

## Example

**Theorem 0.1.** Let  $A$  be an embedded  $n$ -manifold with boundary  $\partial A$ . Let  $v$  be a direction such that  $h_v^A : A \rightarrow \mathbb{R}$  is a Morse function. Let the interval decomposition of the  $k$ -dimensional extended persistent homology of  $h_v^{\partial A} : \partial A \rightarrow \mathbb{R}$  be

$$\text{XPH}_k(\partial A, h_v) = \bigoplus_{[b_i, d_i] \in S_X} \mathcal{I}_{[b_i, d_i]}.$$

Let  $J_A^k$  be the subset of intervals  $[b_i, d_i]$  such that  $b_i = (h_v(p), \text{ord})$  for some  $p \in \text{Crit}(h_v^A, (k, +1))$ , or  $b_i = (h_v(p), \text{rel})$  for some  $p \in \text{Crit}(h_v^A, (n - k - 1, -1))$ . Then

$$\text{XPH}_k(A, h_v) = \bigoplus_{[b_i, d_i] \in J_A^k} \mathcal{I}_{[b_i, d_i]}.$$

**Example 1.** Consider Figure 1. And we will compute  $\text{XPH}_0$ .

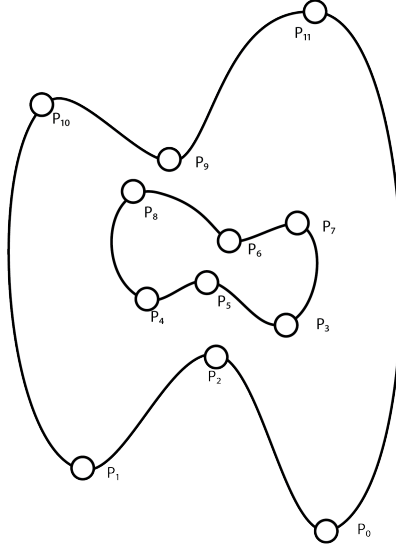


Figure 1: Example figure

**Theorem 0.2.** Let  $A \subset \mathbb{R}^2$  be a 2-dimensional piecewise linear manifold with boundary  $X = \partial$ . Fix  $v \in S^1$ . The 0-dimensional persistent homology of  $h_v^X : X \rightarrow \mathbb{R}$  can be written as

$$\text{PH}_0(X, h_v^X) = \bigoplus_{i=1}^m \mathcal{I}_{[h_v(y_{j_i}), d_i]}$$

where  $y_{j_1}, \dots, y_{j_m}$  are the set of vertex representatives, and  $d_1, \dots, d_m \in \mathbb{R} \cup \infty$ . Here we have only included intervals with positive length.

Let  $J^{\text{ord}}$  be the subset of  $\{1, 2, \dots, m\}$  such that  $d_i$  is finite and  $y_{j_i}$  is  $(+)$ -critical for  $h_v^A$ . Then

$$\text{Ord}_0(A, h_v^A) = \bigoplus_{i \in J^{\text{ord}}} \mathcal{I}_{[(h_v(y_{j_i}), \text{ord}), (d_i, \text{ord})]}$$

Now let  $J^{\text{rel}}$  be the subset of  $\{1, 2, \dots, m\}$  such that  $d_i$  is finite but  $y_{j_i}$  is not  $(+)$ -critical for  $h_v^A$ .

$$\text{Rel}_1(A, h_v^A) = \bigoplus_{i \in J^{\text{rel}}} \mathcal{I}_{[(d_i, \text{rel}), (h_v(y_{j_i}), \text{rel})]}$$

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**Theorem 0.3.**

**Theorem 0.4.** Proposition 4.20. Let  $A \subset \mathbb{R}^n$  be an  $n$ -manifold with boundary  $X = \partial A$ . Let  $v$  be a direction such that  $h_v^A : A \rightarrow \mathbb{R}$  is a Morse function. Let  $\{X_1, \dots, X_k\}$  be the interior boundary components of  $X$  and  $\{Y_1, \dots, Y_l\}$  be the exterior boundary components of  $X$ . Then

$$\text{Ess}_0(A, h_v) = \sum_{j=1}^l \mathcal{I}_{[(\min\{h_v(Y_j)\}, \text{ord}), (\max\{h_v(Y_j)\}, \text{rel})]}$$

and

$$\text{Ess}_{n-1}(A, h_v) = \sum_{i=1}^k \mathcal{I}_{[(\max\{h_v(X_i)\}, \text{ord}), (\min\{h_v(X_i)\}, \text{rel})]}$$