Indeterminate Forms

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INTRODUCTION

- In MAT121's elementary calculus, we were introduced to the concept of limits as a number to which $f(x)$ approaches as x approaches a certain value.
- We also looked at how limits are found by not only right and left limits in tabular presentations but also simple substitution methods.
- But, with simple substitution method some expressions produced meaningless limits. For example, when the limits produces a quotient of 0 and 0, $\left(\frac{0}{0}\right)$. This kind of limits is known as the Indeterminate forms. In this session we will look at various indeterminate forms.

INTRODUCTION

- Consider the following example; $\lim_{x \to 2} \frac{2x^2-5x+2}{5x^2-7x-6}$.
- Substituting 2, is automatically making both the numerator and the denominator be equal to 0, hence, the function has a $\frac{0}{0}$ indeterminate form at $x = 2$.
- But if we factorise, $\lim_{x \to 2} \frac{2x^2-5x+2}{5x^2-7x-6} = \lim_{x \to 2} \frac{(x-2)(2x-1)}{(x-2)(5x+3)} = \lim_{x \to 2} \frac{2x-1}{5x+3}$.
- Here, if we substitute, we will get $\frac{3}{13}$. This is how we handled indeterminate forms in elementary calculus.

INTRODUCTION

- In the cases where the expressions can't be factorized, the indeterminate forms require complicated manipulations. For example, in MAT211's squeeze theorem we found that $\lim_{x\to 0}\frac{\sin x}{x} = 1$ and $\lim_{x\to 0}\frac{1-\cos x}{x} = 0$.
- These are some of the famous and mostly used examples in calculus.
- In this session, we will introduce the l'Hôpital's rule to use in understanding the indeterminate forms.
- Some examples of the indeterminate forms are, $\frac{0}{0}$, $(0)(\pm\infty)$, 1^0, 0^0, ∞^0, $\infty \pm \infty$ where each form has a specific set of rules when handling them.

L'HÔPITAL'S RULE FOR (0/0) AND (∞/∞)

- Suppose that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Given $\lim_{x\to a}\frac{f(x)}{g(x)}$, if $f(x)$ approaches 0 or $\pm\infty$ and $g(x)$ approaches 0 or $\pm\infty$ as x approaches a, where a can be any real number, infinity or negative infinity. Then,

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

- To apply this rule, divide the derivative of f by the derivative of g. Do not fall into the trap of taking the derivative of f/g. The quotient to use is f'/g', not $(f/g)'$. Do not use the quotient rule.

L'HÔPITAL'S RULE FOR (0/0) AND (∞/∞)

Example: Evaluate $\lim_{x\to 0} \frac{\sin x}{x}$

- Substituting directly, gives $\frac{0}{0}$. Applying the l'Hôpital's rule, differentiate $\sin x$ and x separately.

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1}$$

- Substituting at this stage gives $\cos 0 = 1$

Therefore;

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = \frac{\cos 0}{1} = 1$$

Example: Evaluate $\lim_{x\to 0}\frac{3x-\sin x}{x}$.

- Substituting directly, gives $\frac{0-0}{0}=\frac{0}{0}$

Applying the l'Hôpital's rule, differentiate $3x-\sin x$ and x separately.

$$\lim_{x\to 0}\frac{3x-\sin x}{x}=\lim_{x\to 0}\frac{3-\cos x}{1}$$

After substituting 0, we get $\frac{3-\cos 0}{1}=\frac{3-1}{1}=\frac{2}{1}$. Therefore;

$$\boxed{\lim_{x\to 0}\frac{3x-\sin x}{x}=2}$$

L'HÔPITAL'S RULE FOR (0/0) AND (∞/∞)

Example: Evaluate $\lim_{x\to 0}\frac{x-\sin x}{x^3}$.

- Substituting 0 gives 0/0, apply the l'Hôpital's rule.

$$\lim_{x\to 0}\frac{x-\sin x}{x^3}=\lim_{x\to 0}\frac{1-\cos x}{3x^2}$$

- Substituting at this step gives 0/0, apply the l'Hôpital's rule again.

$$\lim_{x\to 0}\frac{x-\sin x}{x^3}=\lim_{x\to 0}\frac{1-\cos x}{3x^2}=\lim_{x\to 0}\frac{\sin x}{6x}$$

- Substituting at this step gives 0/0, apply the l'Hôpital's rule again.

$$\lim_{x\to 0}\frac{x-\sin x}{x^3}=\lim_{x\to 0}\frac{1-\cos x}{3x^2}=\lim_{x\to 0}\frac{\sin x}{6x}=\lim_{x\to 0}\frac{\cos x}{6}=\frac{\cos 0}{6}=\frac{1}{6}.$$

L'HÔPITAL'S RULE FOR (0/0) AND (∞/∞)

Example: Evaluate $\lim_{x\to\left(\frac{\pi}{2}\right)^-} \frac{4\tan x}{1+\sec x}$

- The functions are discontinuous at $\frac{\pi}{2}$ hence we only evaluate the right hand side limit. Substituting $\frac{\pi}{2}$ give an indeterminate form ∞/∞
- Apply the l'Hôpital's rule.

$$\lim_{x\to\left(\frac{\pi}{2}\right)^-} \frac{4\tan x}{1+\sec x} = \lim_{x\to\left(\frac{\pi}{2}\right)^-} \frac{4\sec^2 x}{\tan x \sec x} = \lim_{x\to\left(\frac{\pi}{2}\right)^-} \frac{4\sec x}{\tan x}$$

- The last quotient is also an indeterminate form ∞/∞ but applying the rule will always give another indeterminate form

L'HÔPITAL'S RULE FOR (0/0) AND (∞/∞)

- To handle this, we simply need to apply a couple of trigonometric identities

$$\lim_{x\to\left(\frac{\pi}{2}\right)^-}\frac{4\sec x}{\tan x}=\lim_{x\to\left(\frac{\pi}{2}\right)^-}\frac{4/\cos x}{\sin x/\cos x}=\lim_{x\to\left(\frac{\pi}{2}\right)^-}\frac{4}{\sin x}$$

- This is so because $\sec x=\frac{1}{\cos x}$ and $\tan x=\frac{\sin x}{\cos x}$. Substituting $\frac{\pi}{2}$ at this point we get a perfect limit. Therefore;

$$\lim_{x\to\left(\frac{\pi}{2}\right)^-}\frac{4\tan x}{1+\sec x}=\lim_{x\to\left(\frac{\pi}{2}\right)^-}\frac{4}{\sin x}=\frac{4}{\sin\left(\frac{\pi}{2}\right)}=\frac{4}{1}=4$$

L'HÔPITAL'S RULE FOR $(0 \cdot \infty)$

- Given two functions f and g, if $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$, or $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = 0$. For $\lim_{x\to a} f(x) \cdot g(x)$, we will have indeterminate forms $\infty \cdot 0$ and $0 \cdot \infty$ respectively.
- To understand these indeterminate forms, we have to convert them to the form $0/0$ or ∞/∞ and then apply the l'Hôpital's rule.
- When converting, divide with the reciprocal one function as shown

$$f(x)g(x) = \frac{g(x)}{1/f(x)} \text{ or } f(x)g(x) = \frac{f(x)}{1/g(x)}$$

L'HÔPITAL'S RULE FOR $(0 \cdot \infty)$

Example: Evaluate $\lim_{x\to 0^+} x^2 \ln x$

$$\lim_{x\to 0^+} x^2 \ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x^2} = \frac{\infty}{\infty}$$

- Applying the l'Hôpital's rule,

$$\lim_{x\to 0^+} \frac{\ln x}{1/x^2} = \lim_{x\to 0^+} \frac{1/x}{-2/x^3} = \frac{\infty}{\infty}$$

- The last quotient is also an indeterminate form ∞/∞ but applying the rule will always give another indeterminate form. As such, we simplify algebraically to get the following limit.

L'HÔPITAL'S RULE FOR $(0 \cdot \infty)$

$$\lim_{x\to 0^+}\frac{1/x}{-2/x^3} = \lim_{x\to 0^+}\frac{1}{x}\times\frac{x^3}{-2} = \lim_{x\to 0^+}\frac{x^2}{-2}$$

- This limit is an indeterminate form, hence if we substitute we get the following limit

$$\lim_{x\to 0^+}\frac{x^2}{-2} = \frac{0^2}{-2} = 0$$

- Therefore,

$$\lim_{x\to 0^+} x^2\ln x = \lim_{x\to 0^+}\frac{\ln x}{1/x^2} = \lim_{x\to 0^+}\frac{1/x}{-2/x^3} = \lim_{x\to 0^+}\frac{x^2}{-2} = 0$$

L'HÔPITAL'S RULE FOR $(0 \cdot \infty)$

Example: Evaluate $\lim_{x\to\infty}\left(x\sin\frac{1}{x}\right)$

$$\lim_{x\to\infty}\left(x\sin\frac{1}{x}\right)=\lim_{x\to\infty}\frac{\sin 1/x}{1/x}=\lim_{x\to\infty}\frac{-1/x^2\cdot\cos 1/x}{-1/x^2}$$

- After applying the l'Hôpital's rule, we simplify.

$$\lim_{x\to\infty}\frac{-\cancel{1/x^2}\cdot\cos 1/x}{-\cancel{1/x^2}}=\lim_{x\to\infty}\cos\frac{1}{x}=\cos 0=1$$

- This is so because, $\lim_{x\to\infty}\frac{1}{x}=0$, therefore,

$$\boxed{\lim_{x\to\infty}\left(x\sin\frac{1}{x}\right)=\lim_{x\to\infty}\frac{\sin 1/x}{1/x}=\lim_{x\to\infty}\cos\frac{1}{x}=\cos 0=1}$$

L'HÔPITAL'S RULE FOR $(0 \cdot \infty)$

- The same question can be solved without using the l'Hôpital's rule follows.
- Given $\lim_{x\to\infty}\left(x\sin\frac{1}{x}\right)$, let $h=\frac{1}{x}$. That means $x=\frac{1}{h}$ and since $\lim_{x\to\infty}\frac{1}{x}=0$; $h\to 0$. Then,

$$\lim_{x\to\infty}\left(x\sin\frac{1}{x}\right)=\lim_{h\to 0}\left(\frac{1}{h}\sin h\right)=\lim_{h\to 0}\frac{\sin h}{h}=1$$

- The limit $\lim_{h\to 0}\frac{\sin h}{h}=1$ was already found in MAT211 – Squeezing theorem and in the early slides of this session.

L'HÔPITAL'S RULE FOR $(\infty - \infty)$

- Given two functions f and g, if $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$. For $\lim_{x\to a}[f(x) - g(x)]$, we will have indeterminate forms of $\infty - \infty$.
- To understand these indeterminate forms, we have to convert them to the form $0/0$ or ∞/∞ and then apply the l'Hôpital's rule.
- To convert them, if both functions are rational, simply find the common denominator and put the functions in a one-fraction form. If the functions are linear, divide them by 1 and multiply by the numerator's conjugate in a rationalising-the-numerator process. In short, we reverse the process of rationalising the denominator we use in surds.

L'HÔPITAL'S RULE FOR $(\infty - \infty)$

Example: Evaluate $\lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right)$

- If $x \to 0^+$, $\frac{1}{\sin x} \to \infty$ and $\frac{1}{x} \to \infty$, i.e. $\frac{1}{\sin x} - \frac{1}{x} = \infty - \infty$
- And, if $x \to 0^-$, $\frac{1}{\sin x} \to -\infty$ and $\frac{1}{x} \to -\infty$, i.e. $\frac{1}{\sin x} - \frac{1}{x} = -\infty + \infty$
- Neither form reveals what happens in the limit. To find out, we first combine the fractions.

$$\lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right) = \lim_{x \to 0} \left(\frac{x - \sin x}{x \sin x} \right) = \frac{0}{0}$$

- Now, we can apply the l'Hôpital's rule.

L'HÔPITAL'S RULE FOR $(\infty - \infty)$

- This follows that,

$$\lim_{x\to 0}\left(\frac{x-\sin x}{x\sin x}\right) = \lim_{x\to 0}\left(\frac{1-\cos x}{1\cdot\sin x + x\cdot\cos x}\right) = \lim_{x\to 0}\left(\frac{1-\cos x}{\sin x + x\cos x}\right) = \frac{0}{0}$$

- Let's apply the rule again.

$$\lim_{x\to 0}\left(\frac{1-\cos x}{\sin x + x\cos x}\right) = \lim_{x\to 0}\left(\frac{\sin x}{\cos x + \cos x - x\sin x}\right) = \lim_{x\to 0}\left(\frac{\sin x}{2\cos x - x\sin x}\right)$$

- After substitution, $\lim_{x\to 0}\left(\frac{\sin x}{2\cos x - x\sin x}\right) = \frac{0}{2} = 0$. therefore;

$$\boxed{\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right) = \lim_{x\to 0}\left(\frac{x-\sin x}{x\sin x}\right) = \lim_{x\to 0}\left(\frac{\sin x}{2\cos x - x\sin x}\right) = 0}$$

L'HÔPITAL'S RULE FOR 1^∞, 0^0 AND ∞^0

- The indeterminate forms 1^∞, 0^0 and ∞^0 arise from expressions such as $f(x)^{g(x)}$. One common method to handle these indeterminate forms is to take natural logarithms.
- If $y = f(x)^{g(x)}$, then, $\ln y = \ln f(x)^{g(x)} = g(x) \ln f(x)$
- The indeterminate form that follows from $\ln y$ is $0 \cdot \infty$ and can be handled using previously described methods.
- If $\lim\limits_{x \to 0} \ln y = L$, then $\lim\limits_{x \to 0} y = \lim\limits_{x \to 0} e^{\ln y} = e^L$. Thus

$$\boxed{\lim_{x \to a} f(x)^{g(x)} = e^L}$$

L'HÔPITAL'S RULE FOR 1^∞, 0^0 AND ∞^0

Example: Evaluate $\lim_{x\to 0}(1+3x)^{1/2x}$

- The indeterminate form is 1^∞. Now, let's follow the steps highlighted above. Starting by writing it like $y = (1+3x)^{1/2x}$

$$\ln y = \ln(1+3x)^{1/2x} = \frac{1}{2x}\ln(1+3x)$$

- Now, this will be changed into $\frac{g(x)}{1/f(x)}$ form.

$$\frac{\ln(1+3x)}{1/(\frac{1}{2x})} = \frac{\ln(1+3x)}{2x}$$

L'HÔPITAL'S RULE FOR 1^{∞}, 0^0 AND ∞^0

- Now introducing limits,

$$\lim_{x\to 0} \ln y = \lim_{x\to 0} \frac{\ln(1+3x)}{2x} = \frac{0}{0}$$

- Now, we can apply the l'Hôpital's rule

$$\lim_{x\to 0} \frac{\ln(1+3x)}{2x} = \lim_{x\to 0} \frac{\left(\frac{3}{1+3x}\right)}{2} = \frac{3}{2}$$

- Consequently, $\lim_{x\to 0} \ln y = \frac{3}{2}$. This takes us to,

$$\lim_{x\to 0} (1+3x)^{\frac{1}{2x}} = \lim_{x\to 0} e^{\ln y} = e^{\frac{3}{2}}$$

L'HÔPITAL'S RULE FOR 1^∞, 0^0 AND ∞^0

Example: Evaluate $\lim_{x\to\infty}\left(\frac{a^{\frac{1}{x}}+b^{\frac{1}{x}}}{2}\right)^x$

- Let $y=\left(\frac{a^{\frac{1}{x}}+b^{\frac{1}{x}}}{2}\right)^x$ and $\ln y=\ln\left(\frac{a^{\frac{1}{x}}+b^{\frac{1}{x}}}{2}\right)^x = x\ln\left(\frac{a^{\frac{1}{x}}+b^{\frac{1}{x}}}{2}\right)=\frac{\ln\left(\frac{a^{\frac{1}{x}}+b^{\frac{1}{x}}}{2}\right)}{1/x}$. Thus,

$$\lim_{x\to\infty}\left(\frac{a^{\frac{1}{x}}+b^{\frac{1}{x}}}{2}\right)^x=\lim_{x\to\infty}\frac{\ln\left(\frac{a^{\frac{1}{x}}+b^{\frac{1}{x}}}{2}\right)}{1/x}=\frac{\ln 1}{0}=\frac{0}{0}$$

- Now, we can apply the l'Hôpital's rule

$$\frac{d}{dx}\left[\ln\left(\frac{a^{\frac{1}{x}}+b^{\frac{1}{x}}}{2}\right)\right]=\frac{\left(-\frac{1}{x^2}a^{\frac{1}{x}}\ln a-\frac{1}{x^2}b^{\frac{1}{x}}\ln b\right)/2}{\left(a^{\frac{1}{x}}+b^{\frac{1}{x}}\right)/2}=\frac{-\frac{1}{x^2}\left(a^{\frac{1}{x}}\ln a+b^{\frac{1}{x}}\ln b\right)}{a^{\frac{1}{x}}+b^{\frac{1}{x}}}$$

L'HÔPITAL'S RULE FOR 1^∞, 0^0 AND ∞^0

- And after differentiating, we get;

$$\lim_{x\to\infty}\frac{\ln\left(\frac{a^{\frac{1}{x}}\ +\ b^{\frac{1}{x}}}{2}\right)}{1/x}=\lim_{x\to\infty}\frac{\frac{-\frac{1}{x^2}\left(a^{\frac{1}{x}}\ln a+b^{\frac{1}{x}}\ln b\right)}{a^{\frac{1}{x}}\ +\ b^{\frac{1}{x}}}}{-1/x^2}=\lim_{x\to\infty}\frac{\left(a^{\frac{1}{x}}\ln a+b^{\frac{1}{x}}\ln b\right)}{a^{\frac{1}{x}}\ +\ b^{\frac{1}{x}}}$$

- After substituting, we get

$$\frac{(a^0\ln a+b^0\ln b)}{a^0\ +\ b^0}=\frac{\ln a+\ln b}{2}=\frac{1}{2}\ln(ab)=\ln(ab)^{\frac{1}{2}}=\ln\sqrt{ab}$$

$$\lim_{x\to\infty}\left(\frac{a^{\frac{1}{x}}\ +\ b^{\frac{1}{x}}}{2}\right)^x=\lim_{x\to 0}e^{\ln y}=e^{\ln\sqrt{ab}}=\sqrt{ab}$$

PRACTICE QUESTIONS

- Evaluate the following limits

1. $\lim_{x \to 0} \frac{\tan x - \sin x}{x^3 \tan x}$

2. $\lim_{x \to 1} \frac{x^2 - 3x + 2}{2x^2 - 2x - 1}$

3. $\lim_{x \to \infty} \frac{\ln x}{x^2}$

4. $\lim_{x \to \left(\frac{\pi}{2}\right)^-} \tan x \ln \sin x$

5. $\lim_{x \to \left(\frac{\pi}{2}\right)^-} (\sec x - \tan x)$

6. $\lim_{x \to 0^+} (1 + 3x)^{\cot x}$

Thank You For Reading!!

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