Linear Algebra

Stavros Klaoudatos

Contents

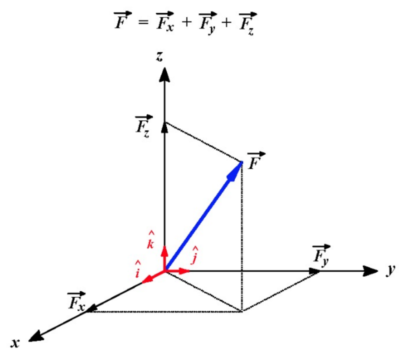
1 Introduction

In this chapter, I will talk to you about Vectors and Vector Spaces, More precisely about Hilbert space, what a vector is, Cross and Dot Products, Bra-Ket Vectors and Inner Products.
These are concepts you need to clearly understand before moving to Quantum Mechanics!

2 What is a Vector?

Vectors are mathematical objects with direction and magnitude! We will discuss about Vectors in 3 Dimensions!
A vector can be analyzed in its 3 components, the x component, the y component and the z component!

An example of a vector is a Force. A force has both magnitude and direction and is represented by a 3 Dimensional Object!



3 Unit Vectors

From Mathematics we have been used to using the unit of 1! Every number in each of components of a vector is a multiple of 1.
In physics we have seen a using units of time or speed or Newtons. Generally we use three letters to symbolize the unit of each axis! These letter are:

$$\hat{i}\ for\ x$$

$$\hat{j}\ for\ y$$

$$\hat{k}\ for\ z$$

These unit vectors, are vectors with a **magnitude of 1**, and the corresponding direction! This is a simple way of showing the direction of a vector!
For example, suppose a vector $\overrightarrow{V}$
and let's say we are given that

$$\overrightarrow{V}_x = 10$$

$$\overrightarrow{V}_y = -20$$

$$\overrightarrow{V}_z = 50$$

(1)

Using what we just learned, we can rewrite it like this:

$$\overrightarrow{V} = 10\hat{\imath} - 20\hat{\jmath} + 50\hat{k} \tag{2}$$

and that provides all the information about that vector, both magnitude and direction!
These Unit Vectors are orthonormal, meaning they are all perpendicular to each other! This holds also true for more than three Dimensions! Such a Space is called a Hilbert Space and is exactly that!
If you know the components of a vector, you can calculate the total magnitude of the vector using the Pythagorean Theorem!

$$\overrightarrow{V} = \sqrt{(10\hat{\imath})^2 + (-20\hat{\jmath})^2 + (50\hat{k})^2} \tag{3}$$

4 Column Vectors

A vector can also be simply represented by a column vector! There is not much to say, you simply arrange the components in a column starting on top with the x component and ending down with the z component

$$\overrightarrow{V} = \begin{pmatrix} \overrightarrow{V}_x \\ \overrightarrow{V}_y \\ \overrightarrow{V}_z \end{pmatrix} = \overrightarrow{V}_x\hat{\imath} + \overrightarrow{V}_y\hat{\jmath} + \overrightarrow{V}_z\hat{k} \tag{4}$$

or

$$\overrightarrow{V} = \begin{bmatrix} \overrightarrow{V}_x \\ \overrightarrow{V}_y \\ \overrightarrow{V}_z \end{bmatrix} = \overrightarrow{V}_x\hat{\imath} + \overrightarrow{V}_y\hat{\jmath} + \overrightarrow{V}_z\hat{k} \tag{5}$$

5 Operations

Vector Operations are a bit different from the classical operation we are used to!
To add Vectors, you can simply add the components of the Vectors and the either leave it be or calculate the magnitude of the new Vector with the Pythagorean Theorem.
You can Multiply Vectors in two ways: The Dot Product, and the Cross Product

5.1 Dot product

Multiplying a Vector by a scalar $a$is very easy:

$$a \cdot \overrightarrow{V} = a \cdot \begin{pmatrix} \overrightarrow{V}_x \\ \overrightarrow{V}_y \\ \overrightarrow{V}_z \end{pmatrix} = \begin{pmatrix} a\overrightarrow{V}_x \\ a\overrightarrow{V}_y \\ a\overrightarrow{V}_z \end{pmatrix} \tag{6}$$

This is the usual way of multiplying! When multiplying two Vectors say n and m, the result is

$$\overrightarrow{n} \cdot \overrightarrow{m} = (n)(m)(\cos\theta) \tag{7}$$

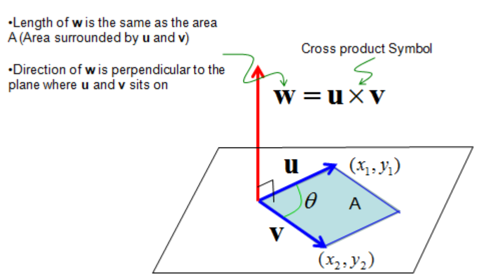
where θ is the angle the two vectors form.
Or, a maybe simpler way of multiplying vectors, is to do the following:

$$\overrightarrow{n} \cdot \overrightarrow{m} = \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} \cdot \begin{pmatrix} m_x \\ m_y \\ m_z \end{pmatrix} = \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} \cdot m_x + \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} \cdot m_y + \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} \cdot m_z \tag{8}$$

An now we can treat each component of m as a scalar and perform the initial multiplication!

5.2 Cross Product

Now the cross product of two vectors is a lot harder and way more complex! The out put of the cross product is a vector that is perpendicular to plane the two initial vectors form, and its magnitude is equal to the area of the plane!



it is denoted like this

$$\overrightarrow{V} \times \overrightarrow{F} \tag{9}$$

It is mostly used in Multivariate Calculus and Electromagnetism as well as in Fluid Dynamics, between a vector and the Del Operator ∇, but more on that in the future!
Tom calculate the cross product, you need to create a matrix and do the following

operations:

$$\overrightarrow{V} \times \overrightarrow{f} = \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ V_x & V_y & V_z \\ F_x & F_y & F_z \end{bmatrix} = [(V_y \cdot F_z) - (V_z \cdot F_y)]\hat{\imath} + [(V_z \cdot F_x) - (V_x \cdot F_z)]\hat{\jmath} + [(V_x \cdot F_y) - (V_y \cdot F_x)]\hat{k} \tag{10}$$

And then you continue solving it according to the components!

6 Bra-Ket Vectors

6.1 Ket Vectors

In Quantum Mechanics, a vector space, is composed by ket-vectors, or simply kets. Ket vectors are also complex vectors, meaning their components are complex numbers! A ket Vector is denoted like this: $|A\rangle$

Here are the Axioms about Ket Vectors:
Suppose $z, w \in C$

Sum of Kets is also a Ket:

$$|A\rangle + |B\rangle = |C\rangle \tag{11}$$

Addition is commutative:

$$|A\rangle + |B\rangle = |B\rangle + |A\rangle \tag{12}$$

Addition is Associative:

$$(|A\rangle + |B\rangle) + |C\rangle = |A\rangle + (|B\rangle + |C\rangle) \tag{13}$$

There is a unique vector 0 such that when added to any ket, it returns the same ket:

$$|A\rangle + 0 = |A\rangle \tag{14}$$

There is a unique ket $-|A\rangle$ for any ket $|A\rangle$, such that their sum is equal to 0:

$$|A\rangle + (-|A\rangle) = 0 \tag{15}$$

Given any ket $|A\rangle$, and any complex number z, you can multiply them to get a new ket:

$$|zA\rangle = z|A\rangle + |B\rangle \tag{16}$$

The distributive property holds:

$$z(|A\rangle + |B\rangle) = z|A\rangle + z|B\rangle \tag{17}$$

6.2 Bra Vectors

In the same way complex numbers have conjugates, a complex vector space has a dual version which is essentially the complex conjugate vector space.
For every ket vector $|A\rangle$, there is a bra vector $\langle A|$ in the dual space!
Generally the axioms are the same and hold true.
But here is something very important for the future that you shouldn't forget:

$$z|A\rangle \Longleftrightarrow \langle A|z^* \tag{18}$$

In general, if a Ket is represented by:

$$|A\rangle = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \tag{19}$$

where a_i are complex numbers, then the Corresponding Bra is represented by:

$$\langle A| = \begin{pmatrix} a_1{}^* & a_2{}^* & a_3{}^* \end{pmatrix} \tag{20}$$

6.3 Inner Product

The inner product for Bras and Kets is analogous to the dot product.
It is denoted like this:

$$\langle A|B\rangle \tag{21}$$

The result of this operation is a complex number
Here are again some Axioms about the inner product:
They are linear:

$$\langle C|\left(|A\rangle + |B\rangle\right) = \langle C|A\rangle + \langle C|B\rangle \tag{22}$$

Interchanging bras and kets corresponds to complex conjugation:

$$\langle A|B\rangle = (\langle A|B\rangle)^* \tag{23}$$

Normalised Vector:

A Normalized vector is a vector that suits the condition:

$$\langle A|A\rangle = 1 \tag{24}$$

Orthogonal Vectors :

Two Vectors are Orthogonal if they satisfy the condition:

$$\langle A|B\rangle = 0 \tag{25}$$

For example $|i\rangle$ and $|j\rangle$ are normalized and orthogonal to each other:

$$\langle i|j\rangle = \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = 1\cdot 0 + 0\cdot 1 = 0 \tag{26}$$

7 Linear Operators

An operator is a function that takes as input a vector and outputs a different vector!

There is an Operator called **Identity Operator**, which is denoted by I.
When it acts on a vector, it returns the same exact vector:
(for 3 dimensions

$$I = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} \tag{27}$$

$$I\,|A\rangle = |A\rangle \tag{28}$$

We will denote the Operator with the letter M for practicality's' sake.

$$M\,|A\rangle = |B\rangle \tag{29}$$

A Linear Operator must give a unique output for every vector in a vector space.

When it acts on a multiple of an input vector, it gives the same multiple of the output vector.

$$Mz\,|A\rangle = z\,|B\rangle \tag{30}$$

When it acts on a sum of vectors, the results are simply added to together.

$$M(|A\rangle + |B\rangle) = M|A\rangle + M|B\rangle \tag{31}$$

A linear operator is a matrix! It is NxN dimensions, based on the dimensions of the space, or in other words, **the amount of orthonormal basis vectors**. For us, it is 3!

M = $\begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}$ (32)

Now to put it in perspective, here is a general computation involving $|A\rangle$, $|B\rangle$ and the operator M

M = $\begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}$, $|A\rangle = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ (33)

$$M|A\rangle = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \tag{34}$$

$$b_1 = m_{11} \cdot a_1 + m_{12} \cdot a_2 + m_{13} \cdot a_3 \tag{35}$$

$$b_2 = m_{21} \cdot a_1 + m_{22} \cdot a_2 + m_{23} \cdot a_3 \tag{36}$$

$$b_3 = m_{31} \cdot a_1 + m_{32} \cdot a_2 + m_{33} \cdot a_3 \tag{37}$$

8 Eigenvalues and Eigenvectors

Before we start off, I want to say that I won't be covering thses in depth, since it requires deeper knowledge and more math, so I am simply going to introduce you to concepts!
When an Operator acts on a vector,**it changes its direction**, **but, If a vector undergoes linear transformation and its direction is constant,**
it's called an Eigenvector $|\lambda\rangle$

$$M|\lambda\rangle = \lambda|\lambda\rangle \tag{38}$$

In English that means that an eigenvector of an operator, when fed to the same operators, comes out as the same vector, just with a different magnitude, which is equal to the product of that vector with a complex number λ, which brings us to the next concept: **Eigenvalues** The eigenvalue(s) is the value that λ is multiplied by after the transformation! An operator can have multiple eigenvalues or eigenvectros. For example, the eigenvalues of the Identity Operator I are simply all the vectors! Since

$$I|A\rangle = |A\rangle \tag{39}$$

So the eigenvectors of I are all the vectors, and the eigenvalue is 1!
Another Example, this time in 2 dimensions
Suppose

$$M = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \tag{40}$$

and a vector

$$|\lambda\rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{41}$$

Is fed to M,

$$M|\lambda\rangle = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Longleftrightarrow$$

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 \cdot 1 + 2 \cdot 1 \\ 2 \cdot 1 + 1 \cdot 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

(42)

Which is obvious now how the vector didn't change its direction, and that it was simply multiplied by 3! Or in other words:

$$M|\lambda\rangle = 3|\lambda\rangle \tag{43}$$

Which means that 3 is the eigen value!!!
Operators in general act the same way on bras!

9 Hermitian Conjugation

Due to the Complex Conjugation of Complex Numbers, we said that

$$Z|A\rangle = |B\rangle \tag{44}$$

Doesn't mean that

$$\langle A|Z = \langle B| \tag{45}$$

,but

$$\langle A|Z^* = \langle B| \tag{46}$$

This obviously applies for z that have an imaginary part, since the conjugate of a real number, is the number itself.
That is for numbers! What happens to Operators!
Here comes **Hermitian Conjugation**
This is an operation on operators.
It is denoted with a dagger †. We denote the Hermitian Conjugate of M like this $M^\dagger$.
When performing such a conjugation, you **conjugate all the elements** of the operator, and then **transpose them**:
$M^\dagger = [M^*]^T$

$$M = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} \Longleftrightarrow M^\dagger = \begin{pmatrix} {m_{11}}^* & {m_{21}}^* & {m_{31}}^* \\ {m_{12}}^* & {m_{22}}^* & {m_{32}}^* \\ {m_{13}}^* & {m_{23}}^* & {m_{33}}^* \end{pmatrix} \tag{47}$$

That means that

$$\langle A|M^\dagger = \langle B| \tag{48}$$

9.1 Hermitian Operators

In Quantum Mechanics, observables are represented by linear operators, that satisfy:
$M = M^\dagger$
These operators, are called Hermitian Operators.
Overall Hermitian Operators are Operators that are equal to their Hermitian Conjugate!
The eigenvalues of Hermitian Operators are all real!
Will denote the Hermitian Operators with the Letter L.
Now I wont go into the derivation of the eigenvalues being all real numbers, but it is easy to do and a nice exercise!

9.2 The Fundamental Theorem

The eigenvectors of a Hermitian Operator are a complete set, meaning that any vector that the operator can generate can be expanded as a sum of its eigen vectors!

If λ_1 and λ_2 are two unequal eigenvalues of a Hermitian Operator, then their corresponding eigenvectors are orthogonal

If two eigenvalues λ_1 and λ_2, are equal, the corresponding eigenvectors can be orthogonal(When different eigen vectors have the same eigen value, it is called degeneracy)

In other words, the eigenvectors of Hermitian Operators form orthonormal basis

10 Outer Product

Although this topic belongs to the bra-ket section, in order to explain it mathematically, we to first explain all the previous sections!
Given a bra $\langle\phi|$ and a ket $|\psi\rangle$ we can also form their outer product, which we denote like:
$|\psi\rangle\langle\phi|$
The outer product **is not** a number. Instead, it is a

linear operator!
Lets consider what happens, when $|\psi\rangle\langle\phi|$ acts on a ket $|A\rangle$:
$|\psi\rangle\langle\phi||A\rangle$
The operation of the outer product is very simple, and defined as:
$|\psi\rangle\langle\phi||A\rangle = |\psi\rangle\langle\phi|A\rangle$
In other words, we take the **inner product**, of $\langle\phi|$ with $|A\rangle$, whose result is a **complex number**, and then multiply $|\psi\rangle$, by that result, which outputs an **operator**.

10.1 Projection Operators

A special case of the outer product, is the outer product of a ket with its own bra!
Assuming that $|\psi\rangle$ is normalized, this operator is called a **projection operator**.
$|\psi\rangle\langle\psi||A\rangle = |\psi\rangle\langle\psi|A\rangle$
The result is obviously always proportional to $|\psi\rangle$
1) Projection Operators are Hermitian

2) The vector $|\psi\rangle$, is an eigenvector of its own projection operator, with an eigenvalue of 1.
3) Any vector orthogonal to $|\psi\rangle$ is also an eigenvector, but with eigenvalue 0.
That means that the eigen values of a projection operator are 1 or 0, and there is only one eigenvector with eigenvalue 1, while all the others have 0.
4) The Square of a projection operator, is the operator itself.

$(|\psi\rangle\langle\psi|)^2 = |\psi\rangle\langle\psi|$
5) The trace of a projection operator is 1. This is obvious, since the trace of a Hermitian operator is the sum of its eigenvalues, and a projection eigenvalue, has all of its eigenvalues equal to 0, except from one that is equal to 1.
The Trace of an Operator, or any square matrix, is defined as the sum of its diagonal elements. Using the Notation Tr for trace, we define the trace of an operator L as:

$$Tr\, L = \sum_i \langle i|L|i\rangle \tag{49}$$

6) If we add all the projection operators for a basis system, we obtain the identity operator:

$$\sum_i |i\rangle\langle i| = I \tag{50}$$

Finally, a very important theorem about projection operators and expectation values

7) The expectation value of any observable L in state $|\psi\rangle$, is given by:

$$\langle\psi|L|\psi\rangle = Tr\ |\psi\rangle\langle\psi|L \tag{51}$$