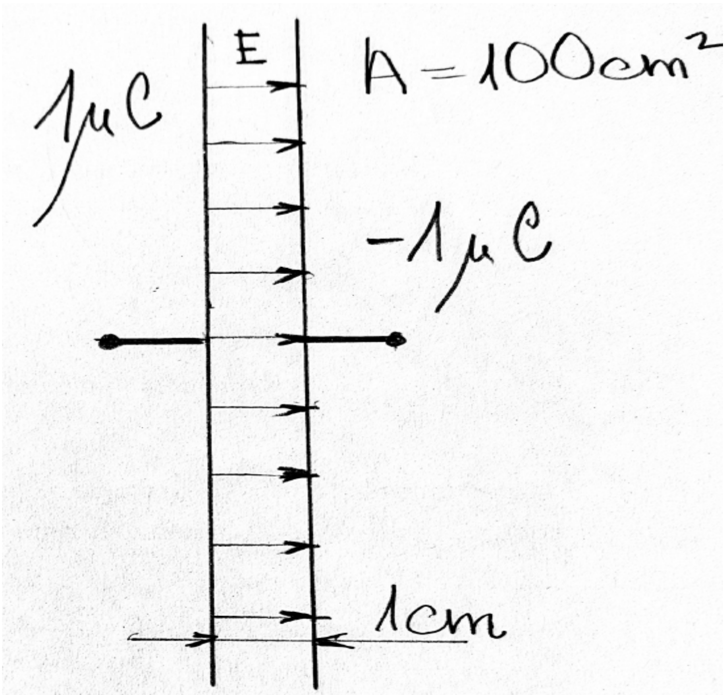
Problem E7

There is a regular plate capacitor with the geometry as follows: Area: $A := 100 cm^2$, separation $d := 1 cm$. The plates have got opposite charges of $Q := 1 \mu C$, and are fully disconnected.
a./ Find the attractive force between the plates. Use the principle of the virtual work.
b./ Find the pulling tension and the electrostatic energy density between the plates.

Solution:



a./
Principle of the virtual work: Magnitude of the force can be determined by differentiating the potential energy (E_{pot}) according to the (x) position.

$$F = \frac{dE_{pot}(x)}{dx}$$

The voltage of the capacitor:

$$U = \frac{Q}{C}$$

Energy of the capacitor is:

$$E_{pot} = \frac{1}{2} QU = \frac{1}{2} \frac{Q^2}{C}$$

Here we must use the above formula of the energy, since the only constant parameter is the charge, due to the disconnected state of the capacitor.

Capacitance of a regular plate capacitor: Here A is the surface, and x is the separation.

$$C = \varepsilon_0 \frac{A}{x}$$

Using this in the above formula:

$$E_{pot} = \frac{1}{2}\frac{Q^2}{C} = \frac{Q^2}{2}\frac{x}{\varepsilon_0 A}$$

Principle of virtual work:

$$F = \frac{dE_{pot}(x)}{dx} = \frac{d}{dx}\left(\frac{Q^2}{2A\varepsilon_0}x\right) = \frac{Q^2}{2A\varepsilon_0}$$

$$F = \frac{Q^2}{2A\varepsilon_0}$$

Let us check the dimensions.

$$N = ? = A^2 s^2 \frac{1}{m^2}\frac{Vm}{As} = \frac{VAs}{m} = \frac{J}{m} = N$$

Q.E.D.

Numerical:

$$F = \frac{Q^2}{2A\varepsilon_0} = \frac{10^{-12}}{2\cdot 10^{-2}\cdot 8{,}86\cdot 10^{-12}} = \frac{10^2}{2\cdot 8{,}86} = 5{,}64N$$

The F force is obviously a pulling force, since opposite charges are facing each other.

This approximation is valid until the separation is significantly smaller than the lateral dimensions of the plate. This is true presently, because the ratio is roughly ten.

b./

The pulling tension is a kind of "negative pressure" which does not repel but attracts the matter.

Pulling tension is $\sigma = \frac{F}{A} = \frac{Q^2}{2A\varepsilon_0}\cdot\frac{1}{A} = \left(\frac{Q}{A}\right)^2 \frac{1}{2\varepsilon_0} = \left(\frac{10^{-6}}{10^{-2}}\right)^2 \frac{10^{12}}{2\cdot 8{,}86} = \frac{10^4}{2\cdot 8{,}86} = 564Pa$

Checkpoint: $\sigma = \frac{F}{A} = \frac{5{,}64N}{10^{-2}m^2} = 564Pa$

Energy density is $\frac{E_{pot}}{A\cdot x} = \frac{Q^2}{2}\frac{x}{\varepsilon_0 A}\cdot\frac{1}{Ax} = \left(\frac{Q}{A}\right)^2 \frac{1}{2\varepsilon_0} = 564\frac{J}{m^3}$

Visibly the pulling tension and the energy density values match perfectly.