Multivariate **Int**egration **Test** **Func**tions - Function References

Klaus Herrmann (Université de Sherbrooke)

September 6, 2024

Contents

Introduction

This note compiles references for the integration test functions implemented in the R package *multIntTestFunc*. For each function there is either a derivation of the exact value of the integral or a reference to a derivation. The functions are sorted by integration domains and Tables 2–6 provide an overview over the test functions for the supported domains at the end of the document. The available integration domains are

- the non-negative real numbers $[0,\infty)^n$ (Section P),
- the Euclidean space $\mathbb{R}^n$ (Section R),
- the standard simplex T_n (Section T),
- the unit hypercube C_n (Section C),
- the unit ball B_n (Section B), and
- the unit sphere S^{n-1} (Section S).

For shorthand notation we use $\cdot$ for inner products, i.e., $\mathbf{a}\cdot\mathbf{b} = \sum_{i=1}^n a_i b_i$ for two vectors in $\mathbb{R}^n$.

P Non-negative real numbers $[0,\infty)^n = \times_{i=1}^n [0,\infty)$

P.1 log-normal density

Consider the function

$$f\colon [0,\infty)^n \to [0,\infty), \mathbf{x} \mapsto f(\mathbf{x}) = \frac{\exp(-((\ln(\mathbf{x})-\mu)^T\Sigma^{-1}(\ln(\mathbf{x})-\mu))/2)}{\prod_{i=1}^n x_i\sqrt{(2\pi)^n\det(\Sigma)}}.$$

This is the density function of the log-normal distribution. Therefore

$$\int_{[0,\infty)^n} f(\mathbf{x})\,\mathrm{d}\mathbf{x} = 1.$$

The n-dimensional vector μ and the symmetric positive definite matrix $\Sigma \in \mathbb{R}^{n\times n}$ are the mean and variance-covariance matrix of a multivariate normal random vector.

P.2 log-t density

Consider the function

$$f\colon [0,\infty)^n \to [0,\infty),$$
$$\mathbf{x} \mapsto f(\mathbf{x}) = \frac{(\prod_{i=1}^n x_i^{-1})\Gamma\left[(\nu+n)/2\right]}{\Gamma(\nu/2)\nu^{n/2}\pi^{n/2}\left|\Sigma\right|^{1/2}}\left[1+\frac{1}{\nu}(\log(\mathbf{x})-\delta)^T\Sigma^{-1}(\log(\mathbf{x})-\delta)\right]^{-(\nu+n)/2}$$

This is the density function of the log-t distribution. Therefore

$$\int_{[0,\infty)^n} f(\mathbf{x})\,\mathrm{d}\mathbf{x} = 1.$$

The n-dimensional vector δ and the symmetric positive definite matrix $\Sigma \in \mathbb{R}^{n\times n}$ are the location and scale matrix of a multivariate log-t random vector. The positive real number ν is the degrees-of-freedom parameter.

R Euclidean Space $\mathbb{R}^n = \times_{i=1}^n \mathbb{R}$

R.1 Gaussian Integral

Consider the function

$$f\colon \mathbb{R}^n \to (0,\infty), \mathbf{x} \mapsto f(\mathbf{x}) = e^{-\|\mathbf{x}\|_2^2}.$$

As a standard result we have $\int_{\mathbb{R}} e^{-x^2}\,\mathrm{d}x = \pi^{1/2}$. This leads to

$$\int_{\mathbb{R}^n} f(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_{\mathbb{R}^n} e^{-\|\mathbf{x}\|_2^2}\,\mathrm{d}\mathbf{x} = \int_{\mathbb{R}^n}\prod_{i=1}^n e^{-x_i^2}\,\mathrm{d}\mathbf{x} = \prod_{i=1}^n \int_{\mathbb{R}^n} e^{-x_i^2}\,\mathrm{d}x_i = \pi^{n/2}.$$

R.2 Floor Norm Integral

For $s > 1$ define the function

$$f\colon \mathbb{R}^n \to (0, \infty), \mathbf{x} \mapsto f(\mathbf{x}) = \frac{\Gamma(n/2+1)}{\pi^{n/2}(1 + \lfloor \|\mathbf{x}\|_2^n \rfloor)^s}.$$

Denote by $B_r(0) = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_2 < r\}$ the open ball centered at zero in $\mathbb{R}^n$. In this case we have for $k = \{1, 2, 3, \dots\}$ that

$$1 + \lfloor \|\mathbf{x}\|_2^n \rfloor = k \Leftrightarrow (k-1)^{1/n} \leq \|\mathbf{x}\|_2 < k^{1/n} \Leftrightarrow \mathbf{x} \in B_{k^{1/n}}(0) \setminus B_{(k-1)^{1/n}}(0).$$

Therefore f is constant on the shells $S_k = B_{k^{1/n}}(0) \setminus B_{(k-1)^{1/n}}(0)$, where we set $S_1 = B_1(0)$, with value $f(\mathbf{x}) = \Gamma(n/2+1)/(\pi^{n/2} k^s)$. Therefore

$$\begin{aligned}
\int_{\mathbb{R}^n} f(\mathbf{x}) \,\mathrm{d}\mathbf{x} = \int_{\uplus_{k \geq 1} S_k} f(\mathbf{x}) \,\mathrm{d}\mathbf{x} &= \sum_{k=1}^{\infty} \int_{S_k} f(\mathbf{x}) \,\mathrm{d}\mathbf{x} = \sum_{k=1}^{\infty} \frac{\Gamma(n/2+1)}{\pi^{n/2} k^s} \mathrm{vol}\,(S_k) \\
&= \sum_{k=1}^{\infty} \frac{\Gamma(n/2+1)}{\pi^{n/2} k^s} \Big(\mathrm{vol}(B_{k^{1/n}}(0)) - \mathrm{vol}(B_{(k-1)^{1/n}}(0)) \Big) \\
&= \sum_{k=1}^{\infty} \frac{\Gamma(n/2+1)}{\pi^{n/2} k^s} \left(\frac{\pi^{n/2}}{\Gamma(n/2+1)} (k^{1/n})^n - \frac{\pi^{n/2}}{\Gamma(n/2+1)} ((k-1)^{1/n})^n \right) \\
&= \sum_{k=1}^{\infty} \frac{1}{k^s} = \zeta(s),
\end{aligned}$$

where ζ is the Riemann zeta function.

R.3 Multivariate normal density

Consider the function

$$f\colon \mathbb{R}^n \to [0, \infty), \mathbf{x} \mapsto f(\mathbf{x}) = \frac{\exp(-((\mathbf{x} - \mu)^T \Sigma^{-1} (\mathbf{x} - \mu))/2)}{\sqrt{(2\pi)^n \det(\Sigma)}}.$$

This is the density function of the multivariate normal distribution. Therefore

$$\int_{\mathbb{R}^n} f(\mathbf{x}) \,\mathrm{d}\mathbf{x} = 1.$$

The n-dimensional vector μ and the symmetric positive definite matrix $\Sigma \in \mathbb{R}^{n \times n}$ are the respective mean and variance-covariance matrix.

R.4 Multivariate t density

Consider the function

$$f\colon \mathbb{R}^n \to [0,\infty), \mathbf{x} \mapsto f(\mathbf{x}) = \frac{\Gamma\left[(\nu+n)/2\right]}{\Gamma(\nu/2)\nu^{n/2}\pi^{n/2}\left|\Sigma\right|^{1/2}}\left[1+\frac{1}{\nu}(\mathbf{x}-\delta)^T\Sigma^{-1}(\mathbf{x}-\delta)\right]^{-(\nu+n)/2}.$$

This is the density function of the multivariate t distribution. Therefore

$$\int_{\mathbb{R}^n} f(\mathbf{x})\,\mathrm{d}\mathbf{x} = 1.$$

The n-dimensional vector δ and the symmetric positive definite matrix $\Sigma \in \mathbb{R}^{n\times n}$ are the respective location vector and scale matrix. The positive real number ν is the degrees-of-freedom parameter.

T Standard Simplex $T_n = \{\mathbf{x} \in \mathbb{R}^n : x_i \geq 0, \|\mathbf{x}\|_1 \leq 1\}$

For a continuously differentiable function $f\colon [0,1] \to \mathbb{R}$ we have, see [5], that

$$\int_{T_n} f(x_1 + \ldots + x_n)\,\mathrm{d}\mathbf{x} = \frac{1}{\Gamma(n)}\int_0^1 f(s)s^{n-1}ds. \tag{1}$$

Equation (1) can be used to construct integrable functions on T_n.

T.1 Dirichlet Integral

For a vector $\boldsymbol{v} \in \mathbb{R}^{n+1}$ with strictly positive entries, i.e., $v_i > 0$, define the function

$$f\colon T_n \to (0,\infty), \mathbf{x} \mapsto f(\mathbf{x}) = \prod_{i=1}^{n} x_i^{v_i-1}(1-x_1-\ldots-x_n)^{v_{n+1}-1}.$$

It can be seen that the integral of f over T_n is the norming constant for the Dirichlet distribution. We therefore have

$$\int_{T_n} f(\mathbf{x})\,\mathrm{d}\mathbf{x} = \frac{\prod_{i=1}^{n+1}\Gamma(v_i)}{\Gamma(\sum_{i=1}^{n+1} v_i)}.$$

T.2 Exponential of Sum

For a constant $c > 0$ define the function

$$f : T_n \to (0, \infty), \mathbf{x} \mapsto f(\mathbf{x}) = e^{-c(x_1 + \ldots + x_n)}.$$

Combining (1) with integration-by-substitution yields

$$\begin{aligned} \int_{T_n} f(\mathbf{x}) \, \mathrm{d}\mathbf{x} &= \frac{1}{\Gamma(n)} \int_0^1 e^{-cs} s^{n-1} \, \mathrm{d}s \\ &= \frac{c^{-n}}{\Gamma(n)} \int_0^c e^{-t} t^{n-1} \, \mathrm{d}t \\ &= \frac{\Gamma(n) - \Gamma(n, c)}{c^n \Gamma(n)}, \end{aligned}$$

where $\Gamma(s, x)$ is the upper incomplete gamma function defined as

$$\Gamma(s, x) = \int_x^\infty t^{s-1} e^{-t} \, \mathrm{d}t.$$

C Unit Cube $C_n = [0, 1]^n$

C.1 Cosine Square

For a vector $\boldsymbol{v} \in \mathbb{R}^n$, $\boldsymbol{v} \neq \mathbf{0}_n$, define the function

$$f : C_n \to \mathbb{R}, \mathbf{x} \mapsto f(\mathbf{x}) = (\cos(\boldsymbol{x} \cdot \boldsymbol{v}))^2.$$

Following [8] one can use the identities $\cos(x) = \Re(e^{ix})$ and $\cos(x)^2 = \frac{1}{2} + \frac{1}{2}\cos(2x)$ to show

$$\int_{C_n} f(\mathbf{x}) \, \mathrm{d}\mathbf{x} = \frac{1}{2} + \frac{1}{2} \cos\left(\sum_{j=1}^n v_j\right) \prod_{j=1}^n \frac{\sin(v_j)}{v_j}.$$

C.2 Floor of Sum

For the function

$$f : C_n \to \mathbb{R}, \mathbf{x} \mapsto f(\mathbf{x}) = \lfloor x_1 + \ldots + x_n \rfloor$$

we have, see [4], that

$$\int_{C_n} f(\mathbf{x}) \, \mathrm{d}\mathbf{x} = \frac{n-1}{2}.$$

C.3 Maximum

For the function

$$f\colon C_n \to \mathbb{R}, \mathbf{x} \mapsto f(\mathbf{x}) = \max(x_1, \ldots, x_n)$$

we have, see [7], that

$$\int_{C_n} f(\mathbf{x})\,\mathrm{d}\mathbf{x} = \frac{n}{n+1}.$$

C.4 Bratley-Fox-Niederreiter I_1

For the function

$$f\colon C_n \to \mathbb{R}, \mathbf{x} \mapsto f(\mathbf{x}) = \prod_{i=1}^{n} |4x_i - 2|$$

we have, see [1, p. 207], that

$$\int_{C_n} f(\mathbf{x})\,\mathrm{d}\mathbf{x} = 1.$$

C.5 Bratley-Fox-Niederreiter I_2

For the function

$$f\colon C_n \to \mathbb{R}, \mathbf{x} \mapsto f(\mathbf{x}) = \prod_{i=1}^{n} i\cos(ix_i)$$

we have, see [1, p. 207], that

$$\int_{C_n} f(\mathbf{x})\,\mathrm{d}\mathbf{x} = \prod_{i=1}^{n} \sin(i).$$

C.6 Bratley-Fox-Niederreiter I_3

For the function

$$f\colon C_n \to \mathbb{R}, \mathbf{x} \mapsto f(\mathbf{x}) = \prod_{i=1}^{n} T_{\nu(i)}(2x_i - 1),$$

where T_k is the Chebyshev polynomial of degree k and $\nu(i) = (i \mod 4) + 1$, we have, see [1, p. 207], that

$$\int_{C_n} f(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0.$$

C.7 Bratley-Fox-Niederreiter I_4

For the function

$$f\colon C_n \to \mathbb{R}, \mathbf{x} \mapsto f(\mathbf{x}) = \sum_{i=1}^{n} (-1)^i \prod_{j=1}^{i} x_j$$

we have, see [1, p. 207], that

$$\int_{C_n} f(\mathbf{x})\,\mathrm{d}\mathbf{x} = -\frac{1}{3}\left(1 - \left(\frac{1}{2}\right)^n\right).$$

C.8 Genz Oscillatory f_1

As proposed in [3, Table 1, p. 84] we can consider the function

$$f\colon C_n \to \mathbb{R}, \mathbf{x} \mapsto f(\mathbf{x}) = \cos\left(2\pi u + \sum_{i=1}^{n} a_i x_i\right),$$

where $u \in \mathbb{R}$ and $(a_1, \ldots, a_n) \in \mathbb{R}^n$ with $a_i \neq 0$. The value of u does not change the difficulty of the integration, while higher values for $|a_i|$ increase the difficulty; see [3, p. 84] for details. We can then compute

$$\int_{C_n} f(\mathbf{x})\,\mathrm{d}\mathbf{x} = \frac{2^n \cos\left(2\pi u + \sum_{i=1}^{n} a_i/2\right) \prod_{i=1}^{n} \sin(a_i/2)}{\prod_{i=1}^{n} a_i}$$

via induction. First we have for $c \in \mathbb{R}$ and $0 \neq \kappa \in \mathbb{R}$ with the standard trigonometric identities $\sin(x+y) - \sin(y) = 2\cos(x/2+y)\sin(x/2)$, $x, y \in \mathbb{R}$, that

$$\int_0^1 \cos(\kappa x + c)\,\mathrm{d}x = \frac{\sin(\kappa + c) - \sin(c)}{\kappa} = \frac{2\cos(\kappa/2 + c)\sin(\kappa/2)}{\kappa}.$$

For $n = 1$ we therefore we have

$$\int_0^1 \cos\left(2\pi u + a_1 x_1\right)\,\mathrm{d}x_1 = \frac{2\cos(a_1/2 + 2\pi u)\sin(a_1/2)}{a_1}.$$

By induction we have for $n \mapsto n+1$ that

$$
\begin{aligned}
&\int_{C_{n+1}} \cos\left(2\pi u + \sum_{i=1}^{n+1} a_i x_i\right) \mathrm{d}\mathbf{x} \\
&\quad = \int_0^1 \int_{C_n} \cos\left(2\pi u + \sum_{i=1}^{n} a_i x_i + \frac{2\pi a_{n+1} x_{n+1}}{2\pi}\right) \mathrm{d}\mathbf{x} \\
&\quad = \int_0^1 \int_{C_n} \cos\left(2\pi \left(u + \frac{a_{n+1} x_{n+1}}{2\pi}\right) + \sum_{i=1}^{n} a_i x_i\right) \mathrm{d}\mathbf{x} \\
&\quad = \int_0^1 \frac{2^n \cos\left(2\pi\left(u + \frac{a_{n+1}x_{n+1}}{2\pi}\right) + \sum_{i=1}^n a_i/2\right) \prod_{i=1}^n \sin(a_i/2)}{\prod_{i=1}^n a_i} \mathrm{d}x_{n+1} \\
&\quad = \frac{2^n \prod_{i=1}^n \sin(a_i/2)}{\prod_{i=1}^n a_i} \int_0^1 \cos\left(2\pi u + a_{n+1}x_{n+1} + \sum_{i=1}^{n} a_i/2\right) \mathrm{d}x_{n+1} \\
&\quad = \frac{2^n \prod_{i=1}^n \sin(a_i/2)}{\prod_{i=1}^n a_i} \frac{2\cos\left(a_{n+1}/2 + 2\pi u + \sum_{i=1}^n a_i/2\right) \sin(a_{n+1}/2)}{a_{n+1}} \\
&\quad = \frac{2^{n+1} \prod_{i=1}^{n+1} \sin(a_i/2)}{\prod_{i=1}^{n+1} a_i} \cos\left(2\pi u + \sum_{i=1}^{n+1} a_i/2\right).
\end{aligned}
$$

B Unit Ball $B_n = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_2 \leq 1\}$

B.1 Norm of Standard Normal Random Vector

The function

$$f\colon B_n \to \mathbb{R}, \mathbf{x} \mapsto f(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}} e^{-\|\mathbf{x}\|_2^2/2}$$

can be seen as the density of a random vector $\boldsymbol{X} = (X_1, \ldots, X_n)$, where all $X_i \sim \mathcal{N}(0,1)$ are independent. We can then rewrite the integral as

$$\int_{B_n} f(\mathbf{x})\, \mathrm{d}\mathbf{x} = \mathbb{P}[\|\boldsymbol{X}\|_2 \leq 1] = \mathbb{P}[\|\boldsymbol{X}\|_2^2 \leq 1] = F_{\chi_n^2}(1),$$

where $F_{\chi_n^2}$ is the distribution function of a chi-square random variable.

B.2 Polynomials

For non-negative integers $a_1, \ldots, a_n$, $a_i \in \{0, 1, 2, \ldots\}$, define the monomial

$$f\colon B_n \to \mathbb{R}, \mathbf{x} \mapsto f(\mathbf{x}) = \prod_{i=1}^{n} x_i^{a_i}.$$

We know from [2] that

$$\int_{B_n} f(\mathbf{x})\,\mathrm{d}\mathbf{x} = \begin{cases} 0, & \text{if at least one } a_i \text{ is odd,} \\ \frac{2\prod_{i=1}^n \Gamma(b_i)}{\Gamma(\sum_{i=1}^n b_i)(n+\sum_{i=1}^n a_i)}, b_i = \frac{a_i+1}{2}, & \text{if all } a_i \text{ are even.} \end{cases}$$

S Unit Sphere $S^{n-1} = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_2 = 1\}$

S.1 Inner Products

For two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ define the function

$$f\colon S^{n-1} \to \mathbb{R}, \mathbf{x} \mapsto f(\mathbf{x}) = (\mathbf{a} \cdot \mathbf{x})(\mathbf{b} \cdot \mathbf{x}).$$

From Proposition 2 in [6] we know

$$\int_{S^{n-1}} f(\mathbf{x})\,\mathrm{d}\mathbf{x} = \frac{\left|S^{n-1}\right|}{n}(\mathbf{a} \cdot \mathbf{b}).$$

For $n > 1$ we have $\left|S^{n-1}\right| = \frac{2\pi^{n/2}}{\Gamma(n/2)}$. For $n = 1$ we have $S^{n-1} = \{-1, 1\}$ and therefore the integral is zero.

S.2 Polynomials

For non-negative integers $a_1, \ldots, a_n$, $a_i \in \{0, 1, 2, \ldots\}$, define the monomial

$$f\colon S^{n-1} \to \mathbb{R}, \mathbf{x} \mapsto f(\mathbf{x}) = \prod_{i=1}^n x_i^{a_i}.$$

For $n > 1$ we know from [2] that

$$\int_{S^{n-1}} f(\mathbf{x})\,\mathrm{d}\mathbf{x} = \begin{cases} 0, & \text{if at least one } a_i \text{ is odd,} \\ \frac{2\prod_{i=1}^n \Gamma(b_i)}{\Gamma(\sum_{i=1}^n b_i)}, b_i = \frac{a_i+1}{2}, & \text{if all } a_i \text{ are even.} \end{cases}$$

For $n = 1$ we have $S^{n-1} = \{-1, 1\}$ and therefore the integral is zero.

Tables of Integration Functions

The following Tables 1–6 provide an overview over the available functions by integration domain. The first column shows the S4 class name within the package, while the last column provides a reference to the relevant subsection in this document.

Pn_	Parameters	$f(\boldsymbol{x}) =$	exact value	Properties	Details
lognormalDensity	$n, \in \mathbb{N}, \mu \in \mathbb{R}^n, \Sigma \in \mathbb{R}^{n\times n}$	$\frac{\exp(-((\ln(\mathbf{x})-\mu)^T\Sigma^{-1}(\ln(\mathbf{x})-\mu))/2)}{\prod_{i=1}^n x_i\sqrt{(2\pi)^n \det(\Sigma)}}$	1.0	C^∞	P.1
logtDensity	$n, \in \mathbb{N}, \delta \in \mathbb{R}^n, \Sigma \in \mathbb{R}^{n\times n}, \nu > 0$	(see P.2)	1.0	C^∞	P.2

Table 1: Functions with integration domain $[0, \infty)^n$.

Rn_	Parameters	$f(\boldsymbol{x}) =$	exact value	Properties	Details
Gauss	$n \in \mathbb{N}$	$\exp(-\left\lVert \boldsymbol{x} \right\rVert_2^2)$	$\pi^{n/2}$	C^∞	R.1
floorNorm	$n \in \mathbb{N},\ s > 1$	$\frac{\Gamma(n/2+1)}{\pi^{n/2}(1+\lfloor \lVert \boldsymbol{x} \rVert_2^n \rfloor)^s}$	$\zeta(s)$	non-continuous	R.2
normalDensity	$n, \in \mathbb{N}, \mu \in \mathbb{R}^n, \Sigma \in \mathbb{R}^{n \times n}$	$\frac{\exp(-((\mathbf{x}-\mu)^T \Sigma^{-1} (\mathbf{x}-\mu))/2)}{\sqrt{(2\pi)^n \det(\Sigma)}}$	1.0	C^∞	R.3
tDensity	$n, \in \mathbb{N}, \delta \in \mathbb{R}^n, \Sigma \in \mathbb{R}^{n \times n}, \nu > 0$	(see R.4)	1.0	C^∞	R.4

Table 2: Functions with integration domain $\mathbb{R}^n$.

standardSimplex_	Parameters	$f(\boldsymbol{x}) =$	exact value	Properties	Details
Dirichlet	$n \in \mathbb{N}$, $\boldsymbol{v} \in (0, \infty)^{n+1}$	$\prod_{i=1}^{n} x_i^{v_i - 1}(1 - \sum_{i=1}^{n} x_i)^{v_{n+1}-1}$	$\frac{\prod_{i=1}^{n+1} \Gamma(v_i)}{\Gamma(\sum_{i=1}^{n+1} v_i)}$	C^∞	T.1
exp_sum	$n \in \mathbb{N}$, $c > 0$	$\exp(-c(x_1 + \ldots + x_n))$	$\frac{\Gamma(n) - \Gamma(n,c)}{\Gamma(n) c^n}$	C^∞	T.2

Table 3: Functions with integration domain T_n.

unitCube_	Parameters	$f(\boldsymbol{x}) =$	exact value	Properties	Details
cos2	$n \in \mathbb{N}$, $\boldsymbol{v} \in \mathbb{R}^n \setminus \mathbf{0}_n$	$(\cos(\boldsymbol{v} \cdot \boldsymbol{x}))^2$	$\frac{1}{2} + \frac{1}{2}\cos(\boldsymbol{v} \cdot \mathbf{1}_n) \prod_{k=1}^{n} \frac{\sin(v_k)}{v_k}$	C^∞	C.1
floor	$n \in \mathbb{N}$	$\lfloor x_1 + \ldots + x_n \rfloor$	$(n-1)/2$	non-continuous	C.2
max	$n \in \mathbb{N}$	$\max(x_1, \ldots, x_n)$	$n/(n+1)$	continuous, non-differentiable	C.3
BFN1	$n \in \mathbb{N}$	$\prod_{i=1}^{n} \lvert 4x_i - 2 \rvert$	1	continuous, non-differentiable	C.4
BFN2	$n \in \mathbb{N}$	$\prod_{i=1}^{n} i \cos(i x_i)$	$\prod_{i=1}^{n} \sin(i)$	C^∞	C.5
BFN3	$n \in \mathbb{N}$	$\prod_{i=1}^{n} T_{\nu(i)}(2x_i - 1)$	0	C^∞	C.6
BFN4	$n \in \mathbb{N}$	$\sum_{i=1}^{n} (-1)^i \prod_{j=1}^{i} x_j$	$-(1 - (1/2)^n)/3$	C^∞	C.7
Genz1	$u \in \mathbb{R}$, $\boldsymbol{a} \in \mathbb{R}^n$, $a_i \neq 0$	$\cos(2\pi u + \boldsymbol{a} \cdot \boldsymbol{x})$	$\frac{2^n \cos\left(2\pi u + \sum_{i=1}^{n} \frac{a_i}{2}\right) \prod_{i=1}^{n} \sin\left(\frac{a_i}{2}\right)}{\prod_{i=1}^{n} a_i}$	C^∞	C.8

Table 4: Functions with integration domain $C_n = [0,1]^n$.

unitBall_	Parameters	$f(\boldsymbol{x}) =$	exact value	Properties	Details
normGauss	$n \in \mathbb{N}$	$\frac{1}{(2\pi)^{n/2}} \exp(-\left\Vert \boldsymbol{x} \right\Vert_2^2 / 2)$	$F_{\chi_n^2}(1)$	C^∞	B.1
polynomial	$n \in \mathbb{N}$, $\boldsymbol{a} \in \{0, 1, 2, \ldots\}^n$	$\prod_{i=1}^n x_i^{a_i}$	(see details)	C^∞	B.2

Table 5: Functions with integration domain B_n.

unitSphere_	Parameters	$f(\boldsymbol{x}) =$	exact value	Properties	Details
innerProduct1	$n \in \mathbb{N}$, $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$	$(\mathbf{a} \cdot \mathbf{x})(\mathbf{b} \cdot \mathbf{x})$	$\frac{\lvert S^{n-1} \rvert}{n}(\mathbf{a} \cdot \mathbf{b})$	C^∞	S.1
polynomial	$n \in \mathbb{N}$, $\boldsymbol{a} \in \{0, 1, 2, \ldots\}^n$	$\prod_{i=1}^{n} x_i^{a_i}$	(see details)	C^∞	S.2

Table 6: Functions with integration domain S^{n-1}.

References

[1] Paul Bratley, Bennett L Fox, and Harald Niederreiter. Implementation and tests of low-discrepancy sequences. *ACM Transactions on Modeling and Computer Simulation*, 2(3):195–213, 1992.

[2] G. Folland. How to integrate a polynomial over a sphere. *The American Mathematical Monthly*, 108(5):446–448, 2001.

[3] Alan Genz. Testing multidimensional integration routines. In *Proc. of international conference on Tools, methods and languages for scientific and engineering computation*, pages 81–94, 1984.

[4] Micah (https://math.stackexchange.com/users/30836/micah). Compute $\int_0^1 \int_0^1 ... \int_0^1 \lfloor x_1 + x_2 + ... + x_n \rfloor dx_1 dx_2 ... dx_n$. Mathematics Stack Exchange. URL:https://math.stackexchange.com/q/532824 (version: 2013-10-20).

[5] Dimitris (https://math.stackexchange.com/users/37229/dimitris). Integration of a function on the simplex. Mathematics Stack Exchange. URL:https://math.stackexchange.com/q/1878128 (version: 2016-08-01).

[6] T. Khan and M. Pinsky. On some integrals over a unit sphere. Technical report, Clemson University, 2003.

[7] lfc (https://math.stackexchange.com/users/253871/lfc). Calculate the following integral. Mathematics Stack Exchange. URL:https://math.stackexchange.com/q/1874340 (version: 2016-07-28).

[8] robjohn (https://math.stackexchange.com/users/13854/robjohn). Putnam exam integral: $\lim_{n\to\infty} \int_0^1 \int_0^1 ... \int_0^1 \cos^2\left(\frac{\pi}{2n}(x_1 + x_2 + ...x_n)\right) dx_1 dx_2 ... dx_n$. Mathematics Stack Exchange. URL:https://math.stackexchange.com/q/733464 (version: 2014-03-31).