Deformtaion Quantization and TYZ

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Contents

1 Question about coefficients of the TYZ expansion

Let (M, ω) be a polarized Kähler manifold with (L, h). We assume the reader is familiar with this setting. It is well known that the associated *Bergman kernel* B_k admits an asymptotic expansion as the tensor power k goes to infinity:

$$B_k(\omega) = \sum_{j=0}^{\infty} k^{n-j} a_j(g_\omega), \tag{1.1}$$

where the coefficients $a_j(g_\omega)$ are smooth functions given by universal polynomials in the curvature of g_ω and its derivatives. By Riemann-Roch-Hirzebruch, we have (for $k \gg 1$)

$$\begin{aligned}
&\sum_{j\geq 0} k^{n-j} \int_M a_j(x) \frac{\omega^n}{n!} \\
&= \int_M B_k(x) \frac{\omega^n}{n!} \\
&= (2\pi)^n \dim H^0(M, L^k) \\
&= \int_M \operatorname{Td}(M) \operatorname{ch}(L^k) \\
&= \sum_{j\geq 0} k^{n-j} \int_M \operatorname{Td}_j(M) \operatorname{ch}_{n-j}(L).
\end{aligned}$$

As a result, we have

$$\int_M a_j(\omega) \frac{\omega^n}{n!} = \int_M \operatorname{Td}_j(M) \operatorname{ch}_{n-j}(L) = \int_M \operatorname{Td}_j(M) \frac{\omega^{n-j}}{(n-j)!} \tag{1.2}$$

for all $j \geq 0$. In particular, the integral of a_j is zero for all $j > n$. It is quite interesting because the formula of a_j does not depend on the dimension yet there is some nontrivial cancellation happening depending on the dimension. Moreover, the formula for a_j is given by a *universal* formula: which means that we can make sense of such functions when there is no Bergman kernel, line bundle L, etc. For example, a_1 is just $1/2$ of the scalar curvature and its integral is $\int_M \operatorname{Td}_1(M) \frac{\omega^{n-1}}{(n-1)!}$ without invoking RRH. Hence we ask the natural question:

Question 1. Is (1.2) remain true when (M, ω) is *not* polarized? In particular,

$$\int_M a_j(\omega) \frac{\omega^n}{n!} = 0 \tag{1.3}$$

for $j > n$?

In this note we answer this question by the theory of *deformation quantization* and Hao Xu's combinatorial approach.

2 Basic definitions on deformation quantization

See [Gut05, Mas14] for more detailed introduction.

Let (M, ω) be a symplectic manifold of dimension $2n$. For $f \in C^\infty(M)$, *Hamiltonian vector field* X_f is defined by

$$\iota_{X_f} \omega = df.$$

Define *Poisson bracket* $\{\,,\,\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ by

$$\{f, g\} = \omega(X_g, X_f).$$

Then $[X_f, X_g] = X_{\{f,g\}}$. Let $C^\infty(M)[[\hbar]]$ be the ring of formal power series with coefficients in $C^\infty(M)$.

Definition 1. A (formal) *star product* (or *deformation quantization*) on M is $\mathbb{C}[[\hbar]]$-bilinear map $\star : C^\infty(M)[[\hbar]] \times C^\infty(M)[[\hbar]] \to C^\infty(M)[[\hbar]]$ written as

$$f \star g = \sum_{k=0}^{\infty} \hbar^k C_k(f,g),$$

where $C_k : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ are bilinear maps called *coefficients*. A star product is required to satisfy:

1. Associativity : $(f \star g) \star h = f \star (g \star h)$;
2. $C_0(f,g) = fg$;
3. $C_1(f,g) - C_1(g,f) = i\{f,g\}$.

When the C_k are bidifferential operators, $\star$ is called *differential* or *local* star product. We will only consider this case.

Example (Moyal-Weyl product on $\mathbb{R}^{2n}$). On $\mathbb{R}^{2n}$ with standard symplectic form, the *Moyal-Weyl product* is defined by

$$f \star_{MW} g(x) = \exp\left(\frac{i\hbar}{2}\delta^{ij}\frac{\partial}{\partial y^i}\frac{\partial}{\partial z^j}\right) f(y)g(z)|_{y=z=x}.$$

This product satisfies the axioms of star product.

A natural question concerns the existence and classification on arbitrary symplectic manifold. There are algebraic approaches by De Wilde-Lecomte, Deligne, Kontsevich, Nest-Tsygan etc. Here we summarize the differential-geometric approaches by Fedosov and Karabegov.

Remark. Note that there are different conventions where one instead requires $\mathbb{R}[[\hbar]]$-linearity and $C_1(f,g) - C_1(g,f) = \{f,g\}$. This difference in convention causes confusion among different papers regarding the constants appearing in various formulas and quantities. The original paper of Fedosov used convention $C_1(f,g) - C_1(g,f) = -i\{f,g\}$. In this note, we try to be as consistent as possible.

3 Symplectic case : Fedosov's construction

The idea is to patch Moyal-Weyl product on each point globally.

Definition 2. The *formal Weyl alebra* W_x corresponding to the symplectic vector space T_xM, is the associative algebra over $\mathbb{C}$ with a unit, its elements being formal series

$$a(y) = \sum_{2k+l\geq 0} \hbar^k a_{k;i_1,\cdots,i_l} y^{i_1}\cdots y^{i_l},$$

where $\hbar$ is a formal parameter, $y = (y^1, \cdots, y^{2n}) \in T_xM$ is a tangent vector, and $a_{k;i_1,\cdots,i_l}$ are covariant tensors. We assign degree 1 and 2 for the variables y^i

and $\hbar$ respectively. The associative product of elements $a, b \in W_x$ is determined by the Weyl rule (c.f. Moyal-Weyl product)

$$a \circ b = \exp\left(\frac{i\hbar}{2}\omega^{ij}\frac{\partial}{\partial y^i}\frac{\partial}{\partial z^j}\right)a(y,\hbar)b(z,\hbar)|_{y=z}.$$

Taking the union of the algebras $W_x, x \in M$, we obtain the *Weyl bundle*. Sections of the Weyl bundle have the form of formal series

$$a(x,y,\hbar) = \sum_{2k+l\geq 0} \hbar^k a_{k;i_1,\cdots,i_l}(x) y^{i_1}\cdots y^{i_l}$$

with $a_{k;i_1,\cdots,i_l}$ are covariant tensor fields on M. The product of two sections can be defined pointwisely:

$$(a \circ b)(x,y,\hbar) = \exp\left(\frac{i\hbar}{2}\omega^{ij}\frac{\partial}{\partial y^i}\frac{\partial}{\partial z^j}\right)a(x,y,\hbar)b(x,z,\hbar)|_{y=z}.$$

We denote the algebra of the sections by $W := C^\infty(M, W)$ to simplify the notation. The center of W is $C^\infty(M)[[\hbar]]$ (i.e. without y).

There is a filtration $W \supset W_1 \supset W_2 \supset \cdots$ with respect to the total degree $2k+l$, i.e. $W_{2k+l} = \{$ formal series starts at degree $2k+l\}$. We also need W-valued differential forms:

$$a = \sum \hbar^k a_{k;i_1,\cdots,i_p;j_1,\cdots,j^q}(x) y^{i_1}\cdots y^{i_l} dx^{j_1}\wedge\cdots\wedge dx^{j_q}$$

where coefficients are covariant tensor fields symmetric with respect to $i_1, \cdots, i_p$ and anti-symmetric with respect to $j_1, \cdots, j_q$. The W-valued differential forms constitute an algebra $W \otimes \Lambda = \bigoplus_{q=0}^{2n}(W \otimes \Lambda^q)$ in which the multiplication is defined by Weyl product of polynomials in y^i and exterior product of differential forms dx^i (dx^i commutes with y^i). This product will be also denoted by $\circ$.

For $a \in W \otimes \Lambda^{q_1}, b \in W \otimes \Lambda^{q_2}$, define $[a, b] = a \circ b - (-1)^{q_1 q_2} b \circ a$. The center of $W \otimes \Lambda$ is $C^\infty(M)[[\hbar]] \otimes \Lambda$ (i.e. without y).

Define two projections of the form $a = a(x, y, dx, \hbar)$ onto the center: $a_0 = a(x, 0, dx, \hbar)$ and $a_{00} = a(x, 0, 0, \hbar)$. In the particular case $a = a(x, y, \hbar) \in W$, we use the notation $\sigma(a)$ for $a_0 = a(x, 0, \hbar)$ and call $\sigma(a)$ the symbol of the section a.

Define two important operators:

$$\delta a = dx^k \wedge \frac{\partial a}{\partial y^k}, \quad \delta^{-1} a_{pq} = \begin{cases} \frac{1}{p+q}\sum_k y^k \iota(\frac{\partial}{\partial_{x^k}}) a_{pq} & \text{if } p+q>0 \\ 0 & p+q=0 \end{cases}$$

where $a_{pq} \in (W_p - W_{p+1}) \otimes \Lambda^q$. Then

$$\delta^2 = 0, \quad (\delta^{-1})^2 = 0, \quad (\delta\delta^{-1} + \delta^{-1}\delta)a = a - a_{00}$$

and

$$\delta a = \frac{i}{\hbar}\left[\omega_{ij} y^i dx^j, a\right].$$

Definition 3. A *symplectic connection* on (M, ω) is a connection ∇ on TM which is torsion-free and satisfies $\nabla\omega = 0$. Such connection always exists and not unique (they form an affine space).

Choose a symplectic connection ∇. It induces a connection on Weyl bundle $\partial : W \otimes \Lambda^q \to W \otimes \Lambda^{q+1}$. In Darboux coordinates, the connection ∂ can be written in the form

$$\partial a = da + \frac{i}{\hbar}\left[\bar{\Gamma}, a\right]$$

where $\bar{\Gamma}$ is defined from Christoffel symbols of ∇ by

$$\bar{\Gamma} = \frac{1}{2}\omega_{ki}\Gamma^k_{jr} y^i y^j dx^r.$$

The curvature of ∂ is then given by

$$\partial^2 a = -\frac{i}{\hbar}\left[R, a\right]$$

with $R = \frac{1}{4}R_{ijkl}y^i y^j dx^k \wedge dx^l$ (R_{ijkl} is the curvature tensor of ∇). Our goal is to modify this connection ∂ so that the curvature becomes zero, i.e. curvature form (?) is given by central element.

Definition 4. A connection D on the Weyl bundle is called *Abelian* if for any section $a \in W \otimes \Lambda$,

$$D^2 a = -\frac{i}{\hbar}\left[\Omega, a\right] = 0.$$

Let

$$D_r := -\delta + \partial + \frac{i}{\hbar}\left[r, a\right]$$

for $r \in W_3 \otimes \Lambda^1$ with $r_0 = 0$. Then the curvature of D_r^2 is given by

$$D_r^2 a = -\frac{i}{\hbar}\left[-\omega + R - \delta r + \partial r + \frac{i}{\hbar}r^2, a\right].$$

Theorem 1 ([Fed94, Fed96])**.** The equation

$$\delta r = R + \partial r + \frac{i}{\hbar}r^2 + \Omega$$

for a given series

$$\Omega = \sum_{k\geq 1}\hbar^k\omega_k$$

where the ω_k are closed 2-forms on M, has a unique solution $r \in W_3 \otimes \Lambda^1$ satisfying the normalization condition $\delta^{-1}r = 0$.

Then $D := D_r$ is Abelian with curvature given by

$$D^2 a = -\frac{i}{\hbar}\left[-\omega - \Omega, a\right].$$

Since D acts as a derivation of $\circ$, the space of flat sections W_D is a subalgebra of W:

$$W_D = \{a \in W : Da = 0\}.$$

Theorem 2 ([Fed94, Fed96])**.** For any $a_0 \in C^\infty(M)[[\hbar]]$, there is a unique section $a \in W_D$ such that $\sigma(a) = a_0$.

By this theorem the map σ gives one-to-one correspondence between W_D and $C^\infty(M)[[\hbar]]$. Denote the inverse map by Q and call it the *quantization procedure*. The Weyl product $\circ$ on W_D can be transported to $C^\infty(M)[[\hbar]]$ yielding star product called *Fedosov's star product*:

$$a \star_{\nabla,\Omega} b = \sigma(Q(a) \circ Q(b)).$$

Fedosov's construction produces every possible star product on M up to equivalence.

Definition 5. The *characteristic class* of a star product $\star_{\nabla,\Omega}$ is the class

$$\mathrm{cl}(\star_{\nabla,\Omega}) = [\omega] + [\Omega] \in H^2(M)[[\hbar]].$$

This class is the same as *Deligne's characteristic class* up to normalization [Neu02].

The map from equivalence classes of star products on (M, ω) to the affine space $[\omega] + \hbar H^2(M)[[\hbar]]$ mapping $[\star]$ to $\mathrm{cl}(\star)$ is a bijection (see [GR99]).

4 Trace and the index theorem

Definition 6. A $\mathbb{C}[[\hbar]]$-linear functional on $C^\infty(M)[[\hbar]]$ is called a *trace* if it vanishes on star commutators, i.e.

$$\mathrm{Tr}(a \star b) = \mathrm{Tr}(b \star a).$$

A trace of a star product always exists and unique up to a constant factor. Moreover, there exists an unique (up to constant) *trace density* $\rho \in C^\infty(M)[[\hbar]]$ satisfying

$$\mathrm{Tr}(a) = \frac{1}{\hbar^n}\int_M a\rho\frac{\omega^n}{n!} = \frac{1}{\hbar^n}\sum_{k\geq 0}\hbar^k\int_M a\rho_k\frac{\omega^n}{n!}.$$

Normalize it so that $\rho_0 \equiv 1$. Star product is called *closed* if trace density is constant $\rho \equiv 1$ [CFS92].

We have an index theorem for deformation quantization, called *algebraic index theorem* (we only state a very special case of it).

Theorem 3 (Algebriac index theorem [Fed96, NT95])**.** Consider $1 \in C^\infty(M)$. Then

$$\mathrm{Tr}(1) = \int_M \exp\left(\frac{[\omega+\Omega]}{\hbar}\right)\hat{A}(M)$$

where $\omega + \Omega$ is a curvature of D_r.

5 Kähler case : Karabegov's construction

Let (M, ω) be a Kähler manifold of complex dimension n. Note that the following construction makes sense when ω is only pseudo-Kähler. This will be important later.

Definition 7. A star product on the Kähler manifold M is called a star product of *separation of variables* type or *anti-Wick* type if, for any open subset $U \subset M$ and functions $a, b, f \in C^\infty(U)$ such that a is holomorphic and b antiholomorphic,

$$a \star f = af, \quad f \star b = fb$$

holds. That is, in the coefficients $C_k, k \geq 1$, in the first argument only derivatives in antiholomorphic and in the second argument only derivatives in holomorphic directions appear.

There is a corresponding notion of *Wick* type, which should be obvious (change the role of holo. and antiholo.). However, there is some discrepancy in the definitions of the Wick and anti-Wick types in the literature.

Definition 8. A *formal deformation* of the Kähler form $\omega = \omega_0$ is a formal series

$$\tilde{\omega} = \omega_0 + \hbar\omega_1 + \hbar^2\omega_2 + \cdots$$

where each ω_k is a closed 2-form on M of type $(1,1)$.

On a contractible coordinate chart U, there exists a formal series

$$\Phi = \Phi_0 + \hbar\Phi_1 + \hbar^2\Phi_2 + \cdots \in C^\infty(M)[[\hbar]]$$

which is a potential of the formal deformation of the Kähler form $\tilde{\omega} = \omega + \hbar\omega_1 + \hbar^2\omega_2 + \cdots$, i.e. $\omega_k = i\partial\bar{\partial}\Phi_k$ for each $k \geq 0$.

Karabegov established the bijection between every star products of separation of variables type and a formal deformations of the Kähler form [Kar96]. Note that this is not only up to equivalence. We omit Karabegov's original construction and present only the resulting statement.

Theorem 4 ([Kar96])**.** Deformation quantization with separation of variables on a Kähler manifold M are in $1-1$ correspondence with the formal deformations of the Kähler form ω_0 on M. The correspondence is as follows.

1. $\star \to \tilde{\omega}$: Given a deformation quantization with separation of variables $\star$, on each chart U there exists a set of formal functions $u^1, \cdots, u^n$ on U such that

$$u^k \star z^l - z^l \star u^k = \hbar\delta^{kl}.$$

Then the formal 2-form $\tilde{\omega}$ is defined by

$$\tilde{\omega} = -i\bar{\partial}\left(\sum_k u^k dz^k\right).$$

It is well-defined, and is a deformation of ω_0.

2. $\star \leftarrow \tilde{\omega}$: Given $\tilde{\omega}$, let Φ be a potential of $\tilde{\omega}$ on U. Then the operators of left $\star$-multiplication on U are characterized by the property that they commute with multiplication by antiholomorphic functions and with the operators $R_{\partial\Phi/\partial\bar{z}^l} = \partial\Phi/\partial\bar{z}^l + \hbar\partial/\partial\bar{z}^l$.

Definition 9. The *Karabegov form* of the star product of separation of variables type $\star$ is the formal 2-form

$$\mathrm{Kar}(\star) = \tilde{\omega}$$

defined above.

As expected, Karabegov form and characteristic class of star product are closely related.

Theorem 5 ([Kar98a, Neu03])**.** For a star product $\star$ of separation of variables type,

$$\mathrm{cl}(\star) = [\mathrm{Kar}(\star)] + \hbar\frac{\mathrm{c}_1(M)}{2}. \tag{5.1}$$

Karabegov's formalism also makes possible to compute the normalized trace density [Kar98b]. Let $(U, \{z_j\}_j)$ be a contractible coordinate chart. Write the trace density on U as

$$\rho\frac{\omega^n}{n!} = e^P idz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge idz^n \wedge d\bar{z}^n.$$

where

$$P = P_0 + \hbar P_1 + \cdots.$$

Then the forms $i\partial\bar{\partial}P$ on each coordinate charts patch into a global formal $(1,1)$-form ω_{tr} on M. On M with the same complex structure, $-\omega_0$ is also a pseudo-Kähler form.

Definition 10. The *dual* of the deformation quantization with separation of variables $\star$ with $\mathrm{Kar}(\star) = \tilde{\omega}$ is a deformation quantization with separation of variables $\tilde{\star}$ on the pseudo-Kähler manifold $(M, -\omega_0)$ with Karabegov form $-\tilde{\omega} + \hbar\omega_{tr}$.

Let Φ be a local potential of $\mathrm{Kar}(\star)$ and Ψ a local potential of $\mathrm{Kar}(\tilde{\star})$. Then, by construction, $\hbar P = \Phi + \Psi$ up to a constant, which can be fixed by normalization. The point of considering the dual is that we can directly identify it without a priori knowing ω_{tr}. Given $\star$, there exists a unique formal differential operator $J : C^\infty(M)[[\hbar]] \to C^\infty(M)[[\hbar]]$ satisfying

$$J(b \star a) = ab$$

for every local holomorphic function a and antiholomorphic function b. Such J is antihomomorphism of algebras from $(C^\infty(M)[[\hbar]], \star)$ to $(C^\infty(M)[[\hbar]], \tilde{\star})$. The inverse operator $I = J^{-1}$ is called the *formal Berezin transform*. I can be directly computed from $\star$ and vice-versa. From the knowledge of the dual, the trace density can then be determined.

Finally, let us mention that a Fedosov type construction can also be adapted to the Kähler case and is capable of reproducing the basic results of Karabegov; see [Neu03].

6 Berezin-Toeplitz deformation quantization

Suppose (M, ω) is a polarized Kähler manifold with (L, h).

Definition 11. For $f \in C^\infty(M)$, the k^{th} *Toeplitz operator* $T_f^{(k)}$ is defined by

$$T_f^{(k)} := \Pi(f\cdot) : H^0(M, L^k) \to H^0(M, L^k).$$

Schlichenmaier proved that these Toeplitz operators define a deformation quantization (of Wick type). See also [MM11].

Theorem 6 ([Sch00])**.** There exists a unique star product

$$f \star g = \sum_{j=0}^{\infty} \hbar^j C_j(f, g)$$

in such a way that for f, g, we have

$$T_f^{(k)} T_g^{(k)} = \sum_{j=0}^{\infty} k^{-j} T_{C_j(f,g)}^{(k)}$$

as $k \to \infty$. We call this star product the *Berezin-Toeplitz deformation quantization* and denote it by $\star_{BT}$.

Its characteristic class is computed first by Hawkins [Haw00]:

$$\mathrm{cl}(\star_{BT}) = [\omega] + \hbar \frac{\mathrm{c}_1(M)}{2}. \tag{6.1}$$

Definition 12. Given star product $\star$, the *opposite* star product $\star^{op}$ is defined by

$$f \star^{op} g := g \star f$$

(it is a star product with respect to the minus of the original symplectic form). Then the characteristic class of $\star^{op}$ is given by

$$\mathrm{cl}(\star^{op}) = -\mathrm{cl}(\star).$$

Since $\star_{BT}$ is of Wick type, its opposite star product $\star_{BT}^{op}$ is of anti-Wick type. Karabegov and Schlichenmaier computed the Karabegov form of $\star_{BT}^{op}$ [KS01]:

$$\mathrm{Kar}(\star_{BT}^{op}) = -\omega - \hbar \mathrm{Ric} \tag{6.2}$$

where Ric denotes the Ricci form $i\mathrm{Ric}_{j\bar{k}} dz^j \wedge d\bar{z}^k$. From this we can recover (6.1) since by (5.1),

$$\mathrm{cl}(\star_{BT}) = -\mathrm{cl}(\star_{BT}^{op}) = -\left(-[\omega] - \hbar\, \mathrm{c}_1(M) + \hbar \frac{\mathrm{c}_1(M)}{2}\right) = [\omega] + \hbar \frac{\mathrm{c}_1(M)}{2}.$$

[KS01, p 70] also identifies formal Berezin transform I of $\star_{BT}^{op}$ and $\tilde{\star}_{BT}^{op}$, the dual of $\star_{BT}^{op}$. We call $\star_B := \tilde{\star}_{BT}^{op}$ the *Berezin star product*. Then

$$\mathrm{Kar}(\star_B) = \omega + i\hbar\partial\bar{\partial}\log B(x) \tag{6.3}$$

where $B(x)$ denotes the formal function obtained from TYZ expansion of the Bergman kernel. That is, for an asymptotic expansion of the form

$$\nu(k) = \sum_{j\geq 0} k^{n-j}\nu_j,$$

formalizer $\mathbb{F}$ gives

$$\mathbb{F}(\nu(k)) = \sum_{j\geq 0} \hbar^j \nu_j \in C^\infty(M)[[\hbar]].$$

Then $B(x)$ is given by $B(x) := \mathbb{F}(B_k(x)) = \sum_{j\geq 0}\hbar^j a_j(x)$ from (1.1).

Now we have three star products at hand, $\star_{BT}, \star_{BT}^{op}$ and $\star_B = \tilde{\star}_{BT}^{op}$, related to each other by

$$\star_{BT} \overset{\text{opposite}}{\longleftrightarrow} \star_{BT}^{op} \overset{\text{dual}}{\longleftrightarrow} \star_B.$$

When three star products are related in this way, they share the same trace density (obvious for the opposite product, and for the dual by construction). Let us compute the trace density of $\star_B$. By (6.2) and (6.3),

$$\rho_B\frac{\omega^n}{n!} = e^{P_B} i dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge i dz^n \wedge d\bar{z}^n$$

with

$$P_B = \frac{1}{\hbar}(\Phi_{\star_{BT}^{op}} + \Psi_{\star_B}) = \log B(x) + \log\det(\omega_{j\bar{k}}).$$

Hence

$$\rho_B\frac{\omega^n}{n!} = B(x)\frac{\omega^n}{n!} \implies \rho_B = B(x).$$

That is, the trace density of the Berezin-Toeplitz deformation quantization is given by the TYZ expansion of the Bergman kernel. This is, of course, expected, since the actual trace of a Toeplitz operator is given by the integral of its symbol multiplied by the Bergman kernel (see [Sch00, Proposition 5.5]). We take this indirect route since all of this can be done when there is no line bundle L and there is no actual Toeplitz operators. This was done by Hao Xu; see the next section.

7 Hao Xu's combinatorial approach

In [Xu12], Xu obtained the closed formula for a_j using the graph theoretic method. Using this, he then recovered the results of [KS01] by purely combinatorial way.

Theorem 7 ([Xu13, Theorem 5.1])**.** On (M,ω), define the star product $\star_B$ of anti-Wick type by the Karabegov form

$$\omega + i\hbar\partial\bar{\partial}\log B(x)$$

with $B(x) := \sum_{j\geq 0}\hbar^j a_j(x)$. Then the Karabegov form of its dual is given by $-\omega - \hbar\mathrm{Ric}$.

Thus we can proceed as before and obtain the trace density of $\star_B$ as $\rho_B = B(x)$. Define $\star_{BT}$ to be the opposite of the dual of $\star_B$ (as before, but now we don't have Toeplitz operators). The characteristic class of $\star_{BT}$ is the same as before. Finally, by the algebraic index theorem (Theorem 3),

$$\begin{aligned}
&\sum_{j\geq 0}\hbar^{j-n}\int_M a_j(x)\frac{\omega^n}{n!}\\
&= \frac{1}{\hbar^n}\int_M B(x)\frac{\omega^n}{n!}\\
&= \mathrm{Tr}_{\star_{BT}}(1)\\
&= \int_M \exp\left(\frac{[\omega] + \hbar\,\mathrm{c}_1(M)/2}{\hbar}\right)\hat{A}(M)\\
&= \int_M \exp\left(\frac{[\omega]}{\hbar}\right)\mathrm{Td}(M)\\
&= \sum_{j\geq 0}\hbar^{j-n}\int_M \mathrm{Td}_j(M)\frac{\omega^{n-j}}{(n-j)!}.
\end{aligned}$$

As a result, we obtain our goal:

$$\boxed{\int_M a_j(x)\frac{\omega^n}{n!} = \int_M \mathrm{Td}_j(M)\frac{\omega^{n-j}}{(n-j)!}}$$

for all $j \geq 0$.

8 Further question

In [Eum26], we showed that if $[\omega]$ is integral, the $1-$forms defined on the space of Kähler potentials $\mathcal{K}_\omega$ defined by

$$\gamma_\varphi^{(j)}(\psi) := \int_M \psi(\Delta_\varphi a_{j-1}(\omega_\varphi) - a_j(\omega_\varphi))\frac{\omega_\varphi^n}{n!} \tag{8.1}$$

for $\psi \in T_\varphi\mathcal{K}_\omega = C^\infty(M,\mathbb{R})$, are closed. Since $\mathcal{K}_\omega$ is simply connected, they can be integrated, and in particular for $j > n+1$, $\gamma^{(j)}$ are integrated to exact cocycles (see [Eum26, Theorem 3.1]). In view of Question 1, we propose the following:

Question 2. Does it remain true that $\gamma^{(j)}$ defined as (8.1) are closed on $\mathcal{K}_\omega$ and integrated to exact cocycles for $j > n+1$, when (M, ω) is *not* polarized?

We hope that this question can also be answered by the theory of deformation quantization.

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