Tensor Products

Stavros Klaoudatos

Contents

1 Labels and Notation 1

2 Representing the Combined System 2

3 Tensor Products in Component Form 3

1 Labels and Notation

Combining smaller systems to create a bigger composite system plays a vital role in physics
An atom for example, is nothing but the collection of nucleons and electrons, each of which has its own system, and when we combine them all, we get that of an atom!
When talking about these combined systems, it is easy to get confused real fast using letters for the labelling of systems, meaning, system A and System B, we will instead use Alice and Bob, and occasionally Charlie, that have become almost universal substitutes for A, B and C.
We will denote Alice's system with S_A, and Bob's with S_B.
Now let's say that we want to combine the two systems into a single, bigger, composite system.
An example could be that Alice's system is a Quantum Mechanical Coin, that has obviously two basis vectors, H and T, denoting Heads and Tails.
Since it is a quantum coin, it can exist in a superposition of the two states:

$$a_H|H\} + a_T|T\} \tag{1}$$

The weird notating is to distinguish Alice's and Bob's ket vectors.
They are in different systems, for now, so we need to be able to understand with whose system we are dealing.

Now, let's assume that Bob's system, is a quantum die, and that its six-dimensional space of states is defined by the basis:

$$|1\rangle\ |2\rangle\ |3\rangle\ |4\rangle\ |5\rangle\ |6\rangle \tag{2}$$

Just like Alice's coin, Bob's die is quantum, which, again, means we can superpose its six-states in the same way:

$$a_1 |1\rangle + a_2 |2\rangle + a_3 |3\rangle + a_4 |4\rangle + a_5 |5\rangle + a_6 |6\rangle \tag{3}$$

2 Representing the Combined System

Now imagine that Alice's and Bob's systems exist and form a single composite system, call it S_{AB}. How do we construct the state-space?
The Answer is with the **Tensor Product** of the two systems S_A and S_B.
We denote that with

$$S_{AB} = S_A \otimes S_B \tag{4}$$

To define S_{AB}, all we need to do is specify its basis vectors.
You probably expected this, but its basis vectors are:

$$|H1\rangle, |H2\rangle, |H3\rangle, |H4\rangle, |H5\rangle, |H6\rangle\ |T1\rangle, |T2\rangle, |T3\rangle, |T4\rangle, |T5\rangle, |T6\rangle \tag{5}$$

These are the basis vectors of S_{AB}. Each state is represented by a double indexes label. On for Alice's subsystem and one for Bob's.
For example, the state $|H2\rangle$, means that the die shows 2 and the coin Heads, and so on.
Each of the kets above, denotes a basis vector of the S_{AB} system. So, in total there are 12 basis vectors.
There are multiple ways to denote them, like: $|H\} \otimes |2\rangle$, or $|H\} |2\rangle$, but we will simply denote them like this:
$|H2\rangle$, just because it is faster and more convenient and comfortable to our eyes!
This also shows that we are talking about one single state, with a two part label.
Again, we can superpose all the states, or maybe some specific states:
$a_{h3} |H3\rangle + a_{t4} |T4\rangle$
As we said earlier, the first half of the label describes the state of Alice's coin, and the second half, Bob's die.
If we want to refer to arbitrary basis vectors, for example like when we denote an eigenvalue of an observable L with λ_i, or in general when we want to simply refer to a basis vector, without specifying which, we will denote them like this:
$|ab\rangle$, or $|a'b'\rangle$, where a and a' represent Alice's state, and b and b', Bob's.
We use a double index to denote a single state.
This holds true in general, but instead of 2 and 6 states, we have N_A and N_B states for Alice's and Bob's systems respectively.

You have probably figured this out by now, but $N_{AB} = N_A N_B$

We can do that for any amount of systems, not only two, but we will not do that here. However, we can clearly see that the equation below is a general equation for j systems!

$$N_j = \prod_i N_{j_i} \tag{6}$$

We have now covered the composition of systems, however, we still need different operators for the different subsystems.
We will denote Alice's set of operators with σ, and Bob's set with τ, just so we don't mix them up.

3 Tensor Products in Component Form

Building tensor products from matrices and column vectors is not hard.
We have used the notation m_{jk} to write any observable M:

$$m_{jk} = \langle j|M|k\rangle \tag{7}$$

where $|j\rangle$ and $|k\rangle$ represent basis vectors. Each combination of these (j and k), produces a different Matrix element.
We will apply this "sandwich formula" to tensor product operators and see what we get, but, because of the double indexed basis, we will need to cycle through $|uu\rangle, |ud\rangle, |du\rangle, |dd\rangle$ to keep things simple, we will use the operator $\sigma_z \otimes I$ as an example.
A we have seen before, $\sigma_z \otimes I$ acts only on Alice half, and does **Absolutely nothing** to Bob's half.
As expected, the result will be a 4x4 Matrix, since we are working in 4 dimensions, with basis vectors $|uu\rangle, |ud\rangle, |du\rangle, |dd\rangle$:

For convenience, I will write $\sigma_z \otimes I$ as $\sigma_z I$

$$\sigma_z \otimes I = \begin{pmatrix} \langle uu|\sigma_z I|uu\rangle & \langle uu|\sigma_z I|ud\rangle & \langle uu|\sigma_z I|du\rangle & \langle uu|\sigma_z I|dd\rangle \\ \langle ud|\sigma_z I|uu\rangle & \langle ud|\sigma_z I|ud\rangle & \langle ud|\sigma_z I|du\rangle & \langle ud|\sigma_z I|dd\rangle \\ \langle du|\sigma_z I|uu\rangle & \langle ud|\sigma_z I|ud\rangle & \langle du|\sigma_z I|du\rangle & \langle du|\sigma_z I|dd\rangle \\ \langle dd|\sigma_z I|uu\rangle & \langle ud|\sigma_z I|ud\rangle & \langle dd|\sigma_z I|du\rangle & \langle dd|\sigma_z I|dd\rangle \end{pmatrix} \tag{8}$$

Now, we need to evaluate these elements. We could allow σ_z to operate on one side and I on the other.
For example, let σ_z on the right and I on the left. Since I does nothing, we only care about the left side and its interaction with σ_z:

$$\sigma_z \otimes I = \begin{pmatrix} \langle uu|uu\rangle & \langle uu|ud\rangle & \langle uu|du\rangle & \langle uu|dd\rangle \\ \langle ud|uu\rangle & \langle ud|ud\rangle & \langle ud|du\rangle & \langle ud|dd\rangle \\ -\langle du|uu\rangle & -\langle ud|ud\rangle & -\langle du|du\rangle & -\langle du|dd\rangle \\ -\langle dd|uu\rangle & -\langle ud|ud\rangle & -\langle dd|du\rangle & -\langle dd|dd\rangle \end{pmatrix} \tag{9}$$

Because the eigenvectors are the orthonormal basis, and using the information from the previous matrix, we see that this matrix produces:

$$\sigma_z \otimes I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{10}$$

That means that we can now write the orthonormal basis as :

$$|uu\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{11}$$

$$|ud\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \tag{12}$$

$$|du\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \tag{13}$$

$$|dd\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{14}$$

, and we can get their eigenvalues from the operation in which the $\sigma_z \otimes I$ acts on them!

I will now show you a rule for combining 2, 2x2 matrices into a 4x4 matrix. This product is also called the Kronecker product!
Suppose we have two, 2x2 matrices, A,B:

$$A \otimes B = \begin{pmatrix} A_{11}B & A_{12}B \\ A_{21}B & A_{22}B \end{pmatrix} \tag{15}$$

And now if we expand all the terms:

$$A \otimes B = \begin{pmatrix} A_{11}B_{11} & A_{11}B_{12} & A_{12}B_{11} & A_{12}B_{12} \\ A_{11}B_{21} & A_{11}B_{22} & A_{12}B_{21} & A_{12}B_{22} \\ A_{21}B_{11} & A_{21}B_{12} & A_{22}B_{11} & A_{22}B_{12} \\ A_{21}B_{21} & A_{21}B_{22} & A_{22}B_{21} & A_{22}B_{22} \end{pmatrix} \tag{16}$$

In general, the Kronecker product of an m x n matrix with an p x q matrix, is an mp x nq matrix!
This also applies to column and row vectors!

The tensor product of two 2x1 column vectors, is a 4x1 column vector.

$$\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} \otimes \begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} \\ a_{11}b_{21} \\ a_{21}b_{11} \\ a_{21}b_{21} \end{pmatrix} \tag{17}$$

We can use these rules to compute a lot of different things, like $|d\rangle \otimes |d\rangle$, or even $\sigma_x \otimes \tau_x$ and other combinations!