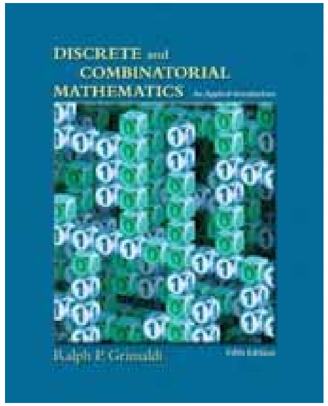
# Discrete Mathematics

## *-- Chapter 2: Fundamentals of Logic*




*Hung-Yu Kao (高宏宇)*
*Department of Computer Science and Information Engineering,*
*National Cheng Kung University*

# Outline

- Basic Connectives and Truth Tables
- Logical Equivalence: The Law of Logic
- Logical Implication: Rule of Inference
- The Use of Quantifiers
- Quantifiers, Definitions, and the Proofs of Theorems

# 2.1 Basic Connectives and Truth Tables

- Statement 敘述 (Proposition 命題): are declarative sentences that are either true or false, but not both.
- Primitive Statement (原始命題)
  - *Examples*
    - ***p***: 'Discrete Mathematics' is a required course for sophomores.
    - ***q***: Margaret Mitchell wrote '*Gone with the Wind*'.
    - ***r***: 2+3=5.
    - "What a beautiful evening!" (not a statement)
    - "Get up and do your exercises." (not a statement)
  - No way to make them simpler

*"The number x is an integer." is a statement ?*

# 2.1 Basic Connectives and Truth Tables

- New statements can be obtained from primitive statements in two ways
  - Transform a given statement $p$ into the statement $\neg p$, which denotes its *negation* and is read “Not $p$”. 非p (***Negation statements***)
  - Combine two or more statements into a *compound* statement, using *logical connectives*. (***Compound statements*** *複合敘述*)

## 2.1 Basic Connectives and Truth Tables

- Compound Statement (Logical Connectives)
  - Conjunction: $p \wedge q$ (read "p and q")
  - Disjunction : $p \vee q$ (read "p or q")
  - Exclusive or : $p \underline{\vee} q$.
  - Implication: $p \rightarrow q$,
    "$p$ implies $q$".

    *p: hypothesis*
    *q: conclusion*
  - Biconditional: $p \leftrightarrow q$
    - "$p$ if and only if $q$" (若且為若)
    - "$p$ iff $q$"
    - "$p$ is necessary and sufficient for $q$".

$p \rightarrow q$ is also called,
- If $p$, then $q$
- $p$ is sufficient for $q$
- $p$ is a sufficient condition for $q$
- $q$ is necessary for $p$
- $q$ is a necessary condition for $p$
- $p$ only if $q$
- q whenever p

# 2.1 Basic Connectives and Truth Tables

- The truth and falsity of the compound statements based on the truth values of their components (primitive statements).

*We do not want a true statement to lead us into believing something that is false.*

| $p$ | $\neg p$ |
|---|---|
| 0 | 1 |
| 1 | 0 |

| $p$ | $q$ | $p \wedge q$ | $p \vee q$ | $p \veebar q$ | $p \rightarrow q$ | $p \leftrightarrow q$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | **1** | **1** |
| 0 | 1 | 0 | 1 | 1 | **1** | **0** |
| 1 | 0 | 0 | 1 | 1 | **0** | **0** |
| 1 | 1 | 1 | 1 | 0 | **1** | **1** |

Truth Tables

# Writing Down Truth Tables

- " True, False" is preferred (less ambiguous) than "1, 0"
- List the elementary truth values in a consistent way (from F…F to T…T or from T…T to F…F).
- They get long: with n variables, the length is $2^n$.
- In the end we want to be able to reason about logic without having to write down such tables.

# 2.1 Basic Connectives and Truth Tables

- $p \to q$ is equivalent with $(\neg p \vee q)$
- It's not relative about the causal relationship
  - *If Discrete Mathematics' is a required course for sophomores, then Margaret Mitchell wrote 'Gone with the Wind'.* **is true**
  - *If "2+3=5", then "4+2=6"* **is true**
  - *If "2+3=6", then "2+4=7"* **is true**

- "Margaret Mitchell wrote '*Gone with the Wind*'*(q)*, and 2+3 $\neq$5 *(not r)*, the 'Discrete Mathematics' is a required course for sophomores *(p)*.
  - $q \wedge (\neg r \to p)$

Table 2.3

| $p$ | $q$ | $r$ | ① $\neg r$ | ② $\neg r \to p$ | ③ $q \wedge (\neg r \to p)$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 0 | 0 |
| 0 | 0 | 1 | 0 | 1 | 0 |
| 0 | 1 | 0 | 1 | 0 | 0 |
| 0 | 1 | 1 | 0 | 1 | 1 |
| 1 | 0 | 0 | 1 | 1 | 0 |
| 1 | 0 | 1 | 0 | 1 | 0 |
| 1 | 1 | 0 | 1 | 1 | 1 |
| 1 | 1 | 1 | 0 | 1 | 1 |

# 2.1 Basic Connectives and Truth Tables

- **Ex 2.1:** Let $s$, $t$, and $u$ denote the primitive statements.
  - $s$: Phyllis goes out for a walk.
  - $t$: The moon is out.
  - $u$: It is snowing.

- English sentences for compound statements.
  - $(t \wedge \neg u) \rightarrow s$:
    - **If** the moon is out **and** it is not snowing, **then** Phyllis goes out for a walk.
  - $t \rightarrow (\neg u \rightarrow s)$ :
    - **If** the moon is out, **then if** it is not snowing Phyllis goes out for a walk.
  - $\neg (s \leftrightarrow (u \vee t))$ :
    - **It is not the case** that Phyllis goes out for a walk **if and only if** it is snowing **or** the moon is out.

*Same?*

# 2.1 Basic Connectives and Truth Tables

- Let $s$, $t$, and $u$ denote the primitive statements.
  - $s$: Phyllis goes out for a walk.
  - $t$: The moon is out.
  - $u$: It is snowing.

- Reversely, examine the logical form for given English sentences.
  - "Phyllis will go out walking if and only if the moon is out."
    - $s \leftrightarrow t$
  - If it is snowing and the moon is not out, then Phyllis will not go out for a walk."
    - $(u \wedge \neg t) \rightarrow \neg s$
  - It is snowing but Phyllis will still go out for a walk.
    - $u \wedge s$

# 2.1 Basic Connectives and Truth Tables

- **Ex 2.3:**
  - Decision (selection) structure
    - In computer science, the **if-then** and **if-then-else** decision structure arise in high-level programming languages such as Java and C++.
    - E.g., "if $p$ then $q$ else $r$," $q$ is executed when $p$ is true and $r$ is executed when $p$ is false.

# 2.1 Basic Connectives and Truth Tables

- **Tautology**, $T_0$: If a compound statement is true for all truth value assignments for its component statements.
  - Example: $p \lor \neg p$
- **Contradiction**, $F_0$: If a compound statement is false for all truth value assignments for its component statements.
  - Example: $p \land \neg p$

- Examples
  - "2 = 3–1" is **not** a tautology, but "2=1 or 2≠1" is;
  - "1+1=3" is **not** a contradiction, but "1=1 and 1≠1" is.

## 2.1 Basic Connectives and Truth Tables

- An **argument** starts with a list of given statements called premises (*hypothesis*) and a statement called the conclusion of the argument.
  - $(p_1 \wedge p_2 \wedge \dots \wedge p_n) \to q$

## 2.2 Logical Equivalence: The Laws of Logic

- **Logically equivalent**, $s_1 \Leftrightarrow s_2$: When the statement $s_1$ is true (false) if and only if $s_2$ is true (false).

$$\neg p \vee q \Leftrightarrow p \rightarrow q$$

$$(p \rightarrow q) \wedge (q \rightarrow p) \Leftrightarrow p \leftrightarrow q$$

*the same truth tables*

| $p$ | $q$ | $\neg p$ | $\neg p \vee q$ | $p \rightarrow q$ | $q \rightarrow p$ | $(p \rightarrow q) \wedge (q \rightarrow p)$ | $p \leftrightarrow q$ |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 |

## 2.2 Logical Equivalence: The Laws of Logic

- Negation, $\underline{\vee}$ (exclusive)

**Table 2.8**

| $p$ | $q$ | $p \veebar q$ | $p \vee q$ | $p \wedge q$ | $\neg(p \wedge q)$ | $(p \vee q) \wedge \neg(p \wedge q)$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| 0 | 1 | 1 | 1 | 0 | 1 | 1 |
| 1 | 0 | 1 | 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 1 | 1 | 0 | 0 |

## 2.2 Logical Equivalence: The Laws of Logic

- Logically equivalent examples
  - $p \Leftrightarrow (p \vee p)$
    - "1+1=2" $\Leftrightarrow$ "1+1=2 or 1+1=2"
  - $\neg\neg p \Leftrightarrow p$
    - "He did not not do it" $\Leftrightarrow$ "He did it"
  - $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$
  - "If $p$ then not $p$" $\Leftrightarrow$ "not $p$"

# The Laws of Logic (1/2)

1) Law of Double Negation : $\neg\neg p \Leftrightarrow p$

2) DeMorgan's Laws : $\begin{cases} \neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q \\ \neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q \end{cases}$

3) Commutative Laws : $\begin{cases} p \vee q \Leftrightarrow q \vee p \\ p \wedge q \Leftrightarrow q \wedge p \end{cases}$

4) Associative Laws : $\begin{cases} p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r \\ p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r \end{cases}$

5) Distributive Laws : $\begin{cases} p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r) \\ p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r) \end{cases}$

# The Laws of Logic (2/2)

$$6)\ \text{Idempotent Laws}: \begin{cases} p \vee p \Leftrightarrow p \\ p \wedge p \Leftrightarrow p \end{cases}$$

$$7)\ \text{Identity Laws}: \begin{cases} p \vee F_0 \Leftrightarrow p \\ p \wedge T_0 \Leftrightarrow p \end{cases}$$

$$8)\ \text{Inverse Laws}: \begin{cases} p \vee \neg p \Leftrightarrow T_0 \\ p \wedge \neg p \Leftrightarrow F_0 \end{cases}$$

$$9)\ \text{Domination Laws}: \begin{cases} p \vee T_0 \Leftrightarrow T_0 \\ p \wedge F_0 \Leftrightarrow F_0 \end{cases}$$

$$10)\ \text{Absorption Laws}: \begin{cases} p \vee (p \wedge q) \Leftrightarrow p \\ p \wedge (p \vee q) \Leftrightarrow p \end{cases}$$

## 2.2 Logical Equivalence: The Laws of Logic

- DeMorgan's Laws: $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$

$$\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$$

| $p$ | $q$ | $p \wedge q$ | $\neg(p \wedge q)$ | $\neg p$ | $\neg q$ | $\neg p \vee \neg q$ | $p \vee q$ | $\neg(p \vee q)$ | $\neg p \wedge \neg q$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 1 | 1 |
| 0 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 0 |
| 1 | 0 | 0 | 1 | 0 | 1 | 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |

# The Principle of Duality

- Definition 2.3: Let $s$ be a statement. Dual of $s$, denoted $s^d$, is the statement obtained from $s$ by replacing each occurrence of $\wedge$ and $\vee$ by $\vee$ and $\wedge$, respectively, and each occurrence of $T_0$ and $F_0$ by $F_0$ and $T_0$, respectively.
  - E.g. $s: \quad (p \wedge \neg q) \vee (r \wedge T_0)$

    $s^d: \quad (p \vee \neg q) \wedge (r \vee F_0)$ *Keep negation!*

- **The Principle of Duality**: Let $s$ and $t$ be statements that contain no logical connectives other than $\wedge$ and $\vee$.
  If $s \Leftrightarrow t$, then $s^d \Leftrightarrow t^d$.

# Substitution Rules

- **Rule 1**: Suppose that the compound statement $P$ is a **tautology**.
  - If $p$ is a primitive statement that appears in $P$ and we replace each occurrence of $p$ by the same statement $q$, then the resulting compound statement $P_1$.is also a tautology.
- **Rule 2**: Let $P$ be a compound statement where $p$ is an arbitrary statement that appears in $P$, and let $q$ be a statement such that $q \Leftrightarrow p$.
  - Suppose that in $P$ we replace **one or more** occurrences of $p$ by $q$. Then this replacement yields the compound statement $P_1$. Under these circumstances $P_1 \Leftrightarrow P$.

# Simplification of Compound Statements

- **Ex 2.11:**
  - $P$: $(p \rightarrow q) \rightarrow r$, and because $(p \rightarrow q) \Leftrightarrow \neg p \vee q$
  - from the second substitution rule

    if $P_1$: $(\neg p \vee q) \rightarrow r$, *then* $P_1 \Leftrightarrow P$

# Simplification of Compound Statements

- **Ex 2.12:**
  - Negate and simplify the compound statement $(p \vee q) \to r$

1. $(p \vee q) \to r \Leftrightarrow \neg(p \vee q) \vee r$ (substitution rule)
2. $\neg[(p \vee q) \to r] \Leftrightarrow \neg[\neg(p \vee q) \vee r]$ (Negating)
3. $\neg[\neg(p \vee q) \vee r] \Leftrightarrow \neg\neg(p \vee q) \wedge \neg r$ (DeMorgan's Laws)
4. $\neg\neg(p \vee q) \wedge \neg r \Leftrightarrow (p \vee q) \wedge \neg r$ (Law of double Negation)
5. $\neg[(p \vee q) \to r] \Leftrightarrow (p \vee q) \wedge \neg r$

# Simplification of Compound Statements

- **Ex 2.13:**
- $p$: Joan goes to Lake George
- $q$: Mary pays for Joan's shopping spree.
- $p \to q$: If Joan goes to Lake George, then Mary will pay for Joan's shopping spree.
- The negation of $p \to q$ :
  - One way: $\neg (p \to q)$. *It is not the case that if Joan goes to Lake George, then Mary will pay for Joan's shopping spree.*
  - Another way: $p \wedge \neg q$. Joan goes to Lake George, but Mary does not pay for Joan's shopping spree.

$$
\begin{aligned}
& \neg(p \to q) \\
\Leftrightarrow\ & \neg(\neg p \vee q) \\
\Leftrightarrow\ & \neg\neg p \wedge \neg q \\
\Leftrightarrow\ & p \wedge \neg q
\end{aligned}
$$

# Relevant Statements to Implication Statement

- **Ex 2.15:**

| $p$ | $q$ | $p \to q$ | $\neg q \to \neg p$ | $q \to p$ | $\neg p \to \neg q$ |
|---|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 1 | 1 |
| 0 | 1 | 1 | 1 | 0 | 0 |
| 1 | 0 | 0 | 0 | 1 | 1 |
| 1 | 1 | 1 | 1 | 1 | 1 |

  - $p \to q \Leftrightarrow (\neg q \to \neg p)$
  - $q \to p \Leftrightarrow (\neg p \to \neg q)$
- Contrapositive of $p \to q$ : $\underline{\neg q \to \neg p}$
- Converse of $p \to q$ : $\underline{q \to p}$
- Inverse of $p \to q$ : $\underline{\neg p \to \neg q}$

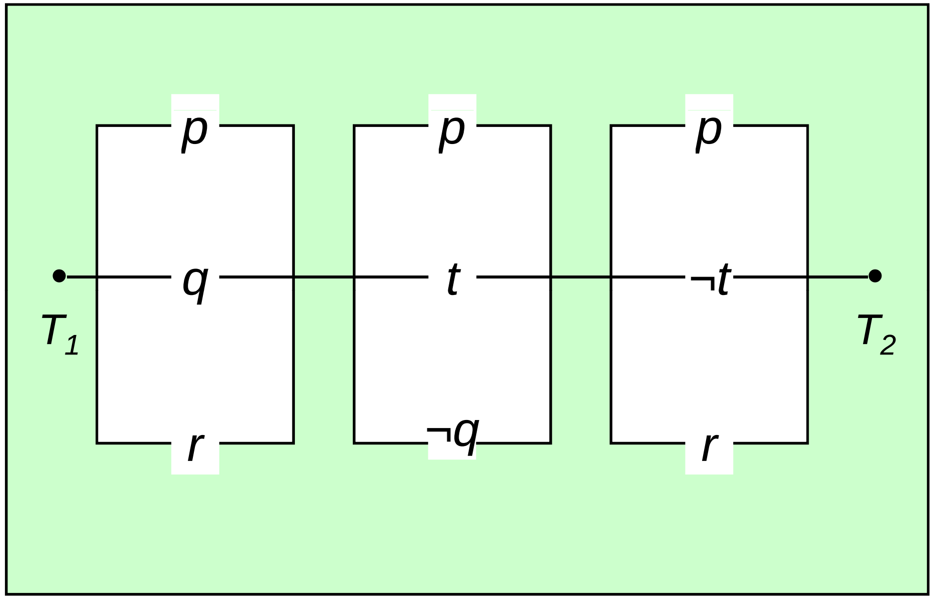
## Simplification of Compound Statements

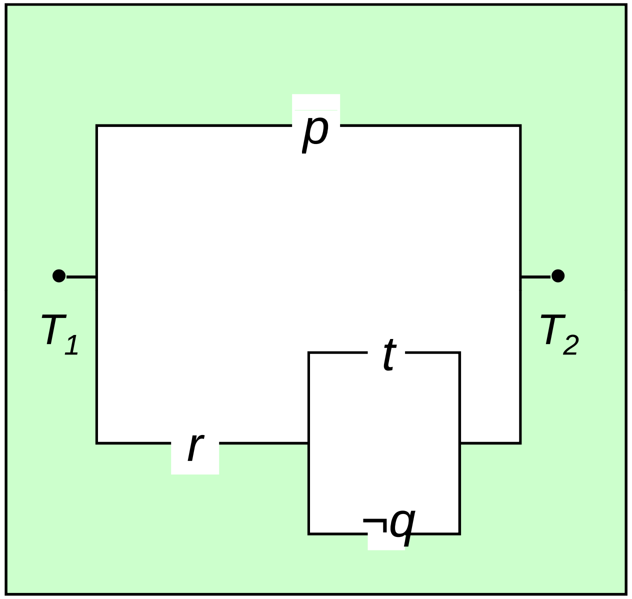
- **Ex 2.16:** How to simply express the compound statement $(p \vee q) \wedge \neg(\neg p \wedge q)$?

| | |
|---|---|
| $(p \vee q) \wedge \neg(\neg p \wedge q)$ | Reasons |
| $\Leftrightarrow (p \vee q) \wedge (\neg\neg p \vee \neg q)$ | (DeMorgan's Law) |
| $\Leftrightarrow (p \vee q) \wedge (p \vee \neg q)$ | (Law of Double Negation) |
| $\Leftrightarrow p \vee (q \wedge \neg q)$ | (Distributive Law of $\vee$ over $\wedge$) |
| $\Leftrightarrow p \vee F_0$ | (Inverse Law) |
| $\Leftrightarrow p$ | (Identity Law) |

# Simplification of Compound Statements

- **Ex 2.18:** A switching network is made up of wires and switches connecting two terminals $T_1$ and $T_2$




$$(p \vee q \vee r) \wedge (p \vee t \vee \neg q) \wedge (p \vee \neg t \vee r) \Leftrightarrow p \vee [r \wedge (t \vee \neg q)]$$ *Practice!*

# Simplification of Compound Statements

$$(p \vee q \vee r) \wedge (p \vee t \vee \neg q) \wedge (p \vee \neg t \vee r)$$

$$\Leftrightarrow p \vee [(q \vee r) \wedge (t \vee \neg q) \wedge (\neg t \vee r)]$$

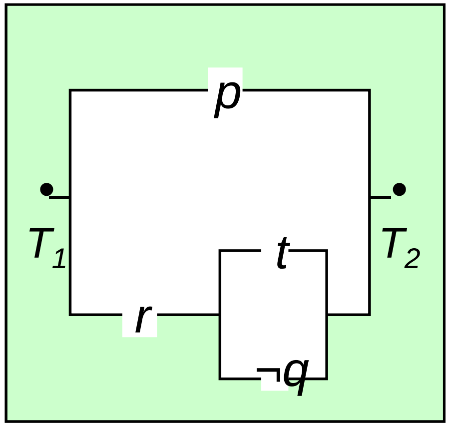
$$\Leftrightarrow p \vee [((q \wedge \neg t) \vee r) \wedge (t \vee \neg q)]$$

$$\Leftrightarrow p \vee [((q \wedge \neg t) \vee r) \wedge \neg(\neg t \wedge q)]$$

$$\Leftrightarrow p \vee [((q \wedge \neg t) \wedge \neg(\neg t \wedge q)) \vee (\neg(\neg t \wedge q) \wedge r)]$$

$$\Leftrightarrow p \vee [F_0 \vee (\neg(\neg t \wedge q) \wedge r)]$$

$$\Leftrightarrow p \vee [r \wedge (t \vee \neg q)]$$



# 2.3 Logic Implication: Rules of Inference

- Argument: $(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \rightarrow q$
  Premiss: $p_1, p_2, \ldots, p_n$
  Conclusion: $q$

- **Ex 2.19:**
  - $p$: Roger studies.
  - $q$: Roger plays racketball.
  - $r$: Roger passes discrete mathematics.

  - $p_1$: Roger studies, then he will pass discrete mathematics.
  - $p_2$: If Roger don't play racketball, then he'll study.
  - $p_3$: Roger failed discrete mathematics.
  - Determine whether the argument $(p_1 \wedge p_2 \wedge p_3) \rightarrow q$ is valid.

$$p_1 : p \rightarrow r$$
$$p_2 : \neg q \rightarrow p$$
$$p_3 : \neg r$$

# Logic Implication: Rules of Inference

- Examine the truth table for the implication

  $[(p \to r) \wedge (\neg q \to p) \wedge \neg r] \to q$

| | | | $p_1$ | $p_2$ | $p_3$ | $(p_1 \wedge p_2 \wedge p_3) \to q$ |
|---|---|---|---|---|---|---|
| $p$ | $q$ | $r$ | $p \to r$ | $\neg q \to p$ | $\neg r$ | $[(p \to r) \wedge (\neg q \to p) \wedge \neg r] \to q$ |
| 0 | 0 | 0 | 1 | 0 | 1 | 1 |
| 0 | 0 | 1 | 1 | 0 | 0 | 1 |
| 0 | 1 | 0 | 1 | 1 | 1 | 1 |
| 0 | 1 | 1 | 1 | 1 | 0 | 1 |
| 1 | 0 | 0 | 0 | 1 | 1 | 1 |
| 1 | 0 | 1 | 1 | 1 | 0 | 1 |
| 1 | 1 | 0 | 0 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 | 1 | 0 | 1 |

$p_1 : p \to r$

$p_2 : \neg q \to p$

$p_3 : \neg r$

- $(p_1 \wedge p_2 \wedge p_3) \to q$ is a valid argument.

# Logic Implication: Rules of Inference

- Definition 2.4: $p$ logically implies $q$, $p \Rightarrow q$.
  - If $p$, $q$ are arbitrary statements such that $p \to q$ is a tautology.
- Rules of inference ← *Use instead of constructing the huge truth table*
  - Using these techniques will enable us to consider only the cases wherein all the premises are true.
  - Development of a step-by-step validation of how the conclusion $q$ logically follows from the premises $p_1, p_2, \ldots, p_n$ in an implication of the form

$$(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \to q$$

# Logic Implication: Rules of Inference

- Rule of Detachment (分離)
  - Modus Ponens: the method of affirming
- Example: $[p \wedge (p \rightarrow q)] \rightarrow q$

| $p$ | $q$ | $p \rightarrow q$ | $p \wedge (p \rightarrow q)$ | $[p \wedge (p \rightarrow q)] \rightarrow q$ |
|---|---|---|---|---|
| 0 | 0 | 1 | 0 | 1 |
| 0 | 1 | 1 | 0 | 1 |
| 1 | 0 | 0 | 0 | 1 |
| 1 | 1 | 1 | 1 | 1 |

- Tabular form:

$$\begin{array}{l} p \\ p \rightarrow q \\ \hline \therefore q \end{array}$$

*therefore*

# Logic Implication: Rules of Inference

- Law of the Syllogism (演繹推理)
  - Tabular form:

$$\begin{array}{l} p \to q \\ q \to r \\ \hline \therefore\ p \to r \end{array}$$

- Modus Tollens: Method of Denying
  - Tabular form:

$$\begin{array}{l} p \to q \\ \neg q \\ \hline \therefore\ \neg p \end{array}$$

# Rules of Inference

Table 2.19

| Rule of Inference | Related Logical Implication | Name of Rule |
|---|---|---|
| 1) $p$, $p \to q$, $\therefore q$ | $[p \wedge (p \to q)] \to q$ | Rule of Detachment (Modus Ponens) |
| 2) $p \to q$, $q \to r$, $\therefore p \to r$ | $[(p \to q) \wedge (q \to r)] \to (p \to r)$ | Law of the Syllogism |
| 3) $p \to q$, $\neg q$, $\therefore \neg p$ | $[(p \to q) \wedge \neg q] \to \neg p$ | Modus Tollens |
| 4) $p$, $q$, $\therefore p \wedge q$ | | Rule of Conjunction |
| 5) $p \vee q$, $\neg p$, $\therefore q$ | $[(p \vee q) \wedge \neg p] \to q$ | Rule of Disjunctive Syllogism |
| 6) $\neg p \to F_0$, $\therefore p$ | $(\neg p \to F_0) \to p$ | Rule of Contradiction |
| 7) $p \wedge q$, $\therefore p$ | $(p \wedge q) \to p$ | Rule of Conjunctive Simplification |
| 8) $p$, $\therefore p \vee q$ | $p \to p \vee q$ | Rule of Disjunctive Amplification |
| 9) $p \wedge q$, $p \to (q \to r)$, $\therefore r$ | $[(p \wedge q) \wedge [p \to (q \to r)]] \to r$ | Rule of Conditional Proof |
| 10) $p \to r$, $q \to r$, $\therefore (p \vee q) \to r$ | $[(p \to r) \wedge (q \to r)] \to [(p \vee q) \to r]$ | Rule for Proof by Cases |
| 11) $p \to q$, $r \to s$, $p \vee r$, $\therefore q \vee s$ | $[(p \to q) \wedge (r \to s) \wedge (p \vee r)] \to (q \vee s)$ | Rule of the Constructive Dilemma |
| 12) $p \to q$, $r \to s$, $\neg q \vee \neg s$, $\therefore \neg p \vee \neg r$ | $[(p \to q) \wedge (r \to s) \wedge (\neg q \vee \neg s)] \to (\neg p \vee \neg r)$ | Rule of the Destructive Dilemma |

# Logic Implication: Rules of Inference

**Ex 2.30:**

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow (r \wedge s) \\ \neg r \vee (\neg t \vee u) \\ p \wedge t \\ \hline \therefore u \end{array}$$

| Steps | Reasons |
|---|---|
| 1) $p \rightarrow q$ | Premise |
| 2) $q \rightarrow (r \wedge s)$ | Premise |
| 3) $p \rightarrow (r \wedge s)$ | Steps (1) and (2) and the Law of the Syllogism |
| 4) $p \wedge t$ | Premise |
| 5) $p$ | Step (4) and the Rule of Conjunctive Simplification |
| 6) $r \wedge s$ | Steps (5) and (3) and the Rule of Detachment |
| 7) $r$ | Step (6) and the Rule of Conjunctive Simplification |
| 8) $\neg r \vee (\neg t \vee u)$ | Premise |
| 9) $\neg(r \wedge t) \vee u$ | Step (8), the Associative Law of $\vee$, and DeMorgan's Laws |
| 10) $t$ | Step (4) and the Rule of Conjunctive Simplification |
| 11) $r \wedge t$ | Steps (7) and (10) and the Rule of Conjunction |
| 12) $\therefore u$ | Steps (9) and (11), the Law of Double Negation, and the Rule of Disjunctive Syllogism |

# Logic Implication: Rules of Inference

$$(\neg p \vee \neg q) \rightarrow (r \wedge s)$$
$$r \rightarrow t$$
$$\neg t$$
$$\therefore p$$

| **Steps** | **Reasons** |
|---|---|
| (1) $r \rightarrow t$ | Premise |
| (2) $\neg t$ | Premise |
| (3) $\neg r$ | (1) and (2), and Method of Denying |
| (4) $\neg r \vee \neg s$ | (3) and Rule of Disjunctive Amplification |
| (5) $\neg(r \wedge s)$ | (4) and DeMorgan's Laws |
| (6) $(\neg p \vee \neg q) \rightarrow (r \wedge s)$ | Premise |
| (7) $\neg(\neg p \vee \neg q)$ | (6) and (5), and Method of Denying |
| (8) $p \wedge q$ | (7), DeMorgan's Laws, and Law of Double Negation |
| (9) $\therefore p$ | (8) and Rule of Conjunctive Simplification |

# Proof by Contradiction

- **Ex 2.35:**
  - Is the right argument valid or invalid?

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow s \\ r \rightarrow \neg s \\ \neg p \veebar r \\ \hline \therefore \neg p \end{array}$$

  - Let conclusion $\neg p$ be false (the argument invalid)
    $\Rightarrow p$ is true
    $\Rightarrow q$ is true
    $\Rightarrow s$ is true
    $\Rightarrow r$ is false
    $\Rightarrow r$ is true ($\neg p \veebar r$, now $p$ is true)
    $\Rightarrow$ contradiction
    $\Rightarrow$ argument is valid

# 2.4 The Use of Quantifiers

- Definition 2.5: A declarative sentence is an **open statement** if
  - it contains one or more variables, and
  - it is not a statement, but
  - it becomes a statement when *the variables in it are replaced by certain allowable choices*.
  - E.g.,
    - $p(x)$: The number $x+2$ is an even integer.
    - $q(x, y)$: The number $y+2$, $x - y$, and $x+2y$ are even integers.
    - For some $x$, $p(x)$. For some $x$, $y$, $q(x, y)$.
    - For some $x$, $\neg p(x)$. For some $x$, $y$, $\neg q(x, y)$.

- **Existential quantifier**: $\exists x$ , "for some $x$", "for at least one $x$" or "there exists an $x$ such that"
- **Universal quantifier**: $\forall x$, "for all $x$", "for any $x$" "for every $x$" or "for each $x$ such that"

# The Use of Quantifiers

- **Ex 2.36:**
  - Given the open statements $p(x): \;\; x \geq 0 \,, \;\; q(x): \;\; x^2 \geq 0$

    $r(x): \;\; x^2 - 3x - 4 = 0, \, s(x): x^2 - 3 > 0$
  - The statement $\forall x \, [\, p(x) \rightarrow q(x)]$ is true
    - For every real number *x*, if $x \geq 0$, then $x^2 \geq 0$.
    - Every nonnegative real number has a nonnegative square.
    - The square of any nonnegative real number is a nonnegative real number.
    - All nonnegative real numbers have nonnegative squares.

  - The statement $\exists x \, [\, p(x) \wedge r(x)]$ is true.
  - The statement $\forall x \, [q(x) \rightarrow s(x)]$ is false. ← *Counterexample!*

# Summarization

Table 2.21

| Statement | When Is It True? | When Is It False? |
|---|---|---|
| $\exists x\ p(x)$ | For some (at least one) $a$ in the universe, $p(a)$ is true. | For every $a$ in the universe, $p(a)$ is false. |
| $\forall x\ p(x)$ | For every replacement $a$ from the universe, $p(a)$ is true. | There is at least one replacement $a$ from the universe for which $p(a)$ is false. |
| $\exists x\ \neg p(x)$ | For at least one choice $a$ in the universe, $p(a)$ is false, so its negation $\neg p(a)$ is true. | For every replacement $a$ in the universe, $p(a)$ is true. |
| $\forall x\ \neg p(x)$ | For every replacement $a$ from the universe, $p(a)$ is false and its negation $\neg p(a)$ is true. | There is at least one replacement $a$ from the universe for which $\neg p(a)$ is false and $p(a)$ is true. |

$\neg\exists x$: $P(x)$ v.s. $\exists x$: $\neg P(x)$

$\neg\forall x$: $P(x)$ v.s. $\forall x$: $\neg P(x)$

# The Use of Quantifiers

- For open statements p(x), q(x), the universally quantified statement $\forall x[p(x) \rightarrow q(x)]$, we define
  - Contrapositive: $\forall x[\neg q(x) \rightarrow \neg p(x)]$
  - Converse: $\forall x[q(x) \rightarrow p(x)]$
  - Inverse: $\forall x[\neg p(x) \rightarrow \neg q(x)]$

# Logical Equivalence and Implication for Quantified Statements

$$\exists x\,[p(x) \wedge q(x)] \Rightarrow [\exists x\; p(x) \wedge \exists x\, q(x)]$$

$$\exists x\,[p(x) \vee q(x)] \Leftrightarrow [\exists x\; p(x) \vee \exists x\, q(x)]$$

$$\forall x\,[p(x) \wedge q(x)] \Leftrightarrow [\forall x\; p(x) \wedge \forall x\, q(x)]$$

$$[\forall x\; p(x) \vee \forall x\, q(x)] \Rightarrow \forall x\,[p(x) \vee q(x)]$$

# The Use of Quantifiers

- Rules for negating statements with one quantifier

a) $\forall x \neg\neg p(x) \Leftrightarrow \forall x\ p(x)$
b) $\forall x \neg[p(x) \wedge q(x)] \Leftrightarrow \forall x\ [\neg p(x) \vee \neg q(x)]$
c) $\forall x \neg[p(x) \vee q(x)] \Leftrightarrow \forall x\ [\neg p(x) \wedge \neg q(x)]$

$$\neg[\forall x\ p(x)] \Leftrightarrow \exists x \neg p(x)$$

$$\neg[\exists x\ p(x)] \Leftrightarrow \forall x \neg p(x)$$

$$\neg[\forall x \neg p(x)] \Leftrightarrow \exists x \neg\neg p(x) \Leftrightarrow \exists x\ p(x)$$

$$\neg[\exists x \neg p(x)] \Leftrightarrow \forall x \neg\neg p(x) \Leftrightarrow \forall x\ p(x)$$

# Commuting Quantifiers

Identical quantifiers commute:

$\exists x \exists y: P(x,y) \Leftrightarrow \exists y \exists x: P(x,y)$ and

$\forall x \forall y: P(x,y) \Leftrightarrow \forall y \forall x: P(x,y)$

But non-identical ones do not, see:

$\exists x \forall y: x=y$ v.s. $\forall y \exists x: x=y$

# The Use of Quantifiers

- **Ex 2.49:**
  - What is the negation of the following statement?

$$\forall x\, \exists y\, [(p(x,y) \wedge q(x,y)) \rightarrow r(x,y)]$$

$$\neg[\forall x\, \exists y\, [(p(x,y) \wedge q(x,y)) \rightarrow r(x,y)]]$$

$$\Leftrightarrow \exists x\, [\neg \exists y\, [(p(x,y) \wedge q(x,y)) \rightarrow r(x,y)]]$$

$$\Leftrightarrow \exists x\, \forall y \neg [(p(x,y) \wedge q(x,y)) \rightarrow r(x,y)]$$

$$\Leftrightarrow \exists x\, \forall y \neg [\neg [p(x,y) \wedge q(x,y)] \vee r(x,y)]$$

$$\Leftrightarrow \exists x\, \forall y [\neg \neg [p(x,y) \wedge q(x,y)] \wedge \neg r(x,y)]$$

$$\Leftrightarrow \exists x\, \forall y [p(x,y) \wedge q(x,y)] \wedge \neg r(x,y)]$$

# Quantifiers, Definitions, and the Proofs of Theorems

- **The Rule of Universal Specification**: If an open statement becomes true for all replacements by the members in a given universe, then that open statement is true for each specific individual member in that universe.

*if* $\forall x\ p(\mathrm{x})$ is true, then $p(x)$ is true for each $a$ in the universe.

  - E.g.

$$\begin{array}{l} \forall x\ [m(x) \rightarrow c(x)] \\ m(a) \\ \hline \therefore c(a) \end{array}$$

| **Steps** | **Reasons** |
|---|---|
| (1) $\forall x[m(x) \rightarrow c(x)]$ | Premise |
| (2) $m(\mathrm{a})$ | Premise |
| (3) $m(\mathrm{a}) \rightarrow \mathrm{c(a)}$ | (1) and the rule of Universal Specification |
| (4) $\therefore \mathrm{c(a)}$ | (2) and (3) and the Rule of Detachment |

# Quantifiers, Definitions, and the Proofs of Theorems

- **The Rule of Universal Generalization**: If an open statement $p(x)$ is proved to be true when x is replaced by **any arbitrarily chosen** element $c$ from our universe, then the universally quantified statement $\forall x\, p(x)$ is true. Furthermore, the rule extends beyond a single variable.
  - **Ex 2.56:**

$$\begin{array}{l} \forall x \ [p(x) \vee q(x)] \\ \forall x \ [(\neg p(x) \wedge q(x)) \rightarrow r(x)] \\ \hline \therefore \forall x \ [\neg r(x) \rightarrow p(x)] \end{array}$$

$$\underline{[\neg r(x) \rightarrow \neg(\neg p(x) \wedge q(x))]}$$

$$\downarrow$$

$$\underline{[\neg r(x) \rightarrow p(x) \vee \neg q(x))]}$$

# Quantifiers, Definitions, and the Proofs of Theorems

$$\begin{array}{l} \forall x\ [p(x) \vee q(x)] \\ \forall x\ [(\neg p(x) \wedge q(x)) \rightarrow r(x)] \\ \hline \therefore \forall x\ [\neg r(x) \rightarrow p(x)] \end{array}$$

- **Ex 2.56:**

| Steps | Reasons |
|---|---|
| 1) $\forall x\ [p(x) \vee q(x)]$ | Premise |
| 2) $p(c) \vee q(c)$ | Step (1) and the Rule of Universal Specification |
| 3) $\forall x\ [(\neg p(x) \wedge q(x)) \rightarrow r(x)]$ | Premise |
| 4) $[\neg p(c) \wedge q(c)] \rightarrow r(c)$ | Step (3) and the Rule of Universal Specification |
| 5) $\neg r(c) \rightarrow \neg[\neg p(c) \wedge q(c)]$ | Step (4) and $s \rightarrow t \Leftrightarrow \neg t \rightarrow \neg s$ |
| 6) $\neg r(c) \rightarrow [p(c) \vee \neg q(c)]$ | Step (5), DeMorgan's Law, and the Law of Double Negation |
| 7) $\neg r(c)$ | Premise (assumed) |
| 8) $p(c) \vee \neg q(c)$ | Steps (7) and (6) and Modus Ponens |
| 9) $[p(c) \vee q(c)] \wedge [p(c) \vee \neg q(c)]$ | Steps (2) and (8) and the Rule of Conjunction |
| 10) $p(c) \vee [q(c) \wedge \neg q(c)]$ | Step (9) and the Distributive Law of $\vee$ over $\wedge$ |
| 11) $p(c)$ | Step (10), $q(c) \wedge \neg q(c) \Leftrightarrow F_0$, and $p(c) \vee F_0 \Leftrightarrow p(c)$ |
| 12) $\therefore \forall x\ [\neg r(x) \rightarrow p(x)]$ | Steps (7) and (11) and the Rule of Universal Generalization |

# Quantifiers, Definitions, and the Proofs of Theorems

- Argument

$$\begin{array}{l} \forall x\ [(j(x) \vee s(x)) \rightarrow \neg p(x)] \\ p(m) \\ \hline \therefore \neg s(m) \end{array}$$

- No junior or senior is enrolled in a physical education class.
- Mary is enrolled in a physical education class.
- Thus Mary is not a senior.

- Establish the validity of this argument

| **Steps** | **Reasons** |
|---|---|
| 1) $\forall x\ [j(x) \vee s(x) \rightarrow \neg p(x)]$ | Premise |
| 2) $p(m)$ | Premise |
| 3) $j(m) \vee s(m) \rightarrow \neg p(m)$ | (1) and Rule of Universal Specification |
| 4) $p(m) \rightarrow \neg(j(m) \vee s(m))$ | (3), $(q \rightarrow t) \Leftrightarrow (\neg t \rightarrow \neg q)$ and Law of Double Negation |
| 5) $p(m) \rightarrow (\neg j(m) \wedge \neg s(m))$ | (4) and DeMorgan's Law |
| 6) $\neg j(m) \wedge \neg s(m)$ | (2) and (5) and Rule of Detachment |
| 7) $\therefore \neg s(m)$ | (6) and Rule of Conjunctive Simplification |

# Quantifiers, Definitions, and the Proofs of Theorems

- Theorem 2.4: If $m$ is an even integer, then $m + 7$ is odd.
  - Proof

1. Since $m$ is even, we have $m = 2a$ for some integer $a$. Then $m + 7 = 2a + 7 = 2a + 6 + 1 = 2(a + 3) + 1$. since $a+3$ is an integer, we know that $m + 7$ is odd.
2. Suppose that $m + 7$ is not odd, hence even. Then $m + 7 = 2b$ for some integer $b$, and $m = 2b – 7 = 2b – 8 + 1 = 2(b – 4) + 1$, where $b – 4$ is an integer. Hence $m$ is odd. (*contraposition method*)
3. Assume that $m$ is even and $m + 7$ is also even. Then $m + 7$ even implies that $m + 7 = 2c$ for some integer $c$. Consequently, $m = 2c – 7 = 2c – 8 + 1 = 2(c – 4) + 1$ with $c – 4$ an integer, so $m$ is odd. Now we have contradiction. So the assumption is false ($m + 7$ is even), and we have $m + 7$ odd. (*contradiction method*)

| | Assumption | Result Derived |
|---|---|---|
| Contraposition | $\neg q(m)$ | $\neg p(m)$ |
| Contradiction | $p(m)$ and $\neg q(m)$ | $F_0$ |