MAT221: CALCULUS II

Transcendental Functions

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Contents

Exponential & Logarithmic Functions

Introduction

- Firstly, it is well known that Exponential and Logarithmic Functions are inverses of each other. That is, they satisfy the fact that;

$$f(f^{-1}(x)) = f^{-1}(f(x)) = x \tag{1}$$

  That means,

$$\log_a(a^x) = a^{\log_a(x)} = x$$

  as long as the base a remains constant through out.
- In addition, we also know how to change logarithms from one base to the other. For example, if you want to change a logarithm of x in base a to base b, we use

$$\boxed{\log_a x = \frac{\log_b x}{\log_b a}} \tag{2}$$

- In MAT121, we looked at the differentiation and integration of the basic exponential and logarithmic functions. Here, we discussed that the derivative of an exponential function $y = e^x$ is e^x. When the base is e, it is called **Natural Exponential Function**.
- For any other base a, we found the derivative of the function $y = a^x$ to be $a^x \ln a$.
- For logarithms, we discussed that the natural logarithmic function of x is present as $\ln x$ as a short form of $\log_e x$.
- We also discussed that the derivative of the function $y = \ln x$ is $\frac{1}{x}$. And for any other general base a we first change it to base e using **Equation 2** and then differentiate which yielded the derivative of $y = \log_a x$ as $\frac{1}{x \ln a}$.
- On **Integration**, we found the reverse processes of the above derivatives. And for most questions we used **integration by substitution** to solve them.

SUMMARY

$$\left.\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{u(x)}\right) = e^{u(x)} \times \frac{\mathrm{d}u}{\mathrm{d}x} \qquad & \frac{\mathrm{d}}{\mathrm{d}x}\left(a^{u(x)}\right) = a^{u(x)} \times \frac{\mathrm{d}u}{\mathrm{d}x} \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\ln u(x)\right) = \frac{1}{u(x)} \times \frac{\mathrm{d}u}{\mathrm{d}x} \qquad & \frac{\mathrm{d}}{\mathrm{d}x}\left(\log_a u(x)\right) = \frac{1}{u(x)\ln a} \times \frac{\mathrm{d}u}{\mathrm{d}x} \end{aligned}\right\} \text{Derivatives} \tag{3}$$

$$\left.\begin{aligned} \int e^{u(x)}\,du = e^{u} + c \qquad & \int a^{u(x)}\,du = \frac{a^{u(x)}}{\ln a} + c \\ \int \frac{1}{u(x)}\,du = \ln|u(x)| + c \end{aligned}\right\} \text{Integrals} \tag{4}$$

Logarithmic Differentiation

- The other thing that we looked at in MAT121 is the **Quotient Rule** which dealt with the differentiation of rational functions. However, some rational functions are so complicated such that using the quotient seems almost impossible.
- For example, consider the function below;

$$y = \frac{x^5}{(1-10x)\sqrt{x^2+2}}$$

  The above function can be solved using the quotient rule but the process will be very complicated to follow. Therefore we will be using the **Logarithmic Differentiation.**
- In this process of differentiation, we will be introducing natural logarithms both sides and solve the function implicitly (*Check* ***Implicit differentiation too***).

- Given $y = \frac{x^5}{(1-10x)\sqrt{x^2+2}}$, introduce natural logarithms both sides.

$$\ln(y) = \ln\left(\frac{x^5}{(1-10x)\sqrt{x^2+2}}\right)$$

- We then apply the rules of indices and logarithms where we know that if the expressions are multiplying, then, their logs are adding and if the expressions are dividing, then their logs are subtracting, and a power is a product of a log.

$$\ln y = \ln x^5 - \ln\left((1-10x)\sqrt{x^2+2}\right) = \ln x^5 - \left[\ln(1-10x) + \ln(x^2+2)^{\frac{1}{2}}\right]$$

$$\ln y = \ln x^5 - \ln(1-10x) - \frac{1}{2}\ln(x^2+2)$$

- After this, we will differentiate implicitly; each term on its own using **equation 3** where in simplified terms, for $y = \ln u$, $y' = \frac{u'}{u}$.

- Let's now use the implicit differentiation to differentiate $\ln y = \ln x^5 - \ln(1-10x) - \ln\sqrt{x^2+2}$ term by term as follows:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\ln y) = \frac{1}{y} \times \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y'}{y} \text{ since } \frac{\mathrm{d}y}{\mathrm{d}x} = y'$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\ln x^5) = \frac{5\cancel{x^4}}{\cancel{x^5}} = \frac{5}{x}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\ln(1-10x)) = \frac{-10}{1-10x}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}\ln(x^2+2)\right) = \frac{1}{\cancel{2}} \times \frac{\cancel{2}x}{x^2+2} = \frac{x}{x^2+2}$$

- From here, we just need to combine everything and simplify if possible.

- Therefore, the derivative of $y = \frac{x^5}{(1-10x)\sqrt{x^2+2}}$ is

$$\frac{y'}{y} = \frac{5}{x} - \left(\frac{-10}{1-10x}\right) - \frac{x}{x^2+2} = \frac{5}{x} + \frac{10}{1-10x} - \frac{x}{x^2+2}$$

- If we multiply y both sides,

$$y' = \left(\frac{5}{x} + \frac{10}{1-10x} - \frac{x}{x^2+2}\right) \times y$$

- But $y = \frac{x^5}{(1-10x)\sqrt{x^2+2}}$, if we substitute we get the final derivative of y as,

$$\therefore \boxed{\frac{\mathrm{d}y}{\mathrm{d}X} = \left(\frac{5}{x} + \frac{10}{1-10x} - \frac{x}{x^2+2}\right)\left(\frac{x^5}{(1-10x)\sqrt{x^2+2}}\right)}$$

- We can see here that it's slightly simpler to do the differentiation using logarithms that it could be if we have used the product and quotient rules. In addition, the answers are also simplified which improves our understanding to the process.
- The same approach is also used to differentiate expressions in which functions are raised to other functions. Thus

$$\boxed{\text{If } y = f(x)^{g(x)}, \text{ then, } \ln y = \ln\left(f(x)^{g(x)}\right) = g(x)\ln f(x)} \tag{5}$$

- Then, we apply **implicit differentiation** and use the product rule on the right hand side to finish the differentiation process.

Example (Differentiate $y = x^x$)

Solution

- Firstly, we can happily see that both power rule $y' = nx^{n-1}$ and the exponential differentiation $y' = a^x \ln a$ won't work. So, let's introduce the logarithms.

$$\ln y = \ln x^x = x \ln x$$

- Here, we will differentiate $\ln y$ implicitly and $x \ln x$ using the product rule.

$$\frac{1}{y} \cdot \frac{\mathrm{d}x}{\mathrm{d}y} = 1 \cdot \ln x + \cancel{x} \cdot \left(\frac{1}{\cancel{x}}\right) = \ln x + 1$$

- Multiplying by y both sides yields

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = (\ln x + 1)x^x}$$

Example (Given that $y = (1-3x)^{\cos(x)}$, find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$)

Solution

- Introduce logarithms

$$\ln y = \ln\left[(1-3x)^{\cos(x)}\right] = \cos(x)\ln(1-3x)$$

- Differentiate

$$\frac{1}{y}\cdot\frac{\mathrm{d}y}{\mathrm{d}x} = -\sin(x)\cdot\ln(1-3x)+\cos(x)\cdot\frac{-3}{(1-3x)} = -\left[\sin(x)(1-3x)+\frac{3\cos(x)}{(1-3x)}\right]$$

$$\therefore \boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = -\left[\sin(x)(1-3x)+\frac{3\cos(x)}{(1-3x)}\right]\left[(1-3x)^{\cos(x)}\right]}$$

- Before we look more examples, let's revise some of the basic differentiation formulas to avoid confusion.

$$\frac{\mathrm{d}}{\mathrm{d}x}(c) = 0 \qquad \text{Constant Differentiation}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(x^n) = nx^{n-1} \qquad \text{Power Rule}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(a^x) = a^x \ln x \qquad \text{Derivative of an Exponential Function}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(x^x) = x^x(1 + \ln x) \qquad \text{Logarithmic Differentiation}$$

- It's very useful to keep on eye on the theorems we use to avoid costly mistakes since a lot of things look similar but are mathematically different.

Example (Differentiate $f(x) = (5-3x^2)^7\sqrt{6x^2+8x-12}$)

Solution

$$\ln f(x) = \ln\left[(5-3x^2)^7\sqrt{6x^2+8x-12}\right] = \ln(5-3x^2)^7 + \ln(6x^2+8x-12)^{\frac{1}{2}}$$

$$\ln f(x) = 7\ln(5-3x^2) + \frac{1}{2}\ln(6x^2+8x-12)$$

$$\frac{f'(x)}{f(x)} = 7\left(\frac{-6x}{5-3x^2}\right) + \frac{1}{2}\left(\frac{12x+8}{6x^2+8x-12}\right)$$

$$f'(x) = f(x)\left(\frac{-42x}{5-3x^2} + \frac{6x+4}{6x^2+8x-12}\right)$$

$$\therefore \boxed{f'(x) = (5-3x^2)^7\sqrt{6x^2+8x-12}\left[\frac{-42x}{5-3x^2} + \frac{6x+4}{6x^2+8x-12}\right]}$$

Example (Differentiate $y = (2x - e^{8x})^{\sin 2x}$)

Solution

$$\ln y = \ln\left(2x - e^{8x}\right)^{\sin 2x} = \sin 2x \ln(2x - e^{8x})$$

$$\frac{y'}{y} = (2\cos 2x) \cdot \ln(2x - e^{8x}) + \sin 2x \cdot \left(\frac{2 - 8e^{8x}}{2x - e^{8x}}\right)$$

$$y' = \left(2\cos 2x \ln(2x - e^{8x}) + \frac{\sin 2x(2 - 8e^{8x})}{2x - e^{8x}}\right) \times y$$

$$\therefore \boxed{y' = \left[2\cos 2x \ln(2x - e^{8x}) + \frac{\sin 2x(2 - 8e^{8x})}{2x - e^{8x}}\right]\left[(2x - e^{8x})^{\sin 2x}\right]}$$

Example (Differentiate $y = \frac{\sqrt{5x+8}\sqrt[3]{1-9\cos 4x}}{\sqrt[4]{t^2+10t}}$.)

$$\ln y = \ln\left(\frac{\sqrt{5x+8}\sqrt[3]{1-9\cos 4x}}{\sqrt[4]{t^2+10t}}\right) = \ln(5x+8)^{\frac{1}{2}} + \ln(1-9\cos 4x)^{\frac{1}{3}} - \ln(t^2+10t)^{\frac{1}{4}}$$

$$\ln y = \frac{1}{2}\ln(5x+8) + \frac{1}{3}\ln(1-9\cos 4x) - \frac{1}{4}\ln(t^2+10t)$$

$$\frac{y'}{y} = \frac{1}{2}\cdot\frac{5}{5x+8} + \frac{1}{\cancel{3}^{1}}\cdot\frac{\cancel{36}^{12}\sin 4x}{1-9\cos 4x} - \frac{1}{\cancel{4}^{2}}\cdot\frac{\cancel{2t+10}^{t+5}}{t^2+10t}$$

$$y' = \left[\frac{5}{2(5x+8)} + \frac{12\sin 4x}{1-9\cos 4x} + \frac{t+5}{2(t^2+10t)}\right]\times y$$

$$\therefore \boxed{y' = \left[\frac{5}{2(5x+8)} + \frac{12\sin 4x}{1-9\cos 4x} + \frac{t+5}{2(t^2+10t)}\right]\left[\frac{\sqrt{5x+8}\sqrt[3]{1-9\cos 4x}}{\sqrt[4]{t^2+10t}}\right]}$$

Logarithmic Integration

- So far, we have seen that the derivative of $\ln x$ is $\frac{1}{x}$. And let's we want to integrate $\frac{1}{x}$ using the power rule; $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$.
- Firstly we will write $\frac{1}{x}$ as x^{-1} using the rules of indices. If we apply the power rule here,

$$\int x^{-1}\,dx = \frac{x^{-1+1}}{-1+1} = \frac{x^0}{0} = \frac{1}{0}$$

  which is undefined.
- Now, knowing the fact that the integral is an antiderivative of a given function, we can happily conclude from **Equation 4** that $\ln x$ is an antiderivative of $\frac{1}{x}$ on the interval $(-\infty, 0) \cup (0, \infty)$ that doesn't contain 0. Therefore;

$$\boxed{\int \frac{1}{u}\,du = \ln|u| + C}$$

Example (Apply **Equation 4** to evaluate $\int_1^e \frac{1}{x}\,dx$)

Solution

- We will firstly find the indefinite integral then handle the limits later. Thus,

$$\int \frac{1}{x} = \ln|x| + C$$

- Bringing the limits in,

$$\int_1^e \frac{1}{x}\,dx = [\ln x]_1^e = \ln e - \ln 1 = 1 - 0 = 1$$

from rules of logarithms, we know that $\ln e = \log_e e = 1$ and $\ln 1 = 0$.

Example (Evaluate $\int \frac{3x^2}{x^3+5}\,dx$.)

Solution

- We will be using **Integration by Substitution**. The idea is to eliminate the a certain part of the expression by substituting a derivative of another part of the given expression. Here, let $u = x^3 + 5$, after differentiating $\frac{\mathrm{d}u}{\mathrm{d}x} = 3x^2$. Making dx a subject of the formula, we get $dx = \frac{du}{3x^2}$. Hence we will substitute u and dx;

$$\int \frac{\cancel{3x^2}}{u} \cdot \frac{du}{\cancel{3x^2}} = \int \frac{1}{u} du = \ln u + C$$

- And substitution u back,yields

$$\int \frac{3x^2}{x^3 + 5}\,dx = \ln|x^3 + 5| + C$$

Example (Evaluate $\int \tan x \, dx$)

Solution

- Here, we will first have to apply the trig identity, $\tan x = \frac{\sin x}{\cos x}$.

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx$$

$$\boxed{u = \cos x \text{ and } du = -\sin x \, dx}$$

- Thus;

$$\int \frac{\sin x}{\cos x} \, dx = -\int \frac{du}{u} = -\ln|u| + C$$

- Substituting u back;

$$\int \tan x \, dx = -\ln|\cos x| + C$$

Example (Integrate $\int \frac{dx}{x\ln x}$.)

Solution

- Firstly, we need to make a choice of u that will eliminate dx as shown

$$\text{Let } u = \ln x; \text{ Then } du = \frac{1}{x}\,dx$$

- Integrating the resulting expression yields the following

$$\int \frac{dx}{x\ln x} = \int \frac{du}{u} = \ln|u| + C$$

- Substituting u back,

$$\int \frac{dx}{x\ln x} = \ln|\ln x| + C$$

Example (Evaluate $\int \frac{\sin 3\theta}{1+\cos 3\theta}\, dx$.)

$$u = 1 + \cos 3\theta \qquad du = -\sin 3\theta\, dx$$

$$\int \frac{\sin 3\theta}{1+\cos 3\theta}\, dx = -\int \frac{du}{u} = -\ln|u| + C$$

$$\therefore \int \frac{\sin 3\theta}{1+\cos 3\theta}\, dx = -\ln|1+\cos 3\theta| + C$$

Example (Evaluate $\int \cot x\, dx$)

$$\cot x = \frac{\cos x}{\sin x} \qquad u = \sin x, \text{ and } du = \cos x dx$$

$$\int \cot x\, dx = \int \frac{\cos x}{\sin x}\, dx = \ln|\sin x| + C$$

Trigonometric Functions

Derivatives of Inverse Trigonometric Functions

- Mathematically, the six basic trigonometric functions do not have inverses because their graphs repeat periodically and do not pass the horizontal line test.
- However, inverses of trigonometric functions are widely used in finding angles that are associated with specific ratios. These inverses usually give angles which are in the range $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. For example, consider

$$\boxed{\sin x = \frac{1}{2}; \quad x = \sin^{-1}\frac{1}{2} = 30^\circ}$$

- When differentiating these functions, careful attention has to be taken to avoid confusing them with normal function as their behaviour is not even the same with the inverses of other functions.

- To derive the derivatives of inverse functions, we will use the formula for finding derivatives of general functions. Thus if $f(x)$ and $g(x)$ are inverses of each other, then

$$\boxed{g'(x) = \frac{1}{f'(g(x))}} \tag{6}$$

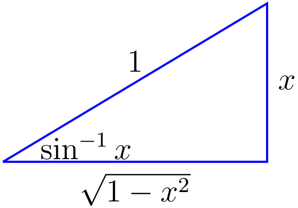
- And always remember from **equation 1** that, two functions $f(x)$ and $g(x)$ are inverses if and only if $f(g(x)) = g(f(x)) = x$.
- In this section, we will derive the derivatives of the inverses for all the six trigonometric functions.
- Before we start diving into the derivations, it might be important to note that the inverses are written in two forms, that is, $\sin^{-1} x$ can also be written as $\arcsin x$.

The Derivative of $y = \sin^{-1} x$

- From previous knowledge, we know that if $y = \sin^{-1} x$, then, $x = \sin y$. Since, these two functions are inverses of each other, their derivative will follow **Equation 6**. Thus,

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin^{-1} x) = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\cos y} = \frac{1}{\cos(\sin^{-1} x)}$$

- To solve for the equivalence of $\cos(\sin^{-1} x)$, we can use a lot of methods. One of the methods is transforming the question into a triangle. Since, $\cos y = x$, we can also write this as $\cos y = \frac{x}{1}$. Using **SOHCAHTOA**, x is the **opposite** side and 1 is the **hypotenuse** to angle y of a right-angled triangle. Consider the figure below,



$$\boxed{\cos(\sin^{-1} x) = \sqrt{1 - x^2}}$$

- Here, the third side, $\sqrt{1 - x^2}$ has been found by using the Pythagoras theorem.And therefore,

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}(\arcsin x) = \frac{1}{\sqrt{1 - x^2}}} \tag{7}$$

- Another method to find the same result is to use implicit differentiation and trigonometric identities.

$$\text{Given } y = \sin^{-1} x \text{ then, } x = \sin y$$

- And if you differentiate $x = \sin y$ implicitly,

$$1 = \cos y \times \frac{\mathrm{d}y}{\mathrm{d}x} \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\cos y}$$

- Using $\cos^2 y + \sin^2 y = 1$, we know that $\cos^2 y = 1 - \sin^2 y$ and $\cos y = \sqrt{1 - \sin^2 y}$. Thus

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{1 - \sin^2 y}}$$

- Since $\sin y = x$, $\sin^2 y = x^2$, $\sqrt{1-\sin^2 y} = \sqrt{1-x^2}$. And after we substitute,

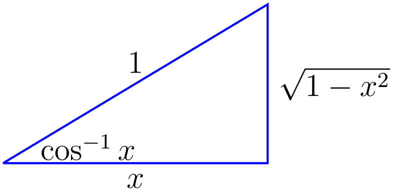
$$\frac{\mathrm{d}}{\mathrm{d}x}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$$

The Derivative of $y = \cos^{-1} x$

- Using the inverse derivative equation,

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cos^{-1} x) = \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{\sin y} = -\frac{1}{\sin(\cos^{-1} x)}$$

- For $\cos y = x$, let x be the adjacent side and 1 be the hypotenuse. Consider the figure below



$$\boxed{\sin(\cos^{-1} x) = \sqrt{1 - x^2}}$$

- Therefore, if we substitute this third side we get the final derivative of arccos as follows.

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}(\arccos x) = -\frac{1}{\sqrt{1 - x^2}}}$$

- Using identities;

$$\text{Given } y = \cos^{-1} x, \text{ then } x = \cos y$$

$$1 = -\sin y \times \frac{\mathrm{d}y}{\mathrm{d}x} \text{ and } \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{\sin y}$$

- But $\sin^2 y + \cos^2 y = 1$, hence, $\sin y = \sqrt{1 - \cos^2 y}$ and $\sin y = \sqrt{1 - x^2}$ because it has been shown from the question that when $y = \cos^{-1} x$, $x = \cos y$.

$$-\frac{1}{\sin y} = -\frac{1}{\sqrt{1 - \cos^2 y}} = -\frac{1}{\sqrt{1 - x^2}}$$

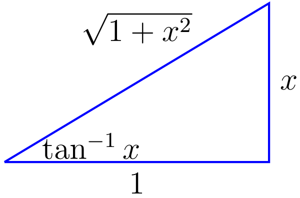
$$\therefore \boxed{\frac{\mathrm{d}}{\mathrm{d}x}(\arccos) = -\frac{1}{\sqrt{1 - x^2}}} \tag{8}$$

The Derivative of $y = \tan^{-1} x$

- Firstly, we will use the inverse functions derivative equation.

$$\frac{\mathrm{d}}{\mathrm{d}x}(\tan^{-1} x) = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sec^2 y} = \frac{1}{\sec^2(\tan^{-1} x)}$$

- After drawing a triangle in the figure below to represent $\tan y = \frac{x}{1}$,



- Since $\sec x = \frac{1}{\cos x}$, using **SOHCAHTOA**, $\sec x = \dfrac{\text{Hypotenuse}}{\text{Adjacent}}$. In this case,

$$\sec(\tan^{-1} x) = \frac{\sqrt{1+x^2}}{1} = \sqrt{1+x^2}$$

$$\text{Therefore;}\ \sec^2(\tan^{-1} x) = (\sqrt{1+x^2})^2 = 1+x^2$$

$$\therefore \boxed{\frac{\mathrm{d}}{\mathrm{d}x}(\arctan x) = \frac{1}{1+x^2}} \tag{9}$$

- You can try to use the identities to verify this as well. In addition, the derivations of the reciprocal functions, **Secant, Cosecant and Cotangent** have been intentionally left out for your practice.

Derivatives of Inverse Trig Functions in Summary

$$
\begin{array}{l|l}
\dfrac{\mathrm{d}}{\mathrm{d}x}(\sin^{-1} u) = \dfrac{1}{\sqrt{1-u^2}} \cdot \dfrac{\mathrm{d}u}{\mathrm{d}x} & \dfrac{\mathrm{d}}{\mathrm{d}x}(\cos^{-1} u) = -\dfrac{1}{\sqrt{1-u^2}} \cdot \dfrac{\mathrm{d}u}{\mathrm{d}x} \\
\dfrac{\mathrm{d}}{\mathrm{d}x}(\tan^{-1} u) = \dfrac{1}{1+u^2} \cdot \dfrac{\mathrm{d}u}{\mathrm{d}x} & \dfrac{\mathrm{d}}{\mathrm{d}x}(\cot^{-1} u) = -\dfrac{1}{1+u^2} \cdot \dfrac{\mathrm{d}u}{\mathrm{d}x} \\
\dfrac{\mathrm{d}}{\mathrm{d}x}(\sec^{-1} u) = \dfrac{1}{|u|\sqrt{u^2-1}} \cdot \dfrac{\mathrm{d}u}{\mathrm{d}x} & \dfrac{\mathrm{d}}{\mathrm{d}x}(\csc^{-1} u) = -\dfrac{1}{|u|\sqrt{u^2-1}} \cdot \dfrac{\mathrm{d}u}{\mathrm{d}x}
\end{array}
$$

$$\text{Derivatives} \tag{10}$$

Example (Differentiate $y = \sin^{-1}(x^3)$ with respect to x)

Solution

- We will use chain rule to differentiate this. Thus;

$$\text{Let } u = x^3; \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 3x^2$$

- This leaves us with $y = \sin^{-1} u$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{\sqrt{1-u^2}} \times 3x^2 = \frac{3x^2}{\sqrt{1-(x^3)^2}}$$

$$\therefore \boxed{\frac{\mathrm{d}}{\mathrm{d}x}(\sin^{-1} x^3) = \frac{3x^2}{\sqrt{1-x^6}}}$$

Example (Differentiate $y = \sec^{-1}(e^x)$ with respect to x)

Solution

$$\text{Let } u = e^x \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = e^x$$

- Since $\frac{\mathrm{d}}{\mathrm{d}x}(\sec^{-1} u) = \frac{1}{|u|\sqrt{u^2 - 1}} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}$; Then,

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sec^{-1}(e^x)) = \frac{1}{|\cancel{e^x}|\sqrt{(e^x)^2 - 1}} \cdot \cancel{e^x}$$

$$\therefore \boxed{\frac{\mathrm{d}}{\mathrm{d}x}(\sec^{-1}(e^x)) = \frac{1}{\sqrt{e^{2x} - 1}}}$$

Example (Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ if $y = x^2(\sin^{-1} x)^3$)

Solution

$$\text{Let } u = \sin^{-1} x \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{\sqrt{1 = -x^2}}$$

- This leaves us with $y = x^2u^3$. Using **Product rule**;

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2xu^3 + 3x^2u^2\frac{\mathrm{d}u}{\mathrm{d}x}$$

- Substitute everything back;

$$\therefore \boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = 2x(\sin^{-1} x)^3 + 3x^2(\sin^{-1} x)^2\frac{1}{\sqrt{1 - x^2}}}$$

Integrals Involving Inverse Trigonometric Functions

- The derivatives on **Equation 10**, are very useful in integrating some expressions. Looking at the derivatives correctly, pairs of the derivatives are similar.
- As such, only the positive derivatives are used for the integrals and the negative ones can be solved using the same positive derivatives.
- Below are the integral functions we will use.

$$\boxed{\begin{aligned} &\int \frac{1}{\sqrt{1-u^2}}\,du = \sin^{-1} u + C \\ &\int \frac{1}{1+u^2} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}\,du = \tan^{-1} u + C \\ &\int \frac{1}{|u|\sqrt{u^2-1}}\,du = \sec^{-1} u + C \end{aligned}} \tag{11}$$

Example (Evaluate $\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$)

Solution

- We will integration by substitution

$$\text{Let } u = e^x; \quad \frac{\mathrm{d}u}{\mathrm{d}x} = e^x. \quad \text{Then } dx = \frac{du}{e^x}$$

$$\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx = \int \frac{\cancel{e^x}}{\sqrt{1-u^2}} \times \frac{du}{\cancel{e^x}} = \int \frac{1}{\sqrt{1-u^2}}\,du = \sin^{-1} u + C$$

- Substituting u back gives

$$\boxed{\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx = \sin^{-1}(e^x) + C}$$

Example (Evaluate $\int \frac{1}{1+3x^2}\, dx$.)

Solution

- The only challenging part that we can encounter is finding the u-substitution. But the idea is to make sure that the terms in the radical are squared. For example, $3x^2$ as a perfect square will be $(\sqrt{3}x)^2$. Thus

$$u = \sqrt{3}\, x \qquad du = \sqrt{3}\, dx \qquad \text{and } dx = \frac{du}{\sqrt{3}}$$

$$\int \frac{1}{1+3x^2}\, dx = \int \frac{1}{1+u^2} \frac{du}{\sqrt{3}} = \frac{1}{\sqrt{3}} \int \frac{1}{1+u^2}\, du = \frac{1}{\sqrt{3}} \tan^{-1} u + C$$

- Substituting u back,

$$\boxed{\int \frac{1}{1+3x^2}\, dx = \frac{1}{\sqrt{3}} \tan^{-1}\left(\sqrt{3}\, x\right)} + C$$

Example (Evaluate $\int \frac{dx}{a^2+x^2}$ where $a \neq 0$ is a constant.)

Solution

- If you look at **Equation 11** closely, you will note that, the first term of the denominators is always 1. Therefore, we need to change this term to 1 by factorising a^2 on the denominator.

$$\int \frac{1}{a^2\left(1+\frac{x^2}{a^2}\right)}\,dx = \int \frac{1}{a^2\left(1+\left(\frac{x}{a}\right)^2\right)}\,dx$$

$$\text{Let } u = \frac{x}{a}; \qquad du = \frac{1}{a}\,dx \qquad \text{and } dx = a\,du$$

- After we substitute these,

$$\int \frac{1}{a^{\cancel{2}}\left(1+u^2\right)}\cancel{a}\,du = \frac{1}{a}\int \frac{1}{1+u^2}\,du$$

Example (Evaluate $\int \frac{dx}{a^2+x^2}$ where $a \neq 0$ is a constant.)

Continued...

- After the constant, $\left(\frac{1}{a}\right)$, has been taken out, we can integrate the remain part.

$$\frac{1}{a}\int \frac{1}{1+u^2}\,du = \frac{1}{a}\tan^{-1} u + C$$

- And here, we will substitute u back into the original expression.

$$\therefore \boxed{\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C}$$

- This process can be used to integrate slightly complicated expressions. Similar processes done for the other integrals give a set usable equations we can use to simplify the work.
- Here are the equations to use whenever the constant $a \neq 0$ is not equal to 1;

$$\begin{aligned}
\int \frac{du}{\sqrt{a^2 - u^2}} &= \sin^{-1} \frac{u}{a} + C \\
\int \frac{du}{a^2 + u^2} &= \frac{1}{a} \tan^{-1} \left(\frac{u}{a}\right) + C \\
\int \frac{du}{u\sqrt{u^2 - a^2}} &= \frac{1}{a} \sec^{-1} \left|\frac{u}{a}\right| + C
\end{aligned} \tag{12}$$

Example (Evaluate $\int \frac{dx}{\sqrt{2-x^2}}$)

Solution

- There two approaches we can use to solve this question. Firstly, we can simply use the formulas on **equation 12**, find the necessary substitutions and integrate, or we can go through the integration process step by step.
- Firstly, let's use the formula. Thus let's make both the terms inside the radical squared.

$$\int \frac{dx}{\sqrt{2-x^2}} = \int \frac{dx}{\sqrt{\left(\sqrt{2}\right)^2 - x^2}}$$

$$\text{Here, } a = \sqrt{2} \quad \text{and} \quad u = x; \text{ Since } \int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} + C,$$

$$\int \frac{dx}{\sqrt{2-x^2}} = \sin^{-1}\left(\frac{x}{\sqrt{2}}\right) + C$$

Example (Evaluate $\int \frac{dx}{\sqrt{9-x^2}}$)

Solution

- Here let's use the step by step approach. (*But you can use any method*). Start with factorising 9 to remain with 1 on the first term of the radical.

$$\int \frac{dx}{\sqrt{9-x^2}} = \int \frac{dx}{\sqrt{9\left(1-\frac{x^2}{9}\right)}}$$

- We will take out 9 from the root and using the rules of surds, it will be 3 out. Then we will find u and proceed with substitution.

$$= \int \frac{dx}{3\sqrt{1-\left(\frac{x}{3}\right)^2}} \quad u = \frac{x}{3} \text{ and } du = \frac{dx}{3}$$

Example (Evaluate $\int \frac{dx}{\sqrt{9-x^2}}$)

Continued//....

- Here, we will substitute any necessary expressions and proceed to integrate.

$$= \int \frac{\cancel{3}du}{\cancel{3}\sqrt{1-u^2}} = \int \frac{du}{\sqrt{1-u^2}} = u + C$$

- Since we know $u = \frac{x}{3}$ we will substitute this back..

$$\therefore \boxed{\int \frac{dx}{\sqrt{9-x^2}} = \sin^{-1}\left(\frac{x}{3}\right) + C}$$

Example (Evaluate $\int \frac{dy}{y\sqrt{5y^2-3}}$)

Solution

- Factorise 3 and take it out of the radical.

$$\int \frac{dy}{y\sqrt{5y^2-3}} = \int \frac{dy}{y\sqrt{3\left(\frac{5y^2}{3}-1\right)}} = \int \frac{dy}{\sqrt{3}y\sqrt{\left(\frac{\sqrt{5}y}{\sqrt{3}}\right)^2-1}}$$

$$\text{Let } u = \frac{\sqrt{5}y}{\sqrt{3}}, \qquad \text{then } dy = \frac{\sqrt{3}\,du}{\sqrt{5}}$$

- Let's do the substitutions here:

$$= \int \frac{\frac{\cancel{\sqrt{3}}\,du}{\sqrt{5}}}{\cancel{\sqrt{3}}\,y\sqrt{(u^2-1)}} = \int \frac{du}{\sqrt{5}\,y\sqrt{(u^2-1)}}$$

Example (Evaluate $\int \frac{dy}{y\sqrt{5y^2-3}}$)

Continued/....

- Here, we have eliminated y from the inside of the radical and we are only left with y to eliminate.

$$\text{Since } u = \frac{\sqrt{5}y}{\sqrt{3}}, \text{ then, } y = \frac{\sqrt{3}\,u}{\sqrt{5}}, \text{ Thus:}$$

$$\int \frac{du}{\cancel{\sqrt{5}} \cdot \frac{\sqrt{3}\,u}{\cancel{\sqrt{5}}}\sqrt{(u^2-1)}} = \int \frac{du}{\sqrt{3}\,u\sqrt{(u^2-1)}} = \frac{1}{\sqrt{3}} \int \frac{du}{u\sqrt{(u^2-1)}}$$

$$\therefore \boxed{\int \frac{dy}{y\sqrt{5y^2-3}} = \frac{1}{\sqrt{3}} \int \frac{du}{u\sqrt{(u^2-1)}} = \frac{1}{\sqrt{3}} \sec^{-1}\left|\frac{\sqrt{5}\,y}{\sqrt{3}}\right| + C}$$

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Closing Remarks

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