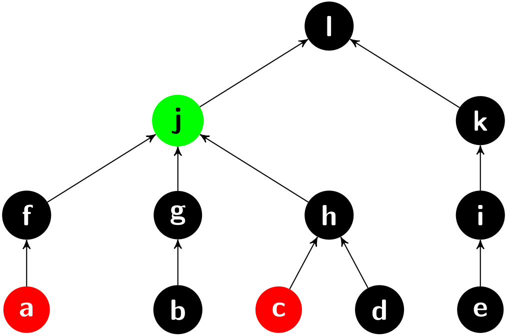
PAINFULLY SLOW: The Offline Least Common Ancestor Problem

Matthew Kilgore

May 5, 2016

Example



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  from x to y is such that x_i is x_{i+1}'s parent.
- The least common ancestor of x and y is the vertex v furthest from the root such that x and y are descendants of v.
- Alternatively, the least common ancestor of x and y is the vertex v on the unique path from x to y closest to the root.

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Least Common Ancestor Problem

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Offline Least Common Ancestor Problem

Given a rooted tree $T = (V, E)$ and m pairs of vertices $x \neq y \in V$, efficiently compute the least common ancestor for each of the m pairs x and y.

[PAUSE FRAME]

Who?

R. E. Tarjan

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R. E. Tarjan

H. N. Gabow

Initial Results

Theorem

We can solve the Offline Least Common Ancestor Problem in time $O((m+n)\alpha(m+n,n))$, where α is an inverse of the Ackermann function.

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... problem, but it is [redacted] use the set union [redacted]y. Let T be a tree with root r and let $pairs = \{\{v_i, w_i\} \mid 1 \le i \le m\}$ be a set of m vertex pairs. We wish to compute LCA(v_i, w_i) for each pair. The following algorithm carries out the computation.

```
procedure LCA;
    begin
        for each {v, w} ∈ pairs do unmark {v, w} od;
        for each v ∈ V do create a set {v} named v od;
        SEARCH(r)
    end LCA;
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                                   else lca(v, w) := FIND(w) fi od
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Umm...

What on earth are UNION and FIND?

[FORCED DISCUSSION FRAME]

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- $\textsc{Create}(x)$: adds $\{x\}$ to the data structure.
- $\textsc{Union}(x, y)$: merges the sets in the data structure containing x and y.
- $\textsc{Find}(x)$: returns a fixed representative element in the set in the data structure containing x.

Example

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$$\{\mathbf{x}\}$$

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[PAUSE FRAME]

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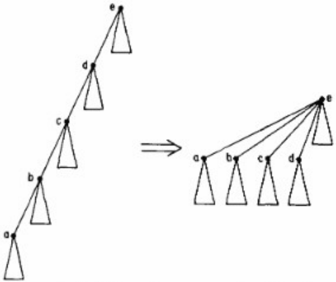

FIG 1. A *FIND* on element a, with collapsing. Triangles denote subtrees Collapsing converts tree T into tree T'

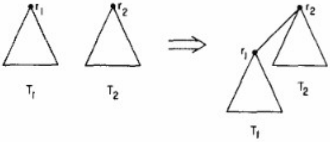

FIG 2 Union of two trees. Root r_1 of T_1 has a descendants; root r_2 of T_2 has b descendants. Root of new tree has $a + b$ descendants.

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if x $=$ y
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if x.rank $<$ y.rank
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Given a family of disjoint sets partitioning n elements, a sequence of $m \geq n$ $\textsc{Find}$s and $n - 1$ $\textsc{Union}$s take time $\Theta(m\alpha(m, n))$, where α is an inverse of the Ackermann function.

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Proof.

Eight dense pages of black magic. $\square$

[PAUSE FRAME]

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Let's parse this with an example.

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- b, d
- f, e

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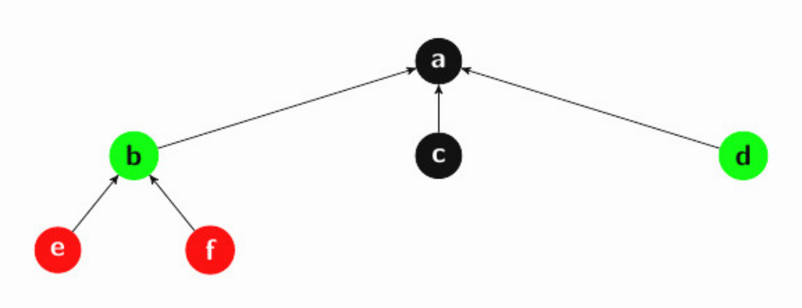

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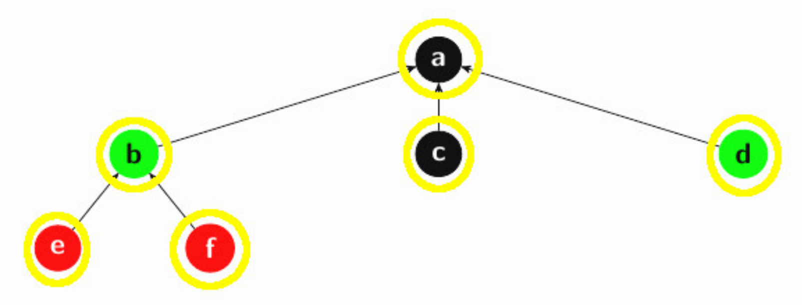

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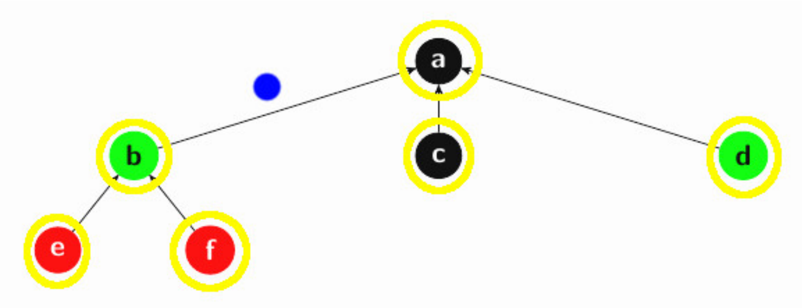

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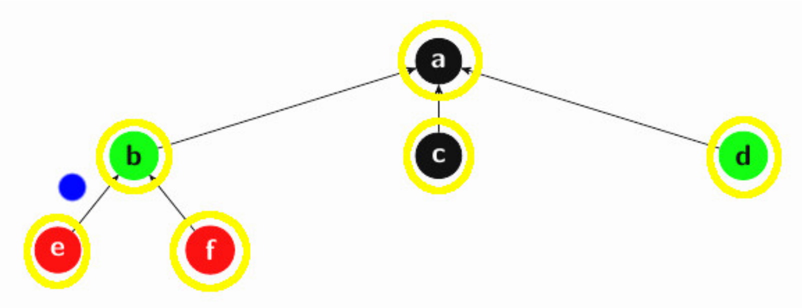

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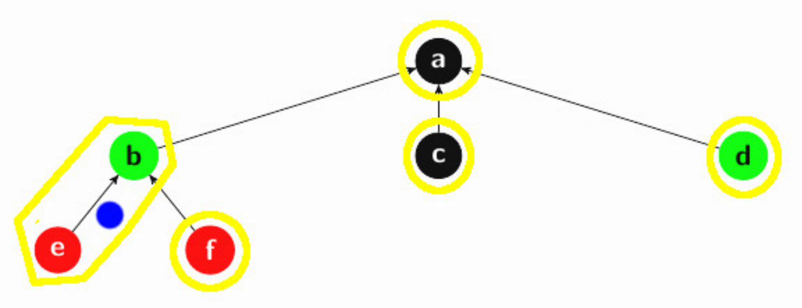

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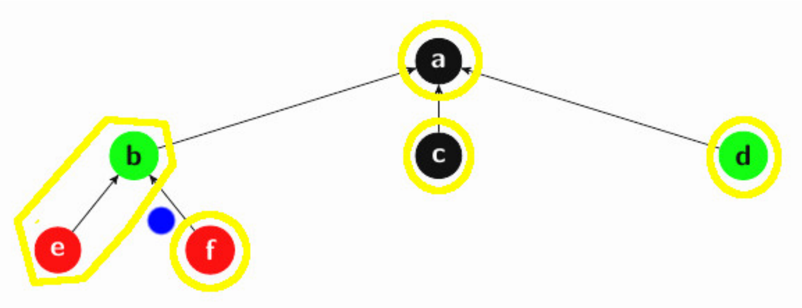

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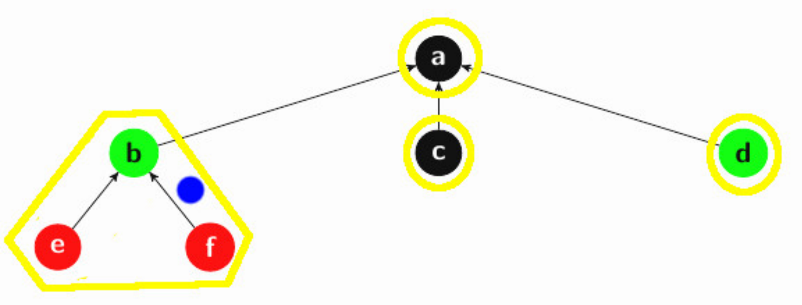

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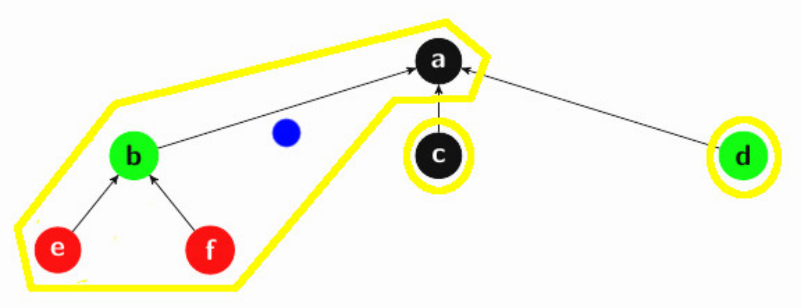

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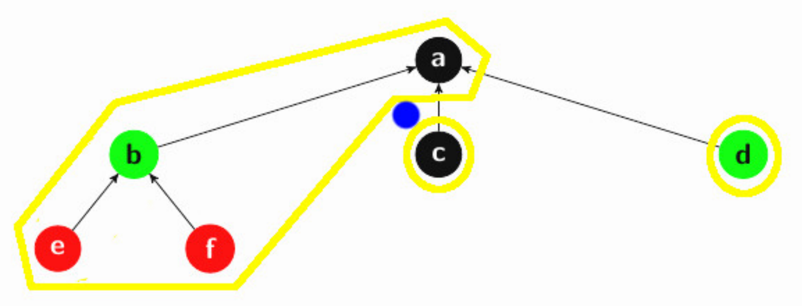

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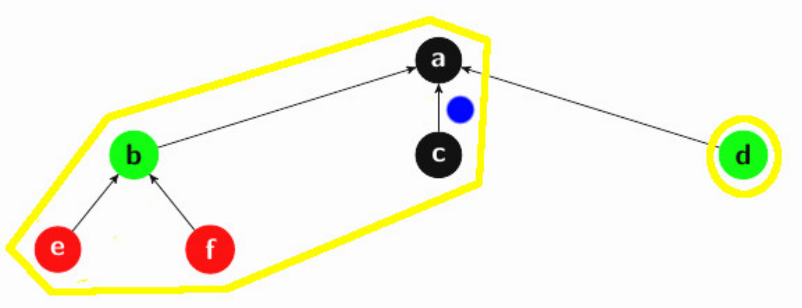

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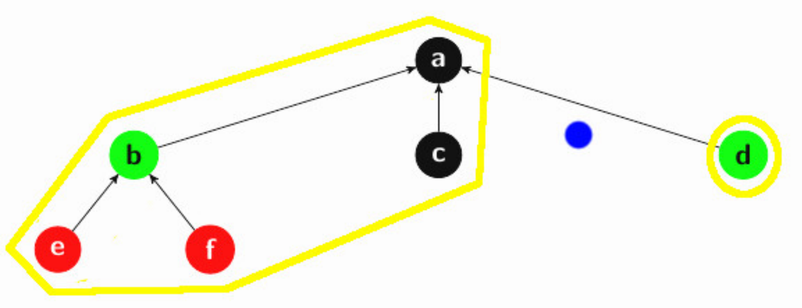

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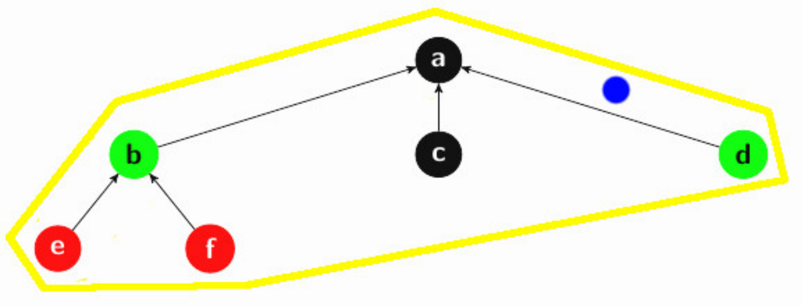

[PAUSE FRAME]

Modernising the Code

TarjanOLCA(u)

```
u.ancestor ← u
for v ∈ u.children do
    TarjanOLCA(v)
    Union(u,v)
    Find(u).ancestor ← u
u.colour ← black
for v such that {u,v} ∈ pairs do
    if v.colour = black
        Least Common Ancestor of u and v is Find(v).ancestor
```

[FORCED DISCUSSION FRAME]

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