HYPERBOLIC FUNCTIONS

JOSOPHAT MAKAWA

INTRODUCTION

- Hyperbolic functions are a special case of exponential functions.
- These functions are formed by taking the combination of mainly two exponential functions; e^x and e^{-x}
- The hyperbolic function of sine, denoted as $\sinh x$, and the hyperbolic function of cosine, denoted $\cosh x$, are defined by:

$$\sinh x = \frac{e^x - e^{-x}}{2} \text{ and } \cosh x = \frac{e^x + e^{-x}}{2}$$

where x is a real number.

- $\sinh x$ is read as "cinch x" or "the hyperbolic sine of x" and $\cosh x$ is read as "kosh x" or "the hyperbolic cosine of x"

INTRODUCTION

- From the definitions of $\sinh x$ and $\cosh x$, we can deduce the following hyperbolic functions:

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad \coth x = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

- The terms “tanh,” “sech,” and “csch” are pronounced “tanch,” “seech,” and “coseech,” respectively.

IDENTITIES OF HYPERBOLIC FUNCTIONS

- The identities of hyperbolic functions are similar to those of trigonometric functions in many forms as shown below

$\sinh(-x) = -\sinh x$	$\cosh(-x) = \cosh x$
$\cosh^2 x - \sinh^2 x = 1$	$\tanh^2 x + \operatorname{sech}^2 x = 1; \coth^2 x - \operatorname{csch}^2 x = 1$
$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$	$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$
$\sinh(x-y) = \sinh x \cosh x - \cosh x \sinh y$	$\cosh(x-y) = \cosh x \cosh y - \sinh x \sinh y$
$\sinh 2x = 2 \sinh x \cosh x$	$\cos 2x = \cosh^2 x + \sinh^2 x$
$\sinh^2 x = \frac{1}{2}(\cosh 2x - 1)$	$\cosh^2 x = \frac{1}{2}(\cosh 2x + 1)$

IDENTITIES OF HYPERBOLIC FUNCTIONS

Prove that $\cosh^2 x - \sinh^2 x = 1$.

- Since we know that $\sinh x = \frac{e^x - e^{-x}}{2}$ and $\cosh x = \frac{e^x + e^{-x}}{2}$,

$$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x + e^{-x}}{2}\right)^2$$

$$= \frac{e^{2x} + 2e^x e^{-x} + e^{-2x}}{4} - \frac{e^{2x} - 2e^x e^{-x} + e^{-2x}}{4}$$

$$= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} = \frac{e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}}{4}$$

$$= \frac{2+2}{4} = \frac{4}{4} = 1$$

DERIVATIVES OF HYPERBOLIC FUNCTIONS

- Derivatives of hyperbolic functions can be obtained by expressing the functions in terms of e^x and e^{-x}. After this, differentiating these combinations provides the respective derivatives.

- For example, consider $\frac{d}{dx}[\cosh x] = \frac{d}{dx}\left(\frac{e^x+e^{-x}}{2}\right)$.

$$= \frac{1}{2} \times \frac{d}{dx}(e^x + e^{-x}) = \frac{1}{2}(e^x - e^{-x}) = \frac{e^x - e^{-x}}{2} = \sinh x$$

- And consider $\frac{d}{dx}[\sinh x] = \frac{d}{dx}\left(\frac{e^x-e^{-x}}{2}\right) = \frac{1}{2} \times \frac{d}{dx}(e^x - e^{-x})$

$$= \frac{1}{2}(e^x + e^{-x}) = \frac{e^x + e^{-x}}{2} = \cosh x$$

DERIVATIVES OF HYPERBOLIC FUNCTIONS

- For $\tanh x$, the following approach will be used
- Since $\tanh x = \frac{\sinh x}{\cosh x}$, we will use the logarithmic differentiation or quotient rule. In this example, we will use logarithmic differentiation.

Let $y = \frac{\sinh x}{\cosh x}$, thus, $\ln y = \ln \frac{\sinh x}{\cosh x} = \ln \sinh x - \ln \cosh x$.

since $\ln u = \frac{u'}{u}$, $\frac{y'}{y} = \frac{\cosh x}{\sinh x} - \frac{\sinh x}{\cosh x} = \frac{\cosh^2 x - \sinh^2 x}{\sinh x \cosh x} = \frac{1}{\sinh x \cosh x}$

Multiply by y both sides. $y' = \frac{1}{\sinh x \cosh x} \times y = \frac{1}{\cancel{\sinh x} \cosh x} \times \frac{\cancel{\sinh x}}{\cosh x}$

$$= \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x$$

DERIVATIVES OF HYPERBOLIC FUNCTIONS

Below is the summery of all the derivatives of the hyperbolic functions

1. $\frac{d}{dx}[\sinh u] = \cosh u \times \frac{du}{dx}$
2. $\frac{d}{dx}[\cosh u] = \sinh u \times \frac{du}{dx}$
3. $\frac{d}{dx}[\tanh u] = \operatorname{sech}^2 u \times \frac{du}{dx}$
4. $\frac{d}{dx}[\coth u] = -\operatorname{csch}^2 u \times \frac{du}{dx}$
5. $\frac{d}{dx}[\operatorname{sech} u] = -\operatorname{sech} u \tan u \times \frac{du}{dx}$
6. $\frac{d}{dx}[\operatorname{csch} u] = -\operatorname{csch} u \coth u \times \frac{du}{dx}$

DERIVATIVES OF HYPERBOLIC FUNCTIONS

Example: Find $f'(x)$ if $f(x) = \cosh(x^2 - 1)$.

- Let $u = x^2 - 1, \frac{du}{dx} = 2x$
- Since $f(x) = \cosh u$ and $\frac{d}{dx}[\cosh u] = \sinh u \times \frac{du}{dx}$;

$$f'(x) = \sinh u \times 2x = 2x \sinh u$$

After substituting u back,

$$\boxed{f'(x) = 2x \sinh(x^2 - 1)}$$

INTEGRALS OF HYPERBOLIC FUNCTIONS

- The integration formulas that corresponds the differentiation above are as follows:

$$\int \sinh u \, du = \cosh u + c \qquad \int \cosh u \, du = \sinh u + c$$

$$\int \operatorname{sech}^2 u \, du = \tanh u + c \qquad \int \operatorname{csch}^2 u \, du = -\coth u + c$$

$$\int \operatorname{sech} u \tanh u \, du = -\operatorname{sech} u + c \qquad \int \operatorname{csch} u \coth u \, du = -\operatorname{csch} u + c$$

INTEGRALS OF HYPERBOLIC FUNCTIONS

Example: Evaluate $\int x^2 \sinh(x^3)\, dx$.

- Let $u = x^3$, $dx = \frac{du}{3x^2}$

$$\int x^2 \sinh(x^3)\, dx = \int x^2 \sinh u \times \frac{du}{3x^2}$$

$$\int \frac{1}{3} \sinh u\, du = \frac{1}{3} \int \sinh u\, du = \frac{1}{3} \cosh u + c$$

$$\therefore \int x^2 \sinh(x^3)\, dx = \frac{1}{3} \cosh(x^3) + c$$

Practice Questions

- Evaluate the following:

1. $\frac{d}{dx}[\sinh(x^2+1)]$
2. $\frac{d}{dx}[\arctan(\tanh x)]$
3. $\frac{d}{dx}[e^{3x}\operatorname{sech} x]$
4. $\frac{d}{dx}[\ln(\sinh 2x)]$
5. $\int \tanh^2 3x \operatorname{sech}^2 3x\, dx$
6. $\int \frac{e^{\sinh x}}{\operatorname{sech} x} dx$
7. $\int \frac{\operatorname{sech}^2 x}{1-2\tanh x} dx$
8. $\int \frac{\cosh(\ln x)}{x} dx$

- Find y' if $x^2 \tanh y = \ln y$

INVERSE HYPEBOLIC FUNCTIONS

- The inverses of hyperbolic functions, are expressed in terms of natural logarithms. This is so because as we all know from the other parts of this session, we tend to express the hyperbolic functions as a combination of exponential functions e^x and e^{-x}.
- For example, below are the log forms for $\sinh^{-1} x$, $\cosh^{-1} x$ and $\tanh^{-1} x$
  - $\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right), x \in \mathbb{R}$
  - $\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right),\ x \geq 1$
  - $\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), -1 < x < 1$

INVERSE HYPEBOLIC FUNCTIONS

- Given $y = \sinh^{-1} x$; $x = \sinh y = \frac{(e^y - e^{-y})}{2}$. Multiply RHS by $\frac{e^y}{e^y}$

$$x = \frac{(e^y - e^{-y})}{2} \times \frac{e^y}{e^y} = \frac{e^{2y} - 1}{2e^y} \text{ multiply by } 2e^y \text{ both sides}$$

$$2xe^y = e^{2y} - 1;\ e^{2y} - 2xe^y - 1 = 0$$

- Using the quadratic formula;

$$e^y = \frac{-(-2x) \pm \sqrt{(2x)^2 - 4(1)(-1)}}{2(1)} = \frac{2x \pm \sqrt{4x^2 + 4}}{2}$$

$e^y = \frac{2x \pm 2\sqrt{x^2+1}}{2} = x \pm \sqrt{x^2 + 1} = x + \sqrt{x^2 + 1}$ (since e^y is always positive.)

Therefore, $y = \ln(x + \sqrt{x^2 + 1})$ when you take natural logs both sides.

DERIVATIVES OF INVERSE HYPEBOLIC FUNCTIONS

- Using a similar process to the one above, we can find all the other inverse hyperbolic functions in natural logarithmic form.
- Differentiating the inverse hyperbolic functions might be done through two different ways.
  1. Using the natural logarithmic forms of the inverse hyperbolic functions. Differentiating these expressions will give the derivative of their respective invers hyperbolic functions.
  2. Using implicit differentiation to differentiate the function part of the hyperbolic functions. After applying a series of identities, find the derivatives of the respective hyperbolic functions.

DERIVATIVES OF INVERSE HYPEBOLIC FUNCTIONS

For example; given $y = \sinh^{-1} x$; we know that $x = \sinh y$. Using implicit differentiation,

$$1 = \cosh y \times \frac{dy}{dx} \text{ thus, } \frac{dy}{dx} = \frac{1}{\cosh y}$$

but $\cosh^2 y - \sinh^2 y = 1$, $\cosh^2 x = 1 + \sinh^2 y$ and $\cosh y = \sqrt{1 + \sinh^2 y}$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+\sinh^2 y}}; \text{ but } \sinh y = x$$

Therefore;

$$\boxed{\frac{d}{dx}[\sinh^2 x] = \frac{1}{\sqrt{1 + x^2}}}$$

DERIVATIVES OF INVERSE HYPEBOLIC FUNCTIONS

- On the same question using logarithms, we know that, if $y = \sinh^{-1} x$, then $y = \ln\left(x + \sqrt{x^2+1}\right)$. Using Chain rule, let $u = x + \sqrt{x^2+1}$.

$$\frac{du}{dx} = 1 + \frac{2x}{2\sqrt{x^2+1}} \text{ and } y = \ln u \text{, thus, } \frac{dy}{du} = \frac{1}{u}$$

$$\text{Hence, } \frac{dy}{dx} = \frac{1}{u} \times \left(1 + \frac{2x}{2\sqrt{x^2+1}}\right)$$

$$= \frac{1}{x+\sqrt{x^2+1}} \times \left(1 + \frac{2x}{2\sqrt{x^2+1}}\right) = \frac{1}{x+\sqrt{x^2+1}}\left(\frac{2\sqrt{x^2+1}+2x}{2\sqrt{x^2+1}}\right)$$

$$= \frac{1}{x+\sqrt{x^2+1}}\left(\frac{2\left(\sqrt{x^2+1}+x\right)}{2\sqrt{x^2+1}}\right)$$

DERIVATIVES OF INVERSE HYPEBOLIC FUNCTIONS

$$= \cancel{\frac{1}{x + \sqrt{x^2+1}}}\left(\frac{2\cancel{\left(\sqrt{x^2+1}+x\right)}}{2\sqrt{x^2+1}}\right)$$

$$= \frac{2}{2\sqrt{x^2+1}} = \frac{1}{\sqrt{x^2+1}}$$

- Since addition is commutative, $\sqrt{x^2+1} = \sqrt{1+x^2}$;

Therefore;

$$\boxed{\frac{d}{dx}\left[\sinh^{-1} x\right] = \frac{1}{\sqrt{1+x^2}}}$$

DERIVATIVES OF INVERSE HYPEBOLIC FUNCTIONS

- Here is the summary of all the derivatives involving inverse hyperbolic functions.

$$\frac{d}{dx}[\sinh^{-1} u] = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$$

$$\frac{d}{dx}[\cosh^{-1} u] = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}; u > 1$$

$$\frac{d}{dx}[\tanh^{-1} u] = \frac{1}{1-u^2}\frac{du}{dx}; |u| < 1$$

$$\frac{d}{dx}[\operatorname{csch}^{-1} u] = -\frac{1}{|u|\sqrt{1+u^2}}\frac{du}{dx}; u \neq 0$$

$$\frac{d}{dx}[\operatorname{sech}^{-1} u] = -\frac{1}{u\sqrt{1-u^2}}\frac{du}{dx}; 0 < u < 1$$

$$\frac{d}{dx}[\coth^{-1} u] = \frac{1}{1-u^2}\frac{du}{dx}; |u| > 1$$

DERIVATIVES OF INVERSE HYPEBOLIC FUNCTIONS

Example: Find $\frac{dy}{dx}$ if $y = \sinh^{-1}(\tan x)$.

- Let $u = \tan x$; $\frac{du}{dx} = \sec^2 x$

$$\frac{dy}{du} = \frac{d}{du}[\sinh^{-1} u] = \frac{1}{\sqrt{1+u^2}}$$

Since, $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$; $\frac{dy}{dx} = \frac{1}{\sqrt{1+u^2}} \times \sec^2 x = \frac{\sec^2 x}{\sqrt{1+\tan^2 x}}$

But $1 + \tan^2 x = \sec^2 x$; $\frac{dy}{dx} = \frac{\sec^2 x}{\sec x}$; Therefore;

$$\boxed{\frac{d}{dx}[\sinh^{-1}(\tan x)] = \sec x}$$

DERIVATIVES OF INVERSE HYPEBOLIC FUNCTIONS

Practice Questions:

1. Differentiate the following with respect to x.
   a) $\sinh^{-1} 5x$
   b) $\tanh^{-1} \sqrt{x}$
   c) $\sinh^{-1} e^{x}$
   d) $\tanh^{-1}(\sin 3x)$
   e) $\ln \cosh^{-1} 4x$
   f) $\cosh^{-1} \ln 4x$

INTEGRALS OF INVERSE HYPEBOLIC FUNCTIONS

- Below are some of the integrals that result into hyperbolic functions.

1. $\int \frac{1}{\sqrt{u^2+a^2}} du = \sinh^{-1}\frac{u}{a} + c,\ a > 0$
2. $\int \frac{1}{\sqrt{u^2-a^2}} du = \cosh^{-1}\frac{u}{a} + c,\ u > a > 0$
3. $\int \frac{1}{a^2-u^2} du = \frac{1}{a}\tanh^{-1}\frac{u}{a} = c,\ a > 0, |u| < a$
4. $\int \frac{1}{u\sqrt{a^2-u^2}} du = \frac{1}{a}\operatorname{sech}^{-1}\frac{|u|}{a} + c,\ a > 0, 0 < |u| < a$

Example: Evaluate $\int \frac{1}{\sqrt{9x^2+25}}\,dx$

$$\int \frac{1}{\sqrt{9x^2+25}}\,dx = \int \frac{1}{\sqrt{25\left(\frac{9x^2}{25}+1\right)}}\,dx = \int \frac{1}{5\sqrt{\left(\frac{9x^2}{25}+1\right)}}\,dx$$

Let $u = \frac{3x}{5}, du = \frac{3}{5}dx$; thus, $dx = \frac{5}{3}du$.

$$\int \frac{1}{5\sqrt{u^2+1}}\,\frac{5}{3}\,du = \int \frac{1}{3\sqrt{u^2+1}}\,du = \frac{1}{3}\int \frac{1}{\sqrt{u^2+1}}\,du = \frac{1}{3}\sinh^{-1}(u) + c$$

Therefore;

$$\int \frac{1}{\sqrt{9x^2+25}}\,dx = \frac{1}{3}\sinh^{-1}\left(\frac{3x}{5}\right) + c$$

INTEGRALS OF INVERSE HYPEBOLIC FUNCTIONS

Example: Evaluate $\int \frac{e^x}{16-e^{2x}}dx$

$$\int \frac{e^x}{16-e^{2x}}dx = \int \frac{e^x}{16\left(1-\frac{e^{2x}}{16}\right)}dx$$

Let $u = \frac{e^x}{4}$; $du = \frac{e^x}{4}dx$ and $dx = \frac{4du}{e^x}$

$$= \int \frac{e^x}{16(1-u^2)}\frac{4du}{e^x} = \int \frac{1}{4(1-u^2)}du = \frac{1}{4}\tanh^{-1}(u) + c$$

Therefore;

$$\boxed{\int \frac{e^x}{16-e^{2x}}dx = \frac{1}{4}\tanh^{-1}\left(\frac{e^x}{4}\right) + c}$$

Practice Questions

1. Evaluate following the integrals

   a) $\int \frac{1}{\sqrt{16x^2-9}}dx$

   b) $\int \frac{\sin x}{\sqrt{1+\cos^2 x}}dx$

   c) $\int \frac{1}{x\sqrt{9-x^4}}dx$

   d) $\int \frac{e^{2x}}{5-e^{2x}}dx$

Thank You For Reading!!

Josophat Makawa

bsc-mat-14-21@unima.ac.mw

+265 999 978 828

+265 880 563 256