Problem set 6.

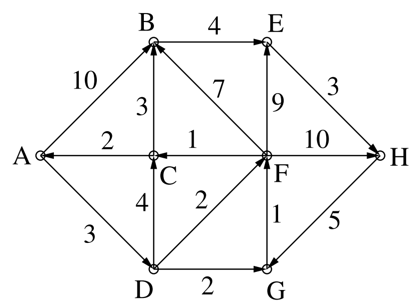
Dijkstra's algorithm.

1. We are given the following adjacency list of a directed graph: $a : b(5), e(6)$, $b : c(4), d(6)$, $c : a(3), d(1)$, $d : e(2)$, $e : c(2), f(1)$, $f : b(3), c(1), d(1)$.
   (a) Using Dijkstra's algorithm, determine the lengths of the shortest paths from a to all other vertices. (No need to justify in words, but one should be able to see the steps, how the set Q updates. Please also show the table for final distances and the parents.)
   (b) We reduce the weight of an edge by 1. For which edges does this not change the distances from a?

2. In a directed graph $G = (V, E)$, some of the vertices are marked as important. Let the set of these vertices be $\emptyset \neq F \subseteq V$. We further know that every edge of the graph has a positive edge weight. For given two vertices $u, v \in F$, we define the shortest distance from u to v as the length of the shortest path that contains no important vertex other than u and v. Assume that the graph is given with its adjacency matrix and importance is stored as a vertex attribute in an array. Give an algorithm with running time $O(|V|^2|F|)$ to determine the distance between all pairs of important vertices.

---

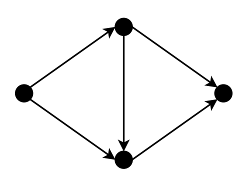
3. Use Dijkstra's algorithm to determine the lengths of the shortest paths from a to all other vertices of the graph. Also provide the shortest paths (as a sequence of vertices). (No need to justify in words, but one should be able to see the steps, how the set Q updates. Please also show the table for final distances and the parents.)



4. Determine the directed graph G with the smallest possible number of edges, so that the given table represents the status of the distance array when the Dijkstra's algorithm is run on G. Specify the contents of P and Q at each stage and the parent of each vertex.

v_1	v_2	v_3	v_4	v_5	v_6
*	2	6	∞	∞	7
*	*	5	9	∞	6
*	*	*	6	9	6
*	*	*	*	8	6
*	*	*	*	7	*

5. Assign edge weights to the edges of the following graph such that in the resulting graph the Dijkstra's algorithm incorrectly computes the lengths of the shortest paths.



6. We are given a city's road network as a weighted directed graph: the vertices are the intersections, the edges are the direct paths between the intersections, and the weights of the edges show the average time it takes to travel the road by car.

   Two intersections in the city, a and b, will be closed next week for road renovations (these cannot be traversed by car). Given two designated vertices in the graph, S and T, we want to decide whether and by how much the time to get from S to T will increase due to the closure of the nodes a and b. (Assume that the average times assigned to the direct roads between intersections do not change due to the closures.) Which algorithm can be applied if we want to solve this problem in $O(n^2)$ running time (where n denotes the number of intersections)?

---

7. We are given a city's road network as a weighted directed graph: the vertices are the intersections, the edges are the direct paths between the intersections, and the weights of the edges show the average time it takes a cyclist to cycle along the road.
   A bicycle courier at vertex f is given the task of delivering two parcels he has with him to the nodes b and c of the city as quickly as possible (in any order). Which algorithm can be applied to determine, with running time $O(n^2)$, the order in which the courier should deliver the parcels and the shortest time in which he can complete his task?

8. In a city, we want to take a truck from point A to point B. We know the road network: given any two intersections, we know if there is a direct road (one that does not pass through another intersection) between them and if so, we also know the maximum height of vehicles that can pass through it without getting stuck at a bridge or an overpass. The roads are two-way, the height does not depend on which direction you want to travel in. Denote the number of intersections by n. After loading the truck and measuring its height, determine in running time $O(n^2)$, whether we can get from a designated intersection A to B.

9. We are given a directed graph G with its adjacency matrix. We know that, with the exception of one negative weight edge, all the other edges have positive weight. There are no negatively weighted cycles in the graph. Give an algorithm with running time $O(n^2)$ to find the shortest paths from a vertex $s \in V(G)$ to all other vertices.

10. We are driving in the countryside, where petrol stations are only available in certain villages. We start from the petrol station in village A and want to get to village B (where there is also a petrol station). We denote the road network with an undirected graph, where the vertices are villages, while edges are the direct roads between neighboring villages. The weights of the edges represent the length of the roads. Assume that the Adjacency list of this graph is given. We are also given the as a vertex attribute, whether a village has a petrol station or not. Assume that there are exactly k villages with a petrol station. Give an algorithm with running time $O(km \log n)$ that determines the shortest path from A to B that never contains a stretch of more than 600 km between two gas stations.