DISCRETE MATHEMATICS

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OBJECTIVES

By the end of this series, you should be able to:

a. solve problems using counting techniques,
b. analyse problems using the pigeon-hole principle,
c. prove graph theory concepts,
d. apply graph theory techniques to real life problems,
e. solve different equations

INTRODUCTION

Discrete mathematics is a branch of mathematics that focuses on countable and distinct objects rather than continuous ones. It has applications in various fields, including computer science, cryptography, logic, and more. Discrete mathematics deals with concepts like sets, relations, functions, graphs, and combinatorics. Combinatorics, in particular, is concerned with counting and arranging objects.

SECTION 1:

Counting Techniques

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PRODUCT RULE

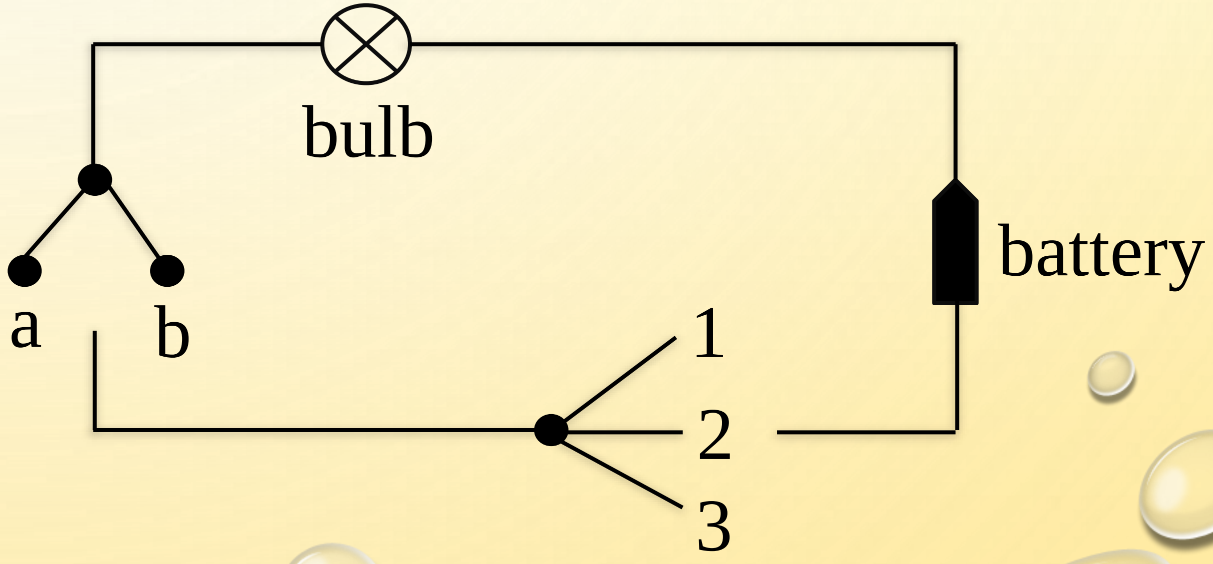
The product rule is a principle in combinatorics that helps you count the number of ways two or more independent tasks can be accomplished sequentially. It's often used when you have a sequence of choices, and you want to determine the total number of outcomes.

If there are **m** ways to do one thing and **n** ways to do another thing, then there are $\mathbf{m} \times \mathbf{n}$ ways to do both things in sequence

COUNTING TECHNIQUES – Product Rule

- If there are two sets and we want to create pairs of elements from first set to elements of the other set, the number of possible pairs is equal to the number of elements in one set times the number of elements in the other set.

Consider the diagram below:



In how many ways can the switches be connected such that the bulb gives light?

- switch a can be connected to switch 1, 2 and 3 on the other side. That is, the connections will be $a1, a2$ and $a3$.
- switch b can be connected to those three other switches too formulating connections b1, b2 and b3.

Using product rule, the total number will be $2 \times 3 = 6$ ways

COUNTING TECHNIQUES – Product Rule

Example 1: How many pairs of numbers an be made from two sets $X = \{1, 2, 3\}$ and $Y = \{4, 5, 6, 7, 8\}$ such that each set has got a member of both sets?

- Since we are given two sets, the total number of ways is the product of elements in either sets. $n(X) \times n(Y)$

$$= 3 \times 5$$

$$= 15 \text{ ways}$$

COUNTING TECHNIQUES – Product Rule

Example 2: How many license plates can you make out of three letters followed by three numerical digits?

- Here we have six events: the first letter, the second letter, the third letter, the first digit, the second digit, and the third digit. The first three events can each happen in 26 ways; the last three can each happen in 10 ways. So the total number of license plates will be $26 \times 26 \times 26 \times 10 \times 1010$ using the product rule

COUNTING TECHNIQUES – Product Rule

Example 3: How many two – digit numbers are there?

- Let's assume that we have two sets.

FIRST DIGIT	SECOND DIGIT

To make sure that we have a two digit number, the first digit can't be a zero. $X = \{1, 2, 3, 4, \ldots, 9\}$ which is 9 ways

The first will have all digits useful. $Y = \{0, 1, 3, 4, \ldots, 9\}$ which is 10 ways.

Total numbers $= n(X) \times n(Y) = 9 \times 10 = 90$ numbers.

COUNTING TECHNIQUES – Product Rule

- Theoretically;

  Two – digit numbers start at 10 and end at 99, from 1 to 9 are single – digit numbers.

  In other words, therefore, two – digit numbers = 99 – 9 = 90.

COUNTING TECHNIQUES – Product Rule

Example 4: How many different ways can you sit people into two chairs chosen from a group 9 people.

- Let the chairs be represented by two boxes.

Chair 1 Chair 2

Since the people will sit on the chairs, the selection is without replacement. That means, once the first person is chosen for the first chair, the selection on the 2^{nd} chair decrease by 1.

By product rule, total ways $= \boxed{9} \times \boxed{(9-1)} = 9 \times 8 = 72$ ways

COUNTING TECHNIQUES – Product Rule

Example 5: How many distinct four-digit numbers are there?

This question is based on the set of the ten numeric symbols as follows. {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}

- Distinct stands for different, that is, we are talking about numbers where no digit in the set is repeated. To achieve this, any chosen digit is removed completely from the set on the next position.
- The other important thing is that zero can’ t be on the first position otherwise it will change to a two-digit number.

COUNTING TECHNIQUES – Product Rule

- 1st digit can be occupied by 1, 2, 3, 4, 5, 6, 7, 8, 9. This is 9 choices
- 2nd digit can be chosen from 0,1, 2, 3, 4, 5, 6, 7, 8, 9. But the digit chosen earlier can't be chosen again. This leaves us with 9 choices.
- 3rd digit can be chosen from 0,1, 2, 3, 4, 5, 6, 7, 8, 9. Here we exclude the first two digits taken earlier (Distinct means that no digit should be repeated). This will let us have 8 choices for 3rd digit an so on.

1st Digit | 2nd Digit | 3rd Digit | 4th Digit

This means that:

$$\boxed{9} \times \boxed{(10-1)} \times \boxed{(10-2)} \times \boxed{(10-3)} = 4\,536$$

SUM RULE

This rule comes into play when you have to make a choice between two or more options that are mutually **exclusive** or **disjoint**. The sum rule helps you count the total number of ways you can make such a choice. It is important that the events must be disjoint: i.e., that there is no way for and to both happen at the same time. For example, a normal deck of 52 cards contains red cards and face cards. However, the number of ways to select a card which is **either red or a face card** is not 26 + 12. This is because there are 6 cards which are both red and face cards. The problem consisting of common elements in sets, have been tackled on Inclusion and Exclusion Principle later in this document.

COUNTING TECHNIQUES – Sum Rule

Definition:

The additive principle states that if event **A** can occur in m ways, and event **B** can occur in exclusive or disjoint n ways, then the event "**A** or **B**" can occur in $m + n$ ways.

COUNTING TECHNIQUES – Sum Rule

Example 1: Suppose you're ordering a pizza and can choose toppings from two different categories: vegetables and meats. There are 4 vegetable toppings and 3 meat toppings. You can either choose toppings from the vegetable category or the meat category.

- ❑ Ways to choose vegetable toppings: 4
- ❑ Ways to choose meat toppings: 3

By the sum rule, the number of topping choices is $4 + 3 = 7$.

COUNTING TECHNIQUES – Sum Rule

In cases of sets, the total number of selections is equal to the union of the two sets. Given **A** and **B** as finite sets, such that no element of the sets is common, the number of ways to choose from either sets is equal to the union of the two sets.

Mathematically, given A and B, such that $\mathrm{A} \cap \mathrm{B} = \{\ \}$,

Number of ways $\mathrm{A} \cup \mathrm{B} = \mathrm{n(A)} + \mathrm{n(B)}; \mathrm{n(A \cap B)} = 0$

$$(\mathrm{A} \cap \mathrm{B}) = \{x \mid x \in \mathrm{A} \text{ and } x \in \mathrm{B}\}$$

NOTE: *The sets should not have common elements.*

COUNTING TECHNIQUES – Sum Rule

Example 2: A class has 50 girls and 150 boys. In how many ways can a teacher choose a representative such that it's either a boy or a girl.

Let A be the set of girls. $A = \{girls\}$

Let B be the set of boys. $B = \{boys\}$

$(A \cap B) = 0$ (Assuming that there is no one who is both)

Number of ways $= n(A \cup B) = |A| + |B| = 50 + 150$

Number of ways $= 200$ ways

PERMUTATIONS

- A permutation is an arrangement of objects in a specific order. In other words, it's a way to rearrange items from a set. The order in which the items are arranged matters in permutations. In some cases, repetition is allowed but for our study, we will stick to permutations where repetition is not allowed. When we talk about repetition, we refer to words like MISSISSIPPI where letters are repeated. On the converse, in permutations without repetition, we refer to words like in the name KENYA where no letter is repeated.

Consider the letters (a, b, c); below are the permutations (arrangements) which can be formed to ensure hat no letter is repeated.

$$abc, acb, bac, bca, cab, cba$$

We know that we have them all listed above - there are 3 choices for which letter we put first, then 2 choices for which letter comes next, which leaves only 1 choice for the last letter. The multiplicative principle says we multiply $3 \cdot 2 \cdot 1 = 6$.

Example 1: How many permutations are there of the letters a, b, c, d, e, f?

There are 6 choices for each choice of first letter, there are 5 choices for the second letter *(we cannot repeat the first letter; we are rearranging letters and only have one of each)*, and for each of those, there are 4 choices for the third, 3 choices for the fourth, 2 choices for the fifth and finally only 1 choice for the last letter. So there are $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$ permutations of the 6 letters.

COUNTING TECHNIQUES – Combinations and Permutations

- A piece of notation is helpful here: $n!$ read "$n!$ factorial", is the product of all positive integers less than or equal to *(for reasons of convenience, we also define 0! to be 1)*. So the number of permutation of 6 letters, as seen in the previous example is 6!

In General:

$$\text{Given } n! = {}^{n}P_{n} = n \bullet (n-1) \bullet (n-2) \bullet \cdots \bullet 3 \bullet 2 \bullet 1$$

COUNTING TECHNIQUES – Combinations and Permutations

In some cases, you may be asked to create fewer permutations than the elements given. For example, people from a group of 10 to sit on 3 chairs.

The first chair will have 10 choices, the second chair 9 and 8 for he third chair. Using product rule, then, total will be $10 \times 9 \times 8 = 720$.

In this case, you are choosing 3 people a group of 10 and putting them on three chairs.

NOTATION: ${}^{n}P_{r}$ means the number of ways to choose r different items from a total of n elements, putting them in r different places.

From college algebra, let's apply Factorial, you find n factorial, and divide by the $(n-r)$ factorial.

$${}^{n}P_{r} = \frac{n!}{(n-r)!}$$

Example 2: How many 4 letter "words" can you make from the letters $\{a, b, c, d, e, f, g, h, i, j\}$ with no repeated letters?

For the first letter, there are 10 choices. For each of those, there are 9 choices for the second letter. Then there are 8 choices for the third letter, and 7 choices for the last letter.

$$^nP_r = 10 \cdot 9 \cdot 8 \cdot 7 = 5\,040$$

- Applying the formula: We are choosing 4 elements from a set of 10 elements.

$$^{n}P_{r} = \frac{n!}{(n-r)!}$$

$$^{10}P_{4} = \frac{10!}{(10-4)!} = \frac{10!}{6!}$$

$$= \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot \cancel{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}}{\cancel{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}} = 10 \cdot 9 \cdot 8 \cdot 7 = 5\,040$$

Example 3: Find ${}^{9}P_{3}$.

$${}^{9}P_{3} = \frac{9!}{(9-3)!}$$

$${}^{9}P_{3} = \frac{9 \times (9-1) \times (9-2) \times (9-3)!}{(9-3)!}$$

$${}^{9}P_{3} = \frac{9 \times 8 \times 7 \times \cancel{6!}}{\cancel{6!}} = 9 \times 8 \times 7 = 504$$

- Example 4: 3 eagles are flying over 365 chambo fish that have been laid for sun drying. The eagles plan to descend each to pin a chambo of its own. In how many ways can they pick 3 fish.

Since the eagles will pick one fish, the possibilities decrease by 1 after every successful pick.

Number ways $\underbrace{365}_{\text{Eagle 1}} \times (\underbrace{365}_{\text{Eagle 2}} - 1) \times (\underbrace{365}_{\text{Eagle 3}} - 2)$

- Applying the formula:

$$ {}^{365}P_3 = \frac{365!}{(365-3)!} = \frac{365!}{362!} $$

$$ {}^{365}P_3 = \frac{365 \times 364 \times 363 \times \cancel{362!}}{\cancel{362!}} $$

$$ {}^{365}P_3 = 365 \times 364 \times 363 $$

$$ {}^{365}P_3 = 48\,228\,180 $$

COMBINATIONS

- Combinations is a technical term meaning for “selections”. We use it to refer to the number of different sets of a certain size that can be selected from a larger collection of objects where order does not matter.
- In this section, we will see that writing AB is the same as writing BA since we deal with subsets of given sets.

COUNTING TECHNIQUES – Combinations and Permutations

- Let us assume that there is a group of 3 lawn tennis players X, Y and Z. A team consisting of two players is to be formed. In how many ways can this be done? It would seem at first glance that there are 6 ways to choose the team–that is XY , XZ, YX, YZ, ZX, ZY.
- However, when we look at the list, only 3 pairs form different teams. X and Y make up same team as Y and X. Here order is not important. In fact, there are only 3 ways in which each team could be constructed. Thus XY , YZ, ZX.

NOTATION: ${}^{n}\mathrm{C}_{r}$ also written as $\binom{n}{r}$ which is read as *"n choose r"*, means the number ways to choose r items from n items. In this case we find total selections and get rid of groups with the same items despite the order. Using permutations and factorial notation:

$$ {}^{n}\mathrm{C}_{r} = \frac{{}^{n}P_{r}}{r!} = \frac{n!}{(n-r)!\,r!} $$

Example 1: How many three person committees can be formed from the five people: A, B, C, D, and E?

In this question we are solving for the number of 3 – member subsets from the five given letters, ${}^{5}C_{3}$.

$$ {}^{5}C_{3} = \frac{{}^{5}P_{3}}{3!} = \frac{5!}{3!\,(5 - 3)!} = \frac{5 \times 4 \times 3 \times 2!}{3! \times 2!} = \frac{5 \times 4 \times 3}{3 \times 2 \times 1} $$

$= \frac{60}{6} =$ 10 committees of three people each.

Example 2: How many different five-card hands can be dealt from a deck of 52 playing cards?

❖ Because the order in which the cards are dealt is not an issue, we are working with combination problem. Thus, using the formula for Combination for n = 52 and r = 5, we have

$$^{52}C_5 = \frac{52!}{(52-5)!\,5!} = \frac{52 \times 51 \times 50 \times 49 \times 48 \times 47!}{5! \times 47!}$$

Continued/…

$$= \frac{52 \times 51 \times 50 \times 49 \times 48}{5 \times 4 \times 3 \times 2 \times 1}$$

$$= 2,598,960$$

Example 3: The sequence $\{x, y, x, y, x, y, y, y, y\}$ has 3x's and 6y′s among them. Find the number of possible 3x's or 6y′s.

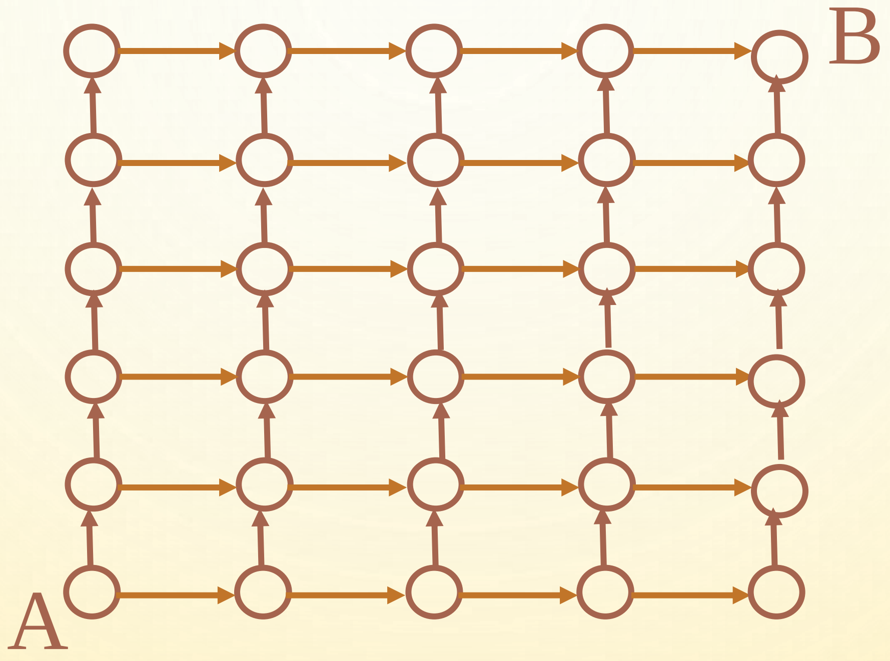
x	y	x	y	x	y	y	y	y

Number of sequences = number of choosing either x's or y′s from a group of 9.

Number of sequences $= {}^9C_3 = {}^9C_6 = 84$

We can, therefore, conclude that ${}^nC_r = {}^nC_{(n-r)}$

Example 4: In a city, suppose that the road network is described by:



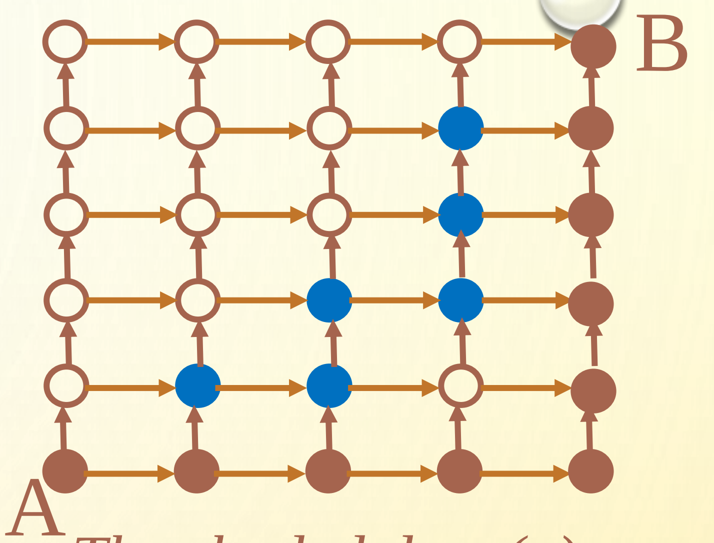
Imagine that the network is defined by horizontal and vertical arrows meeting at nodes/junctions (circles)

COUNTING TECHNIQUES – Combinations and Permutations

Regardless of which path to take from A to B, we make one step horizontally (x) or one step vertically (y) and then combine the subsequent steps.

For example, we have marked two paths on the right

1. $x, x, x, x, y, y, y, y, y$ (brown junctions)
2. $x, y, x, y, x, y, y, x, y$ (blue junctions)



The shaded dots (●) show paths chosen. In this example, we will take horizontal movement as x and vertical one as y. Always consider arrows not dots

- You may have noted that on every path you may choose, you will have 4 x's and 5 y's.
- Thus, we're simply responding the question "*how many paths of length 9 can we make out of 4 x's and 5 y's?*"

This means that we need to choose 4 x – positions and place y's or you can choose 5 y – positions and place x's

$$\binom{x+y}{x,y} = \binom{4+5}{4,5} = {}^{9}\mathrm{C}_4 = {}^{9}\mathrm{C}_5 = 126$$

Hence, we have 126 possible options for shortest routes

BINOMIAL COEFFICIENTS

- Binomial coefficients represent the coefficients of the terms in the expansion of a binomial raised to a power. For example, each and every term in $(x + y)^n$ will have a particular coefficient.

- these particular coefficients are denoted as "n choose k" or $\binom{n}{k}$ given by the formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

- Multiplication is a choice of a term in each bracket.

Consider the process of expanding of $(x+y)^n$, where you may have two terms in the brackets multiplied multiple times. For example, expanding $(x+y)^2$.

$$(x+y)^2 = (x+y)(x+y)$$

(when multiplying, you choose one term from each bracket as shown on the next slide)

When expanding $(x+y)(x+y)$,

1. $x \cdot x$ (choosing x in both brackets)
2. $x \cdot y$ (choosing x in the first and y in the second brackets)
3. $y \cdot x$ (choosing y in the first and x in the second brackets)
4. $y \cdot y$ (choosing y in both brackets)

Each of these choices contributes a term in the final expansion.

$$(x+y)^2 = x \cdot x + x \cdot y + y \cdot x + y \cdot y = x^2 + 2xy + y^2$$

Example 1: Find the coefficient of x^3y^6 in the expansion of $(x+y)^9$.

$$(x+y)^9 = (x+y)(x+y)(x+y)\ldots(x+y)$$

- Since our question has the highest power 9, we can make up to 9 choices.
  1 2 3 up to 9
  In this case, x^3y^6 stands for choosing x from 3 brackets and y from 6 brackets.

The coefficient $= \binom{9}{3}$ or $\binom{9}{6} = 84$; the term is $84x^3y^6$.

Example 2: Find the coefficient of x^3y^3 in the expansion of $(-2\sqrt{x}+3y^{0.75})^{10}$.

We have 10 multiplying brackets.

$\sqrt{x}=x^{\frac{1}{2}}$, so to get x^3:$x^{\frac{1}{2}}\cdot x^{\frac{1}{2}}\cdot x^{\frac{1}{2}}\cdot x^{\frac{1}{2}}\cdot x^{\frac{1}{2}}\cdot x^{\frac{1}{2}}$ (6 choices)

On the hand, $0.75\times 4=3$, (4 choices of $y^{0.75}$ gives y^3)

The corresponding term $(-2\sqrt{x})^6(3y)^4$

Coefficient = $\binom{10}{4}(-2)^6(3)^4=1,088,640$

Binomial Expansions in Sigma Notation

- A binomial expansion for $(x+y)^n$ can be expressed I sigma notation as follows:

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r$$

- As we have seen so far, expansion is a choice of terms from multiplying brackets of binomials. Sigma helps us to add everything up and get the full expression.

- That is,

$$(x+y)^n = (x+y)_1 \times (x+y)_2 \times \cdots \times (x+y)_n$$

- After making the choices, typical term is of the form; $\binom{n}{r} x^{n-r} y^r$ where:
  - y's are coming from r out of n brackets
  - x's are coming from the remaining $(n-r)$ brackets
  - $\binom{n}{r}$ is the binomial coefficient

Example: Expanding $(x + y)^4$ using sigma notation.

$$(x + y)^4 = \sum_{r=0}^{4} \binom{4}{r} x^{4-r} y^r$$

- When $r = 0$: $\binom{4}{0} x^{4-0} y^0 = 1 \cdot x^4 \cdot y^0 = x^4$
- When $r = 1$: $\binom{4}{1} x^{4-1} y^1 = 4 \cdot x^3 \cdot y^1 = 4x^3 y$

- When $r = 2$: $\binom{4}{2} x^{4-2} y^2 = 6 \cdot x^2 \cdot y^2 = 6x^2y^2$
- When $r = 3$: $\binom{4}{3} x^{4-3} y^3 = 4 \cdot x^1 \cdot y^3 = 4xy^3$
- When $r = 4$: $\binom{4}{4} x^{4-4} y^4 = 1 \cdot x^0 \cdot y^4 = y^4$

Adding all terms together:

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

Proving Binomial Identities

Prove that $2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n}$

Since $(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r$, let $x = 1$ and $y = 1$.

$$(1+1)^n = \binom{n}{0} 1^n + \binom{n}{1} 1^{n-1} \cdot 1^n + \binom{n}{2} 1^{n-2} \cdot 1^2 + \cdots + \binom{n}{n} 1^{n-n} \cdot 1^n$$

1 raised to any power is still 1; this implies that

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n}$$

- Let's try to reason better;

Given, $2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{0}$

Let $\mathrm{A} = \{1, 2, 3, 4, \dots, n\}$, how many subsets of A are there?

$\mathrm{A} \subseteq \mathrm{A}, \{\ \} \subseteq \mathrm{A}, \{1\} \subseteq \mathrm{A}$ and so on. From the universal set, $\mathrm{A} = \{1, 2, 3, 4, \dots, n\}$, at every point, there are two options, "either to take it or not" and by product rule:

That is, total number of subsets = $2 \times 2 \times 2 \times \cdots \times 2 = 2^n$

We also know that the number of subsets can be given by $\binom{n}{r}$.

And $\binom{n}{r}$ is choosing r elements from n elements; r can be equal to $\{0, 1, 2, 3, \ldots, n\}$. That is, by sum rule:

$$\binom{n}{r} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n}$$

(there are so many subsets)

Since in all the cases we are counting number of subsets, then,

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n}$$

- Prove that $\binom{2n}{n} = \sum_{r=0}^{n} \binom{n}{r}\binom{n}{n-r} = \binom{n}{r}^2$.

  - $\binom{2n}{n}$ means choosing n items from $2n$ items. We will split this $2n$ into two groups with n items each. That means, if we choose r items from one group, we will choose $n-r$ items in the other where $r = 0, 1, 2, 3, \dots, n$ as shown:

$\binom{n}{r} \times \binom{n}{n-r}$ and by sum rule $\sum_{r=0}^{n} \binom{n}{r}\binom{n}{n-r}$

Since ${}^{n}C_r = {}^{n}C_{n-r}$,

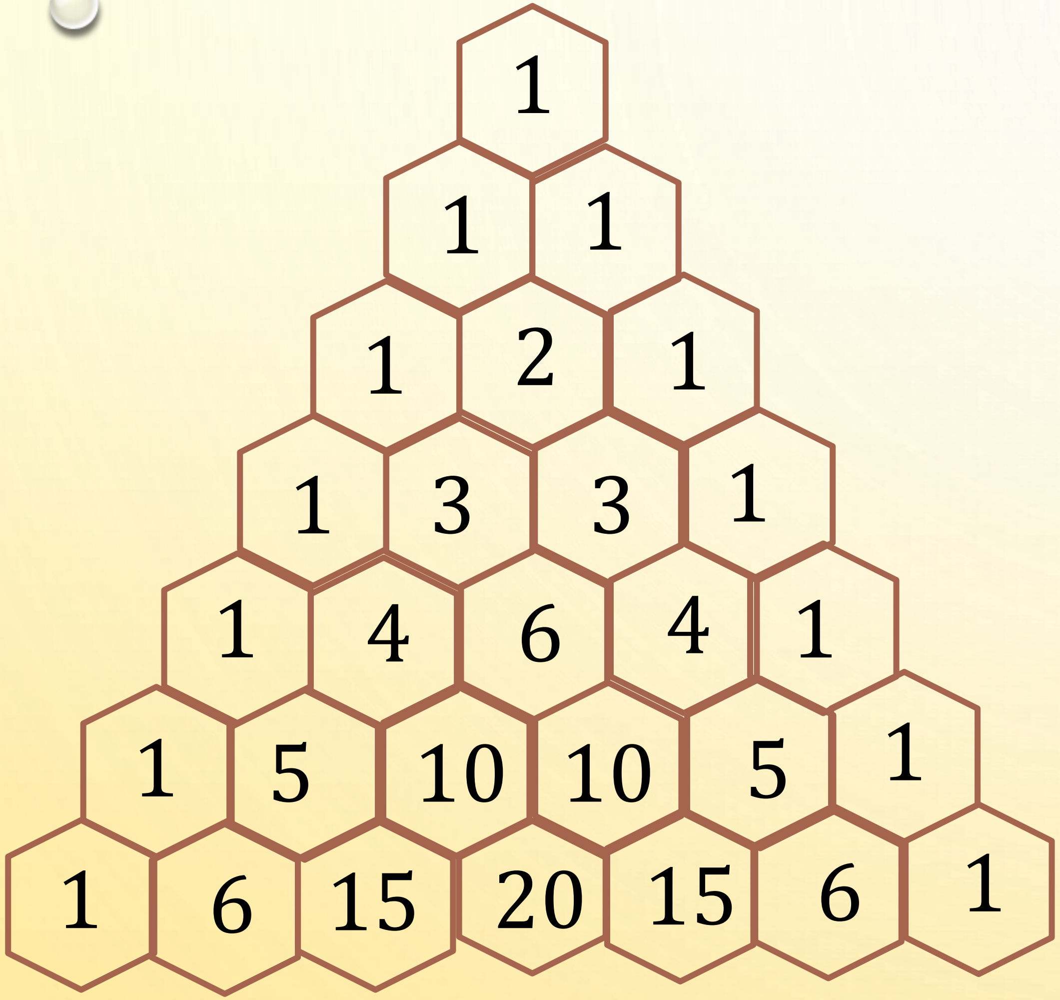
$$\sum_{r=0}^{n}\binom{n}{r}\binom{n}{n-r} = \sum_{r=0}^{n}\binom{n}{r}\binom{n}{r} = \sum_{r=0}^{n}\binom{n}{r}^2$$

(You can still use other sources for some advanced ways of proving this statement and other statements as well)

The Pascal's Triangle

- This an infinite triangular array of numbers which give binomial coefficients of specific combinations.
- Pascal's Triangle is used to expand binomial expressions, such as $(x + y)^n$. The coefficients in the expansion correspond to the numbers in a row of the triangle
- Each subsequent row is constructed by adding the two adjacent numbers from the row above it

- The pascal's triangle looks as follows.



In this diagram, we have the coefficients of respective terms in the expansion of $(x + y)^n$ where n increase from 0 in the first row to 6 in the 6^{th}row.
The next rows are found by adding two close numbers in the preceding row.

We can now like to connect the Pascal's coefficients by finding relationships of coefficients from power to power.

Consider the binomial identity, ${}^{n}C_k = {}^{n-1}C_{k-1} + {}^{n-1}C_k$

By definition of ${}^{n}C_k$, ${}^{n}C_k = \frac{n!}{(n-k)!k!}$: we have

$$ {}^{n-1}C_{k-1} = \binom{n-1}{k-1} = \frac{(n-1)!}{\big(n-1-(k-1)\big)!\,(k-1)!} = \frac{(n-1)!}{(n-k)!\,(k-1)!} $$

And ${}^{n-1}C_k = \binom{n-1}{k} = \frac{(n-1)!}{(n-1-k)!k!}$

Now, in $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$, take the Right Hand Side

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \frac{(n-1)!}{(n-k)!\,(k-1)!} + \frac{(n-1)!}{(n-1-k)!\,k!}$$

You will note that $k(k-1)! = k!$ And $(n-k)(n-k-1)! = (n-k)!.\text{i.e.}\,[5(5-1)! = 5\times 4! = 5!]$ the common denominator will be $(n-k)!\,k!$

$$\frac{(n-1)!}{(n-k)!\,(k-1)!} + \frac{(n-1)!}{(n-1-k)!\,k!} = \frac{[(n-1)!\,k] + [(n-1)!\,(n-k)]}{(n-k)k!}$$

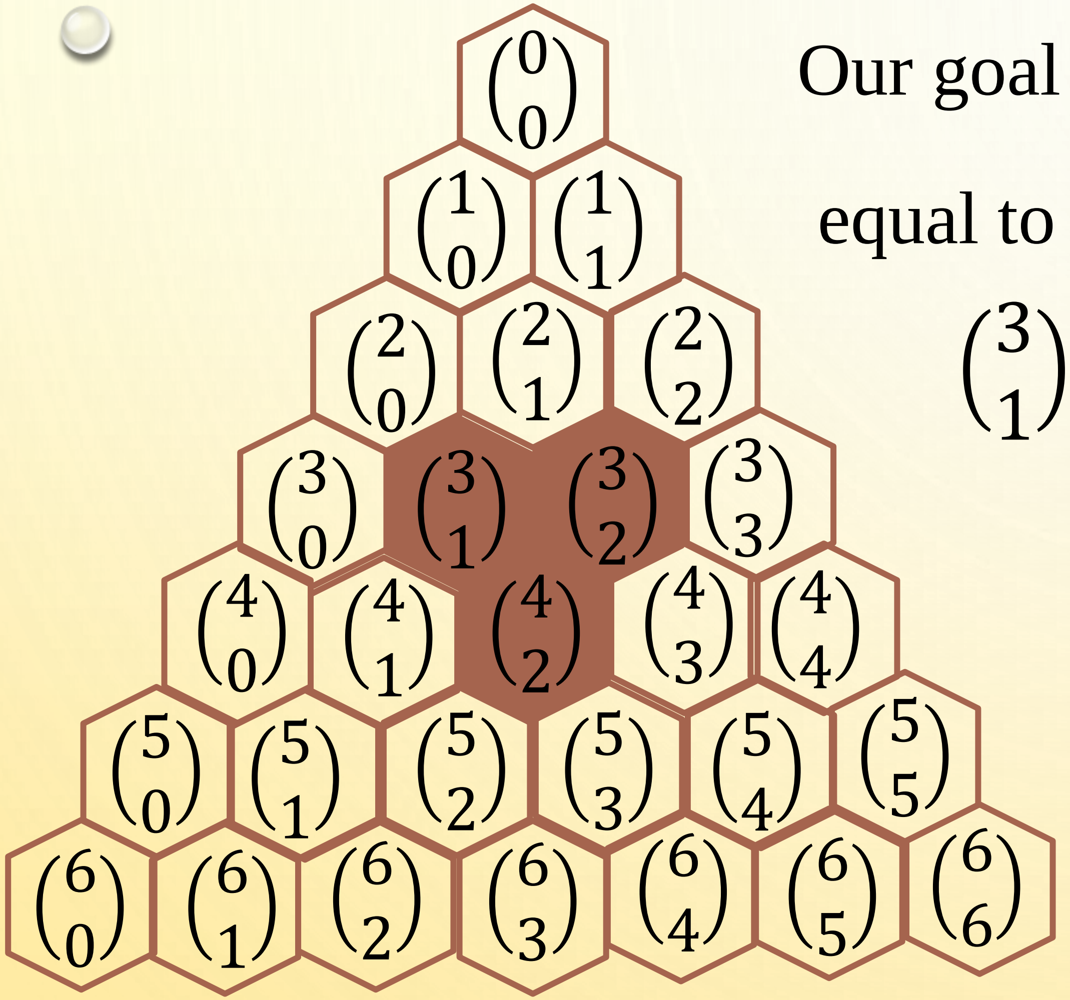
$$\binom{n-1}{k-1} + \binom{n-1}{k} = \frac{(n-1)!(k+n-k)}{(n-k)!k!} \text{ (we factorised the numerator)}$$

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \frac{n(n-1)!}{(n-k)!k!} \text{ (but, } n(n-1)! = n!\text{)}$$

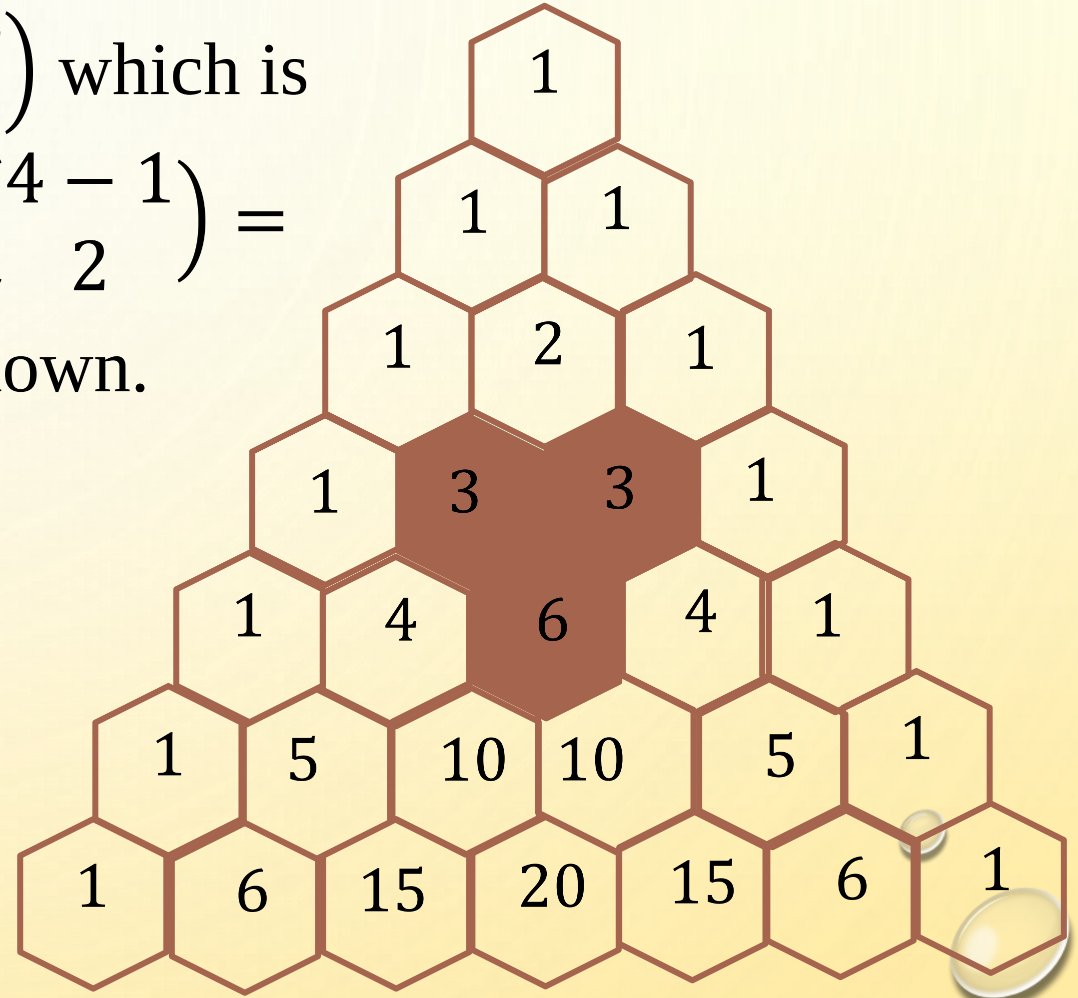
$$\binom{n-1}{k-1} + \binom{n-1}{k} = \frac{n!}{(n-k)!k!} \text{ (and } \frac{n!}{(n-k)!k!} = \binom{n}{k}\text{)}$$

Therefore, $\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$ proved

This just simplifies why we add two close coefficients from the preceding power to find any number. Let’s show this better on the Pascal’s triangle.

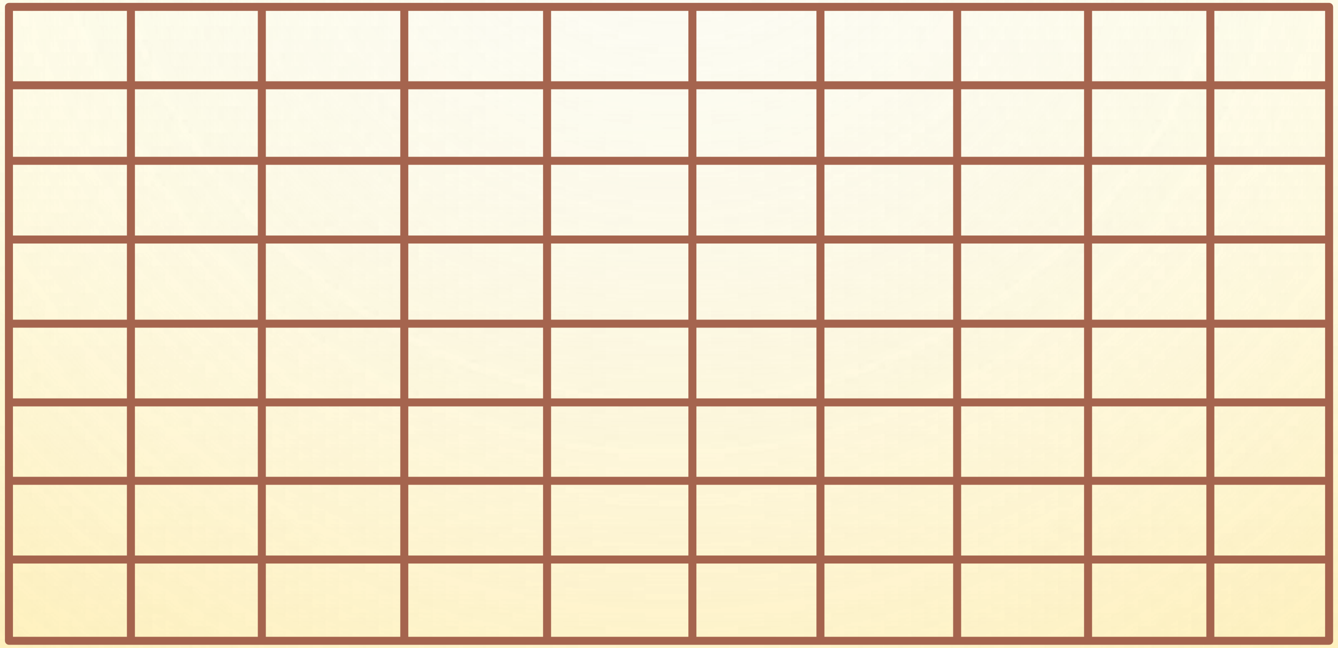


Our goal is to find $\binom{4}{2}$ which is equal to $\binom{4-1}{2-1} + \binom{4-1}{2} =$ $\binom{3}{1} + \binom{3}{2}$ as shown.



NUMBER OF RECTANGLES

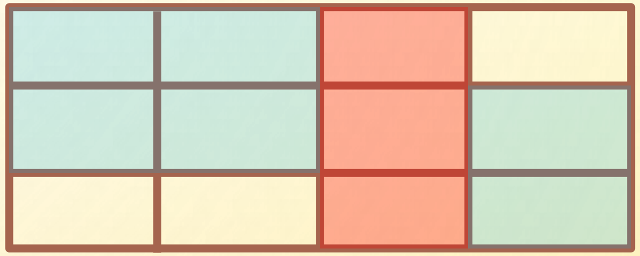
Consider the figure below:



How many rectangles are on this diagram?

- You will note that counting naturally, you will find the number rectangles to be 80 or so. But you should understand there are more rectangles than what you can see. For example;

Despite the inner small rectangles, some of the rectangle examples are the shaded ones.



That is, if you look closely, you are choosing two random vertical lines and two random horizontal lines since any rectangle has to be formed within such metrics.

- In that case, you chose two lines from the vertical lines which, from our question, can be done in $^{11}C_2$ ways.
- You also have to chose two random horizontal lines which can be done in $^{9}C_2$ ways.

Now, using product rule;

$$\text{the total number of rectangles} = {}^{11}C_2 \times {}^{9}C_2$$
$$= 55 \times 36$$
$$= 1980$$

MULTINOMIAL COEFFICIENTS

Multinomial coefficients count the number of ways to distribute

n objects into k distinct categories, where each category can receive a different number of objects. They generalize binomial coefficients for more than two categories. In this part you determine the number of sequences or subsets by choose from group which has more two types of possibilities.

Let **X** be a set of **n** elements. Recall that if we have two colors of paint, say red and blue, and we are going to choose a subset of **k** elements to be painted red with the rest painted blue. Then the number of different ways this can be done is just the binomial coefficient $\binom{n}{k}$. Now suppose that we have three different colors, say red, blue, and green. We will choose some to be colored red, some to be colored blue, and the remaining are to be colored green. In this case, we are choosing multi categories, hence, multinomial coefficients.

Example: Count the number of sequences in three x's, two y's and four z's.

e.g., $x, x, x, y, y, z, z, z, z$

x	x	x	y	y	z	z	z	z

- After this, y will chosen from $9-3$ objects; ${}^{9-3}C_2$
- Lastly, z values will be chosen from the remaining $9-3-2$; ${}^{(9-3-2)}C_4$

Using product rule, the number of sequences is:

$$= {}^{9}C_3 \times {}^{9-3}C_2 \times {}^{(9-3-2)}C_4$$

$$= {}^{9}C_3 \times {}^{6}C_2 \times {}^{4}C_4$$

$$= 1\,260$$

Multinomial coefficients are denoted as $\binom{n}{n_1, n_2, \ldots, n_k}$ where n is the total number of elements $(n_1 + n_2 + \cdots + n_k)$ and $n_1, n_2, \ldots, n_k$ represent the number of objects placed into each of the categories.

Since we are using the product rule, if we have 3 categories;

$$\binom{n}{k_1, k_2, k_3, \ldots} = \binom{n}{k_1} \times \binom{n - k_1}{k_2} \times \binom{n - k_1 - k_2}{k_3} \text{ and so on.}$$

But the last category will always give you 1.

So, the formula for multinomial coefficients is

$$\binom{n}{k_1, k_2, k_3, \ldots, k_r} = \frac{n!}{k_1! \times k_2! \times k_3! \times \cdots \times k_r!}$$

Solving the same question from slide 60 *(click here)*, the number of sequences

$$= \frac{9!}{3! \times 2! \times 4!}$$

$$= \frac{362880}{6 \times 2 \times 24}$$

$$= \frac{362880}{288}$$

$$= 1\,260$$

Example: Find the coefficient of $x^3y^2z^4$ in the expansion of $(x+y+z)^9$.

Since multiplication is a choice of terms from brackets, below is the breakdown of this term:

- x^3 means that x comes from 3 out of 9 brackets
- y^2 means that y comes from 2 out 9 brackets
- z^4means that z comes from 4 out 9 brackets

Using the formula, the coefficient:

$$= \binom{9}{3,2,4}$$

$$= \frac{9!}{3! \times 2! \times 4!}$$

$$= 1,260$$

That is, the full term is $1260x^3y^2z^4$.

Example: Find the coefficient of $x^3y^2z^4$ in the expansion of

$$\left(x^5 + \frac{2}{x^2} + y^{\frac{1}{2}} + z + 3\right)^{11}.$$

- z^4 implies that z is coming from 4 brackets
- $y^2 = y^{\frac{1}{2}} \cdot y^{\frac{1}{2}} \cdot y^{\frac{1}{2}} \cdot y^{\frac{1}{2}}$, i.e. $y^{\frac{1}{2}}$ is chosen from 4 brackets
- $x^5 \times \frac{2}{x^2} = 2x^3$, i.e. Both x^5 and $\frac{2}{x^2}$ come from 1 bracket each
- 3 comes from 1 bracket too.

This is the simplified breakdown

$$\left[(z)^4\left(y^{\frac{1}{2}}\right)^4(x^5)^1\left(\frac{2}{x^2}\right)^1(3)^1\right]$$

The term $= \dfrac{11!}{4!\times4!\times1!\times1!\times1!}\times 2x^3y^2z^4$

$= \dfrac{39916800}{576}\times 2x^3y^2z^4 = 138600x^3y^2z^4$

Therefore, the coefficient is 138600.

INTEGER PART FUNCTION

The integer part function, denoted as $\lfloor x \rfloor$, assigns the greatest integer less than or equal to x. In mathematical notation:

$$\lfloor x \rfloor = \max\{n \in \mathbb{Z} \mid n \leq x\}$$

This means $\lfloor x \rfloor$ is the largest n such that $n \leq x$.

Examples: $\lfloor 2.7 \rfloor = 2$

$\lfloor -3.5 \rfloor = -4$

$\lfloor 5 \rfloor = 5$ (since 5 is already an integer)

Example: Find the number of integers divisible by 3 in the set {1, 2, 3, …, 579}.

The total number will be $\left\lfloor \frac{579}{3} \right\rfloor$ the integer part symbol will help us eliminate decimals if found.

But, the total number of integers $= \lfloor 193 \rfloor = 193$

Therefore: the number of integers is 193.

INCLUSION AND EXCLUSION PRINCIPLE

The inclusion – exclusion principle acts as a continuation to the sum rule we looked at before. In this part we deal cardinality of unions in sets depending on the type of those particular sets. Recall that ways to choose things from two sets A and B was equal to number of elements in set A plus number of elements in set B. But, this statement is only true if and only if sets A and B are disjoint: they don’t have common elements. Now, if they have common elements, the exclusion of repeated is needed. Therefore the use of this principle.

COUNTING TECHNIQUES – Inclusion and Exclusion Principle

From Slide 15, For example, a normal deck of 52 cards contains red cards and face cards. However, the number of ways to select a card which is **either red or a face card** is not 26 + 12 by using the sum rule. This is because there are 6 cards which are both red and face cards. In short, these are not disjoint sets. That means the correct way would be applying inclusion - exclusion principle by simply subtracting 6 from sum 26 + 12.

Mathematically:

Given two sets, A and B, n(A) is the number of elements in set A and n(B) is the number of elements in set B.

And: $n(A \cup B) = n(A) + n(B)$ if $(A \cap B) = \emptyset$ *(disjoint sets)*

But, when the sets are not disjoint, this will mean repeating certain elements. That we should remove repetition by subtracting $n(A \cap B)$. That is,

$$n(A \cup B) = n(A) + n(B) - (A \cap B)$$

By definition, this principle is called Inclusion – exclusion principle because the number $n(A \cup B)$ is found by adding $n(A)$ and $n(B)$ which includes the common terms. And we, then, subtract $(A \cap B)$ which excludes the repetition of the common terms.

If you have been given three or more sets, you simply modify the same equation. For example:

$$\mathbf{n(A \cup B \cup C) = n(A)+n(B)+n(C)-(A \cap B)-(A \cap C)-(C \cap B)+(A \cap B \cap C)}$$

Example 1: Given a set {1, 2, 3, 4, …, 579}, find the number of integers divisible by 2 or 3.

- Let the numbers divisible by 2 be A
- Let the numbers divisible by 3 be B

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

$$= \left\lfloor \frac{579}{2} \right\rfloor + \left\lfloor \frac{579}{3} \right\rfloor - \left\lfloor \frac{579}{6} \right\rfloor = \boxed{386}$$

6 stands for all numbers divisible by both 2 and 3. This is where sets A and B intersect. In this case we use the LCM of the two given numbers.

Example 2: Give the set {1, 2, 3, … , 2000}, how many numbers are divisible by 8 or 9 or 12?

Let A be numbers divisible by 8, B by 9 and C by 12.

n(A ∪ B ∪ C) = n(A)+n(B)+n(C)−(A ∩ B)−(A ∩ C)−(C ∩ B)+(A ∩ B ∩ C)

$$= \left\lfloor\frac{2000}{8}\right\rfloor + \left\lfloor\frac{2000}{9}\right\rfloor + \left\lfloor\frac{2000}{12}\right\rfloor - \left\lfloor\frac{2000}{8\times 9}\right\rfloor - \left\lfloor\frac{2000}{8\times 12}\right\rfloor - \left\lfloor\frac{2000}{9\times 12}\right\rfloor + \left\lfloor\frac{2000}{8\times 9\times 12}\right\rfloor$$

$$= 250 + 222 + 166 - 27 - 20 - 18 + 2$$

$$= 575 \text{ numbers}$$

Example 3: How many sequences of three x's, four y's, five z's and six w's are there where the x's or y's or z's appear together (consecutive among themselves)?

Let A, B and C be sequences where x's or y's or z's appear together respectively.

This means that in a sequence where an element appear consecutively, we are choosing all of them in the sequence; there is only one way of doing so. Lets determine the cardinalities of these sequences/sets.

In our question we looking for $n(A \cup B \cup C)$.

- For n(A) where x's appear together, 1, 4y's, 5z's, 6w's

$$n(A) = \frac{(1+4+5+6)!}{1!\times4!\times5!\times6!} = \frac{16!}{1!\times4!\times5!\times6!} = 10090080$$

- For n(B) where y's appear together, 3x's, 1, 5z's, 6w's

$$n(B) = \frac{(3+1+5+6)!}{3!\times1!\times5!\times6!} = \frac{15!}{3!\times1!\times5!\times6!} = 2522520$$

- For n(C) where z's appear together,

$$n(C) = \frac{(3+4+1+6)!}{3!4!1!6!} = \frac{14!}{3!4!1!6!} = 840840$$

- n(A ∩ B) where x's and y's appear together

$$n(A \cap B) = \frac{(1+1+5+6)!}{1!1!5!6!} = \frac{13!}{1!5!6!} = 72072$$

- n(B ∩ C) where y's and z's appear together

$$n(B \cap C) = \frac{(3+1+1+6)!}{3!1!1!6!} = \frac{11!}{3!1!6!} = 9240$$

- n(A ∩ C) where x's and z's appear together

$$n(A \cap C) = \frac{(1+4+1+6)!}{1!4!1!6!} = \frac{12!}{1!4!6!} = 27720$$

- n(A ∩ B ∩ C) where x's, y's and z's all appear together

$$n(A \cap B \cap C) = \frac{(1+1+1+6)!}{1!1!1!6!} = \frac{9!}{1!6!} = 504$$

Using inclusion – exclusion principle;

n(A∪B∪C) = n(A)+n(B)+n(C)−(A∩B)−(A∩C)−(C∩B)+(A∩B∩C)

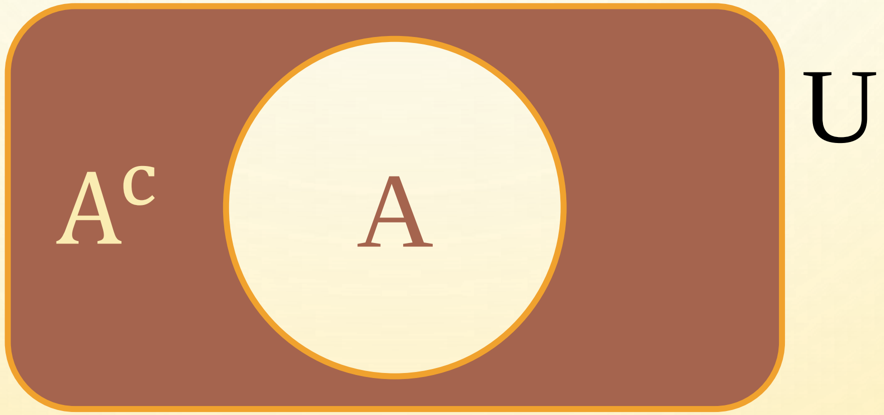
$$= 10090080 + 2522520 + 840840 - 72072 - 9240 - 27720 + 504$$

$$= 103,344,912 \text{ sequences.}$$

NOTE: We have applied multinomials here but you can do a lot.

COUNTING COMPLEMENTS:

Suppose you want to count elements in set A but it is not easy. The other simple approach is by finding the number of elements in the universal set and the number of elements number of elements outside set A. As shown:



That is, $n(A) = n(U) - n(A^c)$ where A^c is the set of elements outside set A but within the universal set.

Example 1: How many permutations of 1, 2, 3, 4, …, 12 are there if 1 comes first, 2 comes fifth and 12 does not come twelfth?

Below is a diagrammatic representation of the question:

1	2	3	4	5	6	7	8	9	10	11	12
1				2							~~12~~

1 and 2 are fixed in their respective points. That means in every permutation, they will be on those positions. As per meaning of permutation, once a digit it can't be used anywhere else.

This place is prohibited for 12. That means all given digits can be used here except 1, 2 and12.

COUNTING TECHNIQUES – Inclusion and Exclusion Principle

Let U = a set of all permutations from 1, 2, to 12’

Let $A \subset U$ such that 1 comes 1^{st}, 2 comes 5^{th} and 12 comes 12^{th}.

$n(U) = {}^{12}P_{12} = 12!$

Now $n(A) = {}^{12-3}P_{12-3} = {}^{9}P_{9}$. We’re subtracting

to show 1, 2, and 12 are on fixed positions.

This our complement as stated on the preceding slide

Lastly, let U* = permutations only 1 comes 1^{st} and 2 comes 5^{th}

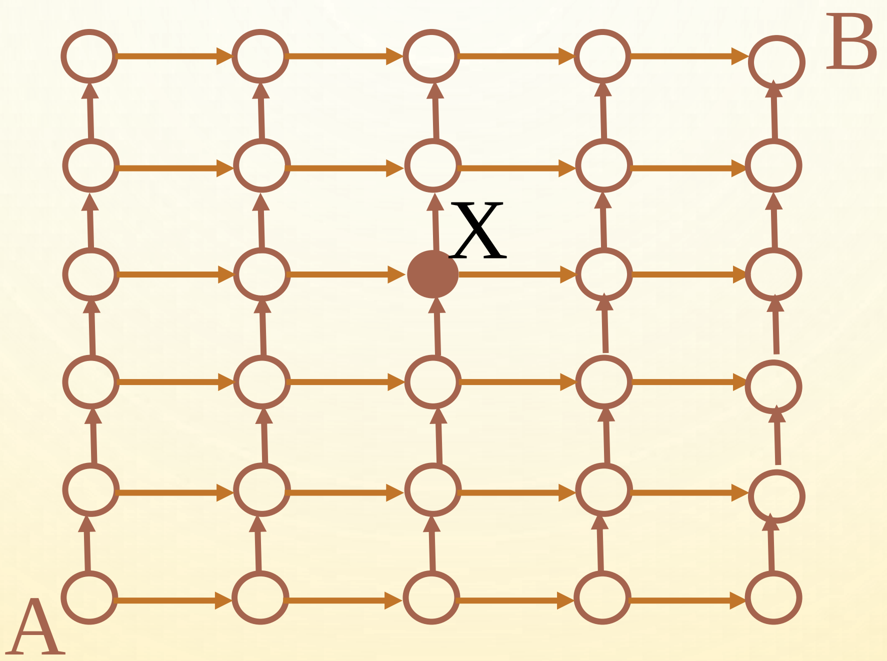
That is, $n(U^*) = {}^{12-2}P_{12-2} = {}^{10}P_{10}$

All this means that

n(U*) – n(A) will be the number of permutations where 12 is not coming on the 12th position but the other (1 and 2) are coming on their respective positions.

$$= {}^{10}P_{10} - {}^{9}P_{9}$$
$$= 3\,265\,920$$

Example 2: Consider the street map below:



Count the number of shortest routes in the above street in moving from A to B that do not pass through junction X.

- Let AB be a set of all routes from A to B.
- AX and XB be routes from A to X and X to B in that order.

n(AB) = $\binom{4+5}{4} = \binom{9}{4} = 126$

n(AX) = $\binom{2+3}{2} = \binom{5}{2} = 10$

n(XB) = $\binom{2+2}{2} = \binom{4}{2} = 6$

To find number of routes from one point to another, you need to add the horizontal movements and the vertical movements then chose any combination you want.

$$\mathbf{n(AB)} = \binom{x+y}{x} = \binom{x+y}{y}$$

COUNTING TECHNIQUES – Inclusion and Exclusion Principle

- Number of routes passing through junction X:

  $= n(AX) \times n(XB)$ by using product rule

  $= 10 \times 6 = 60$

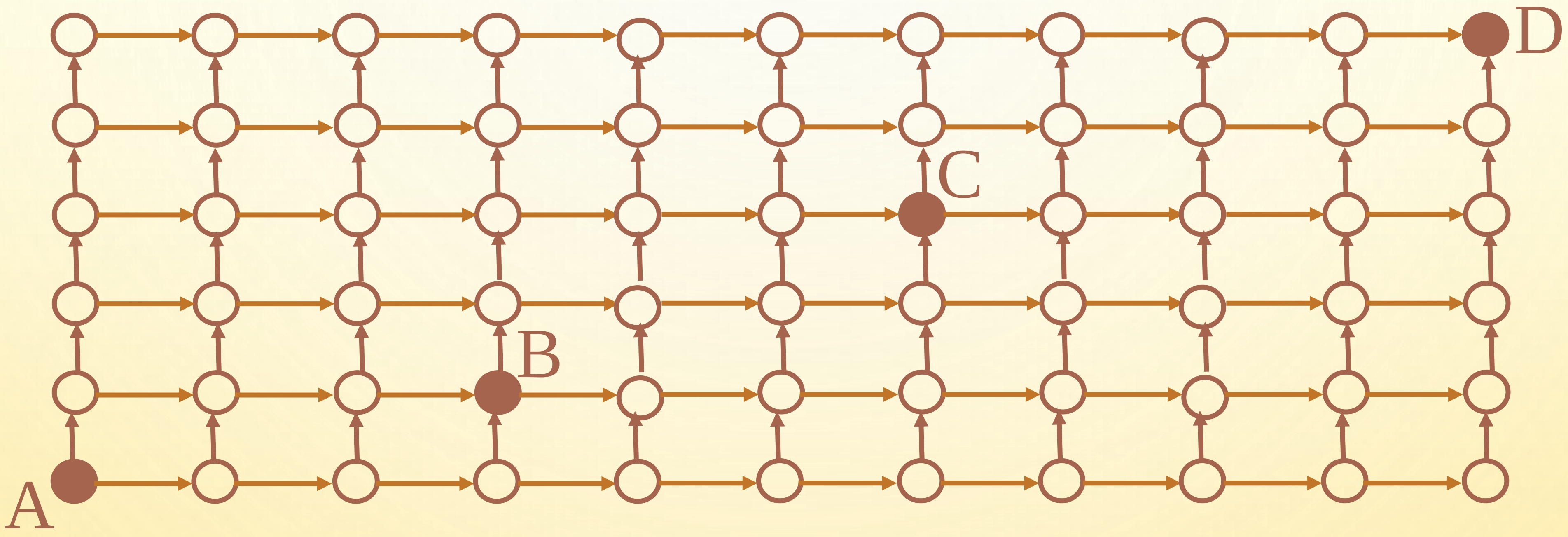
- Therefore, number of routes that do not pass through X:

  = All routes – Routes that passes through X

  $= 126 - 60$

  = 66 routes

Example 3: Consider the following Street map.



Count the number ways to move from A to D through B or C but not both.

- Let AB be a set of different ways to move from A to B
- Let ABD be a set of ways to move from A to D through B
- And so on…

Since we need the number of ways to move from A to D through B or C but not both, the notation can be:

$$n(ABD \cup ACD) - n(ABCD)$$

In this case we are applying the inclusion – exclusion principle. Now, we just need independent numbers and apply here

COUNTING TECHNIQUES – Inclusion and Exclusion Principle

- We can see that:
  - $n(AB) = \binom{3+1}{4} = 4$
  - $n(AC) = \binom{6+3}{6} = 84$
  - $n(BC) = \binom{3+2}{3} = 10$
  - $n(BD) = \binom{7+4}{4} = 330$

- $n(CD) = \binom{4+2}{4} = 15$

Thus;

- $|ABD| = |AB| \times |BD| = 1320$
- $|ACD| = |AC| \times |CD| = 1260$
- $|ABCD| = |AB| \times |BC| \times |CD|$

$$= 600$$

COUNTING TECHNIQUES – Inclusion and Exclusion Principle

And;

- $n(ABD \cup ACD) = |ABD| + |ACD| - |ABD \cap ACD|$

$$= 1320 + 1260 - 600 = 1980$$

Therefore, the needed routes are:

$$n(ABD \cup ACD) - n(ABCD)$$

$$= 1980 - 600$$

$$= 1380$$

- If you checked properly, you may notice that you simply removed the intersection entirely by subtracting it twice on the inclusion - exclusion principle as follows:

We know that $n(ABD) = 1320, n(ACD) = 1260$ and $n(ABD \cap ACD) = n(ABCD) = 600$

$$n(ABD \cup ACD) = n(ABD) + n(ACD) - n(ABCD) - n(ABCD)$$

$$= 1320 + 1260 - 600 - 600 = 1380$$

NOTE: Double exclusion makes sure that there is no common term remaining.

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Thank You For Reading!!

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