Lesson Plans - Polynomials (HS Math)

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1 Introduction to Polynomials

By Mia Petrie

§1.1 Overview

§1.1.1 Objectives

- Students should be able to recite key terms and concepts.
- Students will be able to demonstrate this knowledge by breaking down the parts of a polynomial.
- Students will apply their knowledge of polynomials by successfully adding and subtracting polynomials.

§1.1.2 Materials

Each student will need to have a pencil and paper to take notes and complete worksheets given to them. They will also need some way to connect to online games, whether that be through a Chromebook or their own personal phone. I will need some way of presenting a slideshow (e.g., Computer and projector of sorts), and enough printed-off worksheets for the whole class.

§1.2 Outline

Total lesson length: approx. 50 minutes

Adding and Subtracting Polynomials

- Explanation of today's lesson, and present the learning objectives (Slide 2)
- Review Game (5 minutes), top winners will get candy (Slide 3)
  - This game provides an **incentive** for students to pay attention and learn the terms
- Further explanation of terms/labeling parts of a polynomial (5 minutes) (Slides 4, 5, & 6)
  - Vocab: Terms, coefficients, exponents, variables, degree, and constants.
  - What makes a polynomial?
- Give examples of polynomials and non-polynomials. (5 minutes) (Slide 7)
  - 3 Not polynomials, identify why.

  - 3 real polynomials, identify correctly.
- Standard Form (5 minutes) (Slides 8 & 9)
  - Getting polynomials into standard form
    - Give 3 examples; 2 rearranging and 1 distributing.
- Explain the process of adding Polynomials (10 minutes) (Slides 10 & 11)
  - Like terms (look at exponents)
  - Be aware of signs (positive or negative)
  - Do 3 examples of varying difficulty:
    - Adding, fairly simple, basics
    - Longer, more negative numbers
    - Long, degree 7
- Explain the process of subtracing polynomials
  - Like terms
  - Carrying over the subtraction to everything
  - Do 2 examples of varying difficulty
    - Short and sweet, intro carrying
    - Higher degree, more complex
  - Showing these examples and walking them through problems helps give them the **scaffolding** they need until they are able to complete them on their own.
    - This lesson was mostly the **direct-instruction approach** where the teacher lectures to give the students new information.
- Hand out homework worksheets. Students are free to work in groups or pairs. Encourage them to try to figure it out with their peers without teacher intervention. Will be graded on effort/completion and gone over next class. They have the remaining class time to work on it.
  - Homework helps with **rehearsal**, ensuring they practice the content until they can do it easily and know the processes.

Worksheet Link

Slideshow Link

§1.3 Steps

1. We will start out class by going through a list of what will be discussed in class today. This is located on the provided slideshow. This lesson will utilize the **direct-instruction approach** in order to efficiently introduce the information to the students.

2. Next, we will play a quick review game that covers key terms related to polynomials. Please have all of the students take out a device that they can play Kahoot on (this could be their school Chromebook or phone, whatever they have access to). These should mostly be review, and not new terms. Playing this game, in the beginning, will remind the students of these terms and force them to **retrieve** these terms from their memory. The top three winners will get a small piece of candy, this is a type of **incentive** to encourage students to participate and try their best.
3. After the game we will return to the slideshow and redefine these terms, **rehearsal** of the definitions helps students to remember.
4. The next slide has a polynomial function. Ask the students to examine it and name each part of it (list the terms, coefficients, exponents, what is the variable, and degree?).
5. The next slide has the definition of a polynomial and different types of polynomials. Getting a concrete knowledge of the basics before moving on to the rest of the lesson helps to promote mastery learning.
6. The next slide provides a list of polynomials and non-polynomials. Ask the students to identify what is and isn't based on the previously provided definition.
7. The next 5 slides cover the basics of standard form, converting to standard form, adding polynomials, and subtracting polynomials. Walk the students through each problem on the slide and allow them to ask questions when necessary.
8. The final slide indicates that it is homework time. Pass out the printed-off worksheets. As the slides explain, students are free to work in groups, pairs, or individually. They have the remaining class time to work on it and ask any questions, though encourage them to ask their classmates what they think before going to the teacher.

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2 Polynomial Multiplication

§2.1 Overview

§2.1.1 Objectives

- Students can demonstrate an understanding of how to multiply polynomials.
- Students can check problems with classmates and are able to identify right and wrong answers.
- Students can apply their knowledge to more complex problems.

§2.1.2 Materials

Each student will need to have a pencil and paper to take notes and complete worksheets given to them. I will need some way of presenting a slideshow (ex. Computer and projector of sorts), and enough printed-off worksheets for the whole class.

§2.2 Outline

Total lesson length: approx. 50 minutes

- Explanation of today's lesson, and present the learning objectives (Slide 2)
- Review Homework and Answer any Questions (10 minutes) (Slide 3)
  - This allows for better **mastery learning** as I am able to tell if the students fully understand the material before continuing the lesson.
- Introduce Multiplying Simple/Longer Polynomials (15 minutes) (Slides 4, 5, & 6)
  - Walk through the process
  - Add exponents, combine like terms
  - Do 3 examples
    - Monomial
    - Binomials (with FOIL)
    - A Trinomial and a binomial
- Break into pairs, they can choose partners (5-8 minutes) (Slide 7 & 8)
  - Give each group a long polynomial to work through. ($\sim$degree 5 $\times$ degree 5)
  - Allow them to ask questions and check each other.
    - This is a type of **cooperative learning** that allows students to share ideas and discuss.

- Have each group pair up with another and walk them through each other's problems. Chnage to the next slide that has the answers to each problem, slide 9, so they can check their work. (∼5-8 minutes)
  - Allows for **peer-to-peer** tutoring as the students can help each other if they had any trouble answering the question, and even if they did not they can explain their thought process and how they completely their problem. It is also a type of **formative assessment** as it assesses their knowledge during the lesson and by paying attention to what they are discussing I can determine if they have fully understood the material.
- Hand out homework worksheets. Students are free to work in groups or pairs. Encourage them to try to figure it out with their peers without teacher intervention. Will be graded on effort/completion and gone over next class. They have the remaining class time to work on it. (Cooperative Learning)

Worksheet Link

Slideshow Link

§2.3 Steps

1. This lesson will start off by going over an agenda for today's lesson. This is located in the provided slideshow.
2. Next, we will go over the previous lesson's homework. This is time for the students to ask any questions they may have had and the teacher can walk them through any difficult problems. This is a type of **formative assessment** and allows the teacher to understand if the students understood the previous lesson, and modify the current lesson if needed.
3. Next, we will introduce a new topic, multiplying polynomial equations. The next 3 slides have examples of problems with increasing difficulties. Walk the students through each problem and answer any questions they have. This provides students with **scaffolding** as they are introduced to more difficult problems.
4. The next slide explains a partner activity. Separate the students into pairs and have each pair pick one of the ten problems on the following slide. They will have a few minutes to work through it together. This is a type of **peer-to-peer tutoring** as they are able to help each other.
5. Once all of the pairs appear to have completed their problems, or 8 minutes have passed, ask the pairs to meet with other pairs to discuss their problems and explain what they did to the other groups. Provide them with the answers to their problems (located on the following slide) so that if they do not have the correct answer they can find out why with their new group. This is a type of **cooperative learning** as they are helping each other and bringing together what they have learned, similar to the jigsaw method.

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6. The final slide indicates that it is homework time. Pass out the printed-off worksheets. As the slides explain, students are free to work in groups, pairs, or individually. They have the remaining class time to work on it and ask any questions, though encourage them to ask their classmates what they think before going to the teacher.

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3 Quadratics

By Nathan Diaz

§3.1 Overview

§3.1.1 Objectives

- Students will be able to learn how they could able the quadratic formula to real life situations
- Students able to identify coefficients that are associated with variables
- Students able to repeat the formula naturally

§3.1.2 Duration

- 2 minutes - rapid fire of items that represent the a parabola
- 20 minutes - lecturing through the slides with practice problems
- 25 minutes - collaboration within small group

§3.1.3 Materials

Teachers will need:

- White board
- Dry erase marker
- TI-84 calculator
- Slideshow (and a way to display them)

Students will need:

- Math Notebook
- Pen/pencil
- TI-84 calculator

§3.2 Outline

- First draw out what a parabola (an upside down U) is on the white board and ask students what kind of graph it is and what it represents

- After hearing a couple responses from students then introduce the concept of what a parabola is and what it is useful for.
- Introduce the formula of $ax^2 + bx + c$, to later introduce the quadratic formula.
- After grasping the concept of the quadratic formula I will use the questions that I have from my instruction and we will do it step by step.
- They will then work with a partner with three problems that involves using the quadratic formula
- After these are work on then we will review and work on another 3
- Give them the worksheet for homework and in class work.

§3.3 Instructions

1. Before the class starts, open the slides (link) and start playing the playlist of the classroom and check students in for attendance. After the bell rings, begin class.
2. With continuing this lesson you will use the **direct-instruction approach** where you will take control of the classroom and lecture the students. The lesson will be structured as a 20 minute instruction, 5-7 minutes of the students trying a problem then transitioning back to a 20 minute instruction to finish it up with 10 minutes of **seatwork** problems which will also be homework.
3. Switch to the next slide and ask the students "What is this", and allow the students to raise their hands and answer on what a parabola is. If a student answers correctly and says a parabola, enforce **positive reinforcement** as it will encourage the student to participate again.
4. Then introduce how the standard quadratic equation explaining what each coefficient represents with each variable
5. Introducing a real life problem of how physical objects or sports are involved with the quadratic will follow **problem-based learning** can better understand the concept of what x represents.
6. Then Introduce the quadratic formula and compare the standard equation to what is the quadratic formula to show how each variable is represented
7. The students will partner up and try to do the first three problems where the solutions will be whole numbers or fractions and not be as challenging.
8. Then solve one of the problems that the students had a confusion on, if the students understand then move on to the next column
9. Allow the students to work on the problem for 7-10 minutes.
10. Then work out the problems with the student on white board and ask them questions of placement of each coefficient.
11. After working out the problems, provide the students with seatwork which will also be their homework for the night.

4 Polynomial Factoring I

By Nathan Diaz

§4.1 Overview

§4.1.1 Objectives

- Students will be able to learn how to factor polynomial equations with different value coefficients
- Students able to identify coefficients that are associated with variables
- Students able to factor polynomials naturally

§4.1.2 Duration

- 2 minutes - Ask students what is the best possible way to solve the x for a specific polynomial
- 20 minutes - lecturing through writing examples and working through a problem by doing it step by step
- 25 minutes - collaboration within small group

§4.1.3 Materials

Teachers will need:

- White board
- Dry erase marker

Students will need:

- Math Notebook
- Pen/pencil
- TI-84 calculator

§4.2 Outline

- First write a polynomial equation on the board and allow students to solve it without any guidance
- Then ask students to explain their way of solving

- Increase the challenge by changing the exponents and different coefficient and allow them to solve the problems and solve the step by doing it step by step.
- After multiple examples of this, allow them to work on the homework

§4.3 Instructions

- Before the class starts, write the equation beginning the playlist of the classroom and check students in for attendance. After the bell rings, begin class.
- When students are seated and have their notebooks, pencils and calculators, they will be given 3-5 minutes on how to solve a quadratic equation in order to be exposed to seeing exponents and being able to find the values. With this **mastery learning** we will be able to continue to a new topic.
- With this we transition to involving new equations such as $6x^2 + 2x + 8$ where we allow the student to try to solve the equation.
- After the students haves solve the problem, we will introduce them to special factorizations: $x^2 + 2xy + y^2 = (x + y)^2$ and $x^2 - y^2 = (x - y)(x + y)$.
- With this, we expect students to have some **prior knowledge** on basic polynomial arithmetic.
- I transition to **direct instruction** and teach the student on how to factor polynomials
  - Introducing the Greatest Common Factor (G.C.F)
  - Along with this we will review how to add and subtract polynomials
- We then transition to the students learning on a **learner centered approach**. This also be implemented with **cooperative learning** where students can work on problems together.
- The student will partner up to solve problems on two worksheets.
  - Link 1
  - Link 2

5 Synthetic Division

By Faith Conopeotis

§5.1 Overview

§5.1.1 Objectives

Students will be able to:

- Define synthetic division
- Recognize when to divide polynomials using synthetic division
- Learn how to divide polynomials using synthetic division

§5.1.2 Duration

This lesson will be split up into three parts (roughly 45 min):

- Lecture: 25 min
- Group work: 10 min
- Individual work: 10 min

§5.1.3 Materials

Teacher will need:

- Computer & Projector
- Whiteboard & Markers
- Worksheet (Link)
- Worksheet Answer Key (Link)
- Slideshow (Link)
- Video (Link)

Students will need:

- Pencil/Pen
- Computer
- Notebook/Paper
- Worksheet (pass out to them)
- Calculator (optional)

§5.2 Lesson Instructions

- Before class beings, have the slideshow presented on the screen and the worksheet printed out.
- As the students walk in, have the printed worksheet out for them to pick up before going to their seats
- Before starting the slideshow, have the students take out their notebooks to take notes in during the presentation. Make sure to go at a good pace so that they can write what they want down in enough time.
- Start the slideshow: We want to take on a **direct instruction approach** during this lesson. Therefore, we want to present the information to the class and guide the students through instructions. We want to be explicit and explain to the students the content we want to teach them.
  - Explain what today's lesson is and the learning objectives
  - Explain what Synthetic Division is and why you use it
  - Start going through the steps of synthetic division
    - Make sure to ask if anyone has any questions after every step so that everyone is on the same page
  - Explain the difference between long division and synthetic division
- Have them pull out their worksheets and start working with the people with them at their assigned seats for ten minutes.
  - Set group work is a way of enforcing **cooperating learning**. It's important at the start of learning for students to build off of each other and work together to work out problems. Whenever a new concept is introduced, cooperative learning is an effective way for students to understand faster with the help of other students.
  - Having the students work on this worksheet will allow **mastery learning**. The practice problems that have the same quick steps every time, will allow students to master this concept fast after enough problems have been solved.
- After ten minutes, have students direct their attention back to the screen and explain that they are going to watch a video on synthetic division on when the coefficient is greater than 1
  - Put the example of $3x + 1$ on the whiteboard to explain what the divisor looks like in this situation
- Explain that they are going to open their computers and watch the attached video in the assignment for the day for ten minutes (Mention taking notes while watching the video)
  - Students will use their **prior knowledge** from the slideshow to comprehend the different situation involving synthetic division

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- After ten minutes or when there are a couple minutes left of class, have the class direct their attention back to the screen one last time to explain the homework. Let them know that they need to finish any problems they didn't get to on the worksheet and finish taking notes on the video.
- At the end of class, collect the worksheets from the students that finished them before you dismiss them.

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6 Remainder and Factor Theorem

By Faith Conopeotis

§6.1 Overview

§6.1.1 Objectives

Students will be able to:

- Evaluate polynomials using the remainder theorem
- Utilize the factor theorem to solve a polynomial equation

§6.1.2 Duration

This lesson will take roughly 45 minutes long.

§6.1.3 Materials

Teacher will need:

- Computer & Projector
- Whiteboard & Markers
- Worksheet (Page 1) (Page 1 KEY) (Page 2 with KEY at the bottom)
- Slideshow (Link)
- Video (Link, also in slideshow)

Students will need:

- Pencil/Pen
- Notebook/Paper
- Worksheet (pass out to them)
- Calculator (optional)

§6.2 Lesson Instructions

- Before class begins, have the slideshow presented on the screen and print out the worksheets.
- As the students walk in, have the worksheets out for them to pick up before going to their seats

- Before starting the slideshow, have the students take out their notebooks to take notes in during the presentation. Make sure to go at a good pace so that they can write what they want down in enough time.
- Start presenting the slideshow
  - Start off with the warm up to review synthetic division
  - Explain the objectives of the day
    - The **performance criteria** should be mentioned to the students during the beginning of the class. It's helpful for the students to understand how it's expected to perform and learn the following concept in order to focus and stay on task
  - Explain to the class that they are going to watch this video to learn about these theorems. Then proceed to play the remainder and factor theorem youtube video for the whole class to see on the projector
  - After watching the video, review with the class what was talked about in the video and ask if anyone has any questions.
  - Then use the whiteboard to write out the problems on slides 7-8 so that you can go through step by step.
  - Go through the example using the remainder theorem
  - Go through the example using the factor theorem
- Move on to group work and have them pull out their worksheets.
  - Group work is a good way for students to utilize **peer tutoring**. It's always beneficial to see students help each other out before the teacher needs to intervene.
- Explain to them before starting that the rest of the worksheets will be for homework.
  - This **negative reinforcement** will hopefully motivate the students to finish their work in class so that they don't have to take away time in their night to finish it. Therefore, if the students are on task, they should get everything done.
- As the students are working in groups, walk around the class to see how your students are doing. It's important to use **scaffolding** when noticing a student or group of students are struggling. All students work at different paces and have concepts come easier to them. Therefore, it's necessary to figure out where the student/s are struggling and go from there to best help them understand.
- When there are a couple minutes of class left, direct their attention back to the screen to explain the homework. Let them know that they need to finish any problems they didn't get to on the worksheet. Also, mention that they should reach out if they need help or plan a meeting to help with any part of the homework/class work.
- At the end of class, collect the worksheets from the students that finished them before you dismiss them.

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7 Rational Root Theorem

by Caleb Chiang

§7.1 Overview

§7.1.1 Objectives

Students will be able to:

- define what a root of a polynomial is
- explain where the results of the Factor and Rational Root Theorems arise from
- classify which integers and rational numbers can possibly be roots of a polynomial

Students are NOT expected to have mastered finding roots of polynomials after this lesson. There should be more time given in future lessons and homework to practice and expand on this skill; this lesson is moreso an overview.

§7.1.2 Outline

Estimated Length: 50 minutes (1 typical class period)

1. Review Factor Theorem (10 minutes) - Do an exercise that involves the Factor Theorem to get students in the right head space and review the main statement and key application of the theorem.
2. Finding Roots In General (10 minutes) - A brief discussion about why we want to find the roots of polynomials, as well as the basic idea behind finding them.
3. Integer Roots (10 minutes) - Theory and practice finding integer roots of polynomials
4. Rational Roots (20 minutes) - Expanding what we learned with integers to rationals

§7.1.3 Materials

- Dry Erase Markers (& Eraser)
- Whiteboard
- Copies of the problem set at the end for the whole class

Students should also have some paper and writing utensils for notes and scratch work.

§7.2 Content & Instructions

Before class begins, write Exercise 2.1 (below) on the whiteboard for students to see when they walk in.

§7.2.1 Factor Theorem Review

We'll begin today with a review of the Factor Theorem from earlier[1].

Exercise 7.2.1. Determine whether $x - a$ is a factor of $f(x)$ for each of the polynomials $f(x)$ and constants a below.

(a) $f(x) = 3x^3 + 2x - 4$, $a = 3$

(b) $f(x) = x^4 - 2x^3 + 3x^2 - 10x + 8$, $a = 2$.

(c) $f(x) = 2x^3 - 2x^2 - 13x + 3$, $a = 3$.

After giving students 2 minutes to work on the exercises, have them compare their solutions with one other student next to them. Emphasize for them to not only compare the "Yes" or "No" answers, but also the thought processes they took to arrive at them. Allot 2 minutes to this discussion before bringing everyone back together.

Now go over the solution to this exercise part by part, first by asking if anyone would like to share the consensus solution between them and their partner. If they are correct, affirm it and continue on; otherwise correct the mistake. The solution is below.

Solution 2.1. By the *Factor Theorem*, a polynomial $f(x)$ is divisible by $x - a$ if and only if $f(a) = 0$. Therefore one way to check these is to plug in $f(a)$. We have:

(a) $f(3) = 3 \cdot 3^3 + 2 \cdot 3 - 4 = 83 \neq 0$

(b) $f(2) = 2^4 - 2 \cdot 2^3 + 3 \cdot 2^2 - 10 \cdot 2 + 8 = 0$

(c) $f(3) = 2 \cdot 3^2 - 2 \cdot 3^2 - 13 \cdot 3 + 3 = 0$

And so the answers are $\boxed{\textbf{No, Yes, Yes}}$.

Another way to do this to simply divide each polynomial by $x - a$. We covered synthetic division recently, so this is also a pretty likely method as it is relevant here. □

If only one of these methods is mentioned for the first two parts, encourage the next pair to present a different method to solve the problem. Then close off this section by reminding everyone of the aforementioned Factor Theorem:

Theorem 7.2.2 (Factor Theorem)

Let a be a constant and f be a polynomial. Then $x - a$ is a factor of $f(x)$ if and only if $f(a) = 0$.

[1] This section is an example of **orienting**, which involves setting up the current lesson by reviewing the previous day's lesson and providing a target that we will aim to hit today.

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§7.2.2 Finding Roots In General

We now turn our attention to finding the roots (also called zeros) of polynomials:

Definition 7.2.3. A **root** of a polynomial $f(x)$ is a number r such that $f(r) = 0$.

Here hint at why the Factor Theorem is relevant; if $f(a) = 0$, then we can factor out $x - a$ from $f(x)$ to get a smaller polynomial to work with! In fact, we can generalize this idea as shown below.

Proposition 7.2.4

If $f(x)$ and $g(x)$ are (nonzero) polynomials such that $g(x)$ is a factor of $f(x)$ and $g(r) = 0$, then $f(r) = 0$ as well.

Before presenting the proof of this, set the students loose to see if they can figure out why this proposition is true. Encourage them to work in small groups while doing so. If they have extra time, ask them to explore whether the converse is true: if every root r of g is also a root of f, then is g a factor of f?

After about 5 minutes, have students share what they discussed. If no one produces a correct proof or line of reasoning, it is provided below.

Proof. If g is a factor of f, then there is a polynomial q such that $f(x) = g(x)q(x)$. Because $g(r) = 0$, we have $f(r) = g(r) \cdot q(r) = 0 \cdot q(r) = 0$. □

Combining this with the Factor Theorem, we see that polynomials can be written as $f(x) = a(x - r_1)(x - r_2)\cdots(x - r_n)$, where $r_1, r_2, \ldots, r_n$ are the roots of f. We will see this in action shortly, but the rough idea why this works is that we can repeatedly divide out $x - r$ whenever we find a new root r without losing any of the remaining ones.

One more question for the group before diving into the methodology:

Question 7.2.5. Why do we care about finding the roots of polynomials, or any function?

There isn't exactly a correct answer to this, but this is a good opportunity to mention some real situations where this is relevant. Polynomials are especially useful for modeling, as many structures have curves similar to graphs of polynomials.

For one specific example, let's say a roller coaster follows a curve $h(t)$, where h is the height above the ground after t seconds. We may need to know, for example, how long into the ride the roller coaster reaches its highest point, or how much time it takes the roller coaster to complete its biggest drop. All of these involve solving an equation of the form $h(t) = k$ where k is a constant, which is the same as finding the roots of $h(t) - k$.[2]

[2] We do this discussion to target the **affective domain** of Bloom's Taxonomy; the aim is to give this lesson some sort of *value* to help students feel a little more emotionally invested.

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§7.2.3 Integer Roots

Now that we've set up the eventual task of finding the roots of a polynomial, let's actually do it. Propose the following problem to work on in small groups[3], and remind students that once they find a root r, they can use synthetic division to get an easier polynomial to work with by factoring out $x - r$.

> **Example 7.2.6**
> Find the roots of the polynomial $f(x) = x^3 + x^2 - 21x - 45$. Hint: all the zeros of f are integers.

Solution 2.6. The answer is -3 (double root) and 5.

As of now, all we can really do is brute force, so with enough trying, we'll find them. Eventually, you'll find $x = -3$ (for example), and divide to get $f(x) = (x+3)(x^2-2x-15)$. We know how to factor qudaratics already: $f(x) = (x+3)(x+3)(x-5)$. Thus by the zero property, the roots are -3 and 5. □

This is a good point to ask how they approached doing the trial-and-error[4]. There are a few observations to aim for here, and ask leading questions towards these if they are not.

1. $f(0)$, $f(1)$, and $f(-1)$ are quite easy to calculate.
2. If r is even, then $f(r)$ must be odd, so it can't be zero!
3. In fact, if r is not a factor of 45, $f(r)$ can't be zero either.

This last point requires a bit more explanation, which we walk through on the board:

Proof. Suppose that $f(r) = 0$, so $r^3 + r^2 - 21r + 45 = 0$, or $-45 = r^3 + r^2 - 21r$.

Obviously $r \neq 0$, as this would imply $45 = 0$, so we are allowed to divide by r. This means $\frac{-45}{r} = r^3 + r^2 - 21r$. Now r is an integer, so the righthand side is an integer. This means that $\frac{-45}{r}$ must be an integer, so r is a factor of 45.

On the contrapositive, if r is *not* a factor of 45, then $f(r) \neq 0$. □

This greatly narrows our search; we only need to check the integers which are factors of 45 (and zero). We will see this idea again shortly with the Rational Root Theorem.

One last point to make, and really it is just a definition:

[3] Most of these problems are presented to be worked on in small groups, utilizing **collaborative learning**, where students work together to, in this case, solve a problem to synthesize and build on one another's knowledge and understanding.

[4] Discussions such as these are implemented with the **constructivist approach** to mathematical teaching in mind. By asking students how they solved a problem and guiding them towards discovering more general facts rather than imposing one correct method, they have the opportunity to think creatively and develop their problem-solving skills.

Definition 7.2.7. The **multiplicity** of a root r is the number of times it is the root of a polynomial. Put another way, if a polynomial $f(x)$ can be written as $(x-r_1)(x-r_2)\cdots(x-r_n)$, the multiplicity of r is the number of r_k equal to r.

So in our previous example, -3 had multplicity 2 because $f(x) = (x+3)(x+3)(x-5)$, but 5 had multiplicity 1.

§7.2.4 The Rational Root Theorem

We now have integer roots in our toolbelt, so we expand to all rational roots. This is a good time to check in that we remember what rational numbers are:

Definition 7.2.8. A **rational number** is a number of the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$.

Now we do a sample problem for groupwork first; similar protocol to before.

Example 7.2.9

Find the roots of the polynomial $f(x) = 12x^3 + 8x^2 - 47x + 20$.

Solution 2.9. The answer is $\frac{1}{2}$, $\frac{4}{3}$, and $-\frac{5}{2}$. We will go over the solution more in depth in the discussion below. □

As before, ask students how they approached this example and be prepared to walk through it. These are the main points we want to hit eventually:

1. $f(0) = 12$ and $f(1) = -7$, so there is a root between 0 and 1 as f crosses zero.
2. None of the factors of 12 work! This implies that the roots are not integers.

These lines of reasoning lead us to try rational roots. So suppose that $r = \frac{p}{q}$ was a root of f such that p and q are relatively prime (that is, they share no common factors). Our goal now will be to deduce anything we can about p and q. Let's plug it in:

$$f\left(\frac{p}{q}\right) = 12\left(\frac{p}{q}\right)^3 + 8\left(\frac{p}{q}\right)^2 - 47\left(\frac{p}{q}\right) + 20 = 0$$

Now q can't be zero, so we can multiply through by q^3:

$$12p^3 + 8p^2q - 47pq^2 + 20q^3 = 0.$$

Now let's do what we did before and isolate one of the terms on the ends. We have:

$$\begin{aligned} 12p^3 &= -8p^2q - 47pq^2 + 20q^3 \\ &= q(-8p^2 - 47pq + 20q^2). \end{aligned}$$

This would imply that $\frac{12p^3}{q} = -8p^2 - 47pq + 20q^2$; in particular it is an integer. For the lefthand side to be an integer, q must be a factor of 12. Make this claim, and ask the class to try an justify this. (It's because q can't divide p^3 since they're relatively prime!)

Now have students work out the other side of this: isolate $20q^3$ this time and work out a similar restriction on p (they should find that p must be a factor of 20). When they finish, have them work out all the possible rational numbers $\frac{p}{q}$ that could possibly be roots of f.

By listing out factors of 12 and 20, we get (the notation here is just the list of possible numerators on top and denominators on the bottom):

$$\frac{\pm 1,\ \pm 2,\ \pm 3,\ \pm 4,\ \pm 6,\ \pm 12}{\pm 1,\ \pm 2,\ \pm 4,\ \pm 5,\ \pm 10,\ \pm 20}.$$

By roughly the same processes as above, one could prove the general Rational Root Theorem, which we can just state now as we have an idea as to why it works.

Theorem 7.2.10 (Rational Root Theorem)
Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial with integer coefficients such that both a_n and a_0 are not zero. If $\frac{p}{q}$ is a fraction in simplest terms and $f\left(\frac{p}{q}\right) = 0$, then p is a factor of a_0 and q is a factor of a_n.

This is a pretty good place to stop and look at the practice exercises now.

§7.3 Problems

These are some exercises and problems that students can think about further (as homework). A printable version of these exercises to pass out is available here.

Exercise 7.3.1. Find all the roots of the following polynomials:

(a) $f(x) = 2x^4 - 6x^3 - 12x^2 + 16x$

(b) $g(t) = t^5 + t^4 - 6t^3 - 14t^2 - 11t - 3$

(c) $h(y) = 30y^3 + 11y^2 - 4y - 1$

(d) $p(x) = 25x^4 + 55x^3 - 192x^2 - 44x + 16$

(You may want to use a calculator for this last one.)

Exercise 7.3.2. Without trying to find its roots, explain why the polynomial $f(x) = 3x^4 + 5x^3 + 7x^2 + 4$ has no positive roots.

Exercise 7.3.3. Consider the polynomial $f(x) = x^4 - 12x^3 + 54x^2 - 108x + 81$. Notice that $f(3) = 0$, but no other factor of 81 is a root of f. Would it be correct to assume that f has no other integer (or rational) roots? Why or why not?

Exercise 7.3.4. Let $f(x) = 6x^3 + 25x^2 + 2x - 8$. Find the quotient and remainder when dividing f by $x - 1$. Explain how the result you get shows that there are no roots of f greater than 1. Then, find all the roots of f.

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8 Fundamental Theorem of Algebra

by Caleb Chiang

§8.1 Overview

§8.1.1 Objectives

Students will be able to:

- Evaluate whether two polynomials are the same with limited information
- Collaborate to solve problems using information they recently learned
- Explain the Fundamental Theorem of Algebra and some of its uses

§8.1.2 Outline

1. Guess the Polynomial Activity (20 minutes)
2. The Theorem Itself (5 minutes)
3. Solving Guess the Polynomial (10 minutes)
4. Problem Solving Time (15 minutes)

§8.1.3 Background

This lesson relies heavily on the previous lessons, so we summarize the relevant content that they have worked on already prior to this one:

- Students have been working with polynomials in general for a while now: adding, subtracting, multiplying, dividing them, and finding their zeros.
- Specifically, students found that polynomials can be written in the form $P(x) = (x - r_1)(x - r_2)\cdots(x - r_n)$, where $r_1, r_2, \ldots, r_n$ are the roots of P.
- Students learned the term *degree* for a polynomial, and may have noticed already that a polynomial has as many roots as its degree.
- Students learned that if $Q(r) = 0$ and $Q(x) \mid P(x)$ for polynomials P and Q, then $P(r) = 0$ as well. In particular, if r is a root of P, then $P(x) = (x - r)T(x)$, where T is a polynomial and $\deg T = \deg P - 1$.
- Students have worked with complex numbers in the past with quadratics.

§8.1.4 Materials

- Whiteboard
- Dry Erase Markers
- Lots of scratch paper and writing utensils for said paper

§8.2 Content & Instructions

§8.2.1 Guess the Polynomial

We'll start class with an activity that should take about 20 minutes in total, which is "Guess the Polynomial":

1. Have students split into pairs (we can assign or just have them turn to the person next to them).
2. Within each pair, have one student make up a random polynomial. This student will be the "Creator". At first, don't make any restrictions onto what polynomials they can choose.
3. Have the Creators evaluate their polynomials at $x = 0, 1, 2,$ and 3 and share the results to their partners, the "Guessers".
4. The Guessers will now try to make a polynomial that satisfies the values the Creator provides. If they guess the right polynomial, congratulations! If not, try again.

Let this stage play out for about 5 minutes; it is meant to be very difficult, so encourage students if they feel that guessing is too hard. As students are doing this, encourage them to note down how they are finding polynomials.

After 5 minutes, we move on to a different stage of this game. Now instead of the Creators being able to pick any polynomial, they must pick a cubic polynomial of the form $ax^3 + bx^2 + cx + d$. Repeat everything else the same way, and ask the Guessers to see if they can figure out the polynomials. If they can, have the pairs swap roles and see if they can still figure it out with new numbers.

After another 10 minutes on this new variation, bring the group back together and ask what everyone thought about the game. Some possible questions to start off are: "Did anyone guess the right polynomial in the first version?" and "Why was the second version of the game easier than the first?" (or if someone disagrees, why did they think the first version was easier?)[1]

The idea is that with the degree restriction, we obtain a system of equations that is solvable by substituting in $x = 0, 1, 2, 3$. This gives four linear equations for four variables (one of them is just $d = d$), which should have a solution!

[1]These instructions asks students to apply **metacognition** to think about their own thinking processes. Doing so helps them reflect on their reasoning, understanding what they did at a deeper level and filling in gaps they missed.

§8.2.2 The Theorem

Let's see if we can solidify this idea[2]. Call back to the fact from last lesson that a polynomial $f(x)$ can be written as $a(x-r_1)(x-r_2)\cdots(x-r_n)$, where $r_1, r_2, \ldots, r_n$ are the roots of f. Ask what the degree of f would be then? (It's n).

Now make note that if f had $n+1$ roots, its degree would have to be $n+1$, as multiplying out the linear factors from earlier would result in a x^{n+1} term. Therefore no polynomials that have degree n can have more than n roots.

This leads us to the **Fundamental Theorem of Algebra**:

> **Theorem 8.2.1** (Fundamental Theorem of Algebra)
>
> If f is a one-variable polynomial and $\deg f = n$, then f has exactly n roots, counting multiple roots as multiple and not one.

Note that these roots need not be rational, or even real for that matter. We will abbreviate this moving forward as **FTA**.

We are now ready to tackle Guess the Polynomial, but in general.

§8.2.3 Solving Guess the Polynomial

Now there is one "exception" to the Fundamental Theorem ("exception" in quotes because it hardly counts), and that is the polynomial $f(x) = 0$. What FTA then tells us is that if $f(x) = 0$ for $n+1$ values of x while supposedly being a degree n polynomial, it must just be the zero polynomial (as otherwise it would violate FTA!).

We'll now work to solve a small case of Guess the Polynomial using FTA now. Break up students into small groups to work together on this one. Allow for at least 5 minutes for this; it can be a bit tricky. If they finish early, have them begin to work on the general case: if $P(x)$ and $Q(x)$ are polynomials of degree at most n, show that if $P(x) = Q(x)$ for at least $n+1$ values of x, then $P(x) = Q(x)$ for all x.

> **Example 8.2.2**
>
> Suppose that $f(x)$ is a cubic polynomial, and that $f(1) = 1$, $f(2) = 8$, $f(3) = 27$, and $f(4) = 64$. Explain why $f(x) = x^3$ using FTA.

Proof. The trick here is to think about $f(x) - x^3 = 0$, as that way we are examining roots. Let $g(x) = f(x) - x^3$. Then $\deg g \geq 3$, as it is a cubic minus another cubic.

[2]This section mostly uses **direct instruction**, in that the teacher directs exactly what is being done and discussed for this section. This is probably the most "**lecture**"-ish part of the lesson, during which the teacher talks directly to students about a topic and students are meant to learn by listening and watching. To keep attention, this section is relatively short, and the key theorem can be written using a big blue box to make it clear that it is important!

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By FTA, g must have at most 3 roots. However, 1, 2, 3, and 4 are all roots of g by the given information, so g must be the zero polynomial. Therefore $f(x) - x^3 = 0$ for all x, so $f(x) = x^3$. $\square$

When we bring everyone together, have some students share what they came up with. There are many ways to think about this explanation, so allow for students to voice their entire thought process. Usually, they'll say something similar to the one provided here, but they might use different words, or explicitly write out $g(x) = (x-1)(x-2)(x-3)(x-4)h(x)$, or so on. These are all totally fine and correct[3]!

The general version of this problem is very similar, so we end off this section with it without proof. We just did the specific case of $P = f(x)$ and $Q = x^3$.

Theorem 8.2.3 (Identity Theorem)

If $P(x)$ and $Q(x)$ are polynomials of degree at most n and $P(x) = Q(x)$ for at least $n + 1$ values of x, then $P(x) = Q(x)$ for all x.

§8.2.4 Problems

The rest of this lesson can be dedicated to problem solving[4]. Write the problems below on the board with space below them for students to write solutions on the board. Divide the class into groups to work on them with each group focusing on one problem first. Emphasize to students to focus on the reasoning they used to arrive at their answers, not just the answers themselves.

Problem 8.2.4. Suppose that f is a polynomial with degree n. Show that the graphs $y = f(x)$ and $y = c$ have at most n intersections, where c is a constant.

Problem 8.2.5. Suppose that f is a quartic (degree 4) polynomial for which $f(-1) = 0$, $f(1) = 0$, $f(2) = 15$, $f(3) = 80$, and $f(4) = 255$. What polynomial(s) could f be?

Problem 8.2.6. Solve the system of equations below:

$$\begin{aligned} a + b + c &= 1 \\ 4a + 2b + c &= 8 \\ 9a + 3b + c &= 27 \end{aligned}$$

Problem 8.2.7. Let f be a polynomial with degree n such that $f(0) = f(1) = \cdots = f(n-1) = 1$ and $f(n) = 0$. What is $f(n+1)$?

[3] This comment uses **pedagogical content knowledge** in particular, pointing out what students are likely to think or try in this situation and noting how we can respond to it (in this case, positively). This considers how what we teach can be perceived by students, which affects how we should teach it.

[4] These problems are chosen to be challenging and require application of the concepts we covered earlier in the lesson. This is an example of **elaboration**, adding on and using concepts to commit them more deeply to memory.