Quantum Mechanics - Quantum States

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1 States and Vectors

In Classical Mechanics, knowing the state of a system, implies that you know everything necessary to predict the future of that system.

In Quantum Mechanics, knowing a quantum states means **knowing as much as possible about how the system is prepared**

it is not certain whether Quantum States are totally unpredictable, but for practical reasons we will assume it is!

2 Representing Spin States

This a very very important lecture

Lets try to represent everything we know about the behavior of spin!

2.1 Possible Spin States

This is where confusion takes center stage!

The Coordinate Space of the Spin, and the general space is a 3 Dimensional Complex Vector Space, meaning it spans over the complex scalar field, or in other words every point on this space is a complex number! The Space, is a Vector space, don't forget this! As we said earlier, the state of the system and the measurements, or the output of σ, is not directly correlated with the state of the spin! In fact, the relation is that the state gives us probabilities about the result of the measurement. We said that the result is either 1 or -1, but the average result is obviously a function of the probabilities. The space of THE states of the system, in which we will focus, isn't the same as the space in which the Spin is oriented! The space of the quantum state is also a 3 Dimensional Complex Space which is also very hard to visualize, since once more, every point in this space is a combination of x, y, i, where i is the imaginary component and x,y are real! For now think of it as a simple 2-Dimensional Space, without including the 3rd dimension

Now here comes the Hard Part:

The basis vectors of this space are up and down denoted

$$|u\rangle \, and \, |d\rangle \tag{1}$$

Now before you think of it as one being the positive and the other being the negative side, remember that they are orthonormal! Meaning they are perpendicular to each other!

We use

$$|u\rangle \, and \, |d\rangle \tag{2}$$

to describe the spin component of the z direction!

For the x and y directions, we use

$$|r\rangle \, and \, |l\rangle \tag{3}$$

and

$$|i\rangle \, and \, |o\rangle \tag{4}$$

denoting right and left, and in and out, respectively!

But, **all possible states are linear supper-positions of up and down**, which means we only need up and down to describe a state!
Generally, all the spin states can be represented in a 2 Dimensional Complex Vector Space!

Knowing these, we can now describe each state to its components:

$$|A\rangle = a_u \, |u\rangle + a_d \, |d\rangle \tag{5}$$

where a_u and a_d are the components along the directions up and down.
(You can think of it as the way you describe a vector in 2D, meaning $V_x i + V_y j$)

Mathematically we can define the components as:

$$a_u = \langle u|A\rangle \tag{6}$$

$$a_d = \langle d|A\rangle \tag{7}$$

That means that $|A\rangle$, can represent any state of the spin prepared in any manner.
and that a_u and a_d are complex numbers

If the spin is prepared for a state $|A\rangle$, and the Apparatus is oriented along z,

$$a_u^* a_u \tag{8}$$

Is the probability that the result of the measurement of $\sigma_z = +1$
The value denote by a, are called **Probability Amplitudes**. The values themselves are not probabilities, but they play the main role in determining probabilities!
The probabilities for a Measurement are given by

$$P_u = \langle A|u\rangle \, \langle u|A\rangle \tag{9}$$

$$P_d = \langle A|d\rangle \, \langle d|A\rangle \tag{10}$$

Before the Measurement, all we have is the state-vector $|A\rangle$, that represents the **potential probabilities**, and not actual values!
I have said a lot of times and I will keep saying it, that the only possible outcomes are 1 or -1.

We mentioned earlier that $|u\rangle$ and $|d\rangle$, are mutually orthogonal, which in the language of mathematics, it means that their inner product is 0.

$$\langle u|d\rangle = 0 \tag{11}$$

and

$$\langle d|u\rangle = 0 \tag{12}$$

Which is pretty obvious since

$$|u\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{13}$$

and

$$|d\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{14}$$

and with some simple linear algebra we see why their product is 0.

You can interpret this in a few ways, but mainly, it means that if the spin is prepared for a state $|u\rangle$, then the probability of the spin to be in a down state is 0.
Generally, if the spin is prepared in a specific state, then the probability of being in another is obviously 0.
Also another pretty obvious fact is that the total probability is equal to 1.

$$a_u^* a_u + a_d^* a_d = 1 \tag{15}$$

That is the equivalent of saying that $|A\rangle$, is a normalized vector, which means that the inner product with itself is 1, something that seems natural if you recall the relationship between bra and ket.

$$\langle A|A\rangle = 1 \tag{16}$$

The State of a system is represented by a unit (normalized) vector, in a vector space. If we put everything together, we see how the squared magnitudes of the components of the state vector, along particular basis vectors, represent probabilities for the experimental outcomes!

3 Along the Other Axes

We said earlier, that we can represent any spin state as a linear superposition of the basis vectors $|u\rangle$ and $|d\rangle$.

3.1 Along the x-axis

Now lets to do this for the vectors $|r\rangle$ and $|l\rangle$ which represent spins prepared along the x-axis
If the spin is initially prepared for $|r\rangle$, and we rotate A (the apparatus) along the z-axis to measure σ_z, there will be equal probabilities for up and down, thus we need to find a combination of up and down that if squared, returns 1/2.
That can be satisfied by

$$|r\rangle = \frac{1}{\sqrt{2}}|u\rangle + \frac{1}{\sqrt{2}}|d\rangle \tag{17}$$

Now if you square each component and add them, you see that $\frac{1}{2} + \frac{1}{2} = 1$
Again right and left are mutually orthogonal and for $|l\rangle$ we have:

$$|l\rangle = \frac{1}{\sqrt{2}}|u\rangle - \frac{1}{\sqrt{2}}|d\rangle \tag{18}$$

3.2 Along the y-axis

Again, as expected, in and out, are mutually orthogonal, and can be expressed in terms of up and down.
In and Out, have the same values as r and left, only that the $|d\rangle$ component has i as its numerator:

$$|i\rangle = \frac{1}{\sqrt{2}}|u\rangle + \frac{i}{\sqrt{2}}|d\rangle \tag{19}$$

and

$$|r\rangle = \frac{1}{\sqrt{2}}|u\rangle - \frac{i}{\sqrt{2}}|d\rangle \tag{20}$$

i is the imaginary number, which was expected to appear since we are talking in a complex vector space.

4 The Parameters

Without a doubt there are a lot of parameters in Quantum Mechanics, but how do we find them?
Lets find this through a generalization of an experiment:
Lets start by pointing A along any unit 3-vector n, and prepare a spin for $\sigma = 1$ along that axis. if $\sigma = -1$, then we can tell that A is along -n!
Thus, there must be a state for every orientation of the unit vector n.
The Spin then, is defined by 2 parameters, namely a_u and a_d, but since these numbers are complex, we need in total 4 real parameters!
But, here come normalization and the phase-factor to save the day, well, actually to half the day:
Since normalization gives us that

$$a_u^* a_u + a_d^* a_d = 1 \tag{21}$$

, when we know the 3 parameters, we can find the 4 using this, which leaves us with 3. However, the physical properties of the state-vector, don't depend on the overall phase factor, making one parameter redundant, which in other words means that we can describe all possible orientations of a spin using two parameters:

$a_u \left|u\right\rangle + a_d \left|d\right\rangle$

5 Spin States as Column Vectors

By definition, and following intuition,

$$\left|u\right\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{22}$$

$$\left|d\right\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{23}$$

Using these we can create vectors for right left, in and out, but for no that is not the priority or the point!

6 A quick Summary

Mutually Orthogonal basis vector pairs
$\langle u|d\rangle = 0$, $\langle r|l\rangle = 0$, $\langle i|o\rangle = 0$

It takes two parameters to specify a spin state and $\left|u\right\rangle$ and $\left|d\right\rangle$ are the basis vectors for representing all spin states

We can represent any orientation, in any axis, as a linear combination of $\left|u\right\rangle$ and $\left|d\right\rangle$

We can represent $\left|u\right\rangle$ and $\left|d\right\rangle$ as column vectors