
Example

Theorem 0.1. Let A be an embedded n -manifold with boundary ∂A . Let v be a direction such that $h_v^A : A \rightarrow \mathbb{R}$ is a Morse function. Let the interval decomposition of the k -dimensional extended persistent homology of $h_v^{\partial A} : \partial A \rightarrow \mathbb{R}$ be

$$\text{XPH}_k(\partial A, h_v) = \bigoplus_{[b_i, d_i] \in S_X} \mathcal{I}_{[b_i, d_i]}.$$

Let J_A^k be the subset of intervals $[b_i, d_i]$ such that $b_i = (h_v(p), \text{ord})$ for some $p \in \text{Crit}(h_v^A, (k, +1))$, or $b_i = (h_v(p), \text{rel})$ for some $p \in \text{Crit}(h_v^A, (n - k - 1, -1))$. Then

$$\text{XPH}_k(A, h_v) = \bigoplus_{[b_i, d_i] \in J_A^k} \mathcal{I}_{[b_i, d_i]}.$$

For simplicity, we will illustrate all possible critical points.

Definition 0.1. For a given vertex, we have 3×2 possibilities for critical points.

If the figure encapsulates volume: If volume surrounds the figure:

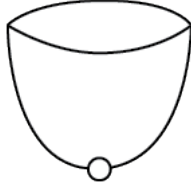


Figure 1: Index= (0,-)

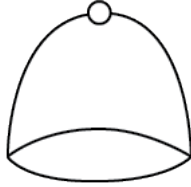


Figure 2: Index= (0,-)

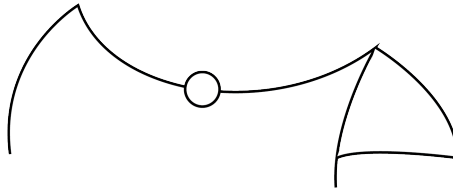


Figure 3: Index= (0,-)

It is well known what the homology groups of the torus are:

1. $H_0(T) = \mathbb{Z}$
2. $H_1(T) = \mathbb{Z} \oplus \mathbb{Z}$
3. $H_2(T) = \mathbb{Z}$

As for the solid torus:

1. $H_0(T) = \mathbb{Z}$
2. $H_1(T) = \mathbb{Z}$
3. $H_2(T) = 0$

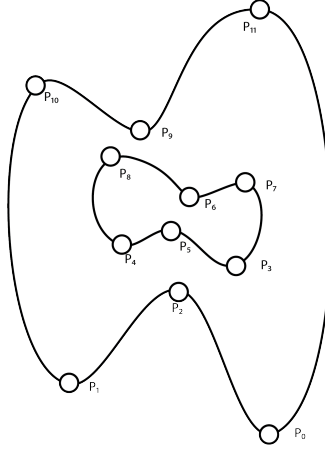


Figure 4: Wonky Torus. This is a "general" slice of a torus with boundary. The circles represent critical points.

Example 1. Consider Figure 1. And we will compute XPH_0 . We will also compare what is happening with the critical points with respect to both A and ∂A .

We will start with the boundary ∂A first.

1. P_0 is a local minimum. It creates a connected component. It is actually the global minimum.
2. P_1 is another local minimum. It creates another class in H_0 .
3. P_2 This is a saddle which joins the two connected components. So we pair (P_1, P_2) .
4. P_3 This is a saddle which looks like a handle. This forms an essential class in H_1 .
5. P_4 adds another handle which forms another class in H_1 .
6. P_5 is a local maximum whichs joins two cycles, so we pair (P_4, P_5) .
7. P_6 is a local minimum, which starts a class in H_0 .
8. P_7 is a saddle, which joins the component, so we pair (P_6, P_7) .
9. P_8 is a saddle, which forms an essential class in H_1 .
10. P_9 is a saddle which looks like a handle, which forms a class in H_1 .
11. P_{10} is a local maximum, which kills a class in H_1 , so we pair (P_9, P_{10}) .
12. P_{11} Is a global maximum, which creates a class in H_2 .

Let us contrast this with the solid torus with boundary A .

1. P_0 is a local minimum. It creates a connected component. It is actually the global minimum.
2. P_1 is another local minimum. It creates another class in H_0 .
3. P_2 This is a saddle which joins the two connected components. So we pair (P_1, P_2) .
4. P_3 Contributes nothing

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- 5. P_4 Contributes nothing
 - 6. P_5 Contributes nothing
 - 7. P_6 Creates connected component.
 - 8. P_7 Merges created component
 - 9. P_8 Creates H1 cycle
 - 10. P_9 Contributes nothing
 - 11. P_{10} Contributes nothing
 - 12. P_{11} Is a global maximum. Contributes nothing

P_3 is where things start becoming different for the solid donut and for its boundary.

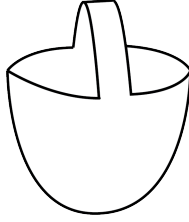


Figure 5: Handle

Theorem 0.2. Let $A \subset \mathbb{R}^2$ be a 2-dimensional piecewise linear manifold with boundary $X = \partial$. Fix $v \in S^1$. The 0-dimensional persistent homology of $h_v^X : X \rightarrow \mathbb{R}$ can be written as

$$\text{PH}_0(X, h_v^X) = \oplus_{i=1}^m \mathcal{I}_{[h_v(y_{j_i}), d_i]}$$

where y_{j_1}, \dots, y_{j_m} are the set of vertex representatives, and $d_1, \dots, d_m \in \mathbb{R} \cup \infty$. Here we have only included intervals with positive length.

Let J^{ord} be the subset of $\{1, 2, \dots, m\}$ such that d_i is finite and y_{j_i} is (+)critical for h_v^A . Then

$$\text{Ord}_0(A, h_v^A) = \oplus_{i \in J^{\text{ord}}} \mathcal{I}_{[(h_v(y_{j_i}), \text{ord}), (d_i, \text{ord})]}$$

Now let J^{rel} be the subset of $\{1, 2, \dots, m\}$ such that d_i is finite but y_i is not (+)-critical for h_v^A .

$$\text{Rel}_1(A, h_v^A) = \oplus_{i \in J^{\text{rel}}} \mathcal{I}_{[(d_i, \text{rel}), (h_v(y_{j_i}), \text{rel})]}$$

Theorem 0.3. Let $A \subset \mathbb{R}^3$ be a 3-dimensional piecewise linear manifold with boundary ∂A . Fix $v \in S^1$. The 0-dimensional persistent homology of $h_v^X : X \rightarrow \mathbb{R}$ can be written as

$$\text{PH}_0(X, h_v^X) = \oplus_{i=1}^m \mathcal{I}_{[h_v(y_{j_i}), d_i]}$$

where y_{j_1}, \dots, y_{j_m} are the set of vertex representatives, and $d_1, \dots, d_m \in \mathbb{R} \cup \infty$. Here we have only included intervals with positive length.

Let J^{ord} be the subset of $\{1, 2, \dots, m\}$ such that d_i is finite and y_{j_i} is (+)critical for h_v^A . Then

$$\text{Ord}_0(A, h_v^A) = \oplus_{i \in J^{\text{ord}}} \mathcal{I}_{[(h_v(y_{j_i}), \text{ord}), (d_i, \text{ord})]}$$

Now let J^{rel} be the subset of $\{1, 2, \dots, m\}$ such that d_i is finite but y_i is not $(+)$ -critical for h_v^A .

$$\text{Rel}_1(A, h_v^A) = \oplus_{i \in J^{\text{rel}}} \mathcal{I}_{[(d_i, \text{rel}), (h_v(y_{j_i}), \text{rel})]}$$

Theorem 0.4. Proposition 4.20. Let $A \subset \mathbb{R}^n$ be an n -manifold with boundary $X = \partial A$. Let v be a direction such that $h_v^A : A \rightarrow \mathbb{R}$ is a Morse function. Let $\{X_1, \dots, X_k\}$ be the interior boundary components of X and $\{Y_1, \dots, Y_l\}$ be the exterior boundary components of X . Then

$$\text{Ess}_0(A, h_v) = \sum_{j=1}^l \mathcal{I}_{[(\min\{h_v(Y_j)\}, \text{ord}), (\max\{h_v(Y_j)\}, \text{rel})]}$$

and

$$\text{Ess}_{n-1}(A, h_v) = \sum_{i=1}^k \mathcal{I}_{[(\max\{h_v(X_i)\}, \text{ord}), (\min\{h_v(X_i)\}, \text{rel})]}$$