Systems and Laws

1.1 Simple Dynamical Systems and the state-space

In Physics, generally we talk about Systems. From simple systems with a single particle that obeys a simple dynamical law, up to complex systems in multiple dimensions with an infinite amount of particles, or N-Particles.

If you know everything about a system at some instant of time, and you know the equations that govern how the systems changes, then you can predict its future or past state. This is a feature of all classical systems. (deterministic and reversible)

Every Classical Laws of Physics and every classical system are reversible and deterministic!

System: A collection of objects, particles, fields, waves, etc., that behave according to a set of equations.

Closed System: A system that is either the entire universe or an isolated, from everything else systems that behave as if nothing else exists.

Open System: A system that has external interactions with other systems.

State-Space: The collection of all possible states occupied by a system.

For example, let's say you have a closed system with a single coin glued on a table showing Heads. The state-space of the coin is equal to 1, since, it can occupy only 1 possible state, namely Heads.

The State Space is a mathematical set, whose elements label the possible states of the system.

The state-space of our example consists of a single point, as said before. We can predict its future state with extreme ease, since nothing ever happens that can change the outcome of our observation of the object's state.

Continuous Systems: The Systems that evolve smoothly, without any discrete jumps or interruptions.

Stroboscopic Systems: The systems that evolve in discrete steps labeled by integers.

Dynamical Systems: The systems that change as time passes.

Dynamical Systems don't consist only of a state-space, but they also entail a dynamical law of motion.

Dynamical Law: A law that tells us the next state of a system, given the current.
You can think of it as a function of time that has as its input the current state of the system, and outputs the next state.

The simplest example of a dynamical law is a loop. Whatever the current state is, the next state will be the same.

Let's say we have a coin so that :

This means that after the first observation, all others will be the same as the first one.

So, the possible histories of observations are

HHHHHHH….

Or

TTTTTTTTT…..

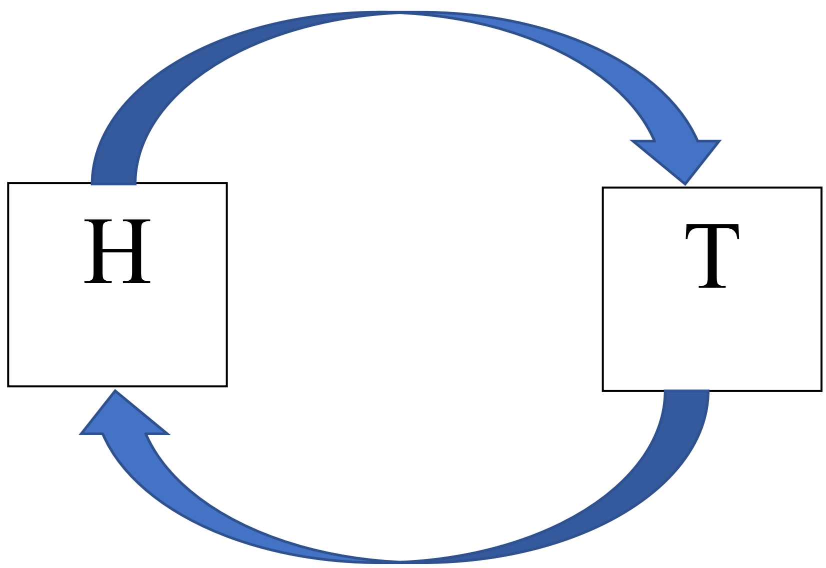
Again, predicting the future is easy! Once you know one state, you can predict all past and future states with ease!

Another example of a dynamical law is the following:

(Closed system with a coin)

Whatever the current state is, the next will be the opposite!

So if our object(Coin), shows heads, the next state is going to be tails.



Once again there are two possible histories:

HTHTHTHTHTH….

THTHTHTHTHT….

Degrees of freedom: The variables that describe a systeμ and that are also linked to the dynamical law of the system.

Our coin has one degree of freedom.

We denote the degrees of freedom with σ()

For example, our system has 2 degrees of freedom

$\sigma = 1$for H

$\sigma =-1$ for T

When we are considering a continuous evolution in time, we symbolize time with t.

When we are considering a discrete evolution in time, we denote time with n.

Since our system is stroboscopic and time evolves discretely(Number of observations), the **state** at **time n** is described by **σ(n).**

So far our equations are:

1st example $\sigma(n) = \sigma(n+1)$

2nd example $\sigma(n+1) = -\sigma(n)$

The future depends completely on the initial state so these laws are **deterministic.**

Cycle: An endlessly repeat pattern.

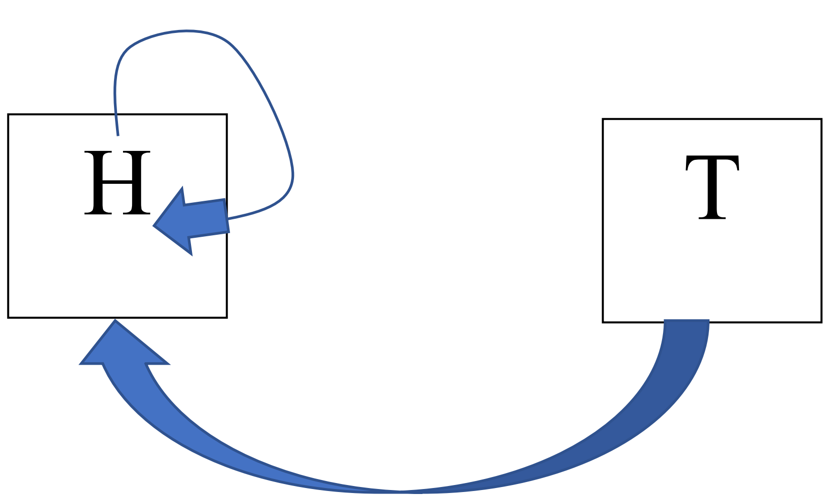
1.2 Rules that are not allowed

In Classical Physics not all laws are allowed. A dynamical law needs to be both reversible and Deterministic.

Reversible: A law that is deterministic in the past and the future.

If in a system, we reverse the “arrows”, it should still be deterministic!

Meaning a system like this is not allowed:



In other words:

The amount of arrows entering a state must be equal to the amount of arrows leaving the state, and must be equal to 1.

The Conservation of Information: Every State has only 1 arrow in and only 1 arrow out.

1.3 Dynamical Systems with infinite number of states.

To describe a system of an infinite number of states, we will use **n** for time, since we talk bout discrete time steps, and **N** for points on the track.

So a history of a system with infinite states and a certain law is a function **N(n)** telling the place along the track **N** at every time **n**.

Example:

…→0→1→2→….

This is a valid system since every state has one arrow pointing into it, and one arrow coming out of it!

We can express this rule as

$$\mathbf{N(n+1) = N(n) + 1}$$

1.4 Cycles and Conservation laws

The State-Space can be divided into multiple cycles. When this happens, the system remains in the cycle it started in. Each cycle has its own dynamical rule, but they are all a part of the state-space.

Conservation Law: Whenever a dynamical law divides the state-space into separate cycles, there is a memory of which cycle they started in.

Each cycle has a numerical value called **Q**.

Whatever the value of **Q** is, it remains the same for all time, since the dynamical law does not allow jumping from one cycle to another.

So **Q** is **conceived** through time.

1.5 The Limits of precision

Resolving Power: The power to distinguish the neighboring values of real numbers.

Real are infinitely dense, meaning that everyone one of them is arbitrary close to an infinite number of neighbors.

Chaotic System: A system in which the time over the system is predictable despite the resolving power limit.

Motion

2.1 Particle Motion

The Position of a particle is specified by the value of each spatial coordinate, namely the x,y,z in the Cartesian.

The Motion of a Particle is defined by its position at every time t.

This means that we can see the three spatial coordinates as functions of time t

x = x(t)

y = y(t)

z = z(t)

Position can also be thought as a vector (tensor-3D-Vector)

Whose components are the above functions of time.

$$\vec{v}(t) = x(t)i + y(t)j + z(t)k$$

The path, or **trajectory**, of a particle is specified by that vector.

Classical Mechanics tries to predict $\vec{v}(t)$ from an initial condition and some dynamical law!

Very important is the role of **velocity**.

Velocity is thought as a vector.

In an infinitesimal displacement of the particle between time t and t+Δt

We say that

$$\vec{v}(t) = x(t)i + y(t)j + z(t)k$$

Becomes

$$\vec{v}(t + \Delta t) = x(t + \Delta t)i + y(t + \Delta t)j + z(t + \Delta t)k$$

Generally if you know calculus,

Velocity **v** is defined as the first time-derivative of position.

In English, that means the rate of change of position with respect to time.

The unit of v is 1 m/s.

$$v_x = \frac{dx}{dt} = \dot{x}$$

$$v_y = \frac{dy}{dt} = \dot{y}$$

$$v_z = \frac{dz}{dt} = \dot{z}$$

Instead of writing all these separate equations and x,y,z , we generally denote the coordinates as x_i

Which translates to

$$v_i = \frac{dx_i}{dt} = \dot{x}_i$$

Since velocity is a vector, it also has a magnitude defined as

$$|\vec{v}| = {v_x}^2 + {v_y}^2 + {v_z}^2$$

This represents how fast the particle moves, without any specific direction. That Magnitude is called **Speed**.

Acceleration is the quantity that describes how quickly velocity changes over time.

If an object travels with a uniform/constant velocity, it experiences no acceleration.

(Later on, in Special Relativity you will learn that the reference frame of an observer that travels with uniform velocity is called IRF, which is short for Inertial Reference Frame.)

A constant velocity implies a constant speed and direction.

You might have guessed it by now, but

Acceleration is the first time derivative of velocity, or the rate of change of velocity with respect to time, or the second time derivative of position, since velocity is its first.

$$a_i = \frac{dv_i}{dt} = \dot{v}_i$$

$$a_i = \frac{d^2 x_i}{dt^2} = \ddot{x}_i$$

2.3 Examples of Motion

Lets say you have a closed system with a single particle in it.

Say that the particle starts to move at t=0 according to:

x(t) = 0

y(t) = 0

$$z(t) = z(0) + v(0) - \frac{1}{2}gt^2$$

The Particle has no motion on the x-y axes but moves along the z-axis.

The constant z(0) and v(0), represent the initial of the position and velocity along the z direction at t=0. We also consider g to be a constant.

So the equations for the velocity are:

$$v_x(t) = 0$$

$$v_y(t) = 0$$

$$v_z(t) = \frac{d}{dt}\left(-\frac{1}{2}gt^2\right)$$

The z component with a little help of easy calculus can be rewritten as :

$$v_z(t) = -gt$$

If you don't understand this, or why the constants z(0) and v(0) were eliminated, I could not suggest you more to visit the Khan academy Calculus playlist on YouTube to learn the needed math. You can access it faster by visiting the math needed sections on the physics page of my site‼‼

Since the x and y components of velocity are 0 at all times, the z component velocity that starts out at t =0 is equal to the whole velocity vector. So, v(0) is the initial condition for velocity.

(Since x(0) = 0, y(0) = 0,z(0) = 0, t =0 so $1/2gt^2$=0)

As time passes, the -gt term becomes non-zero and it will overtake the initial value of velocity, so the particle will be moving along the negative z direction.

Then to Calculate the acceleration, we simple take the time derivative of -gt:

$$a_z = \frac{d(-gt)}{dt} = -g$$

The acceleration along the z axis is constant and negative!

Dynamics

3.1 Aristotle's Law of Motion

For a body to move, some kind of **force** needs to interact with it.

Aristotle tried to write equations of motion but in his era calculus wasn't yet a thing, so he made some wrong conclusions due to that fact.

He wrongly stated that the velocity of an object is proportional to the total applied force. Furthermore, he stated that the Applied force on an object is proportional to its velocity and proportional to its mass.

$$\vec{F} = m\vec{v}$$

With some easy calculus, we can see that this equation isn't illegal. The problem is that they are simply wrong!

3.2 Mass Acceleration and Force

To keep an object moving, the **applied force** needs to overcome that of **friction.**

Law of inertia: An isolated object moving in free space with no forces acting on it, requires nothing to keep moving and needs a force to stop its motion.

Generally, an object that moves through space with no forces acting on it will keep its kinetic energy unless an external force acts upon it.

Different objects ave different inertias.

The quantitative measurement of an object's inertia is its **mass**.

Mass is defined in terms of force and acceleration.

Force is defined by the ability to change the motion of a given mass, and mass is defined by the resistance to that force.

That is Newton's second law of motion.

$$\vec{F} = m\vec{a}$$

Or

$$\vec{F} = m\frac{d\vec{v}}{dt}$$

Force equals mass times the rate of change of velocity.

3.3 Some Examples of solving Newtons equations

Example 1

Consider a particle with no forces acting upon it

$$m\vec{v} = 0$$

Since the velocity components are constant and don't change through time, we can set them to their initial values, 0

$$\boldsymbol{v_x = 0}$$

$$\boldsymbol{v_y = 0}$$

$$\boldsymbol{v_z = 0}$$

So

$$\boldsymbol{v_x(t) = v_x(0)}$$

$$\boldsymbol{v_y(t) = v_y(0)}$$

$$\boldsymbol{v_z(t) = v_z(0)}$$

That transitions us smoothly to Newtons First Law:

Every Object in a stat of uniform velocity tends to remain in that state of motion unless an external force is applied to it.

The first law is simply a special case of the second law, meaning that uniform velocity means that it doesn't change through time so acceleration is equal to 0 which in that case means that there is no force acted upon it that could change its trajectory.

4. Systems of More than one Particle

4.1 System of Particles

We can describe the force that acts upon a certain particle, as the positions of all other particles in a system.

Charges and Masse are **intrinsic properties** of a particle.

That comes from the fact that gravitational forces and electrical forces are proportional to the masses and charges of particles respectively.

Apart from the intrinsic properties, forces also depend on the location of particles.

These locations can be denoted in several ways.

Firstly, the one we will use, for now, is the x_i , y_i , z_i

Which are the coordinates of the ith particle on the cartesian.

Secondly, the other one we will use,

$$\{\vec{\boldsymbol{r}}\}$$

Which denotes the set of all positions, of all particles in a system, or in other words

It represents all the position vectors of all particles in a system.

So as we said earlier,

$$\vec{F} = F(\{\vec{r}\})$$

Once we know the position of every particle, we can use Newtons equations of motion to find the trajectory and expected path of a particle.

(ex. Particle 1)

$$\overrightarrow{F_i}(\{\vec{r}\}) = m_i \cdot \vec{a}_i$$

Since acceleration is the time derivative of velocity, or the second time derivative of position, we can rewrite this as:

$$\overrightarrow{F_i}(\{\vec{r}\}) = m_i \cdot \frac{d^2 r_i}{dt^2}$$

Or in component Form:

$$\overrightarrow{F_i}(\{\vec{x}\}) = m_i \cdot \frac{d^2 x_i}{dt^2}$$

$$\overrightarrow{F_i}(\{\vec{y}\}) = m_i \cdot \frac{d^2 y_i}{dt^2}$$

$$\overrightarrow{F_i}(\{\vec{z}\}) = m_i \cdot \frac{d^2 z_i}{dt^2}$$

4.2 The State Space of a System of Particles

The **state-space** of a system of particles, is the **collection of all possible** states of the system.

The **State** of a System of Particles, consists of the collection of all the positions and velocities of all particles in the system at the given time t.

That shows us that the state space is a 6D space. What 6D space means, although it is really counter-intuitive to think about fitting six perpendicular to each other axes, is that the configuration space (state space) is made up by the 3 position components and 3 velocity components. In general, that means that at every **given time t, each particle's state is given by those 6 components.**

4.3 Momentum and Phase Space

Momentum of a particle is defined as the product of its velocity and mass.

Momentum Space is like the Configuration space, with the only differences being that it is 3D instead of 6D, and instead of **position and velocity,** a state of the system at a given point is given by the 3 **components of momentum**, summing up for the 3D space.

We mentioned the Configuration Space of a system, but in Classical Mechanics, we don't use it that much. What we use is the phase space. Since velocity and momentum are so closely linked, we can use momentum over velocity to define each state. So in simpler words, **phase space = configuration space + momentum space**.

$$\overrightarrow{F}_i(\{\vec{r}\}) = \dot{p}_i$$

And $r_i = \dfrac{p_i}{m}$

4.4 Action, Reaction and the conservation of momentum

Conservation of momentum is basically Newtons 3rd law of motion.

For every action there's an equal, with respect to magnitude, and opposite with respect to direction, reaction.

Suppose we have a closed system where particles interact in pairs. The overall force acting upon a particle at a certain time t, is the sum of all forces exerted upon at that time t.

If we denote the force o++++n particle i **due to particle j** by the symbol $\overrightarrow{f_{ij}}$ then the **total** force acting on particle I is

$$\overrightarrow{F_i} = \sum_{j=0}^{nj} \overrightarrow{f_{ij}}$$

nj being the number of all particle in the system minus the particle i

It follows that

$$\overrightarrow{f_{ij}} = -\overrightarrow{f_{ji}}$$

So we could denote the rate of change of momentum as the sum of any particle I as the sum of all the forces due to all the other particles in the system. If you take a moment and think about it, it something very intuitive.

$$\dot{p}_i = \sum_j \overrightarrow{f_{ij}}$$

So if we take it further it follows that

$$\sum_i \dot{p}_i = \sum_i \sum_j \overrightarrow{f_{ij}}$$

$$\sum_i \dot{p}_i$$

Is the rate of change of the total momentum.

Now according to Newtons 3rd Law, that $\overrightarrow{f_{ij}} = -\overrightarrow{f_{ji}}$

Which means that $\overrightarrow{f_{ij}} + \overrightarrow{f_{ji}} = 0$ so if we substitute this to

$$\sum_i \sum_j \overrightarrow{f_{ij}}$$

We get that

$$\sum_i \dot{p}_i = 0$$

Or $\frac{d}{dt}\sum_i p_i = 0$

And that is the Conservation of momentum‼!

5.Energy

5.1 Force and Potential Energy

Energy is **conserved** in **any form** of energy!

Potential Energy Principle: All forces derive from a potential energy denoted as **V({x})**, where {x} denotes the entire set of 3N coordinates, or the configuration space of all particles in the system.

Generally, the relation between Force and Potential energy is

$V(\{x\}) = -\int F(\{X\})\, dx$ or $-F(x) = \frac{\partial \mathrm{V(x)}}{\partial x}$

If we translate this, what it says is nothing more than, that every force, **is always directed so that it pushes the particle toward the minimum or stationary potential energy.**

It can be gravity, the force of a transformed spring, a pendulum, the effect is the same.

Potential energy is not conserved by itself, but

The sum of Kinetic and Potential energy is. (when talking in a simple system with no fraction or other forms of energy.)

We will denote the Potential energy as V and the Kinetic energy as T.

$$T = \frac{1}{2}mv^2$$

And $V(x)$will refer to V(x_i)

So $E = T + V = \frac{1}{2}mv^2 + \mathrm{V(x)}$

To prove that $\frac{dE}{dt} = 0$ all we need to do is to take the time derivative of T+V so

$\frac{1}{2}mv^2 + \mathrm{V(x)}$ becomes

$mv\dot{v} + \frac{d\mathrm{V(x)}}{dt}$or $mva + \frac{d\mathrm{V(x)}}{dt}$

We can further denote $\frac{d\mathrm{V(x)}}{dt}$ as $\frac{d\mathrm{V(x)}}{dx} \cdot \frac{dx}{dt}$ which translates to

$$\frac{\partial \mathrm{V(x)}}{\partial x} \cdot \mathrm{v}$$

So now we can go back and do

$mva + \frac{d\mathrm{V(x)}}{dx} \cdot \mathrm{v}$ Now if we do some simple math we get

$$\mathrm{v(ma} \ + \frac{\partial \mathrm{V(x)}}{\partial x})$$

Now remember that $-F(x) = \frac{\partial \mathrm{V(x)}}{\partial x}$

We can say from Newtons second law

$$F(x) = \mathrm{ma}$$

That $\dot{E} = -F(x) = \mathrm{v}(\frac{\partial \mathrm{V(x)}}{\partial x} - \frac{\partial \mathrm{V(x)}}{\partial x})$

That shows that $\dot{E} = 0$

6. The Principle of Least Action

6.1 The Transaction to Advanced Mechanics

The principle of least action, which actually is the **principle of stationary action,** is a compact form of the classical laws of physics.

Usually in Classical Mechanics, you are asked to determine the trajectory of a particle. Generally, we need to know 3 things to solve such a problem with Newtons second law:

1) The mass of the given Particle
2) The set of F({x})
3) An initial condition

If there are N coordinates, meaning

X_1 ,X_2 ,… , X_N

Then the initial condition consists of 2N positions and velocities.

You probably are confused and feel that there is no way you can remember and know and do all of this, so here is where **Action and Euler-Langrange** equations come in.

For these all you simply need to know is initial and final positions‼!

6.2 Action and the Langrangian

To Formulate the Action Principle, we need to know:

1) The masses of the particles
2) The Potential Energy

All the Action is, is an integral from the start of the trajectory to the final state of the trajectory.

(example)

Let’s consider a single particle moving on a single dimension, called the x axis. The position of the particle is obviously time dependent so x = x(t).

The kinetic and potential energies are

$T = \frac{1}{2}m\bar{x}^2$and V = V(x)

The Action of the trajectory A of the particle is defined as

$$A = \int_{t0}^{t1} (T - V)\, dt = \int_{t0}^{t1} (\frac{1}{2} m\overline{x}^2 - V(x))\, dt$$

The Langrangian is defined as T-V or $\frac{1}{2} m\overline{x}^2 - V(x)$.

it is denoted as **L**.

The Langrangian is a function of position x_i or simply x, since Potential Energy V depends on it, and a function of $\overline{x}$ since Kinetic Energy T depends on velocity.

So $L = L(x,\overline{x})$

Which in other words means we can define Action as

$$A = \int_{t0}^{t1} L(x,\overline{x})\, dt$$

Action is basically a function of a function, since it is a function of x(t). Our whole goal is to find the equations that minimize the A, hence called **The Principle of Least Action,** although basically we 'stationarize' it.

To sum it all up, the **shortest path between two points,** the path of **Stationary Action.**

Now we will learn how to minimize the Action and find the equations of motion, through the Euler-Langrange equations.

6.3 Derivation of the Euler-Langrange equations

The Euler Langrange equations for the cartesian are:

$$\frac{\boldsymbol{d}}{\boldsymbol{dt}}\frac{\partial \boldsymbol{L}}{\partial \overline{\boldsymbol{x}}}-\frac{\partial \boldsymbol{L}}{\partial \boldsymbol{x}}=0$$

Euler-Langrange Equations Derivation for 1 degree of freedom.

Since time is continuous and doesn't help compute anything, we will substitute it with a stroboscopic time labelled by integers n.

The time between neighboring instants is extremely small. We will call I Δt.

Since the Action is an integral, we can change it into a sum.

$$A=\int_{t0}^{t1} L(x,\overline{x})\,dt=\sum L(x,\overline{x})\Delta\mathrm{t}$$

And we set $\overline{x}=\frac{x_{n+1}-x_n}{\Delta\mathrm{t}}$

So that means we replace the Action with a discrete sum over a small interval and the velocity with the difference of two neighboring points divided by a small time Δt.

Next we will replace position x(t) with the average position between two neighboring instants $\mathrm{x(t)} = \frac{x_{n+1} + x_n}{2}$

That means we can now replace Action with

$$A = \sum_n L(\frac{x_{n+1} - x_n}{\Delta \mathrm{t}}, \frac{x_{n+1} + x_n}{2})\Delta \mathrm{t}$$

Now we want to minimize the Action so we have a result of 0. We will do that by set x_n as a random x say x_8.

Now since there are two terms that contain x_n we need to substitute

$$A = \mathrm{L}\left(\frac{x_9 - x_8}{\Delta \mathrm{t}}, \frac{x_9 + x_8}{2}\right)\Delta \mathrm{t} + \mathrm{L}(\frac{x_8 - x_7}{\Delta \mathrm{t}}, \frac{x_8 + x_7}{2})\Delta \mathrm{t}$$

Next we need to differentiate with respect to x_8 which would result in:

$$\frac{\partial A}{\partial x_8} = \frac{1}{\Delta t}\left(- \frac{\partial L}{\partial \overline{x}}|_{n=9} + \frac{\partial L}{\partial \overline{x}}|_{n=8}\right) + \frac{1}{2}\left(- \frac{\partial L}{\partial x}|_{n=9} + \frac{\partial L}{\partial x}|_{n=8}\right)$$

The $|_{n=8}$ means that you evaluate the function at discrete time n=8

Now we need to see what happens as Δt approaches 0.

The First part:

$$\lim_{\Delta t \to 0} \frac{\left(-\frac{\partial L}{\partial \bar{x}}|_{n=9} + \frac{\partial L}{\partial \bar{x}}|_{n=8}\right)}{\Delta t}$$

If we evaluate this, we see that

$$\lim_{\Delta t \to 0} \frac{\left(-\frac{\partial L}{\partial \bar{x}}|_{n=9} + \frac{\partial L}{\partial \bar{x}}|_{n=8}\right)}{\Delta t} = \frac{d}{dt}\frac{\partial L}{\partial \bar{x}}$$

Next we do the same for the other half, which happens to have an extremely easy limit.

$$\lim_{\Delta t \to 0} \frac{\left(-\frac{\partial L}{\partial x}|_{n=9} + \frac{\partial L}{\partial x}|_{n=8}\right)}{2}$$

Since we said n=9 is equal to n=8 +Δt we see that

The limit is simply

$$\lim_{\Delta t \to 0} \frac{\left(-\frac{\partial L}{\partial x}|_{n=9} + \frac{\partial L}{\partial x}|_{n=8}\right)}{2} = \frac{\partial L}{\partial x}$$

Now if we remember the Principle of least Action, A=0

We can see that

$$\boldsymbol{\frac{d}{dt}\frac{\partial L}{\partial \bar{x}} - \frac{\partial L}{\partial x}} = 0$$

This derivation is the same for more degrees of freedom.

There is an Euler-Langrange equation for each coordinate x_i so we could rewrite them as

$$\frac{\boldsymbol{d}}{\boldsymbol{dt}}\frac{\partial \boldsymbol{L}}{\partial \bar{\boldsymbol{x}}_{\boldsymbol{i}}} - \frac{\partial \boldsymbol{L}}{\partial \boldsymbol{x}_{\boldsymbol{i}}} = 0$$

6.4 More than One Particle and More Dimensions

In total there are N-Dimensions that we call x_i or as we will soon see q_i.

The Orbit is a path through an N-Dimensional space, and time.

The Starting point or origin is the set of $x_i(t_0)$ and the end point is another set $x_i(t_1)$

So now we form the Action and The Langrangian as:

$$L = \sum_i \left(\frac{1}{2} m_i \overline{x_i}^2 - V(\{x\})\right)$$

$$A = \int_{t0}^{t1} L(\{x\},\{\overline{x}\})\, dt$$

So minimizing the Action is actually minimizing a function of a lot of variables, namely all the x_i, so that means we also have a Euler Langrange equation for each and every variable x_i.

6.5 What is Good about Least Action

By using the principle of Least Action, we compress every parameters (mass ,momentum, etc.) into a single function, namely the Langrangian. That means we can summarize the behavior of a particle in a single function.

Chapter 6.6 Generalized coordinates and Momenta

Generally, we can use many coordinate systems to represent any mechanical system.

To over this fact up, we use a generalized notation for coordinates, namely q_i . This means that whatever the coordinate system is, spherical coordinates, phase space, cartesian/configuration space, q_i holds.

The generalized notation for velocity, as expected is $\dot{q}_i$ and so on for acceleration and jerk.

(jerk is the name for the time derivative of acceleration.)

In a general coordinate system the equations of motion are a bit more complicated, but the Action Principle **always applies**.

All systems can be described by a Langrangian.

The Langrangian is always a function of position and velocities

If you know the q_i's and the Langrangian of a system, you have it all.

Now , if we further examine

$$\frac{\partial \boldsymbol{L}}{\partial \dot{\boldsymbol{q}}_i}$$

We see that what it actually means is the time derivative of T(Kinetic Energy), which equals to

$m \cdot \bar{q}$ which is the momentum. So we conclude that

$$\frac{\partial \boldsymbol{L}}{\partial \overline{\boldsymbol{q_i}}} = \boldsymbol{p_i}$$

And now if see the Euler-Langrange equations we realize that

$$\frac{\partial L}{\partial q_i} = \frac{dp_i}{dt}$$

So what that means, and as we will soon learn, is that we can express the laws of motion in terms of momentum and position

q_i and p_i

7. Symmetries and Conservation Laws

7.1 Preliminaries

In modern physics, Conservation laws are really important.

Suppose two particles are traveling in a straight line and they have the same mass, and their positions are q_1 and q_2 respectively.

Their Langrangian is given as

$$L = \frac{1}{2}\left(\dot{q}_1{}^2 + \dot{q}_2{}^2\right) - V(q_1 - q_2)$$

Potential Energy is a function of a combination of particles. In our case $(q_1 - q_2)$,

Let's denote the Time derivative of the Potential Energy

As

$$\frac{d}{dt}V(q_1 - q_2) = V'(q_1 - q_2)$$

So the laws of motion are

$$\dot{p}_1 = -V'(q_1 - q_2)$$

$$\dot{p}_2 = -V'(q_1 - q_2)$$

Now let's see if momentum is conserved!

So let's add them together. $\dot{p}_1 + \dot{p}_2$

Now we said that the Potential is a combination of q_1 and q_2. So lets set it as a linear combination

$$V(q_1 - q_2) = V(a \cdot q_1 - b \cdot q_2)$$

So now the laws of motion are:

$$\dot{p}_1 = -a \cdot V'(a \cdot q_1 - b \cdot q_2)$$

$$\dot{p}_2 = b \cdot V'(a \cdot q_1 - b \cdot q_2)$$

So there is our conservation law:

$\frac{d}{dt}(bp_1+ap_2)=0$

7.2 Examples of Symmetries

Passive change: A **change** that **doesn't affect** the system.

Active change: A **change** that **moves** the system to a **new configuration-space.**

Symmetry: Symmetry is a **coordinate transformation,** that **doesn't affect the Langrangian.**

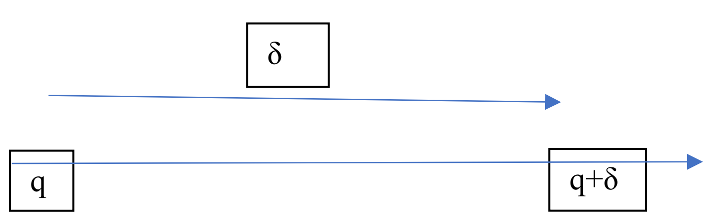
Let's consider a shift in the coordinates of a particle in a closed system from q_i to q'_i.

$q'_i = q'_i(q_i)$

Now let's consider a system with 1 degree of freedom with a Langrangian

$$L = \frac{1}{2}\dot{q}^2$$

Now suppose we make a change in q by shifting it by a small amount δ



Let's assume that δ is simply a constant and does depend on time. If that is the case, then that means that velocity doesn't change. It is equal to the velocity that we would have at that time if we didn't shift our coordinates. Now since velocity doesn't change, the Langrangian doesn't change either!

So $\delta L = 0$

Translation Symmetry: Translation symmetry is called the symmetry of a **system that undergoes a shift in coordinates in space, that is described by adding a constant to the coordinate system.**

You might have seen this coming, but what this essentially means, is that whenever there is a Potential Energy in the system, after such a transformation, the Langrangian changes too.

When the Langrangian doesn't change when there is a shift in the coordinate system, we say that it is invariant over change.

If the change of coordinate system Is simply a rotation, it follows that it doesn't affect the Langrangian.

We also say that δ is an infinitesimal change

Continuous transformation: Transformations which by repeating a process, build up a finite change.

Infinitesimal Transformation: A small shift of the coordinate system that depends on the value for the coordinates. $\delta q_i = f_i(q)\delta$

Continuous Symmetry: A continuous symmetry is an infinitesimal transformation of the coordinates for which the Langrangian doesn't change.

7.3 Consequences of Symmetry

(Hard Part) Optional

Now let's see what happens when we try to compute the change of the Langrangian when we shift q_i and $\dot{q}_i$ by **δ.**

So

$$\delta L = \sum_i \left(\frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i + \frac{\partial L}{\partial q_i}\delta q_i\right)$$

Since we have shown earlier that $\frac{\partial L}{\partial \dot{q}_i}$ is the momentum conjugate to q_i , (p_i) and since we assume that the system evolves along the trajectory that satisfies the Euler-Langrange Equations we set

$$\frac{\partial L}{\partial q_i} = \frac{dp_i}{dt}$$

So now we can rewrite the initial equations as

$$\delta L = \sum_i (p_i \delta \dot{q}_i + \dot{p}_i \, \delta q_i)$$

Now we can use the product rule from Calculus

$$\frac{d(F \cdot G)}{dt} = \dot{F}G + F\dot{G}$$

(BUT IN REVERSE)

So we see that

$$\delta L = \frac{d}{dt} \sum_i (p_i \, \delta q_i)$$

And since we said that

$\delta q_i = f_i(q)\delta$ we see that

$$\delta L = \frac{d}{dt} \sum_i (p_i \, f_i(q))$$

Which is nothing but the proof for the Conservation Law

$$\delta L = \frac{d}{dt} \sum_i (p_i \, f_i(q)) = 0$$

$$Q = \sum_i (p_i \, f_i(q))$$

7.5 Back to examples

For any system of particles, if the Langrangian is invariant over change under simultaneous translation of the positions of all particles, then momentum, is conserved.

And so Newtons 3rd law can be translated to :

Nothing in the Laws of physics changes if everything is simultaneously shifted in space.

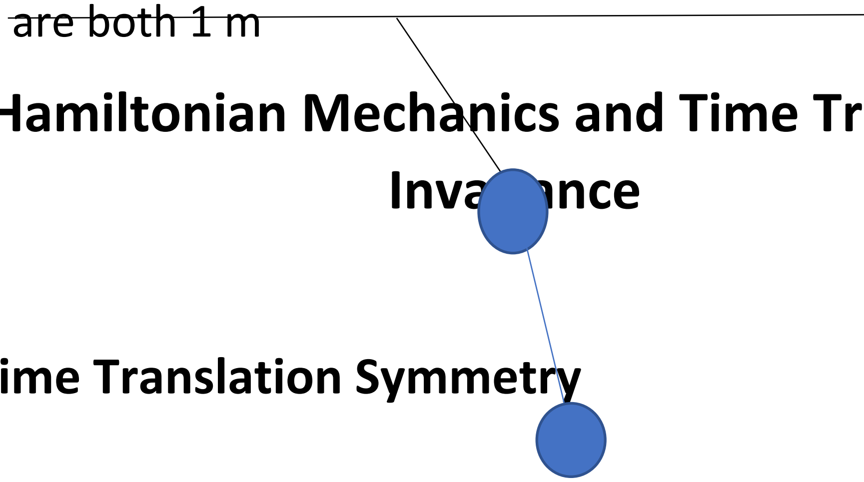
Angular Momentum: A quantity that involves both coordinates and momenta:

$l = yp_x - xp_y$

For any system of particles, if the Langrangian is invariant over change under simultaneous rotation of the positions of all particles, about the origin, angular momentum is conserved.

Exercise: Very Hard Optional Try to find the Laws of motion of a double pendulum :

The masses of the objects are both 1kg and the length of the ropes are both 1 m



8.Hamiltonian Mechanics and Time Translation Invariance

8.1 Time Translation Symmetry

The Symmetry that relates to energy conservation includes a shift of time.

If we conduct an experiment from t_0 to t_1 and the results of that experiment match the results of an experiment that is the same as the first one, with the only difference being that it is conducted in a later time, namely ($t_0 + \Delta t$ to $t_1 + \Delta t$), then we say that the system is invariant under time-translation .

Generally:

If the shift doesn't affect the outcome, we say that it is invariant over *time-translation.*

The Langrangian might vary with time, but **ONLY** because coordinates and velocities vary.

Explicit time dependence: The form of the Langrangian depends on time.

A system is time-translation invariant if there is no explicit time dependence in its Langrangian.

8.2 Energy Conservation and The Hamiltonian

Let's try to see what happens when the value of the Langrangian changes as the system evolves.

$$\frac{dL}{dt} = \sum_i \left(\frac{\partial L}{\partial q_i} \dot{q}_i + \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i \right) + \frac{\partial L}{\partial t_i}$$

Now we can examine each term and replace them so that

$$\frac{\partial L}{\partial q_i} \dot{q}_i = \dot{p}_i \dot{q}_i$$

And

$$\frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i = p_i \ddot{q}_i$$

so combined these give

$$\frac{dL}{dt} = \sum_i (\dot{p}_i \dot{q}_i + p_i \ddot{q}_i) + \frac{\partial L}{\partial t_i}$$

And now if we use the verse of the product rule we:

$$\frac{dL}{dt} = \frac{d}{dt} \sum_i (p_i \dot{q}_i) + \frac{\partial L}{\partial t_i}$$

Now we define the Hamiltonian as

$$H = \sum_i (p_i \dot{q}_i) - L$$

Or in other words H=E and

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t}$$

Now in a system where PE = V(q) The Langrangian is

$$L = \frac{m}{2}\dot{q}^2 - V(q)$$

The Momentum is $p = m\dot{q}$So since

$$H = p\dot{q} - T + V$$

$$H = (m\dot{q})\dot{q} - \frac{m}{2}\dot{q}^2 + V(q)$$

So we see that with simple algebra we get

$$H = \frac{1}{2}m\dot{q}^2 + V(q)$$

Or $$H = T + V$$

If a system is time-translation invariant, H or Energy is conserved.

Energy = Hamiltonian

Hamilton's Equations:

$$\boldsymbol{H = H(\dot{q}_i, p_i)}$$

$$\boldsymbol{\dot{p}_i = -\frac{\partial H}{\partial q_i}}$$

$$\boldsymbol{\dot{q}_i = \frac{\partial H}{\partial p_i}}$$

So by knowing all the values of the coordinates and momenta at any time and the form of the Hamiltonian, we can determine a trajectory throughout phase-space.

8.3 Derivation of Hamilton's Equations