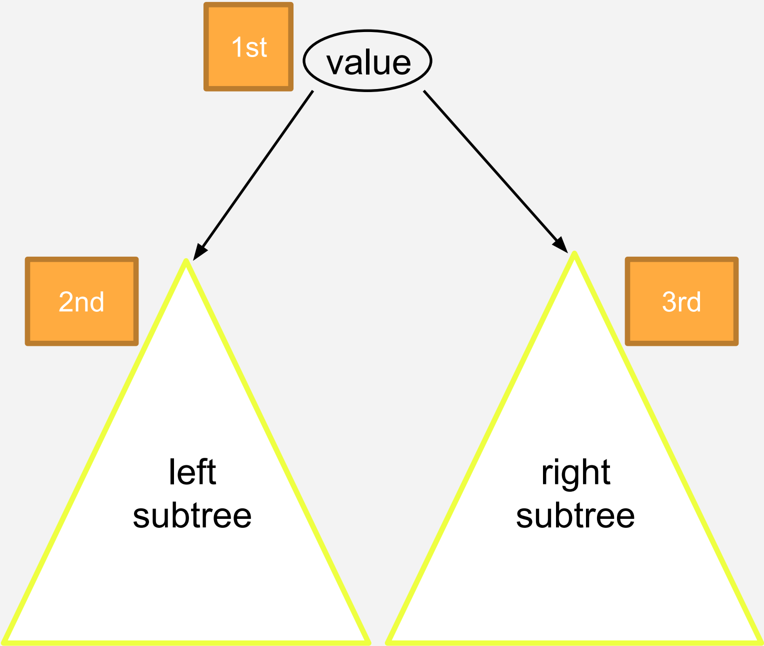
Preorder

- “Pre:” process root before subtrees



Preorder

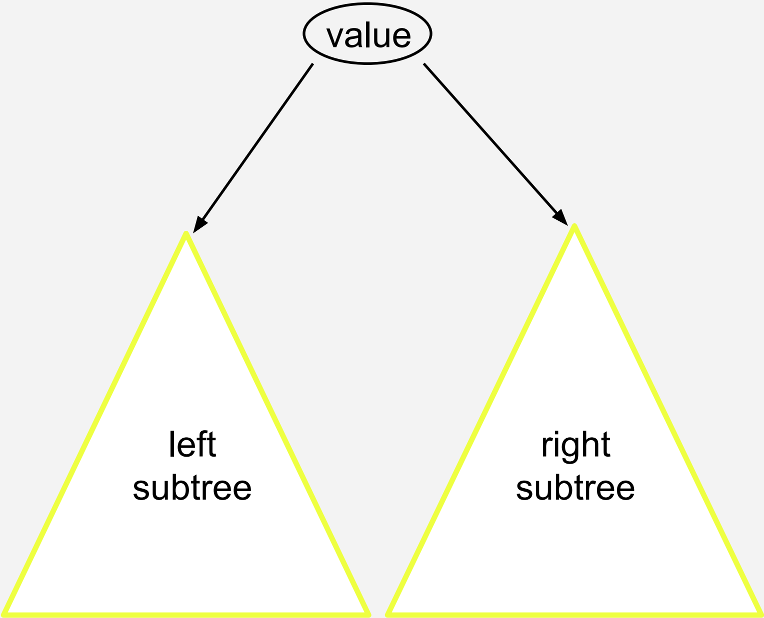
- “Pre:” process root before subtrees



پیمایش میان ترتیب

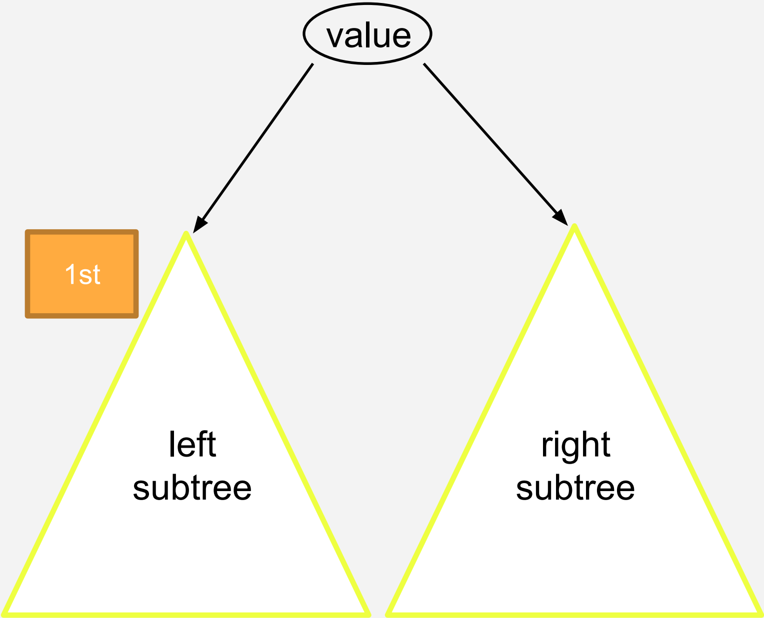
Inorder

- “In:” process root in-between subtrees



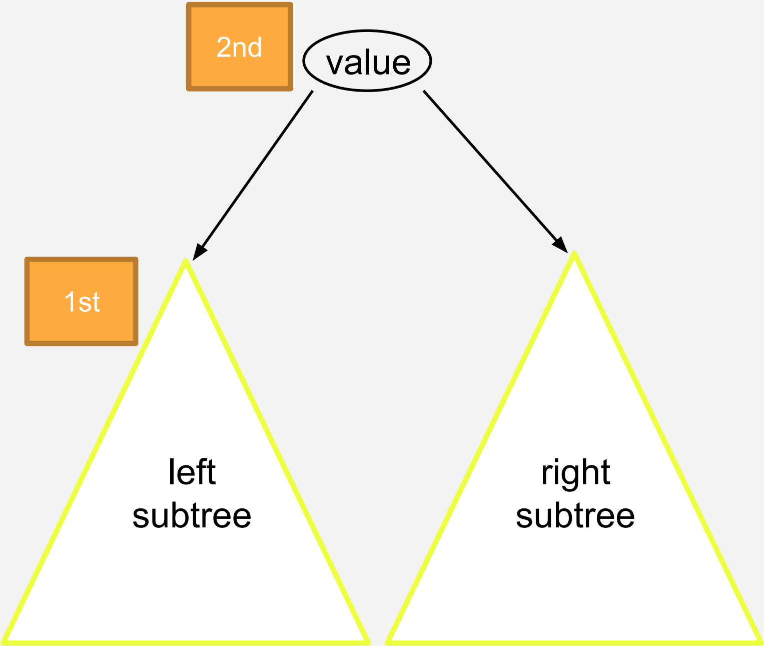
Inorder

- "In:" process root in-between subtrees



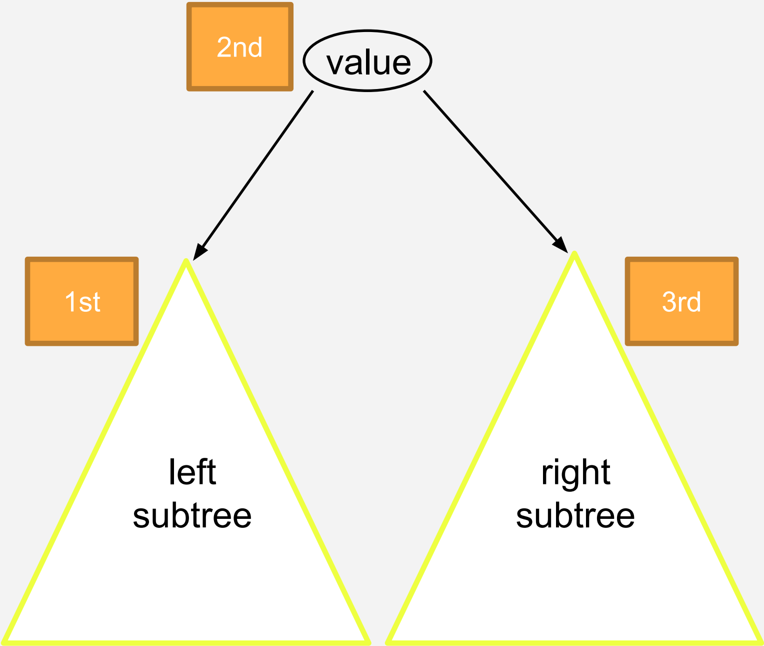
Inorder

- "In:" process root in-between subtrees



Inorder

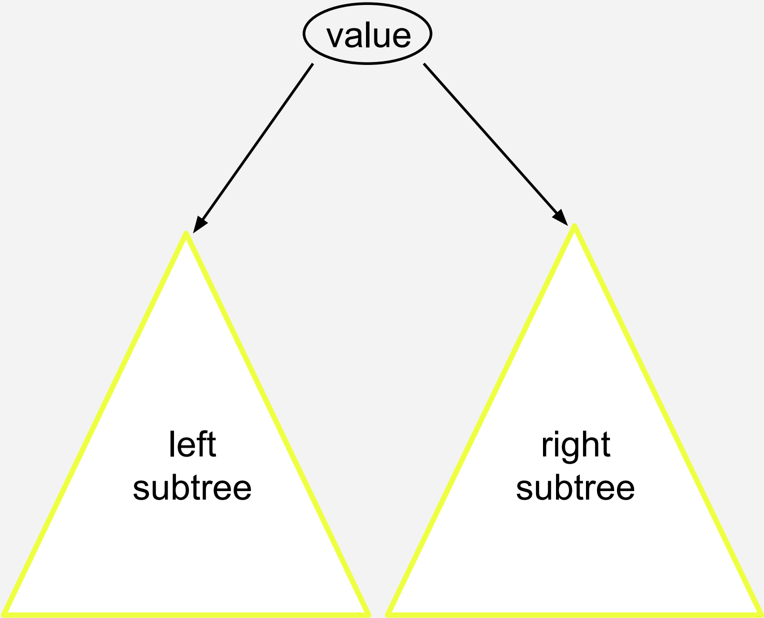
- "In:" process root in-between subtrees



پیمایش پس ترتیب

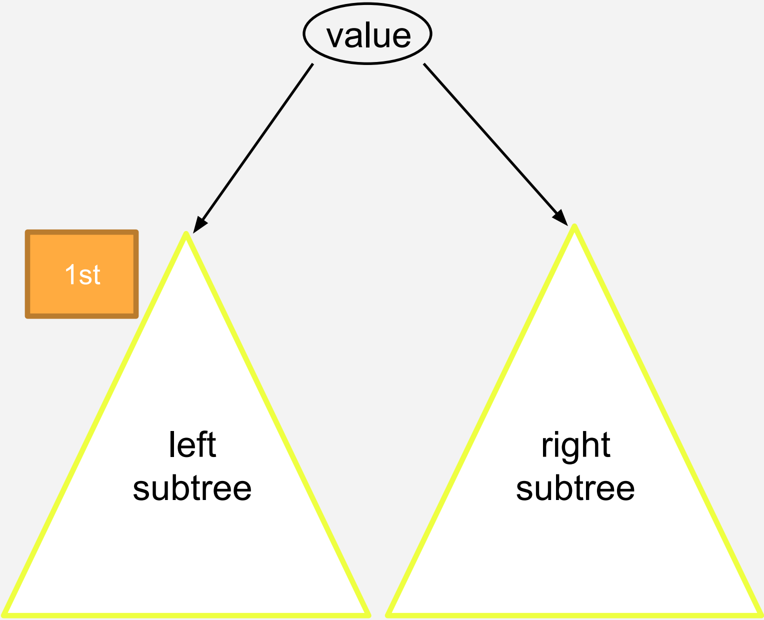
Postorder

- “Post:” process root after subtrees



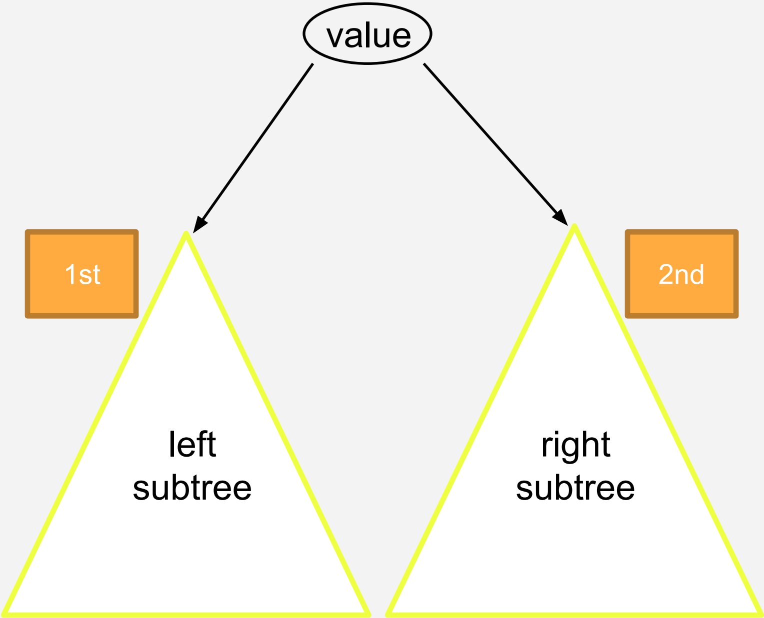
Postorder

- "Post:" process root after subtrees



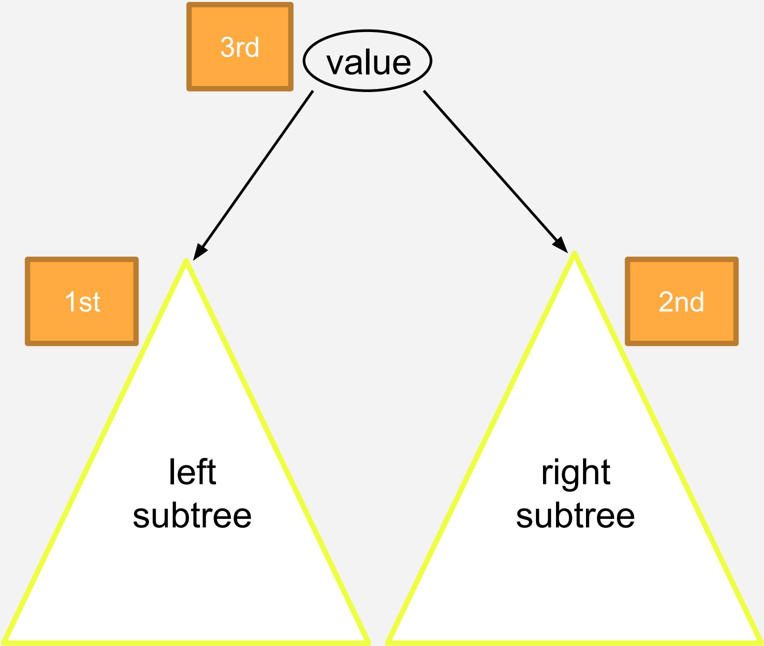
Postorder

- “Post:” process root after subtrees



Postorder

- “Post:” process root after subtrees



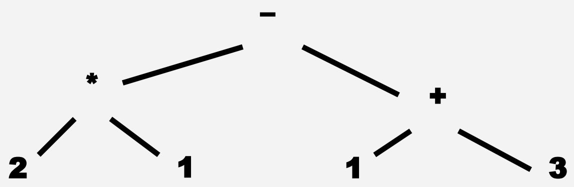
سوال؟

درخت عبارت

نمونه ای از کاربرد درخت و پیمایش آن

Syntax Trees

- Trees can represent expressions (Java, math, …)
- Expression: 2 * 1 – (1 + 3)
- Tree:



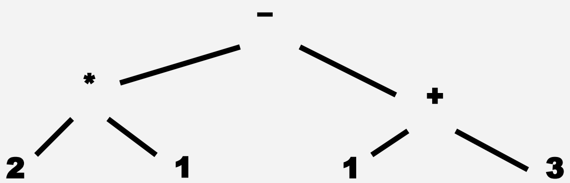
پیمایش پیش ترتیب عبارت

Preorder Traversals of Expression Tree

1- Visit the root

2- Visit the left subtree

3- Visit the right subtree



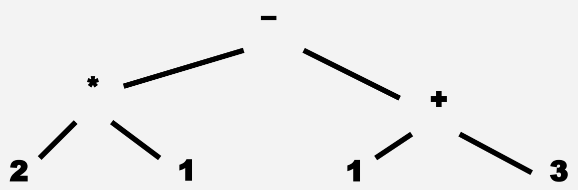
Preorder:

Preorder Traversals of Expression Tree

1- Visit the root

2- Visit the left subtree

3- Visit the right subtree



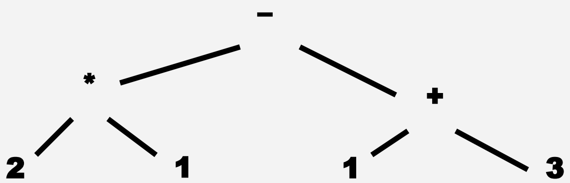
Preorder: -

Preorder Traversals of Expression Tree

1- Visit the root

2- Visit the left subtree

3- Visit the right subtree



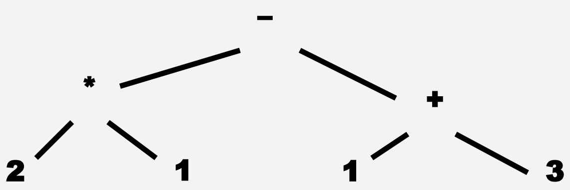
Preorder: - *

Preorder Traversals of Expression Tree

1- Visit the root

2- Visit the left subtree

3- Visit the right subtree



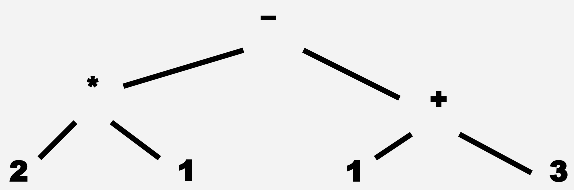
Preorder: - * 2

Preorder Traversals of Expression Tree

1- Visit the root

2- Visit the left subtree

3- Visit the right subtree



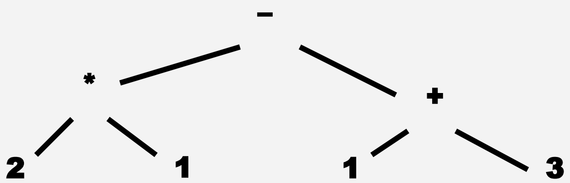
Preorder: - * 2 1

Preorder Traversals of Expression Tree

1- Visit the root

2- Visit the left subtree

3- Visit the right subtree



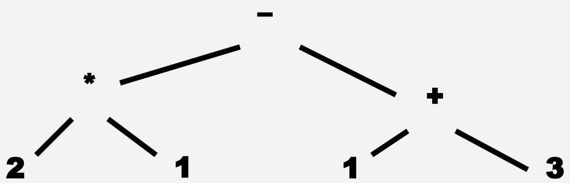
Preorder: - * 2 1 +

Preorder Traversals of Expression Tree

1- Visit the root

2- Visit the left subtree

3- Visit the right subtree



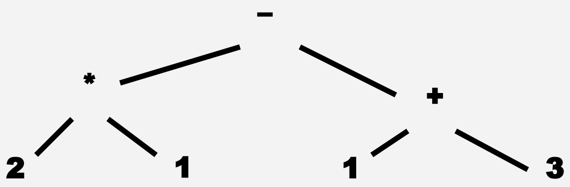
Preorder: - * 2 1 + 1

Preorder Traversals of Expression Tree

1- Visit the root

2- Visit the left subtree

3- Visit the right subtree



Preorder: - * 2 1 + 1 3

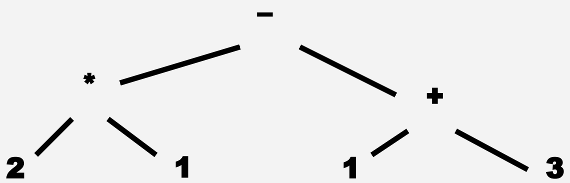
پیمایش پس ترتیب عبارت

Postorder Traversals of Expression Tree

1- Visit the left subtree

2- Visit the right subtree

3- Visit the root



Preorder: - * 2 1 + 1 3

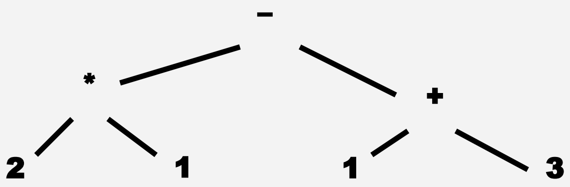
Postorder:

Postorder Traversals of Expression Tree

1- Visit the left subtree

2- Visit the right subtree

3- Visit the root



Preorder: - * 2 1 + 1 3

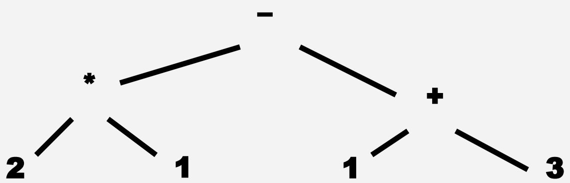
Postorder: 2

Postorder Traversals of Expression Tree

1- Visit the left subtree

2- Visit the right subtree

3- Visit the root



Preorder: - * 2 1 + 1 3

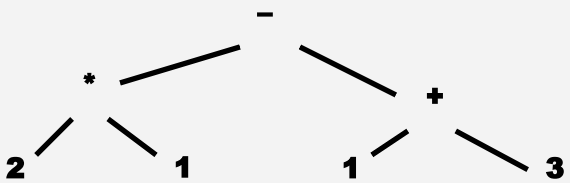
Postorder: 2 1

Postorder Traversals of Expression Tree

1- Visit the left subtree

2- Visit the right subtree

3- Visit the root



Preorder: - * 2 1 + 1 3

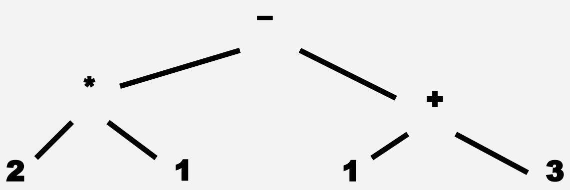
Postorder: 2 1 *

Postorder Traversals of Expression Tree

1- Visit the left subtree

2- Visit the right subtree

3- Visit the root



Preorder: - * 2 1 + 1 3

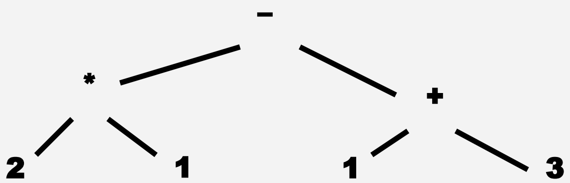
Postorder: 2 1 * 1

Postorder Traversals of Expression Tree

1- Visit the left subtree

2- Visit the right subtree

3- Visit the root



Preorder: - * 2 1 + 1 3

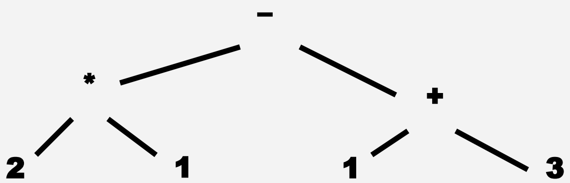
Postorder: 2 1 * 1 3

Postorder Traversals of Expression Tree

1- Visit the left subtree

2- Visit the right subtree

3- Visit the root



Preorder: - * 2 1 + 1 3

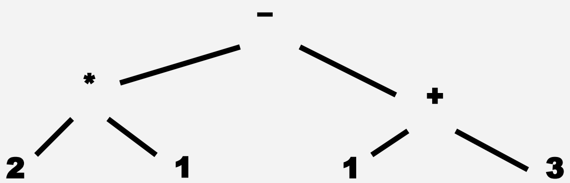
Postorder: 2 1 * 1 3 +

Postorder Traversals of Expression Tree

1- Visit the left subtree

2- Visit the right subtree

3- Visit the root



Preorder: - * 2 1 + 1 3

Postorder: 2 1 * 1 3 + -

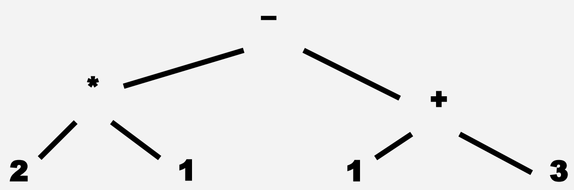
پیمایش میان ترتیب عبارت

Inorder Traversals of Expression Tree

1- Visit the left subtree

2- Visit the root

3- Visit the right subtree



Preorder: - * 2 1 + 1 3

Postorder: 2 1 * 1 3 + -

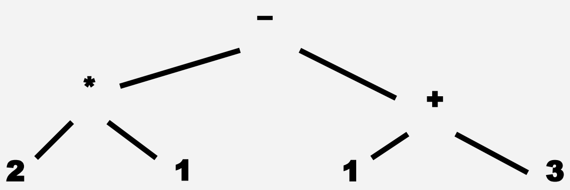
Inorder:

Inorder Traversals of Expression Tree

1- Visit the left subtree

2- Visit the root

3- Visit the right subtree



Preorder: - * 2 1 + 1 3

Postorder: 2 1 * 1 3 + -

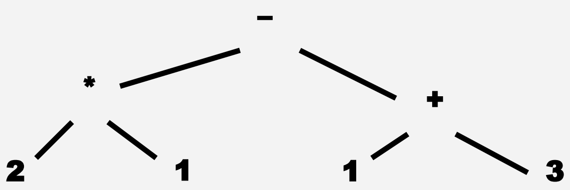
Inorder: 2

Inorder Traversals of Expression Tree

1- Visit the left subtree

2- Visit the root

3- Visit the right subtree



Preorder: - * 2 1 + 1 3

Postorder: 2 1 * 1 3 + -

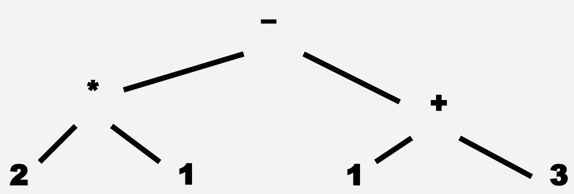
Inorder: 2 *

Inorder Traversals of Expression Tree

1- Visit the left subtree

2- Visit the root

3- Visit the right subtree



Preorder: - * 2 1 + 1 3

Postorder: 2 1 * 1 3 + -

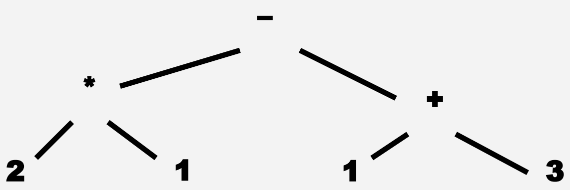
Inorder: 2 * 1

Inorder Traversals of Expression Tree

1- Visit the left subtree

2- Visit the root

3- Visit the right subtree



Preorder: - * 2 1 + 1 3

Postorder: 2 1 * 1 3 + -

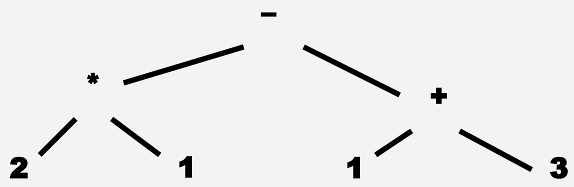
Inorder: 2 * 1 -

Inorder Traversals of Expression Tree

1- Visit the left subtree

2- Visit the root

3- Visit the right subtree



Preorder: - * 2 1 + 1 3

Postorder: 2 1 * 1 3 + -

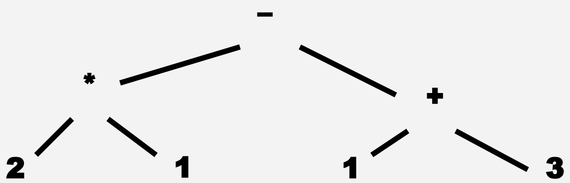
Inorder: 2 * 1 - 1

Inorder Traversals of Expression Tree

1- Visit the left subtree

2- Visit the root

3- Visit the right subtree



Preorder: - * 2 1 + 1 3

Postorder: 2 1 * 1 3 + -

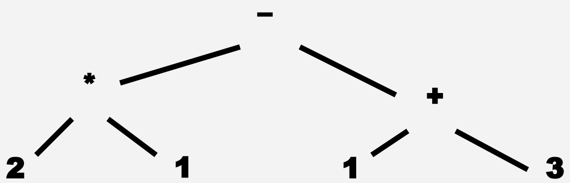
Inorder: 2 * 1 - 1 +

Inorder Traversals of Expression Tree

1- Visit the left subtree

2- Visit the root

3- Visit the right subtree



Preorder: - * 2 1 + 1 3

Postorder: 2 1 * 1 3 + -

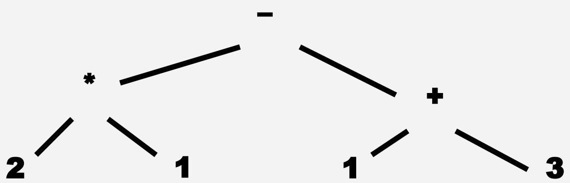
Inorder: 2 * 1 - 1 + 3

Inorder Traversals of Expression Tree

1- Visit the left subtree

2- Visit the root

3- Visit the right subtree



Original expression, except for parenthesis

Preorder: - * 2 1 + 1 3

Postorder: 2 1 * 1 3 + -

Inorder: $\boxed{2 * 1 - 1 + 3}$

سوال؟

Prefix Notation

- Function calls in most programming languages use prefix notation:
  - E.g., **`add(37, 5)`**
- Aka **Polish notation** (PN)
  - In honor of inventor, Polish logician Jan Łukasiewicz
- Some languages (Lisp, Scheme, Racket) use prefix notation for everything
  - Makes the syntax uniform

```
(- (* 2 1) (+ 1 3))

(define (fib n)
  (if (<= n 2)
      1
      (+ (fib (- n 1) (fib (- n 2))))))
```

Postfix Notation

- Some languages (Forth, PostScript, HP calculators) use postfix notation
- Aka **reverse Polish notation** (RPN)

```
2 1 mul 1 3 add sub

/fib { dup
       3 lt
         { pop 1 }
         { dup 1 sub fib exch 2 sub fib add }
       ifelse
     } def
```

Implementing Syntax Tree in Code

```
public interface Expr {
  int eval();
  String inorder();
}
```

Implementing Syntax Tree in Code

```
public interface Expr {
  int eval();
  String inorder();
}

public class Int implements Expr {
  private int v;
  public int eval() { return v; }
  public String inorder() { return " " + v + " "; }
}
```

Implementing Syntax Tree in Code

```
public interface Expr {
  int eval();
  String inorder();
}

public class Int implements Expr {
  private int v;
  public int eval() { return v; }
  public String inorder() { return " " + v + " "; }
}

public class Add implements Expr {
  private Expr left, right;
  public int eval() { return left.eval() + right.eval(); }
  public String inorder() {
    return "(" + left.infix() + "+" + right.infix() + ")";
  }
}
```

سوال؟

بازسازی درخت

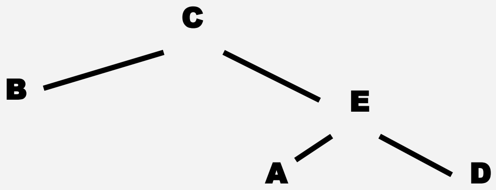
ساخت درخت از روی یک پیمایش آن

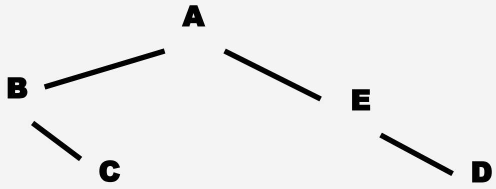
Recover tree from traversal

- Suppose inorder is B C A E D
- Can we recover the tree uniquely?

Recover tree from traversal

- Suppose inorder is B C A E D
- Can we recover the tree uniquely? NO!





Recover tree from traversals

- Suppose inorder is B C A E D
- preorder is A B C D E
- Can we determine the tree uniquely?

Recover tree from traversals

- Suppose inorder is B C A E D
- preorder is A B C D E
- Can we determine the tree uniquely? Yes!

Recover tree from traversals

- Suppose inorder is B C A E D
- preorder is A B C D E
- Can we determine the tree uniquely? Yes!

- What is root?

Recover tree from traversals

- Suppose inorder is B C A E D
- preorder is A B C D E
- Can we determine the tree uniquely? Yes!

- What is root? Preorder tells us: A

Recover tree from traversals

- Suppose inorder is B C A E D
- preorder is A B C D E
- Can we determine the tree uniquely? Yes!

- What is root? Preorder tells us: A
- What comes before/after root A?

Recover tree from traversals

- Suppose inorder is B C A E D
- preorder is A B C D E
- Can we determine the tree uniquely? Yes!

- What is root? Preorder tells us: A
- What comes before/after root A?
    - Inorder tells us:
        - Before: B C
        - After: E D

Recover tree from traversals

- Suppose inorder is B C A E D
- preorder is A B C D E
- Can we determine the tree uniquely? Yes!

- What is root? Preorder tells us: A
- What comes before/after root A?
  - Inorder tells us:
    - Before: B C
    - After: E D
- Now recurse! Figure out left/right subtrees using same technique.

Recover tree from traversals

- Suppose inorder is B C A E D
- preorder is A B C D E
- Root is A; left subtree contains B C; right contains E D

Recover tree from traversals

- Suppose inorder is B C A E D
- preorder is A B C D E
- Root is A; left subtree contains B C; right contains E D

Left:
Inorder is B C
Preorder is B C
- What is root? Preorder: B
- What is before/after B?
  Inorder:
  - Before: nothing
  - After: C

Recover tree from traversals

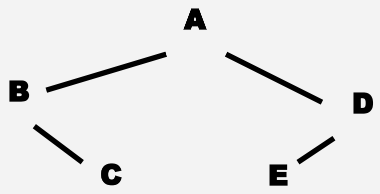
- Suppose inorder is B C A E D
- preorder is A B C D E
- Root is A; left subtree contains B C; right contains E D

Left:
Inorder is B C
Preorder is B C
- What is root? Preorder: B
- What is before/after B?
  Inorder:
  - Before: nothing
  - After: C

Right:
Inorder is E D
Preorder is D E
- What is root? Preorder: D
- What is before/after D?
  Inorder:
  - Before: E
  - After: nothing

Recover tree from traversals

- Suppose inorder is B C A E D
- preorder is A B C D E
- Root is A; left subtree contains B C; right contains E D
- Tree:



سوال؟