EE7330: Network Information Theory **2021**

Lecture Notes 6: Review of information theoretic quantities

Instructor: Shashank Vatedka *Scribe: Kuntal Kokate*

Disclaimer: *These notes have not been subjected to the usual scrutiny reserved for formal publications. Please email the course instructor in case of any errors.*

6.1 Data processing inequality

If $X - Y - Z$ **forms a Markov chain,**

- $I(X;Y) \geqslant I(X;Z)$
- $I(Y;Z) \geqslant I(X;Z)$
- $Pr_{Z|X,Y}(z|x,y) = Pr_{Z|Y}(z,y)$
- $Pr_{X|Y,Z}(x|y,z) = Pr_{X|Y}(x|y)$

6.2 Chain rule of entropy

- $$\begin{aligned} H(X;Y) &= H(X) + H(Y|X) && (6.1)\\ &= H(Y) + H(X|Y) && (6.2) \end{aligned}$$

- $$H(X_1, X_2, ..., X_n) = \sum_{i=1}^{\infty} H(X_i|X_1, X_2..., X_{i-1}) \tag{6.3}$$

6.3 Chain rule of mutual information

- $I(X_1, X_2; Y) = I(X_1; Y) + I(X_2; Y|X_1)$

6.4 Degraded Channels

$\mathbf{q}_{Y|X}$ is stochastically degraded w.r.t. $\mathbf{p}_{Z|X}$ if we can find $\mathbf{r}_{Y|Z}$ such that,

$$\mathbf{q}_{Y|X}(y|x) = \sum_{z \epsilon \mathbb{Z}} \mathbf{r}_{Y|Z}(y|z)\mathbf{p}_{Z|X}(z|x)$$

If $C_{DMC1} \geqslant C_{DMC2} \implies$ channel 2 is degraded wrt channel 1.

HW If $I(X;Z) \geqslant I(X;Y)$, show that $C_{DMC1} \geqslant C_{DMC2}$
Solution:

$$C = max_{\mathbf{p}_x} I(X;Y) \tag{6.4}$$

$$\implies max_{\mathbf{p}_x} I(X;Z) \geqslant max_{\mathbf{p}_x} I(X;Y) \tag{6.5}$$

$$\implies C_{DMC1} \geqslant C_{DMC2} \tag{6.6}$$

6.4.1 Capacity of channels

- $C_{BSC}(p) = 1 - H_2(p)$
- $C_{BEC}(p) = 1 - p$
- $C_{AWGN}(p, \sigma^2) = \frac{1}{2} log(1 + \frac{p}{\sigma^2})$

6.4.2 Examples

Problem 1. Can BSC(0.1) be degraded wrt BSC(0.2)
Solution

$$C_{BSC(0.1)} \geqslant C_{BSC(0.2)}$$

Therefore, BSC(0.2) is degraded wrt BSC(0.1)

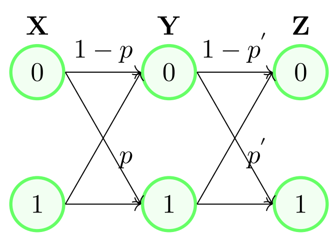

Figure 6.1: Combination of two BSC's

We can write the combination of two BSCs as one BSC($p * p'$).

$$p * p' = p(1 - p') + p'(1 - p) \tag{6.7}$$

$$\implies 0.1 = p'(1 - 0.2) + (1 - p')0.2 \tag{6.8}$$

$$\implies p' = 0.125 \tag{6.9}$$

Therefore we can say that, BSC(0.2) is concantenation of BSC(0.1) and BSC(0.125).

Problem 2. BEC(0.1) vs BEC(0.2)
Solution

$$C_{BEC(0.1)} \geqslant C_{BEC(0.2)}$$

Therefore, BEC(0.2) is degraded wrt BEC(0.1)

$$p'' = (1 - p)p' + p \tag{6.10}$$

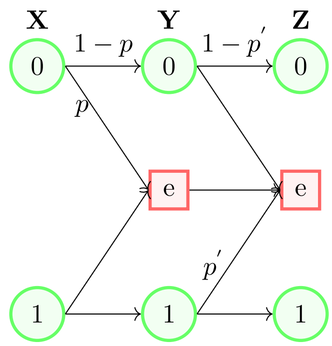

Figure 6.2: Combination of two BEC's

$$0.2 = (1-0.1)p^{'} + 0.1 \tag{6.11}$$

$$\implies p^{'} = \frac{1}{9} \tag{6.12}$$

Problem 3. BSC(0.1) vs BEC(0.1)
Solution

$$C_{BEC(0.1)} = 0.9 \tag{6.13}$$

$$C_{BSC(0.1)} = 0.53 \tag{6.14}$$

Therefore, BSC(0.1) is degraded wrt BEC(0.1).

Problem 4. BSC(0.01) vs BEC(0.5)
Solution

$$C_{BEC(0.5)} = 0.5 \tag{6.15}$$

$$C_{BSC(0.01)} \approx 1 \tag{6.16}$$

Therefore, BEC(0.5) is degraded wrt BSC(0.01).

6.4.3 AWGN channels

Let $\sigma_1^2 \geqslant \sigma_2^2$, then $C_1 \geqslant C_2$. Therefore channel 2 is degraded wrt channel 1.
There exist some σ_3^2 such that,

$$\sigma_1^2 = \sigma_2^2 + \sigma_3^2 \tag{6.17}$$

6.4.4 Uniform channels

Let $\alpha_1 \geqslant \alpha_2$, then $C_2 \geqslant C_1$. Therefore channel 1 is degraded wrt channel 2.
There exist some α_3 such that,

$$\alpha_1 = \alpha_2 + \alpha_3 \tag{6.18}$$