

## Path Connected

**Proposition 0.1.** A disjoint union of locally path-connected spaces is locally path-connected.

**Proposition 0.2.** A locally contractible space is locally path-connected.

**Proposition 0.3.** A CW space is locally contractible.

**Proposition 0.4.** The quotient of a locally path-connected space is locally path-connected.

## Covering Space

**Definition 1.1.** A **universal covering** is a simply connected covering. They are unique up to homeomorphism.

**Proposition 1.1.** A connected topological space has a universal cover iff it is locally path-connected and semilocally simply connected.

**Proposition 1.2.** The number of sheets of a covering space  $p : (\tilde{X}, x_0) \rightarrow (X, x_0)$  with  $X$  and  $\tilde{X}$  path-connected equals the index of  $p_*(\pi_1(\tilde{X}, x_0))$  in  $\pi_1(X, x_0)$ .

**Proposition 1.3.** Let  $p : (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$  be a path-connected covering space of the path-connected, locally path-connected space  $X$ , and let  $H = p_*\left(\pi_1(\tilde{X}, \tilde{x}_0)\right) \subset \pi_1(X, x_0)$ . Then:

1. The covering space  $p : \tilde{X} \rightarrow X$  is normal (regular) if and only if  $H$  is a normal subgroup of  $\pi_1(X, x_0)$ .
2. The group of deck transformations  $\text{Deck}(\tilde{X}/X)$  is isomorphic to the quotient  $N(H)/H$ , where  $N(H)$  is the normalizer of  $H$  in  $\pi_1(X, x_0)$ .

In particular, if  $\tilde{X} \rightarrow X$  is a normal covering, then

$$\text{Deck}(\tilde{X}/X) \cong \pi_1(X, x_0)/H.$$

Hence, for the universal cover  $\tilde{X} \rightarrow X$ , we have

$$\text{Deck}(\tilde{X}/X) \cong \pi_1(X, x_0).$$

**Lemma 1.4.** The number of sheets of the universal covering space is

**Proposition 1.5.** Deck transformations don't have fixed points.

## CW Complexes

**Proposition 2.1.** A CW complex is semi-locally simply connected.

**Proposition 2.2.** A connected CW complex has a universal cover.

**Proposition 2.3.** A covering space of a CW complex is also a CW complex, with cells projecting homeomorphically to cells.

**Proposition 2.4.** If  $X$  is a finite CW complex and if  $Y \rightarrow X$  is a  $n$ -sheeted covering then  $Y$  is a finite CW complex and  $\Xi(Y) = n \cdot \Xi(X)$ .

## Named Theorems

**Definition 3.1.** The Lefschetz Number

$$\Lambda(f) := \sum_n \operatorname{tr}(f_* : H_n(X; \mathbb{Q}) \rightarrow H_n(X; \mathbb{Q}))$$

**Theorem 3.1** (Lefschetz Fixed Point Theorem). If  $X$  is a triangulable space or a retract of a simplicial complex, and if  $f : X \rightarrow X$  is continuous, then if  $\Lambda(f) \neq 0$ ,  $f$  has a fixed point.

## Problems

**Example 1.** Suppose that  $X$  is a finite connected CW complex such that  $\pi_1(X)$  is finite and nontrivial. Prove that the universal covering  $\tilde{X}$  of  $X$  cannot be contractible.

*Proof.* Since  $X$  is a connected CW complex it has a universal cover, which is also a CW complex since  $X$  is a CW complex. Since  $\pi_1(X)$  is finite, the universal cover has a finite number of sheets, and since  $X$  is a finite CW complex, each sheet has finite cells. So  $\tilde{X}$  is a finite CW complex.

Suppose for contradiction that  $\tilde{X}$  is contractible. Then  $H_0(\tilde{X}) = \mathbb{Z}$  and  $H_i(\tilde{X}) = 0$  for  $i > 0$ . Since  $f$  is continuous, and  $\tilde{X}$  is simply connected and hence connected, Let  $p \in \tilde{X}$  and since  $\tilde{X}$  is connected there is only one generator of  $H_0(\tilde{X}; \mathbb{Q})$ ,  $[p]$ . Then  $[p] = 1$ . And  $f_*$  maps  $[p]$  to  $[f(p)]$  but since  $f$  is continuous and  $f(p) \in \tilde{X}$ ,  $[f(p)] = [p]$  hence  $f_*$  is the identity and its trace must be 1. Thus any continuous self map  $f : \tilde{X} \rightarrow \tilde{X}$  has Lefschetz number 1, and thus has a fixed point. Which is a contradiction since  $\pi_1(X)$  is non-trivial and is isomorphic to the group of Deck transformations of  $\tilde{X}$  and thus there is a non-trivial Deck transformation, and Deck transformations don't have fixed points.

Alternatively, by computing  $\chi(\tilde{X})$  using homology and the fact that  $\tilde{X}$  is contractible,  $1 = \chi(\tilde{X}) = |\pi_1(X)| \cdot \chi(X)$  so that since  $\pi_1(X)$  is non-trivial this is impossible.  $\square$

**Example 2.**

*Proof.*  $\square$

**Example 3.**

*Proof.*  $\square$