6. Composite Systems and Entanglement

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Contents

1 Tensor Products **1**
- 1.1 Labels and Notation 1
- 1.2 Representing the Combined System 2

2 Classical Correlation **3**

3 Combining Quantum Systems **5**

4 Two Spins **5**

5 Product States **6**

6 The Parameters for the Product State **7**

7 Entanglement States **7**

8 Entangled Operators and Observables **8**

9 Composite Observables **10**

1 Tensor Products

1.1 Labels and Notation

Combining smaller systems to create a bigger composite system plays a vital role in physics
An atom for example, is nothing but the collection of nucleons and electrons, each of which has its own system, and when we combine them all, we get that of an atom!
When talking about these combined systems, it is easy to get confused real fast using letters for the labelling of systems, meaning, system A and System B, we will instead use Alice and Bob, and occasionally Charlie, that have become almost universal substitutes for A, B and C.
We will denote Alice's system with S_A, and Bob's with S_B.
Now let's say that we want to combine the two systems into a single, bigger, composite system.

An example could be that Alice's system is a Quantum Mechanical Coin, that has obviously two basis vectors, H and T, denoting Heads and Tails.
Since it is a quantum coin, it can exist in a superposition of the two states:

$$a_H|H\} + a_T|T\} \tag{1}$$

The weird notating is to distinguish Alice's and Bob's ket vectors.
They are in different systems, for now, so we need to be able to understand with whose system we are dealing.

Now, let's assume that Bob's system, is a quantum die, and that its six-dimensional space of states is defined by the basis:

$$|1\rangle \ |2\rangle \ |3\rangle \ |4\rangle \ |5\rangle \ |6\rangle \tag{2}$$

Just like Alice's coin, Bob's die is quantum, which, again, means we can superpose its six-states in the same way:

$$a_1 |1\rangle + a_2 |2\rangle + a_3 |3\rangle + a_4 |4\rangle + a_5 |5\rangle + a_6 |6\rangle \tag{3}$$

1.2 Representing the Combined System

Now imagine that Alice's and Bob's systems exist and form a single composite system, call it S_{AB}. How do we construct the state-space?
The Answer is with the **Tensor Product** of the two systems S_A and S_B.
We denote that with

$$S_{AB} = S_A \otimes S_B \tag{4}$$

To define S_{AB}, all we need to do is specify its basis vectors.
You probably expected this, but its basis vectors are:

$$|H1\rangle, |H2\rangle, |H3\rangle, |H4\rangle, |H5\rangle, |H6\rangle \ |T1\rangle, |T2\rangle, |T3\rangle, |T4\rangle, |T5\rangle, |T6\rangle \tag{5}$$

These are the basis vectors of S_{AB}. Each state is represented by a double indexes label. On for Alice's subsystem and one for Bob's.
For example, the state $|H2\rangle$, means that the die shows 2 and the coin Heads, and so on.
Each of the kets above, denotes a basis vector of the S_{AB} system. So, in total there are 12 basis vectors.
There are multiple ways to denote them, like: $|H\} \otimes |2\rangle$, or $|H\} |2\rangle$, but we will simply denote them like this:
$|H2\rangle$, just because it is faster and more convenient and comfortable to our eyes!
This also shows that we are talking about one single state, with a two part label.

Again, we can superpose all the states, or maybe some specific states:
$a_{h3}|H3\rangle + a_{t4}|T4\rangle$
As we said earlier, the first half of the label describes the state of Alice's coin, and the second half, Bob's die.
If we want to refer to arbitrary basis vectors, for example like when we denote an eigenvalue of an observable L with λ_i, or in general when we want to simply refer to a basis vector, without specifying which, we will denote them like this:
$|ab\rangle$, or $|a'b'\rangle$, where a and a' represent Alice's state, and b and b', Bob's.
We use a double index to denote a single state.
This holds true in general, but instead of 2 and 6 states, we have N_A and N_B states for Alice's and Bob's systems respectively.

You have probably figured this out by now, but $N_{AB} = N_A N_B$
We can do that for any amount of systems, not only two, but we will not do that here. However, we can clearly see that the equation below is a general equation for j systems!

$$N_j = \prod_i N_{j_i} \tag{6}$$

We have now covered the composition of systems, however, we still need different operators for the different subsystems.
We will denote Alice's set of operators with σ, and Bob's set with τ, just so we don't mix them up.

2 Classical Correlation

Before we get to quantum entanglement, we will go over what we could call Classical Entanglement.
We will do an experiment, including Alice, Bob, and Charlie.
Imagine that Charlie has two identical boxes, meaning that just by seeing the two boxes, you cannot distinguish them, unless you look at their contents. Now imagine that Charlie has put A check of say 10.000$ in one box, and a check of 10$ in the other one. Now suppose that he used different ink on each check, so that their weight is the exact same!
These are conditions that make it impossible to unambiguously know which box is which just by looking at them.
Next on, imagine that Charlie shuffles them and randomly gives one to Alice and one to Bob.
(For now we won't need Charlie anymore)
Now no matter where Alice is, whatever her speed is (obviously she can't go with c or anywhere close to that), the moment she looks inside the box and sees what check she has, she immediately knows what Bob has.
Emphasize on the immediately! She knows it instantly, faster than c.(Well if you go into biology and see the speed it takes for her to process the information, it probably is slower than c)

You may think that this violates Relativity, specifically its fundamental rule, which states that information can't travel faster than c.
That is obviously false!
the information travels at c in the form of the reflection of the check, but the information includes both her check and Bob's check. What would violate relativity would be if Bob instantly knew his check, when Alice observed hers!
Alice may know Bob's check, but there is no way to tell him what it is faster than the speed of Light!
To be more quantitative, suppose that Charlie, instead of giving them a check in a box, he gives them a $\sigma = 1$ and a $\sigma = -1$.
Now if we assume that Charlie randomly shuffles the the two "boxes", if we repeat the whole experiment a lot of times, in which for example Alice and Bob synchronize their watches so that Alice sees her box 5 seconds before Bob, then they receive the boxes, Alice goes to Greece for example and Bob goes to the moon, everything goes right, they measure and then they meet back in Athens to "compare" their results, we see that the following facts emerge:
Both Alice and Bob will get equally as many $\sigma = 1$ as $\sigma = -1$. Calling Alice's observations σ_A and Bob's σ_B, we see that:
$\langle \sigma_A \rangle = 0$
$\langle \sigma_B \rangle = 0$
and then we obviously see that:

$\langle \sigma_A \sigma_B \rangle = -1$

Note that this is not the product of the averages, but the average of the products!
$\langle \sigma_A \sigma_B \rangle \neq \langle \sigma_A \rangle \langle \sigma_B \rangle$
This, indeed, indicates that Alice's and Bob's Observations are correlated.
In fact, $\langle \sigma_A \sigma_B \rangle - \langle \sigma_A \rangle \langle \sigma_B \rangle$
is called the statistical correlation between Bob's and Alice's observations!
When the statistical correlation is nonzero, then the observations are correlated!
The origin of this correlation is the fact that Alice and Bob where at the same location and Charlie had one of each type of check/σ in the beginning.
If we suppose that the probability distribution of a and b is P(a,b), then,
$P(a,b) = P_a(a)P_b(b)$, means that the variables are **completely uncorrelated**.
In other words, if $P(a,b)$ factorizes like the example above, then the correlation between a and b is 0, or $\langle \sigma_a \sigma_b \rangle - \langle \sigma_a \rangle \langle \sigma_b \rangle = 0$

An example of this, uncorrelated relation between a and b, is that if we suppose that instead of one Charlie, we had two, that had no relation between them, they didn't know each other, or simply one's existence didn't affect in any way the other, and we denoted them by C_A and C_B, and they both had two boxes just like the ones in the previous example, and C_A gave one box to Alice and C_B one to Bob, it is obvious that no matter how far they are or how close, by looking at their own box, they don't know what the box of the other has in it! That implies that even at look at the others doesn't give them any useful information about their check.
They are totally unrelated. In Classical Mechanics, we see that the probability distribution is an incomplete specification of the system's state.
Generally the use of probability in Classical Mechanics, is always associated with the

lack of knowledge relative to the total possible knowledge that someone could have!
Such a knowledge, a complete, implies knowing every part of the system.
Now in order to get to Quantum Mechanics, we need to go past that intuition!

3 Combining Quantum Systems

Charlie's two boxes formed a single classical system formed from two other classical systems.
In Quantum Mechanics you can combine systems with the use of the tensor product we talked earlier.
This time, instead of a die and a coin, suppose that our system is built from two spins.
Just like before, we will use the weird notation $|a\}$ to denote A's states and $|b\rangle$ for Bob's/B's.
W will use S_{AB}, whose basis vectors are again double indexed. We will denote all it basis vector with $|ab\rangle$
Consider some linear operator M, acting on the space of state of the composite system.

$$\langle a'b'|M|ab\rangle = M_{a'b',ab} \tag{7}$$

This represents a Matrix that is formed by sandwiching th operator between the basis vectors!
Each row of the matrix is labelled with a single index a'b', and each column with ab.
The vectors $|ab\rangle$ are orthonormal to each other, meaning their inner product is 0.
Note that it is equal only when ab matches a'b', not when a matches b.
Using the Kronecker delta, we can write this like:

$$\langle ab|a'b'\rangle = \delta_{aa'}\delta_{bb'} \tag{8}$$

That means that the right hand side is 0, unless, a=a' and b=b'.
If they match, then their inner product is 1.
Now since we have the basis vectors, and it is a quantum system, any superposition is allowed.
We can expand any state as:

$$|\Psi\rangle = \sum_{a,b} \psi(a,b)\,|ab\rangle \tag{9}$$

4 Two Spins

Imagine two spins. Alice's and Bob's. Alice and bob each have their own separate apparatus labelled A and B. They are independent and can be oriented along any axis.
Next on, for the spin operators, as we said in 1, we will use σ for Alice and τ for Bob.
Based on all the principles we have laid so far, and the fact that we can represent any spin as a linear combination of $|u\rangle$ and $|d\rangle$, we can make a 2x2 table, comprising for the four basis vectors:

$|uu\rangle, |ud\rangle, |du\rangle, |dd\rangle$
The first part represents the state of σ, and the second, the state of τ.

5 Product States

The most simple state for the composite system is called a product state.
Such a state, is the result of completly independent preparations by Alice and Bob, in which each uses his own Apparatus to prepare a spin.
Alice prepares her spin in

$$a_u|u\} + a_d|d\} \tag{10}$$

and Bob

$$\beta_u |u\rangle + \beta_d |d\rangle \tag{11}$$

We assume that each state is normalized, or that:

$$a_u^* a_u + a_d^* a_d = 1 \tag{12}$$

and

$$\beta_u^* \beta_u + \beta_d^* \beta_d = 1 \tag{13}$$

Which is very very important
The product state defining the system is :

$$|product\ state\rangle = \{a_u|u\} + a_d|d\}\} \otimes \{\beta_u |u\rangle + \beta_d |d\rangle\} \tag{14}$$

where the first factor represents Alice's state, and the second Bob's.
Now if we expand the product and switch to the composite notation, we see that:

$$|product\ state\rangle = a_u\beta_u |uu\rangle + a_u\beta_d |ud\rangle + a_d\beta_u |du\rangle + a_d\beta_d |dd\rangle \tag{15}$$

The main feature of a product state is that each subsystem behaves independently of the other. If Bob does an experiment on his own subsystem, the result would be the same as if Alice's subsystem never existed, and vice versa.
Note that **tensor products** and **product states** are different things.
A *tensor product* us a vector space for studying composite systems.

A *product state*, is a state-vector.

6 The Parameters for the Product State

Lets now see how many parameters it takes to specify such a product state.
Each Factor requires two complex numbers:
a_u and a_d for Alice
and
β_u and β_d for Bob.
That means we need 4 complex Numbers in total, or **8** real numbers.
Due to normalization the required parameters drop to 6.(Look at the earlier lecture to see how)
Because of the overall phases of each state have no physical meaning, we only need 4 parameters to specify a product state!
This isn't very surprising, since it takes 2 for each spin, as we've seen earlier.

7 Entanglement States

The principles of Quantum Mechanics allow us to superpose basis vector in more general ways than just product states.
The most general vector in the composite space of states is:

$$\psi_{uu}\,|uu\rangle + \psi_{ud}\,|ud\rangle + \psi_{du}\,|du\rangle + \psi_{dd}\,|dd\rangle \tag{16}$$

Where we use the wavefunction of the composite system and subscripts to represent the complex coefficients, instead of a and β.
In this case, we have only one normalization condition since the spins are entangled:

$$\psi^*_{uu}\psi_{uu} + \psi^*_{ud}\psi_{ud} + \psi^*_{du}\psi_{du} + \psi^*_{dd}\psi_{dd} = 1 \tag{17}$$

And only 1 phase to ignore.
That results in us needing 6 real parameters, evidently being way richer than a product state, that is prepared by Alice and Bob.
Here something new is happening, namely:

Entanglement

Entanglement is a bit weird.
It is, namely, what Einstein didn't agree with, and we will see later on why and how.
Some states are more entangled than others!
Here is an example of a **maximally entangled state**.
Such a state is called **the singlet state** and it can be written as:

$$|sing\rangle = \frac{1}{\sqrt{2}}(|ud\rangle - |du\rangle) \tag{18}$$

The Singlet State cannot be written as a product state! The same is true for the triplet states:

$$\frac{1}{\sqrt{2}}(|ud\rangle + |du\rangle) \tag{19}$$

$$\frac{1}{\sqrt{2}}(|uu\rangle + |dd\rangle) \tag{20}$$

$$\frac{1}{\sqrt{2}}(|uu\rangle - |dd\rangle) \tag{21}$$

Which are also entangled.
We will soon explain the reason behind their names!

We can some up everything that is important about maximally entangled states in two statements:
1) An entangled State is a complete description of the combined system. No more can be known about it.
2) In a maximally entangled state, nothing is known about the individual subsystems
That is the hard and counter intuitive thing with entanglement, the second statements. We know as much as there is to know about the composite Alice-Bob system, but simultaneously not know anything about the individual subsystems.

8 Entangled Operators and Observables

So far we haven't talked about observables in a two spin system.
We said however, that we need two Apparatuses A and B.
With A we can measure
$\sigma_x, \sigma_y, \sigma_z$ for Alice, and τ_x, τ_y, τ_z.
We represent these observables with Hermitian Operators, in composite state-space, and as if the other observable never existed.
That means that Bob's operator acting on Bob's spin acts exactly the same as if Alice never existed, and vice versa.
Alice's Possible Operator-acting-on-spin results are:
$\sigma_z|u\} = |u\}$
$\sigma_z|d\} = -|d\}$
$\sigma_x|u\} = |d\}$
$\sigma_x|d\} = |u\}$
$\sigma_y|u\} = i|d\}$
$\sigma_y|d\} = -i|u\}$

And as expected, Bob's are:
$\tau_z |u\rangle = |u\rangle$
$\tau_z |d\rangle = -|d\rangle$
$\tau_x |u\rangle = |d\rangle$
$\tau_x |d\rangle = |u\rangle$
$\tau_y |u\rangle = i|d\rangle$
$\tau_y |d\rangle = -i|u\rangle$

Now lets consider how the operators should be defined when acting on product states $|uu\rangle$,$|ud\rangle$,$|du\rangle$ and $|dd\rangle$. When σ acts, it just ignores Bob's Half of the state-label, and vice versa. There are obviously 24 different combinations of operators acting on composite states.
Technically, the operators that act on the composite system are new operators that are given by $\sigma \otimes I$, but it is obvious that nothing changes. The same is true for the vectors, meaning their notation, isn't the same as Bob's vectors. It isn't $|du\rangle$, but $|d\rangle \otimes |u\rangle$.
Knowing this, we can rewrite

$$\sigma_z |du\rangle = -|du\rangle \tag{22}$$

as

$$(\sigma_z \otimes I)(|d\rangle \otimes |u\rangle) = (\sigma_z |d\rangle \otimes I|u\rangle) = (-|d\rangle \otimes |u\rangle) \tag{23}$$

Now we can see clearly why σ_z, for example, acts only on Alice's half.
We will stick to our initial notation for convenience, just know that this is what we mean.

In a product state, any prediction about Bob's half of the system is the same as if it was just a single spin, and Alice's never existed. Obviously the opposite is also true.

Next on, by definition it is given that for every state of a spin, there is some direction for which the spin is +1.
As expected, that implies that **not all spin-component expectation values can be 0**.
More specifically,

$$\langle\sigma_x\rangle^2 + \langle\sigma_y\rangle^2 + \langle\sigma_z\rangle^2 = 1 \tag{24}$$

This is also true for the product state, however, the same can't be said about the singlet state.
In fact, for the $|sing\rangle$ state, it is true that

$$\langle\sigma_x\rangle^2 + \langle\sigma_y\rangle^2 + \langle\sigma_z\rangle^2 = 0 \tag{25}$$

, or simply

$$\langle\sigma_x\rangle = \langle\sigma_y\rangle = \langle\sigma_z\rangle = 0 \tag{26}$$

(As an exercise, you could try to prove this!)

But what does this tell us about the measurements we can make?
As we said in the first few chapters, if the expectation value of σ is 0, then the experimental outcome is equally likely to be +1 and -1.
In other words,
The outcome is completly uncertain!
Even thought we know the exact state-vector, we know nothing about the outcome of any measurement of any component of the spin! That is what we addressed in chapters 1 and 2 as the counter-intuitive relation between state and measurement in a quantum system!
The Principles of Quantum Mechanics, tell us that there is nothing more to know beyond what is encoded in the state-vector.
In other words, we can know everything about a composite system, and nothing about its constituent parts!
That is the weirdness of entanglement that disturbed Einstein!

9 Composite Observables

Suppose we have a Quantum Mechanical Alice-Bob-Charlie setup.
Charlie's role is to prepare two spins in an entangled state, $|sing\rangle$, and then without looking, of affecting the spins at all, to give randomly one to Alice and one to Bob.
Although Alice and Bob know the exact state the combined system is in, they cannot predict unambiguously their individual measurements.
Actually, the fact that the spins are maximally entangled offers some information, however, in order to understand it, we need to consider a wider family of observables that the on that Alice and Bob have, that measures spins separately.
There exist such observables, that can be detected by using both apparatuses and then comparing the results.
Since the operators act on different spins, they commute, meaning Alice can make any measurement on her own spin, without affecting Bob's, and vice versa.
Let's suppose that Alice measures σ_z and Bob, τ_z, and then they meet and multiply their results.
The product $\tau_z\sigma_z$, is an observable is represented mathematically, by σ_z acting on some ket, and subsequently, τ_z acting on the result.
Note that these operations define a new abstract operator with no physical meaning.
If we apply the product $\tau_z\sigma_z$ to the $|sing\rangle$, we get:

$$\tau_z\sigma_z|sing\rangle = \tau_z\sigma_z\frac{1}{\sqrt{2}}(|ud\rangle - |du\rangle) \tag{27}$$

Now if we carry on the calculations, using Pauli matrices, we get:

$$\tau_z |sing\rangle = \tau_z \frac{1}{\sqrt{2}}(|ud\rangle + |du\rangle) \tag{28}$$

And now by applying τ_z, we get:

$$\tau_z \sigma_z |sing\rangle = \frac{1}{\sqrt{2}}(-|ud\rangle + |du\rangle) \tag{29}$$

or simply $-\frac{1}{\sqrt{2}}(|ud\rangle - |du\rangle)$, which is nothing but $-|sing\rangle$
Overall $\tau_z \sigma_z |sing\rangle = -|sing\rangle$
Evidently, the singlet state is an eigenvector of $\tau_z \sigma_z$, with an eigenvalue of -1.
What is the significance of this result??
It is the same as in the classical entanglement experiment. Whenever Alice measure 1, Bob measures -1, and vice versa! The product of their measurements is -1, as we see in the equation above.
If we did the same for $\tau_x \sigma_x$ and $\tau_y \sigma_y$, we would find out the same results!
(Try to come to the same results by carrying out the operations as an exercise.)

Now finally, lets consider another observable. An observable that cannot be measured by Alice and Bob making separate measurements and then comparing them.
Such an observable, can be measured by some Apparatus in Quantum Mechanics, meaning Quantum Mechanics actually insists that such an apparatus can be built.
The Observable is $\vec{\sigma} \cdot \vec{\tau}$
Such an apparatus in very hard to imagine, because it doesn't measure each component separately, that would be impossible due to the fact that different components of the spin don't commute.
This measurement is exactly like the relation between the knowledge about singlet state, and the knowledge of the states of the subsystems!