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## Example

**Theorem 0.1.** Let  $A$  be an embedded  $n$ -manifold with boundary  $\partial A$ . Let  $v$  be a direction such that  $h_v^A : A \rightarrow \mathbb{R}$  is a Morse function. Let the interval decomposition of the  $k$ -dimensional extended persistent homology of  $h_v^{\partial A} : \partial A \rightarrow \mathbb{R}$  be

$$\text{XPH}_k(\partial A, h_v) = \bigoplus_{[b_i, d_i] \in S_X} \mathcal{I}_{[b_i, d_i]}.$$

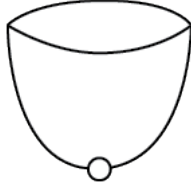
Let  $J_A^k$  be the subset of intervals  $[b_i, d_i]$  such that  $b_i = (h_v(p), \text{ord})$  for some  $p \in \text{Crit}(h_v^A, (k, +1))$ , or  $b_i = (h_v(p), \text{rel})$  for some  $p \in \text{Crit}(h_v^A, (n - k - 1, -1))$ . Then

$$\text{XPH}_k(A, h_v) = \bigoplus_{[b_i, d_i] \in J_A^k} \mathcal{I}_{[b_i, d_i]}.$$

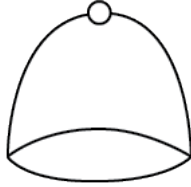
For simplicity, we will illustrate all possible critical points.

**Definition 0.1.** For a given vertex, we have  $3 \times 2$  possibilities for critical points.

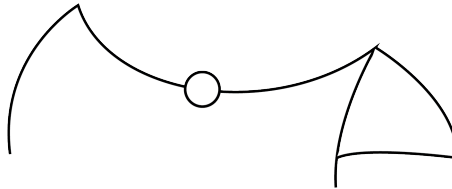
If the figure encapsulates volume: If volume surrounds the figure:



**Figure 1:** Index= (0,-)



**Figure 2:** Index= (0,-)



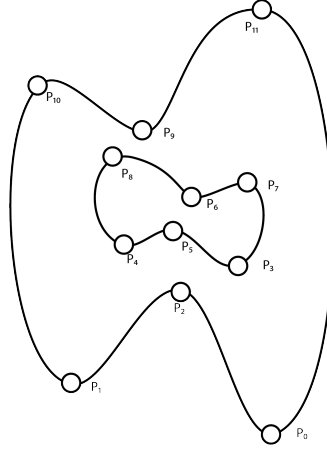
**Figure 3:** Index= (0,-)

It is well known what the homology groups of the torus are:

1.  $H_0(T) = \mathbb{Z}$
2.  $H_1(T) = \mathbb{Z} \oplus \mathbb{Z}$
3.  $H_2(T) = \mathbb{Z}$

As for the solid torus:

1.  $H_0(T) = \mathbb{Z}$
2.  $H_1(T) = \mathbb{Z}$
3.  $H_2(T) = 0$



**Figure 4:** Wonky Torus. This is a "general" slice of a torus with boundary. The circles represent critical points.

**Example 1.** Consider Figure 1. And we will compute  $XPH_0$ . We will also compare what is happening with the critical points with respect to both  $A$  and  $\partial A$ .

We will start with the boundary  $\partial A$  first.

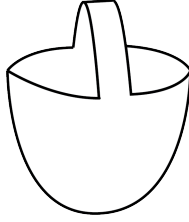
1.  $P_0$  is a local minimum. It creates a connected component. It is actually the global minimum.
2.  $P_1$  is another local minimum. It creates another class in  $H_0$ .
3.  $P_2$  This is a saddle which joins the two connected components. So we pair  $(P_1, P_2)$ .
4.  $P_3$  This is a saddle which looks like a handle. This forms an essential class in  $H_1$ .
5.  $P_4$  adds another handle which forms another class in  $H_1$ .
6.  $P_5$  is a local maximum whichs joins two cycles, so we pair  $(P_4, P_5)$ .
7.  $P_6$  is a local minimum, which starts a class in  $H_0$ .
8.  $P_7$  is a saddle, which joins the component, so we pair  $(P_6, P_7)$ .
9.  $P_8$  is a saddle, which forms an essential class in  $H_1$ .
10.  $P_9$  is a saddle which looks like a handle, which forms a class in  $H_1$ .
11.  $P_{10}$  is a local maximum, which kills a class in  $H_1$ , so we pair  $(P_9, P_{10})$ .
12.  $P_{11}$  Is a global maximum, which creates a class in  $H_2$ .

Let us contrast this with the solid torus with boundary  $A$ .

1.  $P_0$  is a local minimum. It creates a connected component. It is actually the global minimum.
2.  $P_1$  is another local minimum. It creates another class in  $H_0$ .
3.  $P_2$  This is a saddle which joins the two connected components. So we pair  $(P_1, P_2)$ .
4.  $P_3$  Contributes nothing

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- 5.  $P_4$  Contributes nothing
  - 6.  $P_5$  Contributes nothing
  - 7.  $P_6$  Creates connected component.
  - 8.  $P_7$  Merges created component
  - 9.  $P_8$  Creates H1 cycle
  - 10.  $P_9$  Contributes nothing
  - 11.  $P_{10}$  Contributes nothing
  - 12.  $P_{11}$  Is a global maximum. Contributes nothing

$P_3$  is where things start becoming different for the solid donut and for its boundary.



**Figure 5:** Handle

**Theorem 0.2.** Let  $A \subset \mathbb{R}^2$  be a 2-dimensional piecewise linear manifold with boundary  $X = \partial$ . Fix  $v \in S^1$ . The 0-dimensional persistent homology of  $h_v^X : X \rightarrow \mathbb{R}$  can be written as

$$\text{PH}_0(X, h_v^X) = \oplus_{i=1}^m \mathcal{I}_{[h_v(y_{j_i}), d_i]}$$

where  $y_{j_1}, \dots, y_{j_m}$  are the set of vertex representatives, and  $d_1, \dots, d_m \in \mathbb{R} \cup \infty$ . Here we have only included intervals with positive length.

Let  $J^{\text{ord}}$  be the subset of  $\{1, 2, \dots, m\}$  such that  $d_i$  is finite and  $y_{j_i}$  is (+)critical for  $h_v^A$ . Then

$$\text{Ord}_0(A, h_v^A) = \oplus_{i \in J^{\text{ord}}} \mathcal{I}_{[(h_v(y_{j_i}), \text{ord}), (d_i, \text{ord})]}$$

Now let  $J^{\text{rel}}$  be the subset of  $\{1, 2, \dots, m\}$  such that  $d_i$  is finite but  $y_i$  is not (+)-critical for  $h_v^A$ .

$$\text{Rel}_1(A, h_v^A) = \oplus_{i \in J^{\text{rel}}} \mathcal{I}_{[(d_i, \text{rel}), (h_v(y_{j_i}), \text{rel})]}$$

**Theorem 0.3.** Let  $A \subset \mathbb{R}^3$  be a 3-dimensional piecewise linear manifold with boundary  $\partial A$ . Fix  $v \in S^1$ . The 0-dimensional persistent homology of  $h_v^X : X \rightarrow \mathbb{R}$  can be written as

$$\text{PH}_0(X, h_v^X) = \oplus_{i=1}^m \mathcal{I}_{[h_v(y_{j_i}), d_i]}$$

where  $y_{j_1}, \dots, y_{j_m}$  are the set of vertex representatives, and  $d_1, \dots, d_m \in \mathbb{R} \cup \infty$ . Here we have only included intervals with positive length.

Let  $J^{\text{ord}}$  be the subset of  $\{1, 2, \dots, m\}$  such that  $d_i$  is finite and  $y_{j_i}$  is (+)critical for  $h_v^A$ . Then

$$\text{Ord}_0(A, h_v^A) = \oplus_{i \in J^{\text{ord}}} \mathcal{I}_{[(h_v(y_{j_i}), \text{ord}), (d_i, \text{ord})]}$$

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Now let  $J^{\text{rel}}$  be the subset of  $\{1, 2, \dots, m\}$  such that  $d_i$  is finite but  $y_i$  is not  $(+)$ -critical for  $h_v^A$ .

$$\text{Rel}_1(A, h_v^A) = \oplus_{i \in J^{\text{rel}}} \mathcal{I}_{[(d_i, \text{rel}), (h_v(y_{j_i}), \text{rel})]}$$

**Theorem 0.4.** Proposition 4.20. Let  $A \subset \mathbb{R}^n$  be an  $n$ -manifold with boundary  $X = \partial A$ . Let  $v$  be a direction such that  $h_v^A : A \rightarrow \mathbb{R}$  is a Morse function. Let  $\{X_1, \dots, X_k\}$  be the interior boundary components of  $X$  and  $\{Y_1, \dots, Y_l\}$  be the exterior boundary components of  $X$ . Then

$$\text{Ess}_0(A, h_v) = \sum_{j=1}^l \mathcal{I}_{[(\min\{h_v(Y_j)\}, \text{ord}), (\max\{h_v(Y_j)\}, \text{rel})]}$$

and

$$\text{Ess}_{n-1}(A, h_v) = \sum_{i=1}^k \mathcal{I}_{[(\max\{h_v(X_i)\}, \text{ord}), (\min\{h_v(X_i)\}, \text{rel})]}$$