Projections of PL 3-Manifold

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PL-Manifolds

Proposition 0.1. Let X be a PL 3-Manifold. There exists a PL-Manifold that is the topological convex hull of X.

Proposition 0.2. Let X be the convex hull of a PL 3-Manifold. The critical points determine its structure uniquely.

Projections

Let X be a PL 3-Manifold. Let $v \in S^2$. We denote the projection of X in the direction of v by X_v.

Proposition 0.3. Let $x \in X_v$. Then, for any allowable direction $u \in S^2$, $h_u(x) = h_u(y)$ for any $y \in p_v^{-1}(x)$.

Proof. Let $y = (y \cdot v)v + (y - (y \cdot v)v)$. Then, letting $x = (y - (y \cdot v)v)$ be the projection, we write $y = (y \cdot v)v + x$. Now, the allowable directions for the height function of X_v are $u \in S^2$ for which $u \cdot v = 0$, so that $y \cdot u = (y \cdot v)v \cdot u + x \cdot u = x \cdot u$. Therefore we have our conclusion. □

0.1 Convex

Proposition 0.4. Let X be convex. Then, for any $u \in S^2$ such that $u \cdot v = 0$, we have that the diagrams $D_0(X, u) = D_0(X_v, u)$.

Proof. There is a unique off-diagonal point in each diagram with birth at the minimum. □

Corollary 0.5. We can recover $D_0(X,u)$ for any $u \in S^2$.

Proof. Start with any $u \in S^2$, then use any $v \in S^2$ orthogonal to $\square$

Corollary 0.6. Let X be convex. Then the structure of X can be recovered from the projections.

Proof. Since we have $D_0(X,u)$ for all $u \in S^2$ by the previous corollary, and we know that $D_1(X,u)$ and $D_2(X,u)$ are trivial by convexity, we are in the setting of . $\square$

Now, more generally we will not have that equality $D(X,u) = D(X_v,u)$. But in the case of simply connectedness, we have

Proposition 0.7. Let X be convex. Then, it suffices to study a single great circle of directions to recover the structure of X.

Proof. Fix some $w \in S^2$. We identify the set $u \in S^2$ such that $u \cdot w = 0$ with S^1. Pick any $m \in S^2$. Then there will be some $u \in S^1$ for which $m \cdot u = 0$ (explicitely take $\pm m \times w$). Pick one of them, and $D_0(X_u,m) = D_0(X,m)$. $\square$

0.2 Simply Connected

Proposition 0.8. Let X be simply connected. Let $v \in S^2$.

Visibility of Vertices

Definition 0.1. Let X be a PL 3-Manifold. Let $x \in X$ be a critical point. x is said to be **visible** if x is a critical point for some projection p_v and h_u such that $u \cdot v = 0$.

Definition 0.2. A point $x \in X$ is said to be **twice visible** if it is visible from two distinct directions $v_1, v_2 \in S^2$.

Proposition 0.9. Let X be a convex PL 3-Manifold. Every critical point is twice visible.

Proposition 0.10. Let X be a convex PL 3-Manifold. If a critical point is twice visible, it can be uniquely identified.

Visibility of Links

Definition 0.3. visibility of link

Proposition 0.11. Let X be a PL 3-Manifold. If every critical point is twice visible, then one can uniquely determine its structure.

Corollary 0.12. One can recover the convex hull of a PL 3-Manifold from its projections.