Example

Theorem 0.1. Let A be an embedded n-manifold with boundary ∂A. Let v be a direction such that $h_v^A : A \to \mathbb{R}$ is a Morse function. Let the interval decomposition of the k-dimensional extended persistent homology of $h_v^{\partial A} : \partial A \to \mathbb{R}$ be

$$\mathrm{XPH}_k\left(\partial A, h_v\right) = \bigoplus_{[b_i,d_i)\in S_X} \mathcal{I}_{[b_i,d_i)}.$$

Let J_A^k be the subset of intervals $[b_i, d_i)$ such that $b_i = (h_v(p)\,, \mathrm{ord})$ for some $p \in \mathrm{Crit}\left(h_v^A, (k, +1)\right)$, or $b_i = (h_v(p)\,, \mathrm{rel}\ )$ for some $p \in \mathrm{Crit}\left(h_v^A, (n-k-1, -1)\right)$. Then

$$\mathrm{XPH}_k\left(A, h_v\right) = \bigoplus_{[b_i,d_i)\in J_A^k} \mathcal{I}_{[b_i,d_i)}.$$

For simplicity, we will classify all possible critical points.

Definition 0.1. For a given vertex, we have 3×2 possibilities for critical points.

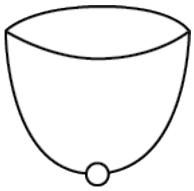

Figure 1: Index= (0,-)

Figure 2: Index= (0,-)

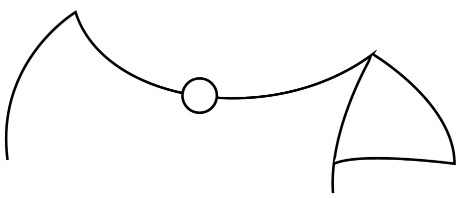

Figure 3: Index= (0,-)

It is well known that the homology groups of the torus are:

1. $H_0(T) = \mathbb{Z}$
2. $H_1(T) = \mathbb{Z} \oplus \mathbb{Z}$
3. $H_2(T) = \mathbb{Z}$

As for the solid torus:

1. $H_0(T) = \mathbb{Z}$
2. $H_1(T) = \mathbb{Z}$
3. $H_2(T) = 0$

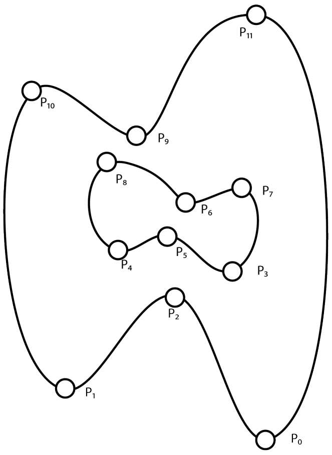

Figure 4: Illustration of multi-headed attention. The two highlighted attention heads have learned to associate *"it"* with different parts of the sentence.

Example 1. Consider Figure 1. And we will compute XPH_0. We will also compare what is happening with the critical points with respect to both A and ∂A.

We will start with the boundary ∂A first.

1. P_0 is a local minimum. It creates a connected component. It is actually the global minimum.
2. P_1 is another local minimum. It creates another class in H_0.
3. P_2 This is a saddle which joins the two connected components. So we pair (P_1, P_2).
4. P_3 This is a saddle which looks like a handle. This forms an essential class in H_1.
5. P_4 adds another handle which forms another class in H_1.
6. P_5 is a local maximum whichs joins two cycles, so we pair (P_4, P_5).
7. P_6 is a local minimum, which starts a class in H_0.
8. P_7 is a saddle, which joins the component, so we pair (P_6, P_7).
9. P_8 is a saddle, which forms an essential class in H_1.
10. P_9 is a saddle which looks like a handle, which forms a class in H_1.
11. P_{10} is a local maximum, which kills a class in H_1, so we pair (P_9, P_{10}).
12. P_{11} Is a global maximum, which creates a class in H_2.

Let us contrast this with the solid torus with boundary A.

1. P_0 is a local minimum. It creates a connected component. It is actually the global minimum.
2. P_1 is another local minimum. It creates another class in H_0.
3. P_2 This is a saddle which joins the two connected components. So we pair (P_1, P_2).
4. P_3
5. P_4

6. P_5
7. P_6
8. P_7
9. P_8
10. P_9
11. P_{10}
12. P_{11} Is a global maximum. Homology does not change.

P_3 is where things start becoming different for the solid donut and for its boundary.

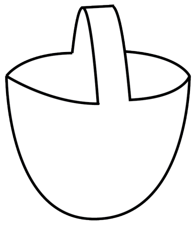

Figure 5: Handle

Theorem 0.2. Let $A \subset \mathbb{R}^2$ be a 2-dimensional piecewise linear manifold with boundary $X = \partial$. Fix $v \in S^1$. The 0-dimensional persistent homology of $h_v^X : X \to \mathbb{R}$ can be written as

$$\mathrm{PH}_0\left(X, h_v^X\right) = \oplus_{i=1}^m \mathcal{I}_{\left[h_v\left(y_{j_i}\right), d_i\right)}$$

where $y_{j_1}, \ldots y_{j_m}$ are the set of vertex representatives, and $d_1, \ldots d_m \in \mathbb{R} \cup \infty$. Here we have only included intervals with positive length.

Let J^{ord} be the subset of $\{1, 2, \ldots m\}$ such that d_i is finite and y_{j_i} is (+)critical for h_v^A. Then

$$\mathrm{Ord}_0\left(A, h_v^A\right) = \oplus_{i \in J^{\text{ord}}} \mathcal{I}_{\left[\left(h_v\left(y_{j_i}\right), \text{ ord }\right), \left(d_i, \text{ ord }\right)\right)}$$

Now let J^{rel} be the subset of $\{1, 2, \ldots m\}$ such that d_i is finite but y_i is not (+)-critical for h_v^A.

$$\mathrm{Rel}_1\left(A, h_v^A\right) = \oplus_{i \in J \text{rel}} \mathcal{I}_{\left[\left(d_i, \text{rel}\right), \left(h_v\left(y_{j_i}\right), \text{ rel }\right)\right)}$$

Theorem 0.3. Let $A \subset \mathbb{R}^3$ be a 3-dimensional piecewise linear manifold with boundary ∂A. Fix $v \in S^1$. The 0-dimensional persistent homology of $h_v^X : X \to \mathbb{R}$ can be written as

$$\mathrm{PH}_0\left(X, h_v^X\right) = \oplus_{i=1}^m \mathcal{I}_{\left[h_v\left(y_{j_i}\right), d_i\right)}$$

where $y_{j_1}, \ldots y_{j_m}$ are the set of vertex representatives, and $d_1, \ldots d_m \in \mathbb{R} \cup \infty$. Here we have only included intervals with positive length.

Let J^{ord} be the subset of $\{1, 2, \ldots m\}$ such that d_i is finite and y_{j_i} is (+)critical for h_v^A. Then

$$\mathrm{Ord}_0\left(A, h_v^A\right) = \oplus_{i \in J^{\text{ord}}} \mathcal{I}_{\left[\left(h_v\left(y_{j_i}\right), \text{ ord }\right), \left(d_i, \text{ ord }\right)\right)}$$

Now let J^{rel} be the subset of $\{1, 2, \ldots m\}$ such that d_i is finite but y_i is not (+)-critical for h_v^A.

$$\mathrm{Rel}_1\left(A, h_v^A\right) = \oplus_{i \in J \text{rel}} \mathcal{I}_{\left[\left(d_i, \text{rel}\right), \left(h_v\left(y_{j_i}\right), \text{ rel }\right)\right)}$$

Theorem 0.4. Proposition 4.20. Let $A \subset \mathbb{R}^n$ be an n-manifold with boundary $X = \partial A$. Let v be a direction such that $h_v^A : A \to \mathbb{R}$ is a Morse function. Let $\{X_1, \ldots X_k\}$ be the interior boundary components of X and $\{Y_1, \ldots Y_l\}$ be the exterior boundary components of X. Then

$$\mathrm{Ess}_0\left(A, h_v\right) = \sum_{j=1}^{l} \mathcal{I}_{[(\min\{h_v(Y_j)\}, \mathrm{ord}), (\max\{h_v(Y_j)\}, \text{ rel }))}$$

and

$$\mathrm{Ess}_{n-1}\left(A, h_v\right) = \sum_{i=1}^{k} \mathcal{I}_{[(\max\{h_v(X_i)\}, \mathrm{ord}), (\min\{h_v(X_i)\}, \text{ rel }))}$$