Example

Theorem 0.1. Let A be an embedded n-manifold with boundary ∂A. Let v be a direction such that $h_v^A : A \to \mathbb{R}$ is a Morse function. Let the interval decomposition of the k-dimensional extended persistent homology of $h_v^{\partial A} : \partial A \to \mathbb{R}$ be

$$\mathrm{XPH}_k\left(\partial A, h_v\right) = \bigoplus_{[b_i, d_i) \in S_X} \mathcal{I}_{[b_i, d_i)}.$$

Let J_A^k be the subset of intervals $[b_i, d_i)$ such that $b_i = (h_v(p)\,, \mathrm{ord})$ for some $p \in \mathrm{Crit}\left(h_v^A, (k, +1)\right)$, or $b_i = (h_v(p)\,, \text{ rel })$ for some $p \in \mathrm{Crit}\left(h_v^A, (n-k-1, -1)\right)$. Then

$$\mathrm{XPH}_k\left(A, h_v\right) = \bigoplus_{[b_i, d_i) \in J_A^k} \mathcal{I}_{[b_i, d_i)}.$$

Example 1. Consider Figure 1. And we will compute XPH_0.

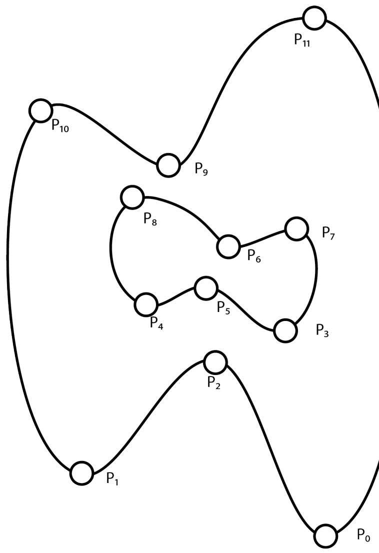

Figure 1: Example figure

Theorem 0.2. Let $A \subset \mathbb{R}^2$ be a 2-dimensional piecewise linear manifold with boundary $X = \partial$. Fix $v \in S^1$. The 0-dimensional persistent homology of $h_v^X : X \to \mathbb{R}$ can be written as

$$\mathrm{PH}_0\left(X, h_v^X\right) = \oplus_{i=1}^m \mathcal{I}_{\left[h_v\left(y_{j_i}\right), d_i\right)}$$

where $y_{j_1}, \ldots y_{j_m}$ are the set of vertex representatives, and $d_1, \ldots d_m \in \mathbb{R} \cup \infty$. Here we have only included intervals with positive length.

Let J^{ord} be the subset of $\{1, 2, \ldots m\}$ such that d_i is finite and y_{j_i} is $(+)$critical for h_v^A. Then

$$\mathrm{Ord}_0\left(A, h_v^A\right) = \oplus_{i \in J^{\text{ord}}} \mathcal{I}_{\left[\left(h_v\left(y_{j_i}\right), \text{ ord }\right), \left(d_i, \text{ ord }\right)\right)}$$

Now let J^{rel} be the subset of $\{1, 2, \ldots m\}$ such that d_i is finite but y_i is not $(+)$-critical for h_v^A.

$$\mathrm{Rel}_1\left(A, h_v^A\right) = \oplus_{i \in J\text{rel}} \mathcal{I}_{\left[\left(d_i, \text{rel}\right), \left(h_v\left(y_{j_i}\right), \text{ rel }\right)\right)}$$

Theorem 0.3. Let $A \subset \mathbb{R}^3$ be a 3-dimensional piecewise linear manifold with boundary ∂A. Fix $v \in S^1$. The 0-dimensional persistent homology of $h_v^X : X \to \mathbb{R}$ can be written as

$$\mathrm{PH}_0\left(X, h_v^X\right) = \oplus_{i=1}^m \mathcal{I}_{\left[h_v\left(y_{j_i}\right), d_i\right)}$$

where $y_{j_1}, \ldots y_{j_m}$ are the set of vertex representatives, and $d_1, \ldots d_m \in \mathbb{R} \cup \infty$. Here we have only included intervals with positive length.

Let J^{ord} be the subset of $\{1, 2, \ldots m\}$ such that d_i is finite and y_{j_i} is (+)critical for h_v^A. Then

$$\mathrm{Ord}_0\left(A, h_v^A\right) = \oplus_{i \in J^{\text{ord}}} \mathcal{I}_{\left[\left(h_v\left(y_{j_i}\right), \text{ ord }\right), \left(d_i, \text{ ord }\right)\right)}$$

Now let J^{rel} be the subset of $\{1, 2, \ldots m\}$ such that d_i is finite but y_i is not (+)-critical for h_v^A.

$$\mathrm{Rel}_1\left(A, h_v^A\right) = \oplus_{i \in J\text{rel}} \mathcal{I}_{\left[\left(d_i, \text{rel}\right), \left(h_v\left(y_{j_i}\right), \text{ rel }\right)\right)}$$

Theorem 0.4. Proposition 4.20. Let $A \subset \mathbb{R}^n$ be an n-manifold with boundary $X = \partial A$. Let v be a direction such that $h_v^A : A \to \mathbb{R}$ is a Morse function. Let $\{X_1, \ldots X_k\}$ be the interior boundary components of X and $\{Y_1, \ldots Y_l\}$ be the exterior boundary components of X. Then

$$\mathrm{Ess}_0\left(A, h_v\right) = \sum_{j=1}^{l} \mathcal{I}_{\left[\left(\min\{h_v(Y_j)\}, \text{ord}\right), \left(\max\{h_v(Y_j)\}, \text{ rel }\right)\right)}$$

and

$$\mathrm{Ess}_{n-1}\left(A, h_v\right) = \sum_{i=1}^{k} \mathcal{I}_{\left[\left(\max\{h_v(X_i)\}, \text{ord}\right), \left(\min\{h_v(X_i)\}, \text{ rel }\right)\right)}$$