ساختمان داده و الگوریتم ها

مبحث ششم:
مرتب سازی سریع

سجاد شیرعلی شهرضا

پائیز 1402

شنبه، 22 مهر 1402

Based on Stanford CS161 Slides from Summer 2020 by Karey Shi

اطلاع رسانی

- بخش مرتبط کتاب برای این جلسه: 5
- امتحانک اول
  - دوشنبه (پس فردا)، 24 مهر 1402
  - به صورت حضوری در کلاس
  - در ساعت کلاس
- ساعت کلاس حل تمرین:
  - پنج شنبه ها، ساعت 10 تا 11:30
  - به صورت مجازی
  - آغاز از هفته گذشته!
- تمرین اول:
  - قرار گرفته در سایت درس
  - مهلت تحویل: ساعت 8 صبح شنبه هفته آینده، 29 مهر 1402

مرتب سازی سریع

یک نمونه واقعی و کاربردی از الگوریتم های تصادفی

QUICKSORT OVERVIEW

EXPECTED RUNNING TIME

$O(n \log n)$

WORST-CASE RUNNING TIME

$O(n^2)$

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EXPECTED RUNNING TIME

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WORST-CASE RUNNING TIME

$O\ (n^2)$

In practice, it works great! It's competitive with MergeSort (& often better in some contexts!), and it runs *in place* (no need for lots of additional memory)

QUICKSORT: THE IDEA

Let's use DIVIDE-and-CONQUER again!

Select a pivot *at random*

Partition around it

Recursively sort L and R!

QUICKSORT: THE IDEA

Select a pivot

3	2	7	6	1	5	4	8

Pick this pivot uniformly at random!

QUICKSORT: THE IDEA

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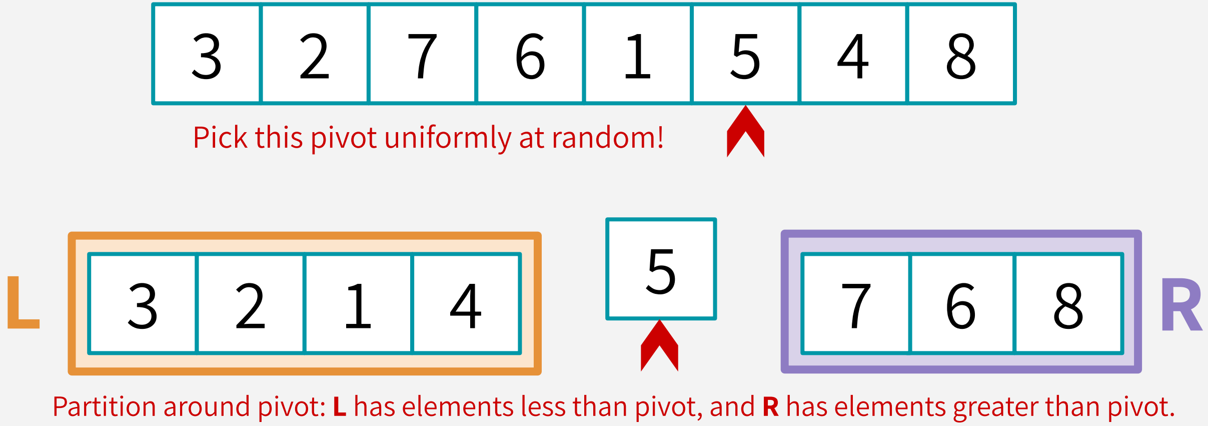
Partition around it



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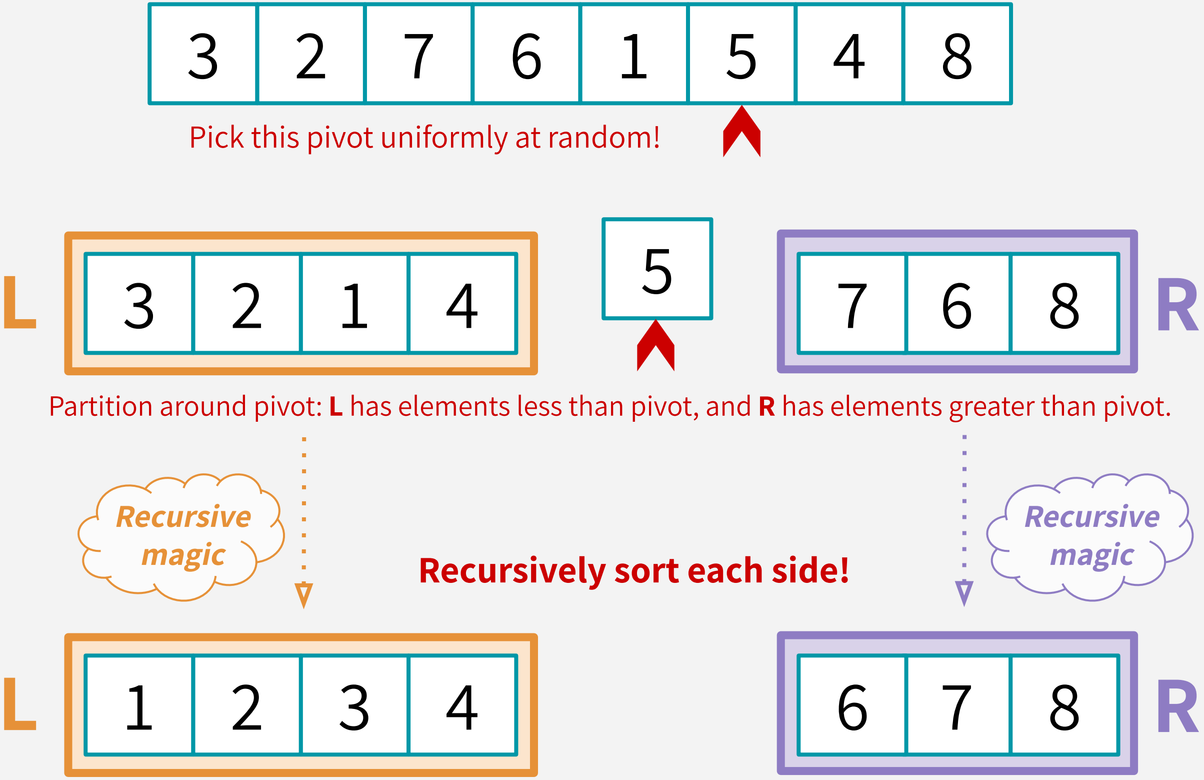

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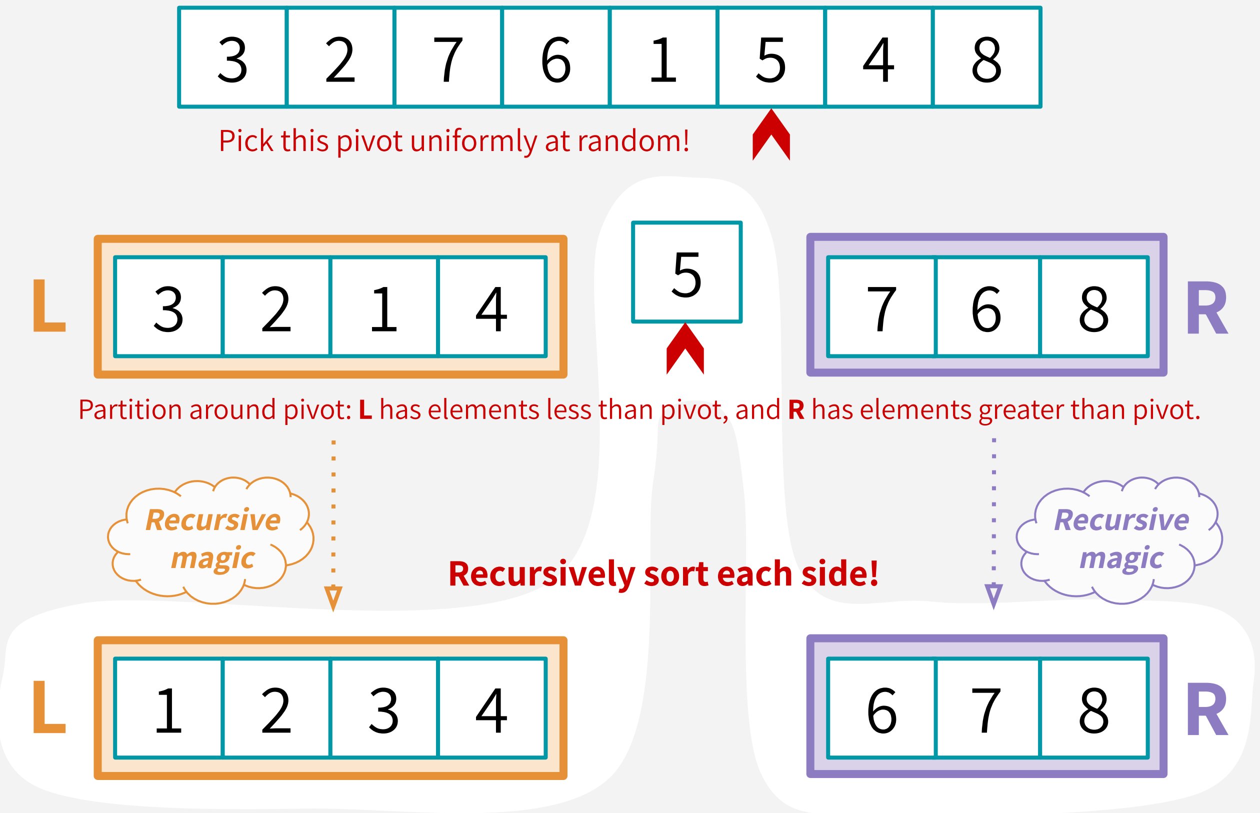

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RECURRENCE RELATION

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Recurrence Relation for QUICKSORT

$$T(n) = T(|L|) + T(|R|) + O(n)$$
$$T(0) = T(1) = O(1)$$

IDEAL RUNTIME?

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In an ideal world, the pivot would split the array exactly in half, and we'd get:

$$T(n) = T(n/2) + T(n/2) + O(n)$$

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In an ideal world:

$$T(n) = 2 \cdot T(n/2) + O(n)$$

$$T(n) = \mathbf{O(n \log n)}$$

Recurrence Relation for QUICKSORT

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With the unluckiest randomness, the pivot would be either min(A) or max(A):

$$T(n) = T(0) + T(n-1) + O(n)$$

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in(A) or max(A):

$$T(n) = T(0) + T(n-1) + O(n)$$

With the worst "randomness"

$$T(n) = T(n\text{-}1) + O(n)$$

$$T(n) = \mathbf{O(n^2)}$$

(recursion tree/table or substitution method!)

سوال؟

EXPECTED RUNTIME = O(n log n)

AN **INCORRECT** PROOF!

Lemma: $E[|L|] = E[|R|] = (n-1)/2$

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(because L and R make up everything except the pivot)

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(Solving for E[|L|])

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Why is this wrong?

EXPECTED RUNTIME = O(n log n)

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Why is this wrong?

Well, for starters, we can use the exact same argument to prove something false...

LET'S START WITH QUICKSORT AND ...

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```

INTRODUCING **SLOW**SORT!

```
SLOWSORT(A):
    if len(A) <= 1:
        return                randomly choose either!
    pivot = either max(A) OR min(A)
    PARTITION A into:
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Recurrence Relation for SLOWSORT

$$T(n) = T(|L|) + T(|R|) + O(n)$$
$$T(0) = T(1) = O(1)$$

One of **|L|** or **|R|** is *always* n-1

$$T(n) = T(0) + T(n-1) + O(n)$$
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Recurrence Relation for SLOWSORT

$$T(n) = T(|L|) + T(|R|) + O(n)$$
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Same recurrence relation as QUICKSORT!

And we also still have:

$E[|L|] = E[|R|] = (n-1)/2$

SLOWSORT

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        retur
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Recurrence Relation for
SORT

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ce relation as
ORT!
still have:
[|R|] = (n-1)/2

RED FLAG:

We could use the exact same (incorrect) proof to prove that **SLOWSort** has expected runtime **O(n log n)**, when it actually has expected runtime of **$\Theta(n^2)$**...

EXPECTED RUNTIME = O(n log n)

AN **INCORRECT** PROOF:

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- Therefore, the expected running time is $O(n \log n)$!

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EXPECTED RUNTIME = O(n log n)

AN

Basically:

E[f(x)**] is *not necessarily* the same as** f(**E[**x**]**)

e.g. $E[X^2]$ is not the same as $(E[X])^2$

We were reasoning about T(**E[**x**]**) instead of **E[**T(x)**]**

Why is this wrong?

EXPECTED RUNTIME = O(n log n)

Instead, to prove that the expected runtime of QuickSort is O(n log n), we're going to count the **number of comparisons** that this algorithm performs, and take the expectation of that!

How many times are any two items compared?

EXPECTED RUNTIME = O(n log n)

Inste… …ort is

O(n … …sons

that … … that!

Will see it in
Algorithm Design Course!

How… …red?

سوال؟

QUICKSORT

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Worst case runtime:
$O(n^2)$

Expected runtime:
$O(n \log n)$

BETTER WORST CASE RUNTIME

- Select a better pivot
  - Ideally, split the array into two equal parts
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سوال؟

مرتب سازی سریع در عمل

چگونگی پیاده سازی (و آیا واقعا کسی از آن استفاده می کند؟)

IMPLEMENTING QUICKSORT

In practice, a more clever approach (Lomuto) is used to implement PARTITION, so that the entire QuickSort algorithm can be implemented “in-place”

(i.e. via swaps, rather than constructing separate L or R subarrays)

AN EXAMPLE IN-PLACE PARTITION

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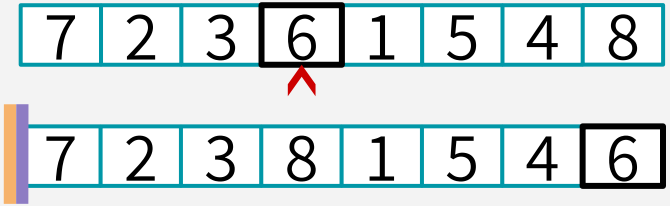
7	2	3	6	1	5	4	8

Choose pivot & swap
with last element so
pivot is at the end.

AN EXAMPLE IN-PLACE PARTITION



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Initialize and

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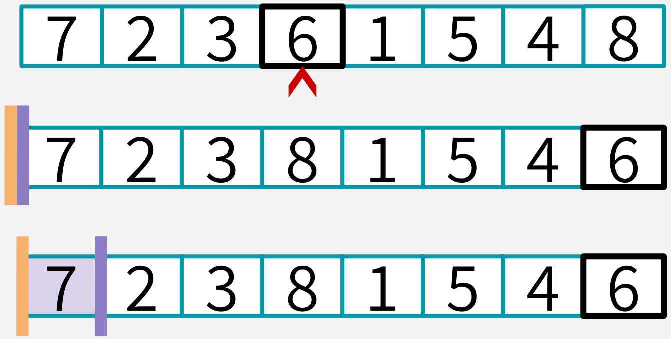

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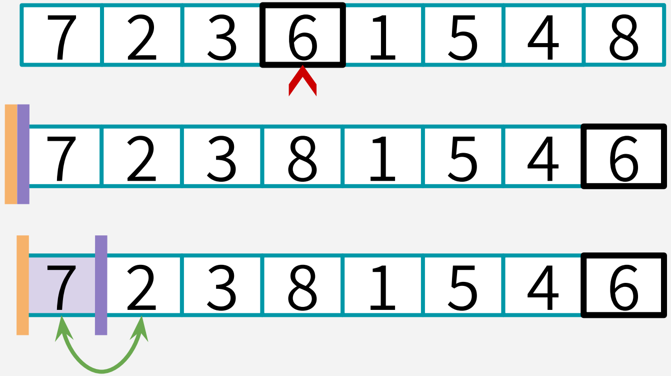

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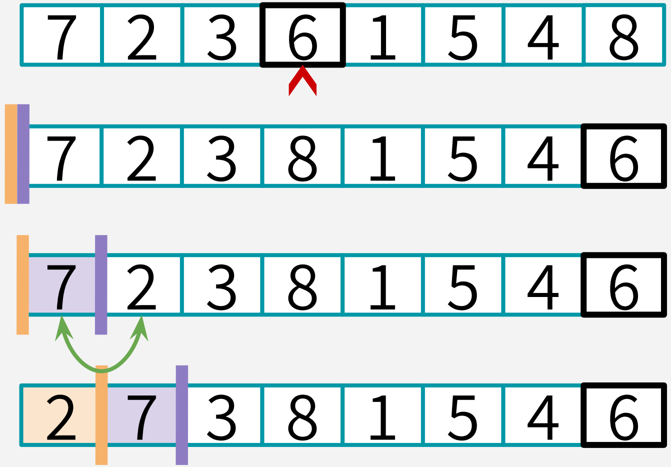

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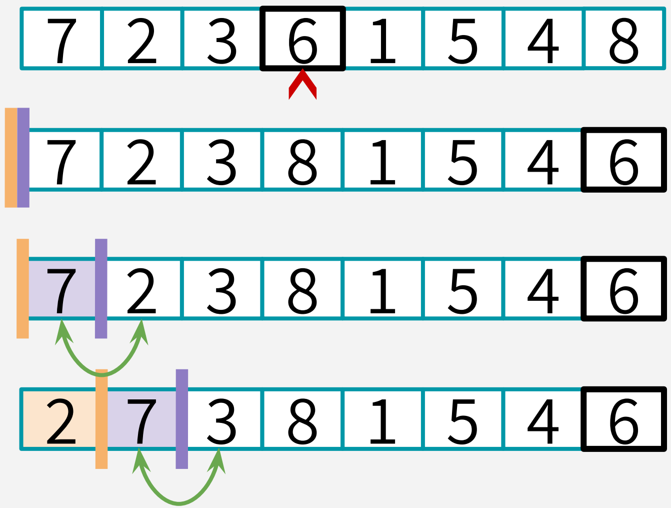

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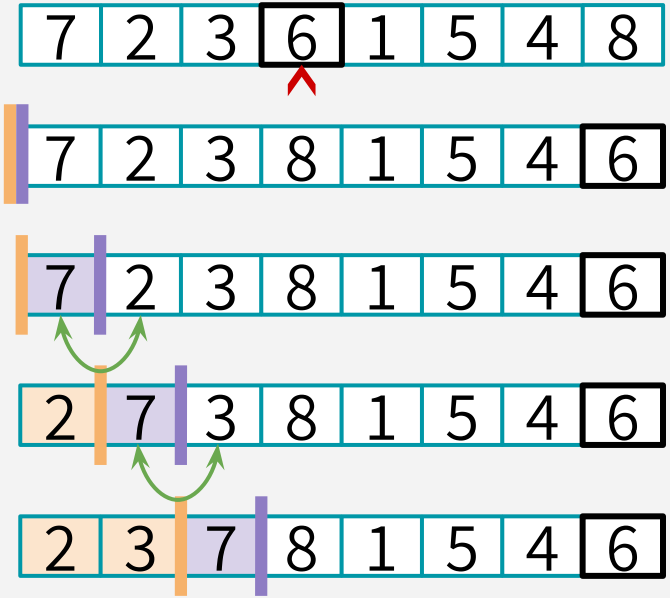

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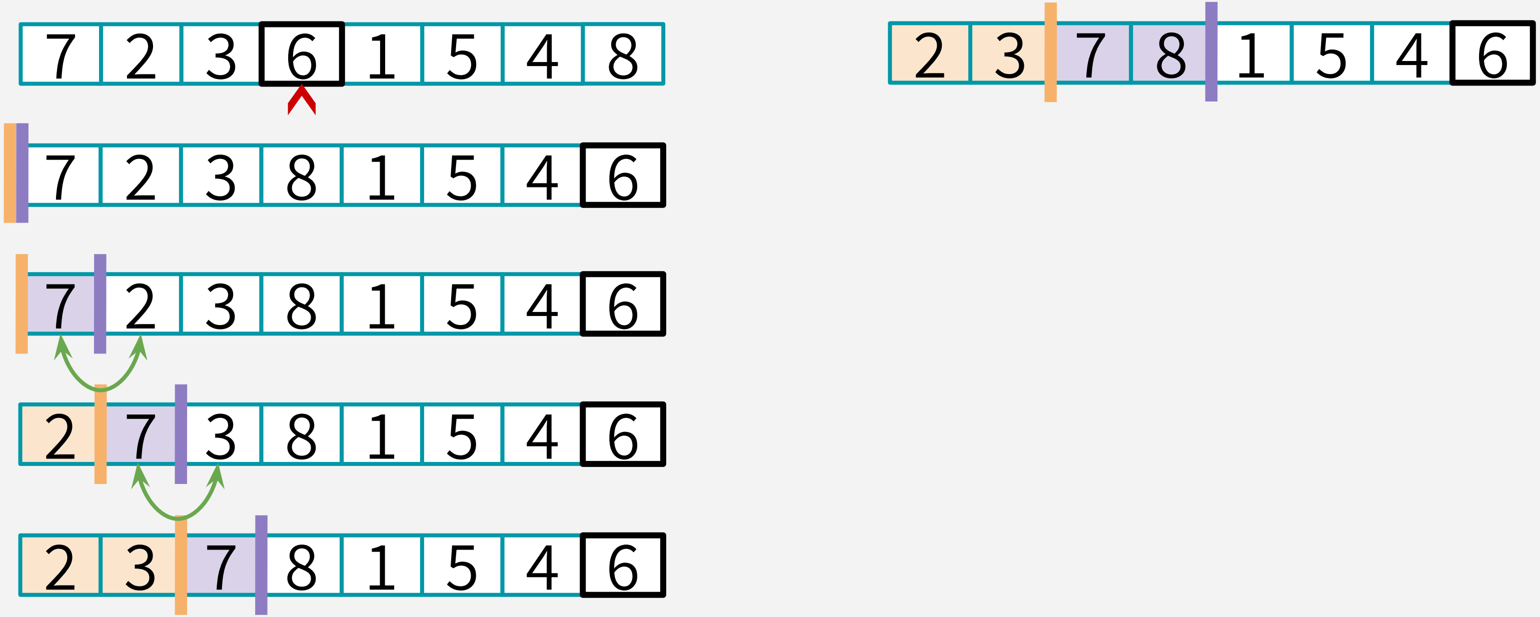

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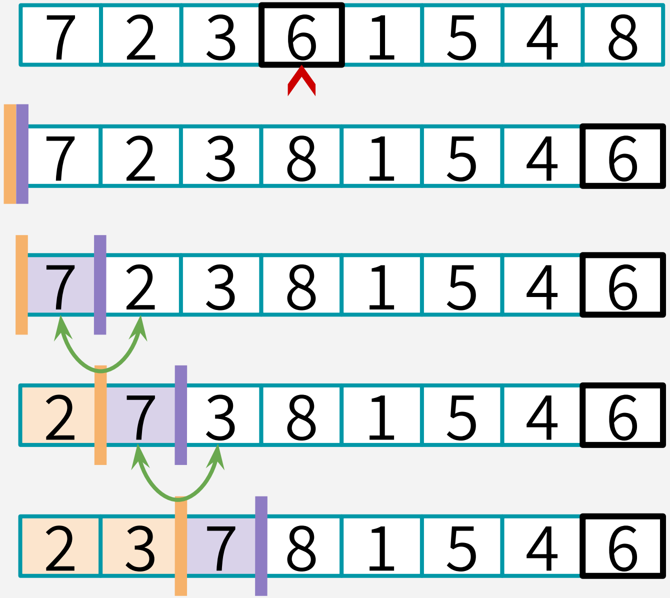

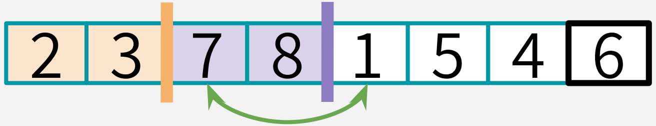

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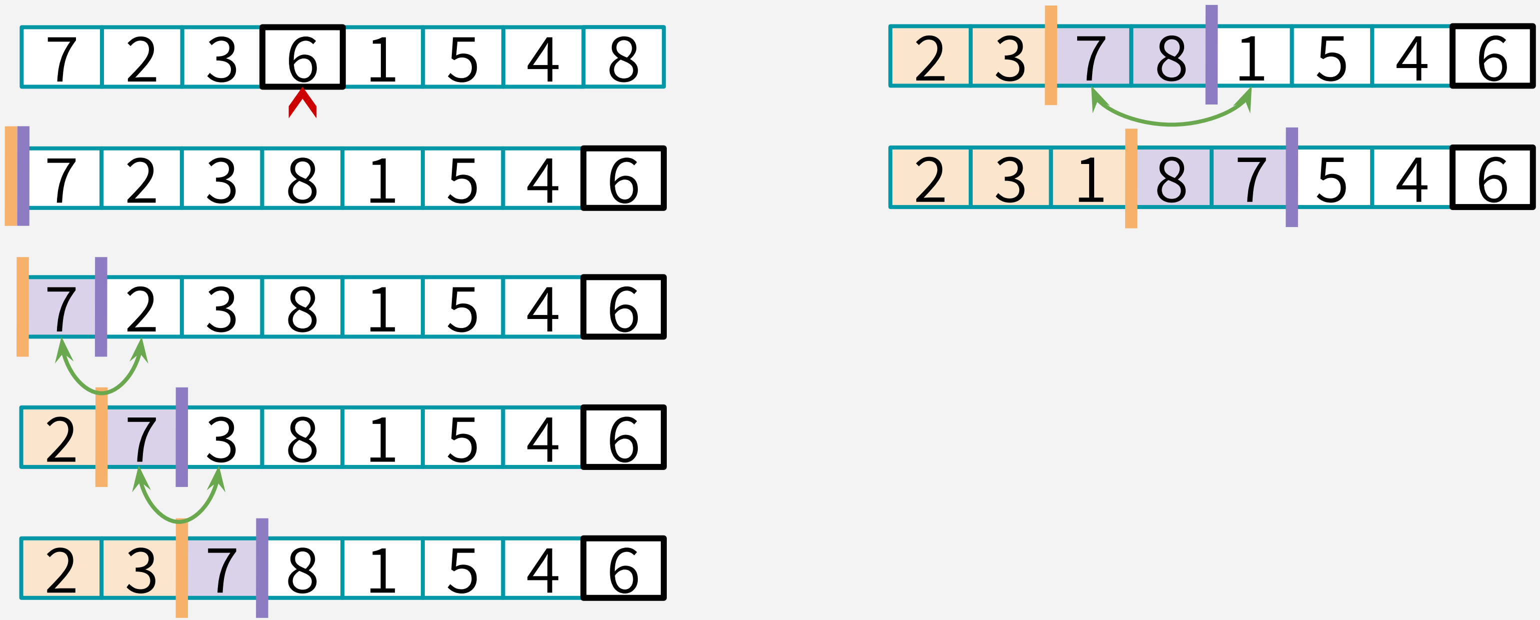

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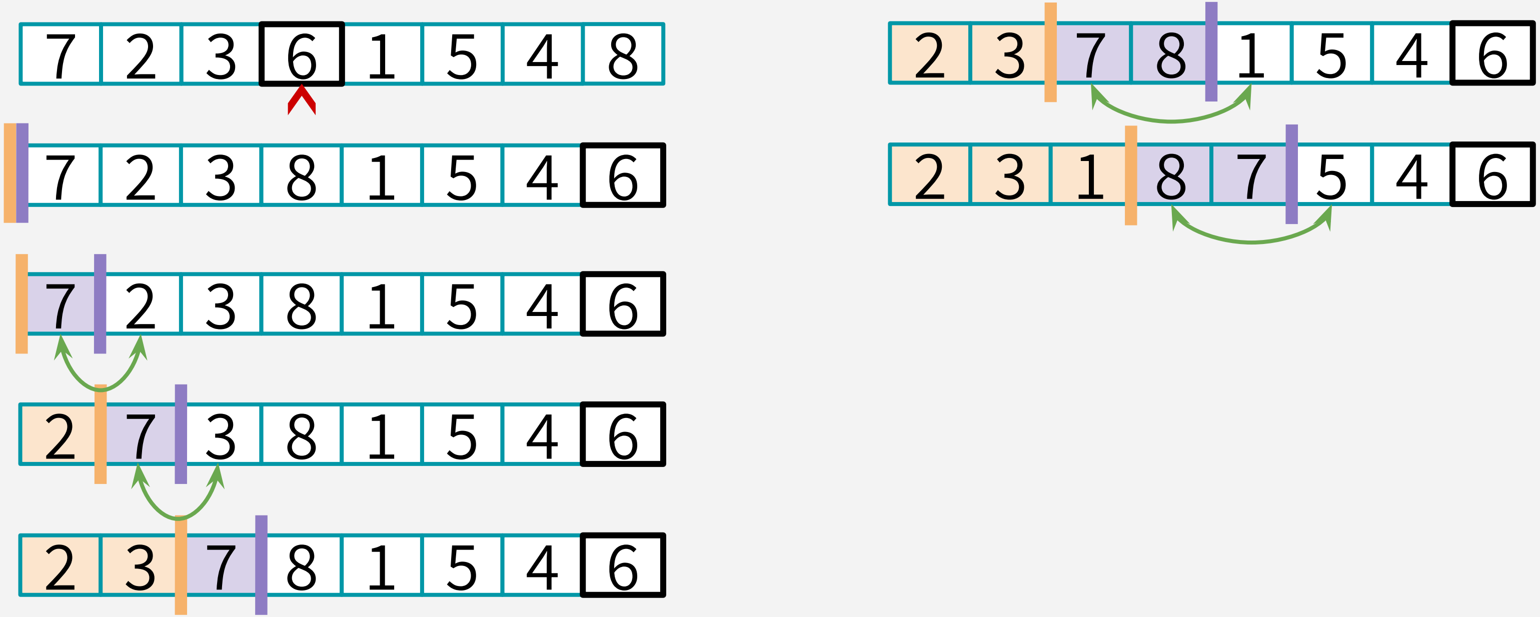

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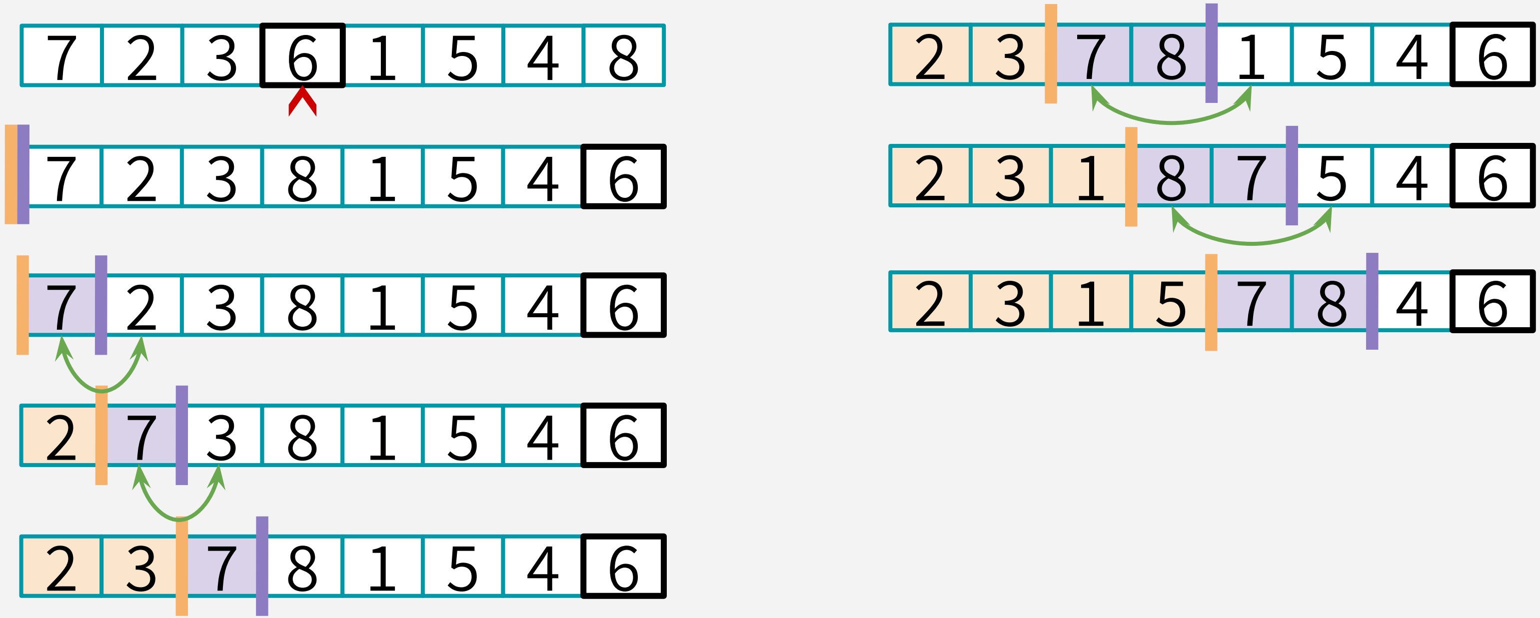

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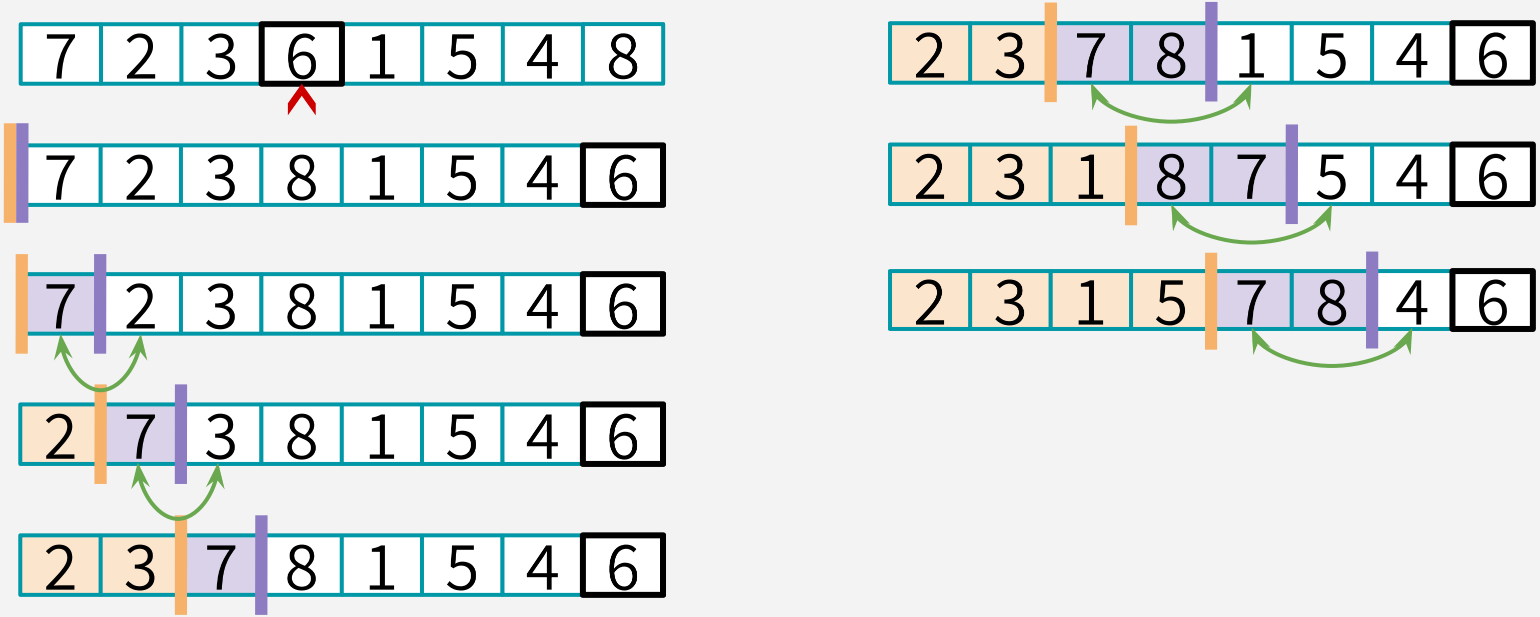

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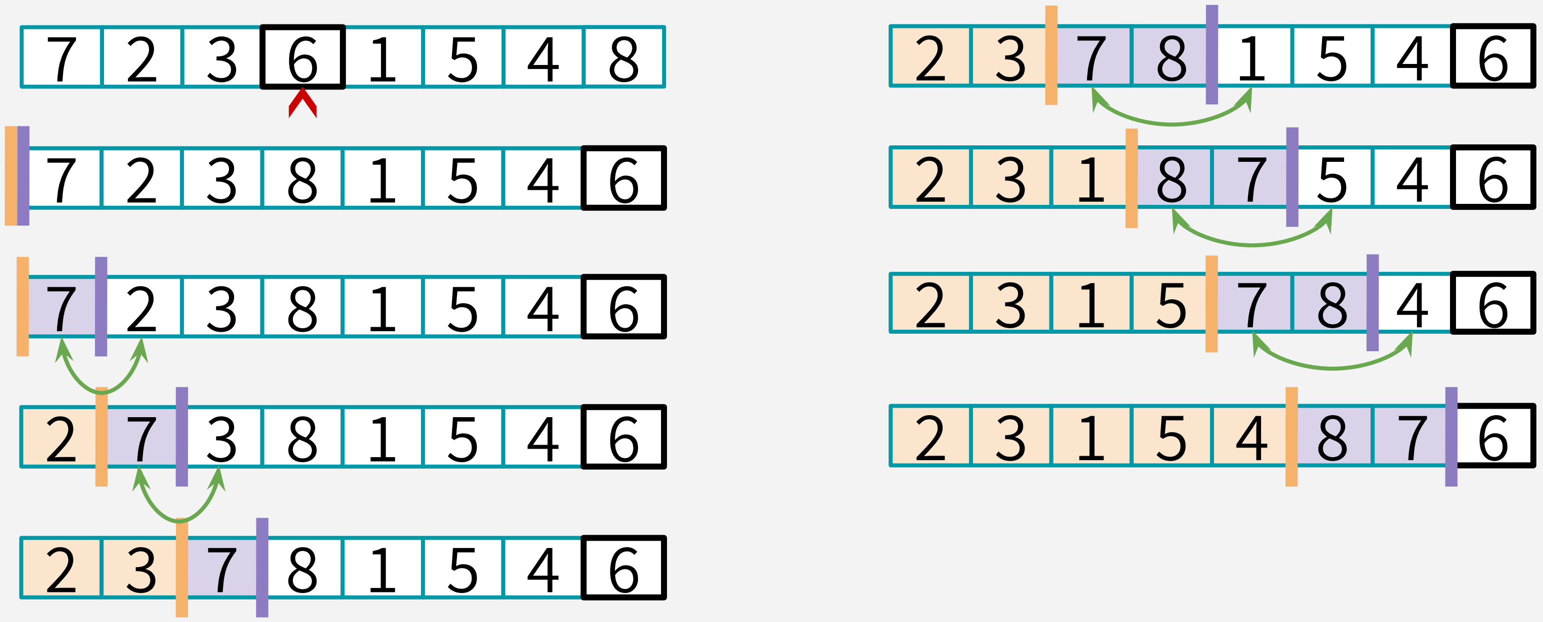

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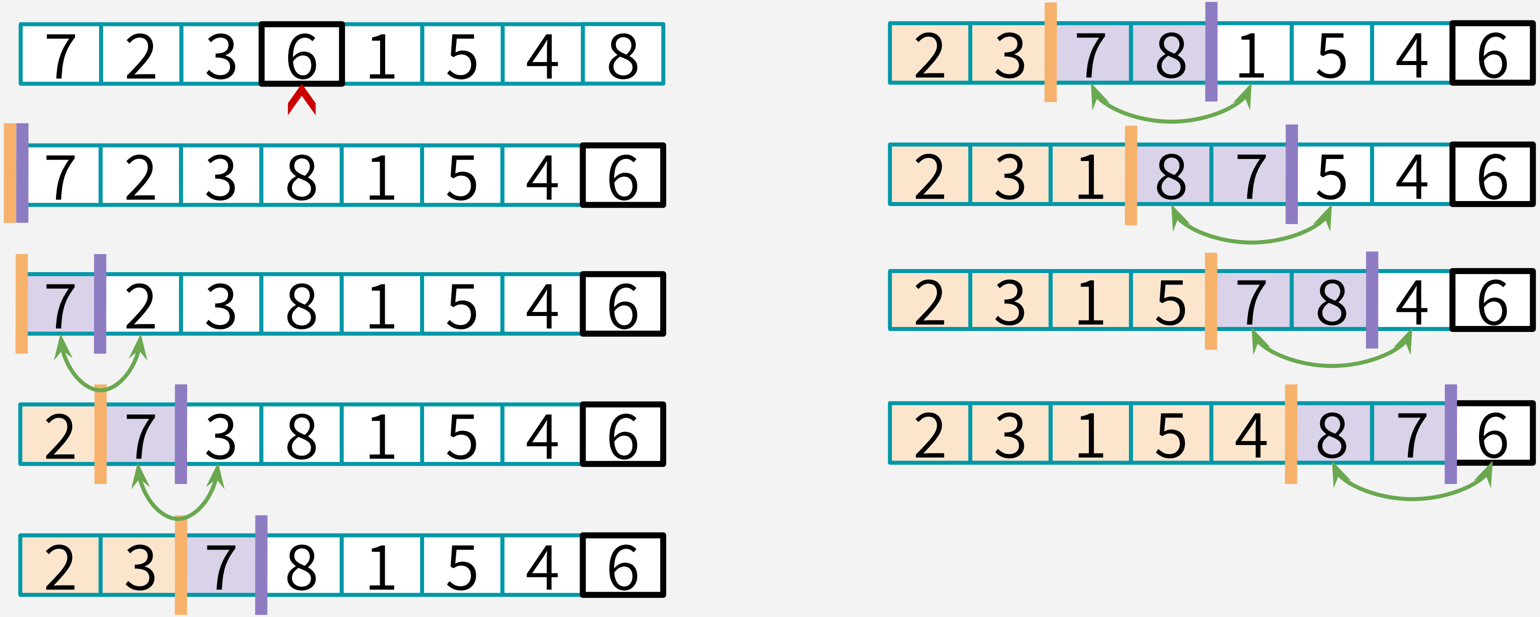
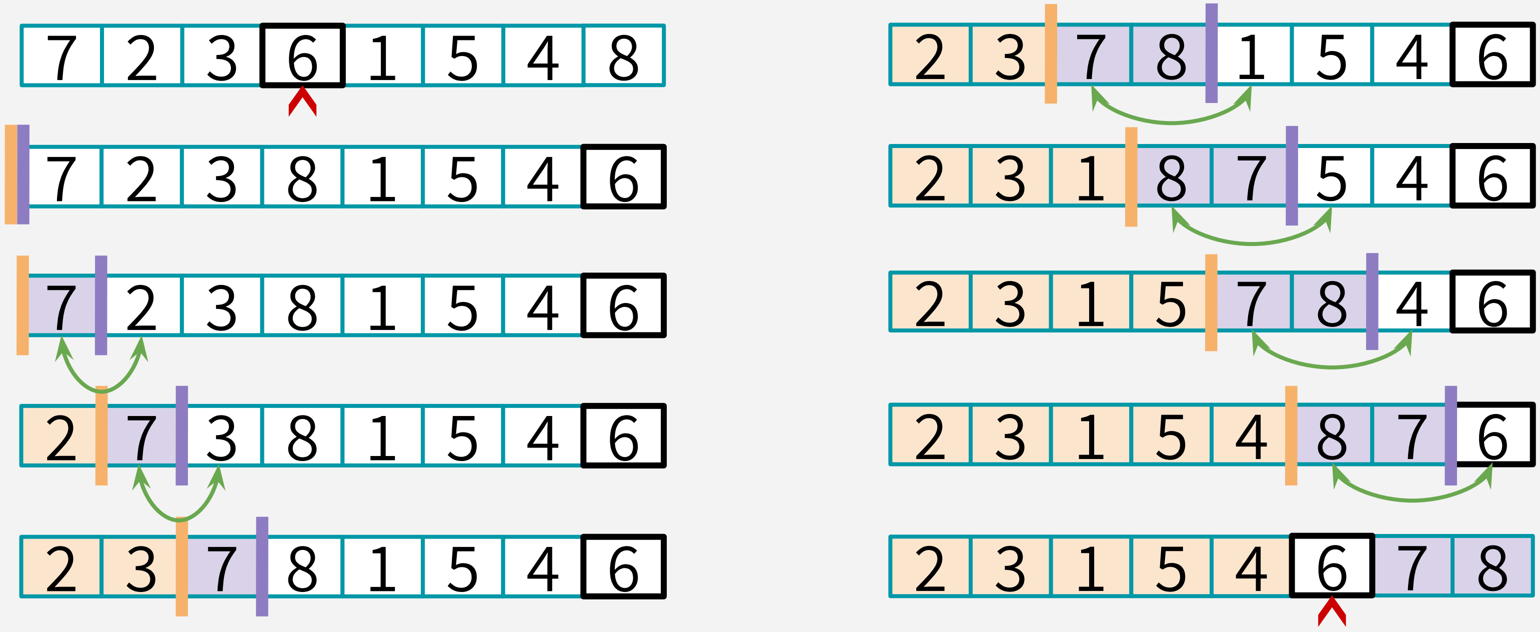
AN EXAMPLE IN-PLACE PARTITION



Choose pivot & swap with last element so pivot is at the end. ⇨ Initialize and ⇨ Increment until it sees something smaller than pivot, **swap** the things ahead of the bars & increment both bars ⇨ Repeat until the bar reaches the end, then swap the pivot into the right place.

AN EXAMPLE IN-PLACE PARTITION



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Initialize and

Increment until it sees something smaller than pivot, **swap** the things ahead of the bars & increment both bars

Repeat until the bar reaches the end, then swap the pivot into the right place.

IMPLEMENTING QUICKSORT

There's another in-place partition algorithm called Hoare Partition that's even more efficient as it performs less swaps.

What we just saw is known as Lomuto method.

(you're not responsible for knowing it in this class)

QUICKSORT vs. MERGESORT

	QuickSort (random pivot)	**MergeSort** (deterministic)
Runtime	**Worst-case: $O(n^2)$** **Expected: $O(n \log n)$**	**Worst-case: $O(n \log n)$**
Used by	Java (primitive types), C (qsort), Unix, gcc…	Java for objects, perl
In-place? **(i.e. with $O(\log n)$ extra memory)**	Yes, pretty easily!	Easy if you sacrifice runtime ($O(n\log n)$ MERGE runtime). Not so easy if you want to keep runtime & stability.
Stable?	No	Yes
Other Pros	Good cache locality if implemented for arrays	Merge step is really efficient with linked lists

سوال؟