FYS4150 Project 3

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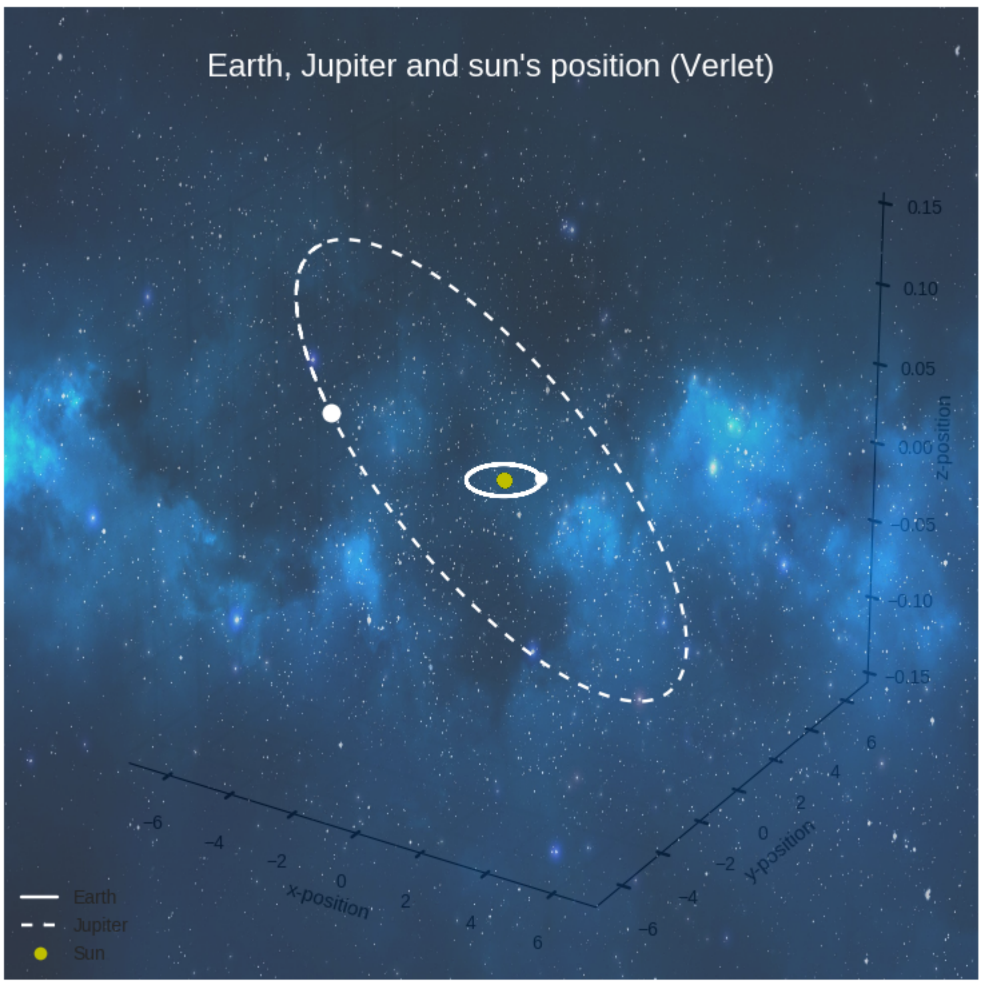


Abstract

In this project we create a model of our solar system using numerical methods for solving ordinary differential equations, namely the Euler and Velocity Verlet method. The methods are compared in efficiency and stability for a simple two-particle system. The results found the Verlet method to be more stable, so we keep it as our method of choice for the remaining simulations. Several tests of physical properties are performed, including but not limited to conservation of energy and momentum, escape velocity and relativistic corrections. Moreover, these test yielded results which further deepened our trust in the Verlet method. The source code and benchmarks of the project can be found on GitHub: `https://github.com/Kjernlie/solar-system`. Note that this work has been a collaboration between Ingrid A. V. Holm (FYS4150) and Johannes K. Kjernlie (FYS3150). We will deliver separate reports, since we are in different courses.

Contents

Introduction

The aim of this project is to model the solar system. We embark upon this task by considering a 2-body system which consists of the Sun and the Earth. Two methods for solving ordinary differential equations, the Euler method and the Velocity Verlet method, are compared in efficiency and stability. We then proceed to add a third planet to our system, Jupiter, and do calculations related to the center of mass, variation of planet mass, escape velocity and conserved quantities. Once variables and our preferred algorithm are established and implemented, we model the entire solar system. Finally, we look into the possibilities of adding effects of special relativity.

Physical Background

The relative motion of the planets in a solar system is mainly governed by Newton's law of gravitation. For example, the force between the Sun and the Earth is

$$F_G = \frac{GM_\odot M_{\mathrm{Earth}}}{r^2}, \tag{1}$$

where $M_\odot$ is the mass of the Sun, M_{Earth} is the mass of the Earth, G is the gravitational constant and r is the distance between the Earth and the Sun. The mass of the Sun is much larger than that of the Earth. Neglecting the motion of the Sun is therefore a reasonable assumption for this two-body system. Furthermore, we assume that the orbit of the Earth around the Sun is co-planar, and we take this to be the xy-plane. Using Newton's second law on the motion of the Earth we get

$$\frac{d^2x}{dt^2} = \frac{F_{G,x}}{M_{\mathrm{Earth}}}, \tag{2a}$$

$$\frac{d^2y}{dt^2} = \frac{F_{G,y}}{M_{\mathrm{Earth}}}. \tag{2b}$$

Here, $F_{G,x}$ and $F_{G,y}$ are components of the gravitational force F_G in the x and y direction, respectively. Introducing $x = r\cos(\theta)$, $y = r\sin(\theta)$ and

$$r = \sqrt{x^2 + y^2} \tag{3}$$

we can rewrite Eq. (2a) and (2b) as

$$F_x = -\frac{GM_\odot M_E}{r^2}\cos(\theta) = -\frac{GM_\odot M_E}{r^3}x, \tag{4a}$$

$$F_y = -\frac{GM_\odot M_E}{r^2}\sin(\theta) = -\frac{GM_\odot M_E}{r^3}y, \tag{4b}$$

According to Newton's third law, when one body exerts a force on a second body, the second body simultaneously exerts a force equal in magnitude and opposite in direction on the first body. Fig. (1) illustrates how we have defined the direction of the forces in our simulations.

The equations describing the forces can be rewritten as four coupled differential equations

$$\frac{dv_x}{dt} = -\frac{GM_\odot}{r^3}x, \tag{5a}$$

$$\frac{dv_y}{dt} = -\frac{GM_\odot}{r^3}y, \tag{5b}$$

$$\frac{dx}{dt} = v_x, \tag{6a}$$

$$\frac{dy}{dt} = v_y. \tag{6b}$$

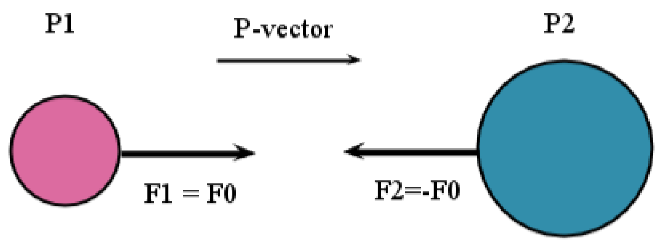


Figure 1: The force F_0 is defined in the direction $\vec{p} = \vec{p}_2 - \vec{p}_1$, where $\vec{p}_i$ are the coordinates of planet i. The force $F_{1\to2}$ from 1 on 2 is then $-F_0$, and $F_{2\to1}$ is $+F_0$.

For circular motion, $a = \frac{v^2}{r}$, the force obeys the following relation

$$F_G = \frac{M_{\text{Earth}} v^2}{r} = \frac{G M_\odot M_{\text{Earth}}}{r^2}, \tag{7}$$

where v is the velocity of Earth. We will use astronomical units with 1 AU = $1.5 * 10^{11}$ (One astronomical unit is the mean distance from the centre of the earth to the centre of the sun.), so that $r = 1$ AU. From Eq. (7) we get

$$G M_\odot = v^2 r, \tag{8}$$

and from the fact that the velocity of Earth is $v = 2\pi r/\text{yr} = 2\pi \text{AU}/\text{yr}$ for circular motion, we get

$$G M_\odot = v^2 r = 4\pi^2 \frac{(\text{AU})^3}{\text{yr}^2}. \tag{9}$$

Escape velocity

In our initial simulation we assume that the Earth begins at a distance 1 AU from the sun, with velocity $|v| = 2\pi$ for circular motion. The Earth can escape from the Sun's gravitational pull when the kinetic energy equals or exceeds the potential energy. This yields the following expression for the escape velocity

$$\begin{aligned} E_{kinetic} &\geq E_{potential} \\ \frac{1}{2} M_{Earth} v^2 &\geq \frac{G M_\odot G_{Earth}}{r} \\ v &\geq \sqrt{2 G M_\odot} \\ v &\geq 2\sqrt{2}\pi \ \frac{\text{Au}}{\text{yr}} \\ v &\geq 8.885766 \ \frac{\text{Au}}{\text{yr}} \end{aligned} \tag{10}$$

Center of Mass and Momentum

We can choose the origin of the coordinate system to be the center of mass, which is defined as

$$\mathbf{R} = \frac{1}{M}\sum_{i=1}^{n} m_i \mathbf{r}_i, \tag{11}$$

where m_i and $\mathbf{r}_i$ are the mass and position of particle i, and M is the total mass. Another restriction we can impose on our system is zero total momentum. The total momentum is defined as

$$\mathbf{p}_{total} = \sum_{i=1}^{n} m_i \mathbf{v}_i. \tag{12}$$

We impose conservation of momentum (discussed in the next section) and choose the initial velocity of the Sun to be

$$\mathbf{v}_{Sun} = -\frac{1}{M_\odot}\sum_{j=1}^{n} m_j \mathbf{v}_j. \tag{13}$$

where m_j and v_j are the mass and velocity of the planet j.

Numerical Approximation

We introduce and use two numerical methods for solving the system of coupled differential equations: the Euler method and the Velocity Verlet method.

Euler's method

The Euler method is a well-know and simple numerical method for solving ordinary differential equations. We derive Euler's method from the Taylor expansions of position and velocity. Neglecting $\mathcal{O}(h^2)$-terms yields

$$x_{i+1} = x_i + hx_i^{(1)} + \mathcal{O}(h^2), \tag{14}$$

$$v_{i+1} = v_i + hv_i^{(1)} + \mathcal{O}(h^2), \tag{15}$$

where h is the step size. With $v^{(1)} = a$ and $x^{(1)} = v$, we can now express the approximations for velocity and position as

$$v_{i+1} = v_i + ha_i, \tag{16}$$

$$x_{i+1} = x_i + hv_i, \tag{17}$$

where $a_i = -\frac{4\pi^2}{r_i^3}x_i$. Note that the Euler method gives an approximation error of the order $\mathcal{O}(h^2)$ for every step, but that the global error will be of the order of $N\mathcal{O}(h^2) \approx \mathcal{O}(h)$ since it is the sum over all steps, N [2]. The number of FLOPs for the Euler method is $6N$.

Velocity Verlet method

The Verlet algorithms are widely used in fields like molecular dynamics because of their numerical stability and easy implementation [2]. As for the Euler method, the Verlet algorithm is derived by Taylor expanding the position and velocity:

$$x_{i+1} = x_i + hx_i^{(1)} + \frac{h^2}{2}x_i^{(2)} + \mathcal{O}(h^3), \tag{18}$$

$$v_{i+1} = v_i + hv_i^{(1)} + \frac{h^2}{2}v_i^{(2)} + \mathcal{O}(h^3), \tag{19}$$

where h is the step size. Furthermore, we can find the Taylor expansion for the derivative of the velocity

$$v_{i+1}^{(1)} = v_i^{(1)} + hv_i^{(2)} + \mathcal{O}(h^2), \tag{20}$$

$$hv_i^{(2)} \approx v_{i+1}^{(1)} - v_i^{(1)}. \tag{21}$$

With Eq. (21) we can now rewrite Eq. (19) as

$$v_{i+1} = v_i + \frac{h}{2}\left(v_{i+1}^{(1)} + v_i^{(1)}\right) + \mathcal{O}(h^3). \tag{22}$$

Using $v^{(1)} = a$ and $x^{(1)} = v$ the approximations for velocity and position become

$$x_{i+1} = x_i + hv_i + \frac{h^2}{2}a_i, \tag{23}$$

$$v_{i+1} = v_i + \frac{h}{2}\left(a_{i+1} + a_i\right), \tag{24}$$

where $a_i = -\frac{4\pi^2}{r_i^3}x_i$. Note that the term a_{i+1} depends on the position at x_{i+1}. This makes it necessary to calculate the position at the updated time, x_{i+1}, before the computation of v_{i+1} [2]. The number of FLOPs in the Verlet algorithm is $11N$.

Unit tests

Energy and angular momentum conservation

In order to test conservation of energy and angular momentum we used a small code snippet that saves the energies for all time steps in a matrix. It then finds the absolute value of the largest error

$$\epsilon_i = \frac{|E_{i,max} - E_{i,min}|}{|E_{i,max}|}, \tag{25}$$

where $i = 0, 1, 2$ are the total, kinetic and potential energies.

Why should energies and angular momentum be conserved? For circular motion and $d\vec{r}_{sun} = 0$, the speed of the Earth $|v_{Earth}|$ and the distance between the Earth and the Sun are constant. Since the potential and kinetic energies are functions of these, their magnitudes should be conserved

$$E_k = \frac{1}{2}m|\vec{v_{earth}}|^2, \tag{26}$$

$$E_p = \frac{Gm_{earth}m_{sun}}{r}. \tag{27}$$

Angular momentum is a function of both velocity and position. For circular motion the velocity is always perpendicular to the radial vector, so this is also a conserved quantity

$$L = m\vec{r} \times \vec{v}. \tag{28}$$

Perihelion precession of Mercury

At this point we should ask ourselves - *how well does our model represent the actual planet orbits?* The planets move around the sun in ellipses. Some perturbations of this motion occur due to the gravitational pull of other planets. These cause the orientation of the ellipses to rotate very slowly in space, as illustrated in Fig. (2). For most planets the observed orbits, which include these small perturbations, can be explained using Newtonian physics alone. In fact, the only planet for which a significant difference between prediction and observation occurs is Mercury, the smallest, innermost planet of the solar system. The predicted angular velocity of the orientation is off by 43 arc seconds (angle $\times\frac{1}{3600}$) per century [1]. Albeit a subtle difference, this caused quite the collective head scratching. We now know the explanation lies in *the theory of general relativity.* Because Mercury lies so close to the Sun, it orbits a region in which spacetime is disturbed by the Sun's mass. We will not go into details (as we do not know them), but introduce an expression for the force between the Sun and Mercury with a relativistic correction

$$F_G = \frac{GM_\odot M_{Mercury}}{r^2}[1 + \frac{3l^2}{r^2c^2}]. \tag{29}$$

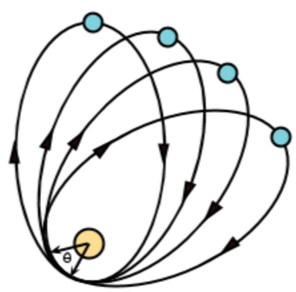


Figure 2: Mercury's orbit around the sun

Results

The masses of the planets and the Sun, which will use for simulations in this project is given in Tab. (1).

Two-body system (Earth-Sun)

Fig. (3) and (4) show simulations of the Earth's orbit around the Sun using the Euler method and the Velocity Verlet method. The Sun was set in origo,

Planet	Mass (kg)
Sun	$M_{\mathrm{sun}} = M_{\odot} = 2 \times 10^{30}$
Earth	$M_{\mathrm{Earth}} = 6 \times 10^{24}$
Jupiter	$M_{\mathrm{Jupiter}} = 1.9 \times 10^{27}$
Venus	$M_{\mathrm{Venus}} = 4.9 \times 10^{24}$
Saturn	$M_{\mathrm{Saturn}} = 5.5 \times 10^{26}$
Mercury	$M_{\mathrm{Mercury}} = 2.4 \times 10^{23}$
Uranus	$M_{\mathrm{Uranus}} = 8.8 \times 10^{25}$
Neptun	$M_{\mathrm{Neptun}} = 1.03 \times 10^{26}$
Pluto	$M_{\mathrm{Pluto}} = 1.31 \times 10^{22}$

Table 1: Table presenting the masses of the planets and the Sun in the solar system

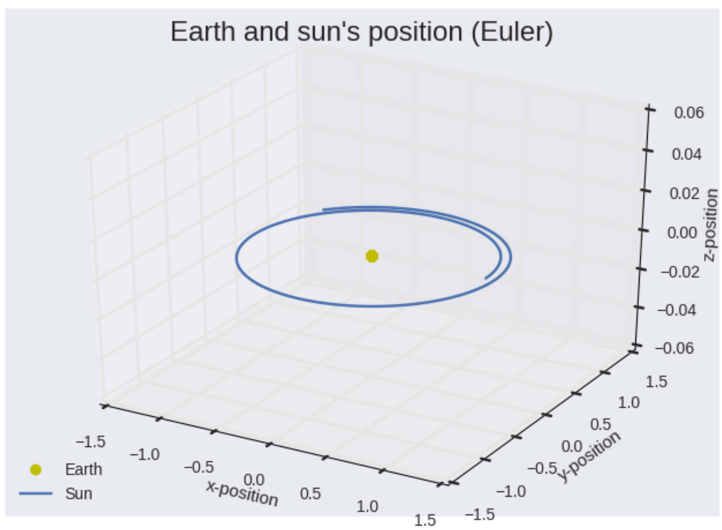


Figure 3: The orbit of the Earth around Sun computed with Euler's method is used with $N = 1500$ and $dt = 0.001$.

and the Earth given an initial velocity of $v_y = 2\pi$ [AU/yr]. These parameters yield a circular, co-planar orbit around the Sun.

Moreover, we are interested in investigating the stability of the Euler and Verlet methods. This was done by varying the step size, dt. Our results are presented in Fig. (5) and (6) for Euler's method and the Verlet method, respectively. From Fig. (5) we see that the Euler method is not very stable as

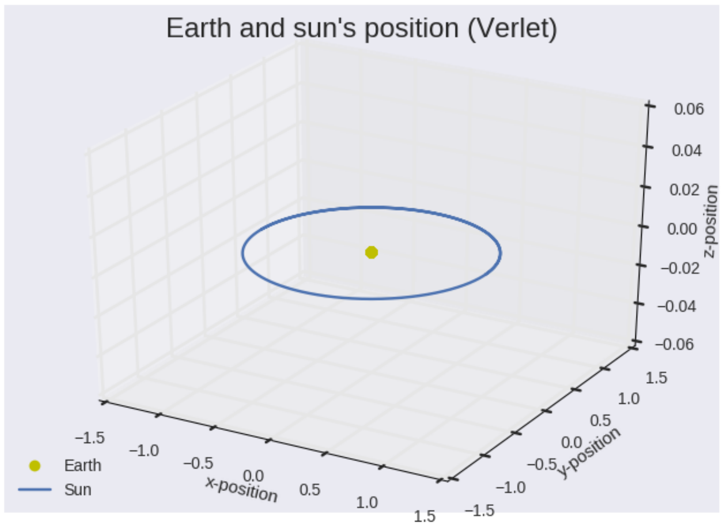


Figure 4: The orbit of the Earth around the Sun computed with Verlet's method for $N = 1500$ and $dt = 0.001$.

a function of increasing dt. In fact, it starts to diverge already for $dt = 0.001$. For $dt = 0.01$ and $dt = 0.1$ the motion diverges so much that the results are useless, and so the algorithm is an unfeasible choice for our application.

Fig. (6) shows that the Verlet method is more stable as a function of dt. The accuracy is high both for $dt = 0.001$ and $dt = 0.01$. For $dt = 0.1$ there is a slight divergence, but still a good approximation in comparison to the Euler's method for the same time step.

Moreover, we are interested in comparing the CPU times of the two method. In Tab. 2 the CPU times for the methods in question is listed. We see that the Euler method have the smaller CPU times of the two. This makes sense because it uses a lower number of FLOPS, which we mentioned in the section *Numerical Approximation*.

Method	CPU times (s)
Euler method	0.009409
Verlet method	0.010226

Table 2: Table showing CPU times for the Euler method and the Verlet method. An average over 10 runs was taken.

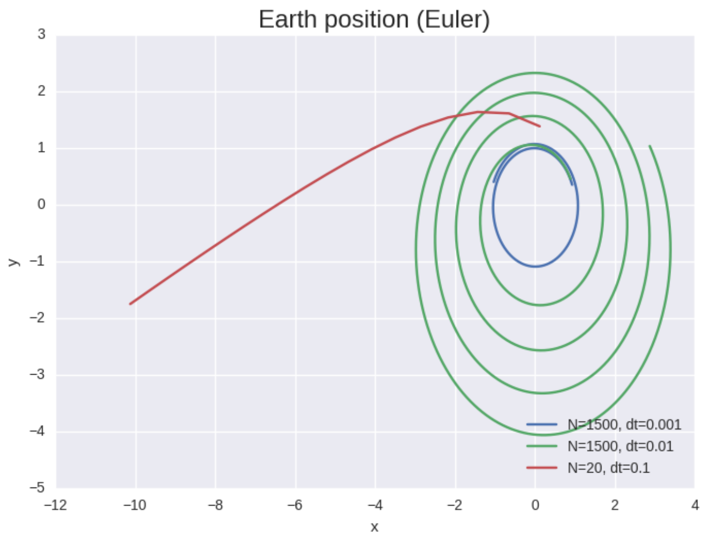


Figure 5: The orbit of the Earth around the Sun have been computed with Euler's method with various values of dt and N, to show the stability of the Euler method

The escape velocity of the Earth, as described in the *Physical Background* section, was also calculated. Fig. (7) shows the position of the Earth as it escapes the Sun. v_{escape} was found by trial and error, estimated to be between 8.88 and 8.89 AU/yr, which is in good agreement with the value we found analytically in Eq. (10). Note that the Sun's marker is made large for visual purposes, so the Earth's orbit *does not* in fact pass through the Sun.

Three-body system (Sun, Earth and Jupiter)

In all computations of the three-body system and the solar system in the next section we have used initial velocities and positions given by NASA (`http://ssd.jpl.nasa.gov/horizons.cgi#top`) from Oct. 13. The mass center is set to the solar system barycenter. Although we did define origo in the center of mass and set total momentum equal to zero, this yielded results very similar to the ones based on NASA's coordinates. Therefore, after a talk with the group teachers at the computer laboratory, we deemed it unnecessary to include the computations with the calculated mass center.

In Fig. (8) the three-body system containing Jupiter, Earth and the Sun has been plotted. We are interested in the effects the additional planet, Jupiter, might have on the Earth's orbit. Thus, Fig. (9) shows the Earth's orbit for both the two-body and three-body system. There were no visible differences,

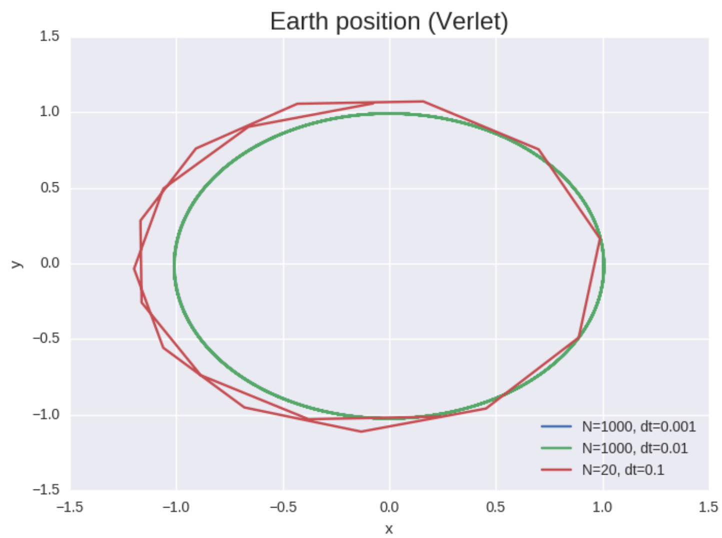


Figure 6: The orbit of the Earth around the Sun have been computed with Verlet's method with various values of dt and N, to show the stability of the Verlet method

which indicates that the mass of Jupiter is small compared to the mass of the Sun, and so has no mentionable effects on Earth.

To observe possible effects of the mass of Jupiter we therefore increased it to $10 \times m_{Jupiter}$ and $1000 \times m_{Jupiter}$. Fig. (10) shows the resulting three-body system. In Fig. (10a) we see that Jupiter starts to diverge away from the Sun. Effects on the Earth's orbit have been plotted in Fig. (11) for the three-body system with $m_{Jupiter}$ and $10 \times m_{Jupiter}$, where we see that Earth's orbit also diverges.

From Fig. 10b it is clear that a large $m_{Jupiter}$ heavily affects the other bodies in the system. The mass of Jupiter is now of the same order of magnitude as the mass of the Sun, meaning that their accelerations are also similar. Jupiter and the Sun start cirulating eachother, driven by their initial velocities and the attractive forces between them. It is worth mentioning that the combined effects of Jupiter and Saturn seem to shoot Earth into space.

Further, we want to analyze the stability of the velocity Verlet method for the three-body system. In Fig. (12) we have plotted the three-body system for three different step sizes, dt. To check the stability of the method we have plotted the positions to Jupiter and Earth for the first rotation around the sun, against a rotation at a later time. If the method is stable we would get the same orbit in both cases. We see in Fig. (12a) that for $dt = 0.001$ the

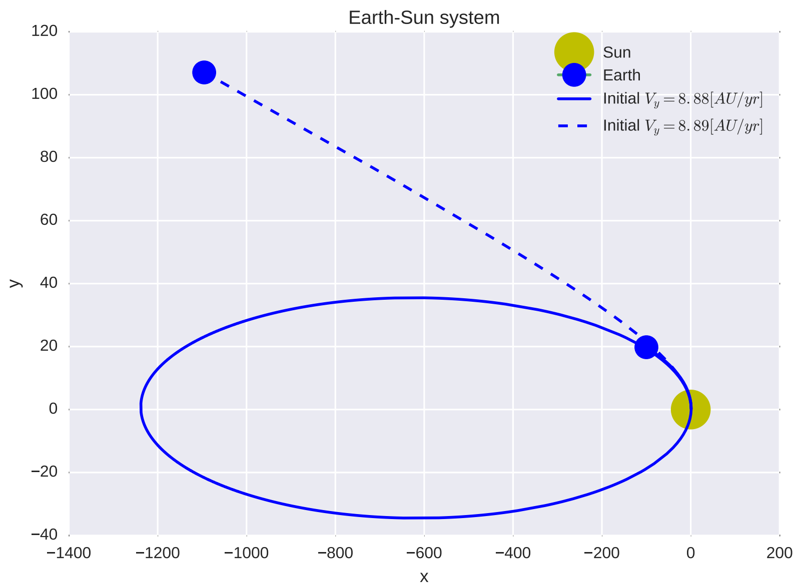


Figure 7: The orbit of the Earth in the Earth-Sun system is plotted for two inital velocity values for Earth, $V_y = 8.88[AU/yr]$ and $V_y = 8.89[AU/yr]$, with 250000 and 1550000 time steps, respectively. The figure shows with the the inital velocity has to be for the Earth to escape the sun.

method looks very stable. Also, from Fig. (12b) with $dt = 0.01$ the method seems to look stable. However, for $dt = 0.1$ in Fig. (12c) we see that the method has become unstable. The orbits for both Earth and Jupiter at the later time have been displaced from the initial orbit.

Solar System

The full solar system can be seen in Fig. (13) and a slightly more artistic version can be seen in Fig. (14). Our full solar system includes the Sun, Earth, Jupiter, Mars, Venus, Saturn, Mercury, Uranus, Neptun and Pluto.

Perihelion precession of Mercury

We simulated the perihelion precession of Mercury and obtained the angle θ over 20 years as output. The result can be seen in Fig. (15). Here, the horizontal axis represent the number of orbital periods of Mercury, where one orbital period is approximately 88 days. Large CPU times at small step sizes make it unfeasible to run the simulation for more years. The step size used was $dt = 1E-8$. Using the *scipy* linear regression tool *linregress* on the calculated points we generated the function

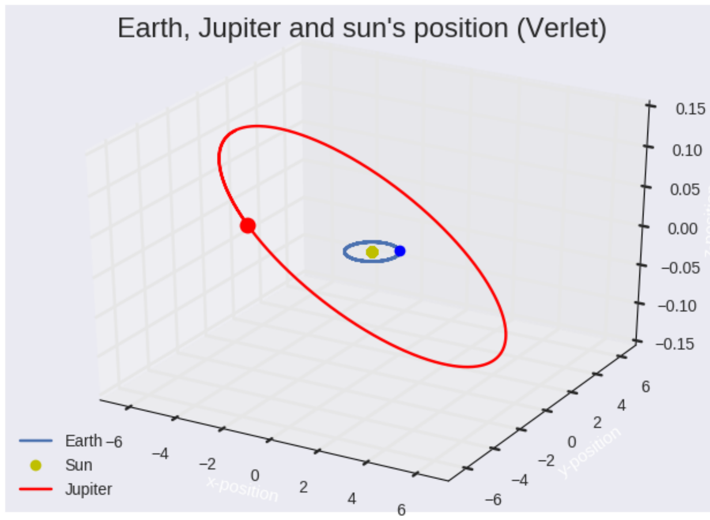


Figure 8: The orbit of Jupiter and Earth around the Sun is plotted. The orbit is computed using the Verlet method with $N = 14000$ and $dt = 0.001$.

$$\theta = 2.06967759501 \cdot 10^{-6} t + 5.13086531842 \cdot 10^{-7}. \tag{30}$$

Since we want to now the value after a century we insert $t = 100$ into the equation and find

$$\theta = 0.00020748 \text{ rad}. \tag{31}$$

We know that the observed value of the perihelion precession is $\theta_{obs} = 43$ arc seconds or $\theta_{obs} = 0.00020847$ [rad] per century, so our number is reasonable. To get a better estimate it would be necessary to use a lower step size, with the drawback of higher computational times.

To make sure that the step size we have used is sufficiently small we computed the perihelion precession for a the unperturbed Newtonian force with $dt = 1E-8$, and saw that it was in the order of magnitude $1E-7$, indicating that ours was a reasonable step size.

Conclusions

In this project we built a model for the solar system using ordinary differential equations. We investigated two numerical methods for solving Eq. (5) and (6). For the simple two-body system containing the Earth and the Sun, Euler's

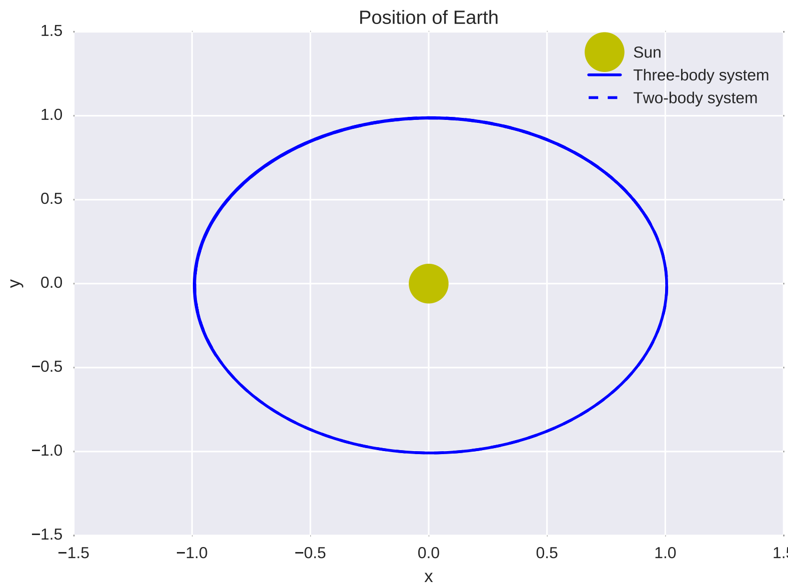


Figure 9: The position of the Earth has been plotted for both the Sun-Earth system and the Sun-Earth-Jupiter system, to find the effect of Jupiter on Earth's orbit. The Verlet method with $N = 1500$ and $dt = 0.001$ was used.

method was found to be inferior to the Verlet method. The Verlet method was therefore used in the rest of the project. Also, we found the velocity needed for the Earth to escape the Sun in our program, and compared it to the escape velocity found analytically. The comparison yielded that our program was a close fit to the analytical value, which deepened our trust in the Verlet method.

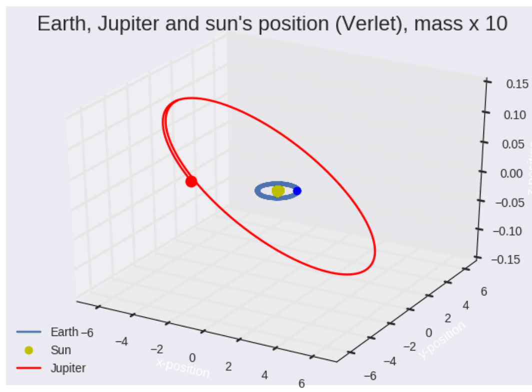
We then introduced Jupiter as a third planet to our system, and found that the effect on the orbit of the Earth was virtually non-existing. By increasing $m_{Jupiter}$ we observed that large values could influence the orbit of Earth and, subsequently, that of the Sun. We found this interesting, as it showed how small the largest planet in our solar system actually is compared to the Sun. Eventually we implemented the rest of planets in our solar system to get a full model. For further work on this project it would be fun to include the planet's moons and other celestial bodies, to get a accurate representation of our solar system.

Lastly, we investigated the perihelion precession of Mercury. As this is one important tests of the general theory of relativity, it was interesting to see that we could closely match the observed value with our simulated values using the relativistic correction to the Newtonian gravitation force. If we had the possibility to use a larger computational force, it would have been exciting to test whether our program could ”exactly” estimate the perihelion precession. With the limited resources available to us during this project, it was not feasible. Nevertheless, it is an exciting note for further work.

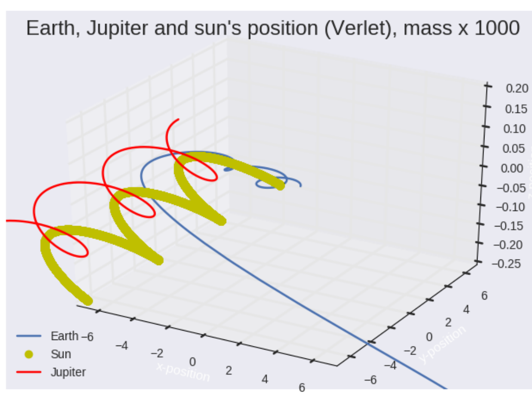
References

[1] Owen Biesel. The precession of mercury’s perihelion. 2008.

[2] Morten Hjorth-Jensen. Computational physics. *Lecture notes*, 2015.



(a) $M_{Jupiter} * 10$



(b) $M_{Jupiter} * 1000$

Figure 10: The three-body system containing Jupiter, Earth and the Sun have been plotted for various masses of Jupiter. Velocity Verlet with $N = 14000$ and $dt = 0.001$ is used to perform the calculations.

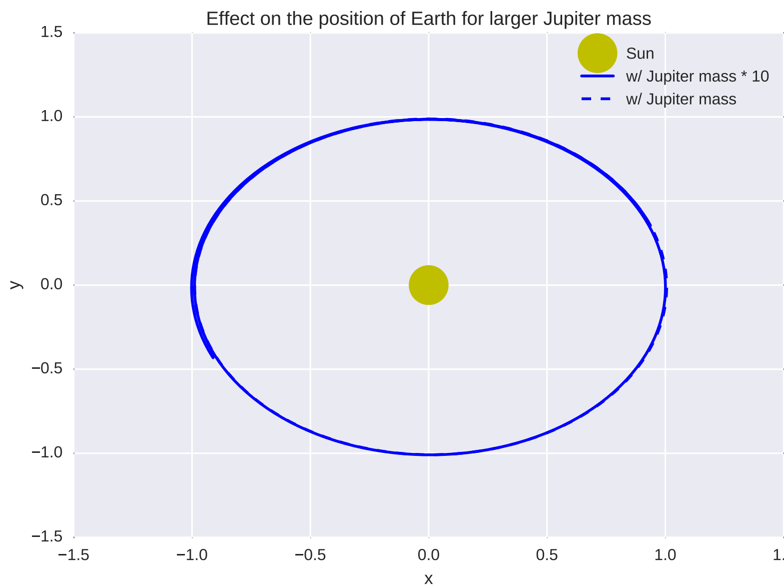
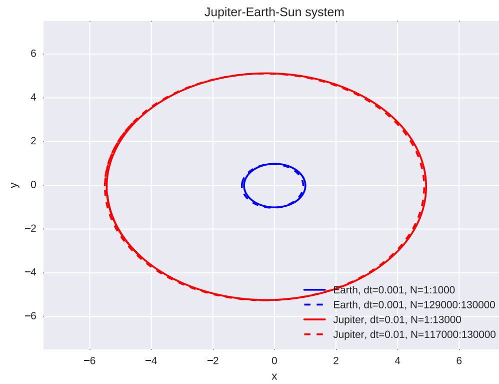
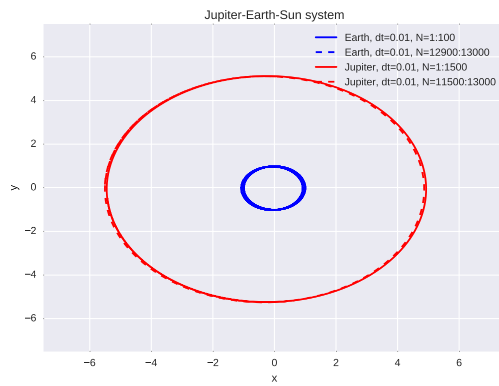


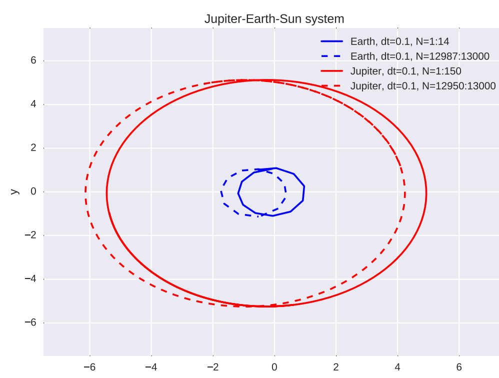
Figure 11: The position of the Earth has been plotted for the Sun-Earth-Jupiter system, for regular mass of Jupiter and ten times the mass of Jupiter to find the effect of the mass Jupiter on Earth's orbit. The Verlet method with $N = 1500$ and $dt = 0.001$ was used.



(a) $dt = 0.001$



(b) $dt = 0.01$



(c) $dt = 0.1$

Figure 12: Stability of the velocity Verlet method for the three-body system containing Jupiter, Earth and the Sun. Various values of the step size, dt, is used with appropriate Ns.

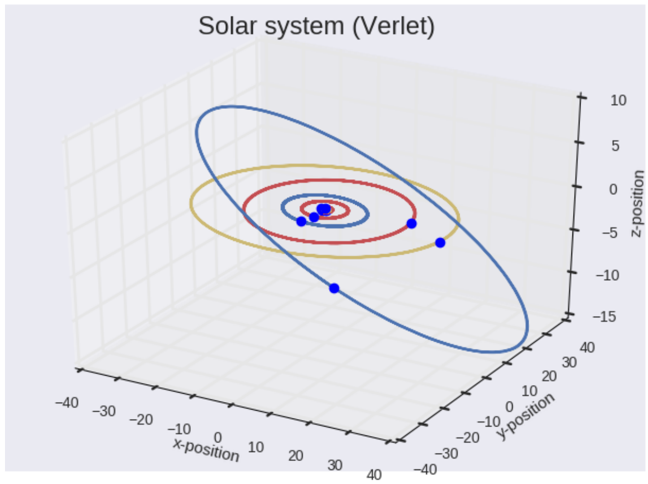


Figure 13: The full solar system is plotted using the Verlet method.

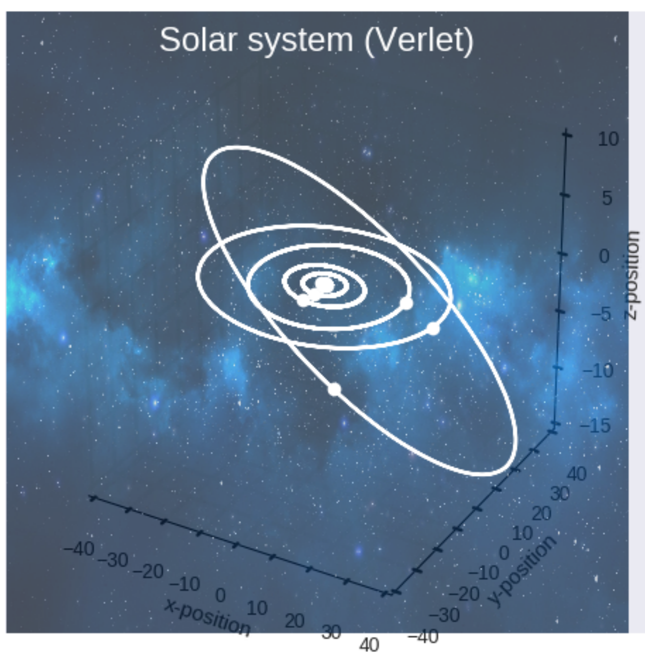


Figure 14: The full solar system is plotted using the Verlet method.

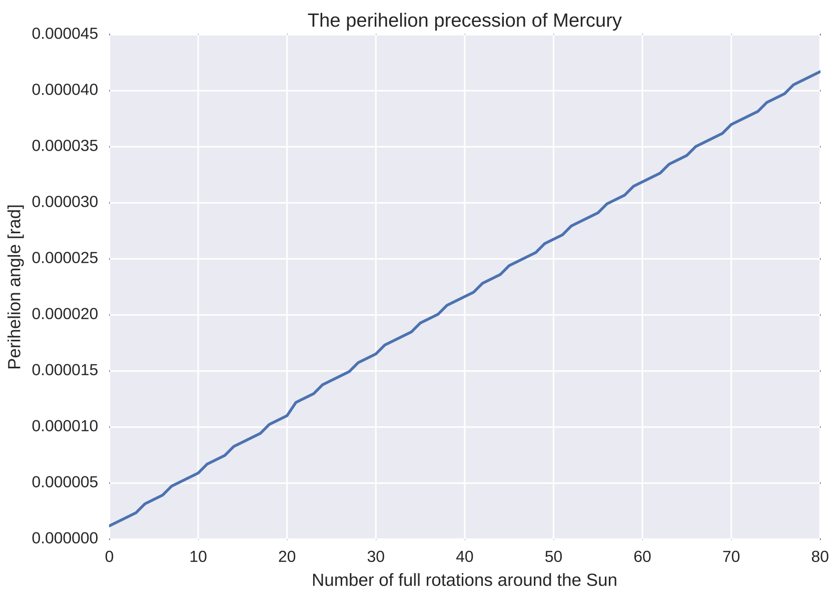


Figure 15: Perihelion precession of Mercury