REDUCTION FORMULA
and
TRIGONOMETRIC SUBSTITUTIONS

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INTRODUCTION

- A Reduction Formula for an integral is a formula which connects an integral linearly with another integral of the same type, but of the lower degree.
- Consider $\int \sin^5 x \cos x \, dx$; if we let $u = \sin x$, $\int u^5 \, dx = \frac{1}{6}\sin^6 x + c$ settles the problem.
- This is not always possible according to the nature of integrands. For example, the problem $\int \sin^5 x \, dx$; there is no way u-substitution can work. Hence we apply the method below.
- Reduction formula is generally obtained by repeated application of integration by parts.

INTEGRATION BY PARTS

Recall, integration by parts;

- Consider a product of two functions of uv, to find the derivative using product rule,

$$d(uv) = v \cdot d(u) + u \cdot d(v)$$

- If we take integrals both sides,

$$\int d(uv) = \int udv + \int vdu \text{ but } \left(\int d(uv) = uv\right)$$

- Make $\int udv$ subject of the formula,

$$\boxed{\int udv = uv - \int vdu}$$

REDUCTION FORMULA SINE

Reduction Formula for R $\sin^n x\, dx$; n ≥ 2

In order to apply integration by parts, we split $\sin^n x$ in two

parts $\sin^n x = (\sin^{n-1} x)(\sin x)$

$dv = \sin x\, dx$	$u = \sin^{n-1} x$
$v = -\cos x$	$du = (n-1)\sin^{n-2} x \cos x\, dx$

Since, $\int u dv = uv - \int v du$,

$$\int \sin^n x\, dx = (-\cos x \sin^{n-1} x) - \int (-\cos x)\left((n-1)\sin^{n-2} x \cos x\, dx\right)$$

REDUCTION FORMULA SINE

$$\int \sin^n x\,dx = -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x \cos^2 x\,dx$$

- But, $\cos^2 x = 1 - \sin^2 x;$

$$\int \sin^n x\,dx = -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x\,(1-\sin^2 x)dx$$

$$\int \sin^n x\,dx = -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x\,dx - (\mathrm{n}-1)\int \sin^n x\,dx$$

$$\mathrm{n}\int \sin^n x\,dx = -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x\,dx$$

Therefore; $\int \sin^n x\,dx = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$

REDUCTION FORMULA SINE

Example: Use the formula above to evaluate $\int \sin^4 x\, dx$.

- Applying the formula yields:

$$\int \sin^4 x\, dx = -\frac{1}{4}\cos x \sin^3 x + \frac{3}{4}\int \sin^2 x\, dx$$

- Apply the formula again on the second part of the right hand side.

$$\int \sin^2 x\, dx = -\frac{1}{2}\cos x \sin x + \frac{1}{2}\int 1 dx = -\frac{1}{2}\cos x \sin x + \frac{1}{2}x$$

$$\int \sin^4 x\, dx = -\frac{1}{4}\cos x \sin^3 x + \frac{3}{4}\left[-\frac{1}{2}\cos x \sin x + \frac{1}{2}x\right] + \mathrm{c}$$

$$\int \sin^4 x\, dx = -\frac{1}{4}\cos x \sin^3 x - \frac{3}{8}\cos x \sin x + \frac{3}{8}x + \mathrm{c}$$

REDUCTION FORMULA OF COSINE

- The same method has to be applied when deriving the reductio n formula for cosine.
- That is, by splitting $\cos^n x$ into $\cos^{n-1} x$ and $\cos x$ and applying the integration by parts method of integration.
- A similar expression to the one found earlier on reduction formula for sine will be found.

REDUCTION FORMULA OF SINE AND COSINE

Practice Questions

Evaluate the following

1. $\int \cos^n x \, dx$
2. $\int \sin^3 x \, dx$
3. $\int \cos^4 x \, dx$

INTEGRALS INVOLVING

POWERS OF TRIG. FUNCTIONS

ODD POWERS OF SINE AND COSINE

- Our goal here is to reduce the power into a simplified form that allows integration by substitution.
- In this method, you write the integral of $\sin^n x$ and $\cos^n x$ as $\sin^{n-1} x \sin x$ and $\cos^{n-1} x \cos x$ respectively.
- Since n is odd, $(n-1)$ will always be even. Hence, the use of the Pythagorean identity, $\cos^2 x + \sin^2 x = 1$ and the u-substitution of $u = \cos x$ or $u = \sin x$ settles the integrand.

ODD POWERS OF SINE AND COSINE

Example: Evaluate $\int \sin^5\theta\, d\theta$.

- We know that $\int \sin^5\theta\, d\theta = \int \sin^4\theta \sin\theta\, d\theta$

$$\int \sin^4\theta \sin\theta\, d\theta = \int (\sin^2\theta)^2 \sin\theta\, d\theta \text{ since } \sin^2 x = 1 - \cos^2 x$$

$$= \int (1-\cos^2\theta)^2 \sin\theta\, d\theta = \int (1 - 2\cos^2\theta + \cos^4\theta) \sin\theta\, d\theta$$

- Let $u = \cos\theta$; $du = -\sin\theta\, d\theta$; $d\theta = \frac{du}{-\sin\theta}$

$$-\int (1 - 2u^2 + u^4) du = -\left(u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right) + c$$

ODD POWERS OF SINE AND COSINE

- Substituting u back;

$$-\left(u-\frac{2}{3}u^3+\frac{1}{5}u^5\right)+c=-\left(\cos\theta-\frac{2}{3}\cos^3\theta+\frac{1}{5}\cos^5\theta\right)+c$$

- Therefore;

$$\int \sin^5\theta\, d\theta = -\cos\theta+\frac{2}{3}\cos^3\theta-\frac{1}{5}\cos^5\theta+\mathrm{c}$$

- For an odd power of cosine apply the same concepts

EVEN POWERS OF SINE AND COSINE

- For powers that are already even, we apply the double angle formulas for cosine to simplify the integrand.
- Recall, the double angle formulas for cosine as follows:
  1. $\cos 2x = \cos^2 x - \sin^2 x$
  2. $\cos 2x = 2\cos^2 x - 1$
  3. $\cos 2x = 1 - 2\sin^2 x$
- The u-substitution of angle multiples deals with the integral such that $\int \sin nx \, dx = \frac{-1}{n} \cos nx + c$ and $\int \cos nx \, dx = \frac{1}{n} \sin nx + c$.

EVEN POWERS OF SINE AND COSINE

Example: Evaluate $\int \cos^4 x\, dx$

- Since $\cos^4 x$ is in terms of cosine form, we will substitute the identity 2 from the preceding slide. That is, $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$.

$$\int \cos^4 x\, dx = \left(\frac{1}{2}(1 + \cos 2x)\right)^2 dx = \int \frac{1}{4}(1 + 2\cos 2x + \cos^2 2x)dx$$

- Manipulating the same identity we get $\cos^2 2x = \frac{1}{2}(1 + \cos 4x)$

$$= \int \frac{1}{4}\left(1 + 2\cos 2x + \frac{1}{2}(1 + \cos 4x)\right) dx$$

EVEN POWERS OF SINE AND COSINE

$$\int \frac{1}{4}\left(1 + 2\cos 2x + \frac{1}{2} + \frac{1}{2}\cos 4x\right) dx = \int \left(\frac{3}{8} + \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x\right) dx$$

- Therefore;

$$\int \cos^4 x\, dx = \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + c$$

- PROOF: Evaluate $\int \frac{1}{a}\sin nx\, dx$

let $u = nx$ and $du = ndx$. Thus, $\int \frac{1}{a\times n}\sin u\, du = -\frac{1}{an}\cos nx + c$

$\int \cos nx\, dx$ left for practice.

Practice questions

1. $\int \sin^3 x \, dx$
2. $\int \cos^6 x \, dx$
3. $\int \sin^5 x \, dx$

PRODUCTS OF POWERS OF SINE AND COSINE

In some questions, you might be given an expression which is the product of sine and cosine. The question comes in this form: $\int \sin^m x \cos^n x \, dx$.

- The approach to this questions changes depending on which one between m and n is even or old.
- In this integral if the exponent on the sines (m) is odd we can strip out one sine, convert the rest to cosines and then use the substitution $u = \cos x$. Likewise, if the exponent on the cosines (n) is odd we can strip out one cosine and convert the rest to sines using and the use the substitution $u = \sin x$.

PRODUCTS OF POWERS OF SINE AND COSINE

- If both of them are odd, just chose one function ($\sin^m x$ or $\cos^n x$) to dissolve into the other.
- In the case where both of them are even, the technique we used in the first descriptions simply won't work and in fact there really isn't any one set method for doing these integrals. Each integral is different and in some cases there will be more than one way to do the integral.

PRODUCTS OF POWERS OF SINE AND COSINE

Example: Evaluate $\int \sin^6 x \cos^3 x \, dx$.

- In this case the exponent on the sine is even while the exponent on the cosine is odd. So, this time we’ll strip out a cosine and convert the rest to sines.

$$\int \sin^6 x \cos^3 x \, dx = \int \sin^6 x \cos^2 x \cos x \, dx \text{ but } \cos^2 x = 1 - \sin^2 x$$

$$= \int \sin^6 x \, (1 - \sin^2 x) \cos x dx = \int (\sin^6 x - \sin^8 x) \cos x \, dx$$

$$\text{Let } u = \sin x \, ; du = cosxdx$$

$$\int (u^6 - u^8) du = \frac{1}{7} u^7 - \frac{1}{9} u^9 + c = \frac{1}{7} \sin^7 x - \frac{1}{9} \sin^9 x + c$$

PRODUCTS OF POWERS OF SINE AND COSINE

Example: Evaluate $\int \sin^2 x \cos^2 x \, dx$.

Solution 1:

- Write $\sin^2 x$ and $\cos^2 x$ in terms of double angle formulae

$$\int \sin^2 x \cos^2 x \, dx = \int \left(\tfrac{1}{2}(1+\cos 2x)\right)\left(\tfrac{1}{2}(1-\cos 2x)\right) dx \text{ (diff. of squares)}$$

$$= \int \tfrac{1}{4}(1-\cos^2 2x)\, dx = \int \tfrac{1}{4}\left(1-\tfrac{1}{2}(1+\cos 4x)\right) dx \text{ (since } \cos^2 2x = 1+\cos 4x)$$

$$\int \frac{1}{4}\left(1-\frac{1}{2}-\frac{1}{2}\cos 4x\right) dx = \int \left(\frac{1}{8}-\frac{1}{8}\cos 4x\right) dx = \frac{1}{8}x - \frac{1}{32}\sin 4x + c$$

Solution 2:

$$\int \sin^2 x \cos^2 x \, dx = \int (\sin x \cos x)^2 dx$$

- Since $2\sin x \cos x = \sin 2x$, $\sin x \cos x = \frac{1}{2}\sin 2x$

$$\int (\sin x \cos x)^2 dx = \int \left(\frac{1}{2}\sin 2x\right)^2 dx \text{ but } \sin^2 2x = \frac{1}{2}(1 - \cos 4x)$$

$$= \int \frac{1}{4}\left(\frac{1}{2}(1 - \cos 4x)\right) dx = \int \left(\frac{1}{8} - \frac{1}{8}\cos 4x\right) dx$$

Therefore;

$$\int \sin^2 x \cos^2 x \, dx = \frac{1}{8}x - \frac{1}{32}\sin 4x + c$$

PRODUCTS OF POWERS OF SINE AND COSINE

Practice questions

Evaluate the following

1. $\int \sin^3 x \cos^4 x \, dx$
2. $\int \sin^4 x \cos^2 x \, dx$

PRODUCT OF MULTIPLE ANGLES OF SINE AND COSINE

- Upon looking at products of powers of sine and cosine, our attention draws to products of sine and cosine which involve multiple angles.
- That is, $\int \sin nx \cos mx \, dx$ where m and n are integers. This type of integrals use the trigonometric product formulae also known as the product-to-sum identities. Recall the product formulae,

1. $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$
2. $\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$
3. $\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$
4. $\cos A \sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$

PRODUCT OF MULTIPLE ANGLES OF SINE AND COSINE

Example: Evaluate $\int \sin 5x \cos 3x \, dx$.

- On this will use $\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$ as follows:

$$\int \sin 5x \cos 3x \, dx = \int \frac{1}{2}[\sin(5x+3x) + \sin(5x-3x)] \, dx$$

$$= \int \frac{1}{2} \sin 8x \, dx + \int \frac{1}{2} \sin 2x \, dx \text{ but } \int \frac{1}{a} \sin nx \, dx = -\frac{1}{an} \cos nx + c$$

$$= \frac{-1}{2 \times 8} \cos 8x - \frac{1}{2 \times 2} \cos 2x + c$$

Therefore;

$$\int \sin 5x \cos 3x \, dx = c - \frac{1}{16} \cos 8x - \frac{1}{4} \cos 2x$$

PRODUCT OF MULTIPLE ANGLES OF SINE AND COSINE

Practice Questions

Evaluate the following

1. $\int cos\,8x\,cos\,3x\,\mathrm{d}x$
2. $\int \cos 6x \sin 2x\,\mathrm{d}x$

POWERS OF TANGENT AND SECANT

- Powers of tangent functions go hand and hand with powers of secant functions as shown below:

$$\int \tan^m x \sec^n x \, dx$$

- Differences in approach comes due to changes in the nature of the powers of tangent and secant respectively.
- $\boldsymbol{m}$ and $\boldsymbol{n}$ can either be even or old depending on the question given.

EVEN POWER FOR SECANT

- In one of the situations, secant will have an even power and tan will have an odd or even power, that is, n is even.
- If this happens you write the integral as

$$\int \tan^m x \sec^{n-2} x \sec^2 x \, dx$$

- Then, you have to write $\sec^{n-2} x$ in terms as $\tan x$ using the identity $\sec^2 x = 1 + \tan^2 x$. If $n - 2$ is greater than 2, you will express it as power multiple of since it will be even. Letting $u = \tan x$, the integrand is simplified and $\sec^2 x$ gets eliminated.

EVEN POWER FOR SECANT

Example: Evaluate $\int \tan^2 x \sec^4 x \, dx$.

Firstly, we have to simplify $tan^2 x \, sec^4 x$ using the rules provided.

$$\tan^2 x \sec^4 x = \tan^2 x \sec^{4-2} x \sec^2 x = \tan^2 x \sec^2 x \sec^2 x$$

But $sec^2 x = 1 + tan^2 x$, we substitute on one $sec^2 x$ and simplify.

$$= \tan^2 x \, (1 + \tan^2 x) \sec^2 x = (\tan^2 x + \tan^4 x) \sec^2 x$$

This means that $\int \tan^2 x \sec^4 x \, dx = \int (\tan^2 x + \tan^4 x) \sec^2 x \, dx$.

Using substitution method of integration, let $u = \tan x$.

EVEN POWER FOR SECANT

$$du = \sec^2 x\, dx; dx = \frac{du}{\sec^2 x}$$

Since the remaining function is $\int (u^2 + u^4) \sec^2 x\, dx$*, substitute* dx.

$$= \int (u^2 + u^4) \sec^2 x \times \frac{du}{\sec^2 x} = \int (u^2 + u^4) du = \frac{1}{3}u^3 + \frac{1}{5}u^5 + c$$

After substituting u back, $\int \tan^2 x \sec^4 x\, dx = \frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x + c$.

EVEN POWER FOR SECANT

Below are the points we should take closer attention at.

- When simplifying the function, on the first step we are having $\int \tan^m x \sec^{n-2} x \sec^2 x \, dx$ which is found by applying rules of indices.
- The other thing we should see is that since n is even, $n - 2$ will always be even. Hence if it is greater than 2, the provided identity will always work e.g. $\sec^{6-2} x \sec^2 x = \sec^4 x \sec^2 x = (1 + \tan^2 x)^2 \sec^2 x$.
- The idea of taking out $\sec^2 x$ from $\sec^m x$ is brought into account to enable elimination through substituting the differentiation of $\tan x$ which is $\sec^2 x$.

ODD POWERS OF TANGENT

- Once the power secant is odd, the preceding method does not work. In this case we consider another situation of having an odd power of tangent.
- In this case you write the integral $\int \tan^m x \sec^n x\, dx$ where m is odd as shown below:

$$\int \tan^{m-1} x \sec^{n-1} x \tan x \sec x\, dx$$

- Once this happen, we know that $m - 1$ is now even. That is, we will apply $\sec^2 x = 1 + \tan^2 x$ but this time to write $\tan x$ in terms of $\sec x$. Then the substitution of $u = \sec x$ simplifies the integrand.

ODD POWERS OF TANGENT

Example: Evaluate $\int \tan^3 x \sec^3 x \, dx$.

Firstly, we have to simplify $\tan^3 x \sec^3 x$ using the rules provided.

$$\tan^3 x \sec^3 x = \tan^{3-1} x \sec^{3-1} x \tan x \sec x = \tan^2 x \sec^2 x \tan x \sec x$$

But $\sec^2 x - 1 = \tan^2 x$, we substitute on $\tan^2 x$ and simplify.

$$= (\sec^2 x - 1) \sec^2 x \tan x \sec x = (\sec^4 x - \sec^2 x) \tan x \sec x$$

Thus, $\int \tan^3 x \sec^3 x \, dx = \int (\sec^4 x - \sec^2 x) \tan x \sec x \, dx$

Let $u = \sec x$, $du = \tan x \sec x \, dx$; $dx = \dfrac{du}{\tan x \sec x}$

ODD POWERS OF TANGENT

Since the remaining function is $\int (u^4 - u^2)\tan x \sec x\, dx$, *substitute* dx.

$$= \int (u^2 + u^4)\tan x \sec x \times \frac{du}{\tan x \sec x} = \int (u^4 - u^2)du$$

$$= \frac{1}{5}u^5 - \frac{1}{3}u^3 + c$$

After substituting u back, $\int \tan^3 x \sec^3 x\, dx = \frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + c$.

EVEN POWERS OF TAN, ODD POWER OF SEC

- If n is odd and m is even, this situation do not match any of the presented method.
- In such a case, the use other methods like integration by parts should be used instead.

POWERS OF TANGENT AND SECANT

Practice questions

1. Integrate $\tan^3 x \sec^5 x$ with respect to x.
2. Evaluate the following integrals

   a. $\int \tan^2 x \sec^6 x \, dx$

   b. $\int \cot^3 x \csc^3 x \, dx$

   c. $\int \tan^5 x \sec x \, dx$

Integrals involving

TRIGONOMETRIC SUBSITUTIONS

TRIGONOMETRIC SUBSITUTION

- Integrals involving trigonometric functions may grow complicated to an extent that we apply the identities to simplify them. This is what we have been so far.
- But in some occasions, this is not always the story. We might tend to encounter some integrands involving radicals which are seemingly impossible to integrate.
- For example, integrands like $\sqrt{a^2 - x^2}, \sqrt{a^2 + x^2}$ and $\sqrt{x^2 - a^2}$ which have no clear way to use in the integration process.
- Therefore, the substitution of a specific trig function helps to eliminate the radicals and usually simplifies the integrand.

TRIGONOMETRIC SUBSITUTION

- Before we dive deep into the process, you might wish to recall the Pythagorean trig identity, $\cos^2 x + \sin^2 x = 1$ which gives birth to $1 + \tan^2 x = \sec^2 x$ and $\cot^2 x + 1 = \csc^2 x$.

Table below shows specific trig substitution to match each given radical.

Expression Given	Trig Substitution
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$

POINTS TO NOTE

- When making substitutions we assume that θ is in the range of their respective inverse functions.
- If the radicals appear in the denominators, always add a condition restricting the value of the denominator to never be equal to zero. For example, if $\sqrt{a^2 - x^2}$ is a denominator always say "where $|x| = a$.

TRIGONOMETRIC SUBSITUTION

Example: Evaluate $\int \frac{dx}{x^2\sqrt{16-x^2}}$.

Let $a = \sqrt{16} = 4$ and considering the radical, $x = a\sin\theta = 4\sin\theta$, thus;

$$\int \frac{dx}{x^2\sqrt{16-x^2}} = \int \frac{dx}{(4\sin\theta)^2\sqrt{16-(4\sin\theta)^2}} = \int \frac{1}{16\sin^2\theta\sqrt{16(1-\sin^2\theta)}}dx$$

$$= \int \frac{1}{16\sin^2\theta\sqrt{16\cos^2\theta}}dx \text{ since } 1-\sin^2\theta = \cos^2\theta$$

$$= \int \frac{1}{16\sin^2\theta\,(4\cos\theta)}dx$$

Since $x = 4\sin\theta\,, dx = 4cos\theta d\theta$. This will eliminate dx and allow us to integrate with respect to theta (θ).

Eliminate dx and simplify.

$$= \int \frac{1}{16\sin^2\theta\,(4\cos\theta)} \times 4\cos\theta\, d\theta = \frac{1}{16\sin^2\theta} d\theta = \int \frac{1}{16}\csc^2\theta\, d\theta$$

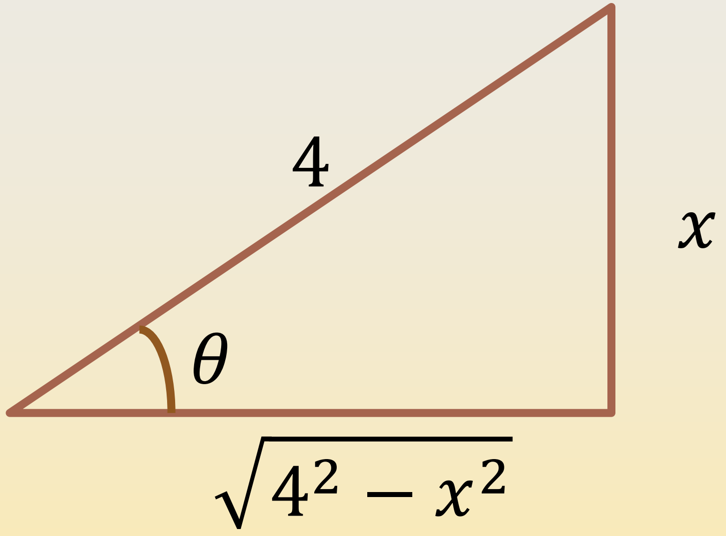
Therefore, it is simple to integrate $\csc^2\theta$.

$$= -\frac{1}{16}\cot\theta + C$$

It is now necessary to get back to the original variable, x. Since $x = 4\sin\theta\,, \sin\theta = \frac{x}{4}$.

Therefore, $\theta = \sin^{-1}\frac{x}{4}$

Using the triangle below,



$$\cot\theta = \frac{1}{\tan\theta} = \frac{adiacent}{opposite}$$

$$\text{Therefore } \cot\theta = \frac{\sqrt{16-x^2}}{x}$$

Finally,

$$\int \frac{dx}{x^2\sqrt{16-x^2}} = -\frac{\sqrt{16-x^2}}{16x} + C$$

TRIGONOMETRIC SUBSITUTION

POINTS TO NOTE

- The idea mostly lies on selecting the correct substitution. Every other thing is just mere algebra.
- Always remember to change the integral back to the original variable, x.

Practice questions

1. Evaluate the following integrals.

   a. $\int \frac{1}{\sqrt{4+x^2}} dx$

   b. $\int \frac{\sqrt{x^2-9}}{x} dx$

   c. $\int \frac{1}{x^4\sqrt{x^2-3}} dx$

Thank You For Reading!!

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