My Derivations on Tensor Calculus

Stavros Klaoudatos

Abstract

This review paper presents my derivations of certain Tensor Calculus quantities.

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1 Tensor Calculus

1.1 Curved 2D Arc Length

Find the Arc Length of the curve R, (created by a position vector) parametrized by λ, in a 2D curve space that forms a sphere, which lives in a 3D Cartesian Space, where $\lambda \to (u = \lambda, v = \lambda)$, from $\lambda = 0 to \lambda = 2$

$(u, v) \to (X(u, v), Y(u, v), Z(u, v))$

where, the parametrisation for the sphere is given by:

$X = cos(v)sin(u)$
$Y = sin(v)sin(u)$
$Z = cos(u)$

The formula for Arc Length, is given by:

$$L_A = \int_a^b \| \frac{d\vec{R}}{d\lambda} \| d\lambda \tag{1}$$

and where $\|\frac{d\vec{R}}{d\lambda}\|^2 = \frac{d\vec{R}}{d\lambda} \cdot \frac{d\vec{R}}{d\lambda}$

Solution:

We first expand the derivative of R with respect to λ using the multivariable chain rule:

$$\frac{d\vec{R}}{d\lambda} \cdot \frac{d\vec{R}}{d\lambda} = (\partial_X R\ d_\lambda X + \partial_Y R\ d_\lambda Y + \partial_Z R\ d_\lambda Z) \cdot (\partial_X R\ d_\lambda X + \partial_Y R\ d_\lambda Y + \partial_Z R\ d_\lambda Z) \tag{2}$$

But since our sphere lives in a Cartesian coordinate system, it means that:

$$\partial_{c^i} \vec{R} \cdot \partial_{c^j} \vec{R} = \delta_{ij} \tag{3}$$

where c^i are the Cartesian coordinates.

Using that, it is obvious, that:

$$\frac{d\vec{R}}{d\lambda} \cdot \frac{d\vec{R}}{d\lambda} = (\frac{dX}{d\lambda})^2 + (\frac{dY}{d\lambda})^2 + (\frac{dZ}{d\lambda})^2 \tag{4}$$

Plugging in the parametrisation equations for the sphere, we get that:

$$\frac{dX}{d\lambda} = -sin(\lambda)sin(\lambda) + cos(\lambda)cos(\lambda) = cos(2\lambda) \tag{5}$$

(By Trig Identity)

$$\frac{dY}{d\lambda} = \frac{d}{d\lambda}sin^2(\lambda) = cos(\lambda)sin(\lambda) + sin(\lambda)cos(\lambda) = sin(2\lambda) \tag{6}$$

and finally:

$$\frac{dZ}{d\lambda} = -sin(\lambda) \tag{7}$$

Using that:

$$\|\frac{d\vec{R}}{d\lambda}\|^2 = \frac{d\vec{R}}{d\lambda} \cdot \frac{d\vec{R}}{d\lambda} \tag{8}$$

and

$$\frac{d\vec{R}}{d\lambda} \cdot \frac{d\vec{R}}{d\lambda} = (\frac{dX}{d\lambda})^2 + (\frac{dY}{d\lambda})^2 + (\frac{dZ}{d\lambda})^2 \tag{9}$$

We get that:

$$\|\frac{d\vec{R}}{d\lambda}\|^2 = (cos(2\lambda))^2 + (sin(2\lambda))^2 + (-sin(\lambda))^2 \tag{10}$$

which by trigonometry identities yields:

$$\|\frac{d\vec{R}}{d\lambda}\|^2 = 1 + sin^2(\lambda) \tag{11}$$

This give us this somewhat difficult integral to solve:

$$\int_0^1 \sqrt{1+(sin(\lambda))^2} d\lambda \tag{12}$$

This is known to be the Elliptic of x,-1. We can use this knowledge to find out that the answer is

$$E(\lambda,-1)|_{\lambda=0}^{\lambda=1} = 1.12388... \tag{13}$$

1.2 Geodesics in Curved 2D Spaces

Geodesic: The Straightest Possible Path in a curved surface/space that also minimizes the distance between two points.
In curved space, a straight path a 0 tangential acceleration when we travel along it at a constant speed.

$$\frac{d^2\vec{R}}{d\lambda^2} = (\frac{d^2\vec{R}}{d\lambda^2})^{normal} + (\frac{d^2\vec{R}}{d\lambda^2})^{tangential} \tag{14}$$

We again have $\vec{R}(u,v) \to (X(u,v), Y(u,v), Z(u,v))$ and we us the partial derivative definition of basis vectors:

$$\vec{e_u} = \frac{\partial\vec{R}}{\partial u} \tag{15}$$

The difference between $\frac{\partial\vec{R}}{\partial u}$ and $\frac{\partial\vec{R}}{\partial\lambda}$ is that one is the vectors tangent to **the surface**, while the other are the vectors tangent **to the curve**, parametrised by λ
The respective acceleration vector is $\frac{d^2\vec{R}}{d\lambda^2}$ Using the multivariable chain rule, it is obvious that:

$$\frac{d\vec{R}}{d\lambda} = \frac{dc^i}{d\lambda} \frac{\partial\vec{R}}{\partial c^i} \tag{16}$$

Using the product rule, and the equation above, it is very easy to expand the acceleration equation:

$$\frac{d^2\vec{R}}{d\lambda^2} = \frac{d}{d\lambda}(\frac{dc^i}{d\lambda} \frac{\partial\vec{R}}{\partial c^i}) \tag{17}$$

$$\frac{d^2\vec{R}}{d\lambda^2} = \frac{d^2c^i}{d\lambda^2} \frac{\partial\vec{R}}{\partial c^i} + \frac{dc^i}{d\lambda}(\frac{d}{d\lambda} \frac{\partial\vec{R}}{\partial c^i}) \tag{18}$$

We can now rewrite the second part of the right hand side of the equation using very simply multivariable calculus rules as follows:

$$\frac{d^2\vec{R}}{d\lambda^2} = \frac{d^2c^i}{d\lambda^2} \frac{\partial\vec{R}}{\partial c^i} + \frac{dc^i}{d\lambda} \frac{dc^j}{d\lambda} \frac{\partial^2\vec{R}}{\partial c^i \partial c^j} \tag{19}$$

We now have a 3D Vector, with components c^1, c^2, and $\hat{n}$

What we can now do with the 2nd degree partial derivative of R is, to expand it in component form, and try to find its coefficients:

$$\frac{\partial^2 \vec{R}}{\partial c^i \partial c^j} = \Gamma^1_{i\ j} \frac{\partial \vec{R}}{\partial c^1} + \Gamma^2_{i\ j} \frac{\partial \vec{R}}{\partial c^2} + L_{i\ j} \hat{n} \tag{20}$$

We have now divided everything in normal and tangential components! The L, which is the Second Fundamental form is the coefficient of the normal component, and the Gammas, are called the Christoffel Symbols, that are the coefficients of the tangential components.
We can now manipulate the equations to find the values of the Christoffel Symbols: Since we live in an orthonormal 3D space, the normal component times any of the other two basis vectors will be equal to zero, as they perpendicular. Using this, we find that:

$$\frac{\partial^2 \vec{R}}{\partial c^i \partial c^j} \cdot \frac{\partial \vec{R}}{\partial c^l} = (\Gamma^k_{i\ j} \frac{\partial \vec{R}}{\partial c^k} + L_{i\ j} \hat{n}) \cdot \frac{\partial \vec{R}}{\partial c^l} \Longleftrightarrow \tag{21}$$

$$\frac{\partial^2 \vec{R}}{\partial c^i \partial c^j} \cdot \frac{\partial \vec{R}}{\partial c^l} = \Gamma^k_{i\ j} (\frac{\partial \vec{R}}{\partial c^k} \cdot \frac{\partial \vec{R}}{\partial c^l}) \tag{22}$$

Recalling the definition of the Metric Tensor, it is trivial that:

$$\Gamma^k_{i\ j} (\frac{\partial \vec{R}}{\partial c^k} \cdot \frac{\partial \vec{R}}{\partial c^l}) = \Gamma^k_{i\ j} g_{k\ l} \tag{23}$$

and therefore

$$\frac{\partial^2 \vec{R}}{\partial c^i \partial c^j} \cdot \frac{\partial \vec{R}}{\partial c^l} = \Gamma^k_{i\ j} g_{k\ l} \tag{24}$$

With simple calculus and substitutions, we can easily find, what we call the **geodesic equation**:

$$\frac{d^2 c^k}{d\lambda^2} + \Gamma^k_{i\ j} \frac{dc^i}{d\lambda} \frac{dc^j}{d\lambda} = 0 \tag{25}$$

By the definition of the normal component, and the cross product, as well as with the use of the contravariant metric tensor and its definition, we can easily now find the equations that describe the coefficients:

$$\Gamma^k_{i\ j} = \frac{\partial^2 \vec{R}}{\partial c^i \partial c^j} \cdot \frac{\partial \vec{R}}{\partial c^l} g^{l\ k} \tag{26}$$