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The Capital Asset Pricing Model

Birth of a Model

The capital asset pricing model was the work of financial economist (and later, Nobel laureate in economics) William Sharpe, set out in his 1970 book `Portfolio Theory and Capital Markets`. His model starts with the idea that individual investment contains two types of risk:

- Systematic Risk. These are market risks that cannot be diversified away. `Interest rates`, `recessions` and `wars` are examples of systematic risks.
- Unsystematic Risk. Also known as `specific risk` this risk is specific to individual stocks and can be diversified away as the investor increases the number of stocks. It represents the component of a stock's return that is not correlated with general market moves.

Inventors

- The CAPM was introduced by Jack Treynor, William Sharpe, John Lintner and Jan Mossin independently, building on the earlier work of Harry Markowitz on diversification and modern portfolio theory.
- Sharpe, Markowitz and Merton Miller jointly received the 1990 `Nobel Memorial Prize in Economics` for this contribution to the field of financial economics.
- Fischer Black (1972) developed another version of CAPM, called Black CAPM or zero-beta CAPM, that does not assume the existence of a riskless asset.

Idea

The general idea behind CAPM is that investors need to be compensated in two ways: `time value of money` and `risk`.

- The time value of money is represented by the risk-free r_f rate in the formula and compensates the investors for placing money in any investment over a period of time.
- The risk is calculated by taking a risk measure β that compares the returns of the asset to the market over a period of time and to the market premium $r_M - r_f$: the return of the market in excess of the risk-free rate.

Facts

the CAPM takes into account the non-diversifiable market risks or beta (β) in addition the expected return of a risk-free asset.

While CAPM is accepted academically, there is empirical evidence suggesting that the model is not as profound as it may have first appeared to be.

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Its assumptions have been criticized from the start as being
too unrealistic for investors in the real world.
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Assumptions

- All investors are risk averse by nature.
- Investors have the same time period to evaluate information.
- There is unlimited capital to borrow at the risk-free rate of return.
- Investments can be divided into unlimited pieces and sizes.
- There are no taxes, inflation or transactions costs.

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Equilibrium in security markets

- Investors hold market portfolio (all traded assets)
- Risky and risk-free assets, `capital market line` : the best attainable capital allocation line
- The risk premium on the market portfolio

$$E(r_M) - r_f$$

- The risk premium on individual assets

$$E(r_i) - r_f = \frac{\sigma_{iM}}{\sigma_M^2}[E(r_M) - r_f]$$

to be proved later

Why market portfolio

- Identical Markowitz analysis, same assets, arrive at same optimal portfolio
- Price adjustment process
- Passive strategy is efficient : the tangency portfolio is the market portfolio

Risk premium

- The CAPM is built on the insight that the appropriate risk premium on an asset will be determined by its contribution to the risk of investors' overall portfolios.
- Portfolio risk matters

Find the contribution

- Portfolio risk and Portfolio variance
- The contribution is the sum of weighted covariance

> Recall the calculation of variance

Contribution

Portfolio Weights	w_1	w_2	. . .	w_{GM}	. . .	w_n
w_1	Cov(r_1,r_1)	Cov(r_1,r_2)	. . .	Cov(r_1,r_{GM})	. . .	Cov(r_1,r_n)
w_2	Cov(r_2,r_1)	Cov(r_2,r_2)	. . .	Cov(r_2,r_{GM})	. . .	Cov(r_2,r_n)
•	•	•		•		•
•	•	•		•		•
•	•	•		•		•
w_{GM}	Cov(r_{GM},r_1)	Cov(r_{GM},r_2)	. . .	Cov(r_{GM},r_{GM})	. . .	Cov(r_{GM},r_n)
•	•	•		•		•
•	•	•		•		•
•	•	•		•		•
w_n	Cov(r_n,r_1)	Cov(r_n,r_2)	. . .	Cov(r_n,r_{GM})	. . .	Cov(r_n,r_n)

$$w_{GM}[w_1\mathrm{Cov}(r_1,r_{GM}) + w_2\mathrm{Cov}(r_2,r_{GM}) + \cdots + w_{GM}\mathrm{Cov}(r_{GM},r_{GM}) + \cdots + w_n\mathrm{Cov}(r_n,r_{GM})]$$

Single asset's contribution

GM's contribution to variance $= w_{GM}Cov(r_{GM}, r_M)$

As we know that: $r_M = \sum_{k=1}^{n} w_k r_k$

Then

$$Cov(r_{GM}, r_M) = Cov(r_{GM}, \sum_{k=1}^{n} w_k r_k) = \sum_{k=1}^{n} w_k Cov(r_{GM}, r_k)$$

Find the risk premium

- Market price of risk

$$\frac{E(r_M) - r_f}{\sigma_M^2}$$

- if an investor changes his/her position by a size of ϵ, negative position in risk-free assets and long position in market portfolio
- The new return is then $r_M + \epsilon(r_M - r_f)$

Find the risk premium

- The incremental expected rate of return

$$\Delta E(r) = \epsilon[E(r_M) - r_f]$$

- The variance

$$\sigma^2 = (1+\epsilon)^2\sigma_M^2 = \sigma_M^2 + (2\epsilon + \epsilon^2)\sigma_M^2$$

- Since ϵ^2 can be negligible, the incremental variance is

$$\Delta\sigma^2 = 2\epsilon\sigma_M^2$$

Find the risk premium

- Marginal price of risk

$$\frac{\Delta E(r)}{\Delta \sigma^2} = \frac{E(r_M) - r_f}{2\sigma_M^2}$$

- Suppose investor was to invest ϵ in GM stock with same arrangement, that yields

$$\Delta E(r) = \epsilon[E(r_{GM}) - r_f]$$

$$\Delta \sigma^2 = \epsilon^2 \sigma_{GM}^2 + 2\epsilon Cov(r_{GM}, r_M)$$

Find the risk premium

rearrange:

$$E(r_{GM}) - r_f = \frac{Cov(r_{GM}, r_M)}{\sigma_M^2}[E(r_M) - r_f]$$

or

$$E(r_{GM}) = r_f + \beta_{GM}[E(r_M) - r_f]$$

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Expected return-to-beta relationship
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Expected return-beta relationship

- In a general case, a portfolio with n assets has

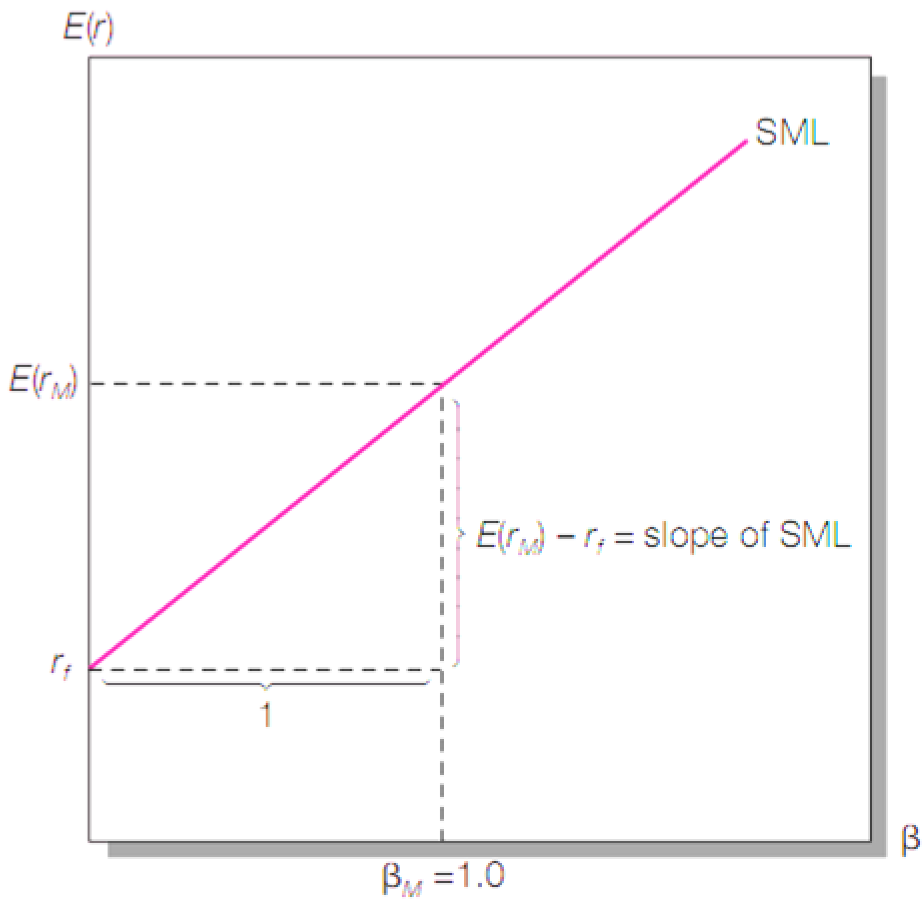
$$
\begin{array}{rrcl}
 & w_1E(r_1) & = & w_1r_f + w_1\beta_1[E(r_M) - r_f] \\
+ & w_2E(r_2) & = & w_2r_f + w_2\beta_2[E(r_M) - r_f] \\
+ & \cdots & = & \cdots \\
+ & w_nE(r_n) & = & w_nr_f + w_n\beta_n[E(r_M) - r_f] \\
\hline
 & E(r_P) & = & r_f + \beta_P[E(r_M) - r_f]
\end{array}
$$

where $\beta_p = \sum_i w_i\beta_i$

Security market line

The expected return-beta relationship

All assets and portfolio must lie on the SML



Security market line

The SML line function

$$E(r_i) = a + b\beta_i$$

The intercept refers to the return of risk-free assets

$$E(r_f) = a + b * 0 = a = r_f$$

The second point which must be on the SML line is the market portfolio with its β equals to 1

$$E(r_M) = a + b * 1 = a + b$$

Therefore

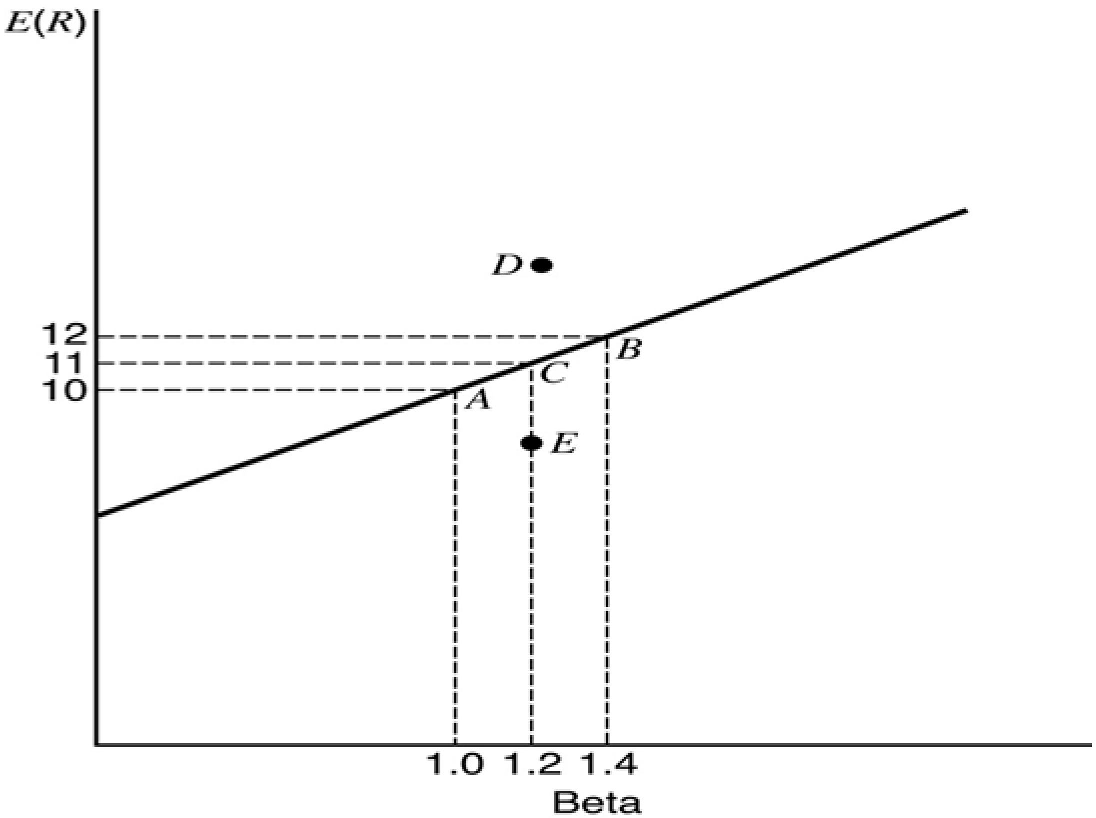
$$E(r_i) = r_f + \beta_i(E(r_M) - r_f)$$

Security market line

- Risk premiums as a function of asset risk
- Risk measured by contributioin of the asset to the portfolio variance (β)
- Valid for both efficient portfolios and individual assets
- `fairly price`

Security market line

- The difference between the fair and actually expected return is called the stock's `alpha`
- Asset managers can start from a passive portfolio and adjust weights of stocks by their `alphas`



Proof of CAPM

Recall the objective function and the results:

$$S_p = \frac{E(r_p) - r_f}{\sigma_p}$$

$$\frac{dS_p}{dW_i} = -(\lambda \sum_{j=1}^{N} W_j \sigma_{ij}) + E(r_i) - r_f = 0$$

$$\lambda = \frac{E(r_p) - r_f}{\sigma_p}$$

Proof of CAPM

$$\frac{dS_p}{dW_i} = \frac{(\frac{\partial E(r_p)}{\partial W_i} - r_f)\sigma_p - (E(r_p) - r_f)\frac{\partial \sigma_p}{\partial W_i}}{\sigma_p^2} = 0$$

$$\frac{\partial E(r_p)}{\partial W_i} = E(r_i)$$

$$\frac{\partial \sigma_p}{\partial W_i} = \frac{\partial \sum_{i=1}^{N} W_i \sum_{j=1}^{N} W_j \sigma_{ij}}{\partial W_i} = \sum_{j=1}^{N} W_j \sigma_{ij}$$

$$E(r_i) - r_f - \frac{E(r_p) - r_f}{\sigma_p} \sum_{j=1}^{N} W_j \sigma_{ij} = 0$$

Proof of CAPM

The market portfolio rate of return can be expressed by

$$r_M = \sum r_i W_i$$

$$Cov(r_i, r_M) = \sum_{j=1}^{N} W_j Cov(r_i, r_j)$$

Why ?

Proof of CAPM

As

$$\lambda Cov(r_i, r_M) = E(r_i) - r_f$$

$$\lambda \sigma_M^2 = E(r_M) - r_f$$

Therefore:

$$E(r_i) = r_f + \frac{E(r_m) - r_f}{\sigma_M^2} Cov(r_i, r_M) = r_f + \beta_i (E(r_M) - r_f)$$

Price and CAPM

CAPM is also applicable to Price (as well as return)

P_i current price of asset at time i

P_M price of market portfolio

Y_i market value of assets (i.e. future price of asset P)

Y_M market value of market portfolio

therefore we have

$$R_i = \frac{Y_i - P_i}{P_i} = \frac{Y_i}{P_i} - 1$$

Price and CAPM

According to CAPM model, we have

$$\frac{Y_i}{P_i} - 1 = r_f + (\frac{Y_i}{P_i} - 1 - r_f)\frac{Cov(R_i R_M)}{\sigma_M^2}$$

where

$$\frac{Cov(R_i R_M)}{\sigma_M^2} = E[(\frac{Y_i - P_i}{P_i} - \frac{\overline{Y}_i - P_i}{P_i})(\frac{Y_M - P_M}{P_M} - \frac{\overline{Y}_M - P_M}{P_M}$$

$$= \frac{1}{P_i P_M} Cov(Y_i Y_M)$$

Price and CAPM

Subsitute into original formula

$$\frac{\overline{Y}_i}{P_i} = R_f + \left(\frac{\overline{Y}_M}{P_M} - R_f\right)\frac{\frac{1}{P_iP_M}Cov(Y_iY_M)}{\frac{1}{P_M^2}Var(Y_M)}$$

where $R_f = 1 + r_f$

Re-arrange

$$\overline{Y}_i = R_fP_i + (\overline{Y}_M - R_fP_M)\frac{Cov(Y_iY_M)}{Var(Y_M)}$$

or

$$P_i = \frac{1}{R_f}[\overline{Y}_i - (\overline{Y}_M - R_fP_M)\frac{Cov(Y_iY_M)}{Var(Y_M)}]$$

Unstandard CAPM ***

1. No short sell
2. No riskfree loan
3. Personal tax
4. Non-tradable assets
5. Heterogeneous expectation
6. No price taking
7. Multi-period CAPM
8. Multi-beta CAPM