

بهرینه سازی با جدول کارنو

Karnaugh Map

Karnaugh Map

- Method of graphically representing the truth table that helps visualize adjacencies

2-variable K-map

B \ A	0	1
0	0	2
1	1	3

3-variable K-map

C \ AB	00	01	11	10
0	0	2	6	4
1	1	3	7	5

4-variable K-map

CD \ AB	00	01	11	10
00	0	4	12	8
01	1	5	13	9
11	3	7	15	11
10	2	6	14	10

Karnaugh Map

- One cell in K-map = one row in truth table
- One cell = a minterm (or a maxterm)
- Multiple-cell areas = standard terms

B \ A	0	1
0	0	2
1	1	3

C \ AB	00	01	11	10
0	0	2	6	4
1	1	3	7	5

Diagram illustrating a 2x4 Karnaugh Map for variables A, B, and C. The map is labeled with AB on the horizontal axis and C on the vertical axis. The cells are numbered 0 through 7. A bracket labeled 'A' spans the columns 11 and 10. A bracket labeled 'B' spans the columns 01 and 11. A bracket labeled 'C' spans the rows 0 and 1.

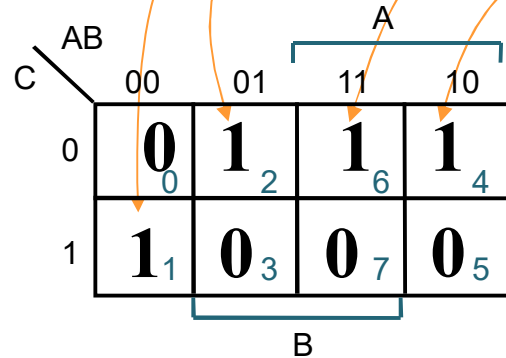
CD \ AB	00	01	11	10
00	0	4	12	8
01	1	5	13	9
11	3	7	15	11
10	2	6	14	10

Diagram illustrating a 4x4 Karnaugh Map for variables A, B, C, and D. The map is labeled with AB on the horizontal axis and CD on the vertical axis. The cells are numbered 0 through 15. A bracket labeled 'A' spans the columns 11 and 10. A bracket labeled 'B' spans the columns 01 and 11. A bracket labeled 'C' spans the rows 11 and 10. A bracket labeled 'D' spans the rows 01 and 11.

Karnaugh Map

$$f_1(A,B,C) = m_1 + m_2 + m_4 + m_6$$

$$= A'B'C + A'BC' + AB'C' + ABC'$$



A	B	C		f_1
0	0	0	m0	0
0	0	1	m1	1
0	1	0	m2	1
0	1	1	m3	0
1	0	0	m4	1
1	0	1	m5	0
1	1	0	m6	1
1	1	1	m7	0

Karnaugh Map

- Numbering Scheme: 00, 01, 11, 10

Gray Code: only a single bit changes from one code word to the next code word.

B \ A	0	1
	0	2
1	1	3

C \ AB	00	01	11	10
	0	2	6	4
1	1	3	7	5

CD \ AB	00	01	11	10
	0	4	12	8
01	1	5	13	9
11	3	7	15	11
10	2	6	14	10

Two-Variable Map (cont.)

- Any two adjacent cells in the map differ by ONLY one variable, which appears complemented in one cell and uncomplemented in the other.

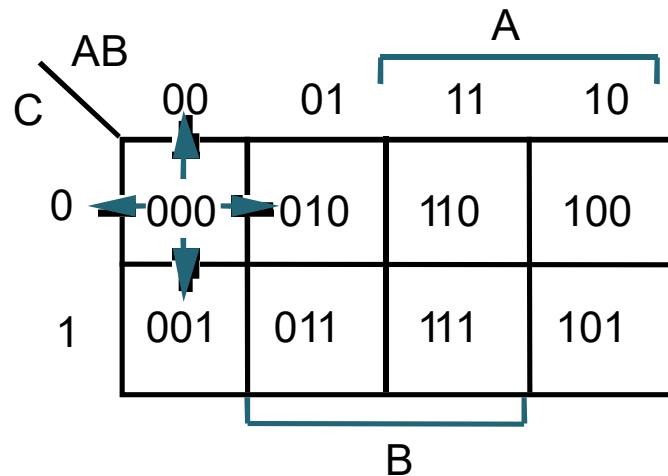
- **Example:**

- $m_0 (=A'B')$ is adjacent to $m_1 (=A'B)$ and $m_2 (=AB')$ but NOT $m_3 (=AB)$

		A	
		0	1
B	0	0	2
	1	1	3

Karnaugh Map

➤ Adjacencies in the K-Map

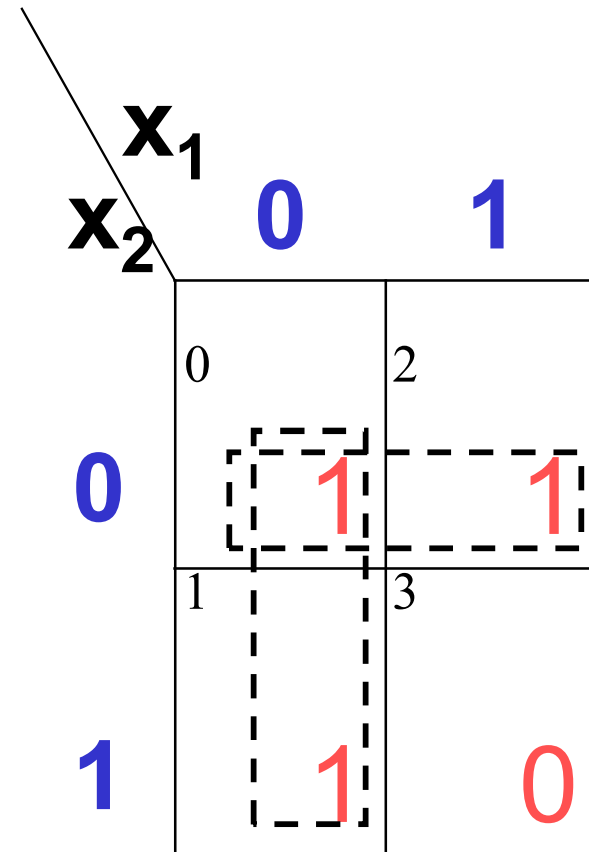


Wrap from first to last column

Top row to bottom row

2-Variable Map -- Example

- $f(x_1, x_2) = x_1'x_2' + x_1'x_2 + x_1x_2'$
 $= m_0 + m_1 + m_2$
- 1's placed in K-map for specified minterms m_0, m_1, m_2
- Grouping (ORing) of 1s allows simplification
- What (simpler) function is represented by each dashed rectangle?
 - $m_0 + m_1 = x_1'x_2' + x_1'x_2 = x_1'(x_2' + x_2) = x_1'$
 - $m_0 + m_2 = x_1'x_2' + x_1x_2' = x_2'(x_1' + x_1) = x_2'$
- Simplified form: $f(x_1, x_2) = x_1' + x_2'$
- Note m_0 is covered twice



Minimization as SOP using K-map

- Enter 1's in the K-map for each product term (minterm) in the function
- Group *adjacent* K-map cells containing 1's to obtain a product with fewer variables
 - Groups must be in power of 2 (2, 4, 8, ...)
- Handle "boundary wrap"
- Answer may not be unique

Minimization as SOP

		A	
		0	1
B	0	0	1
	1	0	1

A asserted, unchanged
B varies

B complemented, unchanged
A varies

$$F = ?$$

$$F = A$$

		A	
		0	1
B	0	1	1
	1	0	0

$$G = ?$$

$$G = B'$$

		A			
		00	01	11	10
Cin	0	0	0	1	0
	1	0	1	1	1

$$\text{Cout} = ?$$

$$\text{Cout} = AB + BC_{in} + AC_{in}$$

		A			
		00	01	11	10
C	0	0	0	1	1
	1	0	0	1	1

$$F(A,B,C) = ?$$

$$F(A,B,C) = A$$

More Examples

		A			
AB		00	01	11	10
C	0	1	0	0	1
	1	0	0	1	1

Diagram showing a Karnaugh map for function F(A,B,C). The map is a 2x4 grid with columns labeled AB (00, 01, 11, 10) and rows labeled C (0, 1). The values are: (0,0)=1, (0,1)=0, (0,2)=0, (0,3)=1, (1,0)=0, (1,1)=0, (1,2)=1, (1,3)=1. Groupings are shown: a vertical group of two 1s in column AB=00, a horizontal group of two 1s in row C=0, and a horizontal group of two 1s in row C=1.

Why not group m4 and m5?

$$F(A,B,C) = \sum m(0,4,5,7)$$

$$F = B' C' + A C$$

		A			
AB		00	01	11	10
C	0	0	1	1	0
	1	1	1	0	0

Diagram showing a Karnaugh map for function F'(A,B,C). The map is a 2x4 grid with columns labeled AB (00, 01, 11, 10) and rows labeled C (0, 1). The values are: (0,0)=0, (0,1)=1, (0,2)=1, (0,3)=0, (1,0)=1, (1,1)=1, (1,2)=0, (1,3)=0. Groupings are shown: a horizontal group of two 1s in row C=0, and a horizontal group of two 1s in row C=1.

F' simply replaces 1's with 0's and vice versa

$$F'(A,B,C) = \sum m(1,2,3,6)$$

$$F' = B C' + A' C$$

Simplification

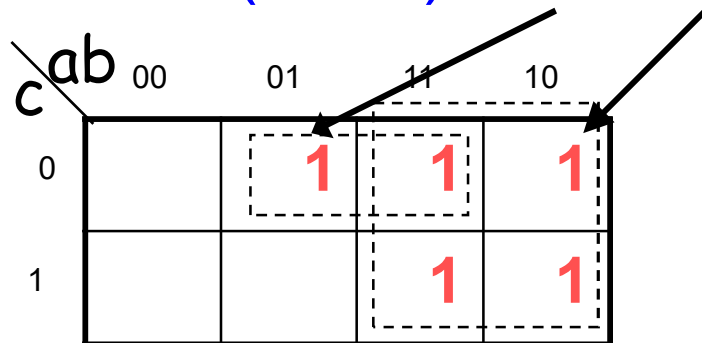
- Enter minterms of the Boolean function into the map, then group terms

- **Example:**



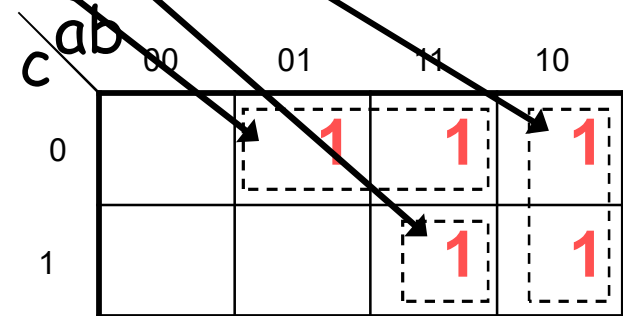
$$f(a,b,c) = bc' + abc + ab'$$

- Result: $f(a,b,c) = bc' + a$



Karnaugh map for $f(a,b,c)$ with variables c (vertical) and ab (horizontal). The map shows minterms 3, 5, 6, and 7. A dashed box groups the top-right two cells (minterms 3 and 5) for the term bc' . Another dashed box groups the bottom-right two cells (minterms 6 and 7) for the term $abc + ab'$.

$c \backslash ab$	00	01	11	10
0		1	1	1
1			1	1



Karnaugh map for $f(a,b,c)$ with variables c (vertical) and ab (horizontal). The map shows minterms 3, 5, 6, and 7. A dashed box groups the top-right two cells (minterms 3 and 5) for the term bc' . Another dashed box groups the bottom-right two cells (minterms 6 and 7) for the term $abc + ab'$.

$c \backslash ab$	00	01	11	10
0		1	1	1
1			1	1

4-Variable Map

AB \ CD		A			
		00	01	11	10
C	00	1	0	0	1
	01	0	1	0	0
	11	1	1	1	1
	10	1	1	1	1
		B			

$$F(A,B,C,D) = \sum m(0,2,3,5,6,7,8,10,11,14,15)$$

F = ?

AB \ CD		AB			
		00	01	11	10
CD	00	m_0	m_4	m_{12}	m_8
	01	m_1	m_5	m_{13}	m_9
	11	m_3	m_7	m_{15}	m_{11}
	10	m_2	m_6	m_{14}	m_{10}

Four-variable Map Simplification

- One square represents a minterm of 4 literals.
- A rectangle of 2 adjacent squares represents a product term of 3 literals.
- A rectangle of 4 squares represents a product term of 2 literals.
- A rectangle of 8 squares represents a product term of 1 literal.
- A rectangle of 16 squares produces a function that is equal to logic 1.

4-Variable Map

		A			
AB		00	01	11	10
CD	00	1	0	0	1
	01	0	1	0	0
	11	1	1	1	1
	10	1	1	1	1
C		B			

$$F(A,B,C,D) = \sum m(0,2,3,5,6,7,8,10,11,14,15)$$

$$F = C + A' B D + B' D'$$

- Find the **smallest number** of the **largest possible subcubes** that cover the ON-set (for SoP)

Simplify for POS

K-map Method: covering zeros to get product of sums form

		A			
AB		00	01	11	10
CD	00	1	0	0	1
	01	0	1	0	0
	11	1	1	1	1
	10	1	1	1	1

Labels: C is on the left, D is on the right, B is at the bottom.

$$F = (B' + C + D) (A' + C + D') (B + C + D')$$

Alternative approach:

Replace F by F', 0's become 1's and vice versa

$$F' = B C' D' + A C' D + B' C' D$$

$$(F')' = (B C' D' + A C' D + B' C' D)'$$

$$F = (B' + C + D) (A' + C + D') (B + C + D')$$

Don't Cares

Don't Cares can be treated as 1's or 0's, whichever is more advantageous

AB \ CD		A			
		00	01	11	10
C	00	0	0	X	0
	01	1	1	X	1
	11	1	1	0	0
	10	0	X	0	0

Diagram showing a 4x4 Karnaugh map for function F(A,B,C,D). The map is labeled with variables A, B, C, and D. The map shows the following values: (0,0,0,0)=0, (0,0,0,1)=0, (0,0,1,0)=X, (0,0,1,1)=0, (0,1,0,0)=1, (0,1,0,1)=1, (0,1,1,0)=X, (0,1,1,1)=1, (1,0,0,0)=1, (1,0,0,1)=1, (1,0,1,0)=0, (1,0,1,1)=0, (1,1,0,0)=0, (1,1,0,1)=X, (1,1,1,0)=0, (1,1,1,1)=0. The map is partitioned into groups: a group of four 1s (0,1,0,0), (0,1,0,1), (1,0,0,0), (1,0,0,1) is highlighted with an orange box; a group of four 1s (0,1,0,0), (0,1,0,1), (0,1,1,0), (0,1,1,1) is highlighted with a blue box; a group of four 1s (0,1,0,0), (0,1,0,1), (1,0,0,0), (1,0,0,1) is highlighted with a blue box; a group of four 1s (0,1,0,0), (0,1,0,1), (1,0,0,0), (1,0,0,1) is highlighted with a blue box.

$$F(A,B,C,D) = \sum m(1,3,5,7,9) + \sum d(6,12,13)$$

$$F = A'D + B' C' D \quad \text{w/o don't cares}$$

$$F = A'D + C' D \quad \text{w/ don't cares}$$

In Product of Sums form: $F = D (A' + C')$

**Same answer as above,
but fewer gates**

AB \ CD		A			
		00	01	11	10
C	00	0	0	X	0
	01	1	1	X	1
	11	1	1	0	0
	10	0	X	0	0

Diagram showing a 4x4 Karnaugh map for function F(A,B,C,D). The map is labeled with variables A, B, C, and D. The map shows the following values: (0,0,0,0)=0, (0,0,0,1)=0, (0,0,1,0)=X, (0,0,1,1)=0, (0,1,0,0)=1, (0,1,0,1)=1, (0,1,1,0)=X, (0,1,1,1)=1, (1,0,0,0)=1, (1,0,0,1)=1, (1,0,1,0)=0, (1,0,1,1)=0, (1,1,0,0)=0, (1,1,0,1)=X, (1,1,1,0)=0, (1,1,1,1)=0. The map is partitioned into groups: a group of four 1s (0,1,0,0), (0,1,0,1), (1,0,0,0), (1,0,0,1) is highlighted with an orange box; a group of four 1s (0,1,0,0), (0,1,0,1), (0,1,1,0), (0,1,1,1) is highlighted with a blue box; a group of four 1s (0,1,0,0), (0,1,0,1), (0,1,1,0), (0,1,1,1) is highlighted with a blue box; a group of four 1s (0,1,0,0), (0,1,0,1), (0,1,1,0), (0,1,1,1) is highlighted with a blue box.

Order Dependency

- Order is important

AB \ CD	00	01	11	10
00	0	0	1	0
01	1	1	1	0
11	0	1	1	1
10	0	1	0	0

Fact: we will get different results depending on the order of groupings.

Question: can we make the approach order-independent?

Answer: yes, as can be seen in the next slides.

Definition of Terms

- **Implicant:**
 - single element of the ON-set or any group of elements that can be combined together in a K-map (= adjacency plane)
- **Prime Implicant (PI) (maximal PI):**
 - implicant (a circled set of 1-cells) satisfying the combining rule, such that if we try to make it larger (covering twice as many cells), it covers one or more 0s.
- **Distinguished 1-cell:**
 - an input combination that is covered by only one PI.
- **Essential Prime Implicant (EPI):**
 - a PI that covers one or more distinguished 1-cells.

Implicant, PI and EPI

AB CD		A			
		00	01	11	10
C	00	0	1	1	0
	01	1	1	1	0
	11	1	0	1	1
	10	0	0	1	1
		B			

Diagram illustrating a 4-variable Karnaugh map for variables A, B, C, and D. The map shows the following values:

CD \ AB	00	01	11	10
00	0	1	1	0
01	1	1	1	0
11	1	0	1	1
10	0	0	1	1

The map is partitioned into groups by color-coded boxes:

- Green boxes (Essential Prime Implicants):** $A'B'D$ (covering cells (0,1), (1,1), (1,0)) and $A'C$ (covering cells (0,1), (1,1), (1,0)).
- Red boxes (Other Prime Implicants):** BC' (covering cells (0,1), (1,1)), AC (covering cells (0,1), (1,1), (1,0)), AB (covering cells (0,1), (1,1)), and $B'CD$ (covering cells (1,1), (1,0)).

6 Prime Implicants:

$A' B' D$, $B C'$, $A C$, $A' C' D$, $A B$, $B' C D$

Essential

Minimum cover =

First: cover **EPIs**

Then: minimum number of **PIs**

$$= B C' + A C + A' B' D$$

Implicant, PI and EPI

AB \ CD		A			
		00	01	11	10
C	00	0	0	1	0
	01	1	1	1	0
	11	0	1	1	1
	10	0	1	0	0

Diagram illustrating a 4x4 Karnaugh map for variables A, B, C, and D. The map shows the following values:

AB \ CD	00	01	11	10
00	0	0	1	0
01	1	1	1	0
11	0	1	1	1
10	0	1	0	0

Groupings are indicated by colored boxes:

- Red boxes highlight the Prime Implicants (PIs): BD , ABC' , ACD , $A'BC$, and $A'C'D$.
- Blue boxes highlight the Essential Prime Implicants (EPIs): BD , ACD , $A'BC$, and $A'C'D$.

5 Prime Implicants:

$BD, ABC', ACD, A'BC, A'C'D$

essential

Minimum cover =

First: cover **EPIs**

Then: minimum number of **PIs**

$$= ABC' + ACD + A'BC + A'C'D$$

More Examples

AB \ CD		A			
		00	01	11	10
C	00	0	0	0	0
	01	0	1	1	0
	11	1	1	1	1
	10	1	0	1	1
		B			

Diagram illustrating a 4x4 Karnaugh map for variables A, B, C, and D. The map shows the following values:

CD \ AB	00	01	11	10
00	0	0	0	0
01	0	1	1	0
11	1	1	1	1
10	1	0	1	1

Groupings are indicated by colored lines:

- Red lines:** Grouping (01, 11) for C=01 and (11, 10) for C=11, representing the prime implicant $B'D$.
- Orange lines:** Grouping (01, 11) for C=01 and (11, 10) for C=10, representing the prime implicant CD .
- Blue lines:** Grouping (11, 10) for C=01 and (11, 10) for C=10, representing the prime implicant AC .
- Yellow lines:** Grouping (11, 10) for C=11 and (11, 10) for C=10, representing the prime implicant $B'C$.

Prime Implicants:

$BD, CD, AC, B'C$

essential

$$= BD + AC + B'C$$

Example

What if there are don't cares in the k-map?

Example: $f(A,B,C,D) = m(4,5,6,8,9,10,13) + d(0,7,15)$

AB \ CD		A			
		00	01	11	10
C	00	X	1	0	1
	01	0	1	1	1
	11	0	X	X	0
	10	0	1	0	1

Initial K-map

AB \ CD		A			
		00	01	11	10
C	00	X	1	0	1
	01	0	1	1	1
	11	0	X	X	0
	10	0	1	0	1

Primes around $A' B C' D'$

AB \ CD		A			
		00	01	11	10
C	00	X	1	0	1
	01	0	1	1	1
	11	0	X	X	0
	10	0	1	0	1

Primes around $A B C' D$

Example: Continued

AB \ CD		A			
		00	01	11	10
C	00	X	1	0	1
	01	0	1	1	1
	11	0	X	X	0
	10	0	1	0	1

Diagram showing prime implicants (circles) around the 1s in the Karnaugh map. The prime implicants are labeled A, B, C, and D.

**Primes around
A B' C' D'**

AB \ CD		A			
		00	01	11	10
C	00	X	1	0	1
	01	0	1	1	1
	11	0	X	X	0
	10	0	1	0	1

Diagram showing essential prime implicants (circles) around the 1s in the Karnaugh map. The prime implicants are labeled A, B, C, and D.

**Essential Primes
with Min Cover**
(each element covered once)

5-Variable K-Map

Y Z		W X			
		00	01	11	10
Y	00	0	4	12	8
	01	1	5	13	9
	11	3	7	15	11
	10	2	6	14	10

V=0

Y Z		W X			
		00	01	11	10
Y	00	16	20	28	24
	01	17	21	29	25
	11	19	23	31	27
	10	18	22	30	26

V=1

5-Variable K-Map

The diagram illustrates a 5-variable Karnaugh map (K-Map) for variables A, B, C, and D. It is structured as two 4x4 grids, one for A=0 (top) and one for A=1 (bottom). The vertical axis for each grid is labeled DE (00, 01, 11, 10) and the horizontal axis is labeled BC (00, 01, 11, 10). The cells are numbered 0-15 for A=0 and 16-31 for A=1.

A	DE \ BC	00	01	11	10
0	00	0	4	12	8
0	01	1	5	13	9
0	11	3	7	15	11
0	10	2	6	14	10
1	00	16	20	28	24
1	01	17	21	29	25
1	11	19	23	31	27
1	10	18	22	30	26

5-Variable K-Map

BC

DE 00 01 11 10

A=0

00				1
01		1	1	
11		1	1	
10	1			1

BC

DE 00 01 11 10

A=1

00				1
01	1	1	1	
11	1	1	1	
10				

$$f(A,B,C,D,E) = \sum m(2,5,7,8,10,13,15,17,19,21,23,24,29,31)$$

BC

DE 00 01 11 10

A=0

00				1
01		1	1	
11		1	1	
10	1			1

BC

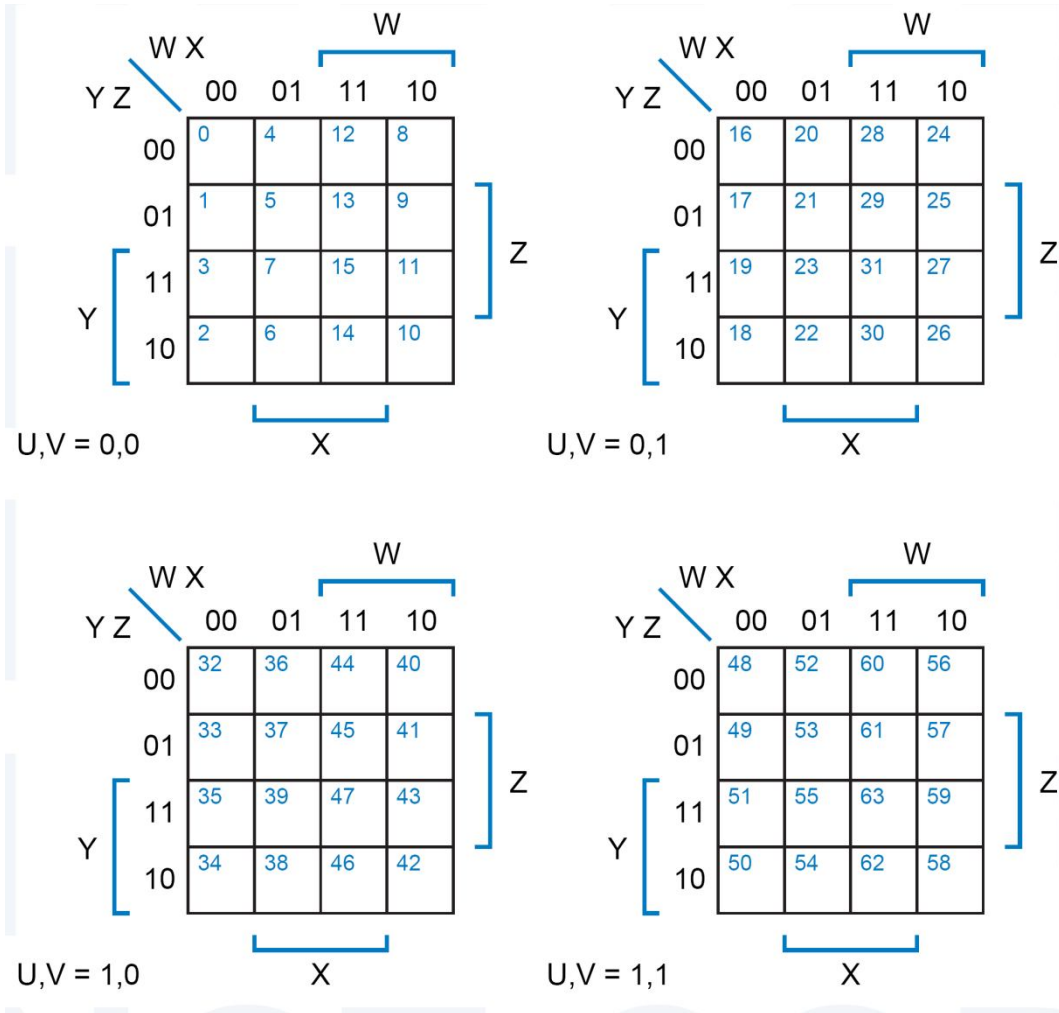
DE 00 01 11 10

A=1

00				1
01	1	1	1	
11	1	1	1	
10				

$$= CE + AB'E + BC'D'E' + A'C'DE'$$

6-Variable K-Map



7-Variable K-Map



8-Variable K-Map



Implementation by NAND gates only

- **NAND:**

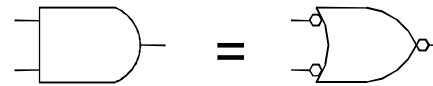
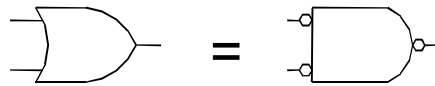
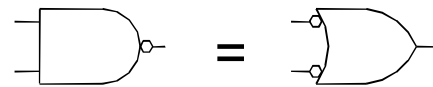
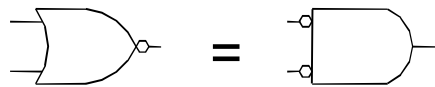
- Universal gate
- Can replace gates by equivalent NAND circuit.
 - Large circuit (many gates)

New Symbols for AND/OR

- **DeMorgan's Law:**

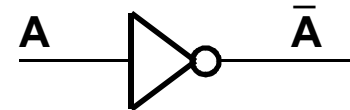
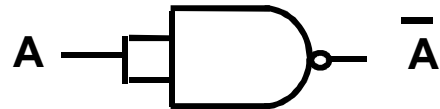
➤ $(a + b)' = a' b'$ $(a b)' = a' + b'$

➤ $a + b = (a' b')'$ $(a b) = (a' + b')'$



NAND-only and NOR-only implementations for NOT

➤ $(a \cdot a)' = a'$

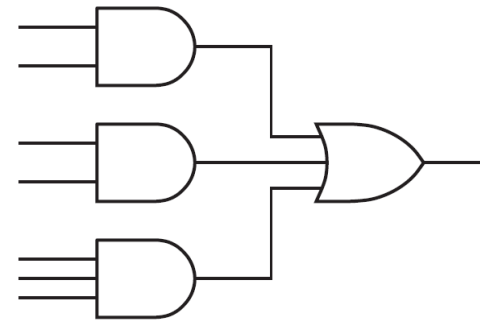


➤ $(a + a)' = a'$

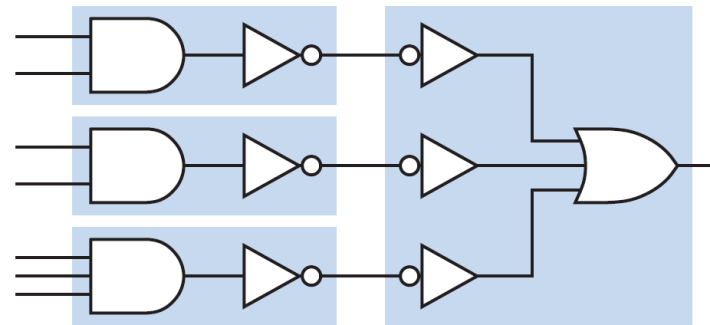


Finding NAND-only Implementation

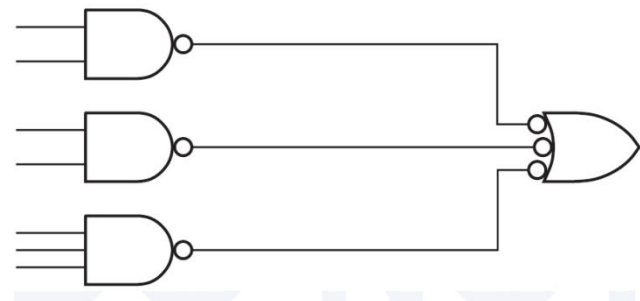
- 1st step: Find Sum-of-Product form



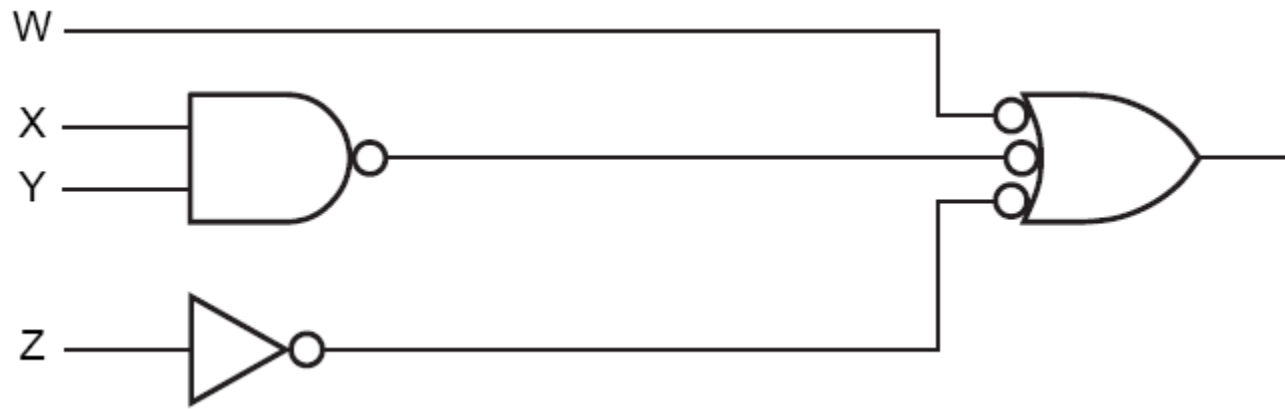
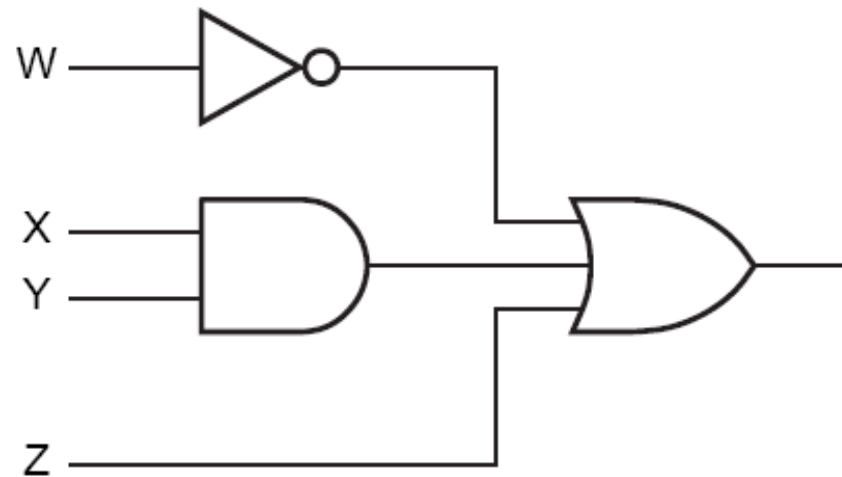
- 2nd step: Add double inverters



- 3rd step: Identify equivalent NANDs

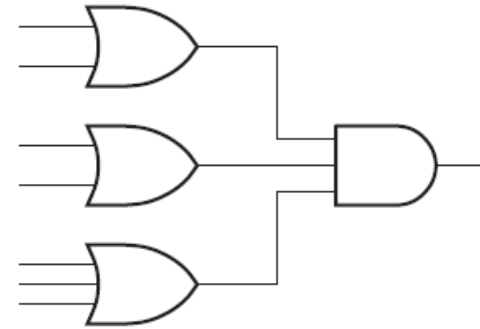


Another Example

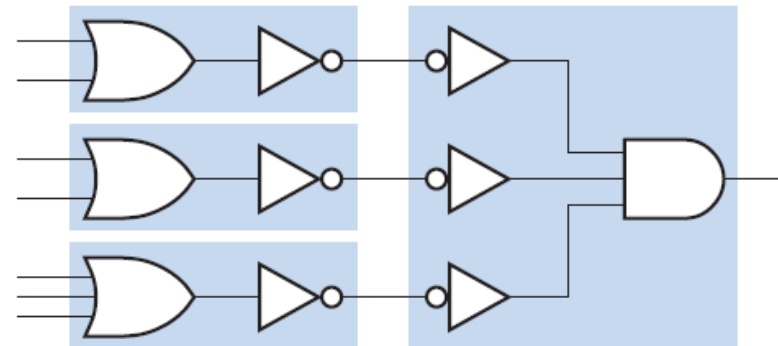


NOR-Only Implementation

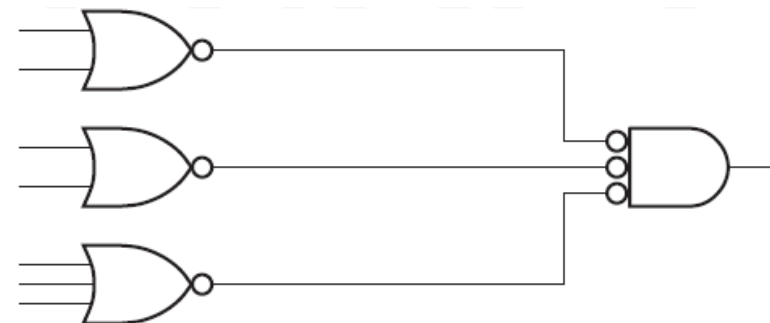
- 1st step: Find Product-of-Sums form



- 2nd step: Add double inverters



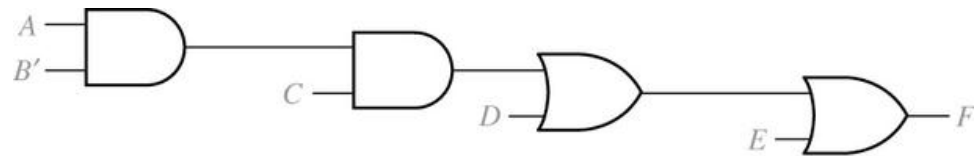
- 3rd step: Identify equivalent NORs



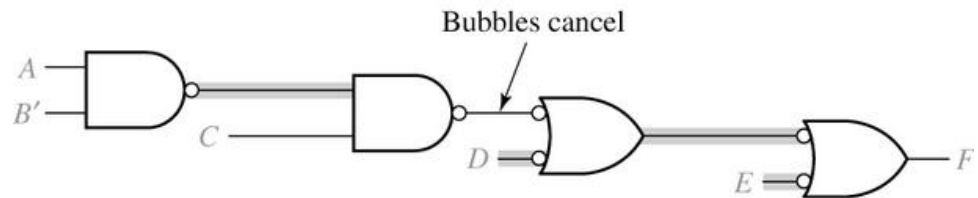
NAND-Only Implementation

• Multi-Level Circuits

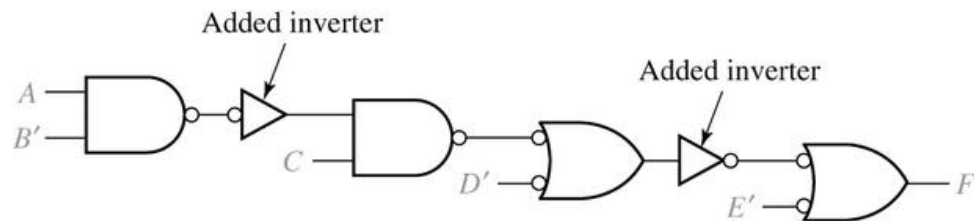
- Convert AND/OR gates to proper NAND gates
 - AND \rightarrow AND-NOT symbol
 - OR \rightarrow NOT-OR symbol
- Bubbles must cancel each other;
- otherwise, insert a NAND inverter.
- Take care of appropriate input literals.



(a) AND-OR network



(b) First step in NAND conversion

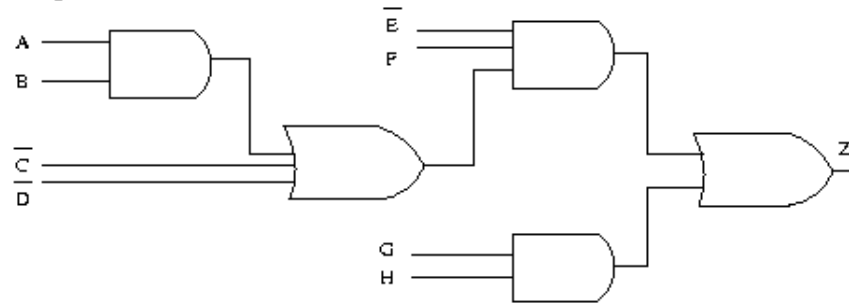


(c) Completed conversion

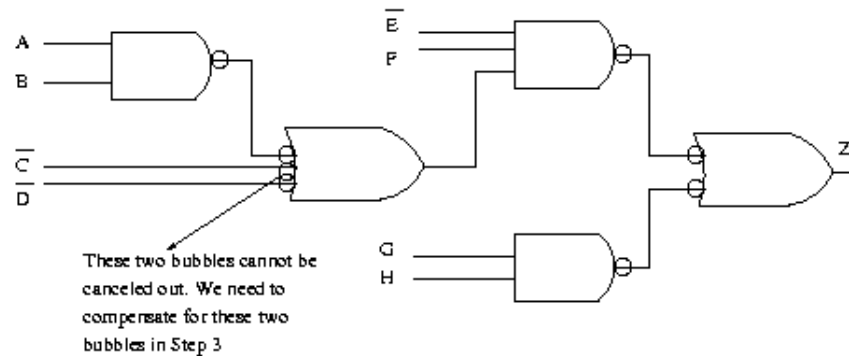
NAND-Only Implementation

- Example:

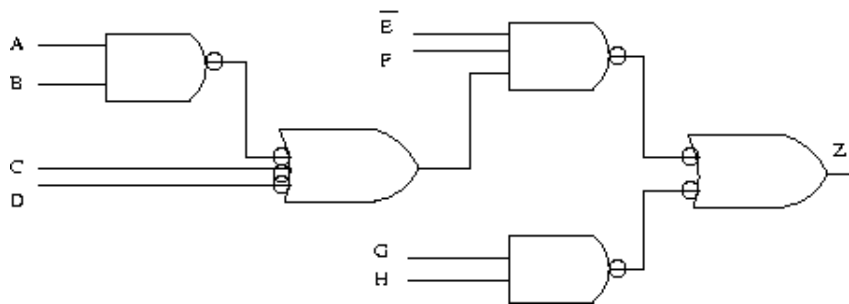
Step 1



Step 2

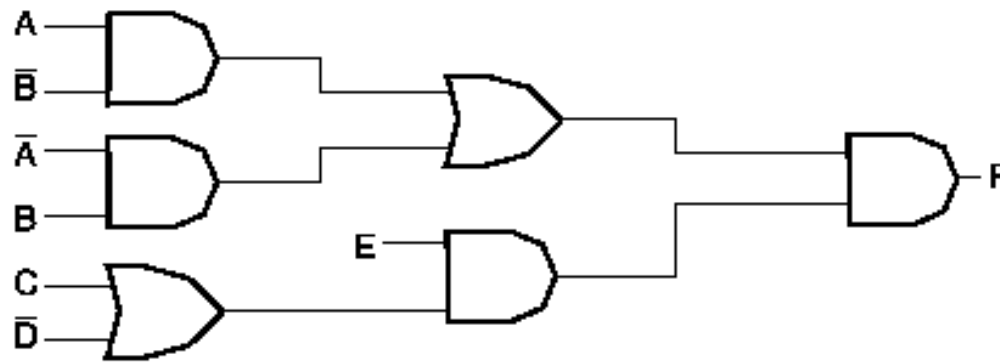


Step 3

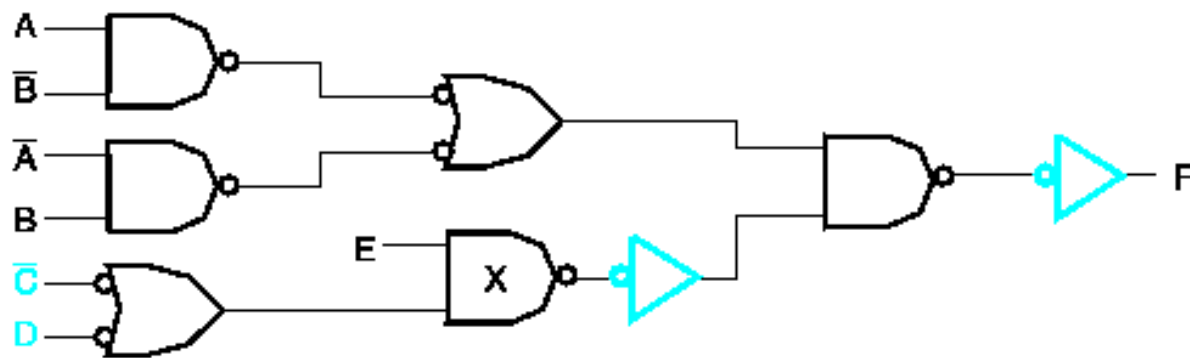


NAND-Only Implementation

- Another Example:



(a) AND – OR gates

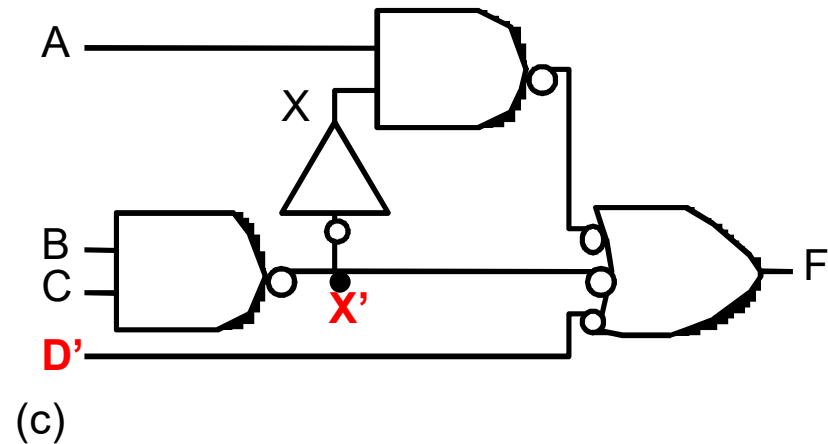
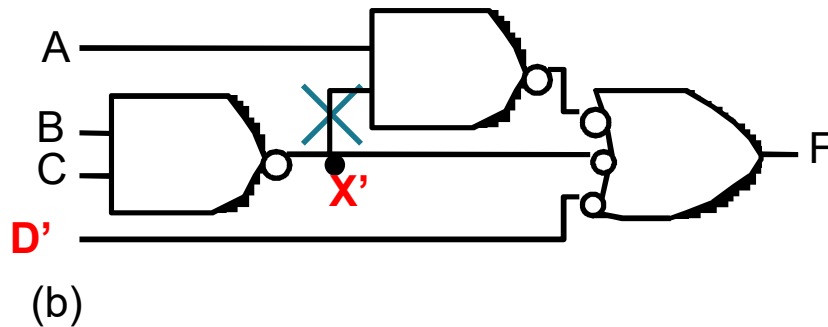
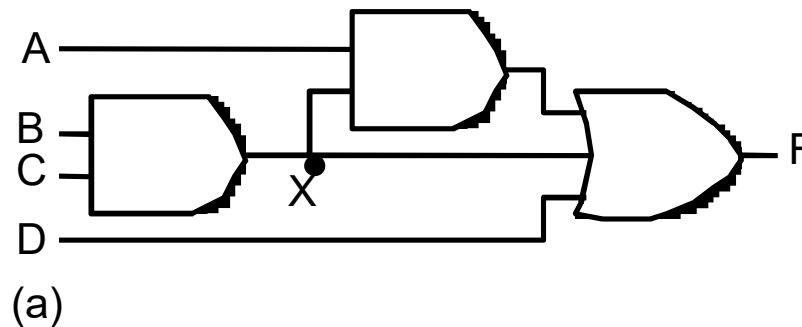


(b) NAND gates

Fig. 2-32 Implementing $F = (A\bar{B} + \bar{A}B)E(C + \bar{D})$

NAND-Only Implementation

- **Be careful about branches:**
 - Gates with multi-fanouts



NOR-Only Implementation

- **NOR-Only:**

- Use “Duality” for the last several slides.

