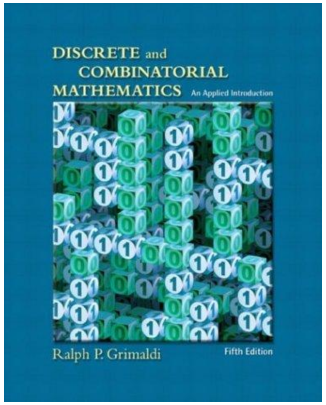
Discrete Mathematics

-- *Chapter 10: Recurrence Relations*



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First glance at “recurrence”

$$F_{n+2} = F_{n+1} + F_n$$

$$a_{n+1} = 3a_n$$

$$a_n = A*3^n$$

Outline

- The first-order linear recurrence relation
- The second-order linear homogeneous recurrence relation with constant coefficients
- The nonhomogeneous recurrence relation
- The method of generating Functions

The First-Order Linear Recurrence Relation

- The equation $a_{n+1} = 3a_n$ is a recurrence relation with constant coefficients. Since a_{n+1} *only depends on its immediate predecessor*, the relation is said to be *first order*.
- The expression $a_0 = A$, where A is a constant, is referred to as an initial condition.

- The **unique** solution of the recurrence relation $a_{n+1} = da_n$, where $n \geq 0$, d is a constant, and $a_0 = A$, is given by $a_n = Ad^n$.

The First-Order Linear Recurrence Relation

- **Ex 10.1** : Solve the recurrence relation $a_n = 7a_{n-1}$, where $n \geq 1$ and $a_2 = 98$.
  - $a_n = a_0(7^n)$, $a_2 = 98 = a_0(7^2) \Rightarrow a_0 = 2$, $a_n = 2(7^n)$.

- **Ex 10.2** : A bank pay 6% annual interest on savings, compounding the interest monthly. If we deposit \$1000, how much will this deposit be worth a year later?
  - $p_{n+1} = (1.005)p_n$, $p_0 = 1000 \Rightarrow p_n = p_0(1.005)^n$
  - $p_{12} = 1000(1.005)^{12} = \$1061.68$

The First-Order Linear Recurrence Relation

- The recurrence relation $a_{n+1} - da_n = 0$ is called linear because each term appears to the first power.

- **Ex 10.4** : Find a_{12} if $a_{n+1}^2 = 5a_n^2$ where $a_n > 0$ for $n \geq 0$ and $a_0 = 2$.
  - Let $b_n = a_n^2$. Then $b_{n+1} = 5b_n$ (linear) for $n \geq 0$ and $b_0 = 4 \Rightarrow b_n = 4 \cdot 5^n$

Homogeneous and Nonhomogeneous

- The general first-order linear recurrence relation with constant coefficients has the form
  $a_{n+1} + ca_n = f(n)$.
  - $f(n) = 0$, the relation is called homogeneous.
  - Otherwise, it is called nonhomogeneous.
- **Ex 10.5** : Let a_n denote the number of comparisons needed to sort n numbers in bubble sort, we find the recurrence relation
  - $a_n = a_{n-1} + (n - 1)$, $n \geq 2$, $a_1 = 0$

```
procedure BubbleSort(n: positive integer; x_1, x_2, x_3, ..., x_n: real numbers)
begin
   for i := 1 to n - 1 do
      for j := n downto i + 1 do
         if x_j < x_{j-1} then
            begin        {interchange}
               temp := x_{j-1}
               x_{j-1} := x_j
               x_j := temp
            end
end
```

Figure 10.2

at each interation i:
puts the minimum element in cells [i..n]
in the i-th cell

sorts increasingly

The First-Order Linear Recurrence Relation

- **Ex 10.6** : In Example 9.6 we sought the generating function for the sequence 0, 2, 6, 12, 20, 30, 42,…, due to the observation $a_n = n^2 + n$. If we fail to see this, alternatively

$$\begin{aligned}
a_1 - a_0 &= 2 \\
a_2 - a_1 &= 4 \\
a_3 - a_2 &= 6 \\
\vdots \quad\; \vdots \;\; &\;\; \vdots \\
a_n - a_{n-1} &= 2n.
\end{aligned}$$

$$\begin{aligned}
a_n - a_0 &= 2 + 4 + 6 + \cdots + 2n = 2(1 + 2 + 3 + \cdots + n) \\
&= 2[n(n+1)/2] = n^2 + n.
\end{aligned}$$

10.2
The Second-Order Linear Homogeneous Recurrence Relation with Constant Coefficients

- Linear recurrence relation of order k:
  - $C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} + \ldots + C_k a_{n-k} = f(n),\ n \geq 0.$
- Homogeneous relation of order 2:
  - $C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} = 0,\ n \geq 2.$
- Substituting $a_n = cr^n$ into the equation, we have
  - $C_0 cr^n + C_1 cr^{n-1} + C_2 cr^{n-2} = 0,\ n \geq 2.$
  - Characteristic equation: $C_0 r^2 + C_1 r + C_2 = 0,\ n \geq 2.$
- The roots r_1, r_2 of this equation are called characteristic roots.
- Three cases for the roots:

(A) distinct real roots

(B) complex conjugate roots

(C) equivalent real roots

Case (A): Distinct Real Roots

- **Ex 10.9** : Solve the recurrence relation $a_n + a_{n-1} - 6a_{n-2} = 0$, $n \geq 2$, and $a_0 = -1$ and $a_1 = 8$.
  - **Solution**

Let $a_n = cr^n$

$r^2 + r - 6 = 0 \Rightarrow r = 2, -3$

$a_n = c_1(2)^n + c_2(-3)^n$

$-1 = a_0 = c_1 + c_2$

$8 = a_1 = 2c_1 - 3c_2 \Rightarrow c_1 = 1, c_2 = -2$

$\Rightarrow a_n = (2)^n - 2(-3)^n$

$a_n=2^n$ and $a_n=(-3)^n$ are both solutions

Linearly independent solutions

Distinct Real Roots

- **Ex 10.10** : Solve Fibonacci relation, $F_{n+2} = F_{n+1} + F_n$, $n \geq 0$, $F_0 = 0$, $F_1 = 1$.
  - **Solution**

    Let $F_n = cr^n$,

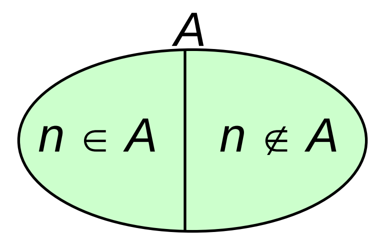
    $r^2 - r - 1 = 0 \Rightarrow \qquad r = (1 \pm \sqrt{5})/2$

$$F_n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right], \qquad n \geq 0.$$

Distinct Real Roots

- **Ex 10.11** : For $n \geq 0$, let $S = \{1, 2, \ldots, n\}$, and let a_n denote the number of subsets of S that contains no consecutive integers. Find and solve a recurrence relation for a_n.
  - **Solution**
  - $a_0 = 1$ and $a_1 = 2$ and $a_2 = 3$ and $a_3 = 5$ (Fibonacci?)
  - If $A \subseteq S$ and A is to be counted in a_n, there are two cases
    - (1) $n \in A$, then $A - \{n\}$ would be counted in a_{n-2}
    - (2) $n \notin A$, then A would be counted in a_{n-1}
  - $a_n = a_{n-1} + a_{n-2}$, $n \geq 2$



$$a_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+2}\right], \qquad n \geq 0.$$

Exhaustive and mutually disjoint

Distinct Real Roots

- **Ex 10.12** : Suppose we have a $2 \times n$ chessboard. We wish to cover such a chessboard using 2×1 vertical dominoes or 1×2 horizontal dominoes.

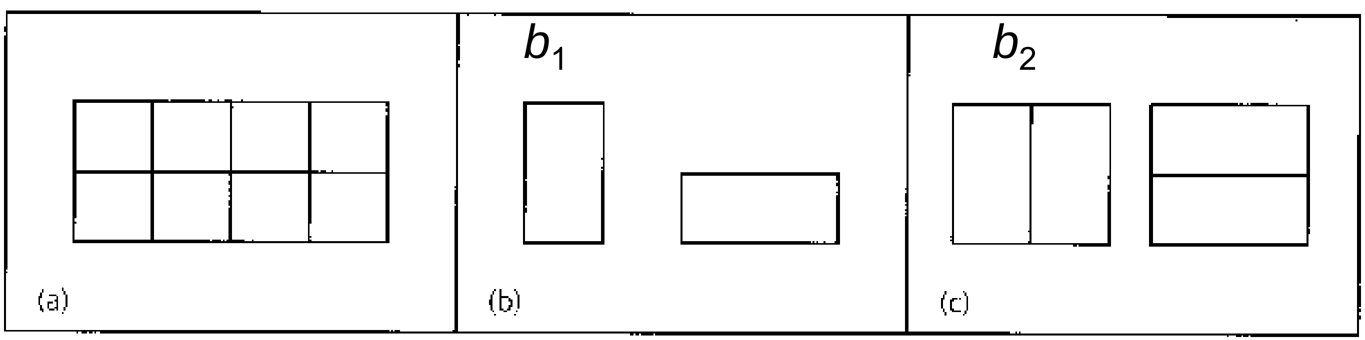
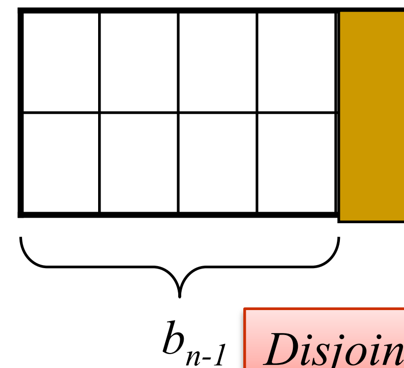


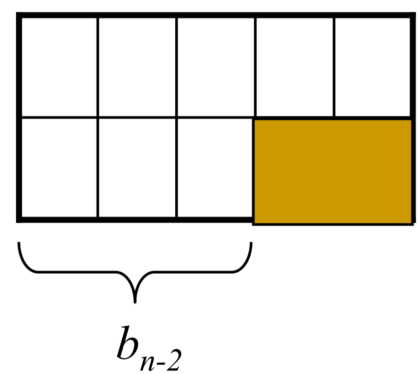
Figure 10.5

Distinct Real Roots

- Let b_n count the number of ways we can cover a $2\times n$ chessboard by using 2×1 vertical dominoes or 1×2 horizontal dominoes.
- $b_1=1$ and $b_2=2$
- For $n \geq 3$, consider the last (nth) column of the chessboard
  1) By one 2×1 vertical domino: Now the remaining $2\times(n-1)$ subboard can be covered in b_{n-1} ways.
  2) By two 1×2 horizontal dominoes, place one above the other: Now the remaining $2\times(n-2)$ subboard can be covered in b_{n-2} ways.
- $b_n = b_{n-1} + b_{n-2}$, $n \geq 3$, $b_1=1$ and $b_2=2$
  $\Rightarrow b_n = F_{n+1}$



Disjoint?



Distinct Real Roots

$$p_n = 2p_{n-2}, \qquad n \geq 3, \qquad p_1 = 1, \qquad p_2 = 2.$$

Substituting $p_n = cr^n$, for $c, r \neq 0$, $n \geq 1$, into this recurrence relation, the resulting characteristic equation is $r^2 - 2 = 0$. The characteristic roots are $r = \pm\sqrt{2}$, so $p_n = c_1(\sqrt{2})^n + c_2(-\sqrt{2})^n$. From

$$1 = p_1 = c_1(\sqrt{2}) + c_2(-\sqrt{2})$$
$$2 = p_2 = c_1(\sqrt{2})^2 + c_2(-\sqrt{2})^2$$

we find that $c_1 = \left(\frac{1}{2} + \frac{1}{2\sqrt{2}}\right)$, $c_2 = \left(\frac{1}{2} - \frac{1}{2\sqrt{2}}\right)$, so

$$p_n = \left(\frac{1}{2} + \frac{1}{2\sqrt{2}}\right)(\sqrt{2})^n + \left(\frac{1}{2} - \frac{1}{2\sqrt{2}}\right)(-\sqrt{2})^n, \qquad n \geq 1.$$

we consider n even, say $n = 2k$

$$p_n = \left(\frac{1}{2} + \frac{1}{2\sqrt{2}}\right)(\sqrt{2})^{2k} + \left(\frac{1}{2} - \frac{1}{2\sqrt{2}}\right)(-\sqrt{2})^{2k}$$
$$= \left(\frac{1}{2} + \frac{1}{2\sqrt{2}}\right)2^k + \left(\frac{1}{2} - \frac{1}{2\sqrt{2}}\right)2^k = 2^k = 2^{n/2}$$

Second- or Higher-Order Recurrence Relation

- **Ex 10.18** : Solve $2a_{n+3} = a_{n+2} + 2a_{n+1} - a_n$, $n \geq 0$, $a_0=0$, $a_1=1$, $a_2=2$
  - Let $a_n = cr^n$
  - Characteristic equation: $2r^3-r^2-2r+1=0 \Rightarrow r = 1, 1/2, -1$
  - $a_n = c_1(1)^n + c_2(-1)^n + c_3(1/2)^n$

  - From $a_0=0$, $a_1=1$, $a_2=2$, derive $c_1=5/2$, $c_2=1/6$, $c_3=-8/3$
  - $a_n = (5/2)+(1/6)(-1)^n+(-8/3)(1/2)^n$

Second- or Higher-Order Recurrence Relation

- **Ex 10.19** : We want to tile a $2\times n$ chessboard using two types of tiles shown in Figure 10.8.

a_2: 2×2 chessboard

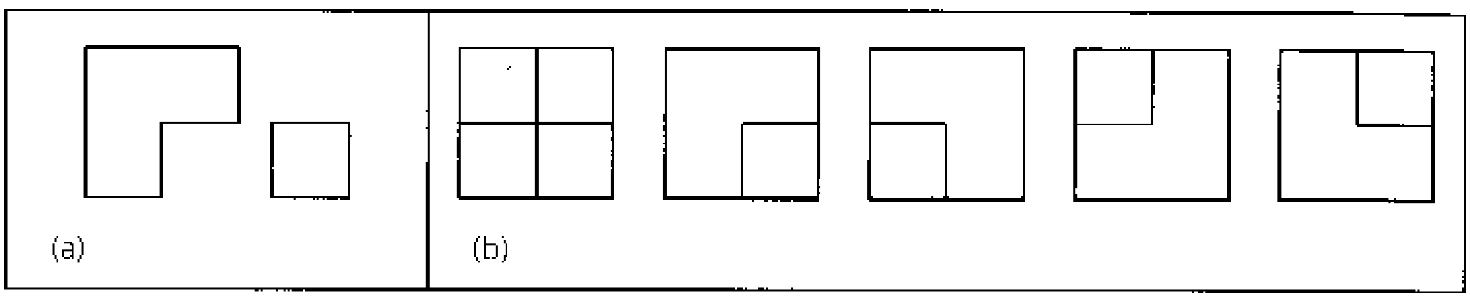
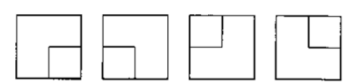


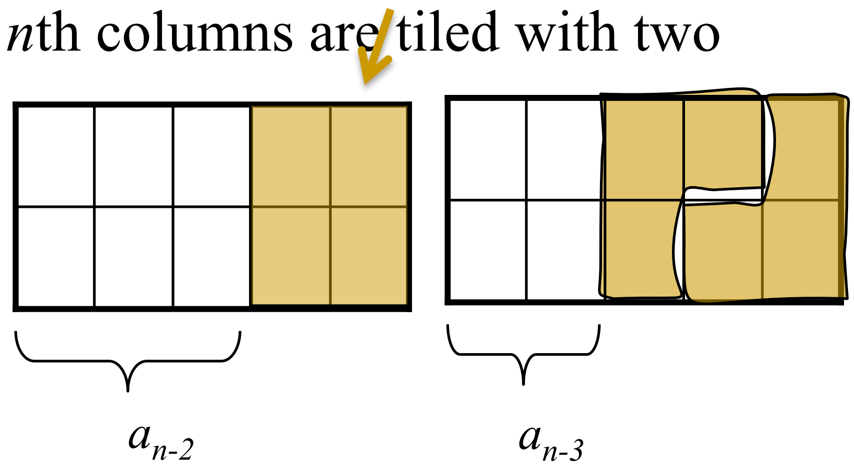
Figure 10.8

a_3: 2×3 chessboard

Second- or Higher-Order Recurrence Relation

- Let a_n count the number of such tilings.
- a_1=1 and a_2=5 and a_3=11
- For $n \geq 4$, consider the last column of the chessboard
  1) the nth column is covered by two 1×1 tiles $\Rightarrow a_{n-1}$
  2) the (n -1)st and the nth column are tiled with one 1×1 tile and a larger tile $\Rightarrow 4a_{n-2}$
  3) the (n - 2)nd, (n - 1)st and the nth columns are tiled with two large tiles $\Rightarrow 2a_{n-3}$
- $a_n = a_{n-1} + 4a_{n-2} + 2a_{n-3}$, $n \geq 4$





Case (B): Complex Roots

- DeMoivre‵s Theorem:
  $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

- If $z = x + iy \in \mathbf{C} \Rightarrow z = r(\cos\theta + i\sin\theta), r = \sqrt{x^2 + y^2}, \dfrac{y}{x} = \tan\theta$, for $x \neq 0$

  If $x = 0, \begin{cases} y > 0, z = yi = yi\sin(\pi/2) = y(\cos(\pi/2) + i\sin(\pi/2)) \\ y < 0, z = |y|\sin(3\pi/2) = |y|(\cos(3\pi/2) + i\sin(3\pi/2)) \end{cases}$

  By DeMoivre's Theorem, $z^n = r^n(\cos n\theta + i\sin n\theta)$

Complex Roots

- Ex 10.20 : Determine $(1+\sqrt{3}i)^{10}$
  - Solution

Complex number $1+\sqrt{3}i$ is represented

as the point $(1,\sqrt{3})$ in the xy-plane

$r=\sqrt{1^2+(\sqrt{3})^2}=2, \theta=\pi/3$

$1+\sqrt{3}i=2(\cos(\pi/3)+i\sin(\pi/3))$

$(1+\sqrt{3}i)^{10}=2^{10}(\cos(10\pi/3)+i\sin(10\pi/3))=2^{10}(\cos(4\pi/3)+i\sin(4\pi/3))$

$=2^{10}((-1/2)-(\sqrt{3}/2)i)=(-2^9)(1+\sqrt{3}i)$

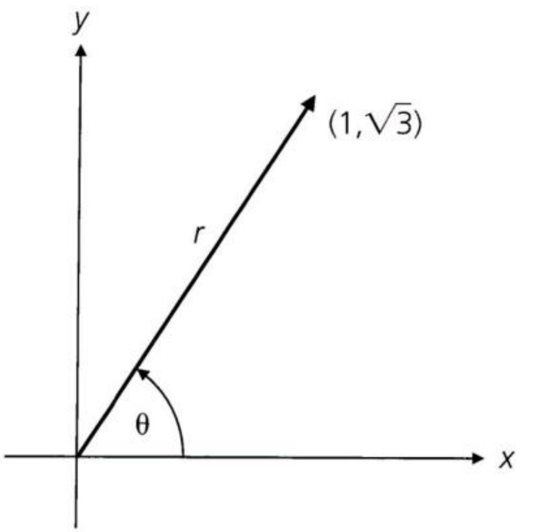


Figure 10.9

Complex Roots

- **Ex 10.21** : Solve the recurrence relation $a_n = 2(a_{n-1} - a_{n-2})$ where $n \geq 2$, $a_0 = 1$, $a_1 = 2$.
  - Let $a_n = cr^n$
  - $r^2 - 2r + 2 = 0 \Rightarrow r = 1 \pm i$
  - $1 + \mathrm{i} = \sqrt{2}(\cos(\pi/4) + i \sin(\pi/4))$
  - $1 - \mathrm{i} = \sqrt{2}(\cos(\pi/4) - i \sin(\pi/4))$

$$
\begin{aligned}
a_n &= c_1(1+i)^n + c_2(1-i)^n \\
&= c_1[\sqrt{2}(\cos(\pi/4) + i\ \sin(\pi/4))]^n + c_2[\sqrt{2}(\cos(-\pi/4) + i\ \sin(-\pi/4))]^n \\
&= c_1(\sqrt{2})^n(\cos(n\pi/4) + i\ \sin(n\pi/4)) + c_2(\sqrt{2})^n(\cos(-n\pi/4) + i\ \sin(-n\pi/4)) \\
&= c_1(\sqrt{2})^n(\cos(n\pi/4) + i\ \sin(n\pi/4)) + c_2(\sqrt{2})^n(\cos(n\pi/4) - i\ \sin(n\pi/4)) \\
&= (\sqrt{2})^n[k_1 \cos(n\pi/4) + k_2 \sin(n\pi/4)],
\end{aligned}
$$

where $k_1 = c_1 + c_2$ and $k_2 = (c_1 - c_2)i$.

$$1 = a_0 = [k_1 \cos 0 + k_2 \sin 0] = k_1$$

$$2 = a_1 = \sqrt{2}[1 \cdot \cos(\pi/4) + k_2 \sin(\pi/4)], \text{ or } 2 = 1 + k_2, \text{ and } k_2 = 1.$$

The solution for the given initial conditions is then given by

$$a_n = (\sqrt{2})^n[\cos(n\pi/4) + \sin(n\pi/4)], \qquad n \geq 0.$$

Complex Roots

- **Ex 10.22** : Let a_n denote the value of the $n \times n$ determinant D_n
  - $a_1 = b$, $a_2 = 0$ and $a_3 = -b^3$
  - $D_n = bD_{n-1} - b^2 D_{n-2}$
  - $a_n = ba_{n-1} - b^2 a_{n-2}$

$$\begin{vmatrix} b & b & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ b & b & b & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & b & b & b & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b & b & b & \cdots & 0 & 0 & 0 & 0 & 0 \\ . & . & . & . & . & \cdots & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & \cdots & b & b & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & b & b & b & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & b & b & b \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & b & b \end{vmatrix}$$

$$a_1 = |b| = b \quad \text{and} \quad a_2 = \begin{vmatrix} b & b \\ b & b \end{vmatrix} = 0 \quad (\text{and} \quad a_3 = \begin{vmatrix} b & b & 0 \\ b & b & b \\ 0 & b & b \end{vmatrix} = -b^3).$$

$$= b\underbrace{\begin{vmatrix} b & b & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ b & b & b & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & b & b & b & \cdots & 0 & 0 & 0 & 0 \\ . & . & . & . & \cdots & . & . & . & . \\ 0 & 0 & 0 & 0 & \cdots & b & b & b & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & b & b & b \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & b & b \end{vmatrix}}_{\text{(This is } D_{n-1}.\text{)}} - b\begin{vmatrix} b & b & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & b & b & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & b & b & b & \cdots & 0 & 0 & 0 & 0 \\ . & . & . & . & \cdots & . & . & . & . \\ 0 & 0 & 0 & 0 & \cdots & b & b & b & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & b & b & b \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & b & b \end{vmatrix}$$

If we let $a_n = cr^n$ for $c, r \neq 0$ and $n \geq 1$, the characteristic equation produces the roots $b[(1/2) \pm i\sqrt{3}/2]$.

Hence

$$\begin{aligned} a_n &= c_1[b((1/2) + i\sqrt{3}/2)]^n + c_2[b((1/2) - i\sqrt{3}/2)]^n \\ &= b^n[c_1(\cos(\pi/3) + i \sin(\pi/3))^n + c_2(\cos(\pi/3) - i \sin(\pi/3))^n] \\ &= b^n[k_1 \cos(n\pi/3) + k_2 \sin(n\pi/3)]. \end{aligned}$$

$b = a_1 = b[k_1 \cos(\pi/3) + k_2 \sin(\pi/3)]$, so $1 = k_1(1/2) + k_2(\sqrt{3}/2)$, or $k_1 + \sqrt{3}\, k_2 = 2$.

$0 = a_2 = b^2[k_1 \cos(2\pi/3) + k_2 \sin(2\pi/3)]$, so $0 = (k_1)(-1/2) + k_2(\sqrt{3}/2)$, or

$$k_1 = \sqrt{3}\, k_2.$$

Hence $k_1 = 1$, $k_2 = 1/\sqrt{3}$ and the value of D_n is

$$b^n[\cos(n\pi/3) + (1/\sqrt{3}) \sin(n\pi/3)].$$

Case (C): Repeated Real Roots

- **Ex 10.23** : Solve the recurrence relation
  $a_{n+2} = 4a_{n+1} - 4a_n$ where $n \geq 0$, $a_0=1$, $a_1=3$
  - **Solution**

    Let $a_n = cr^n$

    $a_n = c_1 2^n + c_2 2^n$?

    X

    $r^2 - 4r + 4 = 0 \Rightarrow r = 2 \Rightarrow 2^n$ and 2^n are not independent solutions, need another independent solution

    So, try $g(n)2^n$, where $g(n)$ is not a constant

    $\Rightarrow g(n+2)2^{n+2} = 4g(n+1)2^{n+1} - 4g(n)2^n$

    $\Rightarrow g(n+2) = 2g(n+1) - g(n) \Rightarrow g(n) = n, \therefore n2^n$ is a solution

    $a_n = c_1(2^n) + c_2(n2^n)$ with $a_0=1$, $a_1=3$

    $a_n = 1(2^n) + (1/2)(n2^n)$

Repeated Real Roots

- In general, if

  $$C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} + \ldots + C_k a_{n-k} = 0$$

  with r, a characteristic root of multiplicity m, then the part of the general solution that involves the root r has the form

  $$A_0 r^n + A_1 n r^n + A_2 n^2 r^n + A_3 n^3 r^n + \ldots + A_{m-1} n^{m-1} r^n$$

$$= (A_0 + A_1 n + A_2 n^2 + A_3 n^3 + \ldots + A_{m-1} n^{m-1}) r^n$$

Repeated Real Roots

- **Ex 10.24** : Let p_n denote the probability that at least one case of measles is reported during the nth week after the first recorded case. School records provide evidence that $p_n = p_{n-1} - (0.25)p_{n-2}$, for $n \geq 2$. Since $p_n = 0$ and $p_1 = 1$, if the first case is recorded on Monday, March 3, 2003, when did the probability for the occurrence of a new case decrease to less than 0.01 for the first time?
  - **Solution**

    Let $p_n = cr^n$

    $r^2 - r + (1/4) = 0 \Rightarrow r = 1/2$

    $p_n = (c_1 + c_2 n)(1/2)^n \Rightarrow c_1 = 0$ and $c_2 = 2 \Rightarrow p_n = n2^{-n+1}$

    $p_n < 0.01 \Rightarrow$ the first n is 12, the week of May 19, 2003.

10.3 The Nonhomogeneous Recurrence Relation

- $a_n + C_1 a_{n-1} = f(n),\ n \geq 1,$
- $a_n + C_1 a_{n-1} + C_2 a_{n-2} = f(n),\ n \geq 2$
- Let $\boldsymbol{a_n^{(h)}}$ denote the general solution of the associated homogeneous relation.
- Let $\boldsymbol{a_n^{(p)}}$ denote a solution of the given nonhomogeneous relation. (particular solution)
- Then $\boldsymbol{a_n = a_n^{(h)} + a_n^{(p)}}$ is the general solution of the recurrence relation.

The Nonhomogeneous Recurrence Relation

- **Ex 10.26** : Solve the recurrence relation
  $a_n - 3a_{n-1} = 5(7^n)$ for $n \geq 1$ and $a_0 = 2$.
  - **Solution**

    The solution for $a_n - 3a_{n-1} = 0$ is $a_n^{(h)} = c(3^n)$.

    Since $f(n) = 5(7^n)$, let $a_n^{(p)} = A(7^n)$

    $\Rightarrow A(7^n) - 3A(7^{n-1}) = 5(7^n) \Rightarrow A = 35/4$

    $a_n^{(p)} = (35/4)7^n = (5/4)7^{n+1}$.

    The general solution is $a_n = a_n^{(h)} + a_n^{(p)} = c(3^n) + (5/4)7^{n+1}$

    So, $a_n = (-1/4)(3^{n+3}) + (5/4)7^{n+1}$

The Nonhomogeneous Recurrence Relation

- **Ex 10.27** : Solve the recurrence relation $a_n - 3a_{n-1} = 5(\mathbf{3}^n)$ for $n \geq 1$ and $a_0 = 2$.
  - **Solution**

    Let $a_n^{(h)} = c(3^n)$.

    Since *$a_n^{(h)}$ and $f(n)$ are not linearly independent*, let $a_n^{(p)} = Bn(3^n) \Rightarrow Bn(3^n) - 3B(n-1)(3^{n-1}) = 5(3^{\mathrm{n}}). \Rightarrow B=5.$

    The general solution is $\mathrm{a}_{\mathrm{n}} = a_n^{(h)} + a_n^{(p)} = c(3^n) + (5)n3^n$

    $a_n = (2 + 5n)(3^n)$

Solution for the Nonhomogeneous First-Order Relation

- $a_n + C_1 a_{n-1} = kr^n$.
  - If r^n is not a solution of the homogeneous relation $a_n + C_1 a_{n-1} = 0$, then $a_n^{(p)} = Ar^n$.
  - If r^n is a solution of the homogeneous relation, then $a_n^{(p)} = Bnr^n$.

Solution for the Nonhomogeneous Second-Order Relation

- $a_n + C_1 a_{n-1} + C_2 a_{n-2} = kr^n$.
  - If r^n is not a solution of the homogeneous relation, then $a_n^{(p)} = Ar^n$.
  - If $a_n^{(h)} = c_1 r^n + c_2 r_1^{\,n}$, where $r \neq r_1$, then $a_n^{(p)} = Bnr^n$.
  - If $a_n^{(h)} = (c_1 + c_2 n) r^n$, then $a_n^{(p)} = Cn^2 r^n$.

The Nonhomogeneous Recurrence Relation

- **Ex 10.28** : The Towers of Hanoi.
  - Let count the minimum number of moves it takes to transfer n disks from peg 1 to peg 3.
  - $a_{n+1} = 2a_n + 1$
    - Transfer the top n disks from peg 1 to peg 2, need a_n moves.
    - Transfer the largest disk from peg 1 to peg 3, need 1 moves.
    - Transfer the n disks on peg 2 onto the largest disk, need a_n moves.

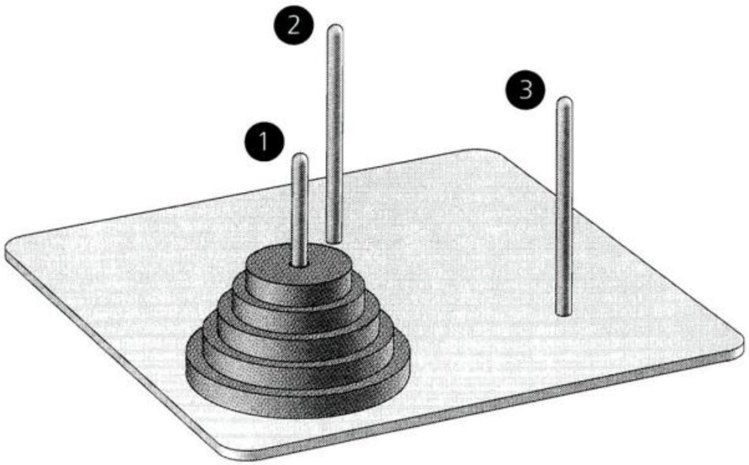


Figure 10.11

For $a_{n+1} - 2a_n = 1$, we know that $a_n^{(h)} = c(2^n)$. Since $f(n) = 1 = (1)^n$ is not a solution of $a_{n+1} - 2a_n = 0$, we set $a_n^{(p)} = A(1)^n = A$ and find from the given relation that $A = 2A + 1$, so $A = -1$ and $a_n = c(2^n) - 1$. From $a_0 = 0 = c - 1$ it then follows that $c = 1$, so $a_n = 2^n - 1$, $n \geq 0$.

The Nonhomogeneous Recurrence Relation

- **Ex 10.31** : For the alphabet = {0,1,2,3}, how many strings of length n contains an even number of 1"s.
  - Let a_n count those strings among the 4^n strings.

    Consider the nth symbol of a string of length n

    1. The nth symbol is 0, 2, 3 $\Rightarrow 3a_{n-1}$
    2. The nth symbol is 1 $\Rightarrow$ there must be an odd number of 1"s among the first n -1 symbols $\Rightarrow 4^{n-1} - a_{n-1}$

    $a_n = 3a_{n-1} + (4^{n-1} - a_{n-1}) = 2a_{n-1} + 4^{n-1}$

    $a_n^{(h)} = c(2^n)$, $a_n^{(p)} = A(4^{n-1})$

    $a_1 = 3 \rightarrow a_n = 2^{n-1} + 2(4^{n-1})$

The Nonhomogeneous Recurrence Relation

- **Ex 10.34** : Solve the recurrence relation $a_{n+2} - 4a_{n+1} + 3a_n = -200$ for $n \geq 0$ and $a_0 = 3000$ and $a_1 = 3300$.
  - **Solution**

    $a_n^{(h)} = c_1(3^n) + c_2(\mathbf{1^n})$.

    Let $a_n^{(\mathrm{p})} = A\mathbf{n} \Rightarrow A(n+2) - 4A(n+1) + 3An = -200$

    $\Rightarrow a_n^{(p)} = 100n$.

    $a_n = a_n^{(h)} + a_n^{(p)} = c_1(3^n) + c_2(1^n) + 100n$

    $\Rightarrow a_n = 100(3^n) + 2900 + 100n$

Particular Solutions to Nonhomogeneous Recurrence Relation

- $C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} + \ldots + C_k a_{n-k} = f(n)$

(1) If $f(n)$ is a constant multiple of one of the forms in the first column of Table 10.2 $\Rightarrow a_n^{(p)}$ in the second column .

(2) When $f(n)$ comprises a sum of constant multiples of terms.

- E.g., $f(n) = n^2 + 3\sin 2n \Rightarrow a_n^{(p)} = (A_2 n^2 + A_1 n + A_0) + (A \sin 2n + B\cos 2n)$

(3) If a summand $f_1(n)$ of $f(n)$ is a solution of the associated homogeneous relation.

- If $f_1(n)$ causes this problem, we multiply the trial solution $(a_n^{(p)})_1$ corresponding to $f_1(n)$ by the smallest power of n, say n^s, for which no summand of $n^s f_1(n)$ is a solution of the associated homogeneous relation. Thus, $n^s(a_n^{(p)})_1$ is the corresponding part of $a_n^{(p)}$.

Table 10.2

	$a_n^{(p)}$
c, a constant	A, a constant
n	$A_1 n + A_0$
n^2	$A_2 n^2 + A_1 n + A_0$
$n^t, t \in \mathbf{Z}^+$	$A_t n^t + A_{t-1} n^{t-1} + \cdots + A_1 n + A_0$
$r^n, r \in \mathbf{R}$	Ar^n
$\sin \theta n$	$A \sin \theta n + B \cos \theta n$
$\cos \theta n$	$A \sin \theta n + B \cos \theta n$
$n^t r^n$	$r^n (A_t n^t + A_{t-1} n^{t-1} + \cdots + A_1 n + A_0)$
$r^n \sin \theta n$	$Ar^n \sin \theta n + Br^n \cos \theta n$
$r^n \cos \theta n$	$Ar^n \sin \theta n + Br^n \cos \theta n$

Particular Solutions to Nonhomogeneous Recurrence Relation

- **Ex 10.36** : For n people at a party, each of them shakes hands with others.
  - a_n counts the total number of handshakes:

    $$a_{n+1} = a_n + n,\ n \geq 2,\ a_2 = 1$$
  - $a_n^{(h)} = c(1^n) = c.$
  - Let $a_n^{(p)} = A_1 n + A_0$
  - By the third remark stated above, multiplying $a_n^{(p)}$ by n^1, then $a_n^{(p)} = A_1 n^2 + A_0 n$
  - $A_1 = ½$, $A_0 = -½ \Rightarrow a_n^{(p)} = (½)n^2 + (-½)n.$
  - $a_n = a_n^{(h)} + a_n^{(p)} = c + (½)n^2 + (-½)n \Rightarrow c = 0$
  - $a_n = (½)n(n-1)$

Particular Solutions to Nonhomogeneous Recurrence Relation

- **Ex 10.37** : $a_{n+2} - 10a_{n+1} + 21a_n = f(n),\ n \geq 0$
- $a_n^{(h)} = c_1(3^n) + c_2(7^n)$.

Table 10.3

$f(n)$	$a_n^{(p)}$
5	A_0
$3n^2 - 2$	$A_3n^2 + A_2n + A_1$
$7(11^n)$	$A_4(11^n)$
$31(r^n),\ r \neq 3, 7$	$A_5(r^n)$
$6(3^n)$	$A_6\underline{n3^n}$
$2(3^n) - 8(9^n)$	$A_7\underline{n3^n} + A_8(9^n)$
$4(3^n) + 3(7^n)$	$A_9\underline{n3^n} + A_{10}\underline{n7^n}$

10.4 The Method of Generating Functions

- **Ex 10.38** : Solve the relation $a_n - 3a_{n-1} = n$, $n \geq 1$, $a_0 = 1$.
  - Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function for $a_0, a_1, \ldots, a_n$.

$$\sum_{n=1}^{\infty} a_n x^n - 3\sum_{n=1}^{\infty} a_{n-1} x^n = \sum_{n=1}^{\infty} n x^n.$$

$$(f(x) - a_0) - 3x\sum_{n=1}^{\infty} a_{n-1} x^{n-1} = \sum_{n=1}^{\infty} n x^n \left(= \sum_{n=0}^{\infty} n x^n \right).$$

$$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + \cdots,$$

$$(f(x) - 1) - 3xf(x) = \frac{x}{(1-x)^2}, \quad \text{and} \quad f(x) = \frac{1}{(1-3x)} + \frac{x}{(1-x)^2(1-3x)}.$$

$$f(x) = \frac{1}{1-3x} + \frac{(-1/4)}{(1-x)} + \frac{(-1/2)}{(1-x)^2} + \frac{(3/4)}{(1-3x)}$$
$$= \frac{(7/4)}{(1-3x)} + \frac{(-1/4)}{(1-x)} + \frac{(-1/2)}{(1-x)^2}.$$

We find a_n by determining the coefficient of x^n in each of the three summands.

a) $(7/4)/(1-3x) = (7/4)[1/(1-3x)]$
$= (7/4)[1 + (3x) + (3x)^2 + (3x)^3 + \cdots]$, and the coefficient of x^n is $(7/4)3^n$.

b) $(-1/4)/(1-x) = (-1/4)[1 + x + x^2 + \cdots]$, and the coefficient of x^n here is $(-1/4)$.

c) $(-1/2)/(1-x)^2 = (-1/2)(1-x)^{-2}$
$= (-1/2)\left[\binom{-2}{0} + \binom{-2}{1}(-x) + \binom{-2}{2}(-x)^2 + \binom{-2}{3}(-x)^3 + \cdots\right]$
and the coefficient of x^n is given by $(-1/2)\binom{-2}{n}(-1)^n = (-1/2)(-1)^n\binom{2+n-1}{n} \cdot (-1)^n = (-1/2)(n+1)$.

Therefore $a_n = (7/4)3^n - (1/2)n - (3/4), n \geq 0$.

The Method of Generating Functions

- **Ex 10.39** : Solve the relation
  $a_{n+2} - 5a_{n+1} + 6a_n = 2,\ n \ge 0,\ a_0 = 3,\ a_1 = 7.$
  - Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function for $a_0, a_1, \ldots, a_n$

$$\sum_{n=0}^{\infty} a_{n+2}x^{n+2} - 5\sum_{n=0}^{\infty} a_{n+1}x^{n+2} + 6\sum_{n=0}^{\infty} a_n x^{n+2} = 2\sum_{n=0}^{\infty} x^{n+2}.$$

$$\sum_{n=0}^{\infty} a_{n+2}x^{n+2} - 5x\sum_{n=0}^{\infty} a_{n+1}x^{n+1} + 6x^2\sum_{n=0}^{\infty} a_n x^n = 2x^2\sum_{n=0}^{\infty} x^n.$$

$$(f(x) - a_0 - a_1x) - 5x(f(x) - a_0) + 6x^2 f(x) = \frac{2x^2}{1-x},$$

$$(1-5x+6x^2)f(x) = 3-8x+\frac{2x^2}{1-x} = \frac{3-11x+10x^2}{1-x},$$

from which it follows that

$$f(x) = \frac{3-11x+10x^2}{(1-5x+6x^2)(1-x)} = \frac{(3-5x)(1-2x)}{(1-3x)(1-2x)(1-x)} = \frac{3-5x}{(1-3x)(1-x)}.$$

A partial-fraction decomposition (by hand, or via a computer algebra system) gives us

$$f(x) = \frac{2}{1-3x} + \frac{1}{1-x} = 2\sum_{n=0}^{\infty}(3x)^n + \sum_{n=0}^{\infty}x^n.$$

Consequently, $a_n = 2(3^n)+1,\ n \geq 0$.