Mandatory 1

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1 Mandatory 1

1.1 1.2.3 Exact solution

Given the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u \tag{1}$$

and

$$u(x, y, t) = e^{ik_x x + k_y y - \omega t} \tag{2}$$

Show that 2 satisfies 1

I begin by expanding 1 to 3

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u = c^2 \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \tag{3}$$

Derivation with respect to t of 2 twice gives us

$$u''(x, y, t) = \frac{\partial^2 u}{\partial t^2} = -\omega^2 e^{ik_x x + k_y y - \omega t} \tag{4}$$

Repeat with respect to x and y gives 5 and 6

$$\frac{\partial^2 u}{\partial x^2} = -k_x^2 e^{ik_x x + k_y y - \omega t} \tag{5}$$

$$\frac{\partial^2 u}{\partial y^2} = -k_y^2 e^{ik_x x + k_y y - \omega t} \tag{6}$$

These can all be placed in 1 with 4 placed the left hand side, and 5 and 6 on the right side resulting in

$$-\omega^2 e^{ik_x x + k_y y - \omega t} = c^2(-k_x^2 e^{ik_x x + k_y y - \omega t} + -k_y^2 e^{ik_x x + k_y y - \omega t}) \tag{7}$$

Multiplying with $-1/e^{ik_x x + k_y y - \omega t}$ yields

$$\omega^2 = c^2(k_x^2 + k_y^2) \tag{8}$$

Taking the root of both sides gives

$$\omega = c\sqrt{(k_x^2 + k_y^2)} \tag{9}$$

This is the equation for the dispersion relation. Given that ω, k_x, k_y holds, equation 2 satisfies the wave equation 1

1.2 1.2.4 Dispersion coefficient

Assuming $m_x = m_y$ and $C = 1/\sqrt(2)$ we get the discrete exact solution

$$u_{ij}^n = e^{\iota(kh(i+j) - \hat{\omega} n \Delta t)} \tag{10}$$

Where $\hat{\omega}$ is a numerical dispersion coefficient, i.e. the numerical representation of the exact ω.

We want to show that for CFL number $C = 1/\sqrt{2}$ we get $\hat{\omega} = \omega$ by inserting it into the discrete equation

$$\frac{u_{i,j}^{n+1} - 2u_{i,j}^n + u_{i,j}^{n-1}}{\Delta t^2} = c^2(\frac{u_{i+1,j}^n - 2u_{i,j}^n + 2u_{i-1,j}^n}{h^2} + \frac{u_{i,j+1}^n - 2u_{i,j}^n + 2u_{i,j-1}^n}{h^2})$$

(11)

Substituting in 10 gives us

$$LHS = \frac{e^{-\iota\hat{\omega}\Delta t} u_{i,j}^n - 2u_{i,j}^n + e^{\iota\hat{\omega}\Delta t} u_{i,j}^n}{\Delta t^2} \tag{12}$$

and

$$RHS = c^2(\frac{e^{\iota kh}u_{i,j}^n - 2u_{i,j}^n + e^{-\iota kh}u_{i,j}^n}{h^2} + \frac{e^{\iota kh}u_{i,j}^n - 2u_{i,j}^n + e^{-\iota kh}u_{i,j}^n}{h^2}) \tag{13}$$

simplifying we can say $kh = \theta_s$ and $\hat{\omega}\Delta t = \phi$

$$\frac{e^{\phi}u_{i,j}^n - 2u_{i,j}^n + e^{\phi}u_{i,j}^n}{\Delta t^2} = c^2(\frac{e^{\theta}u_{i,j}^n - 2u_{i,j}^n + e^{-\theta}u_{i,j}^n}{h^2} + \frac{e^{\theta}u_{i,j}^n - 2u_{i,j}^n + e^{-\theta}u_{i,j}^n}{h^2}) \tag{14}$$

Factoring out $u_{i,j}^n$

$$u_{i,j}^n\frac{e^{\phi} - 2 + e^{\phi}}{\Delta t^2} = c^2 u_{i,j}^n(\frac{e^{\theta} - 2 + e^{-\theta}}{h^2} + u_{i,j}^n\frac{e^{\theta} - 2 + e^{-\theta}}{h^2}) \tag{15}$$

We can use the relationship $e^{\iota\theta} + e^{-\iota\theta} = 2cos(\theta)$ and get clean up the expression

$$u_{i,j}^n\frac{2cos(\phi) - 2}{\Delta t^2} = c^2 u_{i,j}^n(\frac{2cos(\theta) - 2}{h^2} + u_{i,j}^n\frac{2cos(\theta) - 2}{h^2}) \tag{16}$$

$u_{i,j}^n$ should not be equal to 0 so we factor this out, as well as get the 2 out of there

$$\frac{cos(\phi) - 1}{\Delta t^2} = \frac{1}{2}c^2(\frac{cos(\theta) - 1}{h^2} + \frac{cos(\theta) - 1}{h^2}) \tag{17}$$

Multiplying by Δt^2 and sorting on RHS gives

$$cos(\phi) - 1 = (\frac{c\Delta t}{2h})^2(2cos(\theta) - 1) \tag{18}$$

We say $C = \frac{c\Delta t}{h}$, if we set $C = 1/\sqrt{2}$ the equation becomes much simpler

$$cos(\phi) - 1 = cos(\theta) - 1 \tag{19}$$

Now we add 1 to each side, and replace ϕ and θ by their original expressions

$$cos(\hat{\omega}\Delta t) = 2cos(kh) \tag{20}$$

cos^{-1} gives

$$\hat{\omega}\Delta t = kh \tag{21}$$

This can be written as

$$\hat{\omega} = \frac{kh}{\Delta t} \tag{22}$$

From the analytical case we have

$$\omega = c\sqrt{k_x^2 + k_y^2} = c\sqrt{2k^2} = c\sqrt{2}k \tag{23}$$

Because $C = \frac{c\Delta t}{h}$ and $C = 1/\sqrt{2}$ get $\frac{c\Delta t}{h} = 1/\sqrt{2}$.

This can be rearranged to be $\frac{h}{\Delta t} = \sqrt{2}c$, which can be placed into the former equation for $\hat{\omega}$ wich leaves

$$\hat{\omega} = \frac{kh}{\Delta t} = \sqrt{2}c = \omega \tag{24}$$

Showing that $\hat{\omega} = \omega$ when $C = 1/\sqrt{2}$