Basic Mathematics for Differential Kinematics

Skew Symmetric Matrices (เมทริกสมมาตรเสมือน)

นิยาม : เมตริกซ์ S ที่มีขนาด $n \times n$ เป็น skew symmetric ถ้าและต่อเมื่อ

$$S + S^T = 0$$

$$s_{ij} + s_{ji} = 0$$

$$s_{ii} = 0$$

$$\begin{bmatrix} 0 & s_{12} & \cdots & \\ s_{21} & 0 & & \\ \vdots & & \ddots & \\ & & & \end{bmatrix}$$

Observation :

$$\vec{a} \in \mathbb{R}^3 : \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \qquad \vec{b} \in \mathbb{R}^3 : \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

$$\vec{a} \times \vec{b}$$

$$\vec{a} \times \vec{b} = \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}}_{\text{skew symmetric}} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

Skew Operator ($S(\vec{v})$, $\tilde{\vec{v}}$, $[\vec{v}]_\times$)

เป็น function ที่ input เป็น vector 3×1 ($\vec{v} \in \mathbb{R}^3$)

ex. $\vec{v} = [v_x \; v_y \; v_z]^T$

$$\Rightarrow S(\vec{v}) = \begin{bmatrix} 0 & -v_z & v_y \\ v_z & 0 & -v_x \\ -v_y & v_x & 0 \end{bmatrix}$$

$$\hat{x} = [1 \; 0 \; 0]^T \qquad S(\hat{x}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\hat{y} = [0 \; 1 \; 0]^T \qquad S(\hat{y}) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$\hat{z} = [0 \; 0 \; 1]^T \qquad S(\hat{z}) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Skew Operator Properties

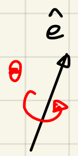
1. $S(\alpha\vec{a} + \beta\vec{b}) = \alpha S(\vec{a}) + \beta S(\vec{b})$

2. $S(\vec{a})\vec{b} = \vec{a} \times \vec{b}$
   $S(\vec{b})^T\vec{a} = \vec{a} \times \vec{b}$

3. $R \in SO_3$; $R(\vec{a} \times \vec{b}) = (R\vec{a}) \times (R\vec{b})$

4. $R\, S(\vec{a})\, R^T = S(R\vec{a})$

Rotation about fixed axis



$$R(\theta) \cdot R(\theta)^T = \mathbb{I}_3$$

$$\frac{d}{d\theta}\left(R(\theta) \cdot (R(\theta))^T\right) = \frac{d(\mathbb{I}_3)}{d\theta}$$

$$R(\theta)\frac{dR(\theta)^T}{d\theta} + R(\theta)^T\frac{dR(\theta)}{d\theta} = 0_{3\times3}$$

$(AB)^T = A^TB^T$

$$R\frac{dR^T}{d\theta} + R^T\frac{dR}{d\theta} = 0_{3\times3}$$

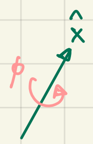
$$\left(R^T\frac{dR}{d\theta}\right)^T + R^T\frac{dR}{d\theta} = 0_{3\times3}$$

$$A^T + A = 0_{3\times3} \Rightarrow A = R^T\frac{dR}{d\theta} \rightarrow (\text{skew-symmetric})$$

$$AR = \cancel{R^T}\frac{dR}{d\theta}\cancel{R}$$

$$\boxed{\frac{dR}{d\theta} = AR}$$

ex.



$\hat{x} = [1 \ 0 \ 0]^T$

$$R_x(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\phi & -s_\phi \\ 0 & s_\phi & c_\phi \end{bmatrix}$$

$$A = \frac{dR}{d\phi}R^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -s_\phi & -c_\phi \\ 0 & c_\phi & -s_\phi \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\phi & s_\phi \\ 0 & -s_\phi & c_\phi \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = S(\hat{x})$$

แกนใดๆ

$\therefore$ $\hat{e} = \hat{x}$ $A = S(\hat{x})$

$\hat{e} = \hat{y}$ $A = S(\hat{y})$

$\hat{e} = \hat{z}$ $A = S(\hat{z})$

$$A = S(\hat{e})$$

$$\frac{dR_{\hat{e}}(\delta)}{d\delta} = AR = S(\hat{e})R_{\hat{e}}(\delta)$$

ขึ้นอยู่กับเวลาใดๆ

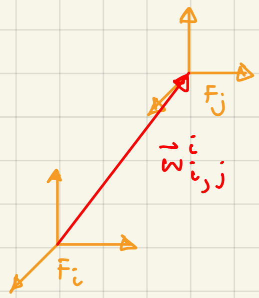
$$\frac{dR(t)}{dt} = A(t)\,R(t) = S(\vec{\omega}(t))\cdot R(t)$$

$$\frac{dR}{dt} = S(\vec{\omega})\cdot R$$

↳ ความเร็วในการหมุนต้องเกี่ยวกับ Rotation Metric

$$R^i_j \longrightarrow \vec{\omega}_{i,j}$$

Full : $\dot{R}^i_j = S(\vec{\omega}^i_{i,j})\, R^i_j$



Transformation of angular velocity ($\vec{\omega}^{\,i}_{i,j}$)

$$\frac{d}{ds}(u\cdot v) = \frac{du}{ds}\cdot v + u\cdot\frac{dv}{ds}$$

$$R^0_2 = R^0_1 R^1_2$$

$$\dot{R}^0_2 = \dot{R}^0_1 R^1_2 + R^0_1 \dot{R}^1_2$$

$$S(\vec{\omega}^0_{0,2}) \cancel{R^0_2 R^{0T}_2} = S(\vec{\omega}^0_{0,1}) R^0_1 \cancel{R^1_2 R^{0T}_2} + R^0_1 S(\vec{\omega}^1_{1,2}) R^1_2 R^{0T}_2$$

$$R^2_0 \qquad R^1_0 \rightarrow R^{0T}_1$$

$$S(\vec{\omega}^0_{0,2}) = S(\vec{\omega}^0_{0,1}) + R^0_1 S(\vec{\omega}^1_{1,2}) R^{0T}_1$$

$$S(\vec{\omega}^0_{0,2}) = S(\vec{\omega}^0_{0,1}) + S(R^0_1 \vec{\omega}^1_{1,2})$$

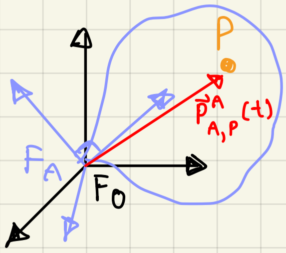
$$\boxed{\vec{\omega}^0_{0,2} = \vec{\omega}^0_{0,1} + R^0_1 \vec{\omega}^1_{1,2}}$$

Ref. frame ยังไม่ตรงกัน

จึงคูณ R เพื่อแปลงให้อยู่ frame เดียวกัน

การบวกกัน Frame of Ref. ต้องตรงกัน

Linear Velocity



$$\vec{p}^{\,0}_{A,P} = R^0_A \, \vec{p}^{A}_{A,P}(t)$$

$$\dot{\vec{p}}^{\,0}_{A,P} = \dot{R}^0_A \, \vec{p}^{A}_{A,P}(t) + R^0_A \, \dot{\vec{p}}^{A}_{A,P}(t)$$

$$= S(\vec{w}^{\,0}_{0,A}) R^0_A \, \vec{p}^{A}_{A,P}(t) + R^0_A \, \dot{\vec{p}}^{A}_{A,P}(t)$$

$S(\vec{a})\vec{b} = \vec{a} \times \vec{b}$

$$= (\vec{w}^{\,0}_{0,A} \times R^0_A \vec{p}^{A}_{A,P}) + R^0_A \, \dot{\vec{p}}^{A}_{A,P}$$

$$= (R^0_A \vec{w}^{A}_{0,A} \times R^0_A \vec{p}^{A}_{A,P}) + R^0_A \, \dot{\vec{p}}^{A}_{A,P}$$

$R(\vec{a} \times \vec{b}) = R(\vec{a}) \times R(\vec{b})$

$$= R^0_A (\vec{w}^{A}_{0,A} \times \vec{p}^{A}_{A,P}) + R^0_A \, \dot{\vec{p}}^{A}_{A,P}$$

$$\dot{\vec{p}}^{\,0}_{A,P} = R^0_A (\vec{w}^{A}_{0,A} \times \vec{p}^{A}_{A,P} + \dot{\vec{p}}^{A}_{A,P})$$

Frame เดียวกัน

Frame ใดก็ได้

$$\boxed{\vec{V}^{\,0}_{A,P} = R^0_A \left[\vec{w}^{A}_{0,A} \times \vec{p}^{A}_{B,P} + \vec{V}^{A}_{B,P} \right]}$$

Velocity of Robot's frame

Twist

$$\vec{\xi}^{\,0}_{0,i} = \begin{bmatrix} \vec{w}^{\,0}_{0,i} \\ \hline \vec{V}^{\,0}_{0,i} \end{bmatrix} \in \mathbb{R}^6 \qquad \vec{w}^{\,0}_{0,i} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}, \quad \vec{V}^{\,0}_{0,i} = \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix}$$

FK

IK

$$\dot{\vec{q}} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

joint velocity

$\hat{z}^i_i = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ การหมุนรอบแกน z เมื่อเทียบกับ frame ตัวเอง

ex. $\vec{\omega}^2_{1,2} = \begin{bmatrix} 0 \\ 0 \\ \dot{q}_2 \end{bmatrix} = \dot{q}_2 \hat{z}^2_2$ → revolute joint, prismatic joint

$\rho_i \in \{0, 1\}$

$\rho_i = 1$ revolute joint

$\rho_i = 0$ prismatic joint

$$\vec{\omega}^i_{i-1,i} = \rho_i \dot{q}_i \hat{z}^i_i \quad \begin{cases} \vec{\omega}^i_{i-1,i} = \dot{q}_i \hat{z}^i_i \\ \vec{\omega}^i_{i-1,i} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{cases}$$

$$\boxed{\vec{\omega}^i_{i-1,i} = \rho_i \dot{q}_i \hat{z}^i_i}$$

Angular Velocity of frame i

$$\vec{\omega}^0_{0,i} = \sum_{j=1}^{i} \rho_j \dot{q}_j \hat{z}^0_j = \rho_1 \hat{z}_1 \dot{q}_1 + \rho_2 \hat{z}_2 \dot{q}_2 + \dots + \rho_i \hat{z}_i \dot{q}_i$$

$$= \left[\rho_1 \hat{z}^0_1 \;\vdots\; \rho_2 \hat{z}^0_2 \;\vdots\; \cdots \;\vdots\; \rho_i \hat{z}^0_i \right] \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_i \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \dot{q}_{i+1} + \dots + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \dot{q}_n$$

$$= \underbrace{\left[\rho_1 \hat{z}^0_1 \;\vdots\; \rho_2 \hat{z}^0_2 \;\vdots\; \cdots \;\vdots\; \rho_i \hat{z}^0_i \;\vdots\; \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \;\vdots\; \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \;\vdots\; \cdots \;\vdots\; \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \right]}_{J^i_\omega(\vec{q})} \underbrace{\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_i \\ \dot{q}_{i+1} \\ \dot{q}_{i+2} \\ \vdots \\ \dot{q}_n \end{bmatrix}}_{\dot{\vec{q}}}$$

$J^i_\omega(\vec{q})$: angular Jacobian Matrix of frame i

$$\vec{\omega}^0_{0,i} = J^i_\omega(\vec{q}) \cdot \dot{\vec{q}}$$

$$J^i_{\omega,j} = \left[J^i_{\omega,1} \;\vdots\; J^i_{\omega,2} \;\vdots\; \cdots \;\vdots\; J^i_{\omega,n} \right] \qquad J^i_{\omega,j} = \begin{cases} \rho_i \hat{z}^0_j \; ; \text{ if } j \le i \\ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \; ; \text{ if } j > i \end{cases}$$

Linear Velocity of Frame i

$$\vec{V}^0_{0,i} = \sum_{j=1}^{i} J_{v,j}\,\dot{q}_j$$

$$J_{v,j} = (1-\rho_j)\hat{z}^0_j + \rho_j \hat{z}^0_j \times (\vec{p}^{\,0}_{0,i} - \vec{p}^{\,0}_{0,j})$$

$$\vec{V}^0_{0,i} = J^i_v(\vec{q}) \cdot \vec{\dot{q}}^0$$

Linear Jacobian

$$J^i_v = [\, J^i_{v,1} \;\vdots\; J^i_{v,2} \;\vdots\; \cdots \;\vdots\; J^i_{v,n} \,]$$

$$J^i_{w,j} = \begin{cases} (1-\rho_j)\hat{z}^0_j + \rho_j \hat{z}^0_j \times (\vec{p}^{\,0}_{0,i} - \vec{p}^{\,0}_{0,j}) \;;\; \text{if } j \leq i \\ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \;;\; \text{if } j > i \end{cases}$$

Reduced Jacobian Matrix & Singularity

ex. 2 DOF RR Robot

$$J^e(\vec{q}) = \begin{bmatrix} \cancel{0} & \cancel{0} \\ \cancel{0} & \cancel{0} \\ \cancel{1} & \cancel{1} \\ \hdashline a & b \\ c & d \\ \cancel{0} & \cancel{0} \end{bmatrix} \text{ in } \mathbb{R}^{6\times 2} \rightarrow \mathbb{R}^{2\times 2} \qquad \dot{\vec{x}} = \begin{bmatrix} \cancel{\dot{r}_x} \\ \cancel{\dot{r}_y} \\ \cancel{\dot{r}_z} \\ \dot{p}_x \\ \dot{p}_y \\ \cancel{\dot{p}_z} \end{bmatrix}$$

$$\dot{\vec{x}} = J^*(\vec{q})\,\dot{\vec{q}} \;\Rightarrow\; \dot{\vec{q}} = J^{-1}(\vec{q})\,\dot{\vec{x}}$$

$$J^{-1} = \frac{1}{\det(J)} \cdot \operatorname{adj}(J)$$

↳ $\det(J^*(\vec{q}))$ must not equal 0

Singularity : สภาวะที่ไม่สามารถควบคุมหุ่นยนต์ได้

Static Analysis

$$\underset{\text{wrench}}{\vec{W}^{0}} = \begin{bmatrix} \vec{n}^{0} \\ \hline \vec{f}^{0} \end{bmatrix} \in \mathbb{R}^{6}$$

$\vec{n}$: moment → $[n_x \ n_y \ n_z]^T$

$\vec{f}$: force → $[f_x \ f_y \ f_z]^T$

$$\underset{\text{joint effort}}{\vec{\tau}} = \begin{bmatrix} \tau_1 \\ \tau_2 \\ \vdots \\ \tau_n \end{bmatrix} \in \mathbb{R}^{n}$$

$W = \vec{w} \cdot \vec{x}$

$W = \vec{\tau} \cdot \vec{q}$

$$\underset{[n\times 1]}{\vec{\tau}} = \underset{[n\times 6]}{J^{e}(\vec{q})^{T}} \ \underset{[6\times 1]}{\vec{w}_e^{0}}$$

$$\vec{w}_e^{0} = \begin{array}{c} \left[\begin{array}{c} \underbrace{\vec{r}_{e,cm}^{\,0} \times \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix}}_{\vec{r} \times \vec{f} \text{ (moment)}} \\ 0 \\ 0 \\ mg \end{array} \right] \end{array} \begin{array}{l} n_x \\ n_y \\ n_z \\ f_x \\ f_y \\ f_z \end{array}$$

$$\boxed{\vec{\tau} = J^{e}(\vec{q}) \, \vec{w}_e^{0}}$$ → ต้อง ref. frame เดียวกัน

$$\vec{\tau}_1 = J^{1}(\vec{q}) \begin{bmatrix} \vec{r}_{1,cm}^{\,0} \times \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix} \\ \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix} \end{bmatrix}$$

$$\vec{\tau}_2 = J^{2}(\vec{q}) \begin{bmatrix} \vec{r}_{2,cm}^{\,0} \times \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix} \\ \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix} \end{bmatrix}$$

$$\vdots$$