Workshop 4

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Proposition 0.1. Let $v_1, \ldots, v_n$ be pairwise orthogonal non-zero vectors. Then they are linearly independent.

Proof. Suppose $c_1 v_1 + \ldots + c_n v_n = 0$. Then, for taking inner product with v_i, we get $c_i v_i \cdot v_i = 0$ since the other components vanish by orthogonality. And since v_i is non-zero, $c_i = 0$. $\square$

Definition 0.1. A set of matrices forms a **reducible set** if there exists a nonzero proper subspace of the underlying vector space that is invariant under all of those matrices.

Proposition 0.2. Do the following set of matrices form a reducible or irreducible set? Explain your answer.

Proof. Yes, $\alpha(1, 1, 1, 1)$ is an invariant subspace. $\square$

Proposition 0.3. A matrix that commutes with every matrix of an irreducible representation must be a multiple of the identity matrix. That is, if D is irreducible and if $AD(g) = D(g)A$ for all $g \in G$, then $A = \alpha I$.

Lemma 0.4. Let $\rho : G \to GL(n, \mathbb{C})$ be a scalar representation. That is, for any g, there exists some $\alpha_g \in \mathbb{C}$ such that $\rho(g) = \alpha_g I$. Then if ρ is irreducible, $n = 1$.

Proof. Suppose $n \geq 1$. Then let W be any 1-dimensional subspace of $\mathbb{C}^n$. Then by definition, for $w \in W$, $\rho(g)w = \alpha_g Iw = \alpha w \in W$ since W is a subspace. So we have a contradiction. $n = 1$. □

Proposition 0.5. Use Schur's lemma to show that an irreducible matrix representation, in $GL(n, \mathbb{C})$, of an abelian group must be one-dimensional.

Proof. Let $\rho : G \to GL(n, \mathbb{C})$ be a an irreducible representation of G. Fix $h \in G$ and let $g \in G$ be arbitrary. Then, since ρ is a group homomorphism, and G is abelian, $\rho(g)\rho(h) = \rho(h)\rho(g)$. Therefore $\rho(h) = \alpha I$ is a scalar multiple of the identity by Schur's Lemma. Since h was arbitrary (albeit fixed), $\rho(h)$ is a scalar multiple of the identity matrix for all $h \in G$. By the Lemma $n = 1$. □

Proposition 0.6 (Schur's Lemma)**.** Let V and W be vector spaces. Let ρ_V and ρ_W be irreducible representations of G on V and W respectively.

1. If V and W are not isomorphic, then there are no non-trivial G-linear maps between them.

2. If $V = W$ finite-dimensional over an algebraically closed field, and if $\rho_V = \rho_W$, then the only nontrivial G-linear maps are the identity, and scalar multiples of the identity.

Proof. □

Definition 0.2. infinitessimal operator

Proposition 0.7. Suppose $T(a)$ is the translation operator that acts on a function $f(x)$ to make it $f(x+a)$. That is: $T(a)f(x) = f(x+a)$. Show that the infinitessimal operator corresponding to the Lie group of translation operators is given by

$$\partial$$

Proof.

$$\left[e^{iap_x} f\right](x) = \sum_{n=0}^{\infty} \frac{(ia\,p_x)^n}{n!} f(x) = \sum_{n=0}^{\infty} \frac{(ia)^n}{n!} \left(-i\frac{d}{dx}\right)^n f(x) = f(x+a).$$

□

Lemma 0.8.

Proof. **Step 1. Define $g(t)$ by $g(t) = f(x+t)$.**

$$g(0) = f(x), \quad \text{and for the } n\text{-th derivative we get}$$
$$g^{(n)}(0) = \frac{d^n}{dt^n}\big[f(x+t)\big]\bigg|_{t=0} = f^{(n)}(x).$$

Step 2. Maclaurin series for $g(t)$.

Because g is sufficiently smooth, its Taylor expansion about 0 is

$$g(t) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!}\, t^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!}\, t^n.$$

Step 3. Substitute $t = a$.

Since $g(t) = f(x+t)$, setting $t = a$ yields

$$f(x+a) = g(a) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!}\, a^n,$$

which is the usual Taylor expansion of $f(x+a)$ about x.

□