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The minimum number of multiplicity 1 eigenvalues among real symmetric matrices whose graph is a nonlinear tree

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Abstract: In the study of eigenvalues, multiplicities, and graphs, the minimum number of multiplicities equal to 1 in a real symmetric matrix with graph G, $U(G)$, is an important constraint on the possible multiplicity lists among matrices in $\mathcal{S}(G)$. Of course, the structure of G must determine $U(G)$, but, even for trees, this linkage has proven elusive. If T is a tree, $U(T)$ is at least 2, but may be much greater. For linear trees, recent work has improved our understanding. Here, we consider nonlinear trees, segregated by diameter. This leads to a new combinatorial construct called a core, for which we are able to calculate $U(T)$. We suspect this bounds $U(T)$ for all nonlinear trees with the given core. In the process, we develop considerable combinatorial information about cores.

Keywords: Eigenvalue; Graph of a matrix; Multiplicity list; Nonlinear tree; $U(T)$

MSC: Primary: 15A18, 05C50 ; Secondary: 15B57

1 Introduction

The graph of an n-by-n real symmetric matrix $A = (a_{ij})$ is an undirected, simple graph $G(A)$ on vertices $\{1, 2, \cdots, n\}$ in which there is an edge $\{i, j\}$ if and only if $a_{ij} \neq 0$. Given a graph G, we consider the set $\mathcal{S}(G)$ of all real symmetric A such that $G(A) = G$. Each element of $\mathcal{S}(G)$ enjoys a list of ordered multiplicities for its eigenvalues, i.e., an ordered partition of n, in which the order corresponds to the numerical order of the underlying, necessarily real, eigenvalues. It is known [10] that if $G = T$ is a tree, then the first and last multiplicity in every ordered list are both 1. We also consider "unordered" multiplicity lists, in which the elements of the partition are simply listed in descending order. For a given graph G, the *catalog* of all such unordered lists that occur among matrices in $\mathcal{S}(G)$ is denoted $\mathcal{L}(G)$, while the ordered catalog is denoted $\mathcal{L}_o(G)$. Of course, determining the catalog, especially for trees, has been the subject of much prior research. The recent book [10] is a good general reference, and it provides a database for $\mathcal{L}(T)$ for all trees on fewer than 13 vertices.

An important constraint on the catalogs for trees is the minimum number of 1's occurring among the lists for a tree T, which we denote as $U(T)$. For any tree, $U(T) \geqslant 2$, and it may well be larger. A high degree vertex (HDV) of a tree is simply one of degree at least 3. A tree is called *linear* if all its HDV's lie on a single induced path of the tree; otherwise it is called "nonlinear" (NL). Any tree with fewer than 4 HDV's is linear, and the first NL tree has 10 vertices (see display in Figure 1).

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There is remarkable information about the catalog of linear trees based upon the so-called linear superposition principle (LSP) [8, 13]. In view of recent work on $U(T)$ for linear trees[2, 3], our interest here is in better understanding $U(T)$ for NL trees.

Of course, it is natural to relate important aspects of multiplicity lists to the structure of a tree. Examples include the following. The *path cover number*, $P(T)$, is the fewest vertex disjoint induced paths of the tree that cover all its vertices. For the 10-vertex NL tree, $P(T) = 4$. It is known that the maximum multiplicity, $M(T)$, appearing in the catalog $\mathcal{L}(T)$, for a tree T, is exactly $P(T)$ [4]. Also, the *diameter*, $d(T)$, of a tree T, is simply the greatest number of vertices in an induced path of T, and the minimum number of distinct eigenvalues (the number of parts in a shortest partition) of a matrix in $\mathcal{S}(T)$ is at least $d(T)$ [5, 9]. We also use diameter to refer to a path in T with length $d(T)$. Moreover, $U(T) \geqslant 2d(T) - n$, in which n is the number of vertices of T [1].

Because there is no known analog of the LSP for NL trees, it is more difficult to understand $U(T)$ for NL trees. So, for NL trees, we develop some new combinatorial ideas, primarily the "core", and relate them to tree structure and to $U(T)$. In the process, we are able to enumerate some tree parameters in terms of cores with a given diameter. In the case of linear trees, prior work has given some detailed information about $U(T)$, mostly using the LSP [1–3]. This includes explicit formulae in the case of 1 or 2 HDV's and results about the possible changes in $U(T)$ with the addition of a vertex. (In particular, vertex addition is done via *pendent vertex addition*, in which a new edge and a vertex pendent at an existing vertex are added, or *edge subdivision*, in which a new vertex of degree 2 is positioned along an existing edge.) If a corresponding incremental result is true in the NL case, a conjecture we make based on new results here would give strong information about $U(T)$ for NL trees.

2 Background

Throughout, we employ standard submatrix notation. Let A be an n-by-n matrix. If $\alpha \subset \{1, \dots, n\}$ is an index set, then $A[\alpha]$ is the principal submatrix of A in the rows and columns indexed by α, and $A(\alpha) := A[\{1, \dots, n\} \backslash \alpha]$. In the case when $\alpha = \{v\}$, we abbreviate $A(\{v\})$ to $A(v)$. Observe that the graph $T[\alpha]$ is the subgraph of T induced by the vertices corresponding to α. If $A \in \mathcal{S}(T)$, then $A[\alpha] \in \mathcal{S}(T[\alpha])$, and we often think of the matrix and graph interchangeably. We write $m_A(\lambda)$ for the multiplicity of the eigenvalue λ in the matrix A (subscript A sometimes omitted). A classical and fundamental theorem for this study is the interlacing inequalities for Hermitian (real symmetric) matrices. An immediate consequence of this theorem is that for any eigenvalue λ and $i \in \{1, \dots, n\}$, $|m_A(\lambda) - m_{A(i)}(\lambda)| \leq 1$. That is, the multiplicity of an eigenvalue of a real symmetric matrix changes by at most 1 if a principal submatrix of size one smaller is extracted.

Let A be a real symmetric matrix whose graph is a tree T. The removal of a vertex v from a tree T corresponds to the removal of a row and column with the same index v from A. When v is deleted from T, a forest of several components $T_1, \dots, T_{deg_v}$ remains. The corresponding matrix is $A(v) = A[T_1] \oplus \cdots \oplus A[T_{\deg_v}]$. The Parter-Wiener, etc. theorem is one of the most important tools in the study of eigenvalues, multiplicities, and graphs. The most general form of the theorem is given in [7].

Theorem 2.1. *Let T be a tree and A a matrix in $\mathcal{S}(T)$. Let $\sigma(A)$ denote the spectrum of A. Suppose that there is a vertex v of T and a real number λ such that $\lambda \in \sigma(A) \cap \sigma(A(v))$. Then*

(1) *there is a vertex u of T such that $m_{A(u)}(\lambda) = m_A(\lambda) + 1$;*

(2) *if $m_A(\lambda) \geq 2$, then $\lambda \in \sigma(A) \cap \sigma(A(v))$ is automatically satisfied and u may be chosen so that $deg_T(u) \geq 3$ and so that there are at least three components T_1, T_2, and T_3 of $T \backslash u$ such that $m_{A[T_i]}(\lambda) \geq 1$, $i = 1, 2, 3$; and*

(3) *if $m_A(\lambda) \geq 1$, then u may be chosen so that $deg_T(u) \geq 2$ and so that there are two components T_1 and T_2 of $T \backslash u$ such that $m_{A[T_i]}(\lambda) \geq 1$, $i = 1, 2$.*

We call a vertex v meeting the requirement in the above theorem a *Parter vertex* of T for λ in A (i.e., v is Parter for λ in A.) In other words, v is Parter for λ if $m_{A(v)}(\lambda) = m_A(\lambda) + 1$ and λ is an eigenvalue of the submatrices corresponding to at least two of the connected components of $T - v$.

Some classes of trees have been studied extensively. A (simple) *star* is a tree on n vertices having a vertex of degree $n - 1$. This vertex is called the *center* of the star. A *generalized star (g-star)* is a tree with at most one HDV; moreover, the HDV (or a degree 2 vertex if there is no HDV) is called the *central vertex* of the g-star. A g-star consists of several paths (*arms*) hanging from the central vertex. For both stars and g-stars, the choice of Parter vertex for any multiple eigenvalue is obvious as there is no other HDV in them.

Lastly, we introduce a construction technique for multiplicity lists with multiple eigenvalues: the *method of eigenvalue assignments* (to subtrees for which possible spectra are known). This technique is an informal method of constructing multiplicity lists using Parter vertices and several coincidences of eigenvalues among various subtrees. A *realization* of an assignment verifies the existence of a desired multiplicity list. Definition 2.2 states the formalization of eigenvalue assignment. More details can be found in [10].

Definition 2.2. Let T be a tree on n vertices and let

$$\left(p_1, p_2, \ldots, p_k, 1^{n-\sum_{i=1}^k p_i}\right)$$

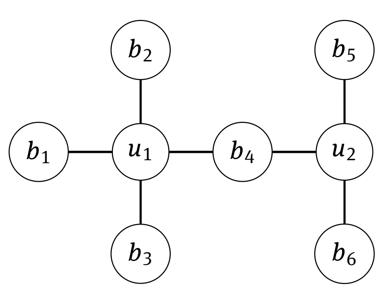
be a non-increasing list of positive integers, with $\sum_{i=1}^k p_i \leqslant n$. The notation 1^l denotes that the last l entries of the list are 1. These will be the desired eigenvalue multiplicities. Note that some of the p_i's may be 1. Then, an *assignment* $\mathcal{A}$ is a collection $\mathcal{A} = \{\mathcal{A}_1, \ldots, \mathcal{A}_k\}$ of k collections $\mathcal{A}_i$ of subtrees of T, corresponding to eigenvalues with multiplicities p_i, with the following properties.

1. (Specification of Parter vertices) For each i, there exists a set V_i of vertices of T such that
   (1a) Each subtree in $\mathcal{A}_i$ is a connected component of $T - V_i$.
   (1b) $|\mathcal{A}_i| = p_i + |V_i|$.
   (1c) For each vertex $v \in V_i$, there exists a vertex x adjacent to v such that x is in one of the subtrees in $\mathcal{A}_i$.
2. (No overloading) We require that no subtree S of T is assigned more than $|S|$ eigenvalues; define $c_i(S) = |\mathcal{A}_i \cap \mathcal{Z}(S)| - |V_i \cap S|$, the difference between the number of subtrees contained in S and the number of Parter vertices in S for the i^{th} multiplicity. Then we require that $\sum_{i=1}^k \max\left(0, c_i(S)\right) \le |S|$ for each $S \in \mathcal{Z}(T)$. If this condition is violated at any subtree, then that subtree is said to be *overloaded*. Notation: if V is a set of vertices and G is a graph, then $V \cap G$ denotes the set of vertices in both V and G. And $\mathcal{Z}(T)$ denotes the collection of all subtrees of T, including T.

Theorem 2.3. *If T is a tree and $q \in \mathcal{L}(T)$ includes multiplicities greater than 1, then there is an assignment for q.*

We use an example from [10] to illustrate this technique.

Example 2.4. Consider the following tree T.



The multiplicity list $(3, 2, 1^3) \in \mathcal{L}(T)$ because of the realizable assignment in which u_1 is Parter for α such that $m(\alpha) = 3$ and u_2 is Parter for β such that $m(\beta) = 2$. To be specific, α appears four times in $T - u_1$: on the three neighbors of u_1 and once on the subtree to the right; β appears three times in $T - u_2$: on the two neighbors of u_2 and once on the subtree to the left. Formally, the assignment is summarized as follows.

$\mathcal{A}_1 = \{\{b_1\}, \{b_2\}, \{b_3\}, \{b_4, b_5, b_6, u_2\}\}$ with $V_1 = \{u_1\}$ for α.

$\mathcal{A}_2 = \{\{b_5\}, \{b_6\}, \{u_1, b_1, b_2, b_3, b_4\}\}$ with $V_2 = \{u_2\}$ for β.

Nevertheless, there is no eigenvalue assignment for the multiplicity list $(2, 2, 2, 1^2)$, hence $(2, 2, 2, 1^2) \notin \mathcal{L}(T)$. In particular, one of the two HDV's (u_1 and u_2) would have to be Parter for two of the multiplicity 2 eigenvalues. However, neither has enough branches of sufficient size to assign the two eigenvalues a total of six times. Furthermore, not $(5, 1^3)$ but $(4, 1^4)$ is possible by letting both u_1 and u_2 be Parter for the same eigenvalue. Also, notice that since $(3, 2, 1^3) \in \mathcal{L}(T)$, $U(T) \leqslant 3$. In fact, according to the database of all trees on fewer than 13 vertices [10], $U(T) = 3$.

3 Cores

3.1 The Notion of Core

We propose a way to classify NL trees by diameter.

Definition 3.1. An NL *core* of diameter $d(\geqslant 5)$ is a NL tree, minimal with respect to the number of vertices. Diameter d cores are not unique ($d \geqslant 6$), but every diameter d NL tree is the result of pendent vertex addition to one or more diameter d cores.

Lemma 3.2. *Any diameter d core contains only 4 HDV's.*

Proof. Suppose the core C had five (or more) HDV's. Choose one, say v, so that the remaining HDV's also do not lie on a single path. A diameter, the longest induced path, can go through at most two neighbors of v. Remove all other neighbors of v in C, making v no longer an HDV, and giving a new tree C'. C' still has diameter d and is still NL. So, C was not a core. □

Now, we give a characterization of diameter d cores.

Proposition 3.3. *An NL tree T with diameter d is a diameter d core if and only if the number of vertices in T is* $d + 5$.

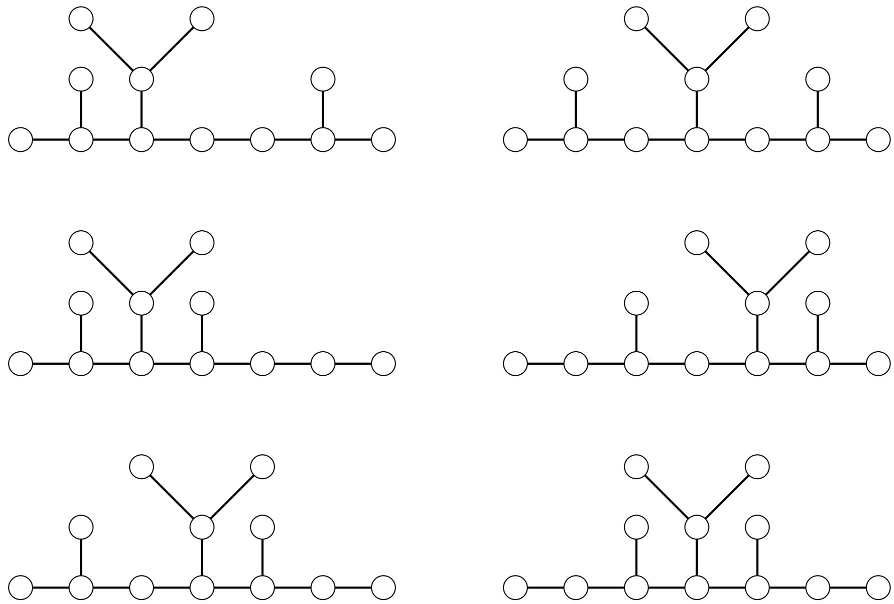
Proof. First, adding five vertices to a path on d vertices can result in an NL tree with diameter d. In particular, we choose three interior vertices on the path, add a pendent vertex to the first and third vertex, and attach a 3-path to the second vertex (with the middle vertex of the 3-path adjacent to it.) Since a core, say T, has the fewest vertices, $n(T) \leqslant d + 5$.

Second, $n(T) \geqslant d + 5$. By Lemma 3.2, among the four HDV's in T, the diameter goes through at most three of them. Then, at least five vertices are not on the diameter – the last HDV that is not on the diameter, say v, at least two of v's neighbors, and one neighbor for each of at least two HDV's that are on the diameter. So, $n(T) \geqslant d + 5$. In conclusion, $n(T) = d + 5$. □

Remark 3.4. We classify NL trees by diameter because cores group NL trees in a systematic way. We remark the following.

1. (Finiteness) For each d, there are finitely many diameter d cores, up to isomorphism.
2. (Infinity) Each diameter d core generates an infinite family of diameter d NL trees via a sequence of pendent additions of vertices without increasing the diameter.
3. (Inclusion) The union of the families generated by all the diameter d cores is the set of all NL trees with diameter d.
4. (Nonuniqueness) A diameter d NL tree may be generated from more than one core.

Example 3.5. The diameter 5 core (see Figure 1) is the only NL tree on 10 vertices, and there are two diameter 6 cores (see Figure 2). We display the six diameter 7 cores, up to isomorphism.



3.2 Counting Non-isomorphic Diameter *d* Cores

We first introduce an algorithm for generating diameter d cores.

Algorithm 3.6. We start with the 10-vertex NL tree (Figure 1) By the minimality of cores and Proposition 3.3, diameter d cores are obtained by adding $d-5$ vertices to the 10-vertex NL tree in a way that the addition of each vertex increases the diameter by 1, which are, in fact, $d-5$ edge subdivisions on exactly *one* diameter. Since the diameter is 5, generating cores involves edge subdivisions on four edges. Let p, q, r, s denote the number of edge subdivisions performed on each of these four edges. Then, (p, q, r, s), an ordered partition of $d-5$ into four parts, each of which is a nonnegative integer (0 is allowed), represents a diameter d core that is obtained by subdividing the first edge of one diameter in the smallest NL tree p times, the second q times, the third r times, and the last s times. Thus, the set of all diameter d cores equals all such ordered partitions of $d-5$ into four parts.

Example 3.7. We revisit the six diameter 7 cores in Example 3.5. By Algorithm 3.6, we consider ordered partitions of $d-5 = 7-5 = 2$. From upper left to lower right, the cores correspond to partitions $(0,0,2,0)$, $(0,1,1,0)$, $(0,0,0,2)$, $(1,1,0,0)$, $(0,1,0,1)$, and $(1,0,1,0)$. One might wonder if some partitions such as $(2,0,0,0)$ are missing. The answer is no because of forward/backward symmetry, which results from orienting the diameter from either end.

Proposition 3.8. *Diameter d cores with ordered partitions* (p, q, r, s) *and* (s, r, q, p) *are isomorphic.*

By Remark 3.4, diameter d cores are finite for each d, so a natural question to ask is the number of distinct diameter d cores. We define $\mathcal{C}(d)$ to be the collection of distinct diameter d cores. And $|\mathcal{C}(d)|$ denotes the number of non-isomorphic diameter d cores.

Proposition 3.9.

$$|\mathcal{C}(d)| = \begin{cases} \dfrac{\binom{d-2}{3}}{2}, & \text{if } d \text{ is even,} \\ \dfrac{\binom{d-2}{3}}{2} + \dfrac{d-3}{4}, & \text{if } d \text{ is odd.} \end{cases}$$

Proof. Among the d vertices along a diameter, we choose any subset of three vertices among the $d-2$ interior vertices of this diameter. Each such selection corresponds to a diameter d core: by considering this subset $\{i, j, k\} \subseteq \{1, 2, \ldots, d-2\}$ in the order $i < j < k$, we attach pendent vertices at each of the vertices corresponding to i and k and attach a 3 -path (v_0, v_1, v_2) to vertex j at v_1. This generates every diameter d core, but up to isomorphism - we have overcounted. If the diameter is oriented horizontally, two diameter d cores constructed this way will be isomorphic if and only if we obtain one from the other by the vertical reflection through the center of the diameter. When d is even, there is no central vertex, and we have thus counted diameter d cores in pairs; hence there are exactly $\binom{d-2}{3}/2$ diameter d cores. When d is odd, there is a central vertex in the diameter d path, and there are thus diameter d cores that are automorphic under the vertical reflection. However, in this case, using the notation above, j must be the middle vertex, and the choice of i determines the choice of k. There are $(d-3)/2$ choices for i, and so there are exactly $(d-3)/2$ diameter d cores that are fixed by the vertical reflection through the center of the longest path, meaning $\binom{d-2}{3}/2$ undercounts $|\mathcal{C}(d)|$ by exactly $((d-3)/2)/2$. The result follows. □

We calculate $|\mathcal{C}(d)|$ up to $d = 20$, and interestingly, we notice that the first differences of this sequence are increasing repeated squares.

d	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\lvert\mathcal{C}(d)\rvert$	1	2	6	10	19	28	44	60	85	110	146	182	231	280	344	408
$\lvert\mathcal{C}(d)\rvert - \lvert\mathcal{C}(d-1)\rvert$	1	1	4	4	9	9	16	16	25	25	36	36	49	49	64	64

Remark 3.10.

$$|\mathcal{C}(d+1)| - |\mathcal{C}(d)| = \begin{cases} (\frac{1}{2}(d-2))^2, & \text{if } d \text{ is even,} \\ (\frac{1}{2}(d-3))^2, & \text{if } d \text{ is odd.} \end{cases}$$

In fact, this sequence of $|\mathcal{C}(d)|$ has appeared elsewhere and been recorded in the On-Line Encyclopedia of Integer Sequence [14] with a generating function

$$\frac{(1+k^2)}{(1-k)^2(1-k^2)^2} \qquad \text{where } k = d-5.$$

Proposition 3.9 provides an explicit formula for this sequence, and observation about first differences in Remark 3.10 can also be additional to the encyclopedia.

3.3 Diameters in a Core

We may also ask how many distinct diameters lie in a given core. We identify cores with different number of diameters and determine the number of diameter d cores with a certain number of diameters. Proof details are in Appendix A.

Lemma 3.11. *In an ordered partition (p, q, r, s), let "$\star$" denote a nonzero integer and "?" some integer that can be zero or nonzero. Then,*

Cores with ordered partition	$(\star, ?, ?, \star)$	$(0, \star, ?, \star)$	$(\star, ?, 0, 0)$ *or* $(0, \star, \star, 0)$	$(0, \star, 0, 0)$
The number of distinct diameters	1	2	4	8

Corollary 3.12. *For each diameter d core ($d \geqslant 6$), the number of distinct diameters is a member of $\{1, 2, 4, 8\}$; moreover, among all the diameter d cores ($d \geqslant 7$), each of $\{1, 2, 4, 8\}$ will occur for at least one of the cores.*

Corollary 3.13. *In $\mathcal{C}(d)$ with $d \geqslant 7$, $|\mathcal{C}(d-2)|$ cores have a unique diameter, $\binom{d-6}{2}$ cores have two diameters, 1 core has eight diameters, and the number of cores with four diameters is*

$$\begin{cases} 1.5d - 7.5 \text{ if } d \text{ is odd,} \\ 1.5d - 8 \text{ if } d \text{ is even.} \end{cases}$$

4 Cores and $U(T)$

After laying the groundwork for cores and exploring their properties, we now discuss $U(T)$ for cores and NL trees.

4.1 Diameter 5 and 6 NL Trees

In this subsection, through an extensive examination, we find that for any diameter 5 or 6 NL tree, $U(T) = 2$. The proof is based on expansion of cores and eigenvalue assignment.

Lemma 4.1. *Suppose $\mathcal{A}$ is a realizable eigenvalue assignment for a tree T, and the multiplicity list realized by $\mathcal{A}$ achieves $U(T)$, then for the tree T' resulting from adding a pendent vertex at a Parter vertex for some multiple eigenvalue, $U(T') \leqslant U(T)$.*

Proof. Suppose a pendent vertex u is added to a Parter vertex v for λ in $\mathcal{A}$ for T, which results in T'. Then, in eigenvalue assignment $\mathcal{A}'$ for T', we assign λ to u and adopt assignment $\mathcal{A}$ for the rest. Notice that besides the assignment of λ to the new vertex u, $\mathcal{A}'$ slightly differs from $\mathcal{A}$ because for other eigenvalues, some subtree need to contain u in $\mathcal{A}'$. However, since T' is larger than T, there is no worry about overloading for $\mathcal{A}'$. Thus, the realizability of $\mathcal{A}$ implies that for $\mathcal{A}'$; so, as no new multiplicity 1 eigenvalue is created by the addition of u, $U(T') \leqslant U(T)$, as desired. $U(T') < U(T)$ happens when a multiplicity 1 eigenvalue becomes multiplicity 2 as a result of the addition; on the other hand, $U(T)$ remains the same when the multiplicity of a multiple eigenvalue increases by 1. $\square$

Theorem 4.2. *If T is a diameter 5 NL tree, then $U(T) = 2$.*

Proof. The 10-vertex NL tree T_0 (see labeling in Figure 1) is the only diameter 5 core, so by Remark 3.4, any diameter 5 NL tree T can be obtained from T_0 via pendent vertex addition. Due to the restriction on diameter, no vertex is more than two edges away from c. So, to expand T_0, the choice of places for vertex addition is limited. In particular, at c, we can add pendent vertices, 2-paths, or simple stars, whereas we can only add pendent vertices to v_i.

T_0 =

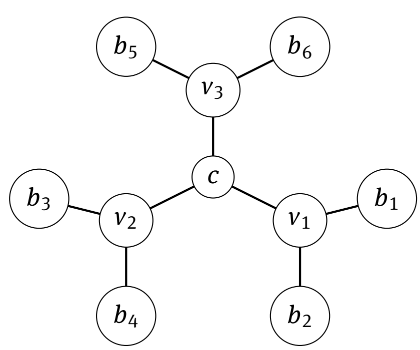


Figure 1: Diameter 5 core

It is known from the database that $(4, 2, 2, 1, 1) \in \mathcal{L}(T_0)$, hence $U(T_0) = 2$. Call the multiple eigenvalues λ, α, and β; an eigenvalue assignment for multiplicity list $(4, 2, 2, 1, 1)$ is as follows.

$\mathcal{A}_1 = \{\{b_1\}, \{b_2\}, \{b_3\}, \{b_4\}, \{b_5\}, \{b_6\}, \{c\}\}$ with $V_1 = \{v_1, v_2, v_3\}$ for λ.

$\mathcal{A}_2 = \mathcal{A}_3 = \{\{v_1, b_1, b_2\}, \{v_2, b_3, b_4\}, \{v_3, b_5, b_6\}\}$ with $V_2 = V_3 = \{c\}$ for α and β.

We will show that as we obtain any diameter 5 NL tree T from T_0, two 1's in this multiplicity list $(4, 2, 2, 1, 1)$ can be maintained by increasing the multiplicity of multiple eigenvalues; therefore, $U(T) = 2$. Although diameter 5 NL trees are infinite, there are only four basic ways to add vertices, and any diameter 5 NL trees can be viewed as adding vertices to T_0 via a combination of these four ways.

The first and second way is about adding one additional vertex to T_0. We can only do so at either c or some v_i. Since c and v_i are all Parter vertices for some multiple eigenvalue, i.e. $\{c, v_1, v_2, v_3\} \subseteq V_1 \cup V_2 \cup V_3$, by Lemma 4.1, trees resulting from such vertex addition have $U(T) = 2$. The third way is to add a 2-path to T_0, say $\{u_1, u_2\}$, which has to be adjacent to c. Then, because $V_2 = V_3 = \{c\}$ for α and β, we assign α and β to this 2-path. As a result, $(4, 3, 3, 1, 1)$ is realized. The fourth way involves adding a simple star to T_0, which again, requires the center of the star to be adjacent to c. We may assume adding a (degenerate) simple star, i.e. a 3-path $\{u_2, u_1, u_3\}$ with u_1 adjacent to c to T_0. This suffices because if u_1 is a Parter vertex for some multiple eigenvalue in this case, Lemma 4.1 also covers stars with more pendent vertices. Multiplicity list $(5, 3, 3, 1, 1)$ can be realized by the following eigenvalue assignment.

$\mathcal{A}_1 = \{\{b_1\}, \{b_2\}, \{b_3\}, \{b_4\}, \{b_5\}, \{b_6\}, \{c\}, \{u_2\}, \{u_3\}\}$ with $V_1 = \{v_1, v_2, v_3, u_1\}$ for λ.

$\mathcal{A}_2 = \mathcal{A}_3 = \{\{v_1, b_1, b_2\}, \{v_2, b_3, b_4\}, \{v_3, b_5, b_6\}, \{u_1, u_2, u_3\}\}$ with $V_2 = V_3 = \{c\}$ for α and β.

Any diameter 5 NL tree T can be dissected into a combination of these four ways of vertex addition to the diameter 5 core T_0; since eigenvalue assignments in these four ways do not conflict with one another, $U(T) = 2$. □

Theorem 4.3. *If T is a diameter 6 NL tree, then $U(T) = 2$.*

Proof. The idea is similar to that for Theorem 4.2, i.e. no new vertex is more than two edges away from any vertex in a diameter 6 core, although the examination is more complicated. There are two diameter 6 cores, T_1 and T_2, and we label their vertices as in Figure 2. $U(T_1) = U(T_2) = 2$ because the multiplicity list $(3, 2, 2, 2, 1, 1)$ is realizable for both of them, according to the database. Call the multiple eigenvalues λ, α, β, and γ. We display the eigenvalue assignment for T_1 and T_2.

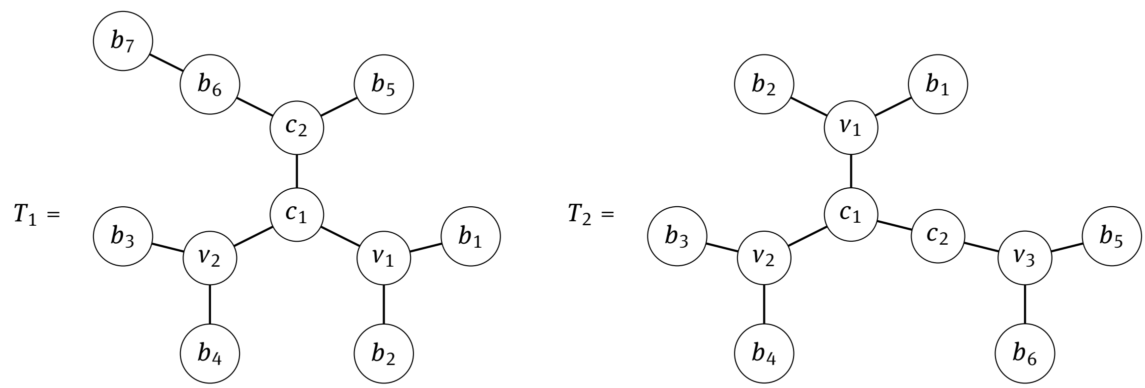


Figure 2: Diameter 6 cores

For T_1: $\mathcal{A}_1 = \{\{b_1\}, \{b_2\}, \{b_3\}, \{b_4\}, \{c_1, c_2, b_5, b_6, b_7\}\}$ with $V_1 = \{v_1, v_2\}$ for λ.
$\mathcal{A}_2 = \mathcal{A}_3 = \{\{v_1, b_1, b_2\}, \{v_2, b_3, b_4\}, \{c_2, b_5, b_6, b_7\}\}$ with $V_2 = V_3 = \{c_1\}$ for α and β.
$\mathcal{A}_4 = \{\{b_5\}, \{b_6, b_7\}, \{c_1, v_1, b_1, b_2, v_2, b_3, b_4\}\}$ with $V_4 = \{c_2\}$ for γ.

For T_2: $\mathcal{A}_1 = \{\{b_1\}, \{b_2\}, \{b_3\}, \{b_4\}, \{c_1\}, \{v_3, b_5, b_6\}\}$ with $V_1 = \{v_1, v_2, c_2\}$ for λ.
$\mathcal{A}_2 = \mathcal{A}_3 = \{\{v_1, b_1, b_2\}, \{v_2, b_3, b_4\}, \{c_2, b_5, b_6, v_3\}\}$ with $V_2 = V_3 = \{c_1\}$ for α and β.
$\mathcal{A}_4 = \{\{b_5\}, \{b_6\}, \{c_1, c_2, v_1, b_1, b_2, v_2, b_3, b_4\}\}$ with $V_4 = \{v_3\}$ for γ.

Any diameter 6 NL tree T can be obtained from T_1 or T_2 by a sequence of pendent vertex addition. We will again show that this multiplicity list $(3, 2, 2, 2, 1, 1)$ can be updated as we obtain T from T_1 or T_2 so that $U(T) = 2$. We explain the eigenvalue assignment strategy for all basic ways to expand T_1 and T_2.

First, for T_1, there are five places to add one additional vertex and two places to add a 2-path or a star. To add one additional vertex to T_1, we can do so at c_1, $v_1(v_2)$, c_2, b_5, or b_6. Because c_1, $v_1(v_2)$, and c_2 are all Parter vertices for some multiple eigenvalue in T_1, Lemma 4.1 applies to these three cases. The only remaining cases are to add a pendent vertex at b_5 or b_6.

1. Adding a pendent vertex u to b_5: we can increase $m(\lambda)$ so as to realize $(4, 2, 2, 2, 1, 1)$. Specifically, an eigenvalue assignment can be as follows:
   $\mathcal{A}_1 = \{\{b_1\}, \{b_2\}, \{b_3\}, \{b_4\}, \{c_1\}, \{b_5, u\}, \{b_6, b_7\}\}$ with $V_1 = \{v_1, v_2, c_2\}$ for λ.
   $\mathcal{A}_2 = \mathcal{A}_3 = \{\{v_1, b_1, b_2\}, \{v_2, b_3, b_4\}, \{c_2, b_5, u, b_6, b_7\}\}$ with $V_2 = V_3 = \{c_1\}$ for α and β.
   $\mathcal{A}_4 = \{\{b_5, u\}, \{b_6, b_7\}, \{c_1, v_1, b_1, b_2, v_2, b_3, b_4\}\}$ with $V_4 = \{c_2\}$ for γ.
2. Adding a pendent vertex u to b_6: we can again increase $m(\lambda)$ so as to realize $(4, 2, 2, 2, 1, 1)$, but via a different eigenvalue assignment:
   $\mathcal{A}_1 = \{\{b_1\}, \{b_2\}, \{b_3\}, \{b_4\}, \{c_1\}, \{b_5\}, \{b_6, b_7, u\}\}$ with $V_1 = \{v_1, v_2, c_2\}$ for λ.
   $\mathcal{A}_2 = \mathcal{A}_3 = \{\{v_1, b_1, b_2\}, \{v_2, b_3, b_4\}, \{c_2, b_5, u, b_6, b_7\}\}$ with $V_2 = V_3 = \{c_1\}$ for α and β.
   $\mathcal{A}_4 = \{\{u\}, \{b_7\}, \{c_1, c_2, b_5, v_1, b_1, b_2, v_2, b_3, b_4\}\}$ with $V_4 = \{b_6\}$ for γ.

Beyond the addition of pendent vertices, we can add a 2-path or a star at either c_1 or c_2 in T_1. On the one hand, since c_1 is a Parter vertex for both α and β, just like c in Theorem 4.2, a similar strategy for eigenvalue assignment when a 2-path or a simple star is added can be adopted. On the other hand, c_2's situation is slightly different because c_2 is only a Parter vertex for γ. First, a tree T resulting from adding a 2-path at c_2 is the same as adding a pendent vertex at b_5 and another at c_2. By the discussion above and Lemma 4.1, $U(T) = 2$. Second, adding a degenerate star (i.e., a 3-path, which suffices for any star together with Lemma 4.1) at c_2 is equivalent to adding a pendent vertex at b_6 and a 2-path at c_2. Then, again, $U(T) = 2$ by previous results. This completes the discussion about all the basic ways of expanding the diameter 6 core T_1.

Now, for the other diameter 6 core, T_2, pendent vertices, 2-paths, and stars can be added. One additional pendent vertex can be added at any nonpendent vertex, and in the eigenvalue assignment for T_2, all of them are a Parter vertex for some multiple eigenvalue, so Lemma 4.1 applies. Then, again, 2-paths and stars can only be adjacent to c_1 or c_2. First, c_1 in T_2 is similar to that in T_1 with regard to its role as a Parter vertex for two multiple eigenvalues in the assignment, so identical strategy is adopted here. Second, when a 2-path is added to c_2 in T_2, it is the same as adding a pendent vertex at b_5 and another at b_6 in T_1. Both additions maintain that $U(T) = 2$. Moreover, the tree resulting from the addition of a degenerate star at c_2 can also be obtained by adding two pendent vertices at b_5 and one pendent vertex at b_6 in T_1. So, by previous results on T_1, we conclude $U(T)$ remain 2 here as well.

We have shown what eigenvalue assignment is like for each basic way of vertex addition to both cores, and any diameter 6 NL tree is obtained by a combination of these. Thus, they all have a multiplicity list without more 1's than their core. So, $U(T) = 2$ for any diameter 6 NL tree. □

When the diameter of an NL tree is as small as 5 or 6, the structure of the tree is greatly limited, despite the fact that there are infinitely many diameter 5 and 6 NL trees. We suspect that $U(T) = 2$ holds true for all diameter 7 NL trees as well; no counterexample is known, but a similar proof to that of 5 or 6 appears complicated. This conjecture is formally stated in Section 4.2, in a more general form for any diameter d NL tree. (However, it is certain that not every diameter 8 NL tree has $U(T) = 2$.)

4.2 $U(T)$ for diameter d cores

We employ the Implicit function theorem (IFT) approach developed in [11] whose purpose is to show a certain multiplicity list is in the catalog of a given tree.

Theorem 4.4. *(Implicit function theorem) Let $f : \mathbb{R}^{n+m} \to \mathbb{R}^n$ be a continuously differentiable function. Suppose that, for $x_0 \in \mathbb{R}^n$ and $y_0 \in \mathbb{R}^m$, $f(x_0, y_0) = 0$ and the Jacobian $\frac{\partial f}{\partial x}(x_0, y_0)$ is invertible. Then there exists a neighborhood $U \subset \mathbb{R}^m$ around y_0 such that $f(x, y) = 0$ has a solution x for any fixed $y \in U$. Furthermore, regarding the solution x as a function of y, gives a function continuous at (x_0, y_0).*

We think of a matrix with graph G as a multivariable function of the entries. Then we build an “initial matrix” $A^{(0)}$ whose graph G_0 is a subgraph of G by edge containment such that the spectrum of $A^{(0)}$ has the multiplicities we desire. This involves specification of a number of determinant conditions (characteristic polynomials). We need to choose as many variables called *implicit entries* as the conditions and obtain a nonsingular Jacobian with respect to these entries. Then we can perturb the edges in $G \backslash G_0$ by some sufficiently small $\epsilon \neq 0$. By letting F be a sufficient set of determinant conditions for the multiplicities, the resulting matrix will have graph G but the same spectrum as $A^{(0)}$. More details can be found in [11].

Theorem 4.5. *Let G be a graph, let $F = (f_k)$ be a vector of r determinant conditions and designate r entries as implicit entries. Suppose a real symmetric matrix $A^{(0)}$ is the direct sum of irreducible matrices $A_1, A_2, \dots, A_p$, and*

1. $F\left(A^{(0)}\right) = 0$;
2. $G\left(A^{(0)}\right)$ *is a subgraph of G;*
3. *if $a_{ij}^{(0)} = 0$, $i \neq j$, then (i, j) is not an implicit entry; and*
4. $\det J\left(A^{(0)}\right)$*, the Jacobian of F with respect to the implicit entries, evaluated at $A^{(0)}$, is nonzero.*

Then there exists a matrix $A \in \mathcal{S}(G)$ such that $F(A) = 0$ and the ordered multiplicities of A are a refinement of the ordered multiplicities of $A^{(0)}$.

The following two lemmas in [11] are helpful with checking condition 4 in Theorem 4.5, using reducibility of $A^{(0)}$. The first one is general, whereas the second addresses a special case in which $A^{(0)}$ is diagonal.

Lemma 4.6. *Let T be a tree and $F = (f_k)_{k=1,\dots,r}$ be defined as above, with r implicit entries identified. Suppose that a real symmetric matrix $A^{(0)}$, whose graph is a subgraph of T, is the direct sum of irreducible matrices $A_1^{(0)}, A_2^{(0)}, \dots, A_p^{(0)}$. Let $J\left(A^{(0)}\right)$ be the Jacobian matrix of F with respect to the implicit entries evaluated at $A^{(0)}$, and suppose that*

1. *every off-diagonal implicit entry in $A^{(0)}$ has a non-zero value;*
2. *for every $k = 1, \dots, r$, $f_k\left(A_l^{(0)}\right) = 0$ for precisely one $l \in \{1, \dots, p\}$; and*
3. *for every $l = 1, \dots, p$, the columns of $J\left(A^{(0)}\right)$ associated with the implicit entries of $A_l^{(0)}$ are linearly independent.*

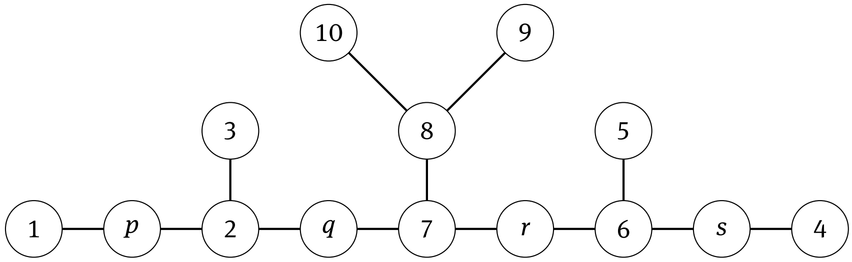
Then $J\left(A^{(0)}\right)$ is nonsingular.

Lemma 4.7. *Let $F = (f_k)$ be a vector of r determinant conditions, and let $A^{(0)}$ be a diagonal matrix. Suppose that for every $k = 1, \dots, r$, $f_k\left(A^{(0)}[l]\right) = 0$ for precisely one $l \in \{1, \dots, n\}$. Take a_{ll} to be an implicit entry if and only if $f_k\left(A^{(0)}[l]\right) = 0$ for some k. If there are then r implicit entries, the Jacobian of F with respect to the implicit entries evaluated at $A^{(0)}$ is nonzero.*

Theorem 4.8. *If T is a diameter d core, then*

$$U(T) = \begin{cases} 2 & \textit{if } d < 7 \\ d-5 & \textit{if } d \geqslant 7 \end{cases}.$$

Proof. If T is a diameter d core such that $d < 7$, then either $d = 5$ or $d = 6$. By Theorem 4.2 and 4.3, $U(T) = 2$. Now, suppose T is a diameter d core such that $d \geqslant 7$ with ordered partition (p, q, r, s), then $p + q + r + s \geqslant 2$. We label vertices in a core as follows. Vertices p, q, r, and s are placeholders for paths with length p, q, r, and s.



To prove $U(T) = d-5$, since we already know $U(T) \geqslant 2d-n = d-5$, for the reverse inequality, it suffices to show that the multiplicity list $(2, 2, 2, 2, 2, 1^{d-5}) \in \mathcal{L}(T)$. Indeed, $(2, 2, 2, 2, 2, 1^{d-5})$ is the only multiplicity list with this many 1's. We will construct an $A = (a_{ij}) \in \mathcal{S}(T)$ with the five distinct multiplicity 2 eigenvalues being $\alpha, \lambda, \beta, \tau$, and γ. Sufficient conditions for having $m_A(\alpha) = m_A(\lambda) = m_A(\beta) = m_A(\tau) = m_A(\gamma) = 2$ can be specified as a vector-valued function. Let $F = (f_k)_{k=1,\dots,15}$ be

$$f_1 = det(A[1,p] - \alpha I_{1+p}) = 0 \tag{1}$$

$$f_2 = a_{3,3} - \alpha = 0 \tag{2}$$

$$f_3 = det(A[4,5,6,7,8,9,10,q,r,s] - \alpha I_{7+q+r+s}) = 0 \tag{3}$$

$$f_4 = det(A[4,s] - \lambda I_{1+s}) = 0 \tag{4}$$

$$f_5 = a_{5,5} - \lambda = 0 \tag{5}$$

$$f_6 = det(A[1,2,3,7,8,9,10,p,q,r] - \lambda I_{7+p+q+r}) = 0 \tag{6}$$

$$f_7 = a_{9,9} - \beta = 0 \tag{7}$$

$$f_8 = a_{10,10} - \beta = 0 \tag{8}$$

$$f_9 = det(A[1,2,3,4,5,6,7,p,q,r,s] - \beta I_{7+p+q+r+s}) = 0 \tag{9}$$

$$f_{10} = det(A[1,2,3,p,q] - \tau I_{3+p+q}) = 0 \tag{10}$$

$$f_{11} = det(A[4,5,6,r,s] - \tau I_{3+r+s}) = 0 \tag{11}$$

$$f_{12} = det(A[8,9,10] - \tau I_3) = 0 \tag{12}$$

$$f_{13} = det(A[1,2,3,p,q] - \gamma I_{3+p+q}) = 0 \tag{13}$$

$$f_{14} = det(A[4,5,6,r,s] - \gamma I_{3+r+s}) = 0 \tag{14}$$

$$f_{15} = det(A[8,9,10] - \gamma I_3) = 0. \tag{15}$$

Any matrix in the kernel of F has the desired multiplicity list. Now we construct a real symmetric matrix $A^{(0)} = (a_{ij}^{(0)})$ with $F(A^{(0)}) = 0$ whose graph is a subgraph of T. First of all, some constraints of F on $A^{(0)}$ are straightforward, such as $a_{3,3}^{(0)} = \alpha$, $a_{5,5}^{(0)} = \lambda$, $a_{9,9}^{(0)} = \beta$, and $a_{10,10}^{(0)} = \beta$ by f_2, f_5, f_7, and f_8. Moreover, subject to f_7, f_8, f_{12}, and $f_{15} = 0$, the three eigenvalues in $A^{(0)}[8,9,10]$ have to be β, τ, and γ. As a result, the submatrix $A[8,9,10]$ is determined, up to a fixed sum of squares of the off-diagonal entries.

$$A[8,9,10] = \begin{bmatrix} \tau+\gamma-\beta & a_{8,9} & a_{8,10} \\ a_{8,9} & \beta & 0 \\ a_{8,10} & 0 & \beta \end{bmatrix} \quad \text{where } a_{8,9}^2 + a_{8,10}^2 = \beta(\tau+\gamma-\beta) - \tau\gamma$$

Thus, we may ignore this branch in T and the corresponding 3-by-3 submatrix in $A^{(0)}$. What remains to be considered are conditions $F \setminus \{f_7, f_8, f_{12}, f_{15}\}$. Similarly, when $p = q = 0$ or $r = s = 0$, the corresponding branch and 3-by-3 submatrix will also be determined with eigenvalues $\{\alpha, \tau, \gamma\}$ or $\{\lambda, \tau, \gamma\}$. Therefore, without loss of generality, we may only consider and construct a submatrix of $A^{(0)}$ whose underlying graph does not have a fully determined branch.

On the one hand, when (i)$d \geqslant 9$, or (ii)$d = 8$ and either $p = q = 0$ or $r = s = 0$, we may choose the initial matrix $A^{(0)}$ to be diagonal, satisfying the conditions in F subject to Lemma 4.7. For example, in the case when $p = 1, q = 1, r = 1$, and $s = 1$, we may choose

$A^{(0)}[1,2,3,4,5,6,7,p,q,r,s] = diag(\alpha,\gamma,\alpha,\lambda,\lambda,\tau,\beta,\tau,\lambda,\alpha,\gamma)$. By Lemma 4.7, $J\left(A^{(0)}\right)$ is nonsingular; hence, by Theorem 4.5, $(2,2,2,2,2,1^{d-5}) \in \mathcal{L}(T)$ as desired.

For diameter 7 and 8 cores that do not meet the above criterion, we have to choose some off-diagonal entries to be implicit entries. Besides the use of diagonal entries for implicit entries, one or two pairs of off-diagonals entries are needed, depending on the core being considered. We can choose from the off-diagonal entries corresponding to edges (2,3) and (5,6). Then we construct $A^{(0)}$ in which the 2-by-2 block consisting of vertices 2 and 3 (respectively, vertices 5 and 6) has eigenvalues τ and γ, i.e. $a^{(0)}_{3,3} = \alpha$, $a^{(0)}_{2,2} = \tau+\gamma-\alpha$, and $a^{(0)}_{2,3} = a^{(0)}_{3,2} = \sqrt{(\tau-\alpha)(\alpha-\gamma)} \neq 0$ (respectively, $a^{(0)}_{5,5} = \lambda$, $a^{(0)}_{6,6} = \tau+\gamma-\lambda$, and $a^{(0)}_{5,6} = a^{(0)}_{6,5} = \sqrt{(\tau-\lambda)(\lambda-\gamma)}$.) Then, by the formulas for entries in the Jacobian provided in [11], the columns of $J\left(A^{(0)}\right)$ associated with these implicit entries are linearly independent because τ and γ are distinct. Now, by Lemma 4.6, $J\left(A^{(0)}\right)$ is again nonsingular. And hence, by Theorem 4.5, $(2,2,2,2,2,1^{d-5}) \in \mathcal{L}(T)$ as desired, completing the proof. □

Lastly, we make a note about a potential upper bound for $U(T)$ for any diameter d NL tree. Since we know $U(T)$ for cores by Theorem 4.8, it is conjectured that $U(T)$ for a core is an upper bound for all trees in the family generated by this core. If we knew the result for linear trees about the change of $U(T)$ upon vertex addition in [3] for NL trees, then Conjecture 4.9 would follow. Note that $d-5$ is $U(T)$ shared by any diameter d core T, from which NL trees of this diameter are built.

Conjecture 4.9. If T is a diameter $d(\geqslant 7)$ NL tree, then $U(T) \leqslant d-5$.

4.3 Determining the Catalog of Cores

The catalog for linear trees is characterized by the LSP, but little is known about that for NL trees. However, with the tools we have developed, we know more about cores.

Lemma 4.10. *For any diameter d core T, $M(T) = P(T) = 4$.*

Proof. $M(T) = P(T)$ holds true for any tree [4]. Secondly, for any diameter d core T, one path cover consists of three paths that start at a pendent vertex, go through one HDV, and end at a pendent vertex, with length $p+3$, $s+3$, and 3, and one path that contains the rest with length $q+r+1$. So, $P(T) \leqslant 4$. Also, $P(T) \geqslant 4$ because T results from the 10-vertex NL tree with path cover number 4 via edge subdivisions, for which the path cover number does not decrease. □

Here we cite a prior result in [12] about M_2, the largest sum of the top two multiplicities over lists in the catalog for T.

Theorem 4.11. *For any tree T on n vertices, $M_2(T) \leqslant n+2-d(T)$.*

Theorem 4.12. *For any diameter $d(\geqslant 7)$ core T,*

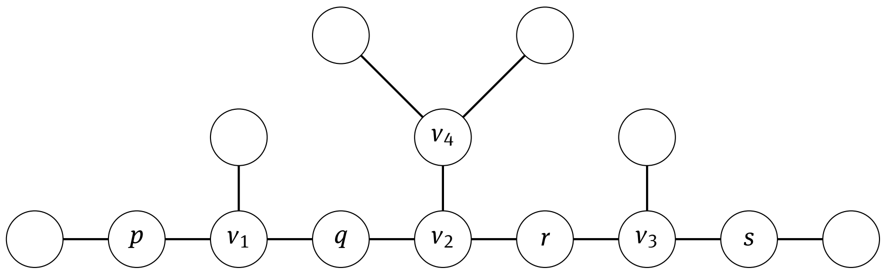
$$\mathcal{L}(T) \subseteq \{4221^{d-3}, 421^{d-1}, 41^{d+1}, 3321^{d-3}, 331^{d-1}, 32221^{d-4}, 3221^{d-2}, 321^{d}, 31^{d+2}, 222221^{d-5}, 22221^{d-3}, 2221^{d-1}, 2221^{d-1}, 221^{d+1}, 21^{d+3}, 1^{d+5}\}.$$

Moreover, for $T \in \mathcal{C}(d)$ in which no interior edges are subdivided, i.e., the ordered partition into four parts (p,q,r,s) has $q=r=0$,

$$\mathcal{L}(T) \subseteq \{4221^{d-3}, 421^{d-1}, 41^{d+1}, 32221^{d-4}, 3221^{d-2}, 321^{d}, 31^{d+2}, 222221^{d-5}, 22221^{d-3}, 2221^{d-1}, 2221^{d-1}, 221^{d+1}, 21^{d+3}, 1^{d+5}\}.$$

Proof. By Lemma 4.10, $M(T) = P(T) = 4$, so any multiplicity list containing a multiplicity greater than 4 is not allowed in $\mathcal{L}(T)$. Next, we rule out some candidate multiplicity lists by citing prior results and/or showing an eigenvalue assignment is impossible. The following are three prior results we will use. First, by Theorem 4.8, $U(T) = d - 5$. Second, the minimum number of distinct eigenvalues in a list is no less than d. Third, by Theorem 4.11, the sum of the largest two multiplicities, $M_2(T) \leqslant n + 2 - d(T) = d + 5 + 2 - d = 7$.

Again, in order to describe eigenvalue assignments more easily, we label vertices in T as follows. Vertices p, q, r, and s are placeholders for paths with length p, q, r, and s, and $p + q + r + s = d - 5$.



First, we start with multiplicity lists with the maximum multiplicity, namely, 4, in it. Any list containing $(4, 4)$ is not possible because $M_2 = 8 \nleqslant 7$. Any list containing $(4, 3)$ is also ruled out because no assignment produces two eigenvalues with multiplicity 4 and 3 simultaneously. To be specific, the multiplicity 4 eigenvalue, say λ, requires three nonadjacent HDV's, either $\{v_1, v_2, v_3\}$ or $\{v_1, v_3, v_4\}$, as Parter vertices. In the former case, v_4 or v_2 can only be a Parter vertex for another eigenvalue with multiplicity at most 2; in the latter case, only v_2 can be a Parter vertex for up to two eigenvalues with multiplicity at most 2. Thus, another multiplicity 3 eigenvalue is not allowed, which rules out any list containing $(4, 3)$. The list $(4, 2, 2)$ is indeed allowed, with plausible eigenvalue assignments suggested by previous discussion. However, any list containing $(4, 2, 2, 2)$ is again impossible because in the case when $\{v_1, v_2, v_3\}$ are Parter vertices for λ, after using v_4 as a Parter vertex for a multiplicity 2 eigenvalue, α, and v_2 for another one, β, the 3-path adjacent to v_2 containing v_4 is "full" against overloading as it has λ, α, and β as its three eigenvalues. On the other hand, in the case when $\{v_1, v_3, v_4\}$ are Parter vertices for λ, only v_2 is available for being a Parter vertex for multiplicity 2 eigenvalues, and it can be a Parter vertex for up to two such eigenvalues, say α and β. A third multiplicity 2 eigenvalue is not allowed because the 3-path is again "full" with λ, α, and β.

Second, we consider multiplicity lists containing 3 as the largest multiplicity. Any list containing $(3, 3, 3)$ is ruled out because an assignment achieving three multiplicity 3 eigenvalues, say α, β, and γ, does not exist. Since no vertex has degree more than three, each of α, β, and γ must have at least two nonadjacent Parter vertices. v_1, v_3, and v_4 can be a Parter vertex for one eigenvalue each because of the limited size of the branches. However, since v_2 and v_4 are adjacent, v_4 has to be a Parter vertex for one of α, β, and γ together with either v_1 or v_3. Without loss of generality, suppose $\{v_1, v_4\}$ are Parter vertices for α, then v_3 can be a Parter vertex for one of β and γ, only if together with v_2 when $r \neq 0$, because otherwise v_2 and v_3 would not be adjacent. As a result, the last eigenvalue has multiplicity at most 2 with v_2 being the only possible Parter vertex given v_1, v_3, and v_4 are fully occupied. Thus, any list containing $(3, 3, 3)$ is not in $\mathcal{L}(T)$. In addition, although the list $(3, 3, 2, 1^{d-3})$ is suggested realizable by the eigenvalue assignment above (only if $q \neq 0$ or $r \neq 0$), any list containing $(3, 3, 2, 2)$ is impossible. Again, multiplicity 3 eigenvalues must have at least two nonadjacent Parter vertices. v_1, v_3, and v_4 can be a Parter vertex for no more than one eigenvalue each. Suppose the most flexible situation when $q \neq 0$ and $r \neq 0$, then for the two multiplicity 3 eigenvalues, say α and β, the choice of Parter vertices can be (i) v_4 with one of v_1 and v_3 for α, and v_2 with the other of v_1 and v_3 for β, or (ii) v_1 and v_2 for α, and v_3 and v_2 for β. In both cases, two more multiplicity 2 eigenvalues are impossible. In particular, in the former case, v_1, v_3, and v_4 are all occupied, and v_2 can be a Parter vertex for at most one multiplicity 2 eigenvalue because the 3-path adjacent to v_2 has already had two eigenvalues, α and β, with α occurring on both pendents and β by the fact that v_2 is a Parter for β. In the latter case, either v_4 or v_2 is available for being a Parter vertex for a multiplicity 2 eigenvalue, but they cannot each be a Parter vertex for two distinct multiplicity 2 eigenvalues. If it happened, the 3-path adjacent to v_2 would have four eigenvalues assigned to it, which is a contradiction. Therefore, any list containing $(3, 3, 2, 2)$ is ruled out by assignment. Lastly, any list containing $(3, 2, 2, 2, 2)$ is not possible because it would contain no more than $n - (3 + 2 + 2 + 2 + 2) = (d + 5) - 11 = d - 6$ multiplicity 1 eigenvalues, whereas we know $U(T) = d - 5$.

In the end, for multiplicity lists containing 2 as the largest multiplicity, any list containing $(2, 2, 2, 2, 2, 2)$ is again ruled out by the fact that $U(T) = d-5$ by Theorem 4.8. Now, the elimination of impossible multiplicity lists is finished. Moreover, for all the lists remaining, there is an eigenvalue assignment. □

In fact, we suspect a stronger statement than Theorem 4.12 is true.

Conjecture 4.13. For a diameter $d(\geqslant 7)$ core $T \in \mathcal{C}(d)$ such that no interior edges are subdivided, i.e., the ordered partition into four parts (p, q, r, s) has $q = r = 0$,

$$\mathcal{L}(T) = \{4221^{d-3}, 421^{d-1}, 41^{d+1}, 32221^{d-4}, 3221^{d-2}, 321^{d}, 31^{d+2}, 222221^{d-5}, 22221^{d-3}, 2221^{d-1}, 2221^{d-1}, 221^{d+1}, 21^{d+3}, 1^{d+5}\}.$$

Moreover, for $T \in \mathcal{C}(d)$ in which some interior edge is subdivided, i.e., the ordered partition into four parts (p, q, r, s) has $q \neq 0$ or $r \neq 0$,

$$\mathcal{L}(T) = \{4221^{d-3}, 421^{d-1}, 41^{d+1}, 3321^{d-3}, 331^{d-1}, 32221^{d-4}, 3221^{d-2}, 321^{d}, 31^{d+2}, 222221^{d-5}, 22221^{d-3}, 2221^{d-1}, 2221^{d-1}, 221^{d+1}, 21^{d+3}, 1^{d+5}\}.$$

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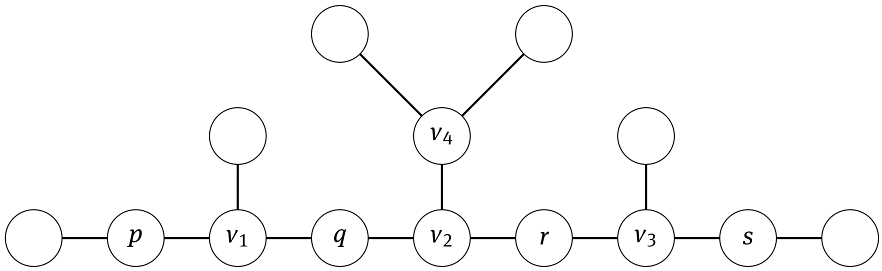
Appendix A Proof for Statements in Section 3.3

Proof of Lemma 3.11. Consider what a diameter d core with ordered partition (p, q, r, s) looks like.



The key to diameters in a core is that they have to contain all the $p + q + r + s$ vertices resulting from edge subdivisions in the 10-vertex NL tree. First, whenever $p \neq 0$ and $s \neq 0$, the longest path in the core is uniquely determined, i.e., the horizontal path at the bottom in the graph. No other path is longer. Also, no matter what q and r are, possibly zero, this longest path contains them. Second, an ordered partition $(0, \star, ?, \star)$ requires either $p = 0$, $q \neq 0$, and $s \neq 0$ or $s = 0$, $r \neq 0$, and $p \neq 0$. Because these are symmetric, it suffices to consider the case when $p = 0$, $q \neq 0$, and $s \neq 0$. Since $q \neq 0$, and $s \neq 0$, most part of diameters is determined; however, vertex v_1 has two pendents (because $p = 0$), either of which can be part of a distinct diameter. Thus, such cores have two diameters. Third, cores with ordered partitions $(\star, ?, 0, 0)$ or $(0, \star, \star, 0)$ have four diameters. On the one hand, $(\star, ?, 0, 0)$ means $p \neq 0$, $r = 0$, and $s = 0$ (the symmetric case is omitted). All the diameters have to contain v_1, $p + 1$ vertices on the left of v_1, v_2, and the q vertices between v_1 and v_2 (possibly none). However, since $r = 0$ and $s = 0$, there are two distinct diameters containing v_4 and either of v_4's pendent vertices and another two distinct diameters containing v_3 and either of v_3's pendent vertices. Thus, such cores have a total of four distinct diameters. On the other hand, ordered partition $(0, \star, \star, 0)$ means $p = s = 0$, and $q, r \neq 0$. This implies that all the diameters contain v_1, v_2, v_3, and the q and r vertices in between. Then four pendent vertices – two of v_1's and two of v_3's – each result in a distinct diameter. Lastly, ordered partition $(0, \star, 0, 0)$ means that $p = r = s = 0$ and $q \neq 0$ (the symmetric case is omitted). Then all diameters have to contain v_1, v_2, and the q vertices in between; in addition, one terminal vertex of the diameters needs to be one of the two pendent vertices of v_1, and the other terminal vertex needs to be one of v_3's and v_4's four pendent vertices. Thus, cores with such ordered partition have eight distinct diameters. □

Proof of Corollary 3.12. Up to forward/backward symmetry, ordered partitions discussed in Lemma 3.11, namely, $(\star, ?, ?, \star)$, $(0, \star, ?, \star)$, $(\star, ?, 0, 0)$, $(0, \star, \star, 0)$, and $(0, \star, 0, 0)$, cover all the possibilities. In particular, $(\star, ?, 0, 0)$ and $(0, \star, 0, 0)$ account for ordered partitions with one nonzero part, $(\star, ?, ?, \star)$, $(0, \star, ?, \star)$, $(\star, ?, 0, 0)$, and $(0, \star, \star, 0)$ account for ordered partitions with two nonzero parts, $(0, \star, ?, \star)$ and $(\star, ?, ?, \star)$ account for ordered partitions with three nonzero parts, and $(\star, ?, ?, \star)$ accounts for ordered partitions with four nonzero parts. Therefore, a diameter d core ($d \geqslant 7$) could only possibly have 1, 2, 4 or 8 distinct diameters. Note that $d \geqslant 6$ guarantees that (p, q, r, s) is a partition of a positive integer; however, for the only diameter 5 core, i.e., with ordered partition $(0, 0, 0, 0)$, have twelve distinct diameters.

Furthermore, for $\mathcal{C}(d)$ such that $d \geqslant 7$, each of $\{1, 2, 4, 8\}$ will occur as the number of distinct diameters for at least one of the cores. In particular, when $d \geqslant 7$, $p + q + r + s \geqslant 2$; also, none of the ordered partitions in form of $(\star, ?, ?, \star)$, $(0, \star, ?, \star)$, $(\star, ?, 0, 0)$, $(0, \star, \star, 0)$, and $(0, \star, 0, 0)$ requires more than two nonzero parts. Thus, each of $\{1, 2, 4, 8\}$ will occur for at least one of the cores. □

Proof of Corollary 3.13. First, the number of cores with a unique diameter is a doubly lagged function of $|\mathcal{C}(d)|$. By Lemma 3.11, such cores have ordered partitions in the form of $(\star, ?, ?, \star)$. In other words, any diameter d core with $p \geqslant 1$ and $s \geqslant 1$ has a unique diameter. Recall Algorithm 3.6, compared to generating all $|\mathcal{C}(d)|$ diameter d cores from $d - 5$ edge subdivisions on four edges in one diameter, we now mandate two edge subdivisions to make $p = 1$ and $s = 1$. Then the rest $d - 7$ edge subdivisions can be applied on any of the four edges, which generates $|\mathcal{C}(d - 2)|$ cores with a unique diameter.

Second, cores with two diameters have ordered partitions in the form of $(0, \star, ?, \star)$. The 0 restricts us to apply $d-5$ edge subdivisions on only three edges; in addition, two $\star$'s requires two edge subdivisions on respective edges and leaves us with $d-7$ edge subdivisions on three edges. This boils down to the number of ways that $d-7$ identical balls can be partitioned into three groups (empty groups are allowed), which is equivalent to inserting two walls into $(d-7)+1$ spaces among a line of $d-7$ objects. Thus, there are $\binom{d-6}{2}$ such partitions. Moreover, isomorphism due to forward/backward symmetry does not arise because of the 0 in the ordered partition. Thus, $\binom{d-6}{2}$ diameter d cores have two diameters.

Third, the number of cores with four diameters is the sum of the number of cores with ordered partitions in the form of $(\star, ?, 0, 0)$ or $(0, \star, \star, 0)$. On the one hand, ordered partition $(\star, ?, 0, 0)$ is similar to $(0, \star, ?, \star)$. So, one edge subdivision is mandated by the $\star$, and the rest $d-6$ edge subdivisions need to be applied to two edges. Thus, there are $\binom{d-6+1}{1} = d-5$ cores with ordered partitions $(\star, ?, 0, 0)$. Again, there is no forward/backward symmetry issue for $(\star, ?, 0, 0)$. On the other hand, cores with ordered partitions $(0, \star, \star, 0)$ have two mandated edge subdivisions by the two $\star$'s and $d-7$ edge subdivisions left, so there are $\binom{d-7+1}{1} = d-6$ ways to partition $d-7$ edge subdivisions to two edges. However, we now encounter the double counting problem due to forward/backward symmetry for order partitions $(0, \star, \star, 0)$. Again, just as Proposition 3.9, parity of d makes a difference. When d is even, $d-7$ is odd, every core is counted exactly twice by the symmetry, so there are $(d-6)/2$ cores with ordered partitions $(0, \star, \star, 0)$; when d is odd, $d-7$ is even, every core is counted exactly twice by the symmetry except the core with ordered partition (p, q, r, s) such that $p = s = 0$ and $q = r$, so there are $(d-6)/2 + 1/2$ cores with ordered partitions $(0, \star, \star, 0)$. To summarize, $\lceil (d-6)/2 \rceil + (d-5)$ diameter d cores have four diameters.

Lastly, the number of cores with eight diameters in $\mathcal{C}(d)$ is 1 because the ordered partition $(0, \star, 0, 0)$ is unique. □