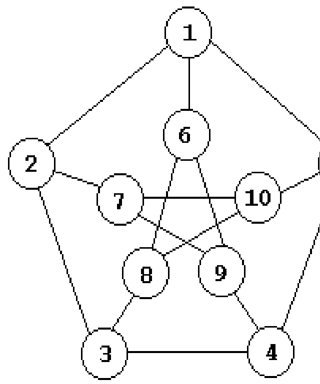
MATH 223: A Graph Decomposition result.

The Petersen graph is a well known graph that often appears as an example or counterexample of graph properties. Given that it is a graph on 10 vertices with each vertex being adjacent to 3 edges, it is possible to ask the question whether the complete graph on 10 vertices can be decomposed into 3 copies of the Petersen graph.



Let A denote the adjacency matrix of the Petersen graph.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

Let J denote the 10×10 matrix of all 1's. Then $J - I$ is the adjacency matrix of the complete graph on 10 vertices.

The matrix A satisfies the remarkable matrix equation

$$A^2 + A = 2I + J.$$

This follows from noticing that any pair of vertices not already joined by an edge are joined by a unique path of length 2; namely there is a unique vertex which is joined to both.

From this equation or directly, we can deduce that $\det(A - \lambda I) = (\lambda - 3)(\lambda - 1)^5(\lambda + 2)^4$ and that $\mathbf{1}$ is an eigenvector of eigenvalue 3. How could one approach this computation without just computing the 10×10 determinant? Note that J has eigenvalue 10 (of multiplicity 1 with eigenvector $\mathbf{1}$) and eigenvalue 0 of multiplicity 9. Thus $2I + J$ has eigenvalue 12 of multiplicity 1 and eigenvalue 2 of multiplicity 9. Now if $\mathbf{x}$ is an eigenvector of A of eigenvalue λ then $\mathbf{x}$ is an eigenvector of $A^2 + A$ of eigenvalue $\lambda^2 + \lambda$. Thsu the possible eigenvectors for A are prescribed. We already know that $\mathbf{1}$ is an eigenvector of A of eigenvalue 3 (or at least it is easy to check). The other 9 eigenvalues must satisfy $\lambda^2 + \lambda = 2$ thus either 1 or -2 with total multiplicity being 9. Now the trace of A, which is 0, is the sum of the eigenvalues and so we deduce that 1 has multiplicity 5 and -2 has multiplicity 4.

The decomposition we are looking for takes the matrix form

$$J - I = A_1 + A_2 + A_3$$

where A_1, A_2, A_3 are each obtained from A by a simultaneous row and column permutation, i.e. $A_i = PAP^T = PAP^{-1}$ where P is a permutation matrix. Each A_i is similar to A. Also, A_i is a symmetric matrix and hence the dimension of an eigenspace of eigenvalue λ is the multiplicity of λ as a root in the characteristic equation for A_i. We note that

$$\det(A_i - \lambda I) = \det(A - \lambda I) = (\lambda - 3)(\lambda - 1)^5(\lambda + 2)^4.$$

For all three matrices A_1, A_2, A_3, we note that $\mathbf{1}$ is an eigenvector of eigenvalue 3 and so each has a 5-dimensional eigenspace of eigenvalue 1 orthogonal to $\mathbf{1}$ and hence contained in $(\text{span}(\mathbf{1}))^\perp$ which is a 9-dimensional space. Now two 5-dimensional subspaces of a 9-dimensional space must have a non zero vector, say $\mathbf{v}$, in their intersection. Thus there is a vector $\mathbf{v}$ that is an eigenvector of eigenvalue 1 for both A_1 and A_2 ($A_1\mathbf{v} = A_2\mathbf{v} = \mathbf{v}$) and moreover $\mathbf{v}$ is orthogonal to $\mathbf{1}$. We compute (using $\mathbf{1} cdot \mathbf{v} = 0$)

$$(J - I)\mathbf{v} = J\mathbf{v} - \mathbf{v} = -\mathbf{v},$$

$$(A_1 + A_2 + A_3)\mathbf{v} = A_1\mathbf{v} + A_2\mathbf{v} + A_3\mathbf{v} = 2\mathbf{v} + A_3\mathbf{v}$$

We deduce that

$$A_3\mathbf{v} = -3\mathbf{v}.$$

This is a contradiction since -3 is not an eigenvalue of A_3.

Thus the desired decomposition cannot exist. This result can be modestly generalized but I introduced it because the problem reviews so many facts from our MATH 223 course.

A different direction to take with the Petersen Graph is to show that for any pair of vertices they are either joined by an edge or a path of length 2 using the fact that there are three eigenvalues for the diagonalizable adjacency matrix A. In particular we can readily show using diagonalizability that $(A - 3I)(A - I)(A + 2I)$ which gives an expression for A^3 in terms of I, A, A^2. Thus

$$\text{span}\{I, A, A^2, A^3, A^4, \ldots\} = \text{span}\{I, A, A^2\}. \quad \dim(\text{span}\{I, A, A^2, A^3, A^4, \ldots\}) = 3.$$

Now we consider powers of A from a different point of view. In particular the i, j entry of A^k counts the number of walks from i to j of k edges. This can be proven by induction if you wish. Thus if we have a pair of vertices i, j joined by a walk of three edges and not one edge or two edges, then the i, j entry of A^3 is nonzero while the i, j entries of I, A, A^2 are zero. Thus

$$A^3 \notin \text{span}\{I, A, A^2\}$$

a contradiction. Thus we conclude that every pair of vertices are joined by an edge or a walk of two edges simply form the fact that A had only 3 eigenvalues. The reverse implication will not work.