Image Processing
lab 4

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Exercise 1 – Skeletons

The morphological skeleton of an image X with structuring element B is defined as:

$$SK(X) = \bigcup_{k=0}^{K} S_k(X)$$

where

$$S_k(X) = (X \ominus kB) - (X \ominus kB) \circ B$$

and $K \in \mathbb{N}$ such that

$$K = max\{k|(X \ominus kB) \neq \emptyset\}.$$

In addition $(X \ominus kB)$ is defined as follows:

$$(X \ominus kB) = \begin{cases} X & \text{if } k = 0 \\ ((\ldots(X \ominus B) \ominus B) \ominus \ldots) \ominus B) & \text{otherwise} \end{cases}$$

Note that in the second case $\ominus B$ is applied k times to X.

a. Let set $B = \hat{B} = \{0\}$ be given.

Claim. $SK(X) = \emptyset$

Proof. Note that for all $z \in \mathbb{Z}^2$ the following holds:

$$(B)_z = \{c|c = b + z, b \in B\} = \{c|c = z\} = \{z\}$$

Then we have by the definition of the erosion of X by B that

$$\begin{aligned} X \ominus B &= \{z|(B)_z \subseteq X\} \\ &= \{z|\{z\} \subseteq X\} \\ &= X. \end{aligned} \tag{1}$$

Similarly we have by the definition of the dilution of X by B that

$$\begin{aligned} X \oplus B &= \{z|(\hat{B})_z \cap X \neq \emptyset\} \\ &= \{z|\{z\} \cap X \neq \emptyset\} \\ &= X. \end{aligned} \tag{2}$$

The skeleton of image X with structuring element B is:

$$\begin{aligned}
SK(X) &= \bigcup_{k=0}^{K} S_k(X) \\
&= \bigcup_{k=0}^{K} ((X \ominus kB) - (X \ominus kB) \circ B) \\
&= \bigcup_{k=0}^{K} ((X \ominus kB) - ((X \ominus kB) \ominus B) \oplus B)
\end{aligned}$$

Now if we apply the definition of $(X \ominus kB)$ and the results of equation 1 and 2 we get:

$$\begin{aligned}
SK(X) &= \bigcup_{k=0}^{K} ((X \ominus kB) - ((X \ominus kB) \ominus B) \oplus B) \\
&= \bigcup_{k=0}^{K} (X - X) \\
&= \bigcup_{k=0}^{K} \emptyset \\
&= \emptyset
\end{aligned}$$

∎

b. Let $X \ominus B = \emptyset$ be given.

Claim. *SK(X) = X*

Proof. Consider the definition of $(X \ominus kB)$:

$$(X \ominus kB) = \begin{cases} X & \text{if } k = 0 \\ ((\dots(X \ominus B) \ominus B) \ominus \dots) \ominus B) & \text{otherwise} \end{cases}$$

Note that $X \ominus B = \emptyset$. In addition for $X = \emptyset$ we have that every other erosion in the definition above is

$$X \ominus B = \emptyset \ominus B = \{z | (B)_z \subseteq \emptyset\} = \emptyset. \tag{3}$$

Hence the previous definition $X \ominus kB$ is equal to:

$$(X \ominus kB) = \begin{cases} X & \text{if } k = 0 \\ \emptyset & \text{otherwise} \end{cases} \tag{4}$$

Now the skeleton of image X with structuring element B is:

$$
\begin{aligned}
SK(X) &= \bigcup_{k=0}^{K} S_k(X) \\
&= \bigcup_{k=0}^{K} ((X \ominus kB) - (X \ominus kB) \circ B) \\
&= \bigcup_{k=0}^{K} ((X \ominus kB) - ((X \ominus kB) \ominus B) \oplus B) \\
&= ((X \ominus 0B) - ((X \ominus 0B) \ominus B) \oplus B) \cup \bigcup_{k=1}^{K} ((X \ominus kB) - ((X \ominus kB) \ominus B) \oplus B)
\end{aligned}
$$

If we use the equation 3, 4 and the definition of $X \oplus B$ we get:

$$
\begin{aligned}
SK(X) &= (X - (X \ominus B) \oplus B) \cup \bigcup_{k=1}^{K} (\emptyset - (\emptyset \ominus B) \oplus B) \\
&= (X - (\emptyset \oplus B)) \cup \bigcup_{k=1}^{K} (\emptyset - (\emptyset \oplus B)) \\
&= \left(X - \{z|(\hat{B})_z \cap \emptyset \neq \emptyset\}\right) \cup \bigcup_{k=1}^{K} \left(\emptyset - \{z|(\hat{B})_z \cap \emptyset \neq \emptyset\}\right) \\
&= (X - \{z|\emptyset \neq \emptyset\}) \cup \bigcup_{k=1}^{K} (\emptyset - \{z|\emptyset \neq \emptyset\}) \\
&= (X - \emptyset) \cup \bigcup_{k=1}^{K} (\emptyset - \emptyset) \\
&= X \cup \bigcup_{k=1}^{K} \emptyset \\
&= X \cup \emptyset \\
&= X
\end{aligned}
$$

■

c.

```
function skfr = IPskeletondecomp (x, b)

  % create the output matrix
  skfr = zeros(size(x), 'uint8');

  k = 1;
  cur = x;
  while any(any(cur)) == 1
    next = IPerode(cur, b);
    % S_k(), since next dilate b is smaller than cur
```

```
    skel = xor(cur, IPdilate(next, b));
    skfr(skel) = k;
    cur = next;
    k++;
  end
end
```

```
function di = IPdilate (x, b)
  bhat = flip(flip(b, 1), 2);
  di = conv2(x, bhat, "same") > 0;
end
```

```
function er = IPerode (x, b)
  xd1 = size(x, 1);
  xd2 = size(x, 2);
  bd1 = size(b, 1);
  bd2 = size(b, 2);
  % pad the image
  a = zeros(size(x) + 2 * size(b));
  a(bd1+1:(bd1 + xd1), bd2+1:(bd2 + xd2)) = x;
  % based on (A erode B)^c = A^c dilate B^hat
  a = ~(IPdilate(~a, flip(flip(b, 1), 2)));
  % unpad
  er = a(bd1+1:(bd1 + xd1), bd2+1:(bd2 + xd2));
end
```

```
function re = IPskeletonrecon (x, b)

  % create the output matrix
  re = zeros(size(x), 'logical');

  re(x == 1) = 1;
  % extract the skeleton subset counts
  maxk = max(x(x > 0));
  if maxk > 1
    for k = 2:maxk
      tmp = zeros(size(x), 'logical');
      tmp(x == k) = 1;
      for i = 1:(k - 1)
        tmp = IPdilate(tmp, b);
      end
      re = or(re, tmp);
    end
  end
end
```

d.
```
x = imread('../images/nutsbolts.tif');
b = ~~[1 1 1; 1 1 1; 1 1 1];
skel = IPskeletondecomp(x, b);
recon = IPskeletonrecon(skel, b);
imwrite(recon, 'recon.png')
% check to test if any logical element differs, if equal the result
    should be 0
any(any(xor(recon, x)))
```

The result of running this script is as follows:

```
ans = 0.
```

This means that the origional and the resulting image are identical. In addition these images can be seen in Figure 1.

Original image the result of the skeleton decomposition and reconstruction

Figure 1: The nutsbolts image and its skeleton decomposition reconstruction.

Exercise 2 – Grey-scale morphology

a. Based on the equation 9.2-1 in the book, the implementation of IPgerode is rather straightforward:

```
function y = IPgdilate (x, b)
#code based on IPfilter of lab 1 ex 3

# cast to double so nan exists
x = double(x);

#get image dimensions
nr = size(x, 1);
nc = size(x, 2);

#now the shifted matrices can be computed:
x_u = [x(2:nr, :); NaN([1 nc])];
x_d = [NaN([1 nc]); x(1:(nr - 1), :)];
x_l = [x(:, 2:nc) NaN([nr 1])];
x_r = [NaN([nr 1]) x(:, 1:(nc - 1))];
x_ul = [x_u(:, 2:nc) NaN([nr 1])];
x_ur = [NaN([nr 1]) x_u(:, 1:(nc - 1))];
x_dl = [x_d(:, 2:nc) NaN([nr 1])];
x_dr = [NaN([nr 1]) x_d(:, 1:(nc - 1))];

#create a new structuring element with 0's for true and NaN for false
c = zeros(size(b));
c(~b) = nan;

#finally the value can be calculated (NaN's are ignored) by taking
    pairwise maxima of matrices
```

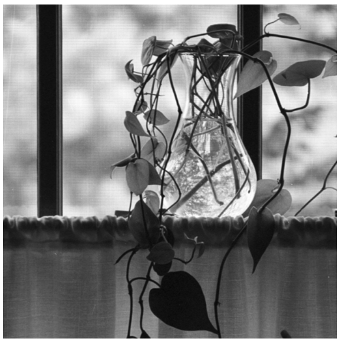
```
y = max(max(max(max(max(max(max(max(
          c(1, 1) + x_dr,  c(1, 2) + x_d), c(1, 3) + x_dl), ...
          c(2, 1) + x_r),   c(2, 2) + x),   c(2, 3) + x_l),  ...
          c(3, 1) + x_ur),  c(3, 2) + x_u), c(3, 3) + x_ul);

# cast the image back to uint8
y = uint8(y);
end
```

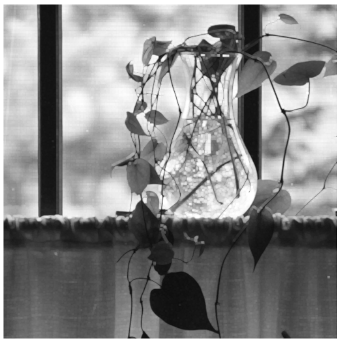
The only trick used here is to use the NaN value to ignore pixels that are outside the picture or not part of the structuring element. This works because $min(NaN, 3) = 3$ and $max(NaN, 3) = 3$. The only problem is that NaN does not exist for uint8's, therefore the image has to be converted to double before the calculations and back afterwards. Also note that maximum/minimum of the elements of the matrices is calculated by taking the the pairwise maxima/minima. The implementation of IPgdilate is based on equation 9.2-2 in the book and the implementation of IPgerode:

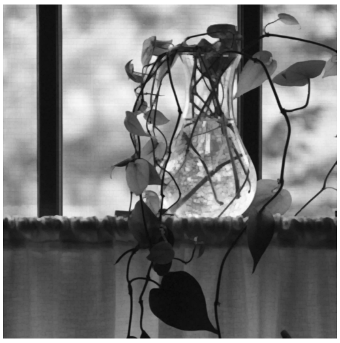
```
function y = IPgerode (x, b)
#code based on IPfilter of lab 1 ex 3

# cast to double so nan exists
x = double(x);

#get image dimensions
nr = size(x, 1);
nc = size(x, 2);

#now the shifted matrices can be computed:
x_u = [x(2:nr, :); NaN([1 nc])];
x_d = [NaN([1 nc]); x(1:(nr - 1), :)];
x_l = [x(:, 2:nc) NaN([nr 1])];
x_r = [NaN([nr 1]) x(:, 1:(nc - 1))];
x_ul = [x_u(:, 2:nc) NaN([nr 1])];
x_ur = [NaN([nr 1]) x_u(:, 1:(nc - 1))];
x_dl = [x_d(:, 2:nc) NaN([nr 1])];
x_dr = [NaN([nr 1]) x_d(:, 1:(nc - 1))];

#create a new structuring element with 0's for true and NaN for false
c = zeros(size(b));
c(~b) = nan;

#finally the value can be calculated (NaN's are ignored) by taking pairwise minima of matrices
y = min(min(min(min(min(min(min(min(
          c(1, 1) + x_dr,  c(1, 2) + x_d), c(1, 3) + x_dl), ...
          c(2, 1) + x_r),   c(2, 2) + x),   c(2, 3) + x_l),  ...
          c(3, 1) + x_ur),  c(3, 2) + x_u), c(3, 3) + x_ul);

# cast the image back to uint8
y = uint8(y);
end
```

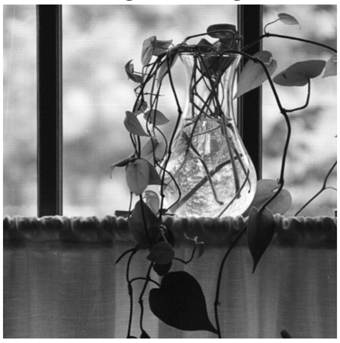
b. The resulting images are shown in figure 2. Like mentioned in example 9.9 of the book the erosion increases the size and intensity of dark features, while decreasing the bright ones. For example the branches of the plant are darker and bigger, while the water in the vase is less bright. This is the result of the minimum operation in the definition of gray-scale erosion. Similarly that example in the dilation bright features are increased in size and intensity, while decreasing the size and intensity of dark features. In this case the branches are thinner and smaller than in the original and the water in the vase is brighter. Also note that smaller features may be more visible by erosion or dilation depending on if they are dark respectivily bright. In addition example 9.10 in the book discusses gray-scale openings and closings. The opening decreases the intensity of bright features smaller than the structuring element, while keeping the dark ones relative the same. Note for example the decreased intensity of the spots in the vase. The closing decreased the intensity of dark features smaller than the structuring element, while keeping the bright ones. Again the dots in the vase show this.


The erosion


Original image


The dilation


The opening


Original image

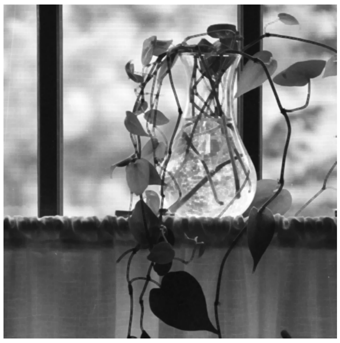

The closing

Figure 2: The grayscale dilation, erosion, opening and closing of vase.tif with a flat box structuring element.

Exercise 3 – Classification

Task distribution

ex1	design	implementation	answers questions	writing report
Klaas	60%	90%	n.a.	50%
Jan	40%	10%	n.a.	50%

ex2	design	implementation	answers questions	writing report
Klaas	50%	30%	25%	25%
Jan	50%	70%	75%	75%

ex3	design	implementation	answers questions	writing report
Klaas	50%	75%	50%	75%
Jan	50%	25%	50%	25%