Determinant line bundle and Quillen metric

Kiyoon Eum

KAIST

GPS

Introduction

- Suppose for some reason you want to define the determinant of the Cauchy–Riemann operator $\bar{\partial}$ on a Riemann surface, depending holomorphically on the complex structure.
- Soon you will realize that it makes not much sense.
- First, determinant is defined on linear operators T from V to itself. $\bar{\partial}$ maps $\Omega^{0,0}(E)$ to $\Omega^{0,1}(E)$.
- Determinant of T is a product of its eigenvalues. If dimension is infinite, it may diverge. In particular, determinant is zero unless it is invertible. $\bar{\partial}$ is not in most cases.
- In this talk, I will explain Quillen's answer to this questions.

Cauchy-Riemann operator

- Let $E \to M$ complex vector bundle over Riemann surfsace M. Let $\Omega^{p,q}(E)$ be the vector space of E-valued (p,q) forms.
- By a *Cauchy-Riemann* or $\bar{\partial}$-operator on E, we mean a differential operator $D : \Omega^{0,0}(E) \to \Omega^{0,1}(E)$ of the form $D = d\bar{z}(\partial_{\bar{z}} + \alpha(z))$ with smooth matrix function $\alpha(z)$.
- Such operators are in one-to-one correspondence with holomorphic structures on the vector bundle E.
- Denote by $\mathcal{A}$ the space of these operators. It is an affine space relative to the complex vector space $\mathcal{B} = \Omega^{0,1}(EndE)$.

Finite dimensional case

- Consider the family of operators $T : V^0 \to V^1$, where V^0 and V^1 are vector spaces of the same finite dimensions.
- Each T induces a map from $\lambda(V^0)$ to $\lambda(V^1)$, where $\lambda(V) := \wedge^{dimV} V$ denotes the highest exterior power of V.
- Hence T determines an element σ_T of the line $\mathcal{L} = \lambda(V^0)^* \otimes \lambda(V^1)$, where the asterisk denotes the dual vector space.
- Upon choosing a generator for this line, σ_T can be identified with a function $\det(T)$, which is holomorphic in T and nonzero exactly when T is invertible.

Infinite dimensional case: an overview

- Now consider $D : \Omega^{0,0}(E) \to \Omega^{0,1}(E)$.
- We cannot take $\lambda(\Omega^{0,0}(E))^* \otimes \lambda(\Omega^{0,1}(E))$: they are infinite dimensional.
- Note that in finite dimension, we can decompose as $V^0 = KerT \oplus W$, $V^1 = CokerT \oplus TW$ so that

$$\mathcal{L} = \lambda(V^0)^* \otimes \lambda(V^1) \simeq \lambda(KerT)^* \otimes \lambda(CokerT).$$

- In the case of D, we know that the kernel and cokernel of D are finite dimensional.
- Thus the line $\mathcal{L}$ should be replaced by the line $\mathcal{L}_D = \lambda(KerD)^* \otimes \lambda(CokerD)$, which now depends on D.

- The family of $\mathcal{L}_D$ forms a holomorphic line bundle $\mathcal{L}$ over $\mathcal{A}$, called the *determinant line bundle*.
- The analogue of the assumption that V^0 and V^1 have the same dimension is $Index(D) = dimKerD - dimCokerD = 0$, which is a topological condition on E, M.
- In this case there is a canonical holomorphic section σ of $\mathcal{L}$ sucht that $\sigma_D \neq 0$ iff D is invertible.
- Then σ can be identified with a holomorphic function $\det(D)$ on $\mathcal{A}$.

Construction of $\mathcal{L}$

- Let $\mathcal{F}$ be the space of Fredholm operators T from Hilbert spaces H^0 to H^1.
- As an open subset of $\mathcal{B}(H^0, H^1)$, $\mathcal{F}$ is a complex Bananch manifold.
- We will construct holomorphic line bundle $\mathcal{L}$ over $\mathcal{F}$ with fiber $\mathcal{L}_T = \lambda(KerT)^* \otimes \lambda(CokerT)$
- Then taking $H^0 = L^2(\Omega^{0,0}(E)), H^1 = W^{1,2}(\Omega^{0,1}(E))$, we have a linear map $\mathcal{A} \to \mathcal{F}, D \mapsto T = D$.
- We define $\mathcal{L}$ over $\mathcal{A}$ by pull-back.

- For any finite-dimensional subspace F of H^1, let U_F be the set of T which satisfies $ImT + F = H^1(CokerT \subset F)$.

- Claim) For each $T \in U_F$, there is a canonical isomorphism

$$\mathcal{L}_T = \lambda(KerT)^* \otimes \lambda(CokerT) \simeq \lambda(T^{-1}F)^* \otimes \lambda(F).$$

- Proof) For each $T \in U_F$, there is an exact sequence

$$0 \to KerT \xrightarrow{\text{id}} T^{-1}F \xrightarrow{\text{T}} F \xrightarrow{\text{proj}} CokerT \to 0.$$

- The set U_F is open, and the family of subspaces $T^{-1}F$ form a holomorphic vector bundle over U_F.

- This defines $\mathcal{L}$ over $\mathcal{F}$ as a holomorphic line bundle.

Canonical section

- Over the connected component of $\mathcal{F}$ consisting of operators of index zero, define a section σ by setting $\sigma_T = 0$ if T is not invertible, $\sigma_T = 1$ under the canonical isomorphism $\mathcal{L}_T = \mathbb{C}$ if T is invertible.
- It is unintuitive that this is even a smooth section. This is because we define $\lambda(0) := \mathbb{C}$, the canonical $\mathbb{C}$.
- Unlike the finite dimensional case, $\mathcal{L}$ is now line *bundle*. So we need to trivialize it.
- After trivialization, σ will be given by holomorphic function on $\mathcal{A}$.
- This is done by defining a flat connection on $\mathcal{L}$ and choosing a trivialization of $\mathcal{L}$ by everywhere flat section ($\mathcal{A}$ is contractible).

Quillen metric

- On holomorphic vector bundle, hermitian metric and holomorphic structure uniquely determine a connection, called Chern connection.
- Goal) define a smooth hermitian metric on $\mathcal{L}$.
- Recall $\mathcal{L}_D = \lambda(KerD)^* \otimes \lambda(CokerD) = \lambda(KerD)^* \otimes \lambda(kerD^*)$
- Suppose we are given hermitian metrics on E, M. They induce hermitian metric on $\Omega^{0,\bullet}(E)$ by L^2 inner product.
- This in turn induces inner products on $KerD, KerD^*$.

- Problem: this is not smooth. Here is why.
- When we define a smooth structure on $\mathcal{L}$, we implicitly cancel out some vectors with possibly different norm. i.e. $v^* \otimes Tv \simeq 1$.
- More concretely, for $a > 0$, let F_a^0 (resp. F_a^1) be the sum of the eigenspaces of the operator D^*D (resp. DD^*) acting on $\Omega^{0,\bullet}(E)$ for eigenvalue $< a$.
- One has a canonical isomorphism $\mathcal{L}_D = \lambda(F_a^0)^* \otimes \lambda(F_a^1)$.
- Under this isomorphism, λ-eigenvector s with $0 < \lambda < a$ is canceled out by Ds, with same eigenvalue.
- For such s, $\lambda|s|^2 = |Ds|^2$.

- To absorb this discontinuity, define a new metric $|| \, ||$ by multiplying L^2 metric with all of positive eigenvalues of D^*D:

$$|| \cdot ||_Q^2 = | \cdot |_{L^2}^2 \prod_{0<\lambda<\infty} \lambda.$$

- If this makes sense, it will be locally described by $|| \cdot ||_Q^2 = | \cdot |_{L^2}^2 \prod_{a<\lambda<\infty} \lambda$ on $\lambda(F_a^0)^* \otimes \lambda(F_a^1)$, which is smooth.
- Of course, this product diverges. But we don't need it to be actually product. We only need it be kind of like product: depends smoothly on λ and

$$\lambda_1 \lambda_2 \lambda_3 \cdots = (\lambda_1 \lambda_2 \cdots \lambda_N) \cdot (\lambda_{N+1} \lambda_{N+2} \cdots).$$

- This can be achieved by zeta function regularization.

- Let $\Lambda : 0 < \lambda_1 \leq \lambda_1 \leq \cdots$. Define zeta function attached to it to be

$$\zeta_\Lambda(s) := \sum_{n\geq 1} \lambda_n^{-s}.$$

- If it converges for $Re(s) \gg 0$, has a meromorphic continuation to $\mathbb{C}$ and has no pole at $s = 0$, define

$$\prod_{n=1}^{\infty} \lambda_n := e^{-\zeta_\Lambda'(0)}.$$

- Then it satisfies the desired properties (eigenvalues of D^*D satisfies those conditions).
- $||\cdot||_Q^2 := |\cdot|_{L^2}^2 e^{-\zeta'_{Spec\Delta}(0)}$ is called the *Quillen metric*.

Quillen curvature formula

- Now we have a smooth metric $||\cdot||_Q$ on $\mathcal{L}$. Associated Chern connection is not flat, but we know its curvature exactly (!).
- Recall $\mathcal{B} = \Omega^{0,1}(EndE)$. Given $B \in \mathcal{B}$, write it as $B = \alpha(z)d\bar{z}$ wrt a local orthonormal frame of E. Let $B^+ = \alpha(z)^* dz \in \Omega^{1,0}(EndE)$.
- Then $\mathrm{Tr}_E(B^+B) \in \Omega^{1,1}(M)$. Define inner product on $\mathcal{B}$ by

$$\frac{i}{2\pi}\int_M tr_E(B^+B).$$

- It determines a Kähler form on the affine space $\mathcal{A}$. That is, fix a base point D_0. Then for $||\cdot||$ denoting the norm on $\mathcal{B}$ induced by that inner product, $i\partial\bar{\partial}q, q(D) := ||D - D_0||$ is a Kähler form on $\mathcal{A}$.

- This Kähler form is exactly the curvature of the determinant line bundle.

Quillen curvature formula [Quillen '85]

The Chern curvature of the Quillen metric $||\cdot||_Q$ is $\partial\bar{\partial}q$.

- Define new metric by $||\cdot||_q^2 = ||\cdot||_Q^2 \times e^q$. Then the associated Chern connection is flat.
- Thus finally, we obtain our goal:

Corollary [Quillen '85]

Given a basepoint D_0, there exists a holomorphic function $\det(D; D_0)$ on $\mathcal{A}$, which is unique up to a scalar of absolute value one, such that

$$||\sigma_D||_q^2 = e^{-\zeta'_{D^*D}(0)} e^{||D-D_0||} = |\det(D; D_0)|^2.$$

Proof of the Quillen curvature formula

- Given D, let ∇ be a Chern connection on E w.r.t. D.
- That is, if $D = d\bar{z}(\partial_{\bar{z}} + \alpha)$ w.r.t. orthonormal frame of E,

$$\nabla = dz(\partial_z - \alpha^*) + d\bar{z}(\partial_{\bar{z}} + \alpha).$$

- Let $F(z, z') : E'_z \to E_z$ be the parallel transport w.r.t. ∇ along the geodesic. Then near diagonal,

$$F(z, z') = e^{(z-z')\alpha^* - (\bar{z}-\bar{z}')\alpha} = 1 + (z - z')\alpha^* - (\bar{z} - \bar{z}')\alpha + \cdots .$$

- Let $r(z, z')$ be the geodesic distance, and write the metric $ds^2 = \rho(z)|dz|^2$.

- Suppose D is invertible. Denote by $G(z,z') = \langle z|D^{-1}|z'\rangle$ the Schwartz kernel of $D^{-1} : \Omega^{0,1}(E) \to \Omega^{0,0}(E)$.
- We want to compute the finite part of G along the diagonal.
- Let a parametrix $G_0(z,z')$ by

$$G_0(z,z') = \frac{i}{2\pi}[dz\partial_z \log r^2(z,z')]F(z,z').$$

- Define the finite part of G along the diagonal by

$$J(z') = \lim_{z\to z'} G(z,z') - G_0(z,z') \in \Omega^{1,0}(EndE).$$

- Here we used duality $(\Lambda^{0,1})^* \simeq \Lambda^{1,0}, 1 \simeq \frac{i}{2}dz \wedge d\bar{z}$.

- Near diagonal,

$$G(z,z') = \frac{i}{2\pi}\frac{dz'}{z-z'}(1 + (z-z')\beta(z') + (\bar{z}-\bar{z}')\gamma(z') + \cdots).$$

- Since $D_z G(z,z') = (\partial_{\bar{z}} + \alpha)G(z,z') = \delta_{z=z'}$, γ must be $-\alpha$.
- Also, since $D \mapsto D^{-1}$ is holomorphic, β depends holomorphically on D.
- Using $r^2 \sim \sqrt{\rho(z)\rho(z')}|z-z'|^2$, we obtain

$$J(z') = \frac{idz'}{2\pi}(\beta - \alpha^* - \frac{1}{2}\log\rho). \tag{0.1}$$

Claim

$$\lim_{t\to 0}\langle z|e^{-t\Delta}G|z\rangle = J(z)$$

uniformly in z. Consequently, for any $B \in \mathcal{B}$,

$$\lim_{t\to 0}\mathrm{Tr}(e^{-t\Delta}D^{-1}B) = \int_M tr(JB).$$

Proof) Use $G = G - G_0 + G_0$ and

$$\lim_{t\to 0}\langle z|e^{-t\Delta}G_0|z\rangle = 0.$$

This follows from asymptotic expansion of the heat kernel

$$e^{-t\Delta}(z,z') \sim \frac{1}{4\pi t}e^{-r(z,z')^2/4t}(F + ta_1 + \cdots).$$

Derivative of the Ray-Singer torsion

- WLOG, assume $Index(D) = 0$ by adding E a vector bundle of the opposite index ($\because$ $Index(D_E \oplus D_F) = Index(D_E) + Index(D_F)$).
- Since invertible operators are dense in $Fred_0$ we only need to consider the holomorphic one-parameter family of invertible $\bar{\partial}$-operators $w \mapsto D_w$.
- The curvature is then computed in terms of canonical section by

$$\bar{\partial}\partial \log ||\sigma||_Q^2 = dwd\bar{w}\, \partial^2_{\bar{w}w}\zeta'(0).$$

- Recall that $\zeta(s) = \mathrm{Tr}(\Delta^{-s}), \Delta = D^*D$.

- First,

$$-\partial_w \zeta(s) = s\,\mathrm{Tr}[\Delta^{-s-1}\partial_w \Delta] = s\,\mathrm{Tr}[\Delta^{-s} D^{-1}\partial_w D]$$

since $\partial_w D^* = (\partial_{\bar{w}} D)^* = 0$.

- Thus using Mellin transform,

$$-\partial_w \zeta(s) = \frac{s}{\Gamma(s)} \int_0^\infty t^{s-1}\,\mathrm{Tr}[e^{-t\Delta} D^{-1}\partial_w D] dt.$$

- Since

$$\mathrm{Tr}[e^{-t\Delta} D^{-1}\partial_w D] = \int_M tr(J\partial_w D) + O(t^\varepsilon)$$

as $t \to 0$ and decay exponentially as $t \to \infty$, we get

- $$-\partial_w \zeta(s) = \frac{s}{\Gamma(s)} \left[\int_0^1 + \int_1^\infty \right] = s \left[\int_M tr(J\partial_w D) + O(s) \right]$$
  as $s \to 0$ ($\because \Gamma(s) = s^{-1} - \gamma + O(s)$).
- Hence
  $$-\partial_w \zeta'(0) = \int_M tr(J\partial_w D).$$
- Now using (0.1), we can compute
  $$\partial^2_{\bar{w}w} \zeta'(0) = \frac{i}{2\pi} \int_M tr(\partial_{\bar{w}} \alpha^* \partial_w D) = \frac{i}{2\pi} \int_M tr(\partial_w D)^+ \partial_w D.$$
- This completes the proof. $\square$

References

- *Determinants of Cauchy-Riemann operators over a Riemann surface* by D. Quillen
- *Lectures on Arakelov geometry* by C. Soulé et al.
- *Analytic torsion and holomorphic determinant bundles I-III* by Bismut-Gillet-Soulé.

Thank you!