浙江工商大學 ZHEJIANG GONGSHANG UNIVERSITY | 金融学院 School of Finance

Portfolio Management

Index Model

Single index model

Why single index?

formidable job for assets managers: too many covariance and imprecision in estimation.

Covariance between securities tends to be positive.

Single index model

Basic form

$$R_i = \alpha_i + \beta_i R_M + e_i$$

	Symbol
1. The variance attributable to the uncertainty of the common macroeconomic factor	$\beta_i^2 \sigma_M^2$
2. The variance attributable to firm-specific uncertainty	$\sigma^2(e_i)$

Basic properties

Variance and covariance

$$\sigma_i^2 = \beta_i^2 \sigma_M^2 + \sigma^2(e_i)$$

$$\mathrm{Cov}(R_i, R_j) = \mathrm{Cov}\,(\alpha_i + \beta_i R_M + e_i, \alpha_j + \beta_j R_M + e_j)$$

$$= \mathrm{Cov}(\beta_i R_M, \beta_j R_M) = \beta_i \beta_j \sigma_M^2$$

Artificial data

Month	Stock return	Market return	R_i	α_i	$\beta_i R_M$	e_i
1	10	4	10	2	6	2
2	3	2	3	2	3	-2
3	15	8	15	2	12	1
4	9	6	9	2	9	-2
5	3 / 40	0/20	3 / 40	2/10	0/30	1/0

Basic properties

By using single-index model, we could have…

$$E(R_p) = \sum_{i=1}^{N} W_i \alpha_i + \sum_{i=1}^{N} W_i \beta_i R_m$$

$$\sigma_p^2 = \sum_{i=1}^{N} W_i^2 \beta_i^2 \sigma_m^2 + \sum_{i=1}^{N} W_i^2 \sigma_{ei}^2 + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} W_i W_j \beta_i \beta_j \sigma_m^2$$

Basic properties

Thus reduce the cost of computation to

n estimates of the expected excess returns, $E(R_i)$
n estimates of the sensitivity coefficients, β_i
n estimates of the firm-specific variances, $\sigma^2(e_i)$
1 estimate for the variance of the (common) macroeconomic factor, σ_M^2,

Risk diversification

$$\sigma_p^2 = \sum_{i=1}^{N} W_i^{\,2} \beta_i^2 \sigma_m^2 + \sum_{i=1}^{N} W_i^{\,2} \sigma_{ei}^2 + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} W_i W_j \beta_i \beta_j \sigma_m^2$$

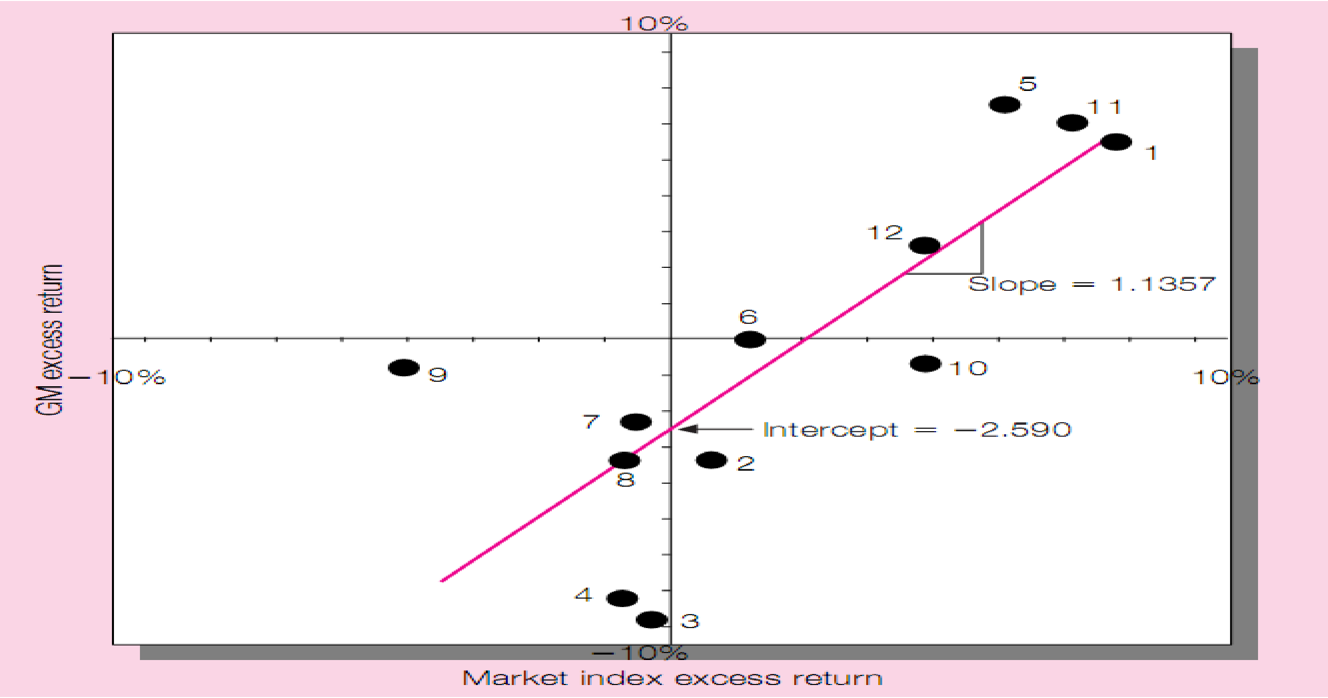
$$= \sum_{i=1}^{N} W_i^{\,2} \sigma_{ei}^2 + \sum_{i=1}^{N} \sum_{j=1}^{N} W_i W_j \beta_i \beta_j \sigma_m^2$$

$$= \sum_{i=1}^{N} W_i^{\,2} \sigma_{ei}^2 + \left(\sum_{i=1}^{N} W_i \beta_i\right)\left(\sum_{j=1}^{N} W_j \beta_j\right) \sigma_m^2$$

$$:= \beta_p^2 \sigma_m^2 + \frac{1}{N} \sum_{i=1}^{N} \frac{1}{N} \sigma_{ei}^2$$

Estimating beta

Scatter diagram



Estimating beta

Month	GM Return	Market Return	Monthly T-Bill Rate	Excess GM Return	Excess Market Return
January	6.06	7.89	0.65	5.41	7.24
February	−2.86	1.51	0.58	−3.44	0.93
March	−8.18	0.23	0.62	−8.79	−0.38
April	−7.36	−0.29	0.72	−8.08	−1.01
May	7.76	5.58	0.66	7.10	4.92
June	0.52	1.73	0.55	−0.03	1.18
July	−1.74	−0.21	0.62	−2.36	−0.83
August	−3.00	−0.36	0.55	−3.55	−0.91
September	−0.56	−3.58	0.60	−1.16	−4.18
October	−0.37	4.62	0.65	−1.02	3.97
November	6.93	6.85	0.61	6.32	6.25
December	3.08	4.55	0.65	2.43	3.90
Mean	0.02	2.38	0.62	−0.60	1.75
Standard deviation	4.97	3.33	0.05	4.97	3.32
Regression results	$r_{GM} - r_f = \alpha + \beta(r_M - r_f)$				
	α	β			
Estimated coefficient	−2.590	1.1357			

Estimating beta

Regression model

$$R_{GMt} = \alpha_{GM} + \beta_{GM} R_{Mt} + e_{GMt}$$

$$e_{GMt} = R_{GMt} - (\beta_{GM} R_{Mt} + \alpha_{GM})$$

Estimating beta

Least mean squared

normal distribution

Covariance between stock return and market return

$$\begin{aligned}\mathrm{Cov}(R_i, R_M) &= \mathrm{Cov}(\beta_i R_M + e_i, R_M) \\ &= \beta_i \mathrm{Cov}(R_M, R_M) + \mathrm{Cov}(e_i, R_M) \\ &= \beta_i \sigma_M^2\end{aligned}$$

Betas may differ

Beta of the security might change

Random error

When securities are combined, errors tend to cancel out.

Betas Tend Toward One

Betas in the forecast period tend to be closer to one (1) than the estimate obtained from historical data

Marshall Blume’s Technique

Correlated one stock up to 50 stock portfolios in order to determine how much information historical betas contain about future betas.

Correlation between historical betas and future betas

1954.7 – 1961.6 and 1961.7 – 1968.6

Estimated beta in two un- overlapped periods

Portfolio Size	Rho	Rho^2
1	.60	.36
2	.73	.53
4	.84	.71
7	.88	.77
10	.92	.85
20	.97	.95
35	.97	.95
50	.98	.96

Marshall Blume's Technique

The method measured directly the adjustment towards one (1) from a given historical series

1948 - 1954	1955 - 1961
Calculate β	Calculate β

$$\beta_{i2} = .343 + 0.677\beta_{i1}$$

Considerations

Modifies the average levels of beta for all stocks.

If the betas increase over the two periods it assumes beta will increase over the next period. This could be undesirable if one does not expect a continuous upward drift.

Vasicek's Technique

An alternative method to Blume would be to take one-half of the historical betas and add it to one-half of the average beta.

While this may move the beta towards the average it is more attractive to adjust each beta based upon the level of sampling error.

Vasicek's Technique

$$\beta_{i2} = \frac{\sigma^2_{\beta i1}}{\sigma^2_{\beta} + \sigma^2_{\beta i1}} \beta + \frac{\sigma^2_{\beta}}{\sigma^2_{\beta} + \sigma^2_{\beta i1}} \beta_{i1}$$

We give larger adjustments to betas with larger error.

Betas and correlation

Betas as forecasters of correlation coefficients

$$\rho_{ij} = \frac{\sigma_{ij}}{\sigma_i \sigma_j} = \frac{\beta_i \beta_j \sigma_m^2}{\sigma_i \sigma_j}$$

Betas and correlation

The ability of the correlation matrix to forecast itself.

The ability of historical betas to forecast the correlation matrix.

via Blume

via Vasicek

Single index model

Ticker Symbol	Security Name	June 1994 Close Price	Beta	Alpha	*R*-SQR	RESID STD DEV-*N*	Standard Error Beta	Standard Error Alpha
GBND	General Binding Corp	18.375	0.52	−0.06	0.02	10.52	0.37	1.38
GBDC	General Bldrs Corp	0.930	0.58	−1.03	0.00	17.38	0.62	2.28
GNCMA	General Communication Inc Class A	3.750	1.54	0.82	0.12	14.42	0.51	1.89
GCCC	General Computer Corp	8.375	0.93	1.67	0.06	12.43	0.44	1.63
GDC	General Datacomm Inds Inc	16.125	2.25	2.31	0.16	18.32	0.65	2.40
GD	General Dynamics Corp	40.875	0.54	0.63	0.03	9.02	0.32	1.18
GE	General Elec Co	46.625	1.21	0.39	0.61	3.53	0.13	0.46
JOB	General Employment Enterpris	4.063	0.91	1.20	0.01	20.50	0.73	2.69
GMCC	General Magnaplate Corp	4.500	0.97	0.00	0.04	14.18	0.50	1.86
GMW	General Microwave Corp	8.000	0.95	0.16	0.12	8.83	0.31	1.16
GIS	General MLS Inc	54.625	1.01	0.42	0.37	4.82	0.17	0.63
GM	General MTRS Corp	50.250	0.80	0.14	0.11	7.78	0.28	1.02
GPU	General Pub Utils Cp	26.250	0.52	0.20	0.20	3.69	0.13	0.48
GRN	General RE Corp	108.875	1.07	0.42	0.31	5.75	0.20	0.75
GSX	General SIGNAL Corp	33.000	0.86	−0.01	0.22	5.85	0.21	0.77

Two-factor model

Dichotomy between systematic and diversifiable risk.

$$R_t = \alpha + \beta_{GDP}GDP_t + \beta_{IR}IR_t + e_t$$

captures differential responses to varing sources of macroeconomic uncertainty.

Multi-factor Model

Stock actually differ in their betas relative to the various macroeconomic factors.

Multifactor model can provide better descriptions of returns.

Why Multifactor model

Factor models are used to predict portfolio behavior, and in conjunction with other tools, to construct customized portfolios with certain desired characteristics, such as the ability to track the performance of indexes or other portfolios

General types of Multifactor model

Macroeconomic factor models use observable economic time series as measures of the factors correlated with security returns.

Fundamental factor models use many of the measurements generated by securities analysts, such as price/earnings ratios, industry membership, company size, financial leverage, dividend yield, etc.

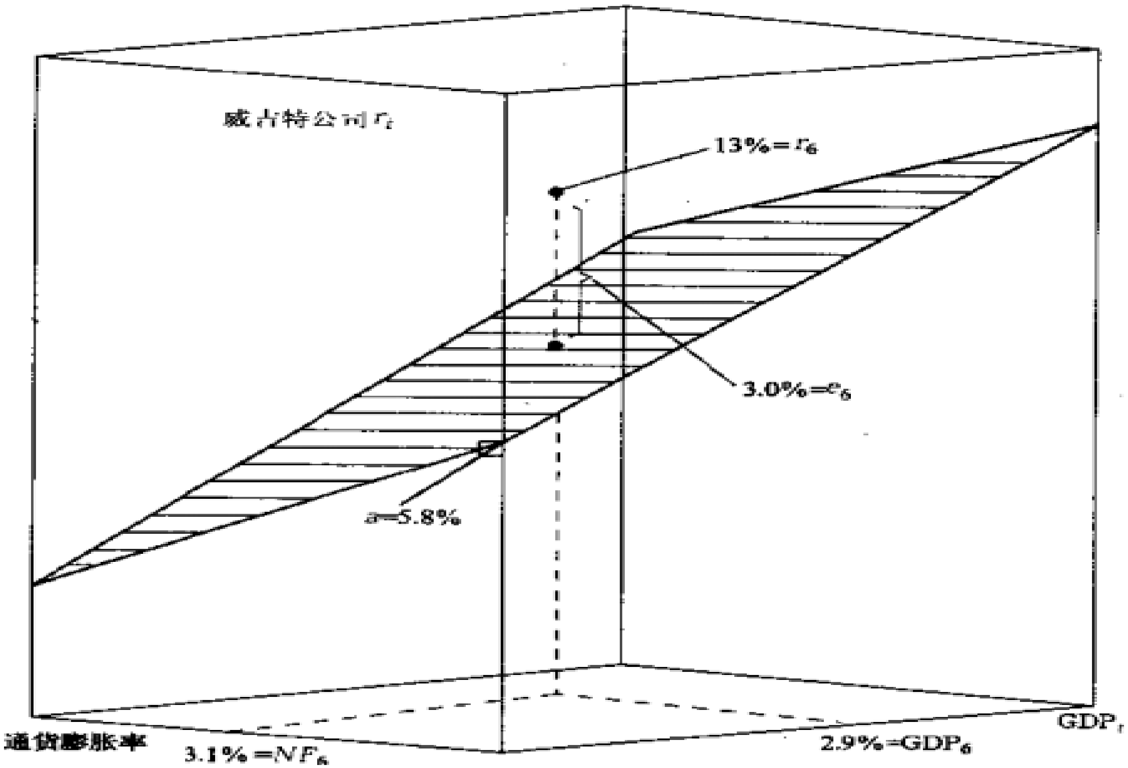
Statistical factor models generate statistical constructs that have no necessary fundamental or macroeconomic analogs, but that explain, in the statistical sense, many of the relationships of security returns from the security return data alone.

Two-factor model

$$R_t = \alpha + \beta_{GDP}GDP_t + \beta_{IR}IR_t + e_t$$

If we have $\beta_{GDP} = 2.2$, $\beta_{IR} = -0.7$ and $\alpha = 5.8\%$

Two-factor model



Two-factor model

We need to estimate

For security

$$\alpha_i, \beta_{i1}, \beta_{i2}, \sigma_{ei}$$

For factors

$$\overline{f}_1, \overline{f}_2, \sigma^2_{f1}, \sigma^2_{f2}, \mathrm{cov}(f_1, f_2)$$

Two-factor model

Expected return

$$\bar{r}_i = \alpha_i + \beta_{i1}\bar{f}_1 + \beta_{i2}\bar{f}_2 + e_i$$

Variance

$$\sigma_i^2 = \beta_{i1}\sigma_{i1}^2 + \beta_{i2}\sigma_{i2}^2 + \sigma_{ei}^2 + 2\beta_{i1}\beta_{i2}\operatorname{cov}(f_1, f_2)$$

Covariance

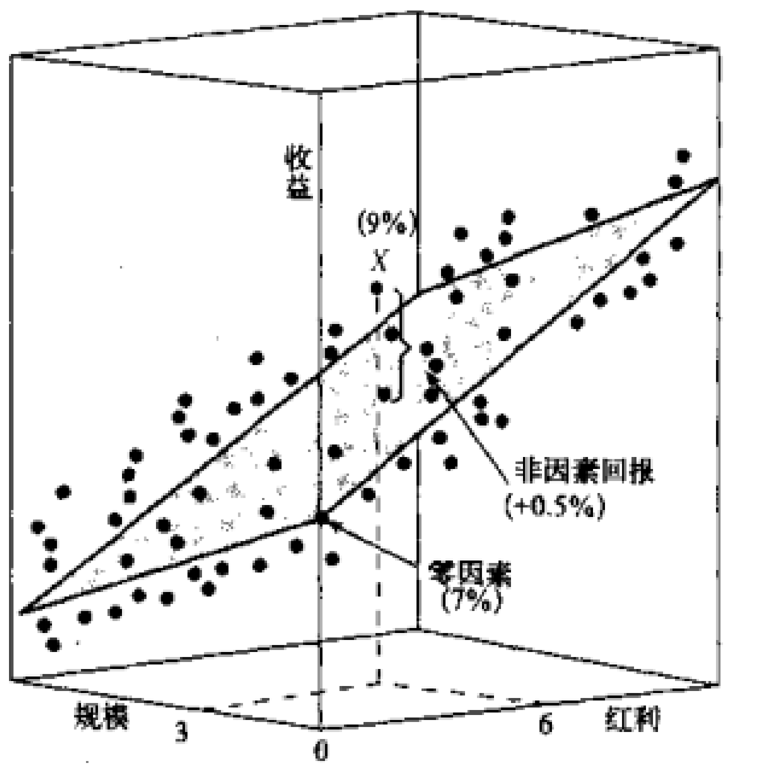
$$\sigma_{ij} = \beta_{i1}\beta_{j1}\sigma_{f1}^2 + \beta_{i2}\beta_{j2}\sigma_{f2}^2 + (\beta_{i1}\beta_{j2} + \beta_{i2}\beta_{j1})\operatorname{cov}(f_1, f_2)$$

Diversification

$$
\begin{aligned}
r_{pt} &= \sum_{i=1}^{N} W_i(\alpha_i + \beta_{i1} f_{1t} + \beta_{i2} f_{2t} + e_{it}) \\
&= \sum_{i=1}^{N} W_i \alpha_i + \sum_{i=1}^{N} W_i \beta_{i1} f_{1t} + \sum_{i=1}^{N} W_i \beta_{i2} f_{2t} + \sum_{i=1}^{N} W_i e_{it} \\
&= \alpha_p + \beta_{p1} F_{1t} + \beta_{p2} F_{2t} + e_{pt}
\end{aligned}
$$

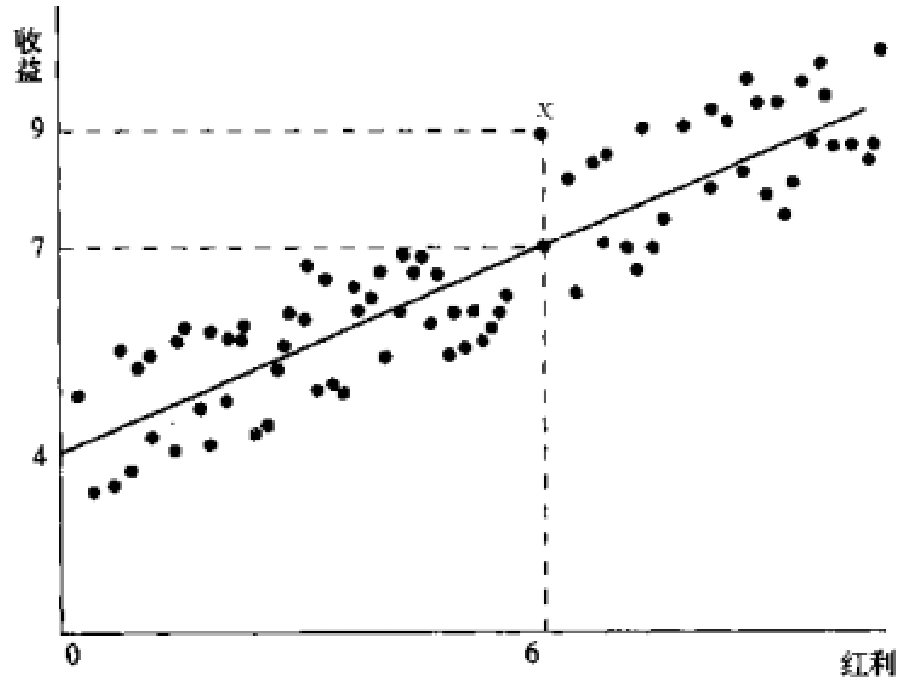
Estimate Two-factor model

For example



Estimate Two-factor model

Should we



Estimate Two-factor model

Makes the factor independent

which makes the expectation, variance and covariance become:

$$\bar{r}_i = \alpha_i + \beta_{i1}\overline{f}_1 + \beta_{i2}\overline{f}_2 + e_i$$

$$\sigma_i^2 = \beta_{i1}\sigma_{i1}^2 + \beta_{i2}\sigma_{i2}^2 + \sigma_{ei}^2$$

$$\sigma_{ij} = \beta_{i1}\beta_{j1}\sigma_{f1}^2 + \beta_{i2}\beta_{j2}\sigma_{f2}^2$$

Industry index model

If a portfolio has N stocks that follow a L factor model.

That means we need 2N+2L+NL estimations

What if stock performance was affected by both market indexes and industry indexes (unique to certain stocks)?

Industry index model

A general multifactor model

$$r_i = \alpha_i + \beta_{im} f_m + \beta_{i1} f_1 + \cdots + \beta_{iL} f_L + e_i$$

where f_m is market factor/index

and f_l are independent industry factors/indexes

Industry index model

The fact is, we would better off assume that some influence to assets/stocks are industry specific

$$r_i = \alpha_i + \beta_{im} f_m + \beta_{iL} f_L + e_i$$

Covariance of two stocks within one industry is

$$\beta_{im}\beta_{jm}\sigma_m^2 + \beta_{iL}\beta_{jL}\sigma_{fL}^2$$

otherwise

$$\beta_{im}\beta_{jm}\sigma_m^2$$

Variants of Multifactor Model

Chen, Roll, Ross

$$R_{it} = \alpha_i + \beta_{iIP} IP_t + \beta_{iEI} EI_t + \beta_{iUI} UI_t + \beta_{iCG} CG_t + \beta_{iGB} GB_t + e_{it}$$

IP = % change in industrial production

EI = % change in expected inflation

UI = % change in unanticipated inflation

CG = excess return of long-term corporate bonds over long-term government bonds

GB = excess return of long-term government bonds over T-bills

Variants of Multifactor Model

Fama, French

$$R_{it} = \alpha_i + \beta_{iM} R_{Mt} + \beta_{i\text{SMB}} \text{SMB}_t + \beta_{i\text{HML}} \text{HML}_t + e_{it}$$

SMB = small minus big: the return of a portfolio of small stocks in excess of the return on a portfolio of large stocks

HML = high minus low: the return of a portfolio of stocks with high ratios of book value to market value in excess of the return on a portfolio of stocks with low book-to-market ratios