Problem set 3.
Dynamic Programming Practice Problems

1. Consider a $n \times n$ chessboard in which we go from the upper left square to the lower right square such that in one step we move to an adjacent square below or to the right. Each square in the chessboard contains an integer and the value of our path is the sum of the numbers in the path.
   (a) Give a procedure using dynamic programming that determines a path with the largest value.
   (b) How should the algorithm in (a) be modified to allow only paths where the numbers follow each other in increasing order?

2. Let $s_1 s_2 \dots s_n$ and $t_1 t_2 \dots t_m$ be sequences of characters of lengths n and m respectively. We want that in a matrix A of size $n \times m$, the entry $A[i, j]$ contains the largest number k for which $s_1 s_2 \dots s_i$ and $t_1 t_2 \dots t_j$ end in the same k characters. (If $s_i \neq t_j$ then this would be 0). Give a procedure that fills the array A in $O(nm)$ steps.

3. We are given two strings of length n and m respectively. We want to find the largest matching substring, i.e. if one text is $a_1 a_2 \cdots a_n$ and the other is $b_1 b_2 \cdots b_m$, then we want to find the maximum number t such that there are indices $1 \leq i \leq n$ and $1 \leq j \leq m$, such that, $a_i a_{i+1} a_{i+2} \dots a_{i+t-1} = b_j b_{j+1} b_{j+2} \dots b_{j+t-1}$ is satisfied. Give an algorithm for this problem using $O(nm)$ steps.

4. Give a dynamic programming algorithm with running time $O(n^2)$ that finds the longest increasing subsequence in a sequence of numbers $a_1, a_2, \dots, a_n$ of length n. For example, the longest increasing subsequence in the sequence $10, 3, 5, 2, 7, 1, 18, 4, 12, 17, 6$ is $3, 5, 7, 12, 17$.

5. We are given two $0-1$ sequences $a_1, a_2, \dots, a_n$, and $b_1, b_2, \dots, b_m$ of lengths n and m respectively. We will now fill a matrix T in the following manner:

   If $0 \leq i \leq n$, then $T[i, 0] = 0$. If $0 \leq j \leq m$, then $T[0, j] = 0$.

   If $1 \leq i \leq n$ and $1 \leq j \leq m$, then $T[i, j] = \begin{cases} T[i-1, j-1] + 1 & \text{if } a_i = b_j \\ \max\{T[i, j-1], T[i-1, j]\} & \text{if } a_i \neq b_j \end{cases}$

   What is the meaning of $T[i, j]$? What property of the two series is given by $T[n, m]$?

6. Each element of a table of size $n \times n$ is a positive integer. We want to go from the bottom left corner of the table to the top right corner by taking one step up or one step to the right in the table. We further want that the elements on our path should be in increasing order. Give an algorithm with running time $O(n^2)$ that determines,
   (a) how many paths satisfy the rules.
   (b) what is the largest value of a path satisfying the rules if the value of a path is the product of the numbers in it.