Connections Between Applicative and Concatenative Tacit Programming

Hashimoto, Daniel Kiyoshi

November 2025 – TalTech, Tallinn, Estonia

About me

I am finishing my Master's Degree

- Universidade Federal do Rio de Janeiro, Brazil
- Bachelor's in Computer Science
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I'm looking for a PhD program

- concatenative languages are similar to string diagrams
- string diagrams for first-order logic

Where I keep my publications and programming projects

- `github.com/Kiyoshi364/static-memory`

My Research Interests

Interesting research topics (probably non-exhaustive):

- alternative programming paradigms and alternative computing models
- theorem proving and proof assistants
- static analysis, type systems, logic systems
- creating and using models

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I like research that *roughly* follows these steps:

1. point at two things;
2. provide a definition of “equality”;
3. claim: “those two things are equal”

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I like research that *roughly* follows these steps:

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(Often, the research starts in a 1-3-2 order)

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Programming without named variables
Succinct programs with emphasis on data flow

We highlight two styles

1.

2.

Tacit (Point-free) Programming

Programming without named variables
Succinct programs with emphasis on data flow

We highlight two styles

1. Applicative Style (higher-order functions)

$$h := \lambda x \,.\, f\,(g\,x) \qquad\qquad h := f \circ g \tag{3}$$

2.

Tacit (Point-free) Programming

Programming without named variables
Succinct programs with emphasis on data flow

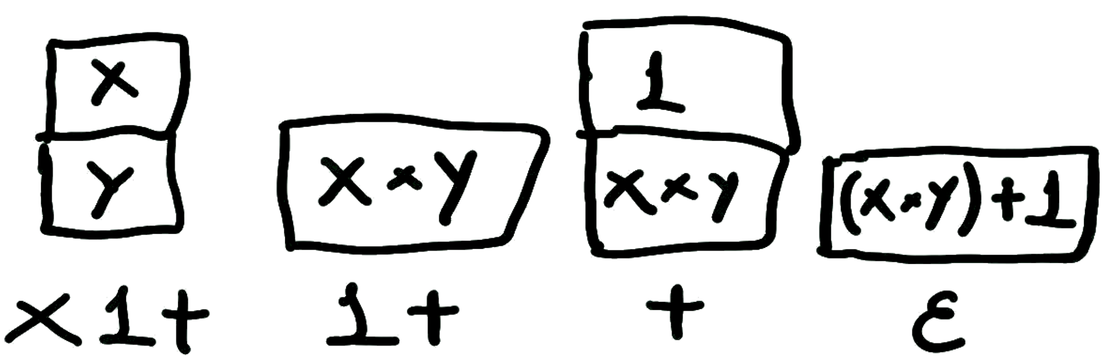
We highlight two styles

1. Applicative Style (higher-order functions)

$$h := \lambda x\,.\,f\;(g\;x) \qquad\qquad h := f \circ g \tag{5}$$

2. Compositive Style (stack-based programming)

$$h := \lambda x\;y\,.(x \times y) + 1 \qquad\qquad h := \times\;1\;+ \tag{6}$$



Tacit (Point-free) Programming

Programming without named variables
Succinct programs with emphasis on data flow

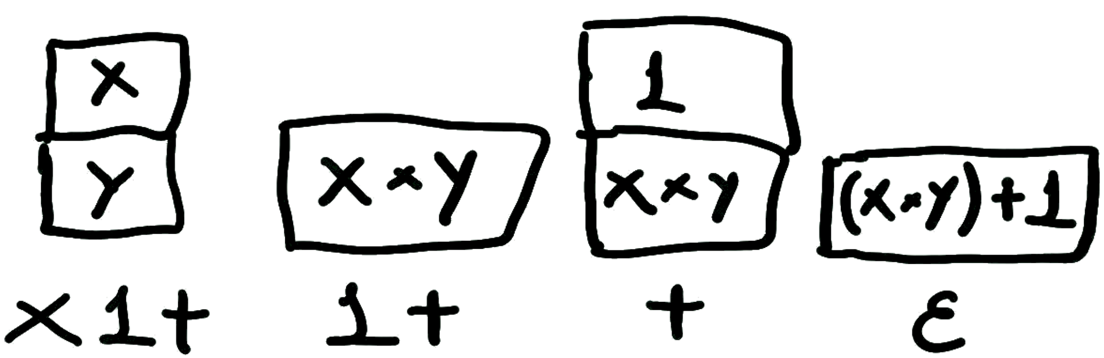
We highlight two styles

1. Applicative Style (higher-order functions)

$$h := \lambda x \,.\, f\,(g\,x) \qquad\qquad h := f \circ g \tag{7}$$

2. Compositive Style (stack-based programming)

$$h := \lambda x\,y\,.(x \times y) + 1 \qquad\qquad h := \times\; 1\; + \tag{8}$$



In both cases, there were no x nor y

Intuition for “Equality”

Example for BCK and $\mathsf{swap}\ \mathsf{zap}$

$$B\,C\,K\,q\,x\,y \qquad \trianglerighteq \qquad y\,x \mid \mathsf{swap}\ \mathsf{zap} \tag{9}$$

Example for BCK and $\mathsf{swap}\;\mathsf{zap}$

$$\begin{aligned} B\,C\,K\,q\,x\,y &\quad\trianglerighteq\quad y\,x \mid \mathsf{swap}\;\mathsf{zap} \\ C\,(K\,q)\,x\,y &\quad\trianglerighteq\quad y\,x \mid \mathsf{swap}\;\mathsf{zap} \end{aligned} \tag{10}$$

Intuition for "Equality"

Example for BCK and `swap zap`

$$\begin{aligned} B\,C\,K\,q\,x\,y &\quad\trianglerighteq\quad y\,x \mid \mathsf{swap}\;\mathsf{zap} \\ C\,(K\,q)\,x\,y &\quad\trianglerighteq\quad y\,x \mid \mathsf{swap}\;\mathsf{zap} \\ K\,q\,y\,x &\quad\trianglerighteq\quad x\,y \mid \mathsf{zap} \end{aligned} \tag{11}$$

Intuition for "Equality"

Example for BCK and `swap zap`

$$\begin{aligned} B\,C\,K\,q\,x\,y &\quad\trianglerighteq\quad y\,x \mid \mathsf{swap}\ \mathsf{zap} \\ C\,(K\,q)\,x\,y &\quad\trianglerighteq\quad y\,x \mid \mathsf{swap}\ \mathsf{zap} \\ K\,q\,y\,x &\quad\trianglerighteq\quad x\,y \mid \mathsf{zap} \\ q\,x &\quad\trianglerighteq\quad x \mid \varepsilon \end{aligned} \tag{12}$$

The programs start related
If I step on one size, the other side is also related after some steps
(This is called (Bi)Simulation, I'll introduce it latter)

Intuition for “Equality”

Example for BCK and `swap zap`

$$\begin{array}{rcl} B\,C\,K\,q\,x\,y & \triangleright & y\,x \mid \texttt{swap zap} \\ C\,(K\,q)\,x\,y & \triangleright & y\,x \mid \texttt{swap zap} \\ K\,q\,y\,x & \triangleright & x\,y \mid \texttt{zap} \\ q\,x & \triangleright & x \mid \varepsilon \end{array} \tag{13}$$

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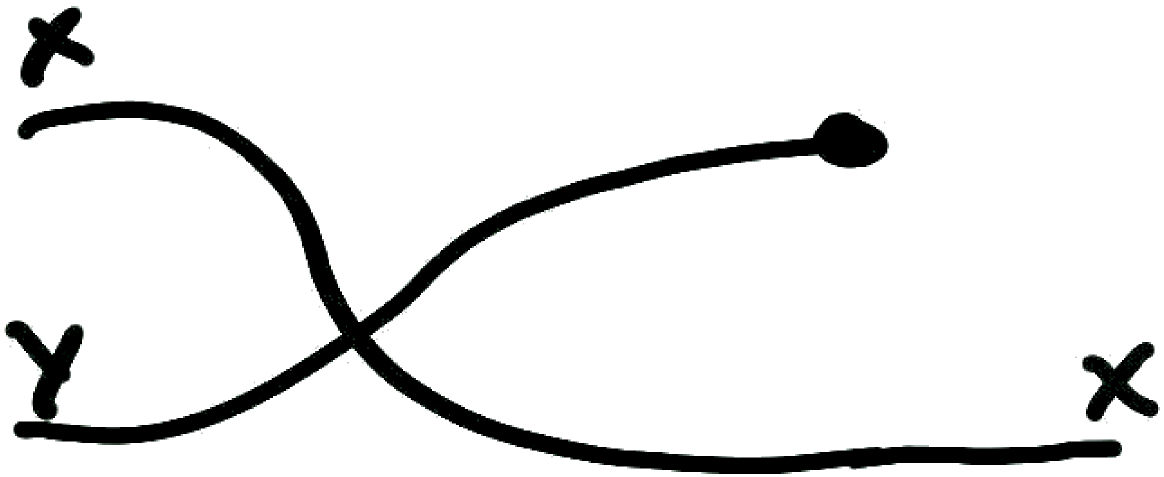

Intuition for "Equality"

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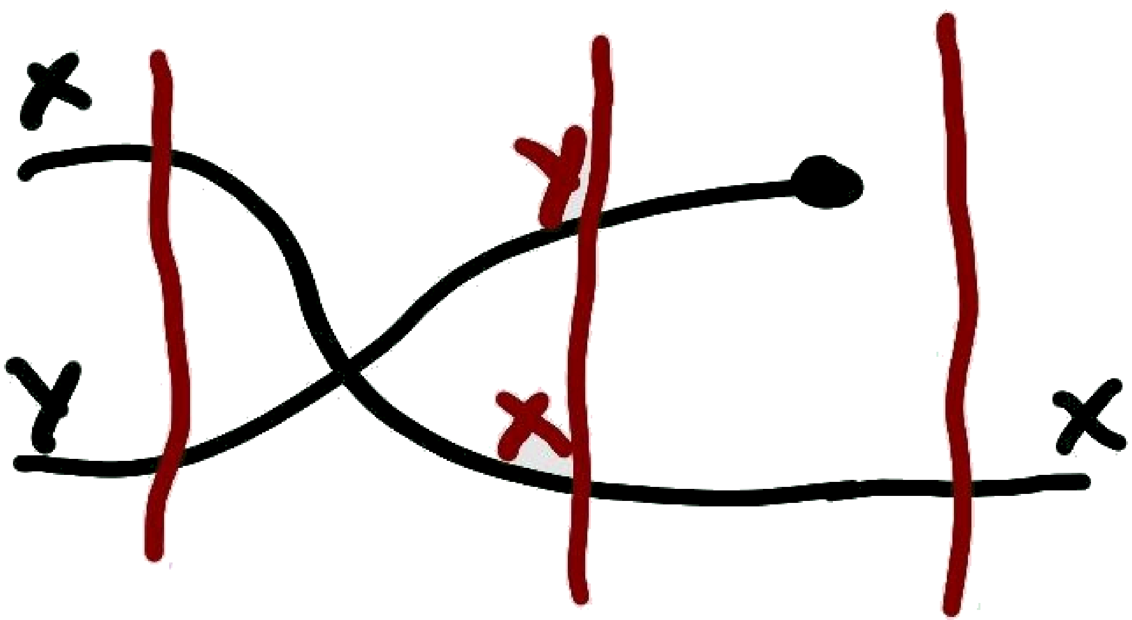

A model for higher-order stack languages Thun (1994), Kerby (2002), Kleffner (2017)

$\mathrm{Cat} = \mathrm{list}(\mathrm{Instr} + \mathrm{Value})$

A model for higher-order stack languages Thun (1994), Kerby (2002), Kleffner (2017)
$\text{Cat} = \text{list}(\text{Instr} + \text{Value})$

Each instruction solves a problem

$$\begin{array}{ll} \text{Effect} & \text{Instructions} \\ \text{Permute} & y\,x\,\texttt{swap} \mapsto x\,y \\ \text{Discard} & x\,\texttt{zap} \mapsto \varepsilon \\ \text{Duplicate} & x\,\texttt{dup} \mapsto x\,x \\ \text{Dequote} & \texttt{[}\rho\texttt{]}\,\texttt{call} \mapsto \rho \\ \text{Quotation Creation} & x\,\texttt{unit} \mapsto \texttt{[}x\texttt{]} \\ \text{Quotation Composition} & \texttt{[}\pi\texttt{]}\,\texttt{[}\rho\texttt{]}\,\texttt{cat} \mapsto \texttt{[}\,\pi\,\rho\,\texttt{]} \\ \text{Stash} & x\,\texttt{[}\rho\texttt{]}\,\texttt{dip} \mapsto \rho\,x \end{array} \tag{17}$$

Recursive/Structural rule

$$\frac{\sigma, \pi : \text{Cat} \qquad \rho_0 \mapsto \rho}{\sigma\,\rho_0\,\pi \mapsto \sigma\,\rho\,\pi} \tag{18}$$

Quotations and Higher-Order Instructions

A quotation is an anonymous function (lambda)
We can put quotations on the stack

Higher-order instructions use and/or create quotations

`call` executes a quotation

$$x\ \texttt{[dup]}\ \texttt{call} \mapsto x\ \texttt{dup} \mapsto x\ x \tag{19}$$

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$$x\ \texttt{[dup]}\ \texttt{call} \mapsto x\ \texttt{dup} \mapsto x\ x \tag{22}$$

`unit` creates a quotation

$$x\ \texttt{unit} \mapsto \texttt{[}x\texttt{]} \tag{23}$$

A quotation is an anonymous function (lambda)
We can put quotations on the stack

Higher-order instructions use and/or create quotations

`call` executes a quotation

$$x\ \texttt{[dup]}\ \texttt{call} \mapsto x\ \texttt{dup} \mapsto x\ x \tag{25}$$

`unit` creates a quotation

$$x\ \texttt{unit} \mapsto \texttt{[}x\texttt{]} \tag{26}$$

`cat` concatenates/composes quotations

$$\texttt{[zap]}\ \texttt{[}\rho\texttt{]}\ \texttt{cat} \mapsto \texttt{[zap}\ \rho\ \texttt{]} \tag{27}$$

> Syntactical concatenation is semantical composition!

Stack Rotations vs The dip Instruction

The standard way is to use rotations for stack shuffling

$$z\;y\;x\;\texttt{dig3}\;\texttt{swap} \mapsto y\;x\;z\;\texttt{swap} \mapsto y\;z\;x \tag{28}$$

The standard way is to use rotations for stack shuffling

$$z\;y\;x\;\texttt{dig3}\;\texttt{swap} \mapsto y\;x\;z\;\texttt{swap} \mapsto y\;z\;x \tag{30}$$

Instead, we use `dip`

`dip` runs a program one level deeper

$$z\;y\;x\;\texttt{[swap]}\;\texttt{dip} \mapsto z\;y\;\texttt{swap}\;x \mapsto y\;z\;x \tag{31}$$

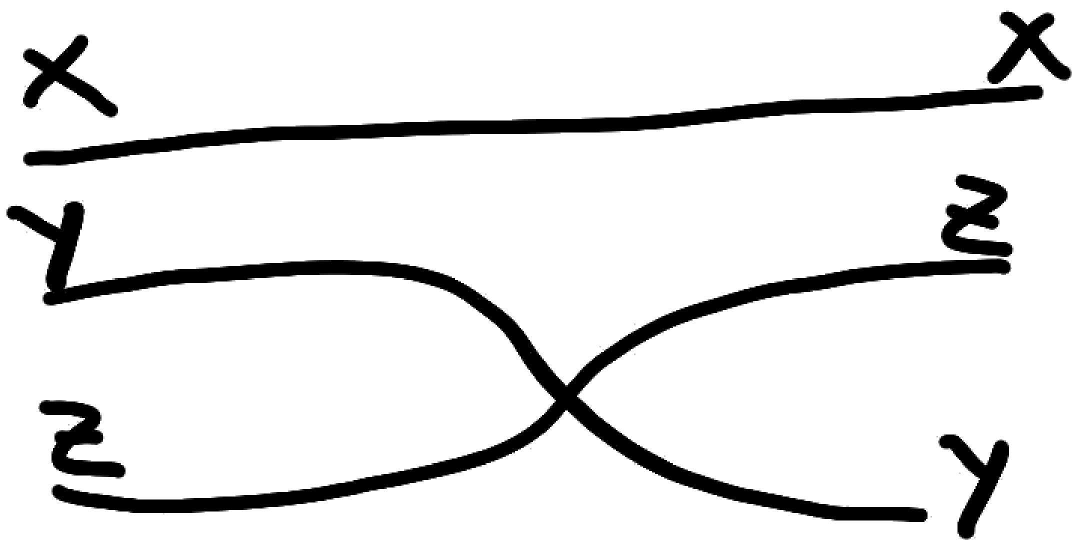
The standard way is to use rotations for stack shuffling

$$z\;y\;x\;\texttt{dig3}\;\texttt{swap} \mapsto y\;x\;z\;\texttt{swap} \mapsto y\;z\;x \tag{32}$$

Instead, we use `dip`
`dip` runs a program one level deeper

$$z\;y\;x\;\texttt{[swap]}\;\texttt{dip} \mapsto z\;y\;\texttt{swap}\;x \mapsto y\;z\;x \tag{33}$$

In the diagrams, `dip` is equivalent to putting an identity on top of the program



First-Order Concatenative Calculus

A model for a stack languages which cannot put functions on the stack

$$y\, x\, \texttt{swap} \mapsto x\, y \qquad\qquad x\, \texttt{zap} \mapsto \varepsilon$$

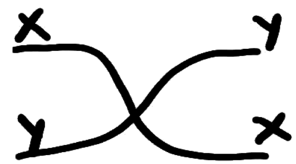
$$x\, \texttt{dup} \mapsto x\, x \qquad\qquad \pi\, \rho$$

$$x\, (\texttt{[}\rho\texttt{]}\, \texttt{dip}) \mapsto \rho\, x \qquad\qquad \varepsilon$$

First-Order Concatenative Calculus and String Diagrams

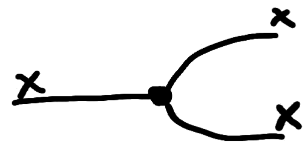
A model for a stack languages which cannot put functions on the stack

$y\; x\; \texttt{swap} \;\mapsto\; x\; y$

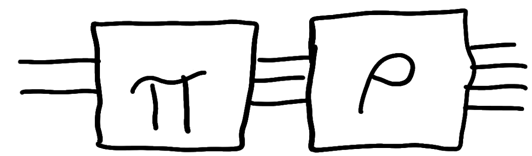


$x\; \texttt{zap} \;\mapsto\; \varepsilon$

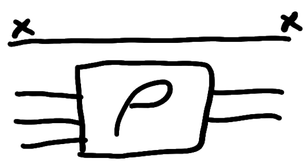
$x\; \texttt{dup} \;\mapsto\; x\; x$



$\pi\; \rho$



$x\; (\texttt{[}\rho\texttt{]}\; \texttt{dip}) \;\mapsto\; \rho\; x$

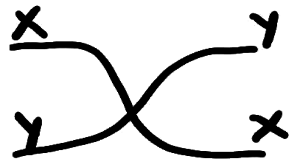


ε

Given a first-order program, I can draw a string diagram
(related to Cartesian Monoidal Categories?)

A model for a stack languages which cannot put functions on the stack

$y\,x\,\texttt{swap} \;\mapsto\; x\,y$



$x\,\texttt{zap} \;\mapsto\; \varepsilon$

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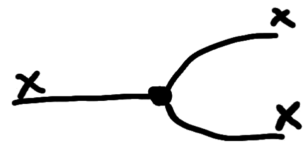
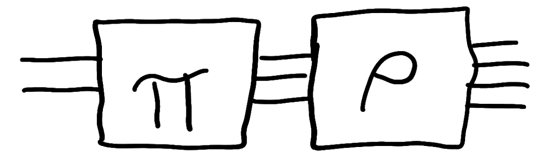
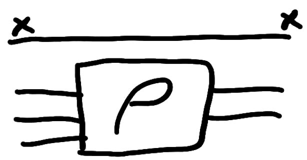


$\pi\,\rho$



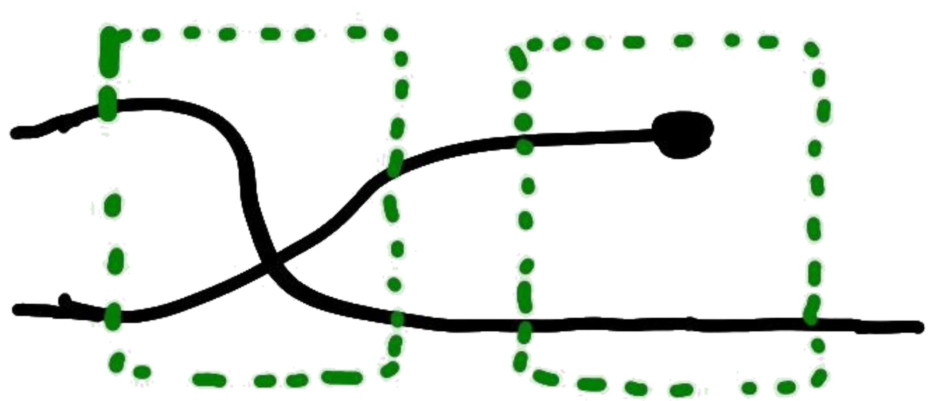
$x\,(\texttt{[}\rho\texttt{]}\,\texttt{dip}) \;\mapsto\; \rho\,x$



ε

Given a first-order program, I can draw a string diagram
(related to Cartesian Monoidal Categories?)

If the arities don't match, one can put extra identities on the bottom



How to Draw (Higher-Order) Concatenative Calculus?

vvv This slide is from before I arrived here vvv

I don't know how to draw an arbitrary Concatenative Program

A quotation (which is in a wire) should become a diagram

For instance:

$$\texttt{swap}\,\texttt{call} \tag{34}$$

Any ideas? (Perhaps, after more higher-order examples)

^^^ This slide is from before I arrived here ^^^

(related to Cartesian Closed Monoidal Category?)

Model for applicative functional languages
"lambda calculus without variables"

Each combinator does an effect

$$
\begin{array}{lcl}
\text{Effect} & & \text{Combinators} \\
\\
\text{Permute} & \mathrm{C} & q\,x\,y \longrightarrow q\,y\,x \\
\text{Discard} & \mathrm{K} & q\,x \longrightarrow q \\
\text{Duplicate} & \mathrm{W} & q\,x \longrightarrow q\,x\,x \\
\text{Identity} & \mathrm{I} & q \longrightarrow q \\
\text{Composition*} & \mathrm{B} & q\,x\,y \longrightarrow q\,(x\,y)
\end{array}
\tag{35}
$$

Created by Schönfinkel (1924), expanded and popularized by Curry (1930)

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\end{array}
\tag{36}
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Smullyan (1985) provided bird names for combinators

Model for applicative functional languages
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Each combinator does an effect

$$\begin{array}{lrllll} \text{Effect} & & \text{Combinators} & & & \\ & & & & & \\ \text{Permute} & \mathrm{C} & q\,x\,y & \longrightarrow & q\,y\,x \\ \text{Discard} & \mathrm{K} & q\,x & \longrightarrow & q \\ \text{Duplicate} & \mathrm{W} & q\,x & \longrightarrow & q\,x\,x \\ \text{Identity} & \mathrm{I} & q & \longrightarrow & q \\ \text{Composition*} & \mathrm{B} & q\,x\,y & \longrightarrow & q\,(x\,y) \end{array} \tag{37}$$

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Smullyan (1985) provided bird names for combinators

Regular Combinators

Regular Combinators keep the first argument q in place, without extra copies
q is a **continuation** (what is the program doing next)
It can reorganize the other arguments in any way

All basic combinators are regular

$$\begin{array}{llll} \mathrm{C}\; q\,x\,y \longrightarrow q\,y\,x & & \mathrm{K}\; q\,x \longrightarrow q \\ \mathrm{W}\; q\,x \longrightarrow q\,x\,x & & \mathrm{I}\; q \longrightarrow q \\ \mathrm{B}\; q\,x\,y \longrightarrow q\,(x\,y) & & \end{array} \tag{38}$$

Regular Combinators

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All basic combinators are regular

$$\begin{array}{llll} \mathrm{C} & q\,x\,y & \longrightarrow & q\,y\,x \\ \mathrm{W} & q\,x & \longrightarrow & q\,x\,x \\ \mathrm{B} & q\,x\,y & \longrightarrow & q\,(x\,y) \end{array} \qquad \begin{array}{llll} \mathrm{K} & q\,x & \longrightarrow & q \\ \mathrm{I} & q & \longrightarrow & q \end{array} \tag{41}$$

BCK is regular

$$B\,C\,K\,q\,x\,y \longrightarrow C\,(K\,q)\,x\,y \longrightarrow K\,q\,y\,x \longrightarrow q\,x \tag{42}$$

Both, CI and WC, are irregular

$$\begin{gathered} C\,I\,q\,x \longrightarrow I\,x\,q \longrightarrow x\,q \\ W\,C\,q\,x \longrightarrow C\,q\,q\,x \longrightarrow q\,x\,q \end{gathered} \tag{43}$$

Regular Combinators by Construction

We can construct regular combinators with these rules:

$$
\begin{aligned}
\mathrm{C}\; q\; x\; y &\longrightarrow q\; y\; x \\
\mathrm{K}\; q\; x &\longrightarrow q \\
\mathrm{W}\; q\; x &\longrightarrow q\; x\; x \\
\mathrm{I}\; q &\longrightarrow q \\
\\
\mathrm{B}\; q\; x\; y &\longrightarrow q\; (x\; y) \\
(B\; \alpha)\; q\; x &\longrightarrow \alpha\; (q\; x) \\
(B\; \alpha\; \beta)\; q &\longrightarrow \alpha\; (\beta\; q) \\
\\
(C\; \alpha\; x)\; q &\longrightarrow \alpha\; q\; x
\end{aligned}
\tag{44}
$$

B also stashes arguments and sequences regular combinators!

C also introduces new values!

We can construct regular combinators with these rules:

$$
\begin{array}{rcl@{\qquad\qquad}rcl}
\mathrm{C}\; q\, x\, y & \longrightarrow & q\, y\, x & y\, x\, \texttt{swap} & \mapsto & x\, y \\
\mathrm{K}\; q\, x & \longrightarrow & q & x\, \texttt{zap} & \mapsto & \varepsilon \\
\mathrm{W}\; q\, x & \longrightarrow & q\, x\, x & x\, \texttt{dup} & \mapsto & x\, x \\
\mathrm{I}\; q & \longrightarrow & q & & \varepsilon & \\
\\
\mathrm{B}\; q\, x\, y & \longrightarrow & q\,(x\, y) & & \times & \\
(B\, \alpha)\; q\, x & \longrightarrow & \alpha\,(q\, x) & x\,(\texttt{[}\rho\texttt{]}\; \texttt{dip}) & \mapsto & \rho\, x \\
(B\, \alpha\, \beta)\; q & \longrightarrow & \alpha\,(\beta\, q) & & \rho\, \pi & \\
\\
(C\, \alpha\, x)\; q & \longrightarrow & \alpha\, q\, x & & x\, \rho &
\end{array}
\tag{45}
$$

Besides B with 0 arguments,
regular combinators map to first-order concatenative

Shuffle combinators keep q in place, without extra copies
and can only shuffle the other arguments (cannot add parenthesis)

$$
\begin{array}{rcl@{\qquad\qquad}rcl}
\mathrm{C}\, q\, x\, y & \longrightarrow & q\, y\, x & y\, x\, \texttt{swap} & \mapsto & x\, y \\
\mathrm{K}\, q\, x & \longrightarrow & q & x\, \texttt{zap} & \mapsto & \varepsilon \\
\mathrm{W}\, q\, x & \longrightarrow & q\, x\, x & x\, \texttt{dup} & \mapsto & x\, x \\
\mathrm{I}\, q & \longrightarrow & q & & \varepsilon & \\
\\
(B\, \alpha)\, q\, x & \longrightarrow & \alpha\, (q\, x) & x\, (\texttt{[}\rho\texttt{]}\, \texttt{dip}) & \mapsto & \rho\, x \\
(B\, \alpha\, \beta)\, q & \longrightarrow & \alpha\, (\beta\, q) & & \rho\, \pi & \\
\\
(C\, \alpha\, x)\, q & \longrightarrow & \alpha\, q\, x & & x\, \rho &
\end{array}
\tag{46}
$$

Shuffle combinators keep q in place, without extra copies
and can only shuffle the other arguments (cannot add parenthesis)

$$\begin{array}{rcl}
\mathrm{C}\,q\,x\,y & \longrightarrow & q\,y\,x \\
\mathrm{K}\,q\,x & \longrightarrow & q \\
\mathrm{W}\,q\,x & \longrightarrow & q\,x\,x \\
\mathrm{I}\,q & \longrightarrow & q \\
\\
(B\,\alpha)\,q\,x & \longrightarrow & \alpha\,(q\,x) \\
(B\,\alpha\,\beta)\,q & \longrightarrow & \alpha\,(\beta\,q) \\
\\
(C\,\alpha\,x)\,q & \longrightarrow & \alpha\,q\,x
\end{array}
\qquad
\begin{array}{rcl}
y\,x\,\texttt{swap} & \mapsto & x\,y \\
x\,\texttt{zap} & \mapsto & \varepsilon \\
x\,\texttt{dup} & \mapsto & x\,x \\
 & \varepsilon & \\
\\
x\,(\texttt{[}\rho\texttt{]}\,\texttt{dip}) & \mapsto & \rho\,x \\
 & \rho\,\pi & \\
\\
 & x\,\rho &
\end{array}
\tag{47}$$

This title is imprecise! We need 2 relations:

1. $(\blacktriangleright)$: ShuffleComb $\times$ FOCat
2. $(\triangleright)$: ShuffleComb&q $\times$ FOCat

1. $(\blacktriangleright) : \text{ShuffleComb} \times \text{FOCat}$

$$C \blacktriangleright \texttt{swap} \qquad K \blacktriangleright \texttt{zap} \qquad W \blacktriangleright \texttt{dup} \qquad \overset{\text{R-EMPTY}}{I \blacktriangleright \varepsilon}$$

$$\frac{\alpha \blacktriangleright \rho}{B\,\alpha \blacktriangleright \texttt{[}\rho\texttt{]}\ \texttt{dip}} \qquad \overset{\text{R-CONCAT}}{\frac{\alpha \blacktriangleright \rho \quad \beta \blacktriangleright \pi}{B\,\alpha\,\beta \blacktriangleright \rho\,\pi}} \qquad \overset{\text{R-PUSH}}{\frac{\alpha \blacktriangleright \rho \quad x_0 \sim x_1}{C\,\alpha\,x_0 \blacktriangleright x_1\,\rho}} \tag{48}$$

2. $(\mathrel{\rhd\!\!\rhd}) : \text{ShuffleComb}\&q \times \text{FOCat}$

1. $(\blacktriangleright)$: ShuffleComb $\times$ FOCat

$$C \blacktriangleright \texttt{swap} \qquad K \blacktriangleright \texttt{zap} \qquad W \blacktriangleright \texttt{dup} \qquad \overset{\text{R-EMPTY}}{I \blacktriangleright \varepsilon}$$

$$\frac{\alpha \blacktriangleright \rho}{B\,\alpha \blacktriangleright \texttt{[}\rho\texttt{]}\ \texttt{dip}} \qquad \overset{\text{R-CONCAT}}{\frac{\alpha \blacktriangleright \rho \quad \beta \blacktriangleright \pi}{B\,\alpha\,\beta \blacktriangleright \rho\,\pi}} \qquad \overset{\text{R-PUSH}}{\frac{\alpha \blacktriangleright \rho \quad x_0 \sim x_1}{C\,\alpha\,x_0 \blacktriangleright x_1\,\rho}} \tag{50}$$

2. $(\triangleright)$: ShuffleComb&q $\times$ FOCat

$$\frac{\alpha \blacktriangleright \rho}{\alpha\,q \triangleright \rho}$$

$$\overset{\text{S-EMPTY}}{q \triangleright \varepsilon} \qquad \overset{\text{S-CONCAT}}{\frac{\hat{\alpha} \triangleright \rho \quad \hat{\beta} \triangleright \pi}{\hat{\alpha}\left\{\hat{\beta}/q\right\} \triangleright \rho\,\pi}} \qquad \overset{\text{S-PUSH}}{\frac{\hat{\alpha} \triangleright \rho \quad x_0 \sim x_1}{\hat{\alpha}\,x_0 \triangleright x_1\,\rho}} \tag{51}$$

- $\hat{\alpha}$ means a combinator with q inside
- $\hat{\alpha}\left\{\hat{\beta}/q\right\}$ substitutes $\hat{\beta}$ for q in $\hat{\alpha}$

Simulation Definition

A relation $(\trianglerighteq)$: $\mathrm{ShuffleComb}\&q \times \mathrm{FOCat}$ is a simulation for relations $(\longrightarrow)$: $\mathrm{ShuffleComb}\&q \times \mathrm{ShuffleComb}\&q$ and $(\mapsto)$: $\mathrm{FOCat} \times \mathrm{FOCat}$ iff

$$\begin{gathered}\forall(\hat{\alpha} : \mathrm{ShuffleComb}\&q)(\rho, \pi : \mathrm{FOCat}),\\ (\hat{\alpha} \trianglerighteq \rho) \wedge (\rho \mapsto^{*} \pi) \Rightarrow \exists\left(\hat{\beta} : \mathrm{ShuffleComb}\&q\right), \left(\hat{\alpha} \longrightarrow^{*} \hat{\beta}\right) \wedge \left(\hat{\beta} \trianglerighteq \pi\right)\end{gathered} \tag{52}$$

A relation $(\vartriangleright) : \text{ShuffleComb}\&q \times \text{FOCat}$ is a simulation for relations $(\longrightarrow) : \text{ShuffleComb}\&q \times \text{ShuffleComb}\&q$ and $(\mapsto) : \text{FOCat} \times \text{FOCat}$ iff

$$\begin{gathered} \forall(\hat{\alpha} : \text{ShuffleComb}\&q)(\rho, \pi : \text{FOCat}), \\ (\hat{\alpha} \vartriangleright \rho) \wedge (\rho \mapsto^{*} \pi) \Rightarrow \exists\left(\hat{\beta} : \text{ShuffleComb}\&q\right), \left(\hat{\alpha} \longrightarrow^{*} \hat{\beta}\right) \wedge \left(\hat{\beta} \vartriangleright \pi\right) \end{gathered} \tag{55}$$

or, with relational composition

$$(\vartriangleright) \ ; \ (\mapsto^{*}) \subseteq (\longrightarrow^{*}) \ ; \ (\vartriangleright) \tag{56}$$

or, with a diagram (unboxed are assumptions, boxed are results)

$$\begin{array}{ccc} \hat{\alpha} & \vartriangleright & \rho \\ \boxed{\downarrow_{*}} & & \mapsto^{*} \\ \boxed{\hat{\beta}} & \boxed{\vartriangleright} & \pi \end{array} \tag{57}$$

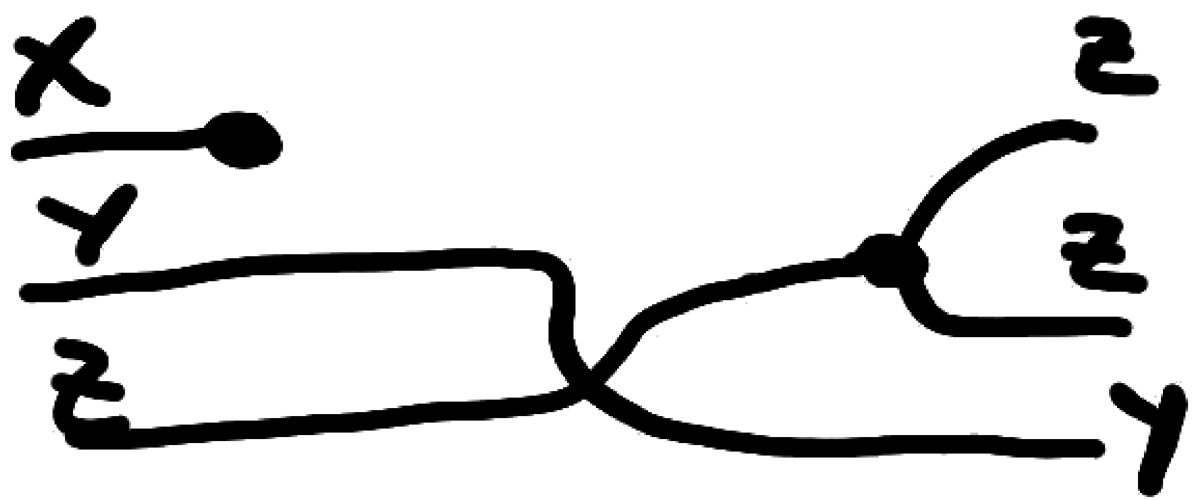
We may need some “bureaucratic steps” on the combinatory side

$$
\begin{array}{rcl}
B\,(B\,K\,C)\,W\,q\,x\,y\,z & \trianglerighteq & z\,y\,x\;\mathsf{zap}\;\mathsf{swap}\;\mathsf{dup} \\
B\,K\,C\,(W\,q)\,x\,y\,z & & \\
K\,(C\,(W\,q))\,x\,y\,z & & \\
C\,(W\,q)\,y\,z & \trianglerighteq & z\,y\;\mathsf{swap}\;\mathsf{dup} \\
W\,q\,z\,y & \trianglerighteq & y\,z\;\mathsf{dup} \\
q\,z\,z\,y & \trianglerighteq & y\,z\,z
\end{array}
\tag{58}
$$

Simulation Example

We may need some “bureaucratic steps” on the combinatory side

$$\begin{array}{rcl} B\,(B\,K\,C)\,W\,q\,x\,y\,z & \triangleright & z\;y\;x\;\mathsf{zap}\;\mathsf{swap}\;\mathsf{dup} \\ B\,K\,C\,(W\,q)\,x\,y\,z & & \\ K\,(C\,(W\,q))\,x\,y\,z & & \\ C\,(W\,q)\,y\,z & \triangleright & z\;y\;\mathsf{swap}\;\mathsf{dup} \\ W\,q\,z\,y & \triangleright & y\;z\;\mathsf{dup} \\ q\,z\,z\,y & \triangleright & y\;z\;z \end{array} \tag{59}$$



Simulation Definition (Again)

A relation $(\vartriangleright) : \text{ShuffleComb\&}q \times \text{FOCat}$ is a simulation for relations $(\longrightarrow) : \text{ShuffleComb\&}q \times \text{ShuffleComb\&}q$ and $(\mapsto) : \text{FOCat} \times \text{FOCat}$ iff

$$
\begin{gathered}
\forall(\hat{\alpha} : \text{ShuffleComb\&}q)(\rho, \pi : \text{FOCat}), \\
(\hat{\alpha} \vartriangleright \rho) \wedge (\rho \mapsto^* \pi) \Rightarrow \exists\left(\hat{\beta} : \text{ShuffleComb\&}q\right), \left(\hat{\alpha} \longrightarrow^* \hat{\beta}\right) \wedge \left(\hat{\beta} \vartriangleright \pi\right)
\end{gathered}
\tag{60}
$$

or, with relational composition

$$
(\vartriangleright) \; ; \; (\mapsto^*) \subseteq (\longrightarrow^*) \; ; \; (\vartriangleright) \tag{61}
$$

or, with a diagram (unboxed are assumptions, boxed are results)

$$
\begin{array}{ccc}
\hat{\alpha} & \vartriangleright & \rho \\
\boxed{\downarrow_*} & & \mathrel{\rotatebox[origin=c]{-90}{$\mapsto$}}_* \\
\boxed{\hat{\beta}} & \boxed{\vartriangleright} & \pi
\end{array}
\tag{62}
$$

“Co”simulation (Simulation of the Converse) Definition

A relation $(\rhd)$: $\mathsf{ShuffleComb}\&q \times \mathrm{FOCat}$ is a simulation for relations $(\longrightarrow)$: $\mathsf{ShuffleComb}\&q \times \mathsf{ShuffleComb}\&q$ and $(\mapsto)$: $\mathrm{FOCat} \times \mathrm{FOCat}$ iff

$$\begin{gathered}\forall\left(\hat{\alpha}, \hat{\beta} : \mathsf{ShuffleComb}\&q\right)(\rho : \mathrm{FOCat}),\\ (\hat{\alpha} \rhd \rho) \land \left(\hat{\alpha} \longrightarrow^{*} \hat{\beta}\right) \Rightarrow \exists(\pi : \mathsf{ShuffleComb}\&q), (\rho \mapsto^{*} \pi) \land \left(\hat{\beta} \rhd \pi\right)\end{gathered} \tag{63}$$

or, with relational composition

$$(\lhd) \;;\; (\longrightarrow^{*}) \subseteq (\mapsto^{*}) \;;\; (\lhd) \tag{64}$$

or, with a diagram (unboxed are assumptions, boxed are results)

$$\begin{array}{ccc} \hat{\alpha} & \rhd & \rho \\ \downarrow_{*} & & \boxed{\downarrow_{*}} \\ \hat{\beta} & \boxed{\rhd} & \boxed{\pi} \end{array} \tag{65}$$

A relation $(\trianglerighteq)$: $\mathrm{ShuffleComb}\&q \times \mathrm{FOCat}$ is a bisimulation for relations $(\longrightarrow)$: $\mathrm{ShuffleComb}\&q \times \mathrm{ShuffleComb}\&q$ and $(\mapsto)$: $\mathrm{FOCat} \times \mathrm{FOCat}$
iff
$(\trianglerighteq)$ is both a simulation and a cosimulation

I strongly believe that $(\trianglerighteq)$ is a bisimulation:

Bisimulation Definition

A relation $(\trianglerighteq)$: $\mathrm{ShuffleComb}\&q \times \mathrm{FOCat}$ is a bisimulation for relations $(\longrightarrow) : \mathrm{ShuffleComb}\&q \times \mathrm{ShuffleComb}\&q$ and $(\mapsto) : \mathrm{FOCat} \times \mathrm{FOCat}$
iff
$(\trianglerighteq)$ is both a simulation and a cosimulation

I strongly believe that $(\trianglerighteq)$ is a bisimulation:
- $(\trianglerighteq)$ is a simulation
  - proved in Rocq—a proof assistant, previously called Coq
- $(\trianglerighteq)$ is a cosimulation
  - strong believes that it is true (no proof yet)

We can extend (▶) and (⊳) to accomodate Higher-Order Concatenative Programs

$$\begin{aligned} C\,I\,q\,\alpha &\longrightarrow^* \alpha\,q \\ C\,B\,q\,\alpha\,x &\longrightarrow^* \alpha\,(q\,x) \\ C\,B\,(C\,I)\,q\,x &\longrightarrow^* q\,(C\,I\,x) \\ C\,(B\,B\,B)\,(C\,B)\,q\,\alpha\,\beta &\longrightarrow^* q\,(C\,B\,\alpha\,\beta) \end{aligned} \qquad \begin{aligned} \texttt{[}\rho\texttt{]}\,\texttt{call} &\mapsto \rho \\ x\,\texttt{[}\rho\texttt{]}\,\texttt{dip} &\mapsto \rho\,x \\ x\,\texttt{unit} &\mapsto \texttt{[}x\texttt{]} \\ \texttt{[}\pi\texttt{]}\,\texttt{[}\rho\texttt{]}\,\texttt{cat} &\mapsto \texttt{[}\,\pi\,\rho\,\texttt{]} \end{aligned} \tag{66}$$

The encoding for quotations that push values and compose quotations:

$$(C\,I\,x)\,q \longrightarrow I\,q\,x \longrightarrow q\,x \tag{67}$$

$$(C\,B\,\alpha\,\beta)\,q \longrightarrow B\,\beta\,\alpha\,q \longrightarrow \beta\,(\alpha\,q) \tag{68}$$

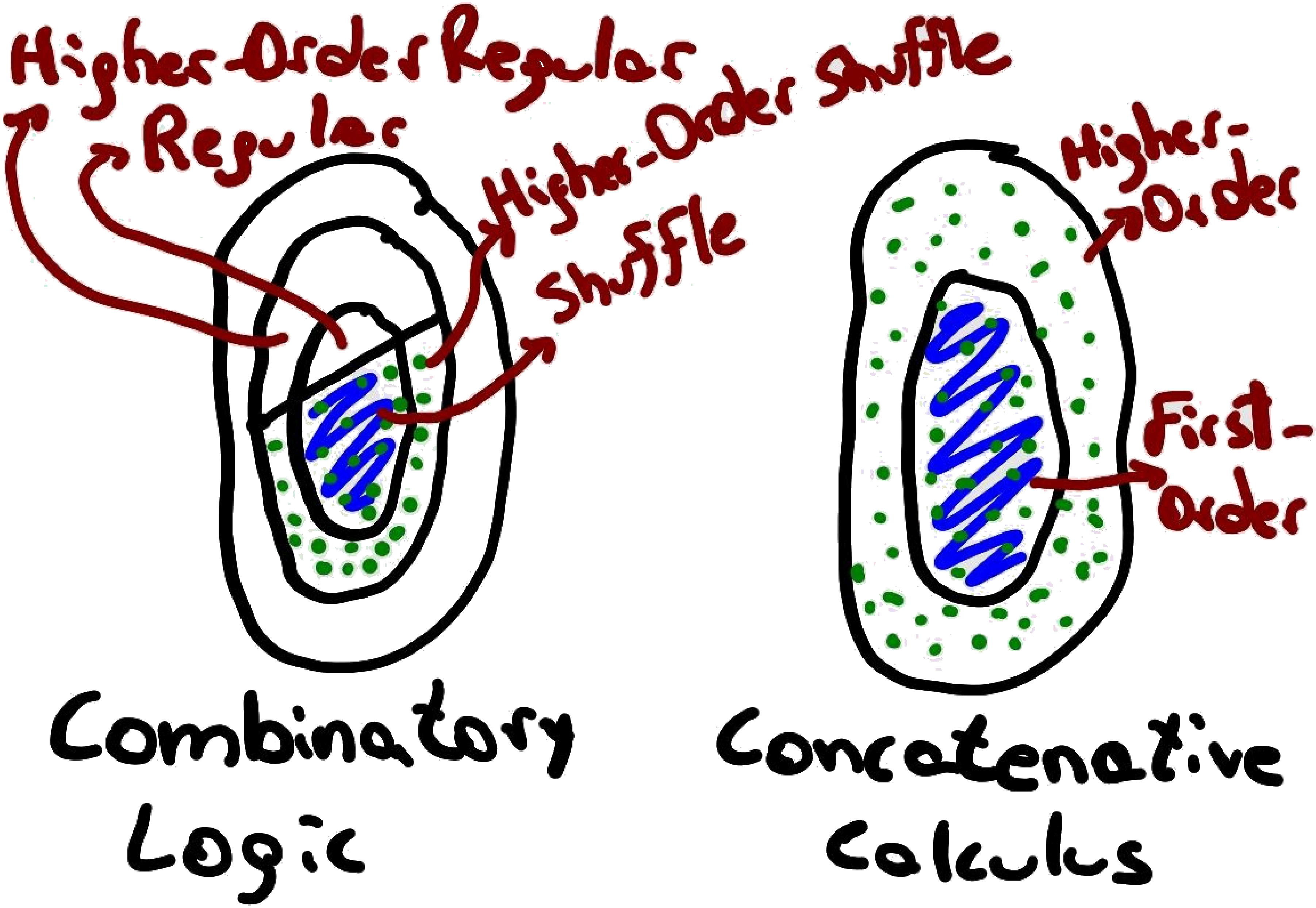
Regular and Shuffle Combinators must keep q at the start, without copies

Higher-order Regular and Shuffle Combinators give some flexibility to that

We consider arguments of a Higher-order Combinator, also a Higher-order Combinator

Thus, q must be "eventually" at the start

$$\begin{gathered} \alpha\,(q\,x)\,y\,z \longrightarrow^* q\,x\,z \\ \alpha_0\,(\alpha_1\,(\dots(\alpha_n\,q\,x_0)\dots)\,x_1)\,x_2\,x_3\dots \longrightarrow^* \alpha_1\,(\dots(\alpha_n\,q\,x_0)\dots)\,x_1\,x_3 \end{gathered} \tag{69}$$


Higher-Order Regular
Regular
Higher-Order Shuffle
Shuffle
Combinatory Logic
Higher-Order
First-Order
Concatenative Calculus