10. The Harmonic Oscillator

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Contents

In Quantum Mechanics, the main components for building a description of the world, are the spins or qubits, that we have seen, and the **Harmonic Oscillator**.
The term Harmonic Oscillatoris used not to describe a specific phenomenon, but as a foundation to understand a general type of phenomena.
For example, such a phenomenon, could be the motion of a particle attached to a spring while hanging from the ceiling. The force that is asked on the moving particle is called **restoring force**, sine it pulls the particle back to its equilibrium position.
Such forces, always have a negative sign, since they always pull th particle towards the opposite direction.
Waves, Electromagnetic Waves, pendulums, the sound, etc. are all examples of systems based on oscillations!

1 The Classical Harmonic Oscillator

We will now work in Classical Mechanics and then switch back to Quantum Mechanics!
Suppose we have a weight hanging from a spring attached to the ceiling.
We will set its origin to be at the equilibrium position, denoted by $y = 0$, or its position

at rest.
In order to describe its motion, we will use the Lagrangian for our computations. The Lagrangian is equal to the Kinetic energy - Potential Energy!
For oscillators, it is known that the Potential energy is $\frac{ky^2}{2}$, where k is the spring constant.
That means that the Lagrangian is equal to:

$$L = \frac{1}{2}m\dot{y}^2 - \frac{ky^2}{2} \tag{1}$$

will now switch from y to x as our coordinate, and we will also make a simple change for our convenience: $x = \sqrt{m}y$, and we will also define another variable $\omega = \sqrt{\frac{k}{m}}$, which is the frequency of our oscillator, making our Lagrangian:

$$L = \frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega^2 x^2 \tag{2}$$

Having made this change of variables, we can describe all the oscillators with this equation. The only way to distinguish them is from the term ω.
Since we work in a single Dimension, we only need one Euler-Lagrange Equation:

$$\frac{\partial L}{\partial x} = \frac{d}{dt}\frac{\partial L}{\dot{x}} \tag{3}$$

Using the definition of the Lagrangian, we can see that

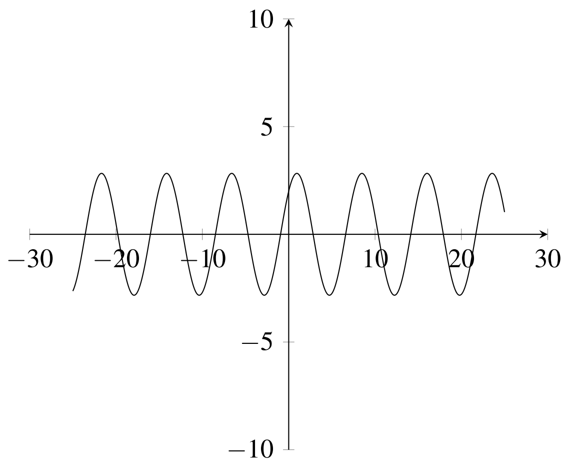
$$\frac{\partial L}{\dot{x}} = \dot{x} \tag{4}$$

Which is by definition the canonical momentum conjugate to x. (This is the very definition of the canonical momentum conjugate).
Then if we follow the Euler-Lagrange, we need to differentiate it one more time, giving us: $\ddot{x}$
The right hand side becomes simply $-\omega^2 x$ after the operations, leaving us with:

$$-\omega^2 x = \ddot{x} \tag{5}$$

If $\ddot{x}$ is the time derivative of canonical momentum, it means that it is equal to mass times acceleration, and by comparing our result to newtons law, ***F = ma***, we see that they are equivalent. The sign is negative, which indicates it is indeed a restoring force.
The General Solution for such an equation is given, as expected by:

$$x = A\cos\omega t + B\sin\omega t \tag{6}$$



Example of the graph of such an equation

2 The Quantum Harmonic Oscillator

Let's now go back to the microscopic scale. We will now try to describe the quantum version of the system we described in Quantum Mechanics.
First of all, we need a wavefunction to represent a particle moving on a line, and also one for every possible state!
$\psi(x)$ must satisfy $\psi^*(x)\psi(x) = P(x)$, where P(x) is the probability density to find the particle at the location x.
Obviously, another requirement, that should come to your mind naturally by now, is that it is normalized to unit total probability:

$$\int_{-\infty}^{+\infty} \psi^*(x)\psi(x)\, dx = 1 \tag{7}$$

In other words, $\psi(x)$ falls to 0 at infinity, so that the integral doesn't blow up. That is also vary obvious, since first of all it needs to be equal to 1, and second, the further away you get, the less the chance to find the particle! That means that at infinity it is practically 0.
Then, we need to know the Hamiltonian, in order to see how the state Ψ changes over time.
We can use the Lagrangian to derive H. We said that $p = \frac{\partial L}{\partial \dot{x}} = \dot{x}$. Knowing that, the Hamiltonian is given by:

$$H = p\dot{x} - L \tag{8}$$

where p is the canonical momentum conjugate to x as we said, or simply $\frac{\partial L}{\partial \dot{x}}$, and L the Lagrangian. The equation 8 is definition of the Hamiltonian for Harmonic Oscillators. Now we could sit here and do all the operations, but we are smarter than that. We know that L = T-V and that H = T+V. That means we can write the Hamiltonian using the

Classical Lagrangian, and then changing from Classical to Quantum Mechanics.

$$H = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega^2 x^2 \tag{9}$$

To change from Classical Mechanics, we need to assign each observable to an operator. Since we don't have a velocity operator, we'll have to get creative and write everything in terms of Canonical momentum conjugate of x and Position. The canonical momentum, as we said is given by $p = \frac{\partial L}{\partial \dot{x}} = \dot{x}$, so we can rewrite the Hamiltonian as:

$$H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 x^2 \tag{10}$$

We can now turn this Classical Hamiltonian, into the Quantum Version by substituting the X and P operators from previous lectures with x and p.
A quick reminder:
$X\psi(x) = x\psi(x)$ and that
$P\psi(x) = -i\hbar\frac{\partial \psi(x)}{\partial x}$ Now substituting everything in, we get the final Hamiltonian:

$$H\psi(x) = -\frac{\hbar^2}{2}\frac{\partial^2 \psi(x)}{\partial x^2} + \frac{1}{2}\omega^2 x^2 \psi(x) \tag{11}$$

Note that we use partial notation because the wave function is also, generally, a function of both position and time. It also means that we are describing the system at a fixed time!

3 The Schrödinger Equation

It is now time to see how the state changes over time. As expected, we will do that with the use of the Time Dependent Schrödinger Equation.

$$i\hbar\frac{\partial |\Psi\rangle}{\partial t} = H|\Psi\rangle \tag{12}$$

or if we divide by $\hbar$

$$i\frac{\partial \psi}{\partial t} = \frac{H\psi}{\hbar} \tag{13}$$

Substituting our Hamiltonian in we get:

$$i\frac{\partial \psi}{\partial t} = -\frac{\hbar}{2}\frac{\partial^2 \psi}{\partial x^2} + \frac{1}{2\hbar}\omega^2 x^2 \psi \tag{14}$$

This tells us, that if we know $\psi(x)$ at a specific time, we can predict its value in the future, and find its past states.
To solve this equation, we start with a known $\psi(x)$ and update it slightly by calculating its derivative. Once done we calculate $\partial_t \psi$. Then we add the result to the wave function and keep repeating. This is the solution to the Schrödinger Equation. Under some specific circumstances, the wave packet will move around like a Harmonic Oscillator.

4 Harmonic Oscillator Energy Levels

Another use of the Hamiltonian, aside from the the Time Dependent Schrödinger Equation, is the calculation of the energy eigenvalues and vectors, through the Time-Independent Schrödinger Equation.

$$H\,|\psi_E\rangle = E\,|\psi_E\rangle \tag{15}$$

The subscript E in ψ_E is to indicate that ψ_E is an energy eigenvector, with a specific eigenvalue E. If we expand the Time-Independent Schrödinger Equation we get:

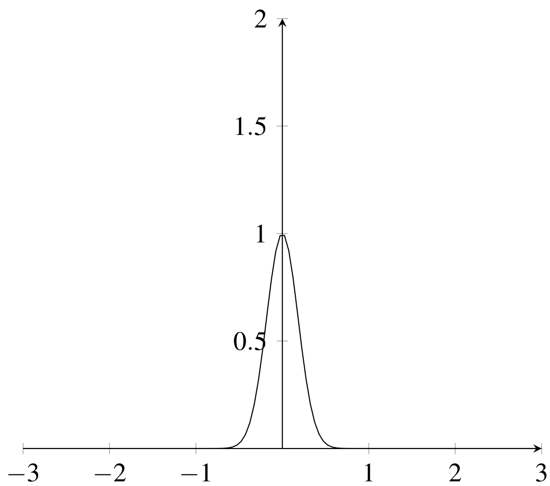
$$-\frac{\hbar^2}{2}\frac{\partial^2\psi_E(x)}{\partial x^2} + \frac{1}{2\hbar}\omega^2 x^2 \psi_E(x) = E\psi_E(x) \tag{16}$$

To find the solution to this equation, we need to find the possible allowable values of E that give us the mathematical solution, and the eigenvectors and possible eigenvalues of the energy.
Here we have a problem. Although there exists a solution for every value E, the most have no realistic physical meaning. Most of the time, $\psi(x)$ blows up, meaning that finding a normalizable solution is very hard and rare.
In reality, most of the E values, grow exponentially as x approaches ∞ or $-\infty$. That is absurd if we think about its physical meaning, and makes no sense. There is obviously a problem with probabilities here and in order to find a correct solution, that both solves the mathematical equation, and makes physical sense, we need to set another requirement for our solutions:
Physical Solutions to the Schrödinger Equation must be *normalizable*.
That constraint has a lot of power and narrows down the possible solutions to very few. Finding such a solution is very rare!

5 The Ground State of the Harmonic Oscillator

If you take a look at the equation 16, or 14, or generally our Hamiltonian you might recognize that there is something weird going on. Since we have the second term, namely $\frac{1}{2\hbar}\omega^2 x^2 \psi_E(x)$, we cannot have a 0 energy.We know from Uncertainty that Momentum and Position don't commute meaning we cannot set them both to 0 to get the minimum energy, thus we deduce that the least possible energy is not 0! It is above 0. That comes from the fact that the operators are squared: X^2 and P^2, meaning they have only positive eigenvalues! That, as we said, results in the Harmonic Oscillator to only be able to have positive eigenvalues.
In Quantum Mechanics, the lowest possible energy level is called the **Ground State** and is denoted by $\psi_0(x)$. *(Subscript 0 just denotes that it is the lowest energy level, not that it is 0)*
There exists a very useful mathematical theorem that helps us a lot with this, but proving it is beyond the scope of this chapter.
The Ground State Wave Function for any potential has no zeros and it's the only energy eigenstate that has no nodes.

That means we all have to do is to find is a node-less solution for some value E.
You can find this in any way! Guess it, calculate it, ask someone. For our purpose, it will be given to us: $\psi(x) = e^{-\frac{\omega}{2\hbar}x^2}$. Its graph looks something like this:



It satisfies all our conditions: 1) It is nodeless 2) It goes to 0 quickly as it moves away from the origin, x=0, that it doesn't blow up.
Let's now plug it in our Hamiltonian and see what happens:
First of all we will start with the term $\frac{\partial^2}{\partial x}$:
Let's begin with the first derivative and then move to the second:

$$\frac{\partial \psi(x)}{\partial x} = \frac{\omega}{2\hbar} 2xe^{-\frac{\omega}{2\hbar}x^2} \tag{17}$$

or simply $\frac{\omega}{\hbar}xe^{-\frac{\omega}{2\hbar}x^2}$.
Now taking the second derivative, we get that:

$$\frac{\partial^2 \psi(x)}{\partial x^2} = \frac{\omega}{\hbar}xe^{-\frac{\omega}{2\hbar}x^2} + \frac{\omega^2}{\hbar^2}x^2e^{-\frac{\omega}{2\hbar}x^2} \tag{18}$$

Next on, if we plug this to the Time-Independent Schrödinger Equation as we initially wanted, while replacing $\psi(x)$ with $e^{-\frac{\omega}{2\hbar}x^2}$ we get that:

$$\frac{\hbar}{2}\omega e^{-\frac{\omega}{2\hbar}x^2} - \frac{1}{2}\omega^2x^2e^{-\frac{\omega}{2\hbar}x^2} + \frac{1}{2}\omega^2x^2e^{-\frac{\omega}{2\hbar}x^2} = Ee^{-\frac{\omega}{2\hbar}x^2} \tag{19}$$

Cancelling now the terms, we get that:

$$\frac{\hbar}{2}\omega e^{-\frac{\omega}{2\hbar}x^2} = Ee^{-\frac{\omega}{2\hbar}x^2} \tag{20}$$

This clearly tells us that the only solution is when $E = \frac{\hbar\omega}{2}$ We now have both Wave Function and Ground State Energy!
$E_0 = \frac{\hbar\omega}{2}$ and $\psi_0(x) = e^{-\frac{\omega}{2\hbar}x^2}$

6 Creation and Annihilation Operators

Throughout these chapters, we have seen the power of operators, especially in the Harmonic oscillators. Having all these operators, means we can compute a lot and different commutators. A clever thing to do, according to Leonard Susskind, is to always compute these commutators. I might seem like a big, long and boring series of operations, which it totally is, but once in a while, we might find very interesting things that can help us describe the system better than without it! If we get very lucky, we could even find a set of operators that close under commutation! We will soon see an example of this!
Lets try to do this with the Harmonic Oscillator. We can rewrite the Hamiltonian in terms of X and P:

$$H = \frac{P^2 + \omega^2 X^2}{2} \tag{21}$$

We know from previous chapters that $[X,P] = i\hbar$. We can use that to construct two new operators, the **Annihilation and Creation Operators**. We will need to make an operator, that when acting on an energy eigenvector, it returns a new eigenvector that has the next energy level.(1 level higher), and one that does the opposite, meaning it returns an eigenvector with a the next lower energy level.
We will call them the creation and annihilation operators respectively.
Let's now see how they work and what they are equal to:

First of all, we have already written the Hamiltonian in the form $H = \frac{P^2+\omega^2X^2}{2}$. now if you have finished the 8th grade and know a bit of complex numbers and their properties, we can clearly see that there is a sum of squares, meaning we can express them using the opposite of the square difference identity, namely : $a^2+b^2 = (a+ib)(a-ib)$. We can now use that to manipulate the Hamiltonian and the identity, to get :

$$H = \frac{1}{2}(P+i\omega X)(P-i\omega X) \tag{22}$$

That is almost correct! The Problem is that P and X don't commute . If we expand everything and do the operations, we see that the equation 22 is not the Hamiltonian, but the $H - \frac{\omega\hbar}{2}$. *(As an easy exercise you can prove this)*. In other words,
$H = \frac{1}{2}(P+i\omega X)(P-i\omega X) + \frac{\omega\hbar}{2}$.
Now, by definition, the two operators, are the $(P+i\omega X)$ and the $(P-i\omega X)$. We denote them as:

$$a^+ = (P+i\omega X) \tag{23}$$

is the creation and

$$a^- = (P-i\omega X) \tag{24}$$

the annihilation. The operators are also called the **raising** and **lowering** operators, because, well that is what they do. I believe it is obvious which is which!
Actually, the equations above are not the *"official"* definitions. The official are:

$$a^+ = \frac{-i}{\sqrt{2\omega\hbar}}(P+i\omega X) \tag{25}$$

and

$$a^{-} = \frac{i}{\sqrt{2\omega\hbar}}(P - i\omega X) \tag{26}$$

Using these operators, we can rewrite the Hamiltonian as:

$$H = \omega\hbar(a^{+}a^{-} + 1/2) \tag{27}$$

The two Operators, are obviously Hermitian! Another Interesting fact is that their commutator is 1. $[a^{-}, a^{+}] = 1$.
(Try to prove this as an exercise)

We can now define our Third Operator: N, using the two operators.
This operator is called the Number Operator, and is defined as $N = a^{+}a^{-}$. that means, we can rewrite the Hamiltonian even simpler!

$$H = \omega\hbar(N + 1/2) \tag{28}$$

Now, something that is very easy to prove, is that the commutators, of these operators, are equal to:
$[a^{-}, a^{+}] = 1$,
$[a^{-}, N] = a^{-}$,
$[a^{-}, N] = a^{+}$ *(Try to prove this as an exercise)*

Now this is what we said earlier! A set of operators that close under commutation! Now, using the set of commutators, we can find the eigenstuff of N, which, as we can clearly see from equation 28, will then give us all the eigenstuff of H. To do that, we will using something like the mathematical induction to prove our hypothesis! Mathematical induction, is a technique for proving hypothesis, What we do, is we form our hypothesis around the variable n, and then, if we prove that the hypothesis, also holds for n+1, it means that it holds for all n. Here, we will do something similar!

Suppose that we have an n and its eigenvector $|n\rangle$.
By definition, $N|n\rangle = n|n\rangle$. Now suppose we have a new vector that is given by $a^{+}|n\rangle$.
Now we could continue this on and on and do all the calculations, do all the proofs, but that is not the goal of this section. The result we would get if we did everything is that

$$N(a^{+}|n\rangle) = (n+1)(a^{+}|n\rangle) \tag{29}$$

In English, that means that that whenever a^{+} acts on the eigenvector $|n\rangle$, it returns an eigenvector, whose eigenvalue is the eigenvalue of n+1. We can write this as:

$$a^{+}|n\rangle = |n+1\rangle \tag{30}$$

This can go on and on, giving us each time $|n+2\rangle$, $|n+3\rangle$,...., $|n+m\rangle$.
It is now obvious why it is called the raising operators. Now you probably made a right guess, namely that the lowering does the exact opposite. It returns $|n-1\rangle$.
However, there is a problem! These two relations, tell us that there exist unending

sequences both above and below n. The problem is that we said that the ground state is positive! It cannot have a negative eigenvalue! So clearly there is a problem here! We need a way to end the sequence that the lowering operator produces, when it reaches the ground state. The Only possible way for this to happen, is there exists a vector, denoted by $|0\rangle$, such that when a^- acts on it, it returns 0. Mathematically,

$$\exists\, |0\rangle : \;\; a^- |0\rangle = 0 \tag{31}$$

(Note that $|0\rangle$ isn't the 0 vector, meaning it has nonzero components) Since $|0\rangle$ has the lowest energy in the system, it is simply the Ground State, and its energy is $E_0 = \frac{\omega\hbar}{2}$ It is also defined as the eigenvector of n with eigenvalue 0. We usually say, that the ground state is **annihilated** by a^-.

This is a small taste of the power of the operators in Quantum Mechanics! We use 3 operators we defined from the Hamiltonian, and used them to find all the possible energy values of H.

Since n is a natural number, the possible energy values as we said are:

$$E_n = \omega\hbar(n + 1/2) \tag{32}$$

or simply $\frac{(2n-1)}{2}\omega\hbar$

The Harmonic Oscillator (energy levels), as well as the Hydrogen Atom, are the first results of Quantum Mechanics, and probably the most important!

7 Wave Functions

We just saw the power of operators, and how easy it is it solve everything with their use. However, Operators are very abstract and aren't very intuitive. Their physical meaning, as to the reason why their are equal to what their equal, is really not intuitive and doesn't make a lot of sense. That can't be said about wavefunctions! Wave functions are a lot more intuitive and easy to understand, as well as to visualize.
We will now try to show everything with the help of the wavefunction!
Let's try to write $a^- |0\rangle = 0$ in terms of the position and momentum operators, and the ground state wavefunction:

$$\frac{i}{\sqrt{2\omega\hbar}}(P - i\omega X)\psi_0(x) = 0 \tag{33}$$

or with the use of basic algebra:

$$(P - i\omega X)\psi_0(x) = 0 \tag{34}$$

Now it is time to work out the operation, and replace the operators P and X, with what we have seen earlier:

We rewrite 34 as

$$P\psi_0(x) - i\omega X\psi_0(x) = 0 \tag{35}$$

$$P\psi_0(x) = i\omega X\psi_0(x) \tag{36}$$

Now rewriting $P\psi_0(x)$ as $-i\hbar\nabla\psi_0(x)$ But since we are talking in one dimension, ∇ becomes ∇_x or simply ∂_x, and X as x, we get:

$$-i\hbar\partial_x\psi_0(x)) = i\omega x\psi_0(x) \tag{37}$$

Now dividing by $-i\hbar$, we get to

$$\partial_x\psi_0(x)) = -\frac{\omega x}{\hbar}\psi_0(x) \tag{38}$$

The solution to this equation is very easy, as you remember it is an exponential, however, we have already calculated that with operators, so we know that it is simply the ground state wavefunction!
$e^{-\frac{\omega}{2\hbar}x^2}$

It should be obvious to you that we are at the bottom of the ladder of energies! We can now use the creation or raising operator to go up a level! From n=0 to n=1.
We now from the definition of the creation operator, that $a^+|n\rangle = |n+1\rangle$. That means, that all we need to do to get to the first excited state, is to calculate:

$$\psi_1(x) = \frac{-i}{\sqrt{2\omega\hbar}}(P + i\omega X)\psi_0(x) \tag{39}$$

or to be more precise,

$$\psi_1(x) = \frac{-i}{\sqrt{2\omega\hbar}}(P + i\omega X)e^{-\frac{\omega}{2\hbar}x^2} \tag{40}$$

Let's solve this! It is pretty easy!

$$\psi_1(x) = \frac{-i}{\sqrt{2\omega\hbar}}(-i\hbar\partial_x e^{-\frac{\omega}{2\hbar}x^2} + i\omega x e^{-\frac{\omega}{2\hbar}x^2}) \tag{41}$$

The derivative of $e^{-\frac{\omega}{2\hbar}}x^2$ is equal to $-\frac{\omega x}{\hbar}e^{-\frac{\omega}{2\hbar}}x^2$ Now if we add this to equation 41, we get

$$\psi_1(x) = \frac{-i}{\sqrt{2\omega\hbar}}(i\omega x e^{-\frac{\omega}{2\hbar}x^2} + i\omega x e^{-\frac{\omega}{2\hbar}x^2}) \tag{42}$$

or simply

$$\psi_1(x) = \frac{-i}{\sqrt{2\omega\hbar}}2i\omega x e^{-\frac{\omega}{2\hbar}x^2} \tag{43}$$

or even simpler

$$\psi_1(x) = \frac{-i}{\sqrt{2\omega\hbar}}2i\omega x\psi_0(x) \tag{44}$$

or the final, simplest:

$$\psi_1(x) = \frac{\sqrt{2\omega\hbar}\, x\psi_0(x)}{\hbar} \tag{45}$$

The main difference between $\psi_1(x)$ and $\psi_0(x)$, is that in the first excited state, there is a term x. In other words there is a node at x=0. **Nodes are simply zeros in the functions!** This is true for all $\psi_n(x)$, meaning $\psi_n(x)$ has n nodes, or n zeros!
That means, that at ∞, The function approaches 0, or in other words, it has ∞ 0. That makes now a lot of sense, and we see what we meant it doesn't blow up, or that it approaches 0 at infinity. This also explains the reason why eigenfunctions of even degrees are symmetric while those of odd, are asymmetric!
When $\psi_n(x)$ goes up a state, so does the number of 0's in the wave function.
Such a function is called Hermite function, or Hermite polynomial That also means, that the higher the energy, the more spread out the wavefunction becomes, and the higher it oscillates, the higher the momentum. That makes obviously a lot of sense. Physically, it means that the mass is moving farther from the equilibrium and moving faster.

Now notice how I said it approaches 0. Well, it turns out, I never quite reaches 0! That means there is a finite chance, although infinitesimal, that the particle will be found outside the boundaries! That is the Quantum Tunneling phenomenon.

7.1 Simple Quantum Tunneling Experiment

There is a way to see this phenomenon in action. You have probably already seen it, but you probably never acknowledged its significance! I am talking about grabbing a glass of water. What does that have to do with quantum tunneling. Well, if you fill a glass with water and grab it with your fingers, if you grab it gently, you won't be able to see the photons passing thought the air barrier between the finger and the glass. However, if you press it hard enough, you will be able to see the photons passing through the air barrier between your hand and the glass. That is quantum tunneling in action!

8 Quantization

This is the end! Our final section together! In my opinion, this section, is one of the most important, and most interesting facts that Quantum Mechanics tells us. And what better way to finish our journey, than with a simple example, that has a lot of applications.
We will talk about electromagnetic waves in a region of space, enclosed by a pair of perfectly reflecting mirrors, that keep them bouncing back and fourth.
Let's suppose that these waves are of wavelength λ. In this example, there are more than just one harmonic oscillators produces by the fields. Thankfully, these oscillators, oscillate independently, which means we can focus on a specific one, namely the one with wavelength λ. The only number that is important in the context of harmonic oscillations is the frequency. That frequency ω, is given by $\omega = \frac{2\pi c}{\lambda}$, where c is the

speed of light.In Quantum Mechanics, we said that the frequency plays a major role when it comes to determining the energy of the system:
$(n+1/2)\hbar\omega$ The second term, $1/2\hbar\omega$, is called the zero point energy, and it doesn't concern us, meaning we can and will ignore it!
so what is left is

$$\frac{2\pi\hbar c}{\lambda} \tag{46}$$

or, in other words, the energy is given by:

$$\frac{2\pi\hbar c}{\lambda}n \tag{47}$$

where n¿0
That means, that the energy of an electromagnetic wave is quantized in units of $\frac{2\pi\hbar c}{\lambda}$.
Now why is this so important you ask? Well, these units are called **photons**. Photon, is also another name for quantized unit of energy in the Quantum Harmonic Oscillator.
Photos are indivisible! If a wave is excited to its n^{th} state, has n photons!!

THE END