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Calculus II

Tutorial 1

Josophat Makawa Chifundo

University of Malawi

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Introduction to Transcendental Functions

This tutorial covers the basics of transcendental functions, including exponential and logarithmic functions, as well as inverse trigonometric functions. It provides a comprehensive overview of the concepts, properties, and applications of these functions in calculus.

Note

This document contains tutorial problems with detailed solutions to help reinforce your understanding of the concepts. Make sure to attempt the problems before looking at the solutions.

1 Logarithmic Differentiation

Logarithmic differentiation is a technique used to differentiate functions that are products or quotients of variables raised to variable powers. It is particularly useful when dealing with functions of the form $y = f(x)^{g(x)}$, where both $f(x)$ and $g(x)$ are functions of x. By taking the natural logarithm of both sides, we can simplify the differentiation process.

Highlight

Key Concept: When differentiating expressions where both the base and exponent are functions of x, logarithmic differentiation often simplifies the process.

Problem

Problem 1.1: Differentiate the function $y = x^x$ with respect to x.

Solution

Solution:

Given the function $y = x^x$.

Since both the base and the exponent involve the variable x, we use logarithmic differentiation.

Step 1: Take the natural logarithm of both sides of the equation.

$$\ln y = \ln(x^x)$$

Step 2: Apply the logarithm property $\ln\left(a^b\right) = b\ln a$ to the right side.

$$\ln y = x \ln x$$

Step 3: Differentiate both sides with respect to x. The left side requires implicit differentiation, while the right side uses the product rule with $u = x$ and $v = \ln x$.

$$\begin{aligned}
\frac{d}{dx}(\ln y) &= \frac{d}{dx}(x\ln x) \\
\frac{1}{y}\cdot\frac{dy}{dx} &= \left(\frac{d}{dx}(x)\right)\ln x + x\left(\frac{d}{dx}(\ln x)\right) \\
\frac{1}{y}\frac{dy}{dx} &= (1)\cdot\ln x + x\cdot\left(\frac{1}{x}\right) \\
\frac{1}{y}\frac{dy}{dx} &= \ln x + 1
\end{aligned}$$

Step 4: Solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = y(\ln x + 1)$$

Step 5: Substitute $y = x^x$:

$$\frac{dy}{dx} = x^x(\ln x + 1)$$

Thus, the derivative is $\boxed{x^x(1 + \ln x)}$.

Problem

Problem 1.2: Find $\dfrac{dy}{dx}$ if $y = \ln(x^2+1)^{\sin x}$.

Solution

Solution:

Let $y = \ln(x^2+1)^{\sin x}$. Take the natural logarithm of both sides:

$$\ln y = \sin x \cdot \ln\left(x^2+1\right)$$

Differentiate both sides with respect to x:

$$\frac{1}{y}\frac{dy}{dx} = \cos x \cdot \ln\left(x^2+1\right) + \sin x \cdot \frac{2x}{x^2+1}$$

Multiply both sides by y and substitute $y = (x^2+1)^{\sin x}$:

$$\frac{dy}{dx} = (x^2+1)^{\sin x}\left[\cos x \cdot \ln\left(x^2+1\right) + \sin x \cdot \frac{2x}{x^2+1}\right]$$

Final derivative: $\boxed{(x^2+1)^{\sin x}\left(\cos x \ln\left(x^2+1\right) + \frac{2x\sin x}{x^2+1}\right)}$.

Problem

Problem 1.3: Differentiate the function $y = (x^2+1)^{\tan x}$ with respect to x.

Solution

Solution:

Let $y = (x^2+1)^{\tan x}$. Take the natural logarithm of both sides:

$$\ln y = \tan x \cdot \ln\left(x^2+1\right)$$

Differentiate both sides with respect to x:

$$\frac{1}{y}\frac{dy}{dx} = \sec^2 x \cdot \ln\left(x^2+1\right) + \tan x \cdot \frac{2x}{x^2+1}$$

Multiply both sides by y and substitute $y = (x^2+1)^{\tan x}$:

$$\frac{dy}{dx} = (x^2+1)^{\tan x}\left[\sec^2 x \cdot \ln\left(x^2+1\right) + \frac{2x\tan x}{x^2+1}\right]$$

Final derivative: $\boxed{(x^2+1)^{\tan x}\left(\sec^2 x \ln\left(x^2+1\right) + \frac{2x\tan x}{x^2+1}\right)}$.

Problem

Problem 1.4: Find $\frac{dy}{dx}$ if $y = \frac{(x^2+3)^5}{(2x-1)^4}$.

Solution

Solution:

Let $y = \frac{(x^2+3)^5}{(2x-1)^4}$. Take the natural logarithm of both sides:

$$\ln y = 5\ln\left(x^2+3\right) - 4\ln(2x-1)$$

Differentiate both sides with respect to x:

$$\begin{aligned}\frac{1}{y}\frac{dy}{dx} &= 5 \cdot \frac{2x}{x^2+3} - 4 \cdot \frac{2}{2x-1} \\ &= \frac{10x}{x^2+3} - \frac{8}{2x-1}\end{aligned}$$

Multiply both sides by y:

$$\frac{dy}{dx} = \frac{(x^2+3)^5}{(2x-1)^4}\left(\frac{10x}{x^2+3} - \frac{8}{2x-1}\right)$$

Final derivative: $\boxed{\frac{(x^2+3)^5}{(2x-1)^4}\left(\frac{10x}{x^2+3} - \frac{8}{2x-1}\right)}$.

Problem

Problem 1.5: Differentiate $y = x^2\sqrt{x+1}\,e^{3x}$ with respect to x.

Solution

Solution:

Let $y = x^2\sqrt{x+1}\,e^{3x}$. Take the natural logarithm of both sides:

$$\ln y = 2\ln x + \frac{1}{2}\ln(x+1) + 3x$$

Differentiate both sides with respect to x:

$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x} + \frac{1}{2(x+1)} + 3$$

Multiply both sides by y:

$$\frac{dy}{dx} = x^2\sqrt{x+1}\,e^{3x}\left(\frac{2}{x} + \frac{1}{2(x+1)} + 3\right)$$

Final derivative: $\boxed{x^2\sqrt{x+1}\,e^{3x}\left(\frac{2}{x} + \frac{1}{2(x+1)} + 3\right)}$.

Problem

Problem 1.6: Find $\dfrac{dy}{dx}$ if $y = (x^{\sin x})^{\cos x}$.

Solution

Solution:

Let $y = (x^{\sin x})^{\cos x}$. Take the natural logarithm of both sides:

$$\begin{aligned} \ln y &= \cos x \cdot \ln\left(x^{\sin x}\right) \\ &= \cos x \cdot \sin x \cdot \ln x \end{aligned}$$

Differentiate both sides with respect to x:

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}[\cos x \cdot \sin x \cdot \ln x]$$

Let $u = \cos x$, $v = \sin x$, $w = \ln x$. Use the product rule:

$$\frac{d}{dx}(uvw) = u'vw + uv'w + uvw'$$

$$u' = -\sin x, \quad v' = \cos x, \quad w' = \frac{1}{x}$$

$$\begin{aligned} \frac{1}{y}\frac{dy}{dx} &= (-\sin x)\sin x \ln x + \cos x \cos x \ln x + \cos x \sin x \cdot \frac{1}{x} \\ &= -\sin^2 x \ln x + \cos^2 x \ln x + \frac{\cos x \sin x}{x} \end{aligned}$$

Multiply both sides by y:

$$\frac{dy}{dx} = (x^{\sin x})^{\cos x}\left[(\cos^2 x - \sin^2 x)\ln x + \frac{\cos x \sin x}{x}\right]$$

Final derivative: $\boxed{(x^{\sin x})^{\cos x}\left((\cos^2 x - \sin^2 x)\ln x + \dfrac{\cos x \sin x}{x}\right)}$.

Problem

Problem 1.7: Find $\dfrac{dy}{dx}$ if $y = e^{x \ln x}$.

Solution

Solution:

Rewrite $y = e^{x \ln x} = x^x$. From Problem 1:

Take natural logarithm:

$$\ln y = x \ln x$$

Differentiate both sides:

$$\frac{1}{y}\frac{dy}{dx} = \ln x + 1$$

Solve for derivative:

$$\frac{dy}{dx} = x^x(\ln x + 1)$$

Final derivative: $\boxed{x^x(1 + \ln x)}$.

Problem

Problem 1.8: Find $\dfrac{dy}{dx}$ if $y = \ln(x^2+1)^{\sin x}$.

Solution

Solution:

Let $y = \ln(x^2+1)^{\sin x}$. Take natural logarithm:

$$\ln y = \sin x \cdot \ln\left(x^2 + 1\right)$$

Differentiate both sides:

$$\frac{1}{y}\frac{dy}{dx} = \cos x \cdot \ln\left(x^2+1\right) + \sin x \cdot \frac{2x}{x^2+1}$$

Multiply by $y = (x^2+1)^{\sin x}$:

$$\frac{dy}{dx} = (x^2+1)^{\sin x}\left[\cos x \ln\left(x^2+1\right) + \frac{2x \sin x}{x^2+1}\right]$$

Final derivative: $\boxed{(x^2+1)^{\sin x}\left(\cos x \ln\left(x^2+1\right) + \frac{2x\sin x}{x^2+1}\right)}$.

Problem

Problem 1.9: Find $\dfrac{dy}{dx}$ if $y = \ln(\sin(\ln x))$.

Solution

Solution:

Let $y = \ln(\sin(\ln x))$. Apply chain rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{\sin(\ln x)} \cdot \cos(\ln x) \cdot \frac{1}{x} \\ &= \frac{\cos(\ln x)}{x \sin(\ln x)}\end{aligned}$$

Final derivative: $\boxed{\dfrac{\cos(\ln x)}{x \sin(\ln x)}}$.

2 Logarithmic Integration

Logarithmic integration is a technique used to evaluate integrals that involve logarithmic functions. It is particularly useful when dealing with integrals of the form $\int \frac{f(x)}{g(x)}\,dx$, where $f(x)$ and $g(x)$ are functions of x. By using integration by parts, partial fractions or substitution, we can simplify the integral and find its value.

Problem

Problem 2.1: Evaluate the integral $\int \frac{1}{x}\,dx$.

Solution

Solution:

Using logarithmic integration:

$$\int \frac{1}{x}\,dx = \boxed{\ln|x| + C}$$

Problem

Problem 2.2: Evaluate the integral $\int \frac{2x}{x^2+1}\,dx$.

Solution

Solution:

Let $u = x^2 + 1 \implies du = 2x\,dx$:

$$\begin{aligned}\int \frac{2x}{x^2+1}\,dx &= \int \frac{1}{u}\,du \\ &= \ln|u| + C \\ &= \boxed{\ln|x^2+1| + C}\end{aligned}$$

Problem

Problem 2.3: Evaluate the integral $\int \frac{1}{3x+2}\,dx$.

Solution

Solution:

Let $u = 3x + 2 \implies du = 3\,dx \implies dx = \frac{du}{3}$:

$$\begin{aligned}\int \frac{1}{3x+2}\,dx &= \int \frac{1}{u}\cdot\frac{du}{3} \\ &= \frac{1}{3}\ln|u| + C \\ &= \boxed{\frac{1}{3}\ln|3x+2| + C}\end{aligned}$$

Problem

Problem 2.4: Evaluate the integral $\int \frac{5}{2x-7}\,dx$.

Solution

Solution:

Let $u = 2x - 7 \implies du = 2\,dx \implies dx = \frac{du}{2}$:

$$\begin{aligned}\int \frac{5}{2x-7}\,dx &= 5\int \frac{1}{u}\cdot\frac{du}{2} \\ &= \frac{5}{2}\ln|u| + C \\ &= \boxed{\frac{5}{2}\ln|2x-7| + C}\end{aligned}$$

Problem

Problem 2.5: Evaluate the integral $\int \frac{1}{x^2-4}\,dx$.

Solution

Solution:

Factor denominator and use partial fractions:

$$\frac{1}{x^2-4} = \frac{A}{x-2} + \frac{B}{x+2}$$

$$1 = A(x+2) + B(x-2)$$

Solving for coefficients:

$$\text{Let } x = 2 : 1 = 4A \implies A = \frac{1}{4}$$
$$\text{Let } x = -2 : 1 = -4B \implies B = -\frac{1}{4}$$

Integrate:

$$\begin{aligned}
\int \frac{1}{x^2-4}dx &= \frac{1}{4}\int \frac{1}{x-2}dx - \frac{1}{4}\int \frac{1}{x+2}dx \\
&= \frac{1}{4}\ln|x-2| - \frac{1}{4}\ln|x+2| + C \\
&= \boxed{\frac{1}{4}\ln\left|\frac{x-2}{x+2}\right| + C}
\end{aligned}$$

Problem

Problem 2.6: Evaluate the integral $\int \frac{3x+5}{x^2+5x+6}\,dx$.

Solution

Solution:

Factor denominator:

$$x^2 + 5x + 6 = (x+2)(x+3)$$

Partial fraction decomposition:

$$\frac{3x+5}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}$$
$$3x + 5 = A(x+3) + B(x+2)$$

Solve for coefficients:

$$\text{Let } x = -3 : 3(-3) + 5 = -B \implies B = 4$$
$$\text{Let } x = -2 : 3(-2) + 5 = A \implies A = -1$$

Integrate:

$$\int \frac{3x+5}{x^2+5x+6}dx = \int \frac{-1}{x+2}dx + \int \frac{4}{x+3}dx$$
$$= -\ln|x+2| + 4\ln|x+3| + C$$
$$= \boxed{4\ln|x+3| - \ln|x+2| + C}$$

Problem

Problem 2.7: Evaluate the integral $\int \frac{2x+1}{x^2+x}\,dx$.

Solution

Solution:

Factor denominator:

$$x^2 + x = x(x+1)$$

Partial fraction decomposition:

$$\frac{2x+1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$
$$2x+1 = A(x+1) + Bx$$

Solve for coefficients:

$$\text{Let } x = 0 : 1 = A \implies A = 1$$
$$\text{Let } x = -1 : -1 = -B \implies B = 1$$

Integrate:

$$\int \frac{2x+1}{x^2+x}dx = \int \frac{1}{x}dx + \int \frac{1}{x+1}dx$$
$$= \boxed{\ln|x| + \ln|x+1| + C}$$

Problem

Problem 2.8: Evaluate the integral $\int \frac{4x-1}{2x^2-x}\,dx$.

Solution

Solution:

Factor denominator:

$$2x^2 - x = x(2x-1)$$

Partial fraction decomposition:

$$\frac{4x-1}{x(2x-1)} = \frac{A}{x} + \frac{B}{2x-1}$$
$$4x - 1 = A(2x-1) + Bx$$

Solve for coefficients:

$$\text{Let } x = 0: \; -1 = -A \implies A = 1$$
$$\text{Let } x = \frac{1}{2}: 1 = \frac{B}{2} \implies B = 2$$

Integrate:

$$\int \frac{4x-1}{2x^2-x}dx = \int \frac{1}{x}dx + \int \frac{2}{2x-1}dx$$
$$\text{Let } u = 2x-1,\; du = 2dx \implies dx = \frac{du}{2}$$
$$\int \frac{2}{2x-1}dx = \int \frac{1}{u}du = \ln|u| + C$$
$$= \ln|2x-1| + C$$
$$\therefore \int \frac{4x-1}{2x^2-x}dx = \boxed{\ln|x| + \ln|2x-1| + C}$$

Problem

Problem 2.9: Evaluate the integral $\int \frac{1}{x^2+4x+3}\,dx$.

Solution

Solution:

Factor denominator:

$$x^2 + 4x + 3 = (x + 1)(x + 3)$$

Partial fraction decomposition:

$$\frac{1}{(x+1)(x+3)} = \frac{A}{x+1} + \frac{B}{x+3}$$
$$1 = A(x+3) + B(x+1)$$

Solve for coefficients:

$$\text{Let } x = -3 : 1 = -2B \implies B = -\frac{1}{2}$$
$$\text{Let } x = -1 : 1 = 2A \implies A = \frac{1}{2}$$

Integrate:

$$\int \frac{1}{x^2+4x+3}dx = \frac{1}{2}\int \frac{1}{x+1}dx - \frac{1}{2}\int \frac{1}{x+3}dx$$
$$= \frac{1}{2}\ln|x+1| - \frac{1}{2}\ln|x+3| + C$$
$$= \boxed{\frac{1}{2}\ln\left|\frac{x+1}{x+3}\right| + C}$$

Problem

Problem 2.10: Evaluate the integral $\int \frac{2x+3}{x^2+3x+2}\,dx$.

Solution

Solution:

Factor denominator:

$$x^2 + 3x + 2 = (x + 1)(x + 2)$$

Partial fraction decomposition:

$$\frac{2x+3}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$
$$2x+3 = A(x+2) + B(x+1)$$

Solve for coefficients:

$$\text{Let } x = -2: \; -1 = -B \implies B = 1$$
$$\text{Let } x = -1: 1 = A \implies A = 1$$

Integrate:

$$\int \frac{2x+3}{x^2+3x+2} dx = \int \frac{1}{x+1} dx + \int \frac{1}{x+2} dx$$
$$= \boxed{\ln|x+1| + \ln|x+2| + C}$$

Problem

Problem 2.11: Evaluate the integral $\int \frac{1}{x^2 - 2x}\, dx$.

Solution

Solution:

Factor denominator:

$$x^2 - 2x = x(x-2)$$

Partial fraction decomposition:

$$\frac{1}{x(x-2)} = \frac{A}{x} + \frac{B}{x-2}$$
$$1 = A(x-2) + Bx$$

Solve for coefficients:

$$\text{Let } x = 0: 1 = -2A \implies A = -\frac{1}{2}$$
$$\text{Let } x = 2: 1 = 2B \implies B = \frac{1}{2}$$

Integrate:

$$\int \frac{1}{x^2 - 2x} dx = -\frac{1}{2} \int \frac{1}{x} dx + \frac{1}{2} \int \frac{1}{x-2} dx$$
$$= \boxed{-\frac{1}{2} \ln|x| + \frac{1}{2} \ln|x-2| + C}$$

Problem

Problem 2.12: Evaluate the integral $\int \frac{3x+2}{x^2+2x}\, dx$.

Solution

Solution:

Factor denominator:

$$x^2 + 2x = x(x+2)$$

Partial fraction decomposition:

$$\frac{3x+2}{x(x+2)} = \frac{A}{x} + \frac{B}{x+2}$$
$$3x + 2 = A(x+2) + Bx$$

Solve for coefficients:

$$\text{Let } x = 0 : 2 = 2A \implies A = 1$$
$$\text{Let } x = -2 : -4 = -2B \implies B = 2$$

Integrate:

$$\int \frac{3x+2}{x^2+2x} dx = \int \frac{1}{x} dx + \int \frac{2}{x+2} dx$$
$$= \boxed{\ln|x| + 2\ln|x+2| + C}$$

Problem

Problem 2.13: Evaluate the definite integral $\int_1^e \frac{1}{x}\, dx$.

Solution

Solution:

Antiderivative of $\frac{1}{x}$:

$$\begin{aligned}\int_1^e \frac{1}{x}dx &= [\ln|x|]_1^e \\ &= \ln e - \ln 1 \\ &= 1 - 0 \\ &= \boxed{1}\end{aligned}$$

Problem

Problem 2.14: Evaluate the definite integral $\int_0^1 \frac{2}{x+2}\,dx$.

Solution

Solution:

Let $u = x + 2 \implies du = dx$:

$$\begin{aligned}\int_0^1 \frac{2}{x+2}dx &= 2\left[\ln|x+2|\right]_0^1 \\ &= 2(\ln 3 - \ln 2) \\ &= 2\ln\left(\frac{3}{2}\right) \\ &= \boxed{2\ln\frac{3}{2}}\end{aligned}$$

Problem

Problem 2.15: Evaluate the definite integral $\int_1^2 \frac{1}{x^2}\,dx$.

Solution

Solution:

Antiderivative of $\frac{1}{x^2}$:

$$\begin{aligned}\int_1^2 \frac{1}{x^2}dx &= \left[-\frac{1}{x}\right]_1^2 \\ &= \left(-\frac{1}{2}\right) - (-1) \\ &= 1 - \frac{1}{2} \\ &= \boxed{\frac{1}{2}}\end{aligned}$$

3 Derivatives of Inverse Trig Functions

The derivatives of inverse trigonometric functions are important in calculus and have various applications in mathematics and physics. Below are some problems and solutions related to the derivatives of inverse trigonometric functions for your practice.

Problem

Problem 3.1: Find $\frac{d}{dx} \arcsin(2x)$.

Solution

Solution:

Let $y = \arcsin(2x)$. By the chain rule:

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{\sqrt{1-(2x)^2}} \cdot 2 \\ &= \boxed{\frac{2}{\sqrt{1-4x^2}}} \end{aligned}$$

Problem

Problem 3.2: Differentiate $y = x \arctan x$ with respect to x.

Solution

Solution:

Using the product rule:

$$\begin{aligned} \frac{dy}{dx} &= \arctan x + x \cdot \frac{1}{1+x^2} \\ &= \boxed{\arctan x + \frac{x}{1+x^2}} \end{aligned}$$

Problem

Problem 3.3: Find $\frac{d}{dx} \arccos\left(\sqrt{x}\right)$ for $0 < x < 1$.

Solution

Solution:

Let $y = \arccos(\sqrt{x})$. By the chain rule:

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-(\sqrt{x})^2}} \cdot \frac{1}{2\sqrt{x}}$$
$$= \boxed{-\frac{1}{2\sqrt{x}\sqrt{1-x}}}$$

Problem

Problem 3.4: Differentiate $y = \arctan\left(\frac{1}{x}\right)$ for $x > 0$.

Solution

Solution:

Let $y = \arctan\left(\frac{1}{x}\right)$. By the chain rule:

$$\frac{dy}{dx} = \frac{1}{1+\left(\frac{1}{x}\right)^2} \cdot \left(-\frac{1}{x^2}\right)$$
$$= \boxed{-\frac{1}{x^2+1}}$$

Problem

Problem 3.5: Find $\frac{d}{dx}\left[x^2 \arcsin x\right]$.

Solution

Solution:

Using the product rule:

$$\frac{d}{dx}\left[x^2 \arcsin x\right] = 2x \arcsin x + x^2 \cdot \frac{1}{\sqrt{1-x^2}}$$
$$= \boxed{2x \arcsin x + \frac{x^2}{\sqrt{1-x^2}}}$$

Problem

Problem 3.6: Differentiate $y = \arccos(3x^2)$ for $|x| < \frac{1}{\sqrt{3}}$.

Solution

Solution:

Let $y = \arccos(3x^2)$. By the chain rule:

$$\begin{aligned}\frac{dy}{dx} &= -\frac{1}{\sqrt{1-(3x^2)^2}} \cdot \frac{d}{dx}(3x^2) \\ &= -\frac{1}{\sqrt{1-9x^4}} \cdot 6x \\ &= \boxed{-\frac{6x}{\sqrt{1-9x^4}}}\end{aligned}$$

Problem

Problem 3.7: Find $\dfrac{d}{dx}\arctan(e^x)$.

Solution

Solution:

Let $y = \arctan(e^x)$. By the chain rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{1+(e^x)^2} \cdot \frac{d}{dx}(e^x) \\ &= \frac{1}{1+e^{2x}} \cdot e^x \\ &= \boxed{\frac{e^x}{1+e^{2x}}}\end{aligned}$$

4 Integrals Involving Inverse Trig Functions

The integrals involving inverse trigonometric functions are important in calculus and have various applications in mathematics and physics. Below are some problems and solutions related to the integrals involving inverse trigonometric functions for your practice.

Problem

Problem 4.1: Evaluate the integral $\int \frac{1}{\sqrt{1-x^2}}\,dx$.

Solution

Solution:

Standard integral involving the inverse sine function:

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \boxed{\arcsin x + C}$$

Problem

Problem 4.2: Evaluate the integral $\int \frac{1}{1+x^2}\,dx$.

Solution

Solution:

Standard integral involving the inverse tangent function:

$$\int \frac{1}{1+x^2}\,dx = \boxed{\arctan x + C}$$

Problem

Problem 4.3: Evaluate the integral $\int \frac{1}{|x|\sqrt{x^2-1}}\,dx$ for $|x| > 1$.

Solution

Solution:

Standard integral involving the inverse secant function:

$$\int \frac{1}{|x|\sqrt{x^2-1}}\,dx = \boxed{\operatorname{arcsec}|x| + C}$$

Problem

Problem 4.4: Evaluate the integral $\int \frac{2}{4+x^2}\,dx$.

Solution

Solution:

Rewrite and apply the inverse tangent formula:

$$\begin{aligned}\int \frac{2}{4+x^2}dx &= 2\cdot\frac{1}{2}\arctan\left(\frac{x}{2}\right)+C\\ &= \boxed{\arctan\left(\frac{x}{2}\right)+C}\end{aligned}$$

Problem

Problem 4.5: Evaluate the definite integral $\int_0^{1/2} \frac{1}{\sqrt{1-x^2}}\,dx$.

Solution

Solution:

Using the antiderivative $\arcsin x$:

$$\begin{aligned}\int_0^{1/2}\frac{1}{\sqrt{1-x^2}}dx &= [\arcsin x]_0^{1/2}\\ &= \arcsin\left(\frac{1}{2}\right)-\arcsin(0)\\ &= \frac{\pi}{6}-0\\ &= \boxed{\frac{\pi}{6}}\end{aligned}$$

Problem

Problem 4.6: Evaluate the integral $\int \frac{1}{\sqrt{9-x^2}}\,dx$.

Solution

Solution:

Using the standard integral formula with $a = 3$:

$$\begin{aligned}\int \frac{1}{\sqrt{9-x^2}}dx &= \arcsin\left(\frac{x}{3}\right) + C \\ &= \boxed{\arcsin\left(\frac{x}{3}\right) + C}\end{aligned}$$

Problem

Problem 4.7: Evaluate the integral $\int \frac{1}{x^2+6x+10}\,dx$.

Solution

Solution:

Complete the square and integrate:

$$x^2 + 6x + 10 = (x+3)^2 + 1$$

$$\int \frac{1}{(x+3)^2+1}dx = \boxed{\arctan(x+3) + C}$$

Problem

Problem 4.8: Evaluate the integral $\int \frac{1}{\sqrt{4x-x^2}}\,dx$ for $0 < x < 4$.

Solution

Solution:

Complete the square and substitute:

$$4x - x^2 = 4 - (x-2)^2$$

$$\begin{aligned}\int \frac{1}{\sqrt{4-(x-2)^2}}dx &= \arcsin\left(\frac{x-2}{2}\right) + C \\ &= \boxed{\arcsin\left(\frac{x-2}{2}\right) + C}\end{aligned}$$

Problem

Problem 4.9: Evaluate the integral $\int \frac{1}{x^2 - 4x + 5}\,dx$.

Solution

Solution:

Complete the square and integrate:

$$x^2 - 4x + 5 = (x-2)^2 + 1$$
$$\int \frac{1}{(x-2)^2 + 1} dx = \boxed{\arctan(x-2) + C}$$

Problem

Problem 4.10: Evaluate the definite integral $\int_0^1 \frac{1}{1+x^2}\,dx$.

Solution

Solution:

Using the antiderivative of $\arctan x$:

$$\begin{aligned}\int_0^1 \frac{1}{1+x^2} dx &= [\arctan x]_0^1 \\ &= \arctan(1) - \arctan(0) \\ &= \frac{\pi}{4} - 0 \\ &= \boxed{\frac{\pi}{4}}\end{aligned}$$

Problem

Problem 4.11: Evaluate the definite integral $\int_1^2 \frac{1}{\sqrt{4-x^2}}\,dx$.

Solution

Solution:

Using the antiderivative of $\arcsin\left(\frac{x}{2}\right)$:

$$\begin{aligned}
\int_1^2 \frac{1}{\sqrt{4-x^2}}dx &= \left[\arcsin\left(\frac{x}{2}\right)\right]_1^2 \\
&= \arcsin(1) - \arcsin\left(\frac{1}{2}\right) \\
&= \frac{\pi}{2} - \frac{\pi}{6} \\
&= \boxed{\frac{\pi}{3}}
\end{aligned}$$

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Contact Information

Email: makawacean6@gmail.com

Phone: +265 999 978 828 / +265 899 990 636

Website: https://www.jmcmath.com