Path Connected

Proposition 0.1. A disjoint union of locally path-connected spaces is locally path-connected.

Proposition 0.2. A locally contractible space is locally path-connected.

Proposition 0.3. A CW space is locally contractible.

Proposition 0.4. The quotient of a locally path-connected space is locally path-connected.

Covering Space

Definition 1.1. A **universal covering** is a simply connected covering. They are unique up to homeomorphism.

Proposition 1.1. A connected topological space has a universal cover iff it is locally path-connected and semilocally simply connected.

Proposition 1.2. The number of sheets of a covering space $p : (\widetilde{X}, x_0) \to (X, x_0)$ with X and $\widetilde{X}$ path-connected equals the index of $p_*(\pi_1(\widetilde{X}, x_0))$ in $\pi_1(X, x_0)$.

Proposition 1.3. Let $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a path-connected covering space of the path-connected, locally path-connected space X, and let $H = p_* \left(\pi_1(\widetilde{X}, \widetilde{x}_0)\right) \subset \pi_1(X, x_0)$. Then:

1. The covering space $p : \widetilde{X} \to X$ is normal (regular) if and only if H is a normal subgroup of $\pi_1(X, x_0)$.
2. The group of deck transformations $\mathrm{Deck}(\widetilde{X}/X)$ is isomorphic to the quotient $N(H)/H$, where $N(H)$ is the normalizer of H in $\pi_1(X, x_0)$.

In particular, if $\widetilde{X} \to X$ is a normal covering, then

$$\mathrm{Deck}(\widetilde{X}/X) \cong \pi_1(X, x_0)/H.$$

Hence, for the universal cover $\widetilde{X} \to X$, we have

$$\mathrm{Deck}(\widetilde{X}/X) \cong \pi_1(X, x_0).$$

Lemma 1.4. The number of sheets of the universal covering space is

Proposition 1.5. Deck transformations don't have fixed points.

CW Complexes

Proposition 2.1. A CW complex is semi-locally simply connected.

Proposition 2.2. A connected CW complex has a universal cover.

Proposition 2.3. A covering space of a CW complex is also a CW complex, with cells projecting homeomorphically to cells.

Proposition 2.4. If X is a finite CW complex and if $Y \to X$ is a n-sheeted covering then Y is a finite CW complex and $\Xi(Y) = n \cdot \Xi(X)$.

Named Theorems

Definition 3.1. The Lefschetz Number

$$\Lambda(f) := \sum_n \mathrm{tr}(f_* : H_n(X;\mathbb{Q}) \to H_n(X;\mathbb{Q}))$$

Theorem 3.1 (Lefschetz Fixed Point Theorem)**.** If X is a triangulable space or a retract of a simplicial complex, and if $f : X \to X$ is continuous, then if $\Lambda(f) \neq 0$, f has a fixed point.

Problems

Example 1. Suppose that X is a finite connected CW complex such that $\pi_1(X)$ is finite and nontrivial. Prove that the universal covering $\widetilde{X}$ of X cannot be contractible.

Proof. Since X is a connected CW complex it has a universal cover, which is also a CW complex since X is a CW complex. Since $\pi_1(X)$ is finite, the universal cover has a finite number of sheets, and since X is a finite CW complex, each sheet has finite cells. So $\widetilde{X}$ is a finite CW complex.

Suppose for contradiction that $\widetilde{X}$ is contractible. Then $H_0(\widetilde{X}) = \mathbb{Z}$ and $H_i(\widetilde{X}) = 0$ for $i > 0$. Since f is continuous, and $\widetilde{X}$ is simply connected and hence connected, Let $p \in \widetilde{X}$ and since $\widetilde{X}$ is connected there is only one generator of $H_0(\widetilde{X}; \mathbb{Q})$, $[p]$. Then $[p] = 1$. And f_* maps $[p]$ to $[f(p)]$ but since f is continuous and $f(p) \in \widetilde{X}$, $[f(p)] = [p]$ hence f_* is the identity and its trace must be 1. Thus any continuous self map $f : \widetilde{X} \to \widetilde{X}$ has Lefschetz number 1, and thus has a fixed point. Whic is a contradiction since $\pi_1(X)$ is non-trivial and is isomorphic to the group of Deck transformations of $\widetilde{X}$ and thus there is a non-trivial Deck transformation, and Deck transformations don't have fixed points.

Alternatively, by computing $\chi(\widetilde{X})$ using homology and the fact that $\widetilde{X}$ is contractible, $1 = \chi(\widetilde{X}) = |\pi_1(X)| \cdot \chi(X)$ so that since $\pi_1(X)$ is non-trivial this is impossible. □

Example 2. Homology of the circle.

Proof. Let v_0, v_1, v_2 be vertices of a triangulation of the circle. Then, □

Example 3.

Proof. □