Exercise-set 7.
Solutions

1. When k committees have at least k members together, for $k = 1, 2, \ldots$ (Hall's condition).

2. a) Yes.
   b) No (H, J, L, M like only B, E, F).

3. a) Count the number of edges between A and B in two ways.
   b) Count the number of edges between X and $N(X)$ in two ways.
   c) Frobenius' theorem.

4. Use Hall's condition for (for the people-chocolates bipartite graph) for $|X| \leq n$ and $|X| \geq n+1$, resp.

5. Use Hall's condition for $|X| \leq \frac{n}{2}$ and $|X| \geq \frac{n}{2}$, resp.

6. There is a non-connected counterexample.

7. In the bipartite graph (days, programs; offers) use Hall's condition for days including consecutive ones or not.

8. Can select the edges greedily or use Hall's condition.

9. a) Use Frobenius' theorem.
   b) Use Hall's theorem or unite the vertices of degree 3 and use exercise 3.

10. No perfect matching: $N(\{a_1, a_2, a_4, a_6, a_8\}) = \{b_2, b_3, b_6, b_8\}$.

11. The (rows, colums; coins) bipartite graph is 4-regular.

12. Hall's condition holds for the (figures, sets; containment) bipartite 4-regular (multi)graph.

13. First: $|A| = |B| = 10$, then check Hall's condition.

14. a) $\nu(G) = \tau(G) = 8$, a minimum covering set is $\{a_2, a_3, a_6, a_8, b_1, b_4, b_7, b_9\}$.
    b) $\nu(G) = \tau(G) = 8$, a minimum covering set is $\{a_1, a_3, a_6, a_9, b_1, b_3, b_6, b_8\}$.

15. $\nu(G) = \tau(G) = 100$, $\rho(G) = 102$, a maximum matching e.g. is $\{\{a_i, b_{i+1}\},\ i = 1, 2, \ldots, 100\}$.

16. a) $\nu(G) = \tau(G) = 6$,
    b) $\nu(G) = \tau(G) = 9$.

17. a) $\nu(G) = \tau(G) = 4$.
    b) $\alpha(G) = 6$.

18. $\nu(G) = \tau(G) = 6,\ \rho(G) = 10$.

19. a) 3 vertices can cover at most 9 edges $\Longrightarrow\ \tau(G) \geq 4$.
    b) $E(G) \geq 9$, 2 vertices can cover at most 8 edges $\Longrightarrow\ \tau(G) \geq 3$.

20. * Construct a graph G: $V(G) =$ squares, and u and v are adjacent $\Longleftrightarrow$ the squares share a side. This graph is bipartite and connected, $\deg(v) \leq 4\ \forall v$. 7 vertices can cover at most 28 edges $\Longrightarrow\ \tau(G) \geq 8$.