5. Uncertainty

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1.1 States that depend on more than one measurable

An important property of spins, is, that if you know the result of a measurement of one spin observable, you cannot observe and learn another one's, while still knowing the other one.
The state of a single spin can be specified by the eigenvalue of a single operator.
Measuring one quantity, destroys the information about the others.
In Quantum Mechanics, you can now the location of a particle in 3 dimensions, meaning all x,y and z, but you cannot simultaneously know these and the momentum of that particle.

Also, in a Two-Spin system, we can know the states of the two qubits simultaneously, even if they are independent.
In that case, we need to conduct multiple measurements to fully characterize the state of the system.
We will need to measure each spin separately ad associate their measurements with two different operators L and M.

A measurement leaves the system in an eigenstate corresponding to the eigenvalue that was measured.
If we measure both spins, in a two-spin system, is both an eigenvector of L and M.
We call this a **simultaneous eigenvector** of L and M.
Since we have two different operator, it follows that we need two different sets of labels for the basis vectors.
We will use λ_i and μ_a
These are also the eigenvectors of L and M.

Generally it is true that:

$$L|\lambda_i,\mu_a\rangle = \lambda_i|\lambda_i,\mu_a\rangle \tag{1}$$

and

$$M|\lambda_i,\mu_a\rangle = \mu_a|\lambda_i,\mu_a\rangle \tag{2}$$

In order to have a basis of simultaneous eigenvectors, L and M must commute. This i very easy to see, and it is true for all basis vectors!
Knowing this, we can use it in reverse to say that **if two observables commute, then there is a complete set of simultaneous eigenvectors of the two observables**, or
if two observables commute, they can be simultaneously measured!

1.2 Wave functions

In a general way, in Quantum Mechanics, the **wave functions** don't have to do anything with waves, not in the way you probably think at least. In Chapters 8-9 we will go deeper and talk a lot about wave functions.
Suppose we have a basis of states for some Quantum System, denoted by $|a,b,c,d,e,f,...\rangle$, where a,b,c,d,e,f,... are the eigenvalues of a complete set of commuting observables A,B,C,D,E,F,...
Next let $|\Psi\rangle$ be a state-vector. We can now express that state vector in terms of a wave function and the orthonormal basis.

$$|\Psi\rangle = \sum_{a,b,c,d,e,f,...} \psi(a,b,c,d,e,f,...)\,|a,b,c,d,e,f,....\rangle \tag{3}$$

The term $\psi(a,b,c,d,e,f,...$ is what we call wave function, and is nothing but all the coefficients of the state-vector:

$$\psi(a,b,c,d,e,f,...) = \langle a,b,c,d,e,f,...|\Psi\rangle \tag{4}$$

However don't forget that in this case here, the system consists of a,b,c,d,e,f,... spins, so that is why the wave function is a function of all these. The wave function belongs

to the system not a specific observable, just like the state vector!
According to the probability principle of Quantum Mechanics, the square magnitude of the wave function is the probability for the commuting observables to have the values a,b,c,d,e,f,....

$$P(a,b,c,d,e,f,...) = \psi^*(a,b,c,d,e,f,...)\psi(a,b,c,d,e,f,...) \tag{5}$$

For example, the probability of the observable A being a and B being b is given by:

$$P(a,b) = \psi^*(a,b)\psi(a,b) \tag{6}$$

The form of the wave function depends on the observables we focus on.
For example, for a single spin,

$$\psi(u) = \langle u|\Psi\rangle \tag{7}$$

$$\psi(d) = \langle d|\Psi\rangle \tag{8}$$

define our σ_z basis
Also, as expected, it s normalized, meaning

$$\sum_{a,b,c,d,e,f,...} \psi^*(a,b,c,d,e,f,...)\psi(a,b,c,d,e,f,...) = 1 \tag{9}$$

1.3 State-Vectors and Wave Functions

The term wave function refers to the collection of the coefficients, or components, a_j of what we have mentioned the components of the state-vector $|A\rangle$.

Confusing is a very likely outcome of the statement above, and the natural question to ask is:
What is the difference between state-vectors and wave functions?
A wave function can represent a state-vector. Their difference is in their physical meaning.
The state-vector sees the coefficients a_j as **coordinates** in a specific basis of eigenvectors, while the wave function, is simply the **collection** of these coefficients.

2 Measurement

Now lets go back to measurements and observables. Suppose we want to measure two observable L and M in a single experiment. This implies that L and M must commute. If they don't, is impossible to have unambiguous knowledge of both.

For now lets go back to a single spin! Any observable of a spin is represented by a 2x2 Hermitian Matrix that has the form:

$$\begin{pmatrix} r & w \\ w^* & r' \end{pmatrix} \tag{10}$$

, where r,r' are real numbers and w is a complex number and w^* is its conjugate!

It takes 3 parameters to specify this observable!

There is also a way to write any Spin Observable, in terms of the 4 Pauli Matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{11}$$

$$\sigma_z = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \tag{12}$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{13}$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{14}$$

The Identity Operator I is an observable, but the only possible value it can output is 1.

So, by ignoring I, the most general observable of a spin, is a superposition of the three: σ_x,σ_y and σ_z

These observables don't commute, so we cannot measure them simultaneously!

3 The Uncertainty Principle

Uncertainty doesn't mean that the result of an experiment is uncertain! If an observable is in an eigenstate, then there is no uncertainty in the result of the measurement of that observable.

However, no matter the state, there is always uncertainty about some observable.

If A and B don't commute, there must be uncertainty in one, if not in both!

Such an example is the famous *Heisenberg Uncertainty Principle*

that originally stated that you cannot simultaneously know both position and momentum of a particle.

4 The Meaning of Uncertainty

There needs to be no uncertainty about what we call uncertainty.

If we suppose that the eigenvalues of an observable A are denoted by a, then, given a state $|\Psi\rangle$,

$$\langle\Psi|A|\Psi\rangle = \sum_a aP(a) \tag{15}$$

That means that P(A) is based on the expectation value. When we say the uncertainty in A, is simply the standard deviation.
$\overline{A} = A - \langle A\rangle$

The probability distribution for $\overline{A}$ is the same as for A, with the only difference, that the average of $\overline{A}$ shifts to 0.
The exact same is true for the eigenvectors and eigenvalues of $\overline{A}$.
The square of the uncertainty is of a is what we call $(\Delta A)^2$, and it is defined by : $(\Delta A)^2 = \sum_a (a - \langle A\rangle)^2 P(a)$ or simply:

$$(\Delta A)^2 = \langle\Psi|\overline{A}^2|\Psi\rangle \tag{16}$$

In other words:
The square of the Uncertainty is the average value of the Operator A^2

5 Cauchy-Swarz Inequality

The uncertainty Principle is nothing but an inequity that tells us that the product if the uncertainties of two observables is something larger than something that has to do with the commutator.
The triangle inequality says that in any vector space, the magnitude of one side of the triangle is less than the sum of the magnitudes of the other two sides, which seems pretty obvious and intuitive.
Knowing this, we can derive that:

$$|X||Y| \geq |X \cdot Y| \tag{17}$$

6 The General Uncertainty Principle

Let $|\Psi\rangle$ be any ket, and suppose we define:
$|X\rangle = A\,|\Psi\rangle$
and
$|Y\rangle = iB\,|\Psi\rangle$
We define the product of the Uncertainties of A and B as:

$$\Delta A \Delta B = \frac{1}{2}|\,\langle\Psi|[A,B]|\Psi\rangle\,| \tag{18}$$

If the commutator of two observables isn't 0, we cannot know both observables simultaneously!
Try to prove the equation 9, using the Triangle Inequality!