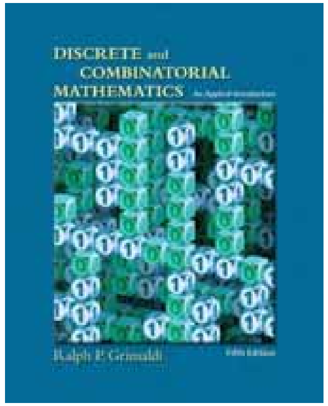
Discrete Mathematics

-- *Chapter 7: Relations: The Second Time Round*



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Outline

- Relations Revisited: Properties of Relations
- Computer Recognition: Zero-One Matrices and Directed Graphs
- **Partial Orders**: Hasse Diagrams
- **Equivalence Relations** and Partitions

7.1 Relations Revisited: Properties of Relations

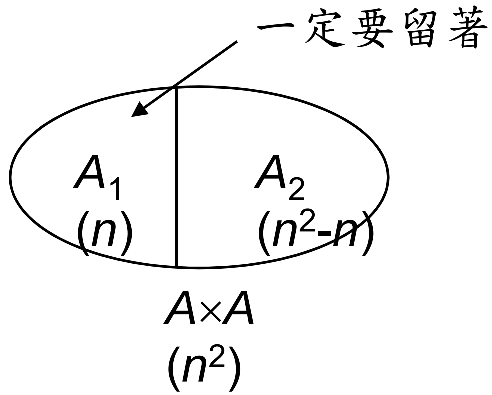
- Definition 7.1: For sets A, B, any subset of A × B is called a (binary) relation from A to B. Any subset of A × A is called a (binary) relation on A .

- **Ex 7.1**
  - Define the relation $\Re$ on the set Z by a$\Re$b, if a $\leq$ b.
  - For x, y$\in$Z and $n\in Z^+$, the modulo *n* relation $\Re$ is defined by x$\Re$y if x - y is a multiple of *n*, e.g., with n=7, 9$\Re$2, -3$\Re$11, but 3 $\not\Re$ 7

- **Ex 7.2** : Language A $\subseteq \Sigma^*$. For x, y$\in$A, define x$\Re$y if x is a prefix of y.

Reflexive

- Definition 7.2: A relation $\Re$ on a set A is called **reflexive** if $(x, x) \in \Re$, for **all $x \in A$**.
- **Ex 7.4** : For $A = \{1, 2, 3, 4\}$, a relation $\Re \subseteq A \times A$ will be reflexive if and only if $\Re \supseteq \{(1, 1), (2, 2), (3, 3), (4, 4)\}$. But $\Re_1 = \{(1, 1), (2, 2), (3, 3)\}$ is not reflexive, $\Re_2 = \{(x, y) | x \leq y, x, y \in A\}$ is reflexive.
- **Ex 7.5** : Given a finite set A with $|A| = n$, we have $|A \times A| = n^2$, so there are 2^{n^2} relations on A. Among them $2^{(n^2-n)}$ are reflexive.
  - $A = \{a_1, a_2, \ldots, a_n\}$
  - $A \times A = \{(a_i, a_j) | 1 \leq i, j \leq n\} = A_1 \cup A_2$
  - $A_1 = \{(a_i, a_i) | 1 \leq i \leq n\}$
  - $A_2 = \{(a_i, a_j) | i \neq j, 1 \leq i, j \leq n\}$



Symmetric

- Definition 7.3: A relation $\Re$ on a set A is called **symmetric** if for **all x, $y \in A$**, $(x, y) \in \Re \Rightarrow (y, x) \in \Re$.

- **Ex 7.6** : $A = \{1, 2, 3\}$
  - $\Re_1 = \{(1, 2), (2, 1), (1, 3), (3, 1)\}$, symmetric, but not reflexive.
  - $\Re_2 = \{(1, 1), (2, 2), (3, 3), (2, 3)\}$, reflexive, but not symmetric.
  - $\Re_3 = \{(1, 1), (2, 2), (3, 3)\}$ and $\Re_4 = \{(1, 1), (2, 2), (3, 3), (2, 3), (3, 2)\}$, both reflexive and symmetric.
  - $\Re_5 = \{(1, 1), (2, 3), (3, 3)\}$, neither reflexive nor symmetric.

Symmetric

- To count the symmetric relations on $A = \{a_1, a_2, \ldots, a_n\}$.
  - $A \times A = A_1 \cup A_2$, $A_1 = \{(a_i, a_i) | 1 \leq i \leq \mathrm{n}\}$, $A_2 = \{(a_i, a_j) | i \neq j, 1 \leq i, j \leq n\}$
  - A_1 contains n pairs, and A_2 contains n^2-n pairs.
  - A_2 contains $(n^2$-$n)/2$ subsets $S_{i,j}$ of the form $\{(a_i, a_j), (a_j, a_i) \mid i < j\}$.
  - So, we have totally $2^n \times 2^{(1/2)(n^2-n)}$ symmetric relations on A.

- If the relations are both reflexive and symmetric, we have $2^{(1/2)(n^2-n)}$ choices.

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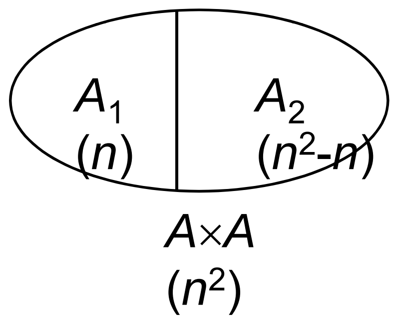


Transitive

- Definition 7.4: A relation $\Re$ on a set A is called **transitive if** $(x, y), (y, z) \in \Re \Rightarrow (x, z) \in \Re$ for all $x, y, z \in A$.

- **Ex 7.8** : Define the relation $\Re$ on the set Z^+ by $a \Re b$ if a divides b. This is a transitive and reflexive relation but not symmetric.
- **Ex 7.9** : Define the relation $\Re$ on the set Z by $a \Re b$ if $a \times b \geq 0$. What properties do they have?
  - Reflexive, symmetric
  - Not transitive, e.g., (3,0),(0,-7) $\in \Re$, but (3,-7) not

Antisymmetric

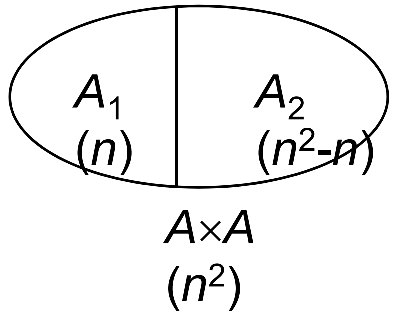
- Definition 7.5: A relation $\Re$ on a set A is called **antisymmetric** if $(x, y) \in \Re$ and $(y, x) \in \Re \Rightarrow x = y$ for all x, $y \in A$.
  - Both a related to b and b related to a if a and b are one and the same element from A

- **Ex 7.11** : Define the relation $(A, B) \in \Re$ if $A \subseteq B$. Then it is an anti-symmetric relation.
- Note that "*not symmetric*" is different from anti-symmetric.

- **Ex 7.12** : A = {1, 2, 3}, what properties do the following relations on A have?
  - $\Re=\{(1, 2), (2, 1), (2, 3)\}$ (not symmetric, not antisymmetric)
  - $\Re=\{(1, 1), (2, 2)\}$ (symmetric and antisymmetric)

Antisymmetric

- To count the antisymmetric relations on $A = \{a_1, a_2, \ldots, a_n\}$.
  - $A \times A = A_1 \cup A_2$, $A_1 = \{(a_i, a_i) | 1 \le i \le \text{n}\}$, $A_2 = \{(a_i, a_j) | i \ne j, 1 \le i, j \le n\}$
  - A_1 contains n pairs, and A_2 contains n^2-n pairs.
  - A_2 contains $(n^2$-$n)/2$ subsets $S_{i,j}$ of the form $\{(a_i, a_j), (a_j, a_i) \mid i < j\}$.
  - Each element in A_1 can be selected or not.
  - Each element in $S_{i,j}$ can be selected **in three alternatives**: ***either (a_i, a_j), or (a_j, a_i), or none*****.**
  - So, we have totally $2^n \times 3^{(1/2)(n^2-n)}$ anti-symmetric relations on A.



Antisymmetric

- **Ex 7.13** : Define the relation $\Re$ on the functions by $f \Re g$ if f is dominated by g (or $f \in O(g)$). What are their properties?
  - Reflexive
  - Transitive
  - not symmetric (e.g., g=n, f =n^{2,} g=O(f), but f $\neq$O(g))
  - not antisymmetric (e.g., g(n)= n, $f(n)$ = n+5, $f \Re g$ and $g \Re f$, *but* $f \neq g$)

Partial Order

- Definition 7.6: A relation $\Re$ is called a partial order (partial ordering relation), if $\Re$ is *reflexive, anti-symmetric and transitive*.

- (A,R) is a **p**artially **o**rdered **set** / poset if R is a partial ordering on A. Typical notation: (A,≤); think "no loops".
- If a≤b or b≤a, the elements a and b are comparable.
- If all pairs are comparable, ≤ is a total ordering or chain.

Partial Order

- **Ex 7.15** : Let A be the set of positive integers divisors of n, the relation $\Re$ on A by $a\Re b$ if a divides b, it defines a *partial order*. How many ordered pairs does it occur in $\Re$.
  - E.g. $A = \{1, 2, 3, 4, 6, 12\}$, $\Re = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 6), (1, 12), (2, 2), (2, 4), (2, 6), (2, 12), (3, 3), (3, 6), (3, 12), (4, 4), (4, 12), (6, 6), (6, 12), (12, 12)\}$
  - If $(a, b) \in \Re$, then $a = 2^m \cdot 3^n$ and $b = 2^p \cdot 3^q$ with $\mathbf{0 \le m \le p \le 2, 0 \le n \le q \le 1}$.
  - Selection of size 2 from a set of size 3, with **repetition**.

$$\binom{3+2-1}{2} = \binom{4}{2} = 6 \text{ for } m, p; \ \binom{2+2-1}{2} = \binom{3}{2} = 3 \text{ for } n, q$$

$$\therefore \text{total} = 6 \cdot 3 = 18 \text{ ordered pairs}$$

  - For

$$n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k} \Rightarrow \text{the number of ordered pairs} = \prod_{i=1}^{k} \binom{(e_i+1)+2-1}{2} = \prod_{i=1}^{k} \binom{e_i+2}{2}$$

Maximal element

Equivalence relation

- Definition 7.7. A relation $\Re$ is called an **equivalence relation**, if $\Re$ is *reflexive, symmetric and transitive*.

- Given an equivalence relation R on A, for each $a \in A$ the equivalence class [a] is defined by $\{x \mid (x,a) \in R\}$.

- **Ex 7.16** (b): If $A = \{1, 2, 3\}$, the following are all equivalence relations
  - $\Re_1 = \{(1, 1), (2, 2), (3, 3)\}$
  - $\Re_2 = \{(1, 1), (2, 2), (3, 3), (2, 3), (3,2)\}$
  - $\Re_3 = \{(1, 1), (1, 3), (2, 2), (3, 1), (3, 3)\}$
  - $\Re_4 = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$

Examples

- **Ex 7.16** (c): For a finite set A, $\boldsymbol{A}\times\boldsymbol{A}$ is the largest equivalence relation on A. If $A = \{a_1, a_2, \ldots, a_n\}$, then the equality relation $\Re = \{(a_i, a_i) | 1 \le i \le n\}$ is the smallest equivalence relation on A.

- **Ex 7.16** (d): Let $A = \{1, 2, 3, 4, 5, 6, 7\}$, $B = \{x, y, z\}$, and $f : A \to B$ be the onto function. $f = \{(1, x), (2, z), (3, x), (4, y), (5, z), (6, y), (7, x)\}$.
  Define the relation $\Re$ on A by $a\Re b$ if $f(a) = f(b)$.
  $\Re$ is reflexive, symmetric, and transitive, so it is an equivalence relation.
  (e.g., f(a)=f(b), f(b)=f(c)=> f(a)=f(c))

- **Ex 7.16** (e): If $\Re$ is a relation on A, then $\Re$ is both an equivalence relation and a partial order relation iff $\Re$ is the equality relation on A.
  - equality relation $\{(a_i, a_i) | a_i \in \mathrm{A}\}$

7.2 Computer Recognition: Zero-One Matrices and Directed Graphs

- Definition 7.8: Let relations $\Re_1 \subseteq A\times B$ and $\Re_2 \subseteq B\times C$. The composite relation $\Re_1 \circ \Re_2$ is a relation defined by $\Re_1 \circ \Re_2 = \{(x, z) \mid \exists y \in B$ such that $(x, y) \in \Re_1$ and $(y, z) \in \Re_2\}$.

  (Note the different ordering with function composition.)
  $f : A \rightarrow B, \quad g : B \rightarrow C, \quad g_\circ f : A \rightarrow C$

- **Ex 7.17** : Consider $\Re_1 = \{(1, x), (2, x), (3, y), (3, z)\}$ and $\Re_2 = \{(w, 5), (x, 6)\}$, and $\Re_3 = \{(w, 5), (w, 6)\}$. $\Re_1 \circ \Re_2 = \{(1, 6), (2, 6)\}$, and $\Re_1 \circ \Re_3 = ?$ $\varnothing$
- **Ex 7.18** : Let A be the set of employees {L. Alldredge,…} at a computer center, while B denotes a set of programming language {C++, Java,…}, and C is a set of projects $\{p_1, p_2, \ldots\}$, consider $\Re_1 \subseteq A\times B$, $\Re_2 \subseteq B\times C$. What is the means of $\Re_1 \circ \Re_2$?

Composite Relation

- Theorem 7.1: $\Re_1 \subseteq A\times B$, $\Re_2 \subseteq B\times C$, and $\Re_3 \subseteq C\times D \Rightarrow \Re_1\circ(\Re_2\circ\Re_3) = (\Re_1\circ\Re_2)\circ\Re_3$
- Definition 7.9. We define the powers of relation $\Re$ by (a) $\Re^1=\Re$; (b) $\Re^{n+1}=\Re\circ\Re^n$.

- **Ex 7.19** : If $\Re = \{(1, 2), (1, 3), (2, 4), (3, 2)\}$, then $\Re^2 = \{(1, 4), (1, 2), (3, 4)\}$, $\Re^3 = ?$ and $\Re^4 = ?$

$\Re^3 = \{(1,4)\}$
and for $n \geq 4$, $\Re^n=\varnothing$

Relation Matrix

- Definition 7.10: An $m \times n$ zero-one matrix $E = (e_{ij})_{m \times n}$ is a rectangular array of numbers arranged in m rows and n columns, where each e_{ij} denotes the entry in the ith row and jth column of E, and each such entry is 0 or 1.
- **Relation matrix**: A relation can be represented by an $m \times n$ zero-one matrix.
- **Ex 7.21** : Consider $\Re_1 = \{(1, x), (2, x), (3, y), (3, z)\}$, $\Re_2 = \{(w, 5), (x, 6)\}$, and $\Re_1 \circ \Re_2$ to be represented by relation matrices?

$$M(\mathscr{R}_1) = \begin{array}{c} \\ (1) \\ (2) \\ (3) \\ (4) \end{array} \begin{array}{c} \begin{array}{cccc} (w) & (x) & (y) & (z) \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}, \qquad M(\mathscr{R}_2) = \begin{array}{c} \\ (w) \\ (x) \\ (y) \\ (z) \end{array} \begin{array}{c} \begin{array}{ccc} (5) & (6) & (7) \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$M(\mathscr{R}_1) \cdot M(\mathscr{R}_2) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{array}{c} \\ (1) \\ (2) \\ (3) \\ (4) \end{array} \begin{array}{c} \begin{array}{ccc} (5) & (6) & (7) \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} = M(\mathscr{R}_1 \circ \mathscr{R}_2).$$

Boolean addition' with 1+1=1

Relation Matrix

- **Ex 7.22**: If $\Re=\{(1, 2), (1, 3), (2, 4), (3, 2)\}$, then what are the relation matrices of $\Re^2$, $\Re^3$ and $\Re^4$?

$$M(\mathscr{R}) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(M(\mathscr{R}))^4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(M(\mathscr{R}))^2 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Relation Matrix

- Let A be a set with $|A| = n$ and $\Re$ be a relation on A. If $M(\Re)$ is the relation matrix for $\Re$, then
  - $M(\Re) = \mathbf{0}$ if and only if $\Re = \phi$.
  - $M(\Re) = \mathbf{1}$ if and only if $\Re = A \times A$.
  - $M(\Re^m) = [M(\Re)]^m$
- Definition 7.11: Let $E = (e_{ij})_{m\times n}$, $F = (f_{ij})_{m\times n}$ be two mxn ero-one matrices. We say that E precedes, or is less than, F, written as $E \le F$, if $e_{ij} \le f_{ij}$ for all i, j.

- **Ex 7.23** : $E \le F$. How many zero-one matrices G do have the results of $E \le G$?

$$E = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad F = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$2^3=8$

Relation Matrix

- Definition 7.12: $I_n = (\delta_{ij})_{n\times n}$ is the $n\times n$ zero-one matrix, where

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases}$$

- Definition 7.13: $A = (a_{ij})_{m\times n}$ is a zero-one matrix, the transpose of A, written A^{tr}, is the matrix $(a^*_{ji})_{n\times m}$ where $a^*_{ji} = a_{ij}$

- **Ex 7.24** :

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \qquad A^{\text{tr}} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

- **Theorem 7.2**: If M denote the relation matrix for $\Re$ on A, then
  - (A) $\Re$ is reflexive if and only if $I_n \leq M$.
  - (B) $\Re$ is symmetric if and only if $M = M^{\text{tr}}$.
  - (C) $\Re$ is transitive if and only if $M^2 \leq M$.
  - (D) $\Re$ is anti-symmetric if and only if $M \cap M^{\text{tr}} \leq I_n$.

Directed Graph

- Definition 7.14. A <u>directed graph</u> can be denoted as $G = (V, E)$, where V is the <u>vertex set</u> and E is the <u>edge set</u>.
  - (a, b): if $a, b \in V$ $(a, b) \in E$, then there is a edge from a to b. Vertex a is called <u>source</u> (origin) of the edge, and b is <u>terminating vertex</u>.
  - (a, a): is called a <u>loop</u>.
- $V = \{1, 2, 3, 4, 5\}$, $E = \{(1, 1), (1, 2), (1, 4), (3, 2)\}$
  - Isolated vertex: vertex 5 in Fig. 7.1.
- Single undirected edge $\{a, b\} = \{b, a\}$ in Fig. 7.2 (b) is used to represent the two directed edges shown in Fig. 7.2 (a).

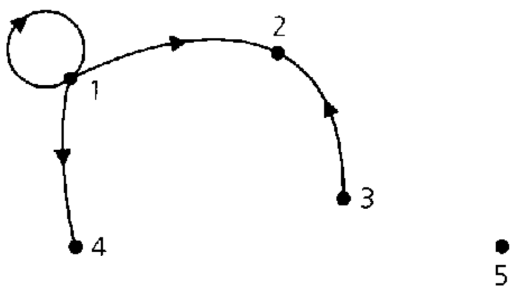


Figure 7.1

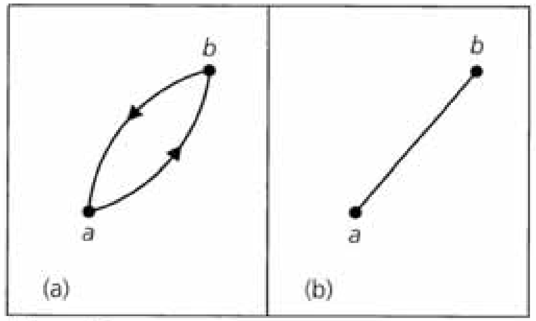


Figure 7.2

Directed Graph

- **Ex 7.26** precedence graph

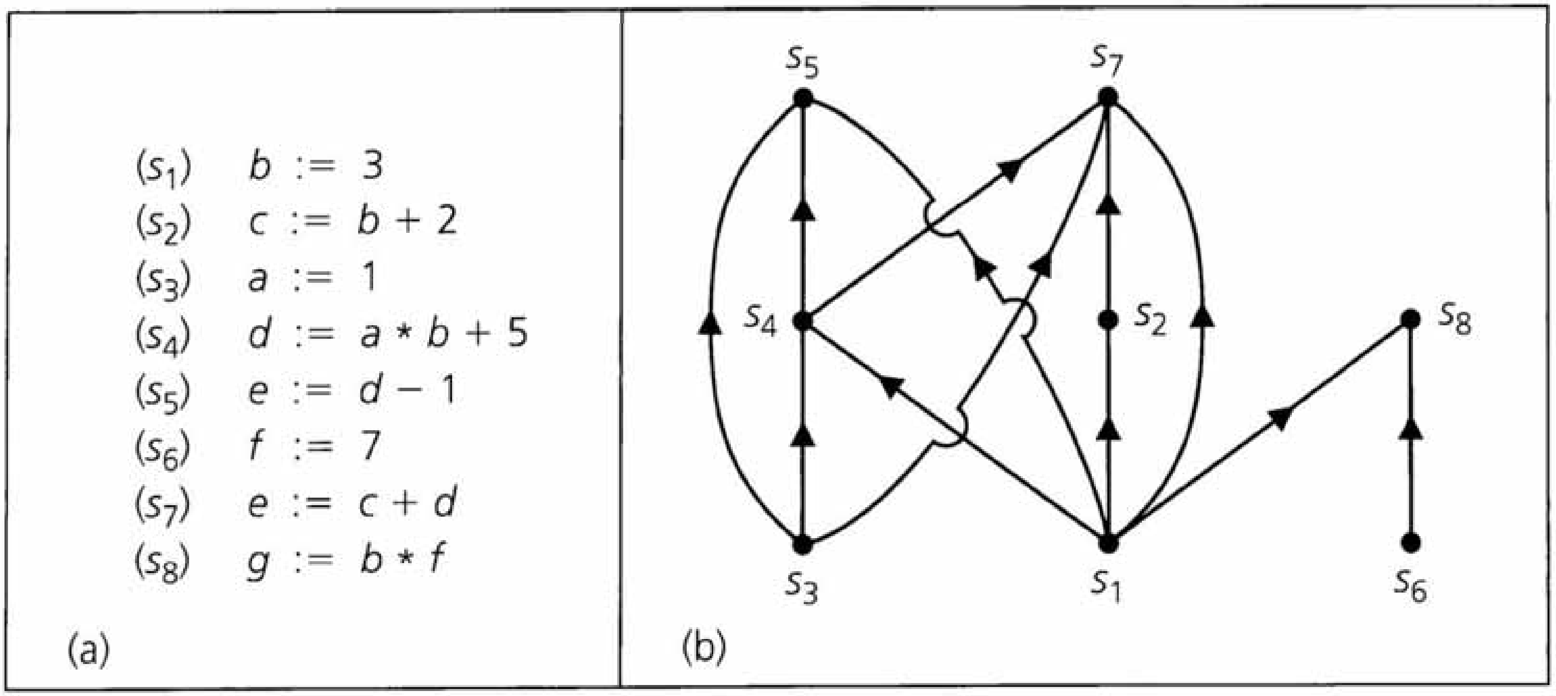


Figure 7.3

Directed Graph

- **Ex 7.27** : R = {(1,1),(1,2),(2,3),(3,2),(3,3),(3,4),(4,2)}
  - directed graph in Fig. 7.4 (a)
  - (associated) undirected graph in Fig. 7.4 (b)
  - path: In the connected graph, any two vertices x, y, with $x \neq y$, there is a path starting at x and ending at y.
  - cycle: a closed path starts and terminates at the same vertex, containing at least three edges.
    - E.g.: {3, 4}, {4, 2}, and {2, 3}

No repeated vertex

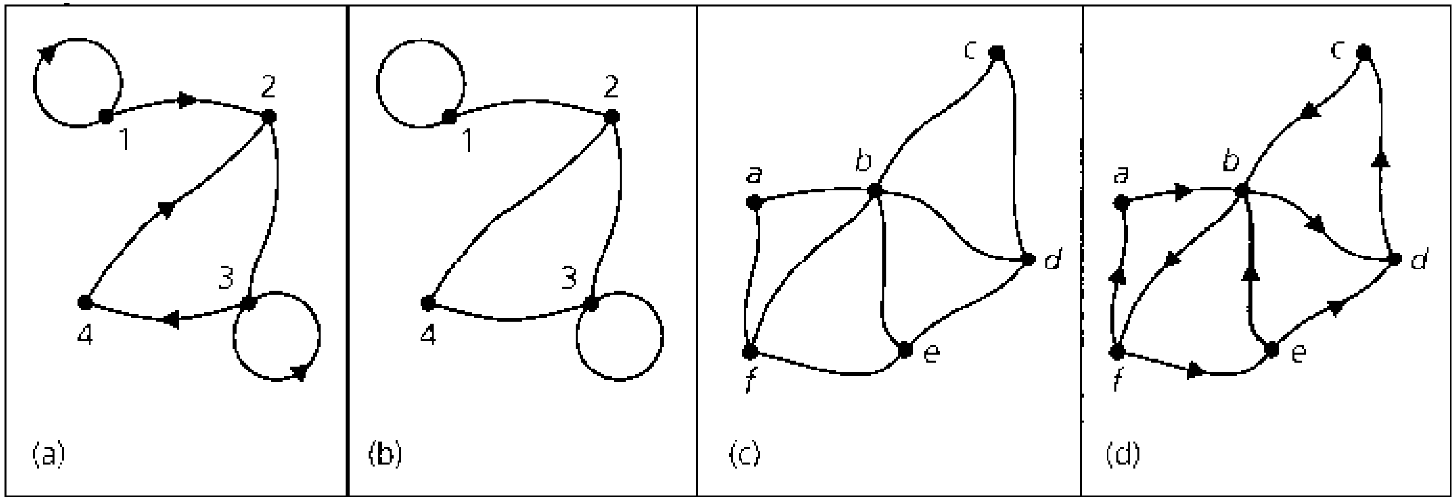


Figure 7.4

Directed Graph

- Definition 7.15: A directed graph G on V is called **strongly connected**, if for all $x, y \in V$, where $x \neq y$, there is a path (in G) of directed edges from x to y.
  - e.g., Fig. 7.5
- **Disconnected graph**: is the union of two connected pieces called the components of the graph.
  - e.g., Fig. 7.6

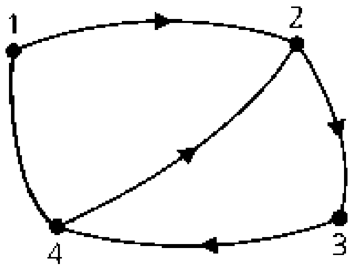


Figure 7.5

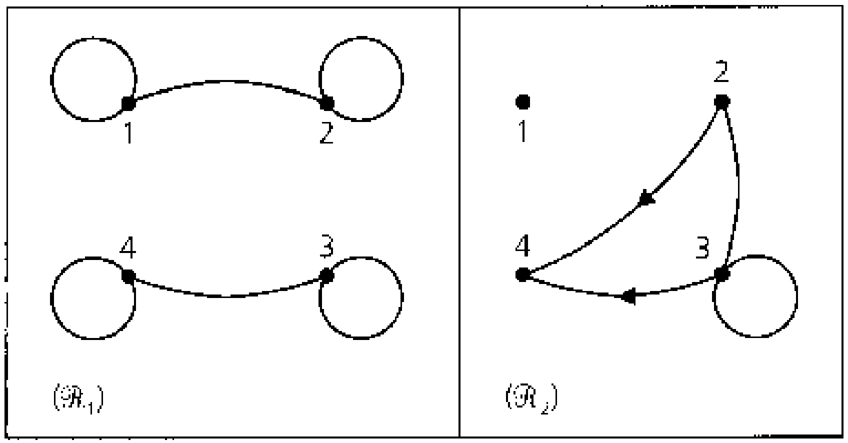


Figure 7.6

Directed Graph

- **Complete graph**: the graphs of undirected graphs that are loop-free and have an edge for every pair of distinct vertices, which are denoted by K_n.
  - e.g., Fig. 7.7

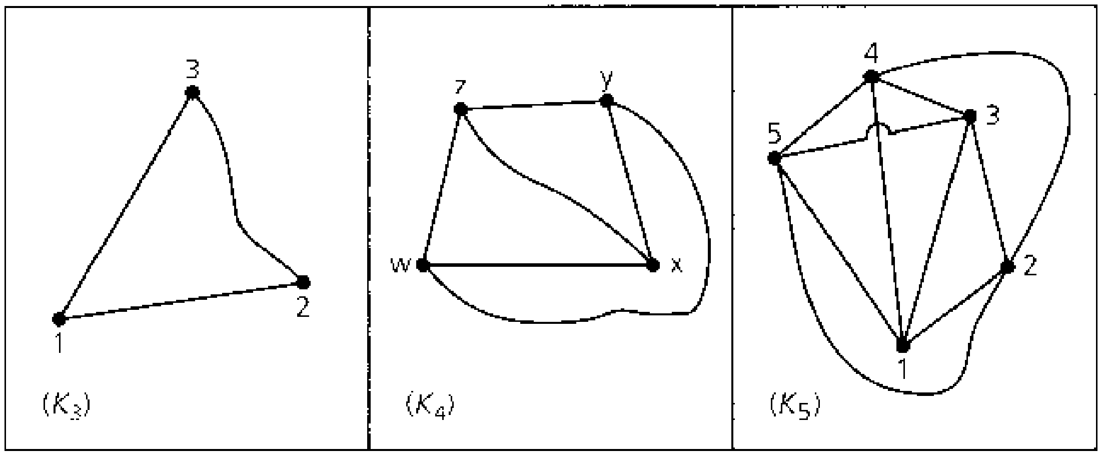
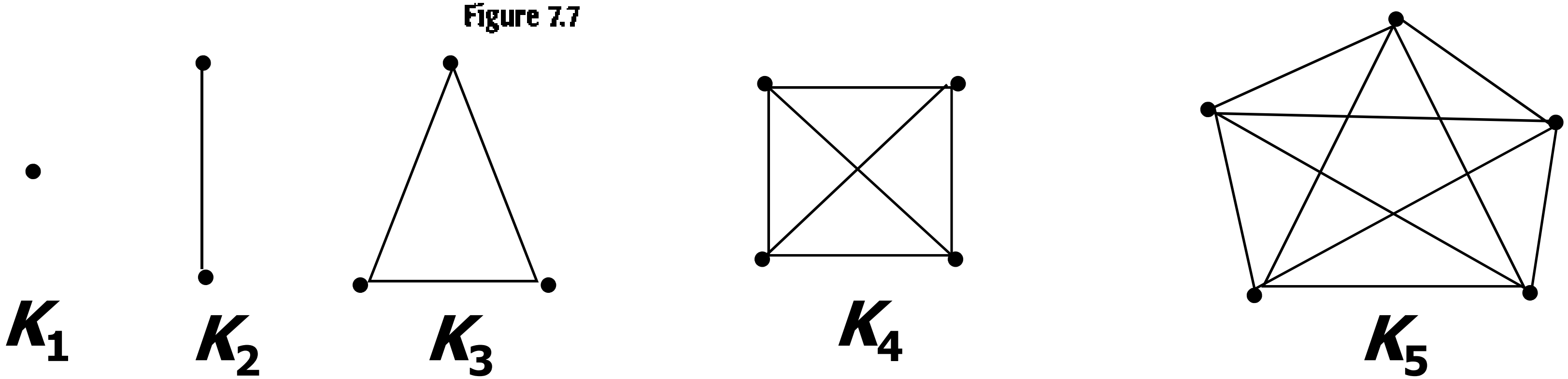


Figure 7.7



Directed Graph

- **Ex 7.30** : $\Re$ is reflexive if and only if its directed graph contains a loop at each vertex.
  - e.g., Fig 7.8, $A = \{1, 2, 3\}$ and $\Re = \{(1,1), (1, 2), (2, 2), (3, 3), (3, 1)\}$
- **Ex 7.31** : $\Re$ is symmetric if and only if its directed graph may be drawn only by loops and undirected edges.
  - e.g., Fig 7.9, $A = \{1, 2, 3\}$ and $\Re = \{(1,1), (1, 2), (2, 1), (2, 3), (3, 2)\}$
- **Ex 7.32** : $\Re$ is anti-symmetric if and only if for any $x \neq y$ the graph contains at most one of the edges (x, y) or (y, x)
  - e.g., Fig 7.10, $A = \{1, 2, 3\}$ and $\Re = \{(1,1), (1, 2), (2, 3), (1, 3)\}$

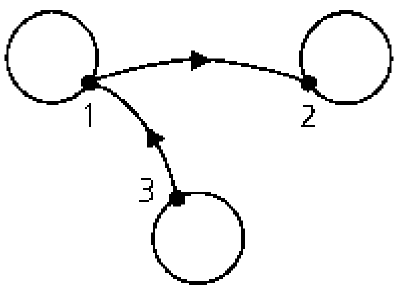


Figure 7.8

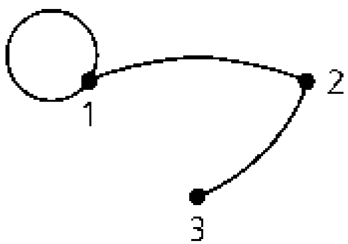


Figure 7.9

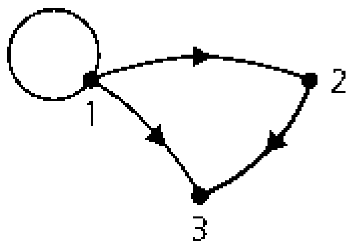


Figure 7.10

Directed Graph

- **Ex 7.32** : $\Re$ is transitive if and only if for all $x, y \in A$, if there is a **path** from x to y in the associated graph, then there is an **edge** (x, y) also.
  - e.g., Fig 7.10, $A = \{1, 2, 3\}$ and $\Re = \{(1, 1), (1, 2), (2, 3), (1, 3)\}$
- **Ex 7.33** : Fig 7.11, a relation is an equivalence relation if and only if its graph is one complete graph augmented by loops at every vertex or consists of disjoint union of complete graphs augmented by loops at each vertex.
  - e.g., Fig 7.11, $A = \{1, 2, 3, 4, 5\}$ and $\Re_1 = \{(1,1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (4, 3), (4, 4), (5, 5)\}$, $\Re_2 = \{(1,1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3), (4, 4), (4, 5), (5, 4) (5, 5)\}$.

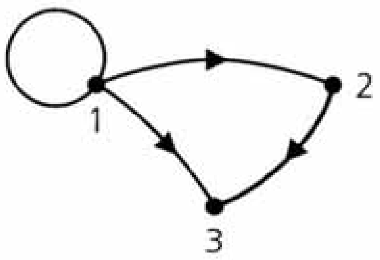


Figure 7.10

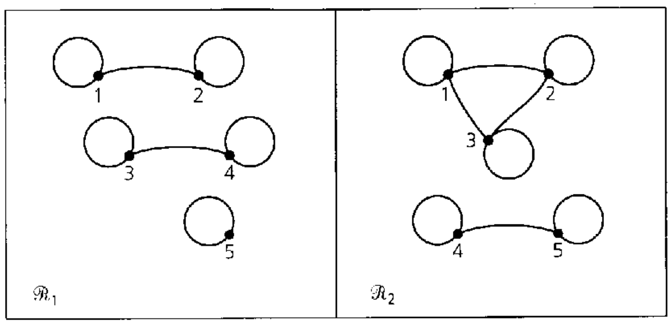


Figure 7.11

Directed Graph

reflexive: loop on each vertex
symmetric: undirected edge + loops
transitive: two paths
equivalence: disjoint union of complete graphs + loops at every vertex

directed graphs ⟷ relations

⇓ ⇓

adjacency matrices ⟷ relation matrices

7.3 Partial Orders: Hasse Diagrams

- Definition: Let A be a set with $\Re$ a relation on A. The pair (A, $\Re$) is called a **partially ordered set**, or poset, if relation $\Re$ on A is partially ordered.
  - If A is called a poset, we understand that there is a partially order $\Re$ on A that makes A into this set.

natural counting: **N**

$x+5=2$: **Z**

$2x+3=4$: **Q**

$x^2-2=0$: **R**

$x^2+1=0$: **C**

Something was lost when we went from **R** to **C**. We have lost **the ability to "order"** the elements in **C**.

2+i < 1+2i ?

7.3 Partial Orders: Hasse Diagrams

- **Ex 7.34** : Let A be the set of courses offered at a college. Define the relation $\Re$ on A by $x\Re y$ if x, y are the same course or if x is a prerequisite for y.

- **Ex 7.35** : Define $\Re$ on $A = \{1, 2, 3, 4\}$ by $x\Re y$ if x divide y. Then $\Re = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (1, 3), (1, 4), (2, 4)\}$ is a partial order, and $(A, \Re)$ is a poset.

- **Ex 7.36** : PERT (Program Evaluation and Review Technique) network is first used by U.S. Navy in 1950.
  - E.g., Let A be the set of tasks that must be performed to build a house. Define the relation $\Re$ on A by $x\Re y$ if x, y are the same task or if x must be performed before y.

Partial Orders: Hasse Diagrams

- **Ex 7.37** : Figure 7.17 (b) illustrates a simpler diagram for (a), called the **Hasse diagram**. The directions are assumed to go from the bottom to the top.

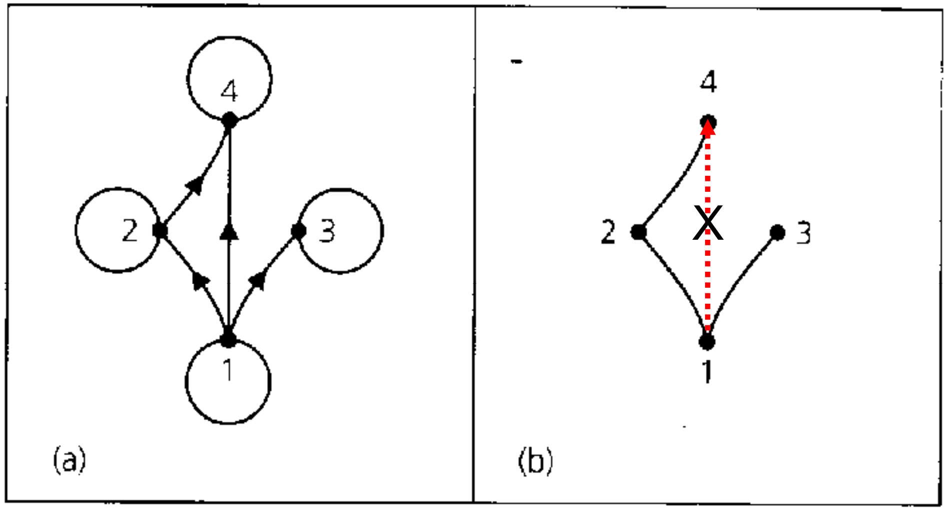
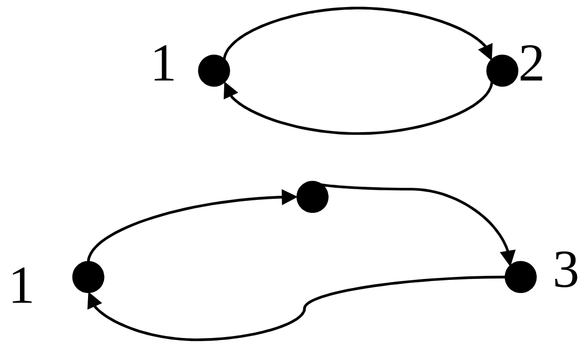


Figure 7.17

not partial order



Hasse Diagram

- If $(A, \Re)$ is a poset, we construct a Hasse diagram for $\Re$ on A by drawing a line segment from x up to y, if
  - $x \Re y$
  - there is no other z such that $x \Re z$ and $z \Re y$. *(in between x and y)*
- **Ex 7.38** : In Fig. 7.18 we have the Hasse diagrams for the following four posets.
  - (a) $\Re$ is the subset relation on A is the power set of $\mathcal{U}$ with $\mathcal{U} = \{1, 2, 3\}$
  - (b), (c), and (d) are the divide relations.

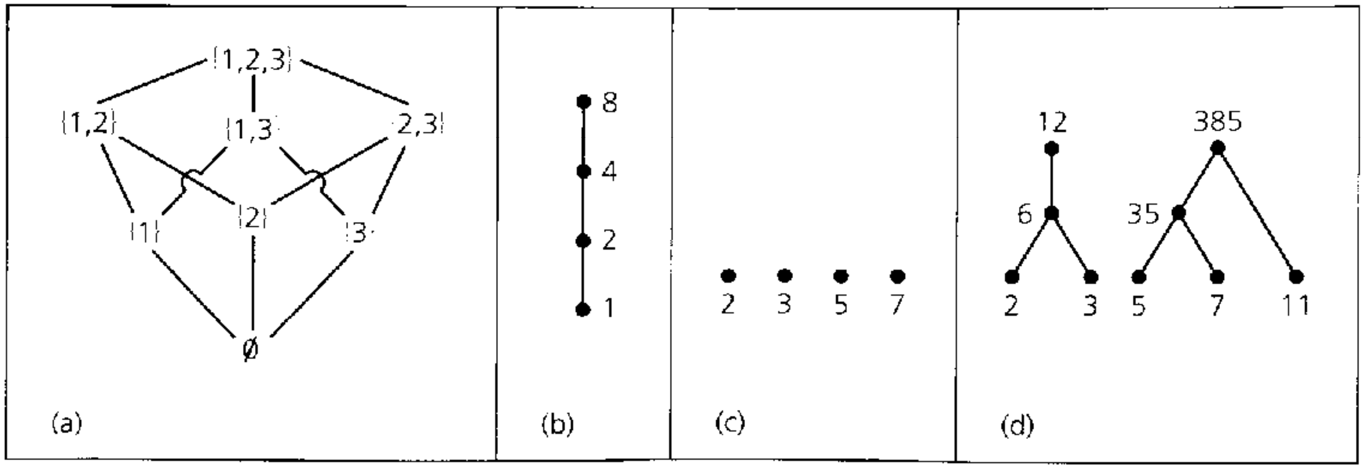


Figure 7.18

Totally Ordered

- Definition 7.16. If $(A, \Re)$ is a poset, we say that A is **totally ordered** (linearly ordered) if for all $x, y \in A$ either $x \Re y$ or $y \Re x$. In this case, $\Re$ is called a total order.
- **Ex 7.40**
  a) On the set **N**, the relation $\Re$ defined by $x \Re y$ if $x \le y$ is a total order.
  b) The subset relation is a partial order but not total order, e.g., $\{1, 2\}, \{1, 3\} \in A$, but $\{1, 2\} \not\subset \{1, 3\}$ or $\{1, 3\} \not\subset \{1, 2\}$.
  c) Fig 7.19 is a total order.

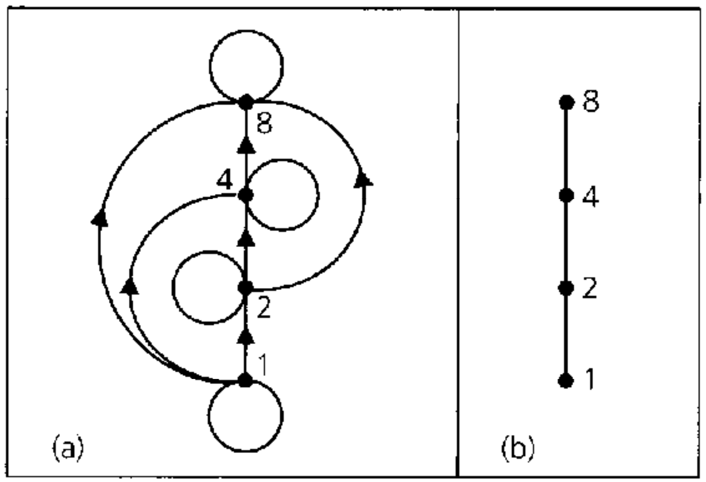


Figure 7.19

Topological Sorting

- Given a Hasse diagram for a partial order relation $\Re$, how to find a total order $\Im$ for which $\Re \subseteq \Im$.

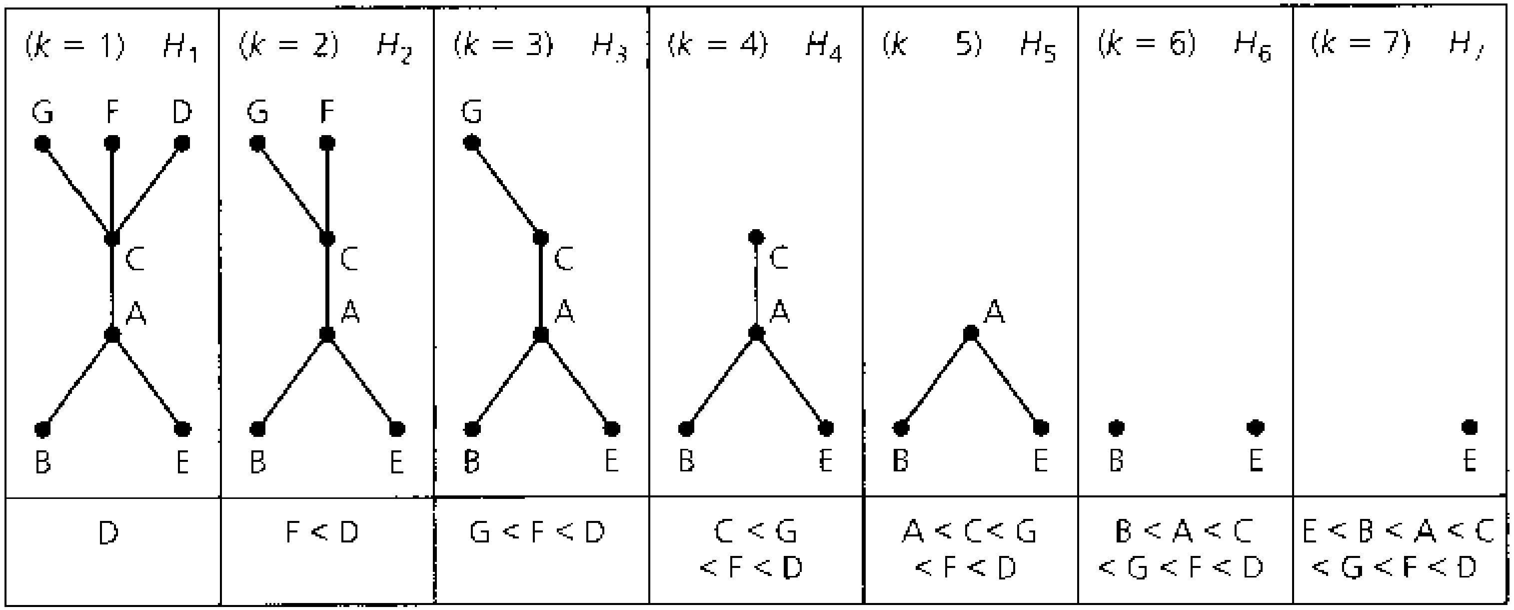


Figure 7.21

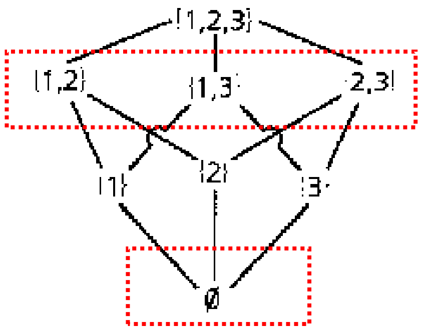
Not unique, 12 answers

Topological Sorting

- For a partial order $\Re$ on a set A with $|A| = n$
  - Step 1: Set $k = 1$. Let H_1 be the Hasse diagram of the partial order.
  - Step 2: Select a vertex v_k in H_k such that no edge in H_k starts at v_k.
  - Step 3: If $k = n$, the process is completed and we have a total order
    $$\Im : v_n < v_{n-1} < \cdots < v_1$$
    that contains $\Re$.
  - If $k < n$, then remove from H_k the vertex v_k and all edges of H_k that terminate at v_k. Call the result H_{k+1}. Increase k by 1 and return to step (2).

Maximal and Minimal

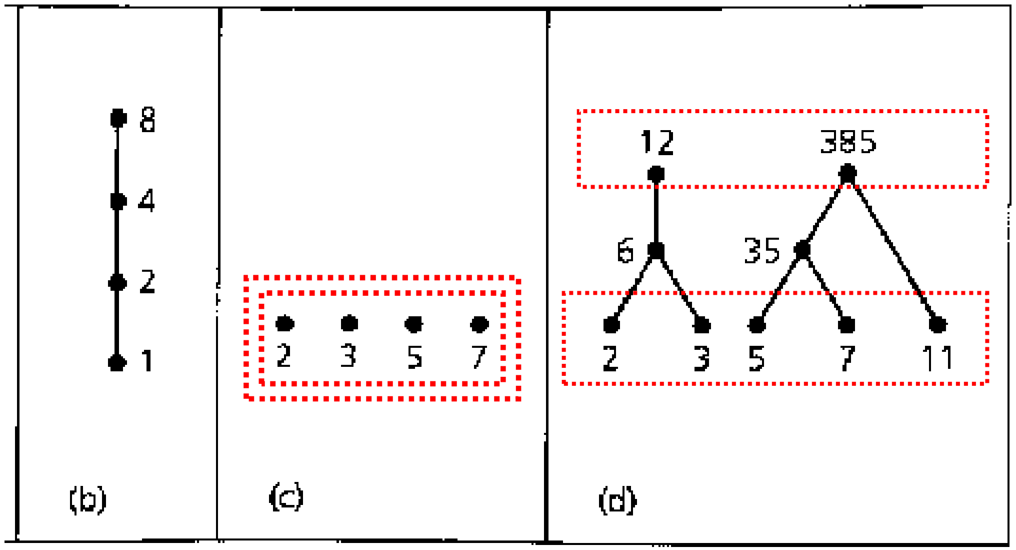
- Definition 7.17: If (A, $\Re$) is a poset, then x is a maximal element of A if for all $a \in A$, $a \neq x \Rightarrow x \not\Re\, a$. Similarly, y is a minimal element of A if for all $b \in A$, $b \neq y \Rightarrow b \not\Re\, y$.
- **Ex 7.42** : $\mathcal{U} = \{1, 2, 3\}$, $A = P(\mathcal{U})$.
  - For the poset (A, $\subseteq$), $\mathcal{U}$ is the maximal and ϕ is the minimal.
  - Let B be the *proper subsets* of $\{1, 2, 3\}$. Then we have multiple maximal elements $\{1, 2\}$, $\{1, 3\}$, and $\{2, 3\}$ for the poset (B, $\subseteq$), and ϕ is still the only minimal element.



- **Ex 7.43** : For the poset ($\mathbf{Z}$, $\leq$), we have neither a maximal nor a minimal element. The poset ($\mathbf{N}$, $\leq$), has no maximal element but a minimal element 0.

Maximal and Minimal

- **Ex 7.44** : How about the poset in (b), (c), and (d) of Fig. 7.18? Do they have maximal or minimal elements?
- Theorem 7.3: If (A, $\Re$) is a poset and A is finite, then A has both a maximal and a minimal element.



Least and Greatest

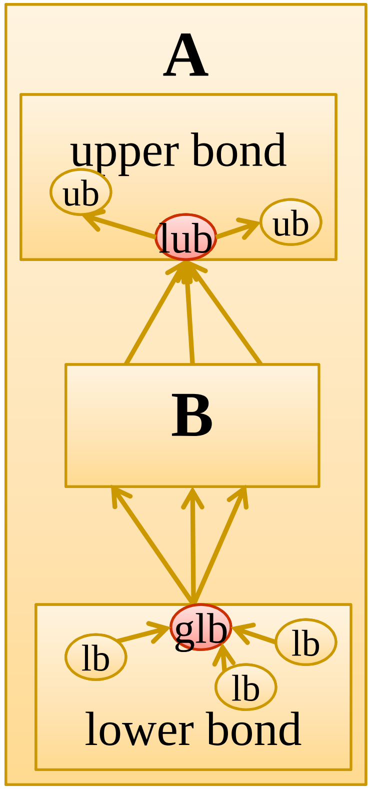
- Definition 7.18: If $(A, \Re)$ is a poset, then x is a **least** element of A if for all $a \in A$, $x \Re a$. Similarly, y is a **greatest** element of A if for all $a \in A$, $a \Re y$.
- **Ex 7.45** : $\mathcal{U} = \{1, 2, 3\}$, $A = P(\mathcal{U})$.
  - For the poset $(A, \subseteq)$, $\mathcal{U}$ is the greatest and ϕ is the least.
  - Let B be the nonempty subsets of $\mathcal{U}$. Then we have $\mathcal{U}$ as the greatest element and three minimal elements for the poset $(B, \subseteq)$, but no least element.
- Theorem 7.4: If poset $(A, \Re)$ has a greatest **or** a least element, then that element is unique.
  - Proof: Assume x and y are both greatest elements.
    Since x is a greatest element, $y \Re x$. Likewise, $x \Re y$ while y is a greatest element. As $\Re$ is antisymmetric, it follows $x = y$.

Lower and Upper Bound

- Definition 7.19: If $(A, \Re)$ is a poset with $B \subseteq A$, then
  - $x \in A$ is called a **lower bound** of B if $x \Re b$ **for all** $\boldsymbol{b \in B}$
  - $y \in A$ is called an **upper bound** of B if $b \Re y$ **for all** $\boldsymbol{b \in B}$

- An element $x' \in A$ is called a *greatest lower bound* (**glb**) of B if for all other lower bounds x'' of B we have $x'' \Re x'$. Similarly, an element $x' \in A$ is called a *least upper bound* (**lub**) of B if for all other upper bounds x'' of B we have $x' \Re x''$.

- Theorem 7.5: If $(A, \Re)$ is a poset and $B \subseteq A$, then B has **at most one** lub (glb).



Lower and Upper Bound

- **Ex 7.47** : Let ***U*** ={1, 2, 3, 4} with A = P(***U***) and let $\Re$ be the subset relation on B. If B = {{1}, {2}, {1, 2}}, then what are the upper bounds of B, lower bounds of B, the greatest lower bound and the least upper bound?
  - Upper bounds: {1, 2}, {1, 2, 3}, {1, 2, 4}, and {1, 2, 3, 4}
  - lub: {1, 2}
  - glb = ϕ

{2, 3, 4} is not.

- **Ex 7.48** : Let $\Re$ be the "≤" relation on A. What are the results for the following cases?
  - A = **R** and B = [0, 1] => lub:1, glb:0
  - A = **R** and $B = \{q \in \mathbf{Q} \mid q^2 < 2\}$ => $\text{lub} : \sqrt{2}, \quad \text{glb} : -\sqrt{2}$
  - A = **Q** and $B = \{q \in \mathbf{Q} \mid q^2 < 2\}$ => ? *No lub and glb*

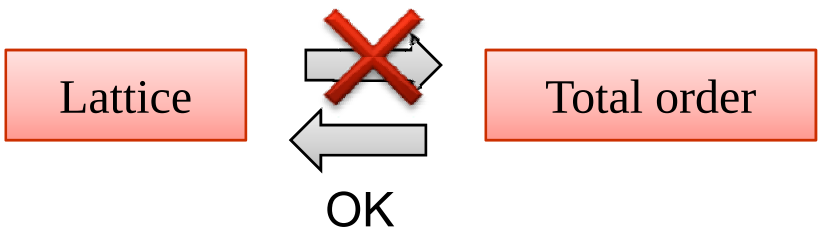
Lattice

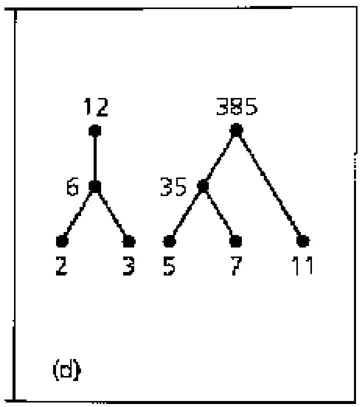
- Definition 7.20. The poset $(A, \Re)$ is called a **lattice** if for *all* $x, y \in A$ the elements lub$\{x, y\}$ and glb$\{x, y\}$ both exist in A.

- **Ex 7.49** : For $A = \mathbf{N}$ and $x, y \in \mathbf{N}$, define $x\Re y$ by $x \le y$. Then lub$\{x, y\}$ = max$\{x, y\}$, glb$\{x, y\}$ = min$\{x, y\}$, and $(\mathbf{N}, \le)$ is a lattice.

- **Ex 7.50** : For the poset $(P(\boldsymbol{U}), \subseteq)$, if $S, T \subseteq \boldsymbol{U}$, we have lub$\{S, T\} = S \cup T$ and glb$\{S, T\} = S \cap T$ and it is a lattice.

- **Ex 7.51**: consider the poset in Example 7.38(d). Here we find that lub$\{2, 3\}$ = 6 exists, but there is no glb for the elements 2 and 3.
  - This partial order is not a lattice.





Partition

- Definition 7.21. Given a set A and index set I, let $\phi \neq A_i \subseteq A$ for $i \in I$.
  - Then $\{A_i\}_{i \in I}$ is a partition of A if (a) $A = \cup_{i \in I} A_i$ and (b) $A_i \cap A_j = \phi$ for $i \neq j$.
  - Each subset A_i is called a cell (block) of the partition.

- **Ex 7.52** : $A = \{1, 2, \ldots, 10\}$
  - $A_1 = \{1, 2, 3, 4, 5\}$, $A_2 = \{6, 7, 8, 9, 10\}$.
  - $A_i = \{i, i+5\}$, $1 \leq i \leq 5$.
- **Ex 7.53** : Let $A = \mathbf{R}$, for each $i \in \mathbf{Z}$, let $A_i = [i, i+1)$. Then $\{A_i\}_{i \in \mathbf{Z}}$ is a partition of $\mathbf{R}$.

Equivalence Class

- Definition 7.22: Let $\Re$ be an equivalence relation on a set A. For each $x \in A$, the **equivalence class** of x, denoted $[x]$, is defined by $[x] = \{y \in A \mid y \Re x\}$

$x \Re y$?

- **Ex 7.54** : Define the relation $\Re$ on **Z** by $x \Re y$ if $4 \mid (x-y)$.
  - $[0] = \{\ldots, -8, -4, 0, 4, \ldots\} = \{4k \mid k \in \mathbf{Z}\}$
  - $[1] = \{\ldots, -7, -3, 1, 5, \ldots\} = \{4k+1 \mid k \in \mathbf{Z}\}$
  - $[2] = \{\ldots, -6, -2, 2, 6, \ldots\} = \{4k+2 \mid k \in \mathbf{Z}\}$
  - $[3] = \{\ldots, -5, -1, 3, 7, \ldots\} = \{4k+3 \mid k \in \mathbf{Z}\}$
- **Ex 7.55** : Define the relation $\Re$ on **Z** by $a \Re b$ if $a^2=b^2$, $\Re$ is an equivalence relation.
  - $[n] = [-n] = \{-n, n\}$

$$Z = \{0\} \cup \left(\bigcup_{n \in Z^+} \{-n, n\} \right)$$

Equivalence Class

- **Theorem 7.6**: If $\Re$ is an equivalence relation on a set A and $x, y \in A$, then
  - (a) $x \in [x]$
  - (b) $x \Re y$ if and only if $[x] = [y]$
  - (c) $[x] = [y]$ or $[x] \cap [y] = \phi$. *(identical or disjoint)*

- **Ex 7.56** :
  - Let $A = \{1, 2, 3, 4, 5\}$, $\Re = \{(1, 1), (2, 2), (2, 3), (3, 2), (3, 3), (4, 4), (4, 5), (5, 4), (5, 5)\}$. $[1] = \{1\}$, $[2] = \{2, 3\} = [3]$, $[4] = \{4, 5\} = [5]$. Then, we have $A = [1] \cup [2] \cup [4]$.
  - Consider an onto function $f: A \rightarrow B$. $f(\{1, 3, 7\}) = x$; $f(\{4, 6\}) = y$; $f(\{2, 5\}) = z$. The relation $\Re$ defined on A by $a \Re b$ if $f(a) = f(b)$.
    - $A = [1] \cup [4] \cup [2] = f^{-1}(x) \cup f^{-1}(y) \cup f^{-1}(z)$.

- **Ex 7.58** : If an equivalence relation $\Re$ on $A = \{1, 2, 3, 4, 5, 6, 7\}$ induces the partition $A = \{1, 2\} \cup \{3\} \cup \{4, 5, 7\} \cup \{6\}$, what is $\Re$?
  - $[1] = \{1, 2\} = [2] = \{(1, 1), (2, 2), (1, 2), (2, 1)\}$
  - $\Re = (\{1,2\} \text{x} \{1,2\}) \cup (\{3\} \text{x} \{3\}) \cup (\{4, 5, 7\} \text{x} \{4, 5, 7\}) \cup (\{6\} \text{x} \{6\})$

$\{(1,1), (2,2)\}$ v.s. $\{1,2\}\text{x}\{1,2\}$

Equivalence and Partition

- **Theorem 7.7**: If A is a set, then
  - (a) any equivalence relation $\Re$ on A induces a partition of A; and
  - (b) any partition of A gives rise to an equivalence relation $\Re$ on A.
- **Theorem 7.8**: For any set A, there is one-to-one correspondence between the set of equivalence relations on A and the set of partitions of A.

- **Ex 7.59** :
  - (a) If A= {1, 2, 3, 4, 5, 6}, how many relations on A are equivalence relations? (identical containers)
    - *a partition of A: a distribution of the (distinct) elements of A into identical containers with no container left empty* $\sum_{i=1}^{6} S(6,i) = 203$
  - (b) How many of the equivalence relations in part (a) satisfy 1, 2 $\in$ [4]?

$$\sum_{i=1}^{4} S(4,i) = 15$$