EE3025 Assignment-1

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Download all python codes from

https://github.com/Kkuntal990/EE3025−DSP/tree/main/assignment_1/code

and latex-tikz codes from

https://github.com/Kkuntal990/EE3025−DSP/blob/main/assignment_1/ee18btech11028.tex

1 Problem

1.1. Defining x(n) and h(n),

$$x(n) = \left\{ \underset{\uparrow}{1}, 2, 3, 4, 2, 1 \right\} \tag{1.1.1}$$

$$h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{1.1.2}$$

1.2. Compute X(k), H(k) and y(n) using FFT and IFFT

2 Solution

2.1. input signal x(n)

$$x(n) = \left\{ \underset{\uparrow}{1}, 2, 3, 4, 2, 1 \right\} \tag{2.1.1}$$

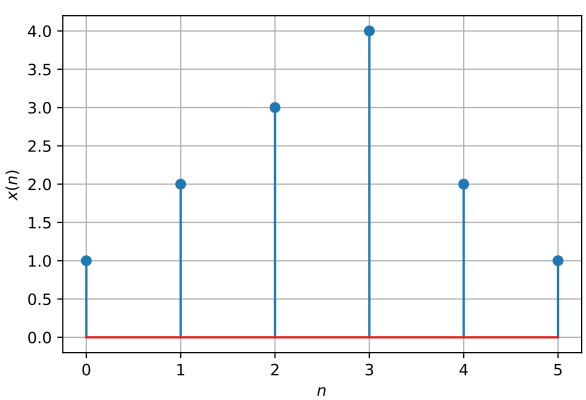

Fig. 2.1: input signal : x(n)

2.2. Impulse Response of the System is

$$h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{2.2.1}$$

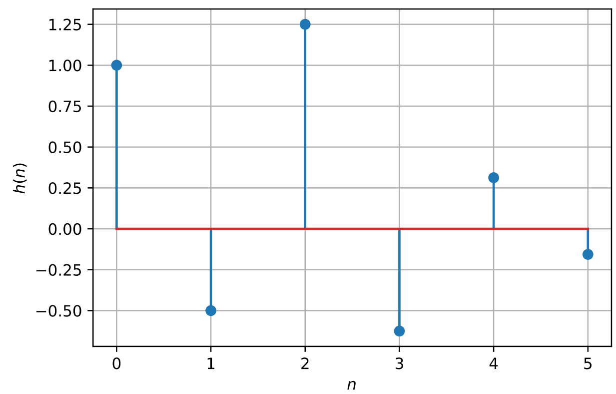

Fig. 2.2: impulse response : h(n)

2.3. FFT of the input signal x(n) is

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}, \quad k = 0, 1 \ldots N-1 \tag{2.3.1}$$

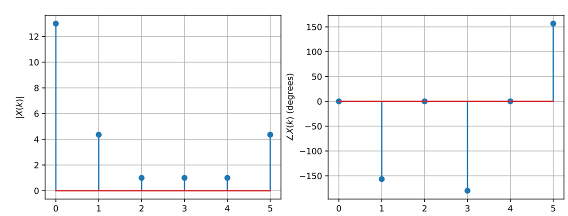

Fig. 2.3: FFT of x(n) : X(k)

2.4. FFT of the impulse response h(n) is

$$H(k) = \sum_{n=0}^{N-1} h(n) e^{-j2\pi kn/N}, \quad k = 0, 1, \ldots, N-1 \tag{2.4.1}$$

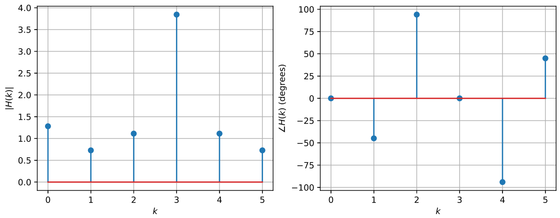

Fig. 2.4: FFT of h(n) : H(k)

2.5. FFT of output Signal y(n) can be computed as

$$Y(k) = X(k)H(k) \tag{2.5.1}$$

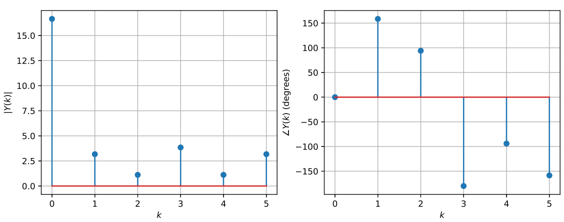

Fig. 2.5: $Y(k) = H(k)X(k)$

2.6. y(n) can be computed by taking IFFT of Y(k)

$$y(n) = \frac{1}{N}\sum_{n=0}^{N-1} Y(k)e^{j2\pi nk/N}, \quad k = 0, 1, \dots, N-1 \tag{2.6.1}$$

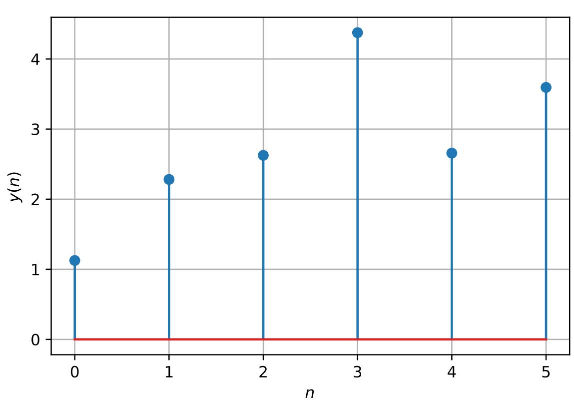

Fig. 2.6: IFFT of Y(k) : y(n)

3 Problem

3.1. Wherever possible, express all the above equations as matrix equations.

4 Solution

4.1. FFT of signal X(n)

$$X(k) \triangleq W_N^{nk} x(n), \quad k = 0, 1, \dots, N-1 \tag{4.1.1}$$

where $W_N^{nk} = e^{-j2\pi kn/N}$ which can be ex– pressed as:

$$\begin{bmatrix} X(0)\\X(1)\\X(2)\\X(3)\\X(4)\\X(5)\end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1\\ 1 & W_N^1 & W_N^2 & W_N^3 & W_N^4 & W_N^5\\ 1 & W_N^2 & W_N^4 & W_N^6 & W_N^8 & W_N^{10}\\ 1 & W_N^3 & W_N^6 & W_N^9 & W_N^{12} & W_N^{15}\\ 1 & W_N^4 & W_N^8 & W_N^{12} & W_N^{16} & W_N^{20}\\ 1 & W_N^5 & W_N^{10} & W_N^{15} & W_N^{20} & W_N^{25}\end{bmatrix} \begin{bmatrix} x(0)\\x(1)\\x(2)\\x(3)\\x(4)\\x(5)\end{bmatrix} \tag{4.1.2}$$

$$\implies \begin{bmatrix} X(0)\\X(1)\\X(2)\\X(3)\\X(4)\\X(5)\end{bmatrix} = \begin{bmatrix} 1+2+3+4+2+1\\ 1+(2)e^{-j\pi/3}+...+(1)e^{-j5\pi/3}\\ 1+(2)e^{-2j\pi/3}+...+(1)(e^{-2j5\pi/3}\\ 1+(2)e^{-3j\pi/3}+...+(1)e^{-3j5\pi/3}\\ 1+(2)e^{-4j\pi/3}+...+(1)e^{-4j5\pi/3}\\ 1+(2)e^{-5j\pi/3}+...+(1)e^{-5j5\pi/3}\end{bmatrix} \tag{4.1.3}$$

On solving,

$$\implies \begin{bmatrix} X(0)\\X(1)\\X(2)\\X(3)\\X(4)\\X(5)\end{bmatrix} = \begin{bmatrix} 13+0j\\ -4-1.732j\\ 1+0j\\ -1+0j\\ 1+0j\\ -4+1.732j\end{bmatrix} \tag{4.1.4}$$

4.2. Similarly,

$$\begin{bmatrix} H(0)\\H(1)\\H(2)\\H(3)\\H(4)\\H(5)\end{bmatrix} = \begin{bmatrix} h(0)+h(1)+h(2)+h(3)+h(4)+h(5)\\ h(0)+h(1)e^{-j\pi/3}+...+h(5)e^{-j5\pi/3}\\ h(0)+h(1)e^{-2j\pi/3}+...+h(5)e^{-2j5\pi/3}\\ h(0)+h(1)e^{-3j\pi/3}+...+h(5)e^{-3j5\pi/3}\\ h(0)+h(1)e^{-4j\pi/3}+...+h(5)e^{-4j5\pi/3}\\ h(0)+h(1)e^{-5j\pi/3}+...+h(5)e^{-5j5\pi/3}\end{bmatrix} \tag{4.2.1}$$

On solving,

$$\implies \begin{bmatrix} H(0)\\ H(1)\\ H(2)\\ H(3)\\ H(4)\\ H(5)\end{bmatrix} = \begin{bmatrix} 1.28125 + 0j,\\ 0.51625 - 0.5141875j,\\ -0.078125 + 1.1095625j,\\ 3.84375 + 0j,\\ -0.071825 - 1.1095625j.\\ 0.515625 + 0.5141875j \end{bmatrix} \tag{4.2.2}$$

4.3. Compute $Y(k)$ using Eq (2.5.1)

$$\begin{bmatrix} Y(0)\\ Y(1)\\ Y(2)\\ Y(3)\\ Y(4)\\ Y(5)\end{bmatrix} = \begin{bmatrix} X(0)\cdot H(0)\\ X(1)\cdot H(1)\\ X(2)\cdot H(2)\\ X(3)\cdot H(3)\\ X(4)\cdot H(4)\\ X(5)\cdot H(5)\end{bmatrix} \tag{4.3.1}$$

Solving,

$$\begin{bmatrix} Y(0)\\ Y(1)\\ Y(2)\\ Y(3)\\ Y(4)\\ Y(5)\end{bmatrix} = \begin{bmatrix} 16.6562 + 0j\\ -2.95312 + 1.16372j\\ -0.07812 + 1.10959j\\ -3.84375 - 9.27556j\\ -0.07812 - 1.10959j\\ -2.95312 - 1.16372j\end{bmatrix} \tag{4.3.2}$$

4.4.

$$y(n) \triangleq (W_N^{nk})^* Y(k), \quad n = 0, 1, \ldots, N-1 \tag{4.4.1}$$

where $(W_N^{nk})^*$ is conjugate of W_N^{nk} from (4.1.1).

$$\begin{bmatrix} y(0)\\ y(1)\\ y(2)\\ y(3)\\ y(4)\\ y(5)\end{bmatrix} = \begin{bmatrix} 1.125 + 0j\\ 2.28125071 + 0j\\ 2.6250019 - 1.11022302\times 10^{-16}j\\ 4.37499667 - 1.47104551\times 10^{-15}j\\ 2.6562481 + 6.10622664\times 10^{-16}j\\ 3.59375262 - 1.60982339\times 10^{-15}j\end{bmatrix} \tag{4.4.2}$$

4.5. Properties :

a) symmetry property :

$$W_N^{k+N/2} = -W_N^k$$

b) Periodicity property :

$$W_N^{k+N} = W_N^k$$

c)

$$W_N^2 = W_{N/2}$$

4.6. Using properties to derive FFT from DFT :

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}, \quad k = 0, 1, \ldots, N-1 \tag{4.6.1}$$

$$= \sum_{n=even} x(n) W_N^{kn} + \sum_{n=odd} x(n) W_N^{kn} \tag{4.6.2}$$

$$= \sum_{m=0}^{2} x(2m) W_N^{2mk} + \sum_{m=0}^{2} x(2m+1) W_N^{(2m+1)k} \tag{4.6.3}$$

using property c, we get,

$$X(k) = \sum_{m=0}^{2} x(2m) W_{N/2}^{mk} + W_N^k \sum_{m=0}^{2} x(2m+1) W_{N/2}^{mk} \tag{4.6.4}$$

$$= X_1(k) + W_N^k X_2(k) \tag{4.6.5}$$

4.7.
- X_1(k) and X_2(k) are 3 point DFTs of x(2m) and x(2m+1) , m=0,1,2.
- X_1(k) and X_2(k) are periodic, Hence X_1(k+3) = X_1(k) and X_2(k+3) = X_1(k).
- By performing this step once we can see that number of operations have been reduced from N^2 to $\frac{N^2}{2}$.

4.8. Using the above properties recursively we have implemented radix-2 Fast-Fourier transform algorithm.

Algorithm	t($N = 128$)	t($N = 2048$)
DTFT	33.2 ms	7.36 s
FFT	1.54 ms	27.6 ms

- $t(N)$ corresponds to average time of execution for sample size of N.
- We can observe that as we increase N, the difference in execution times is drastically increasing.
- From our implementation of radix-2 FFT we can see that complexity is reduced from $O(n^2)$ to $O(n \log n)$

4.9. Taking an example of 8-point function,

$$x(n) = \left\{\underset{\uparrow}{1}, 2, 3, 4, 2, 1, 0, 0\right\} \tag{4.9.1}$$

We know that from 4.1.1,

$$X(k) \triangleq W_N^{nk} x(n), \quad k = 0, 1, \ldots, N-1 \tag{4.9.2}$$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \\ X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{bmatrix} W_8^0 W_8^0 \ W_8^0 \ W_8^0 \ W_8^0 \ W_8^0 \ W_8^0 \ W_8^0 \\ W_8^0 W_8^1 \ W_8^2 \ W_8^3 \ W_8^4 \ W_8^5 \ W_8^6 \ W_8^7 \\ W_8^0 W_8^2 \ W_8^4 \ W_8^6 \ W_8^8 \ W_8^{10} W_8^{12} W_8^{14} \\ W_8^0 W_8^3 \ W_8^6 \ W_8^9 \ W_8^{12} W_8^{15} W_8^{18} W_8^{21} \\ W_8^0 W_8^4 \ W_8^8 \ W_8^{12} W_8^{16} W_8^{20} W_8^{24} W_8^{28} \\ W_8^0 W_8^5 W_8^{10} W_8^{15} W_8^{20} W_8^{25} W_8^{30} W_8^{35} \\ W_8^0 W_8^6 W_8^{12} W_8^{18} W_8^{24} W_8^{30} W_8^{36} W_8^{42} \\ W_8^0 W_8^7 W_8^{14} W_8^{21} W_8^{28} W_8^{35} W_8^{42} W_8^{49} \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \\ x(5) \\ x(6) \\ x(7) \end{bmatrix} \tag{4.9.3}$$

$$\implies \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \\ X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{bmatrix} 13 \\ -3.121 - 6.535j \\ 1j \\ 1.121 - 0.535j \\ -1 \\ 1.121 + 0.535j \\ -1j \\ -3.121 + 6.535j \end{bmatrix} \tag{4.9.4}$$

$$\begin{bmatrix} H(0) \\ H(1) \\ H(2) \\ H(3) \\ H(4) \\ H(5) \\ H(6) \\ H(7) \end{bmatrix} = \begin{bmatrix} W_8^0 W_8^0 \ W_8^0 \ W_8^0 \ W_8^0 \ W_8^0 \ W_8^0 \ W_8^0 \\ W_8^0 W_8^1 \ W_8^2 \ W_8^3 \ W_8^4 \ W_8^5 \ W_8^6 \ W_8^7 \\ W_8^0 W_8^2 \ W_8^4 \ W_8^6 \ W_8^8 \ W_8^{10} W_8^{12} W_8^{14} \\ W_8^0 W_8^3 \ W_8^6 \ W_8^9 \ W_8^{12} W_8^{15} W_8^{18} W_8^{21} \\ W_8^0 W_8^4 \ W_8^8 \ W_8^{12} W_8^{16} W_8^{20} W_8^{24} W_8^{28} \\ W_8^0 W_8^5 W_8^{10} W_8^{15} W_8^{20} W_8^{25} W_8^{30} W_8^{35} \\ W_8^0 W_8^6 W_8^{12} W_8^{18} W_8^{24} W_8^{30} W_8^{36} W_8^{42} \\ W_8^0 W_8^7 W_8^{14} W_8^{21} W_8^{28} W_8^{35} W_8^{42} W_8^{49} \end{bmatrix} \begin{bmatrix} 1 \\ -0.5 \\ 1.25 \\ -0.65 \\ 0.3125 \\ -0.15625 \\ 0.078125 \\ -0.0390625 \end{bmatrix} \tag{4.9.5}$$

$$\implies \begin{bmatrix} H(0) \\ H(1) \\ H(2) \\ H(3) \\ H(4) \\ H(5) \\ H(6) \\ H(7) \end{bmatrix} = \begin{bmatrix} 1.32 \\ 0.858 - 0.514j \\ -0.015 - 0.007j \\ 0.516 + 1.829j \\ 3.96 \\ 0.516 - 1.829j \\ -0.015 + 0.007j \\ 0.858 + 0.514j \end{bmatrix} \tag{4.9.6}$$

So,

$$\begin{bmatrix} Y(0) \\ Y(1) \\ Y(2) \\ Y(3) \\ Y(4) \\ Y(5) \\ Y(6) \\ Y(7) \end{bmatrix} = \begin{bmatrix} X(0) \cdot H(0) \\ X(1) \cdot H(1) \\ X(2) \cdot H(2) \\ X(3) \cdot H(3) \\ X(4) \cdot H(4) \\ X(5) \cdot H(5) \\ X(6) \cdot H(6) \\ X(7) \cdot H(7) \end{bmatrix} \tag{4.9.7}$$

Solving,

$$\implies \begin{bmatrix} Y(0) \\ Y(1) \\ Y(2) \\ Y(3) \\ Y(4) \\ Y(5) \\ Y(6) \\ Y(7) \end{bmatrix} = \begin{bmatrix} 17.16 \\ -6.04 - 4j \\ -0.007 - 0.015j \\ 1.55 + 1.77j \\ -3.96 \\ 1.55 - 1.77j \\ 0.007 + 0.015j \\ -6.04 + 4j \end{bmatrix} \tag{4.9.8}$$

From 4.4.1

$$\implies \begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ y(4) \\ y(5) \\ y(6) \\ y(7) \end{bmatrix} = \begin{bmatrix} 0.53125 \\ 1.69 \\ 3.09 \\ 4.375 \\ 2.773 \\ 3.593 \\ 0.203 \\ 0.8984 \end{bmatrix} \tag{4.9.9}$$