Limits

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Part I

Calculus I

Chapter 1

Limits

1.1 Introduction to Limits

In this chapter, we will learn about limits, their properties, I will provide examples, include exercises, but most importantly, I will explain everything based on intuition and logic, rather than teaching you rules and formulas to learn by heart. This applies for all chapters in this book.
Fist of all, in order to understand limits, you will need to have a good understanding of functions.

The limit of a function f(x), as x approaches some value c, is equal to the value of the function, as x gets very very close to the value c, and it exists only if the value of the limit function is the same approaching from both sides.

$$\lim_{x \to a} f(x) = \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) \tag{1.1}$$

By Approaching, we mean **the values right next to n, but from both sides. That means that we take the values exactly above and exactly below n, and plug them in the function**.

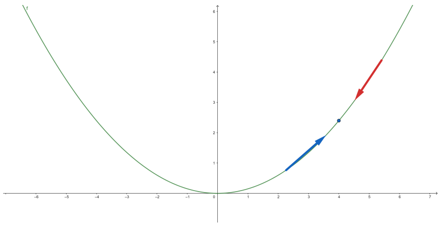


Figure 1.1: Approaching from both sides

Now of course, we won't be taking any 0,000000....1's or any of that sort. This simply has a demonstrative purpose to show what it means to approach a value. However, we will use this when talking about limits of functions as x approaches 0, since exactly below it, it is a negative number, and exactly above it, is a positive number. Why is that a problem you ask. Well, if we try to find the limit of $\frac{1}{x}$ as x approaches 0, we would encounter an example of an undefined limit:

$$\lim_{x \to 0} \frac{1}{x} \tag{1.2}$$

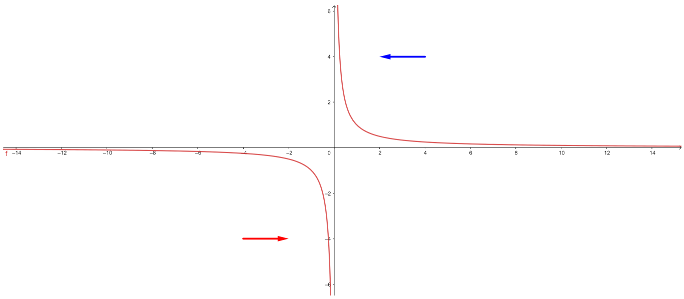
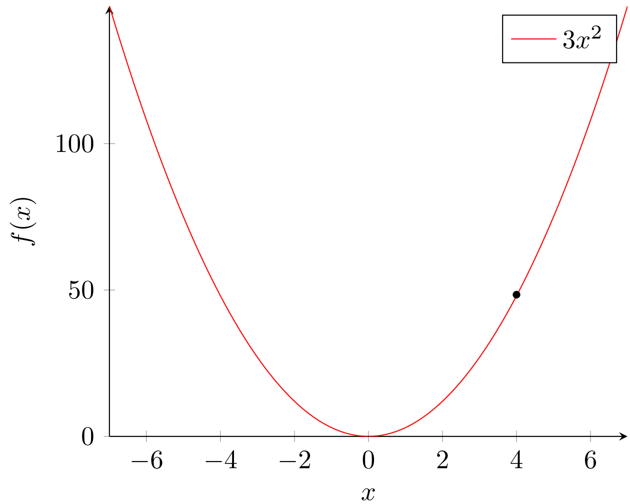


Figure 1.2: Approaching 0 from both sides

We see, that if we plug in -0,0....1, the result is $-\infty$, but if we plug +0,00...1, we get $+\infty$.
When something like this occurs, we say that the limit of the function, as x approaches x=0 doesn't exist.

Before moving on to the formal definition of limits, here are some examples to show you the calculations and build some intuition.
Suppose we have a function f(x), where $f(x) = 3x^2$ and its domain is $\mathbb{R}$.

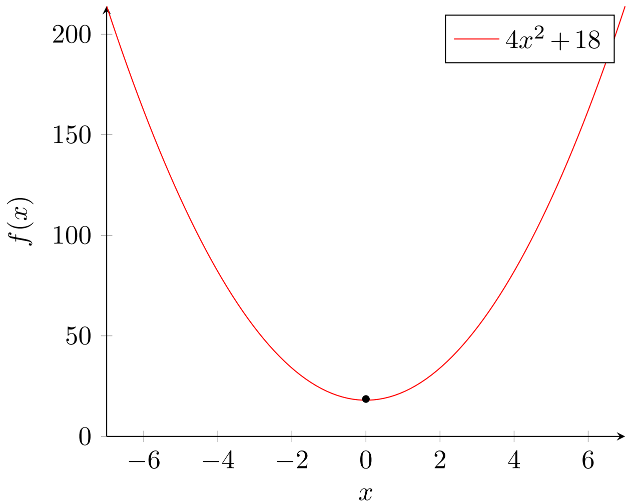


If we wanted to find what happens to this function when x approaches 4, all we have to do is to find

$$\lim_{x \to 4} 3x^2 \tag{1.3}$$

If we now plug in 4, we see that the limit of this function, as x approaches 4 is simply:
$3 \cdot 4^2$ or just 48

Let g(x) $= 7x^2 - 3x^2 + 18$, and it is defined $\forall x \in \mathbb{R}$. Suppose we want to find the limit of g(x), as x approaches 0.
Before evaluating anything, we can easily simplify the function given, to help us out.
By doing $7x^2 - 3x^2$, we are left with
g(x) $= 4x^2 + 18$



Here, it is clear that the

$$\lim_{x\to 0} 4x^2+18 \tag{1.4}$$

is simply 18.
The upper side of the limit is equal to the lower side, and are equal to 0.
Now, given everything so we have said so far, it is logical that the thought has crossed your mind:
So what are the differences between

$$\lim_{x\to 1} f(x) \tag{1.5}$$

and f(1) that make limits so useful and important?

So far, we have only seen a difference in $f(x)=\frac{1}{x}$. The difference is when $f(x_0)$ is not defined. An example of that is $f(x)=\frac{x-6}{x^2-36}$. f(6) for example isn't defined. We see that if we plug in 6, we get a weird $\frac{0}{0}$. Later on, we will see some rules, that we can use to find such limits. This limit for example, using L'Hopital's rule is equal to $\frac{1}{12}$. We can also approximate these limits. We can do this, by simply plugging values slightly above or below the value our variable approaches to.

To specify from which side we approach, the left-hand/lower side, meaning the value(s) exactly below the value we have in mind, and the right-hand/upper side, which are values exactly above, we simply add the negative or positive symbol respectively.

For example,

$$\lim_{y\to 0^+} \frac{1}{y} = +\infty \tag{1.6}$$

, while

$$\lim_{y\to 0^-} \frac{1}{y} = -\infty \tag{1.7}$$

, or another example

$$\lim_{x\to 2^-} \frac{-3}{x-2} = +\infty \tag{1.8}$$

, and

$$\lim_{x\to 2^+} \frac{-3}{x-2} = -\infty \tag{1.9}$$

Another thing to keep in mind, is that there other types of limits that are not defined. We will soon see a rule that will help us find these limits, L'Hopital's rule, but we first need to learn about derivatives. These limits are easy to remember and will be very easy to solve once we learn this rule. Generally, if the limits as x approaches some value c of some function f(x) equal to one of the following, for now we will say it is not defined:

$$\frac{0}{0} \quad or \quad \frac{\pm\infty}{\pm\infty} \quad or \quad \infty - \infty \quad or \quad 0 \cdot \infty ... \tag{1.10}$$

Generally for all indeterminate forms, we can use L'Hopital's rule to find the limits, but for now we will continue with the definitions of limits and their properties.

However, the reason behind why limits are so useful and important, is something you will learn when we will start talking about continuity and derivatives

For now, you can now test your understanding by completing the following exercises.

I will use different variables to change things up but that doesn't change a thing

Evaluate the limits

1)

$$\lim_{x \to 10} \frac{x^2}{3} \tag{1.11}$$

2)

$$\lim_{x \to 3} (x-2)^{2021} \tag{1.12}$$

3)

$$\lim_{x \to -3} 7x^2 + 15x - 2021^{(x+3)} \tag{1.13}$$

4)

$$\lim_{y \to 3} 36y^2 + 36y + 8 \tag{1.14}$$

5)

$$\lim_{\theta \to 11} \frac{11\theta + 23\theta - 69}{21\theta^2 - 13\theta + 53} \tag{1.15}$$

6)

$$\lim_{l \to 4} \frac{l^3 - 64}{l^2 + 2l - 8} \tag{1.16}$$

7)

$$\lim_{x \to 100} \frac{x^{-2} + 3x - 225}{3x^2 + 17x^{(x-100)}} \tag{1.17}$$

8)

$$\lim_{y \to -5} \frac{y^2 + 5y}{\frac{y^3}{5} + y^2} \tag{1.18}$$

1.2 The Delta-Epsilon($\delta - \epsilon$) definition

Let f(x) be a function defined on an open interval around x_0. We say that the limit of f(x) as x approaches x_0 is L.

$$\lim_{x \to x_0} f(x) = L$$

If for every $\epsilon > 0$, there exists a δ , such that for all x:
$0 < |x - x_0| < \delta \Longrightarrow |f(x) - L| < \epsilon$

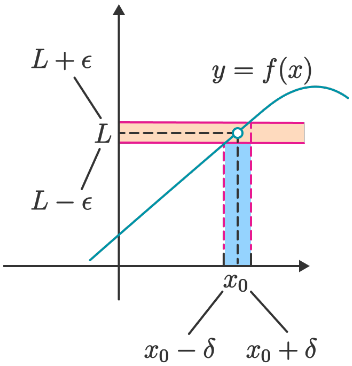


Figure 1.3: This image belongs to brilliant.org

What does this all mean you're asking. Lets take a step back and try to come up with a definition for limits by our own, with everything we have said so far. We can say that: **We can make f(x) get as close as we want to L, which is given by the limit of f(x) as x approaches c, if we can make x sufficiently close to c.** This should make total sense to you and should come as no surprise. Having said this, we can change words for mathematical notation to write everything more condensed.
Firstly, instead of saying "as close as we want", we will denote it with a positive number, ϵ, which, as I just said denotes **how close** we want f(x) to be to L. ($\epsilon > 0$)

After specifying ϵ, we need to find another positive number, which we will denote with δ, such that, if x is within δ of c, then f(x), will be within ϵ of L.($\delta > 0$)

Now using all of this, we can finally write our definition, which is that:
For any given $\epsilon > 0$, we can find a $\delta > 0$, such that if $|x - c| < \delta \implies |f(x) - L| < \epsilon$

1.3 Properties of Limits

Limits, just like anything else in mathematics have properties. There are some specific rules that they follow that we can use in our advantage to simplify them and calculate them. Most, if not all of these properties are very intuitive and come very natural. I will list all of these rules, provide examples, prove them and also provide some exercises on them. In order to prove all of these rules, we will use the delta-epsilon definition of limits. Before going into the proofs, I should warn you, that it would be best if you had a good and clear understanding of the Delta-Epsilon Definition, otherwise you will probably get confused or lost somewhere in the middle of all these Greek letters.
Some limits are weird. This comes from the fact, that when finding a limit, you, well, find its limit. You find the value that is the boundary, but you never actually take the value. You come extremely close, but never take it. How can we use this fact and what it change?

Consider the following example.

Let f(x) = $\frac{x^2-4}{x-2}$ Now lets suppose we want to find its limit as x approaches 2.

$$\lim_{x\to 2}\frac{x^2-4}{x-2} \tag{1.19}$$

You first thought might be that it is not defined, since we get that

$$\lim_{x\to 2}\frac{x^2-4}{x-2} = \frac{0}{0} \tag{1.20}$$

, which is an indeterminate form, and is also wrong. Since x never quite reaches 2, we can use factorisation and cancel out some terms to get our limit:

$$\lim_{x\to 2}\frac{x^2-4}{x-2} = \lim_{x\to 2}\frac{(x+2)(x-2)}{x-2} = \lim_{x\to 2} x+2 = 4 \tag{1.21}$$

This is a very useful thing to have in mind and can help you solve limits that seem to have an indeterminate form.

1.3.1 The Constant Rule for Limits

Throughout your experience with mathematics, you have and will come across many constants. This rule here is for evaluating the limit of a constant. It sounds like something pointless and obvious,

and it is, but nevertheless, we will prove it.
The Constant Rule, states that:

The limit of a constant number b, as x approaches some value a, is equal to the same constant b, or in other words,

$$\lim_{x \to a} b = b \tag{1.22}$$

For example,

$$\lim_{x \to 2021} 52 = 52 \tag{1.23}$$

There isn't much more to it. Lets Prove it.

Proving the Constant Rule

In order to prove this, and all the other rules, we will use the Delta-Epsilon Definition.

Suppose that a,b are constants and that

$$\lim_{x \to a} b = b \tag{1.24}$$

According to the Delta-epsilon definition of Limits, we need to find a $\delta > 0$, such that for every $\epsilon > 0$,where $|b - b| < \epsilon$ whenever $0 < |x - a| < \delta$
Obviously $|b - b| = 0$, and since $\epsilon > 0$, the inequality $|b - b| < \epsilon$ holds for any value of δ, proving our point.

1.3.2 The Identity Rule for Limits

The Identity rule for limits, is by far the easiest to understand. All it simply states is that:

The limit of x as x approaches some value a, is equal to a.

$$\lim_{x \to a} x = a \tag{1.25}$$

Proving the Identity Rule for Limits

In order to prove this, we must find a $\delta > 0$ such that for every $\epsilon > 0$, we get that $|x-a| < \epsilon$ whenever, $0 < |x-a| < \delta$, We can satisfy this condition with $\delta = \epsilon$, thereby proving that the identity is true.

1.3.3 The Scalar Product Rule for Limits

The Scalar product rule helps a lot with cleaning up our equations and keeping in only what is needed. The Scalar Product rule, states that:

If the limit of an function f(x) times a constant k, as x approaches some value c exists, then it is equal with the k^{th} multiple of the limit of that function f(x) as x approaches some value c, or simply

$$\lim_{x \to c} kf(x) = k \cdot \lim_{x \to c} f(x) \tag{1.26}$$

For example,

$$\lim_{x \to 2} 5x^2 = 5 \cdot \lim_{x \to 2} x^2 = 5 \cdot 4 = 20 \tag{1.27}$$

Proving the Scalar Rule

Again, to prove this rule, we will use our beloved delta-epsilon definition for limits.

Suppose that

$$\lim_{x \to a} c \cdot f(x) = c \cdot L \tag{1.28}$$

There exists a $\delta > 0$, such that $|f(x) - L| < \frac{\epsilon}{k}$, whenever $0 < |x-a| < \delta$, for some $k > 0$, so that $|c| < k$. Thus, we get that

$$|c \cdot f(x) - c \cdot L| = |c| \cdot |f(x) - L| < \epsilon \tag{1.29}$$

1.3.4 The Sum Rule for Limits

The next rule that we are going to learn is the Sum Rule . The sum rule is very useful and also very easy to understand. It is very intuitive, but also very helpful, however, it's not that easy to prove.

If the limits, as x approaches some value c, of two functions, call them f(x) and g(x) exist, then the sum of their limits, as x approaches the value c, is equal to the limit, as x approaches that same value c of the sum of the two function.

$$\lim_{x \to c} (f(x) + g(x)) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x) \tag{1.30}$$

Obviously you can do the same when subtracting.

$$\lim_{x \to c} (f(x) - g(x)) = \lim_{x \to c} f(x) - \lim_{x \to c} g(x) \tag{1.31}$$

For example, if we had a function, say $h(x) = 2x^2 + 3x$, we could break it into two separate functions $f(x) = 2x^2$ and $g(x) = 3x$, and calculating, as the limit as x approaches 3 of h(x), we can simply do:

$$\lim_{x \to 3} (2x^2 + 3x) = \lim_{x \to 3} 2x^2 + \lim_{x \to 3} 3x \tag{1.32}$$

Proving the Sum Rule

In order to prove this rule, we will need to use the delta-epsilon definition and come up with some creative ways to rewrite everything.

So first of all, suppose we have two functions, f(x) and g(x), such that their limits, as x approaches some value c, that we denote with L and M respectively, exist:

$$\lim_{x\to c} f(x) = L \quad and \quad \lim_{x\to c} g(x) = M \tag{1.33}$$

What we want to show is that

$$\lim_{x\to c} (f(x) + g(x)) = L + M \tag{1.34}$$

Here comes our definition in to play.
According to the Delta-Epsilon Definition for a limit, in order to prove this, we need to show, that

For any $\epsilon > 0$, we can find a $\delta > 0$, such that

$$If \;\; 0 < |x - c| < \delta, \;\; then \;\; |(f(x) + g(x)) - (L + M)| < \epsilon. \tag{1.35}$$

What we are going to do to solve this, is to break the two functions into their own limits, and just write what we know.
We will use $\delta_1 \;\; and \;\; \epsilon_1$ for f(x) and $\delta_2 \;\; and \;\; \epsilon_2$ for g(x).
We can now rewrite the limit of f(x) as x approach some value c, in the form of the delta-epsilon definition:
For any $\epsilon_1 > 0$, we can find a $\delta_1 > 0$, such that
If $0 < |x - c| < \delta_1$, then $|f(x) - L)| < \epsilon_1$

The same is true for g(x)

For any $\epsilon_2 > 0$, we can find a $\delta_2 > 0$, such that

If $0 < |x - c| < \delta_2$, then $|g(x) - M)| < \epsilon_2$

Now lets consider what would happen if $\epsilon_1 = \epsilon_2 = \frac{\epsilon}{2}$

There need to exist some δ_1 and δ_2, so that,

If $0 < |x - c| < \delta_1$, then $|f(x) - L)| < \frac{\epsilon}{2}$

and

If $0 < |x - c| < \delta_2$, then $|g(x) - M)| < \frac{\epsilon}{2}$ We now suppose that the value δ, was the smaller of the two values we just used, $\delta_1 and \delta_2$, where obviously, $\delta > 0$,since $\delta_1, \delta_2 > 0$. Rewriting everything now should look something like this:

If $0 < |x - c| < \delta$, then $|f(x) - L| < \frac{\epsilon}{2}$ and $|g(x) - M| < \frac{\epsilon}{2}$

Now, we will add the two inequalities, and say that:

If $0 < |x - c| < \delta$, then $|f(x) - L| + |g(x) - M| < \epsilon$

You have probably already learned the Triangle Inequality, which states that: $|A + B| \leq |A| + |B|$, which will help us out tremendously here, because we can write that: If $0 < |x - c| < \delta$, then $|f(x) - L + g(x) - M| \leq |f(x) - L| + |g(x) - M| < \epsilon$ Which gives us the solution, since we can rearrange everything and get that: If $0 < |x - c| < \delta$, then $|f(x) + g(x) - (L + M)| < \epsilon$, which is exactly what we wanted to prove (Eq. 1.20), thus:

$$\lim_{x \to c} (f(x) + g(x)) = L + M \tag{1.36}$$

or

$$\lim_{x \to c} (f(x) + g(x)) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x) \tag{1.37}$$

Solve these limits using the sum rule

1)
$$\lim_{x\to 10} 15x^2 + 23x + 7x \tag{1.38}$$

2)
$$\lim_{x\to 3} 11x^2 - 14x^{(3x-x^2)+15x} \tag{1.39}$$

3)
$$\lim_{x\to -3} 7x^2 + 15x - 2021^{(x+3)} \tag{1.40}$$

4)
$$\lim_{y\to 3} \frac{14y+28+6}{2y} \tag{1.41}$$

5)
$$\lim_{x\to 9} \frac{3x^2+2x-\sqrt{x}}{3} \tag{1.42}$$

1.3.5 The Product Rule for Limits

The Product rule , is a very useful rule too and makes our lives a lot easier, especially in future chapters. The product rule, as its name suggests, is a rule about the product of functions in limits. Specifically, it states that:

If the limits of two functions, as x approaches some value c exist, then, the limit, as x approaches that c, of the product of the two functions, is equal to the product of the limits of the two functions as x approaches that very c.

$$\lim_{x\to c}(f(x)\cdot g(x)) = \lim_{x\to c} f(x)\cdot \lim_{x\to c} g(x) \tag{1.43}$$

For example, if we had a function $h(x) = 15x^3 + 8x$, we could use the three rules we have learned so far to calculate the limit of h(x) as x approaches say 2.

$$\lim_{x\to 2} 15x^3 + 8x = \lim_{x\to 2} 15x^3 + \lim_{x\to 2} 8x = L \tag{1.44}$$

we can now use the scalar rule and the new product rule, just for fun, to rewrite everything as:

$$\lim_{x \to 2}(3x^2) \cdot \lim_{x \to 2} 5x + 8 \lim_{x \to 2} x = L \tag{1.45}$$

Having a oversimplified equation makes it very easy to solve :

$$\lim_{x \to 2} 3x^2 \tag{1.46}$$

becomes 12,

$$\lim_{x \to 2} 5x \tag{1.47}$$

becomes 10,

$$8 \lim_{x \to 2} x \tag{1.48}$$

becomes 16, and now combining everything together, we get that

$$L = 136 \tag{1.49}$$

Proving the Product Rule for Limits

Let there be a f(x) and a g(x), such that

$$\lim_{x\to a} f(x) = L \quad \textit{and} \quad \lim_{x\to a} g(x) = K \tag{1.50}$$

We can now, using the sum and the constant rule, prove that:

$$\lim_{x\to a}(f(x)-L) = \lim_{x\to a} f(x) - \lim_{x\to a} L = L - L = 0 \tag{1.51}$$

and

$$\lim_{x\to a}(g(x)-K) = \lim_{x\to a} g(x) - \lim_{x\to a} K = K - K = 0 \tag{1.52}$$

Next on, for any $\epsilon > 0$, there exists a $\delta_1 > 0$ and a $\delta_2 > 0$, such that

$$|(f(x)-L)-0| < \sqrt{\epsilon} \quad \textit{whenever} \quad 0 < |x-a| < \delta_1 \tag{1.53}$$

and

$$|(g(x)-K)-0| < \sqrt{\epsilon} \quad \textit{whenever} \quad 0 < |x-a| < \delta_2 \tag{1.54}$$

In order to advance, we will define a δ, which is the smaller of the two delta values. We can now say, that

If $0 < |x-a| < \delta$, then

$$|(f(x)-L)(g(x)-K)-0| = |(f(x)-L)(g(x)-K)| < \epsilon \tag{1.55}$$

We have just proven that

$$\lim_{x\to a}(f(x)-L)(g(x)-K) = 0 \tag{1.56}$$

Before actually proving the product rule, we will expand the $(f(x)-L)(g(x)-K)$ product, which gives us:

$$(f(x)-L)(g(x)-K) = f(x)g(x) - Kf(x) - Lg(x) + LK \tag{1.57}$$

We can now rearrange everything, and write the equation as the result of f(x)g(x):

$$f(x)g(x) = (f(x)-L)(g(x)-K) + Kf(x) + Lg(x) - LK \quad (1.58)$$

Now, using what we just wrote, we can finally prove the product rule

$$\lim_{x\to a} f(x)g(x) = \lim_{x\to a} ((f(x)-L)(g(x)-K) + Kf(x) + Lg(x) - LK) \quad (1.59)$$

We can use the sum rule to get:

$$= \lim_{x\to a}(f(x)-L)(g(x)-K) + \lim_{x\to a} Kf(x) + \lim_{x\to a} Lg(x) - \lim_{x\to a} LK \quad (1.60)$$

Everything now makes sense and we can finally see the end of the tunnel:

$$= 0 + \lim_{x\to a} Kf(x) + \lim_{x\to a} Lg(x) - \lim_{x\to a} LK \quad (1.61)$$

Now, using the the constant rule, we clearly see, that

$$= KL + LK - LK = KL \quad (1.62)$$

Which proves the product rule.

Solve these limit using the product rule

1)

$$\lim_{x\to 9} 5x^3\sqrt{x} + 2x^2 \cdot 3\sqrt{x} \quad (1.63)$$

2)

$$\lim_{x\to 16} \frac{10x \cdot \sqrt{\sqrt{x}}}{10x^2} \quad (1.64)$$

3)

$$\lim_{x\to -2} 4x^2 + 32x^3 - 16x + 8x^4 \quad (1.65)$$

4)

$$\lim_{y\to 3} 2x^3 - x^2 + 6x - 3 \quad (1.66)$$

5)

$$\lim_{x\to 2} x^2 + 4x + 2 + \frac{13^{(x-2)^{2021}}}{101^x - 20200 + 10000} \quad (1.67)$$

1.3.6 The Power Rule for Limits

The Power Rule , is yet another useful rule. With this rule, calculating the limits of a function that has powers in it is very easy. For example, calculating the 3rd root of some multiple of x, which of course is simply the function raised to the $\frac{1}{3}^t h$ power, as x approaches some value c may not provide the answer directly, but by manipulating the limit with this rule, you can simplify everything and get the result faster and with less operations. The Power rule, states, that:

If the limit of a function f(x), that is raised to a real number k, as x approaches some value c exists, then it is equal to the limit of f(x) as x approaches that value c, raised to that number k.

$$\lim_{x\to c}(f(x))^k = (\lim_{x\to c} f(x))^k \tag{1.68}$$

where k$\in \mathbb{R}$

For example, suppose we have an $f(x) = \sqrt{\frac{15x^2}{2}+6}$. We could rewrite that as $f(x) = (\frac{15x^2}{2}+6)^{\frac{1}{2}}$. Now suppose we wanted to find the limit of f(x) as x approaches 2, of f(x). We could use the power rule and define a new h(x), such that $f(x) = h(x)^{\frac{1}{2}}$. Now all we do is evaluate the limit:

$$\lim_{x\to 2} h(x)^{\frac{1}{2}} = (\lim_{x\to 2} h(x))^{\frac{1}{2}} = (\lim_{x\to 2} \frac{15x^2}{2}+6)^{\frac{1}{2}} \tag{1.69}$$

and now finally we get that

$$(\lim_{x\to 2} \frac{15x^2}{2}+6)^{\frac{1}{2}} = 36^{\frac{1}{2}} = 6 \tag{1.70}$$

Proving the Power Rule for Limits

Again, to prove this rule we will use the delta-epsilon definition for limits. This is probably the hardest one to prove, because we will have top prove it using mathematical induction . Here, we will prove it for n being an integer , but feel free to try and prove it for Real numbers by yourselves.

Suppose we have a function f(x), whose limit, as x approaches some value a exists. We want to prove, that

$$\lim_{x\to a}(f(x))^n = (\lim_{x\to a} f(x))^n = K^n \tag{1.71}$$

For n = 2, we can use the product rule, and we see that

$$\lim_{x\to c}(f(x))^2 = \lim_{x\to a} f(x)f(x) = \lim_{x\to a} f(x)\lim_{x\to a} f(x) = K \cdot K = K^2 \tag{1.72}$$

We have just proved, that this rule holds for n = 2. Let's now assume that this rule also holds for n-1, or that simply

$$\lim_{x\to c}(f(x))^{n-1} = K^{n-1} \tag{1.73}$$

Then, by using the product rule, we can finally prove the power rule, since we have that:

$$\lim_{x\to c}(f(x))^n = \lim_{x\to c}(f(x)^{n-1}f(x)) = \lim_{x\to c} f(x)^{n-1} \cdot \lim_{x\to c} f(x) \tag{1.74}$$

or simply

$$K^{n-1} \cdot K = K^n \tag{1.75}$$

, which proves, by induction the power rule for integers.

Solve these limit using the Power Rule

1)
$$\lim_{x\to 2}(5x+7x^2)^3 \tag{1.76}$$

2)
$$\lim_{x\to 3}\sqrt{2x^4-13x^2+\sqrt{x^2+7x}} \tag{1.77}$$

3)
$$\lim_{x\to -12}\frac{13x^2+\sqrt{13x^{-2}}}{x^2-\sqrt{x^2+4}} \tag{1.78}$$

4)
$$\lim_{y\to 6}\sqrt{\frac{14x^7+5x^3-\sqrt{\frac{1}{2x^2+3}}}{15x^2-10}} \tag{1.79}$$

5)
$$\lim_{x\to 11}\frac{e^{x^2-11x}\sqrt{17x^2-34x+\frac{x^3-8\sqrt{\frac{12x}{33}}}{e^{\frac{7623}{x}}-693}}}{10x^{-2}\cdot 15x+\sqrt{11x}} \tag{1.80}$$

1.3.7 The Quotient Rule

The Quotient Rule is also a useful tool when it comes to calculating limits. It is not a new rule in the way the previous ones were.

The Quotient rule is a combination of the Product and the Power Rule. All it states, is that, if the following limits exist, and if

$$\lim_{x\to c} f(x)=L \quad \textit{and} \quad \lim_{x\to c} g(x)=K \tag{1.81}$$

$$\Longleftrightarrow \tag{1.82}$$

$$\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c} f(x)\cdot g(x)^{-1}=\lim_{x\to c} f(x)\cdot\lim_{x\to c} g(x)^{-1}=\frac{L}{K} \tag{1.83}$$

Is this extremely easy to prove, and is therefore left, to you, the reader as an exercise. An example of this rule would be, that if we had a function, call it h(x), which is equal to $h(x)=\frac{13x^2+7x}{12x^{838}+38x}$ and we wanted to find its limit as x approaches, say 1. We could divide the function into 2 other functions f(x) and g(x), such that

$$f(x) = 13x^2 + 7x \quad and \quad g(x) = 12x^{838} + 38x \tag{1.84}$$

We can now use the Quotient Rule to find the limit of h(x) as x approaches 1:

$$\lim_{x\to 1} \frac{13x^2 + 7x}{12x^{838} + 38x} = \lim_{x\to 1}(13x^2 + 7x) \cdot (\lim_{x\to 1}(12x^{838} + 38)^{-1} \tag{1.85}$$

which of course leaves us with

$$(13 + 7) \cdot (12 + 38)^{-1} = \frac{20}{50} = \frac{2}{5} \tag{1.86}$$

Now it is your time to practice this rule

Solve these limit using the Quotient Rule

1)

$$\lim_{x\to 3} \frac{13x^2 + 7x}{9x^4 - 7x^3 + 12x^2 - 5x + 20} \tag{1.87}$$

2)

$$\lim_{x\to 2} \frac{13x \cdot 14x^2}{41x - \frac{164}{x} + 3x} \tag{1.88}$$

3)

$$\lim_{x\to 5} \frac{7.2x \cdot 5x^{\frac{1}{2}}}{4x^2 - \frac{125x}{5x} + 2x^2 + 12} \tag{1.89}$$

1.4 Squeeze Theorem

In Calculus, there exists a theorem called the Squeeze Theorem , also known as the sandwich theorem, or the two policemen and a drunk theorem, which is all about the limit of a function. The Squeeze theorem is way of **finding the limit of a function by comparing it with the limits of other functions that satisfy a specific condition**. The story, and idea behind the theorem, is that if two policemen, who are escorting a drunk prisoner to a cell, where one is in front of the prisoner and one behind, forming a sandwich, are both in a cell, then, no matter which path they took, and no matter how fast they got there, the prisoner must also be in the cell.
The mathematical translation , is that:

If we have some functions, f,g and h, that are defined over an interval D, except maybe the point c, and the following relation holds for all x in the Interval D, that are not equal to c:

$$g(x) \leq f(x) \leq h(x) \tag{1.90}$$

and if we suppose that the limits of g(x) and h(x), as x approaches the value, c, are both equal to L, then, the limit of f(x) as x approaches the same value c, must also be equal to L:

$$If \quad \lim_{x\to c} g(x) = L, \quad and \quad \lim_{x\to c} h(x) = L \Longrightarrow \lim_{x\to c} f(x) = L \tag{1.91}$$

The Functions g(x) and h(x), are also called the lower(g(x)) and upper (h(x)) bounds of f(x).
Moreover, c doesn't need to be in the interior of D. It can be and endpoint of D, but in that case, we take either of the right and left limits.

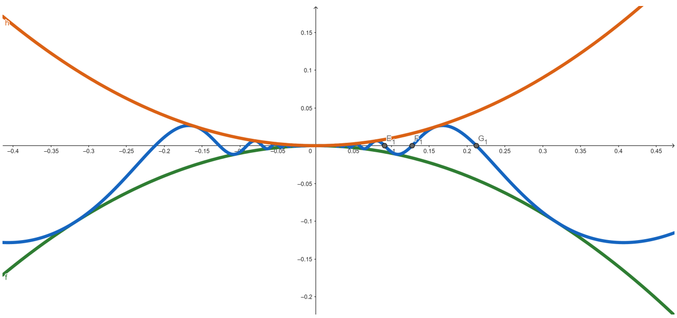


Figure 1.4: Squeeze Theorem Graph example

1.4.1 Proof of the Squeeze Theorem

Let g(x), f(x), and g(x), be three functions such that

$$\forall x \in \mathbb{R}, \quad g(x) \leq f(x) \leq h(x) \tag{1.92}$$

, where:

$$\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L \tag{1.93}$$

For any given $\epsilon > 0$, we can find two δ, namely δ_1 and δ_2, such that:
If

$$0 < |x - a| < \delta_1 \tag{1.94}$$

and

$$0 < |x - a| < \delta_2 \tag{1.95}$$

, then:

$$|g(x) - L| < \epsilon \tag{1.96}$$

and

$$|h(x) - L| < \epsilon \tag{1.97}$$

Now, using the first relation, that $g(x) \leq f(x) \leq h(x)$, we can rewrite the equations (1.90) and (1.91) as

$$L - \epsilon < g(x) \tag{1.98}$$

and

$$L + \epsilon > h(x) \tag{1.99}$$

We can write this, because $h(x) \geq g(x)$, so the only point where they are equal are at a, in which case both equal L. That means that $h(x) \geq L$, so the absolute value $|h(x) - L|$ is always h(x) - L. The opposite is true for g(x), which is the lower boundary of f(x), which, as we said $g(x) \leq h(x) \longrightarrow g(x) \leq L$, which means, that the absolute value $|g(x) - L|$ is always equal to L-g(x). Knowing this, we ca rearrange the two inequalities and get equations (1.92) and (1.93)

Next, we define a new value, δ, to be the smaller of the two δ_1, δ_2.
We can now write, that, if $|x-a| < \delta$, then

$$L-\epsilon < g(x) \leq f(x) \leq h(x) < L+\epsilon \tag{1.100}$$

, or

$$L-\epsilon < f(x) < L = \epsilon \tag{1.101}$$

and now with some basic algebra, we can rewrite it as:

$$-\epsilon < f(x) - L < \epsilon \tag{1.102}$$

, which is equal to

$$|f(x) - L| < \epsilon \tag{1.103}$$

, Proving the Squeeze Theorem

You should be able by now to see why this proves the theorem.

1.4.2 Example of The Squeeze Theorem

We just learned about the squeeze theorem, which is probably the most intuitive theorem in this book, but, he haven't seen any useful applications, when it comes to solving limits, up until now. I will now show you how to calculate a very famous limit using the squeeze theorem.

Proof of sin(x)/x

We will now prove, one of the most famous limits,

$$\lim_{x \to 0} \frac{sin(x)}{x} = 1 \tag{1.104}$$

In order to prove this, we will have to use the Unit Circle :

The angle FES is the angle x. What we want to see is what happens when x approaches 0.
First of all, we will create a relation, so that we can later on use the squeeze theorem.
By definition, we know, that: The area of the triangle EFO, is less or equal than the area of the sector EFO, which less or equal than

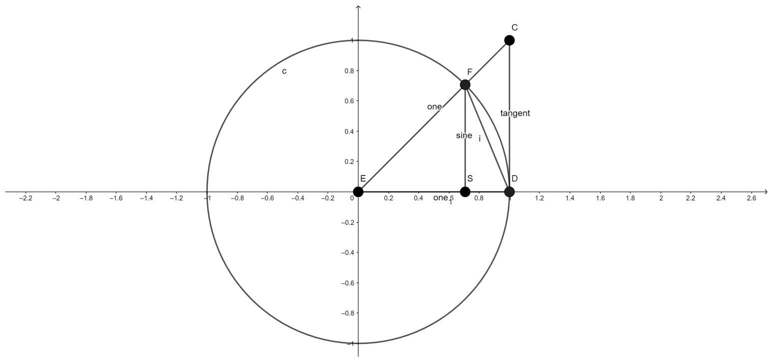


Figure 1.5: Unit Circle

the area of the triangle ECO.
Using basic geometry and trigonometry, we see that, the area of EFO, is equal to $\frac{sin(x)}{2}$, the area of the sector EFO is $\frac{x}{2}$, and finally, the area of the ECO triangle is equal to $\frac{tan(x)}{2}$.

$AreaT = \frac{1}{2} \cdot base \cdot height \ and \ AreaS = r^2 \cdot \frac{angle}{2}$

We can rewrite everything as:

$$\frac{sin(x)}{x} \leq \frac{x}{2} \leq \frac{tan(x)}{2} \tag{1.105}$$

We can now multiply everything with some value to get to a relation that we can use. Intuition tells us to multiply by 2 to lose the denominator, however, we can do something way better that will give us the answer almost immediately. If you remember, we said that we the value of the limit we want to find, doesn't need to necessarily be in the interval that the function(s) are defined, it can be an endpoint of an open interval.
We will multiply all the terms by $\frac{2}{sin(x)} > 0$, which will give us:

$$1 \leq \frac{x}{sin(x)} \leq \frac{1}{cos(x)} \tag{1.106}$$

Now, by simply taking reciprocals, gives us:

$$cos(x) \leq \frac{sin(x)}{x} \leq 1 \tag{1.107}$$

Now, finally, by taking the limits of of each term, as x approaches 0, yields:

$$\lim_{x\to 0} cos(x) \leq \lim_{x\to 0} \frac{sin(x)}{x} \leq \lim_{x\to 0} 1 \tag{1.108}$$

and when evaluating the limits, we get that:

$$1 \leq \lim_{x\to 0} \frac{sin(x)}{x} \leq 1 \tag{1.109}$$

and, According to the Squeeze Theorem, $\frac{sin(x)}{x} = 1$

Prove that

$$\lim_{x\to 0} x^2 cos(\frac{1}{x}) = 0 \tag{1.110}$$

1.5 Limits at ∞

We have seen limits at 0, and how they work, but we haven't talked about infinity at all. First of all, infinity is not a number. Infinity is more of an idea. Therefore, saying that $\frac{1}{\infty} = 0$, which seems logical is wrong. It is not defined. However, what is defined is its limit, or in other words

$$\lim_{x\to\infty} \frac{1}{x} \tag{1.111}$$

which does equal to 0. **Infinity is something beautiful and mathematics, but you need to be cautious when using it and solving limits at infinity.**

1.5.1 Degrees of the Functions

When talking about limits at infinity, the only thing that matters are degrees. If you have a simple function like for example $f(x) = 3x^2 + 12x + 29$, and take its limit at infinity, because the fastest growing term is positive and on the numerator, the limit of the function at infinity is $+\infty$. If we f(x) was instead equal to $f(x) = -x^3 - 9999x^2 + 9999$, its limit at infinity would be $-\infty$, since the faster growing term is in the numerator, and it is negative.

Generally,

When taking the limit of f(x) at infinity, you first divide f(x) in to 2 functions g(x) and h(x), such that $f(x) = \frac{g(x)}{h(x)}$**. Afterwards, you find the degree/faster growing term of each function. If the degree of h(x) is bigger than g(x)'s, then the limit is equal to 0. If g(x)'s is bigger then the limit is equal to** $\pm\infty$**, based on the sign of the fastest growing term. If the Degrees are equal, then the limit is equal to the quotient of the coefficients of the 2 fastest growing terms of the two functions.**

Suppose that $f(x) = ax^n + c_1$ **and** $g(x) = bx^m + c_2$**, where** $a, b \neq 0$
If $n > m$**, then**

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \pm\infty \quad (\textit{depending on the sign of } a) \tag{1.112}$$

If $m > n$**, then**

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0 \tag{1.113}$$

If $n = m$**, then**

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{a}{b} \tag{1.114}$$

Here are some examples:

$$\lim_{x\to\infty} x^3 = \infty \tag{1.115}$$

$$\lim_{x\to\infty} -x^3 + 13x^2 + 7 = -\infty \tag{1.116}$$

$$\lim_{x\to\infty} \frac{x^4 - 17x}{2021x^3 + 2020} = \infty \tag{1.117}$$

$$\lim_{x\to\infty} \frac{-x^3 - x^2}{13x + 5} = -\infty \tag{1.118}$$

$$\lim_{x\to\infty} \frac{2020x^8 + x^2 0}{x^{21}} = 0 \tag{1.119}$$

$$\lim_{x\to\infty} \frac{3x^6 + 7x + 20}{5x^6 - 14x^2 - 2021} = \frac{3}{5} \tag{1.120}$$

But not all limits can be solved as easy, and some can only be approximated. As I just said, infinity is not a number and you cannot substitute in wherever you want. For example, a very famous limit at infinity is Euler's number.
If I asked you to tell me the limit of

$$\lim_{n\to\infty} (1 + \frac{1}{n})^n \tag{1.121}$$

, you would probably do the following:

$$\lim_{n\to\infty} (1 + \frac{1}{n})^n = (1 + \frac{1}{\infty})^\infty = (1+0)^\infty = 1^\infty = 1 \tag{1.122}$$

Which is **Completely Wrong** We frankly do not know what $(1 + \frac{1}{\infty})^\infty$ is equal to, but we can approximate. If we start plugging very big numbers, we will start seeing a very familiar number. That number will be Euler's number, e. As I said, we cannot know what $(1 + \frac{1}{\infty})^\infty$, but we can say that it settles toward 2.71828...

Evaluate the following limits

1)
$$\lim_{x\to\infty} x^2+3x-10 \tag{1.123}$$

2)
$$\lim_{x\to\infty} 3x^2+8x+201-x^3 \tag{1.124}$$

3)
$$\lim_{x\to\infty} \frac{14x^3-82x62+13x-12}{12x^2+6x-1} \tag{1.125}$$

4)
$$\lim_{x\to-\infty} \frac{14x^3-82x62+13x-12}{12x^2+6x-1} \tag{1.126}$$

4)
$$\lim_{x\to\infty} \frac{20x^10+3.9\cdot 10^2021}{x^11} \tag{1.127}$$

5)
$$\lim_{x\to\infty} \frac{12x^5-11x^4+32x^3+3x^3-12x+90}{13x^2-20x^3-\frac{2}{3}x+6x-5x^5} \tag{1.128}$$

1.6 Continuity

Something that will help us a lot in the future and is important to know is whether a function is continuous or not.

We call a function f(x) continuous at the point c, if

$$\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = \lim_{x\to c} f(x) = f(c) \tag{1.129}$$

Note that it is crucial that it is equal to f(c) **A functions f(x), is called continuous on its domain if it is continuous on every point in its domain. If it is not continuous at a specific value a, we say that it is discontinuous at a, or that it has a discontinuity at a.**

Intuitively, a function is continuous if it doesn't require you to lift your pencil while creating its graph.

When it comes to continuity, there is a common misunderstanding when it comes to discontinuities. a function can be continuous on its domain, but not for all Real Numbers. That means, that we call a function discontinuous if it is not defined at some point. For example, $f(x) = \frac{1}{x}$ is continuous over its domain, since 0 is not in its domain. You need to be careful when it comes to continuity and what is your reference point.
Moreover, **Limits and whether a function is continuous or not are two different things. A limit of a function of x, as x approaches some value c can be defined, but that doesn't mean that the function is continuous at that point. In order for a function to be continuous, the following must be true** $\forall c$ in its domain.

$$\lim_{x \to c} f(x) = f(c) \tag{1.130}$$

Right now you might struggle with finding a function that has a discontinuity, because we haven't talked about such functions so far. These functions of course are of the case functions, or for example:

$$f(x) = \begin{cases} x^2 \quad if \quad case \quad 1 \\ 2x \quad if \quad case \quad 2 \\ 3x^2 \quad if \quad case \quad 3 \\ . \\ . \end{cases} \tag{1.131}$$

These functions are the ones that usually have discontinuities, meaning the limit of the function at some point x_0 is different from $f(x_0)$, which the definition for a discontinuity.

For example, lets see one of these functions:
Suppose we have a function f(x), such that:

$$f(x) = \begin{cases} 2x^2 & \text{for } x \neq 1 \\ 1 & \text{for } x = 1 \end{cases} \tag{1.132}$$

Here we see that there is a clear discontinuity at x = 1. If we find the limit of the function f(x) as x approaches 1, it is clear that the result will be 2, however, since the function has an exception for

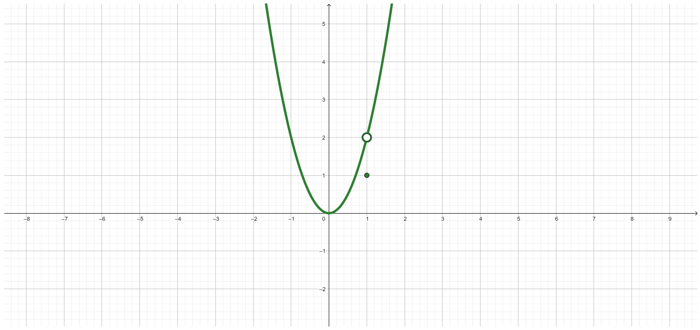


Figure 1.6: Discontinuity at x=1

when x takes the value x, we see that

$$\lim_{x \to 1} f(x) \neq f(1) \quad or \quad 2 \neq 1 \tag{1.133}$$

There are many types of discontinuities. The one displayed by the figure 1.6, is called a removable discontinuity. We will now examine different types of discontinuities and what they mean.

1.6.1 Types of Discontinuities

There are 3 types of discontinuities, all similar in some way and different in another. The main idea, as we said previously is that

$$\lim_{x \to c} f(x) \neq f(c) \tag{1.134}$$

The 3 different discontinuities are the different ways to satisfy the relation above:

Let f(x) be a function defined over an interval D that has a discontinuity at $x = x_0$, and let

$$\lim_{x \to x_0} f(x) = L$$

The Discontinuity is a Removable Discontinuity , if the limit L exists and

$$\lim_{x \to x_0} f(x) \neq f(x_0)$$

The Discontinuity is a Jump Discontinuity , if the Limits L^+ and L^- are defined, but are not equal.

$$L^+ \neq L^-$$

The Discontinuity is an Essential Discontinuity , if at least one of the L^+ and L^- doesn't exist.

Note that for there to be a discontinuity at the point x_0, the point x_0 needs to be in the function's domain!

Here are some discontinuous functions and their graphs.

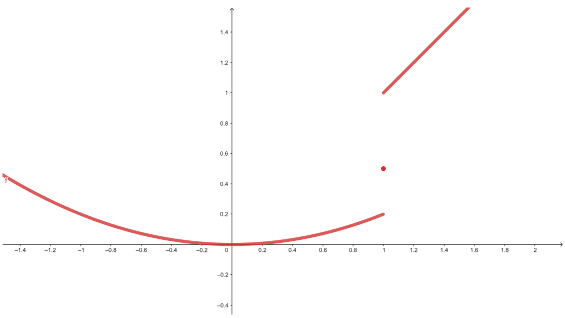


Figure 1.7: Example of Jump Discontinuity at x = 1

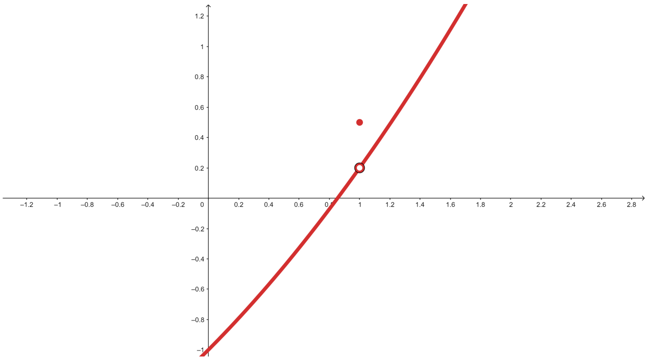


Figure 1.8: Example of Removable Discontinuity at x=1

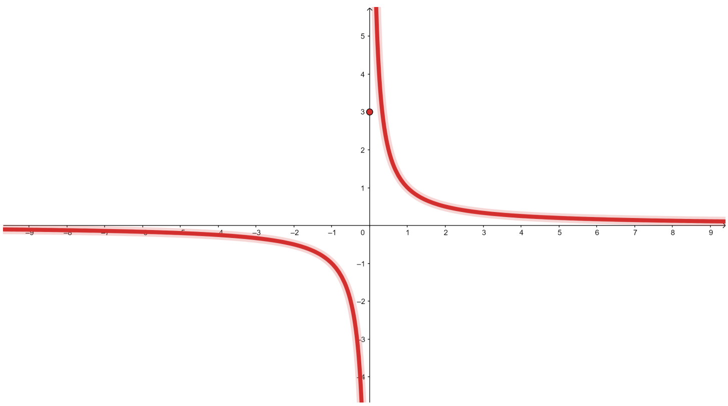


Figure 1.9: Example of Essential Discontinuity at x=0

Find which of these functions have discontinuities, and the type of each discontinuity

1)

$$f(x) = \frac{1}{x} \tag{1.135}$$

2)

$$g(x) = 3x^2 + 17x - \frac{x^2}{x-2} \tag{1.136}$$

3)

$$h(x) = \begin{cases} \frac{6x}{x+1} & \text{for } x \neq 1 \\ 3 & \text{for } x = 1 \end{cases} \tag{1.137}$$

4)

$$k(x) = \begin{cases} \frac{2x}{x^3+11} & \text{for } x \leq 1 \\ 3 & \text{for } x = 1 \\ 12x^2 & \text{for } x \geq 1 \end{cases} \tag{1.138}$$

5)

$$l(x) = \begin{cases} \sin \frac{3}{3x-3} & \text{for } x < 1 \\ 0 & \text{for } x = 1 \\ \frac{1}{x-1} & \text{for } x > 1. \end{cases} \tag{1.139}$$