Graph Theory Exercises 1 - Basics
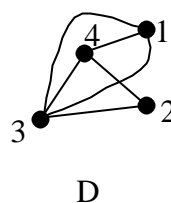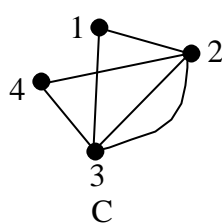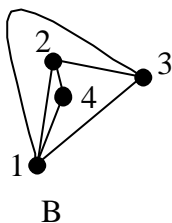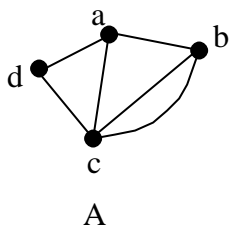
1) Draw the graphs whose vertices and edges are as follows. In each case say if the graph is a simple graph.

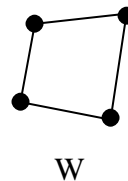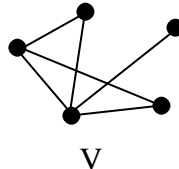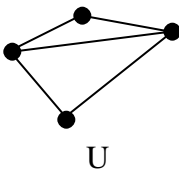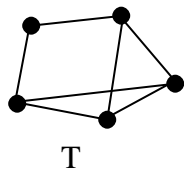   a) $V = \{u, v, w, x\}$, $E = \{uv, vw, wx, vx\}$

   b) $V = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $E = \{12, 22, 23, 34, 35, 67, 68, 78\}$

   c) $V = \{n, p, q, r, s, t\}$, $E = \{np, nq, nt, rs, rt, st, pq\}$

2) Which of graphs B, C and D are isomorphic to graph A? State the corresponding vertices in each isomorphic pair.

3) Which of the graphs P, Q, … W are subgraphs of G?
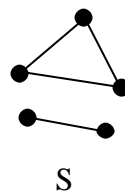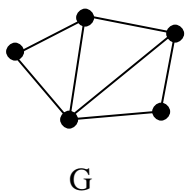
4) Write down the degree sequence of each of the following graphs.
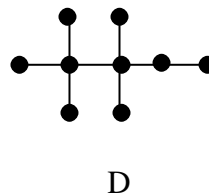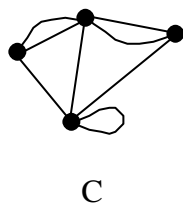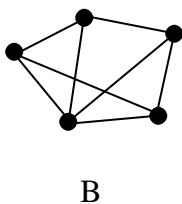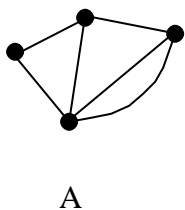
5) Graphs G_1 and G_2 have the same degree sequence - are they necessarily isomorphic? If your answer is no, give a counter example.
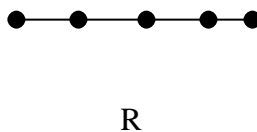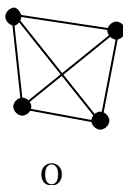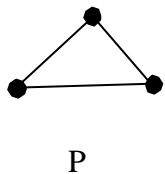
6) Graphs G_1 and G_2 are isomorphic. Do they necessarily have the same degree sequence? If your answer is no, give a counter example.

7) Draw simple connected graphs with the degree sequences

   a) (1, 1, 2, 3, 3, 4, 4, 6)

   b) (3, 3, 3, 3, 3, 5, 5, 5)

   c) (1, 2, 3, 3, 3, 4, 4)

8) Use the Handshaking Lemma to prove that the number of vertices of odd degree in any graph must be even.

9) Complete the following statements with walk/trail/path :-
    a) wvyxvurs is a ____ of length __ between w and s
    b) vxvur is a ____ of length __ between v and r
    c) uvyxvw is a ____ of length __ between u and w
    d) ruvwy is a ____ of length __ between r and y

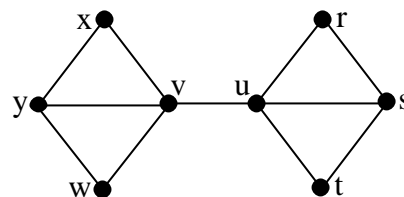10) Write down all the paths between x and t in the graph in question 9. Which edge in this graph is a bridge?

11) Draw an r-regular graph with 6 vertices for $r = 3$ and $r = 4$.

12) Why are there no 3-regular graphs with 5 vertices?

13) Draw the graphs K_5 , N_5 and C_5 .

14) Draw the complete bipartite graphs $K_{2,3}$, $K_{3,5}$, $K_{4,4}$. How many edges and vertices does each graph have? How many edges and vertices would you expect in the complete bipartite graphs $K_{r,s}$.

15) Under what conditions on r and s is the complete bipartite graph $K_{r,s}$ a regular graph?

16) Show that, in a bipartite graph, every cycle has an even number of edges.

17) **This is a more challenging question than the first sixteen**.
    The complement of a simple graph G is the graph obtained by taking the vertices of G (without the edges) and joining every pair of vertices which are not joined in G. For instance
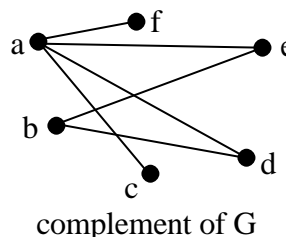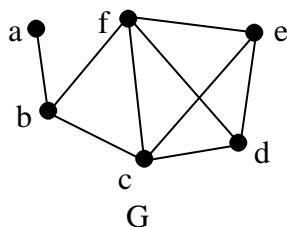
a) Verify that the complement of the path graph P_4 is P_4 .

b) What are the complements of K_4 , $K_{3,3}$, C_5 ?

c) What is the relationship between the degree sequence of a graph and that of its complement?

d) Show that if a simple graph G is isomorphic to its complement then the number of vertices of G has the form 4k or 4k + 1 for some integer k.

e) Find all the simple graphs with 4 or 5 vertices which are isomorphic to their complements.

f) Construct a graph with eight vertices which is isomorphic to its complement.