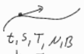
$f: \mathbb{R} \to \mathbb{R} \to$ real value functions

$f: \mathbb{R} \to \mathbb{R}^n \to$ vector functions

$(f_1, f_2, \dots, f_n)$ t, s, T

$r(t) = x(t)i + y(t)j + z(t)k$

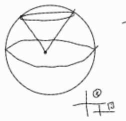


t, s, T, N, B
k, τ

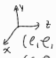
$f: \mathbb{R}^n \to \mathbb{R}$

$\frac{\partial}{\partial x}$; $\iint_R f\, dA$

Δf, $\nabla^2 f$, $\nabla \times (\nabla f)$



(x, y)
(r, θ)



$(e, e, 0)$
(r, θ, z)

$a_1, a_2, a_3, \dots$ $a_n = f(n)$

$\{a_n\}$

$\lim_{n\to\infty} a_n = a$

$\lim_{n\to\infty} \frac{2^n}{n!} = 0$

$\lim_{n\to\infty} \left(\frac{n+2}{n}\right)^{3n-4} = \left(1+\frac{2}{n}\right)^{3n-4} = \left(1+\frac{2}{n}\right)^{3n}\left(1+\frac{2}{n}\right)^{-4}$
$= \left(1+\frac{2}{n}\right)^{3 \cdot 2 \cdot \frac{n}{2}} = e^6$

$\sum_{1}^{\infty} a_n$ $\sum c_n (x-a)^n$ power series

Taylor series

Fourier series

ODE

$F(x, y, y') = 0$

$F(x, y, y', y'') = 0$

$F(x, y, \dots, y^{(n)}) = 0$

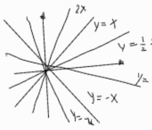
$y' = 3x + 1$

(IVP) $y = x^3 + x + c$

$xy' + y = x^2$

$y(0) = 1$

Ordinary differential equation



$2x$, $y = x$, $y = \frac{1}{2}x$, $y = -\frac{1}{2x}$, $y = -x$, $y = -x$

family of curves $y = Cx$

$y' = C$

$y' = \frac{y}{x}$

$y' = \frac{y}{x} \Rightarrow \frac{dy}{dx} = \frac{y}{x} \Rightarrow \frac{dy}{y} = \frac{dx}{x}$

$\Rightarrow \ln|y| = \ln|x| + \ln|C|$

$\ln|y| = \ln(Cx)$

$y = Cx$

separable O.E.

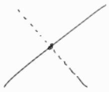
$P(x)\, dx = g(y)\, dy$

first order D.E.

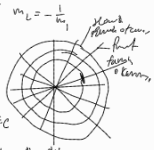
$F(x, y, y') = 0$

$M(x, y)\,dx + N(x, y)\,dy = 0$

$\to p(x)\,dx = g(y)\,dy$ separable D.E



$m_2 = -\frac{1}{m_1}$



second family of curves

first family of curves

F.F.C $\perp$ $\forall$ S.F.C

1. interchange $\frac{dy}{dx}$ with $-\frac{dx}{dy}$

2. eliminate parameter

$y = Cx$

$\Rightarrow y' = C \Rightarrow y' = \frac{y}{x} \Rightarrow \frac{dy}{dx} = \frac{y}{x}$ D.E of F.F.C

$\Rightarrow -\frac{dx}{dy} = \frac{y}{x}$

$\Rightarrow -x\,dx = y\,dy$

$-\frac{x^2}{2} = \frac{y^2}{2} + C$

$\frac{y^2}{2} + \frac{x^2}{2} = C$

$y^2 + x^2 = C$ $\to$ orthogonal trajectories

Ex. Find the orthogonal trajectories for $x^2 + y^2 = C^2$

$2x + 2y\frac{dy}{dx} = 0$

$x + y\frac{dy}{dx} = 0$

$x + y\left(-\frac{dx}{dy}\right) = 0$

$y\left(-\frac{dx}{dy}\right) = -x$

$y\frac{dx}{dy} = x$

$\frac{1}{y}dy = \frac{1}{x}dx$

$y = Cx$

$y = Cx^2 \Rightarrow \frac{dy}{dx} = 2\frac{y}{x^2}x = 2\frac{y}{x}$

$\frac{dy}{dx} = 2Cx$ $\frac{dy}{dx} = 2\frac{y}{x}$

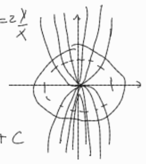
$\Rightarrow \left(-\frac{dx}{dy}\right) = 2\frac{y}{x}$

$-x\,dx = 2y\,dy$

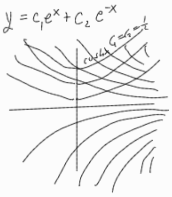
$\frac{2y^2}{2} = -\frac{x^2}{2} + C$

$2y^2 + x^2 = C$

$\frac{x^2}{2} + y^2 = C$



$y = c_1 e^x + c_2 e^{-x}$



$\cosh x$, $c_1 = c_2 = \frac{1}{2}$

$\frac{dy}{dx} = c_1 e^x - c_2 e^{-x}$

$\frac{d^2y}{dx^2} = c_1 e^x + c_2 e^{-x}$

$\frac{d^2y}{dx^2} = \frac{e^{-x}}{2}\left(\frac{dy}{dx} + y\right)e^x + \frac{e^x}{2}\left(y - \frac{dy}{dx}\right)e^{-x}$

$= \frac{(y' + y)}{2} + \frac{(y - y')}{2}$

$= y$ $\boxed{y'' = y}$

$\begin{cases} e^x c_1 + e^{-x} c_2 = y \\ e^x c_1 - e^{-x} c_2 = \frac{dy}{dx} \end{cases}$

$2e^x c_1 = \frac{dy}{dx} + y$

$\boxed{c_1 = \frac{\frac{dy}{dx} + y}{2e^x}}$

$\frac{\frac{dy}{dx} + y}{2e^x} + e^{-x}c_2 = y$

$\Rightarrow e^{-x}c_2 = y - \frac{\frac{dy}{dx} + y}{2e^x}$

$\Rightarrow c_2 = e^x y - \frac{e^{-x}}{2}\left(\frac{dy}{dx} + y\right)$

$\Rightarrow c_2 = \frac{e^x}{2}\left(2y - \frac{dy}{dx} - y\right)$

$\Rightarrow \boxed{c_2 = \frac{e^x}{2}\left(y - \frac{dy}{dx}\right)}$

$y' = f\left(\frac{y}{x}\right)$ $f(\lambda x, \lambda y) = \lambda^n f(x, y)$

$\Rightarrow$ H.D.E

$z = \frac{y}{x} \Rightarrow y = xz \Rightarrow y' = z + x \cdot z'$

$z + xz' = f(z)$

$xz' = f(z) - z$

$x\frac{dz}{dx} = f(z) - z$

$\frac{dx}{x} = \frac{dz}{f(z) - z}$

$xy' - y = x\tan\left(\frac{y}{x}\right)$ $\frac{y}{x} = z$

$y' = \frac{y}{x} + \tan\left(\frac{y}{x}\right)$ $\frac{dx}{x} = \frac{dz}{z + \tan z - z}$

$y' = f\left(\frac{y}{x}\right)$

$f(z) = z + \tan(z)$ $\frac{dx}{x} = \frac{\cos z}{\sin z}dz$

$\ln|x| = \ln|\sin z| + \ln|C|$

$x = C\sin z$

$\boxed{x = C\sin\frac{y}{x}}$

$y' = \frac{y}{x} + \tan\left(\frac{y}{x}\right)$

Seperable D.E

Homogenious D.E

Exact D.E.

$f(x, y) = C$

$df = 0$

$f_x\,dx + f_y\,dy = 0$

$M(x, y)\,dx + N(x, y)\,dy = 0$

$\begin{cases} M = f_x \\ N = f_y \end{cases} \Rightarrow \begin{cases} M_y = f_{xy} \\ N_x = f_{yx} \end{cases}$ $\boxed{M_y = N_x}$

$M\,dx + N\,dy = 0$

$M_y = N_x \Rightarrow$ Exact D.E.

$\begin{cases} f_x = M \\ f_y = N \end{cases} \overset{f(x)}{\Rightarrow} f = \int M(x, y)\,dx + g(y)$

$\to f_y = \frac{\partial}{\partial y}\left(\int M\,dx\right) + g'(y)$

$N = \frac{\partial}{\partial y}\int M\,dx + g'(y) \Rightarrow g(y) = \dots$

$\boxed{f(x, y) = C}$ general solution

$(\underbrace{2x + 2y + 3}_{M})\,dx + (\underbrace{3y^2 + 2x + 1}_{N})\,dy = 0$

$M_y = 2$ $N_x = 2$ find ψ

$M_y = N_x \Rightarrow$ Exact D.E

$\begin{cases} f_x = M \\ f_y = N \end{cases} \Rightarrow \begin{cases} f_x = 2x + 2y + 3 \\ f_y = 3y^2 + 2x + 1 \end{cases}$ $\to f = x^2 + 2yx + 3x + g(y)$

$f_y = 2x + g'(y)$ $\frac{d\psi}{dy}$

$3y^2 + \cancel{2x} + 1 = \cancel{2x} + g'(y)$

$3y^2 + 1 = g'(y)$

$g(y) = y^3 + y$

$f(x, y) = x^2 + 2yx + 3x + y^3 + y = C$

$(y\cos xy + 2)\,dx + (x\cos xy + 1)\,dy = 0$

$M_y = \cos xy - xy\sin(xy)$

$N_x = \cos xy - xy\sin(xy)$

$\begin{cases} f_x = M \\ f_y = N \end{cases} \Rightarrow \begin{cases} f_x = y\cos xy + 2 \\ f_y = x\cos xy + 1 \end{cases}$

$\to f: \int y\cos xy + 2\,dx = \sin xy + 2x + g(y)$

$f_y = x\cos xy + g'(y)$

$x\cancel{\cos xy} + 1 = \cancel{x\cos xy} + g'(y)$

$g'(y) = 1$ $f = c$

$g(y) = y$ $\sin xy + 2x + y = C$

$(\sin x)y' - \cos x \cdot y = \csc^2 x$

$y' - \frac{\cos x}{\sin x} \cdot y = \frac{1}{\sin^3 x}$

$\mu = e^{\int -\frac{\cos x}{\sin x}dx} = e^{-\ln|\sin x|} = \frac{1}{\sin x}$

$I_n = -\cot x\csc^{n-2}x - (n-2)I_n + (n-2)I_{(n-2)}$

$y = \sin x\left[\int \csc^4 x\,dx + c\right]$

$I_n = \int \csc^n x\,dx = -\cot x\csc^{n-2}x - (n-2)\int \csc^{n-2}x\cot^2 x\,dx$

$\begin{cases} \csc^{n-2}x = u \\ \csc^2 x\,dx = dv \end{cases} \Rightarrow \begin{cases} du = -(n-2)\csc^{n-2}x\cot x\,dx \\ v = -\cot x \end{cases}$

$\frac{d}{du}\csc x = -\csc x\cot x$

$\frac{d}{dx}\csc$

$I_n = \frac{-1}{n-1}\cot x\csc^{n-2}x + \frac{(n-2)}{n-1}I_{n-2}$

$y = \sin x\left[\frac{-1}{3}\cot x\csc^2 x + \frac{2}{3}(-\cot x) + c\right]$

Integrating factor

$y\,dx - x\,dy = 0$

$M_y = 1$ $N_x = -1$

$M_y \neq N_x$

not exact

$y\,dx - x\,dy = 0 \,\Big|\cdot\left(\frac{1}{x^2}\right)$

$\frac{y}{x^2}dx - \frac{1}{x}dy = 0$ $z = z(x, y)$

$M_y = \frac{1}{x^2} = N_x = \frac{1}{x^2}$ Exact $\mu = \mu(z)$

$M\,dx + N\,dy = 0$

$M_y \neq N_x$

$\mu M\,dx + \mu N\,dy = 0$

$(\mu M)_y = (\mu N)_x$

$\mu_y M + \mu M_y = \mu_x N + \mu N_x$

(PDE) Partial differential equations

$\mu_y = \mu_z \cdot z_y$

$\mu_x = \mu_z \cdot z_x$

$\mu_z \cdot z_y M + \mu M_y = \mu_z z_x N + \mu N_x = 0$

$\mu_z(z_y M - z_x N) = \mu(N_x - M_y)$

$\frac{d\mu}{\mu} = \left(\frac{N_x - M_y}{z_y M - z_x N}\right)dz$

$\frac{N_x - M_y}{z_y M - z_x N} = \varphi(z)$ $\ell(z) = \frac{N_x - M_y}{z_y M - z_x N}$

$\ln\mu = \int \ell(z)\,dz$

$\mu = e^{\int \ell(z)\,dz}$

$y\,dx - x\,dy = 0 \to N_x = -1,\ M_y = 1$

$z = xy \to z_x = y,\ z_y = x$

$\frac{M_y - N_x}{z_x N - z_y M} = \frac{1+1}{yx - y(-x)} = \frac{2}{2yx} = \frac{1}{yx} = \frac{1}{z} = \ell(z)$

$\mu = e^{\int -\frac{1}{z}dz} = e^{-\ln|z|} = z^{-1} = \frac{1}{z} = \frac{1}{xy}$

$(\underset{M}{y})\,dx\,(\underset{N}{-x})\,dy = 0 \to M_y = 1\ N_x = -1$

$z = x^2 + y^2 \Rightarrow z_x = 2x,\ z_y = 2y$

$\ell = \frac{M_y - N_x}{z_x N - z_y M} = \frac{2}{2x(-x) - 2y(y)}$

$= \frac{2}{2(-x^2 - y^2)} = \frac{-1}{x^2 + y^2} = -\frac{1}{z} = \ell(z)$

$\mu = e^{\int -\frac{1}{z}dz} = \frac{1}{z} = \frac{1}{x^2 + y^2}$

Linear differential equation

$y' + p(x)y = g(x)$

$\frac{dy}{dx} = g(x) - y\,p(x)$

$\underbrace{[g(x) - y\,p(x)]}_{M}dx\,\underbrace{[-1]}_{N}dy = 0$

$z = x \nearrow z_y = 0,\ z_x = 1$

$\frac{N_x - M_y}{z_y M - z_x N} = \frac{0 - (-p(x))}{0 - 1(-1)} = p(x)$

$\mu = e^{\int p(x)\,dx} \to$ integrating factor for linear D.E

$\mu y' + p(x)y\mu = g(x)\mu$

$\mu' = p(x)\,e^{\int p(x)\,dx}$

$e^{\int p(x)\,dx} \cdot y' + y\,p(x)\,e^{\int p(x)\,dx} = g \cdot \mu$

$(\mu y)' = g \cdot \mu$

$\mu y = \int g \cdot \mu\,dx + c$

$y = \frac{1}{\mu}\left[\int g\mu\,dx + c\right]$

$\mu = e^{\int p(x)\,dx}$

$y = \frac{1}{\mu}\left[\int g\mu\,dx + c\right]$

Second order D.E

$F(x, y, y', y'') = 0$

$f(x, y', y'') = 0$ y is missing

$y' = z \Rightarrow y'' = z'$

$f(x, z, z')$

$f(x, z, z') = 0$ (first order)

$G(y, y', y'') = 0$

$y' = z \Rightarrow y'' = z' = \frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} = z\frac{dz}{dy}$

$G(y, z, z\frac{dz}{dy}) = 0$ first order

$w(e^x, xe^x) = \begin{vmatrix} x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^{2x} + \cancel{xe^{2x}} - \cancel{xe^{2x}} = e^{2x} \neq 0$

$\begin{vmatrix} q(x) & xq(x) \\ q'(x) & xq'(x) - q(x) \end{vmatrix}$ $q(x) \neq 0$

$= xq(x)q'(x) + q^2(x) - xq(x)q'(x)$

$= q^2(x) \neq 0$

$y'' + p(x)y' + q(x)y = 0$

if y is a solutions $\Rightarrow$

$y_2 = v_2 y_1$

$y_2 = vy_1 \quad w(y_1, y_2) \neq 0$

$y_2' = v'y_1 + vy_1'$

$y_2'' = (v''y_1 + v_1'y_1') + (v'y_1' + vy_1'')$

$\Rightarrow v_1 y_1'' + 2v_1'y' + v''y_1 + p(x)vy_1' + p(x)v'y_1 + q(x)vy_1 = 0$

$v(y_1'' + p(x)y_1 + q(x)) + (v''y_1 + (2y_1' + p(x)y_1)v') = f(x)$

$y'' + 3y' - x = 0$

$y' = z \quad y'' = z'$

$z' + 3z - x = 0$

$z' + 3z - x =$

$z = \frac{1}{e^{3x}}\left[\int xe^{3x}\,dx + c\right]$

$\int xe^{3x}dx \quad u = x \quad du = dx$

$v = \frac{e^{3x}}{3} \quad dv = e^{3x}dx$

$= \frac{xe^{3x}}{3} - \frac{1}{3}\int e^{3x}dx$

$= \frac{xe^{3x}}{3} - \frac{e^{3x}}{9}dx +$

$z = \frac{1}{e^{3x}}\left[\frac{xe^{3x}}{3} - \frac{e^{3x}}{9} + c_1\right]$

$z = \frac{x}{3} - \frac{1}{9} + C_1 e^{-3x}$

$y = \frac{1}{6}x^2 - \frac{1}{9}x - \frac{1}{3}C_1 e^{-3x} + C_2$

v''

$(2v'y' + v''y_1) + p(x)v'y_1$

$v''y_1 + (2y_1' + (2y_1' + p(x)y_1))v'$

$v''y_1 = (2y_1' + p(x)y_1)y'$

$\frac{v''}{v'}y_1 = 2y_1' + p(x)y_1$

$\frac{v''}{v'} = \frac{2y_1'}{y_1} + p(x)$

$\frac{z'}{z} = -\frac{2y_1'}{y_1} - p(x)$

$\ln(z) = \ln y_1^{-2} - \int p(x)dx$

$z = e^{\ln y_1^{-2}} e^{-\int p(x)dx}$

$v' = \frac{1}{y_1^2} e^{\int -p(x)dx}$

$v = \int \frac{1}{y_1^2} e^{\int -p(x)dx} dx$

$y_h = c_1 y_1 + c_2 y_2$

$w = (y_1, y_2) \neq 0$

Ex

$y'' = 1 + y$

$y' = z \Rightarrow y'' = z\frac{dz}{dy}$

$\frac{z\,dz}{dy} = 1 + y$

$z\,dz = 1 + y\,dy$

$\frac{z^2}{2} = \frac{(1+y)^2}{2}$

$z = \sqrt{(1+y)^2 + C_1}$

$y(0) = -1$

$y'(0) = 0$

$C_1 = 0$

$z = \pm\sqrt{(1+y)^2}$

$z = \pm(1+y)$

$\frac{dy}{dx} = \pm(1+y)$

$\frac{dy}{1+y} = \pm dx$

$\ln|1+y| = \pm x + C_2$

$y'' + p(x)y' + q(x)y = f(x)$

$y'' + py' + qy = f(x)$

Second order differential equation with constant coefficient

$y'' + py' + qy = 0$ / homogenious

$y = e^{\lambda x} \quad y' = \lambda e^{\lambda x} \quad y'' = \lambda^2 e^{\lambda x}$

$\lambda^2 e^{\lambda x} + p\lambda e^{\lambda x} + q$

$e^{\lambda x}(\lambda^2 + p\lambda + q) = 0 \Rightarrow$ $e^{\lambda x} \neq 0$ $\lambda^2 + p\lambda + q = 0$

Characteristic equation of homogenious equation

$\lambda^2 + p\lambda + q = 0$

$\Delta = b^2 - 4ac = p^2 - 4q$

$\Delta > 0 \quad \lambda_1 \neq \lambda_2$

$w = \begin{vmatrix} e^{\lambda_1 x} & e^{\lambda_2 x} \\ \lambda_1 e^{\lambda_1 x} & \lambda_2 e^{\lambda_2 x} \end{vmatrix}$

$= \lambda_2 e^{(\lambda_1 + \lambda_2)x} - \lambda_1 e^{(\lambda_1 + \lambda_2)x}$

$= (\lambda_2 - \lambda_1)e^{(\lambda_1 + \lambda_2)x}$

$\Delta > 0 \Rightarrow y_h = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$

$\Delta = 0 \Rightarrow \lambda_1 = \lambda_2 = \frac{-p}{2} \Rightarrow y_1 = e^{-\frac{p}{2}x}$

$y_2 = v \cdot y_1$

$v = \int \frac{1}{y_1^2} e^{\int -p(x)dx} dx$

$v = \int \frac{1}{e^{-px}} e^{\int -p\,dx} dx$

$v = \int e^{px} e^{-px} dx$

$v = x$

$y_1 = e^{-\frac{p}{2}dx}$

$y_2 = xe^{-\frac{p}{2}x}$

$y_h = c_1 e^{-\frac{p}{2}x} + c_2 xe^{-\frac{p}{2}x}$

$y_1 = y_1(x), \quad y_2 = y_2(x)$

$c_1 y_1 + c_2 y_2 = 0 \quad c_1 = c_2$

$\Rightarrow y_1$ and y_2 are independent

$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = w(y_1, y_2)$

$w(y_1, y_2) \neq 0 \Rightarrow y_1\ y_2$ are independent

$y_1 = \sin x$

$y_2 = \cos x$

$w(y_1 y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \sin x & \cos x \\ \cos & -\sin x \end{vmatrix} = -1$

$\Delta < 0 \Rightarrow \lambda_{1,2} = \alpha \pm \beta i$

$y_1 = e^{(\alpha + \beta i)x} \quad y_2 = e^{(\alpha - \beta i)x}$

$y_1 = \hat{y}_1 + \hat{y}_2 = e^{\alpha x}(\cos\beta + i\sin\beta) + e^{\alpha x}(\cos\beta - i\sin\beta)$

$= e^{\alpha x}\cos\beta$

$y_2 = \frac{\hat{y}_1 - \hat{y}_2}{2} = e^{\alpha x}\sin\beta x$

$\Delta > 0 \Rightarrow y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \quad (\lambda_1 \neq \lambda_2)$

$\Delta = 0 \Rightarrow y_h = e^{-\frac{p}{2}x} + c_2 xe^{-\frac{p}{2}x} \quad y_1 = \lambda_2 = -\frac{p}{2}$

$\Delta < 0 \Rightarrow y_h = c_1 e^{\alpha x}\cos\beta x + c_2 e^{\alpha x}\sin\beta x$

$= \lambda_{1,2} = \alpha \pm \beta i$

$\{a_n\} \to$ sequence $\lim_{n\to\infty} a_n = l \Rightarrow a_n \to l$
$\{a_n\}$ convergent
$\sum a_n$
($\{a_n\}$ is convergent)

$\left(\lim_{n\to\infty} a_n \ \ldots E \text{ or } a_n \to \infty\right)$
$\Rightarrow \{a_n\}$ is divergent

$a_n = (-1)^n$ $\{a_n\}$ diverges to infinity

$a_n = \frac{n^2+2}{3+5n}$ $\lim_{n\to\infty} a_n$ D.N.E

a_n diverges to infinity

$\sqrt[n]{n} \to 1$

$\sqrt[n]{a} \to 1 \quad a>0$

$\left(1+\frac{a}{n}\right)^n \to e^a \quad \left(1+\frac{a}{n}\right)^{bn+c} \to e^{ab}$

$\frac{a^n}{a!} \to 0 \qquad a^n \to \quad |a|<1$

$\frac{\ln n}{n} \to 0$

$$c_1 = \frac{\begin{vmatrix} 1 & \frac{1-\sqrt5}{2} \\ 1 & \left(\frac{1-\sqrt5}{2}\right)^2 \end{vmatrix}}{\begin{vmatrix} \frac{1+\sqrt5}{2} & \frac{1-\sqrt5}{2} \\ \left(\frac{1+\sqrt5}{2}\right)^2 & \left(\frac{1-\sqrt5}{2}\right)^2 \end{vmatrix}}$$

$$= \frac{\left(\frac{1-\sqrt5}{2}\right)\left(\frac{1-\sqrt5}{2}-1\right)}{\left(\frac{1+\sqrt5}{2}\right)\left(\frac{1-\sqrt5}{2}\right)\begin{vmatrix} 1 & 1 \\ \frac{1+\sqrt5}{2} & \frac{1-\sqrt5}{2}\end{vmatrix}}$$

$$= \frac{\left(\frac{1-\sqrt5}{2}\right)\left(\frac{1-\sqrt5}{2}-1\right)}{(-1)\left(\frac{1-\sqrt5}{2}-\frac{1+\sqrt5}{2}\right)} = \frac{1}{(-1)(-\sqrt5)\sqrt5} = \frac{1}{\sqrt5}$$

$$c_2 = \frac{\begin{vmatrix} \frac{1+\sqrt5}{2} & 1 \\ \left(\frac{1+\sqrt5}{2}\right)^2 & 1 \end{vmatrix}}{\begin{vmatrix} \frac{1+\sqrt5}{2} & \frac{1-\sqrt5}{2} \\ \left(\frac{1+\sqrt5}{2}\right)^2 & \left(\frac{1-\sqrt5}{2}\right)^2 \end{vmatrix}} = \frac{\left(\frac{1+\sqrt5}{2}\right)\left(1-\frac{1+\sqrt5}{2}\right)}{(-1)(-\sqrt5)}$$

$c_1 = \frac{1}{\sqrt5} \quad c_2 = -\frac{1}{\sqrt5}$

$$F_1 = \frac{1}{\sqrt5}\left(\frac{1+\sqrt5}{2}\right)^n - \frac{1}{\sqrt5}\left(\frac{1-\sqrt5}{2}\right)^n$$
$$= \frac{1}{\sqrt5}\left(\frac{1+\sqrt5}{2} - \frac{1-\sqrt5}{2}\right) = 1$$

$$c_2 = \frac{1-\sqrt5}{5-\sqrt5} = \frac{\cancel{1-\sqrt5}}{\sqrt5(\cancel{\sqrt5-1})} = \frac{-1}{\sqrt5}$$

$a_n \le a_{n+1} \quad |a_n| < B$
$\Rightarrow a_n \to L$
L.u.b

$a_n \ge a_{n+1}$,
$B < a_n \Rightarrow a_n \to L$
g.l.b

$1, 1, 2, 3, 5, 8$

Fibnocci sequence

$\begin{cases} F_n = F_{n-1} + F_{n-2} & n \ge 3 \\ F_1 = F_2 = 1 \end{cases}$

$F_{n+2} = F_{n+1} + F_n \quad n \ge 1$
recurrence equation with constant coefficients

$F_n = \lambda^n$

$\lambda^{n+2} - \lambda^{n+1} - \lambda^n = 0$

$\lambda^n(\lambda^2 - \lambda - 1) = 0$

$\lambda^2 - \lambda - 1 = 0$

$\lambda = \frac12 \pm \sqrt{\frac54}$

$\lambda_{1/2} = \frac{1\pm\sqrt5}{2}$

$F_n = c_1\lambda_1^n + c_2\lambda_2^n$

$\begin{cases} F_1 = 1 \\ F_2 = 1 \end{cases} \Rightarrow \begin{cases} c_1\left(\frac{1+\sqrt5}{2}\right)^1 + c_2\left(\frac{1-\sqrt5}{2}\right)^1 = 1 \\ c_1\left(\frac{1+\sqrt5}{2}\right)^2 + c_2\left(\frac{1-\sqrt5}{2}\right)^2 = 1 \end{cases}$

$= \begin{cases} -c_1\left(\frac{1+\sqrt5}{2}\right)^2 + c_2 = -\frac{1+\sqrt5}{2} \\ c_1\left(\frac{1+\sqrt5}{2}\right)^2 + c_1\left(\frac{6-2\sqrt5}{4}\right) = 1 \end{cases}$

$c_2\left(\frac{10-2\sqrt5}{4}\right) = \frac{1-\sqrt5}{2}$

$c_2 = \frac{1-\sqrt5}{5-\sqrt5}$

$-c_1\left(\frac{6+2\sqrt5}{4}\right) = -\frac{1+\sqrt5}{2} - \frac{1-\sqrt5}{5-\sqrt5}$

$c_1\left(\frac{6+2\sqrt5}{4}\right) = \frac{1+\sqrt5}{2} + \frac{1-\sqrt5}{5-\sqrt5}$

$c_1\left(\frac{6+2\sqrt5}{4}\right) = \frac{(5-\sqrt5)(1+\sqrt5)+1-\sqrt5}{2(5-\sqrt5)}$

$c_1\left(\frac{6+2\sqrt5}{2}\right) = \frac{\cancel{5}+5\sqrt5-\sqrt5-\cancel{5}+1-\sqrt5}{5-\sqrt5}$

$c_1\left(\frac{6+2\sqrt5}{2}\right) = \frac{1+3\sqrt5}{5-\sqrt5}$

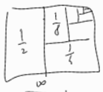
$c_1 = \frac{2(1+3\sqrt5)}{(5-\sqrt5)(6+2\sqrt5)}$

$c_1 = \frac{2+6\sqrt5}{25+5\sqrt5}$

$F_{n+2} + AF_{n+1} + BF_n = 0$

$\begin{cases} \lambda^2 + A\lambda + B\lambda = 0 \Rightarrow \\ F_n = c_1\lambda_1^n + c_2\lambda_2^n \end{cases}$

$\begin{cases} \lambda_1 = \lambda_2 \\ F_n = c_1\lambda_1^n + c_2 n\lambda_2^n \end{cases}$



$\frac12 + \frac1{2^2} + \frac1{2^3} + \ldots \frac1{2^n} - 1$

$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$

$\sum_{n=1}^{10} a_n = a_1 + a_2 \cdots a_{10}$
$= \sum_{n=0}^{4} a_{2n+1} + \sum_{n=1}^{5} a_{2n}$

$\sum_{n=2}^{8} a_n = \sum_{n}^{6} a_{n+1} = \sum_{n=3}^{9} a_{n-1}$

$\sum a_n \quad \sum_{50}^{\infty} a_n$

$S_n = a_1 + a_2 + \ldots + a_n$
$\{S_n\}$
$S_n \to S \Leftrightarrow \sum_{n=1}^{\infty} a_n = S$

if the sequence of partial sums converges then the series converges to the same limit. Otherwise the series is divergent

$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \ldots +$

$S_n = a + ar + ar^2 + \ldots + ar^{n-1}$
$rS_n = ar + ar^2 + \ldots \quad ar^n$

$S_n - rS_n = a - ar^n$
$S_n(1-r) = a\,(1-r^n)$
$S_n = a\frac{1-r^n}{1-r}$

$S_n = \frac{a}{1-r} \quad |r|<1$
$r=1 \Rightarrow S_n = na$
$r=-1 \Rightarrow \lim_{n\to\infty} S_n = \lim \frac{1-(-1)^n}{2}$
D.N.E

Ex.
$\sum_{2}^{\infty} \frac{2^n}{5^n} = \sum_{0}^{\infty}\left(\frac25\right)^{n+2}$

$= \frac{4}{25}\sum_{0}^{\infty}\left(\frac25\right)^n = \frac{4}{25}\frac{1}{1-\frac25} = \frac45\frac13$
$|r| = \left|\frac25\right| < 1 \qquad = \frac{4}{15}$

$\sum_{n=0}^{\infty}\left(\frac{e^2}{10}\right)^n = \frac{1}{1-\frac{e^2}{10}} = \frac{10}{10-e^2}$

$S_n \to S$
$S_{n-1} \to S$
$\boxed{a_n \to 0}$

$\sum^{\infty}(-1)^n \Rightarrow$ divergent
(N^{th} term Test)
$a_n \not\to 0$
$\Rightarrow \sum a_n$ is divergent

n^{th} term Test

$\boxed{\text{if } a_n \not\to 0 \Rightarrow \sum a_n \text{ is divergent}}$

Comparison Test

$\sum a_n$, $a_n \ge 0$
$a_n \le b_n$ $n \ge N$ for some N
$\sum b_n$ is convergent
$\Bigg\} \Rightarrow \sum a_n$ is convergent

$\sum b_n$ is divergent; $(b_n \ge 0)$
$a_n \ge b_n \quad \forall n \ge N$ for some N
$\sum a_n$ is divergent

$\sum_{1}^{\infty} \frac{1}{2^n} \quad f(x) = \frac{1}{x^2}$ 1) continuous 2) positive 3) decreasing

Integral Test

$\int_1^{\infty} \frac{1}{x^2} = \lim_{t\to\infty}\int_1^t \frac{1}{x^2}dx = \lim_{t\to\infty}\frac{-1}{t} - \left(\frac{-1}{1}\right) = 1$

$a_n = f(n)$

$f(x)$ is:
1.) continuous
2.) decreasing
3.) Nonnegative

$\Downarrow$

$\boxed{\int_N^{\infty} f(x)\,dx \text{ and } \sum_N^{\infty} a_n}$
converge / diverge together

p-series

$\sum_1^{\infty} \frac{1}{n^p}$

$p=1 \Rightarrow \sum_1^{\infty} \frac1n$ Harmonic series

$\int_1^{\infty}\frac1x dx = \lim_{t\to\infty} \ln t - \ln 1 = \infty$
divergent $\Rightarrow$ H.S diverges

$\int_1^{\infty} \frac{1}{x^p}dx = \lim_{t\to\infty} \frac{1}{(1-p)t^{p-1}} - \frac{1}{1-p} \begin{cases} \infty & p<1 \\ \frac{1}{p-1} & p>1 \end{cases}$

$1 + \underbrace{\frac12}_{\ge\frac12} + \underbrace{\frac13 + \frac14}_{\ge\frac14+\frac14=\frac12} + \underbrace{\frac15 + \ldots + \frac18}_{\ge\frac18\cdots\frac18=\frac12} + \underbrace{\frac19 + \cdots + \frac1{16}}_{\ge\frac12} + \ldots$

$S_n \ge \underbrace{\frac12 + \frac12 + \cdots + \frac12}_{(k+1)} = (k+1)\frac12$
$n = 2^k$
$n\to\infty \Rightarrow k\to\infty$
$S_n \to \infty$

limit comparison Test

$\sum a_n \quad \sum b_n \quad a_n \ge 0 \quad b_n \ge 0$

$\lim_{n\to\infty}\frac{a_n}{b_n} = \rho \qquad 0<\rho<\infty$
$\Rightarrow \sum a_n \quad \sum b_n$
converge or diverge

$l = 0 \to \sum b_n$ convergent
$\Rightarrow \sum a_n$ is convergent

$l = \infty \to \sum b_n$ divergent
$\Rightarrow \sum a_n$ is divergent

$\sum \frac{1}{n\ln n} \qquad b_n = \frac1n$ (divergent)

$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\frac{1}{n\ln n}}{\frac1n} = \lim_{n\to\infty}\frac{1}{\ln n} \Rightarrow$
Test inconclusive

$b_n = \frac{1}{n^2}$ convergent

$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim\frac{\frac{1}{n\ln n}}{\frac{1}{n^2}} = \lim_{n\to\infty}\frac{n}{\ln n} = \infty$

$\int_2^{\infty}\frac{1}{x\ln x}dx \Rightarrow \int_2^{\infty}\frac1v dv = \ln|\ln x|\Big|_2^{\infty} = \infty$ divergent
$v = \ln x \to x\,dv = dx$

by integral test $\sum \frac{1}{n\ln n}$ diverges

Ratio Test

$\sum a_n$ is a series with positive terms
$b_n = \frac1n$ (divergent)

$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \rho \quad b_n = \frac1n$ (divergent)

$\rho<1 \to$ series converges
$\rho>1 \to$ series diverges
$\rho=1 \to$ Test is inconclusive

Root Test

$\lim_{n\to\infty}\sqrt[n]{a_n} = \ell$

$\rho<1 \to$ convergent
$\rho>1 \to$ divergent
$\rho=1 \to$ inconclusive

$\sum \frac1n$

$\frac{\frac{1}{n+1}}{\frac1n} \to 1 \Rightarrow$ inconclusive

$\sqrt[n]{\frac1n} = \frac{1}{\sqrt[n]{n}} \to 1$ inconclusive

$\sum \frac{1}{3^n}$

$\sqrt[n]{\frac{1}{3^n}} \to \frac13 < 1 \to$ convergent

$\frac{\frac{1}{3^{n+1}}}{\frac{1}{3^n}} \to \frac13 < 1 \to$ convergent

Alternating Series

$\sum_{1}^{\infty} (-1)^{n+1} u_n = u_1 - u_2 + u_3 - + \dots$

if $\begin{cases} u_n \geq 0 \\ u_n \geq u_{n+1} \\ u_n \to 0 \end{cases}$ $\Rightarrow$ the Alternating series convergence (Leibniz's Test)

Leibniz's Test for alternating series

if $u_n \geq u_{n+1}$ $\forall n \geq N$, for some $N \in \mathbb{Z}$

- u_n are all positive $\sum$
- $u_n \to 0$

$\sum_{1}^{\infty} (-1)^{n+1} u_n$ converges

$\sum_{1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + - \dots$

Alternating harmonic series

$u_n > 0$; $u_n \geq u_{n+1}$ $u_n \to 0$

$\sum a_n$, $\sum |a_n|$ ← absolute value series

Def'n.

if the absolute value series converges $\Rightarrow \sum a_n$ is called absolutely convergent.

- if $\sum |a_n|$ diverges but $\sum a_n$ converges then $\sum a_n$ is called a conditionally convergent.

$\sum |a_n| < \infty \Rightarrow \sum a_n < \infty$

Ex.

$\sum_{1}^{\infty} \frac{(-1)^{n+1}}{n}$ is conditionally convergent.

$\sum_{1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ $\sum |a_n|$ is p-series with $p > 1$ → absolutely convergent

$\sum_{1}^{\infty} \frac{\sin(n)}{n\sqrt{n}}$

$\left| \frac{\sin(n)}{n\sqrt{n}} \right| \leq \frac{1}{n^{3/2}}$ ← p series $p > 1$ → by C.T. the abs. value series converges.

→ by A.V.S.T $\sum_{1}^{\infty} \frac{\sin n}{n\sqrt{n}}$ converges.

Estimate the sum

Estimation theorem for Alternating Series.

if $\sum_{1}^{\infty} (-1)^{n+1} u_n$ converges to L.

$E_s = \sum_{1}^{n} (-1)^{k+1} u_k \Rightarrow |L - E_s| < u_{n+1}$

the sign of $L - E_s$ is the same as the sign of $(-1)^{n+1}$

$\sum_{1}^{\infty} \frac{(-1)^{n+1}}{n} \simeq 1 - \frac{1}{2} + \frac{1}{3}$ 3

$\sum_{1}^{\infty} \frac{(-1)^{n+1}}{n} = L$ $|E| = |L - E_s| < \frac{1}{4}$

$L - (1 - \frac{1}{2} + \frac{1}{3}) < 0$

$|E| < 10^{-2}$, $\sum_{1}^{n} (-1)^{k+1} u_k$ $\Rightarrow \frac{1}{n+1} < 10^{-2}$

$\Rightarrow n > 99 \Rightarrow n = 100$

$\sum_{0}^{\infty} \frac{(-1)^n}{n!} = \frac{1}{e}$ $\frac{1}{(n+1)!} < 10^{-2}$

$\Rightarrow n = 4$

$E.S = \sum_{1}^{n} (-1)^{k+1} u_k$

$\sum_5 = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}$

$= \frac{1}{2} - \frac{1}{6} + \frac{1}{24} = \frac{12 - 4 + 1}{24} = \frac{9}{24}$

$L - E_s = -0.007$ $(-1, \frac{1}{5!}$

Power series

$\sum_{0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$

Power series about $x=a$.

$\sum_{0}^{\infty} c_n x^n$ about $x=0$

Ratio test

$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$

$\Rightarrow \lim_{n\to\infty} \left| \frac{c_{n+1}(x-a)^{n+1}}{c_n (x-a)^n} \right| < 1$

$\Rightarrow |x-a| \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| < 1$

$|x-a| < \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right|$

$R = \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right|$

radius of convergence.

$R = \infty \Rightarrow \sum c_n (x-a)^n$ converges everywhere

$\forall x$ $|x-a| < \infty$

$R = 0 \Rightarrow \sum_{0}^{n} c_n (x-a)^n$ converges only at $(x=a)$

$\sum_{0}^{\infty} c_n (x-a)^n$ $a - R < x < a + R$

The endpoints must be tested seperately

$\sum_{0}^{\infty} c_n R^n$ $\sum_{0}^{\infty} c_n (-R)^n$

$R = \lim_{n\to\infty} \frac{1}{\sqrt[n]{c_n}}$

Ex $\sum_{0}^{\infty} x^n$

$R = \lim_{n\to\infty} \frac{c_n}{c_{n+1}} = \lim_{n\to\infty} \frac{1}{1} = 1$

$|x| < 1 \Rightarrow$ convergent

$x = 1 \Rightarrow \sum 1^n$, $x = -1 \Rightarrow \sum (-1)^n$ } both divergent

$\sum x^n$ converges in $(-1, 1)$

geometric series

$\sum_{0}^{\infty} x^n = \frac{1}{1-x}$; $|x| < 1$

Ex $\sum_{1}^{\infty} \frac{x^n}{n}$

$R = \lim_{n\to\infty} \frac{c_n}{c_{n+1}} = \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n+1}} = 1$

$x = 1$

$\sum_{1}^{\infty} \frac{1}{n}$ diverges (H.S)

$x = -1$

$\sum \frac{(-1)^n}{n}$ converges (A.H.S)

$\sum_{1}^{\infty} \frac{x^n}{n}$ $[-1, 1)$

Ex

$\sum_{1}^{\infty} \frac{x^n}{n^2}$ $R = 1$

$x = 1$, $x = -1$

$\sum \frac{1}{n^2}$, $\sum \frac{(-1)^n}{n^2}$ convergent

$\sum_{1} \frac{x^n}{n^2}$ converges in $[-1, 1]$

Ex $\sum_{1}^{\infty} \frac{x^n}{n!}$

$R = \lim_{n\to\infty} \frac{c_n}{c_{n+1}} = \lim_{n\to\infty} \frac{\frac{1}{n!}}{\frac{1}{(n+1)!}} = \infty$

Series converges everywhere

$\sum \frac{(3x)^n}{4^n}$

$R = \lim_{n\to\infty} \frac{1}{\sqrt[n]{\frac{3^n}{4^n}}} = \frac{4}{3}$

$\lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right|$ $\frac{4}{3}$

interval of convergence

$(-\frac{4}{3}, \frac{4}{3})$

the interval of convergence.

$\frac{1}{1-x} = \sum_{0}^{\infty} x^n$; $|x| < 1$

$\frac{1}{1+x} = \sum_{0}^{\infty} (-1)^n x^n$ $|-x| < 1$ $(|x| < 1)$

$\frac{1}{1-x^2} = \sum_{0}^{\infty} (x^2)^n = \sum_{0}^{\infty} x^{2n}$ $|x^2| < 1$ $(|x| < 1)$

$\frac{1}{1+x^2} = \sum_{0}^{\infty} (-1)^n x^{2n}$ $|x| < 1$

$\frac{3}{4-5x} = \frac{3}{4}\left(\frac{1}{1-\frac{5}{4}x}\right) = \frac{3}{4}\sum_{0}^{\infty} (\frac{5}{4}x)^n$

$= \sum_{0}^{\infty} 3 \frac{5^n}{4^{n+1}} x^n$ $|x| < \frac{4}{5}$

$\frac{x^3}{2+5x^2} = \frac{x^3}{2}\left(\frac{1}{1-(-\frac{5}{2}x^2)}\right) = \frac{x^3}{2}\sum_{0}^{\infty} \left[-\frac{5}{2}\right]^n x^{2n}$

$= \sum_{0}^{\infty} \frac{(-1)^n 5^n}{2^{n+1}} x^{2n+3}$; $|x| < \sqrt{\frac{2}{5}}$

$|-\frac{5}{2}x^2| < 1$

$\Rightarrow |x^2| < \frac{2}{5}$ $|x| < \sqrt{\frac{2}{5}}$

$\frac{1}{1+x^2} = \sum_{0}^{\infty} (-1)^n x^{2n}$, $|x| < 1$

$\tan^{-1} x = \sum_{0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ $|x| \leq 1$

$\frac{1}{1+x} = \sum_{0}^{\infty} (-1)^n x^n$, $|x| < 1$

$\ln(1+x) = \sum_{0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$; $-1 < x \leq 1$

$\ln 2 = \sum_{0}^{\infty} \frac{(-1)^n}{n+1} \Rightarrow \sum_{1}^{\infty} \frac{(-1)^n}{n} = -\ln 2$

$\sum_{1}^{\infty} \frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \dots$

$= -\ln 2$

$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{1 - 1 + \frac{x^2}{2}}{x^2} = \lim_{x\to 0} \frac{\frac{x^2}{2}}{x^2} = \frac{1}{2}$

$f(x) = x^2 e^{3x}$

$e^x = \sum_{0}^{\infty} \frac{x^n}{n!} \Rightarrow e^{3x} = \sum_{0}^{\infty} \frac{(3x)^n}{n!}$

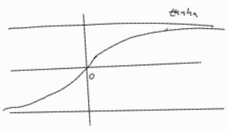
$x^2 e^{3x} = \sum_{0}^{\infty} \frac{3^n x^{n+2}}{n!}$; $|x| < \infty$

$\sin x^2 = \sum_{0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!}$, $|x| < \infty$

$\frac{1}{1-x^2} = \sum_{0}^{\infty} x^{2n}$, $|x| < 1$

$\tan^{-1} x = \sum \frac{x^{2n+1}}{2n+1}$ $-1 < x < 1$

$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ $x \to +\infty \Rightarrow y \to 1$, $x \to -\infty \Rightarrow y \to -1$



$y = \tanh^{-1} x$

$y' = \frac{1}{1-x^2}$

$\frac{1}{1-x} = \sum_{0}^{\infty} x^n$ $|x| < 1$

$\frac{d(1-x)^{-1}}{dx} = \frac{1}{(1-x)^2} = \sum_{1}^{\infty} n x^{n-1}$ $(n=1, \ x \neq 0)$

$(1 + x + x^2 + x^3)' = 0 + 1 + 2x + 3x^2 \dots$

$R = \lim_{n\to\infty} \left(\frac{c_n}{c_{n+1}} \right) = \lim_{n\to\infty} \frac{n}{n+1} = 1$

$\sum_{1}^{\infty} n x^{n-1} \overset{\int}{\Rightarrow} \sum_{0}^{\infty} x^n = \frac{1}{1-x} \overset{\frac{d}{dx}}{\Rightarrow} \frac{1}{(1-x)^2}$

Taylor series:

$f(x) = \sum_{0}^{\infty} c_n (x-a)^n$ $|x-a| < R$

$a - R \overset{?}{<} x \overset{?}{<} a + R$

$x = a \Rightarrow$

$f(a) = c_0$

$f'(x) = \sum_{1}^{\infty} n c_n (x-a)^{n-1}$

$\Rightarrow f'(a) = c_1$ $(x^n)' = n \dots$

$\Rightarrow f^{(n)}(a) = n! c_n$ $(x^n)'' = n(n-1)x^{n-2}$

$(x)^{(n-1)} = n(n-1)\dots(2)x'$

$f(x) = \sum \frac{f^{(n)}(a)}{n!}(x-a)^n$ $(x^n)^{(n)} = n!$

$|x-a| < R$ $(x^n)^{(n+1)} = 0$ $(\sim x_n)$

$R = \lim_{n\to\infty} \left| \frac{(n+1) f^{(n)}(a)}{f^{(n+1)}(a)} \right|$

$e^x = \sum$

$f(x) = e^x \Rightarrow f^{(n)}(x) = e^x$

$f^{(n)}(0) = e^0 = 1$

$e^x = \sum \frac{f^{(n)}(0)}{n!} x^n$, $|x| < R$

$e^x = \sum \frac{1}{n!} x^n$, $|x| < \infty$ $x = 1 \Rightarrow \sum_{0}^{\infty} \frac{1}{n!} = e$

$R = \lim \left| \frac{\frac{1}{n!}}{\frac{1}{(n+1)!}} \right| = \lim_{n\to\infty} (n+1) = \infty$

$\sum_{0}^{\infty} \frac{1}{2^n n!} = \sum_{0}^{\infty} \frac{(\frac{1}{2})^n}{n!} = \sqrt{e}$

$\sum_{0}^{\infty} \frac{(-1)^n}{3^{n+1} n!} = \frac{1}{3} \sum \frac{(-\frac{1}{3})^n}{n!} = \frac{1}{3} e^{-\frac{1}{3}} = \frac{1}{3\sqrt[3]{e}}$

$e^x = \sum \frac{x^n}{n!}$; $|x| < \infty$

$\cosh = \frac{e^x + e^{-x}}{2} = \frac{1}{2}\left(\sum_{0}^{\infty} \frac{x^n}{n!} + \frac{(-x)^n}{n!}\right) = \sum_{0}^{\infty} \frac{x^{2n}}{(2n)!}$ $|x| < \infty$ $= 1 + \frac{x^2}{2!} + \frac{x^4}{4!}$

$\sinh = \frac{e^x - e^{-x}}{2} = \frac{1}{2} \sum_{0}^{\infty} \frac{x^n}{n!} - \frac{(-x)^n}{n!} = \sum_{0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$ $|x| <$

$\cos x = \frac{e^{ix} + e^{-ix}}{2}$

$= \frac{1}{2} \sum_{0}^{\infty} \frac{(ix)^n}{n!} + \frac{(-ix)^n}{n!}$

$= \sum_{0}^{\infty} \frac{(ix)^{2n}}{(2n)!} = \sum_{0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ $|x| < \infty$

$= \sin x = \frac{e^{ix} - e^{-ix}}{2i}$

$= \frac{1}{2i} \sum_{0}^{\infty} \frac{(ix)^n}{n!} - \frac{(-ix)^n}{n!} = \frac{1}{i} \sum_{0}^{\infty} \frac{i^{2n+1} x^{2n+1}}{(2n+1)!}$

$= \sum_{0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$\lim_{x\to 0^+} \frac{1}{x} - \frac{1}{\sin x} = \lim_{x\to 0^+} \frac{\sin x - x}{x \sin x} = \lim_{x\to 0^+} \frac{-\frac{x^3}{3!}}{x(x - \frac{x^3}{6})} = 0$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \dots$

$\sin x \simeq x - \frac{x^3}{3!}$

$e=\sum_0^n \frac{1}{n!} \Rightarrow e^x = P_n(x) + R_n(x)$

$e^1 = P_n(1) + R_n(1)$

$|R_n| \le 10^{-5}$

$f(x)=e^x \quad f^{(n+1)}_{(x)} = e^x$

$10^{-5} = M_{n+1}$

$\max_{0\le t\le 1} |f^{(n+1)}_{(t)}| = e^1 \le 3$

$\frac{1}{n!} \ge \frac{1}{10^5}$ $n! \le 10^5$

$|R_n(1)| \le 10^{-5}$

$\frac{M_{n+1}}{(n+1)!} |1-0|^{n+1}$

$(n+1)! \ge 3\cdot 10^5$

$e \simeq \sum_0^8 \frac{1}{n!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{1}{5040} + \frac{1}{40320}$

$(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}$

$\binom{-1}{3} = (-1)^3$

$\binom{-1}{k} = (-1)^k$

$(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{n}x^n$ $n \in \mathbb{Z}^+$

$(1-x)^{-1} = 1 + \binom{-1}{1}x + \binom{-1}{2}x^2 + \dots = 1 - x + x^2 + \dots = \sum_0^\infty (x)^n$

$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_0^\infty (-x)^n,\ |x|<1$

$(1+x)^{\frac{1}{2}} = 1 + \binom{\frac{1}{2}}{1}x + \binom{\frac{1}{2}}{2}x^2 \dots$

$= 1 + \frac{1}{2}x + \frac{\frac{1}{2}(-\frac{1}{2})}{2\times 1}x^2 + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})x^3}{3\times 2\times 1} + \dots$

$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - + \dots$

$f(x) = \frac{1}{1-x} = (1-x)^{-1}|_0 = 1$

$f'(x) = \frac{1}{1-x} = (1-x)^{-2}|_0 = 1$

$f''(x) = 2(1-x)^{-3}|=2$

$\vdots$

$f^n(x) = n!(1-x)^{-1-n} = \frac{n!}{(1-x)^{n+1}}|n!$

$f(x) = \sum_0^\infty \frac{f^n(0)}{n!}x^n$

$= \sum_0^\infty \frac{n!}{n!}x^n = \sum_0^\infty 1\,n!$

$f(x) = \frac{1}{a+bx}$

$\frac{1}{\sqrt{x+1}\ \ln(\sqrt{x+1})}$

$\frac{1}{\sqrt{x+1}} \to x$

$f'(x) = -(a+bx)^{-1}$

$\vdots$

$f^n(x) = \frac{(-1)^n \cdot b^n \cdot n!}{(a+bx)^{n+1}}$

$f^n(0) = \frac{(-1)^n b^n n!}{a^{n+1}}$

$\frac{1}{a+bx} = \sum_0^\infty \frac{(-1)^n b^n}{a^{n+1}} x^n$

$\int_0^{0.2} \sin(x^2)\,dx =$

$\int_0^{0.2} \sin x^2\,dx \simeq \int_0^{0.2} \left(x^2 - \frac{x^6}{6} + \frac{x^{10}}{120}\right)dx$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \quad \simeq \frac{x^3}{3} - \frac{x^7}{42} + \frac{x^{11}}{1320}\Big|_0^{0.2}$

$\sin(x^2) = x^2 - \frac{x^6}{6} + \frac{x^{10}}{120} =$

$y' + y = x \quad y(0) = 0$

$\lambda + 1 = 0 \quad y = ce^{-x} - 1 + x$

$\lambda = -1$

$y_h = ce^{-x} \quad y(0) = c - 1 = 0$

$c = 1$

$y_p = a_1 x + a_0 \quad y = e^x + x - 1$

$y_p' = a_1$

$a_1 = 1$

$a_0 = -1$

$y = \sum_0^\infty c_n x^n \Rightarrow y' = \sum_1^\infty n c_n x^{n-1}$

$y = \sum_0^\infty c_n x^n \quad y' = \sum_1^\infty n c x^{n-1}$

$x = y + y' = \sum_0^\infty c_n x^n + \sum_1^\infty n c_n x^{n-1}$

$= \sum_1^\infty c_{n-1} x^{n-1} + \sum_1^\infty n c_n x^{n-1}$

$= \sum_1^\infty (n c_n - c_{n-1}) x^{n-1} = 0 + 1x + 0x^2 \dots$

$c_0 = 0$

$n=1 \Rightarrow c_1 + c_0 = 0 \qquad c_3 = \frac{1}{3}c_2 = -\frac{1}{3}\left(\frac{1}{2}\right)$

$n=2 \Rightarrow 2c_2 + c_1 = 1 \qquad = \frac{-1}{3\times 2} = \frac{-1}{3!} = \frac{(-1)^n}{n!}$

$n \ge 3 \Rightarrow n c_n + c_{n-1} = 0$

$c_n = -\frac{1}{n} c_{n-1}$

$y = \sum_2^\infty \frac{(-1)^n x^n}{n!}$

$= e^x - 1 - x$

Fourier Series

$\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx \quad (f \text{ is even})$

$\int_{-a}^a f(x)\,dx = 0 \quad (f \text{ is odd})$

$f(x+T) = f(x)$

$\int_0^T f(x)\,dx = \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\,dx = \int_a^{a+T} f(x)\,dx$

$\int_{-\pi}^{\pi} \underbrace{\sin nx}_{odd}\,\underbrace{\cos nx}_{even}\,dx = \left[\sin^2 x\right]_{-\pi}^{\pi} = \sin^2\pi - \sin^2(-\pi)$
$=$

$\int_{-\pi}^{\pi} \sin nx\,dx = -\left[n\cos x\right]_{-\pi}^{\pi} = n\cos\pi - n\cos(-\pi)$
$= n -$

$\int_{-\pi}^{\pi} \cos nx\,dx = \left[\sin x\right]_{-\pi}^{\pi} = 0$

$\int_{-\pi}^{\pi} \cos mx \cos nx\,dx = \frac{1}{2}\int_{-\pi}^{\pi} [\cos(m-n)x + \cos(m+n)x]\,dx = \begin{cases} 0 & m \ne n \\ \pi & m = n \end{cases}$

$\int_{-\pi}^{\pi} \sin mx \sin nx\,dx = \frac{1}{2}\int_{-\pi}^{\pi} [\cos(m-n)x - \cos(m+n)x]\,dx = \begin{cases} 0 & m \ne n \\ \pi & m = n \end{cases}$

$\sin a \sin b = \frac{1}{2}(\cos(a-b) - \cos(a+b))$

$\cos a \cos b = \frac{1}{2}(\cos(a-b) + \cos(a+b))$

$\int_{-l}^{l} \sin\frac{n\pi}{l}x\,dx \quad T=2l$

$\int_{-l}^{l} \cos\frac{m\pi}{l}x\,dx$

$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} a_0\,dx + \sum_1^\infty a_n \int_{-\pi}^{\pi} \cos\,dx + b_n \int_{-\pi}^{\pi} \sin nx\,dx$

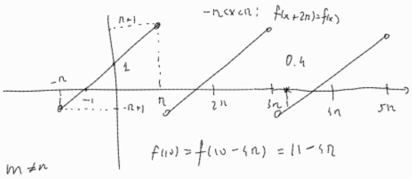
$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \qquad a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \qquad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}x\,dx$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}x\,dx$

$f(x) = x+1 \qquad \pi$

$-\pi < x < \pi: \quad f(x+2\pi) = f(x)$



$f(10) = f(10 - 4\pi) = 11 - 4\pi$

$f(\pi^-) = \lim_{x\to\pi^-} f(x) = \pi + 1$

$f(\pi^+) = \lim_{x\to\pi^+} f(x) = -\pi + 1$

$\text{Average} = \frac{f(\pi^-) + f(\pi^+)}{2} = 1$

At the point of discontinuity $x = x_0$ Fourier series converges to $\frac{f(x_0+) + f(x_0-)}{2}$

$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x + 1\,dx = \frac{1}{2\pi}\left[\frac{x^2}{2} + x\right]_{-\pi}^{\pi} = \frac{1}{2\pi}\left[\frac{\pi^2}{2} + \pi - \left(\frac{\pi^2}{2} - \pi\right)\right] = 1$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x+1)\cos nx\,dx$

d	S
$x+1$	$\cos nx$
1	$\frac{1}{n}\sin nx$
0	$-\frac{1}{n^2}\cos nx$

$= \frac{1}{\pi}\left(\frac{1}{n}(x+1)\sin nx + \frac{1}{n^2}\cos nx\right)\Big|_{-\pi}^{\pi} = 0$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x+1)\sin nx\,dx$

d	S
$x+1$	$\sin nx$
1	$-\frac{1}{n}\cos x$
0	$-\frac{1}{n^2}\sin nx$

$= \frac{1}{2\pi}\left(\frac{-1}{n}(x+1)\cos x + \frac{1}{n^2}\sin nx\right)\Big|_{-\pi}^{\pi}$

$= \frac{-1}{n\pi} 2\pi(-1)^n = -\frac{2}{n}(-1)^n$

$f(x) = 1 + \sum_1^\infty -\frac{2}{n}(-1)^n \sin nx$

$x_0 = \frac{\pi}{2} \to f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} + 1$

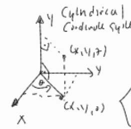
$= 1 + 2\sin\frac{\pi}{2} + \sin 2\left(\frac{\pi}{2}\right) + \frac{2}{3}\sin 3\left(\frac{\pi}{2}\right) - \frac{2}{4}\sin 4\left(\frac{\pi}{2}\right) - \frac{2}{5}\sin 5\left(\frac{\pi}{2}\right)$

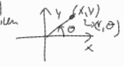
$= 1 + 2 - \frac{2}{3} + \frac{2}{5} - \frac{2}{7} + \frac{2}{9}$

$\frac{\pi}{2} = 2\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right) \Rightarrow \sum_0^\infty \frac{(-1)^n}{2n+1} = \frac{\pi}{4}$

$x_0 = \pi \Rightarrow \frac{f(\pi^+) + f(\pi^-)}{2} = 1 + \sum 0$

Cylindrical Coordinate System

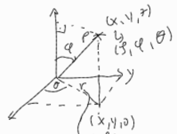




$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}\frac{y}{x} \end{cases}$$

$(x,y,z) \to (r,\theta,z)$ $\quad 0 \le \theta \le 2\pi,\ r \ge 0$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}, \quad \begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}\frac{y}{x} \\ z = z \end{cases}$$



$r = \rho\sin\varphi$

$$\begin{cases} z = \rho\cos\varphi \\ r = \rho\sin\varphi \\ x = \rho\sin\varphi\sin\theta \\ y = \rho\sin\varphi\cdot\cos\theta \end{cases} \quad \rho \ge 0,\ 0 \le \theta < 2\pi,\ 0 \le \varphi \le \pi$$

$$\begin{cases} \rho = \sqrt{x^2+y^2+z^2} \\ \varphi = \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) \\ \theta = \tan^{-1}\left(\frac{y}{x}\right) \end{cases}$$

$(\rho,\varphi,\theta) \sim (r,\theta,z)$

$$\begin{cases} \rho = \sqrt{r^2+z^2} \\ \varphi = \cos^{-1}\left(\frac{z}{\sqrt{r^2+z^2}}\right) \\ \theta = \theta \end{cases} \quad \begin{cases} r = \rho\sin\varphi \\ \theta = \theta \\ z = \rho\cos\varphi \end{cases}$$

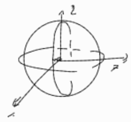
$\sin\varphi = \frac{r}{\rho}$

$\rho = \frac{r}{\sin\theta}$

Ex

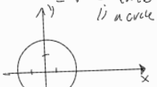
$\rho = 2$

$x^2+y^2+z^2 = 4$



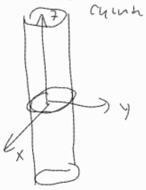
sphere of radius 2

Ex $r = 2$ in 2D is a circle



$x^2+y^2=4$

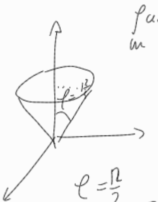
3d $= r = 2$ cylinder



Ex. $\varphi = \frac{\pi}{4}$

cone

ρ and θ are free



$\varphi = \frac{\pi}{2}$

$\cos\varphi = \frac{\sqrt{2}}{2}$

$$\cos\varphi = \frac{z}{\sqrt{x^2+y^2+z^2}} \Rightarrow \left\{\frac{z}{\sqrt{x^2+y^2+z^2}} = \frac{\sqrt{2}}{2}\right\}$$

$$\Rightarrow \frac{z^2}{x^2+y^2+z^2} = \frac{1}{2} \Rightarrow z \ge 0$$

$z^2 = x^2 \quad z \ge 0$

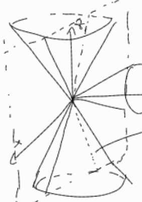
$z^2 = x^2+y^2$

$z = 0 \Rightarrow x^2+y^2 = 0 \Rightarrow x = y = 0 \Rightarrow (0,0,0)$ point

$z = z_0 \Rightarrow x^2+y^2 = z_0^2$ circle

$x = 0 \Rightarrow z^2 = y^2 \Rightarrow z = \pm y$

$y = 0 \Rightarrow z^2 = x^2 \Rightarrow z = \pm x$

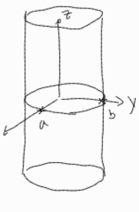


$y = \sqrt{x^2+y^2}$

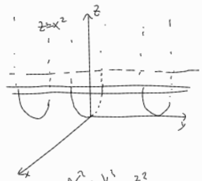
$z = -\sqrt{x^2+y^2}$

$x = +\sqrt{y^2+z^2}$

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



$z = x^2$



$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

$z = z_0 \Rightarrow \frac{x^2}{a^2\left(1-\frac{z_0}{c^2}\right)} + \frac{y^2}{b^2\left(1-\frac{z_0}{c^2}\right)} = 1$ ellipse

$|z_0| < c$

$|z| = c \Rightarrow \frac{x^2}{a^2}+\frac{y^2}{b^2} = 0$

$x = y = 0$

$(0,0,\pm c)$ points

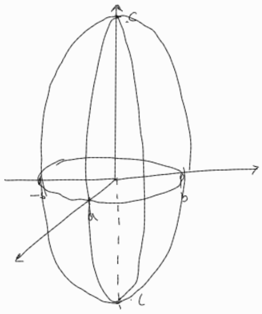
$|z| > c \Rightarrow$ no intersection

$y = 0 \Rightarrow \frac{x^2}{a^2}+\frac{z^2}{c^2} = 1$ ellipse

$y = y_0 \Rightarrow$

$x_0 = 0$

$x = x_0 \Rightarrow$

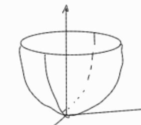


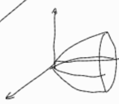
$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

$x = 0 \Rightarrow z = y^2$ parabola

$y = 0 \Rightarrow z = x^2$ parabola

$z_0 \ge 0 \Rightarrow x^2+y^2 = z_0$ circle





$y = 2x^2+3z^2$

$y = y_0 \Rightarrow$ ellipse

$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$

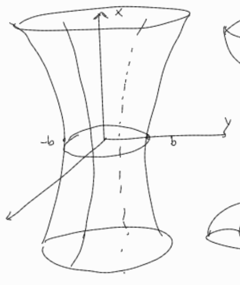
hyperboloid of one sheet

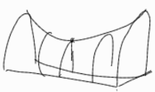
$z_0 = 0 \Rightarrow \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ellipse

$z = z_0 \Rightarrow$ ellipse

$x = 0 \Rightarrow \frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ hyperbola

$y = 0: \frac{x^2}{a^2}-\frac{z^2}{c^2}=1$ hyperbola





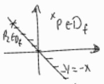
$f(x,y) = \frac{x-y}{x+y}$

$z = f(x,y)$ depends on independent variables

$\frac{dy}{dx} = 0 \quad \frac{dx}{dy} = 0$

$Adx + Bdy = 0 \Rightarrow A = B = 0$

$D_f = \mathbb{R}^2 - \{(x,y) \mid y = -x\}$



$(0,0) \notin D_f$

$\lim_{(x,y)\to(0,0)} f(x,y)$

$\lim_{(x,y)\to(a,b)} f(x,y) = l$

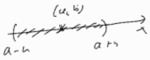
$\forall\varepsilon>0\ \exists\delta>0: \quad \sqrt{(x-a)^2+(y-b)^2} < \delta \Rightarrow |f(x,y)-l| < \varepsilon$

$\forall\varepsilon>0\ \exists\delta>0: \begin{cases} |x-a| < \delta \\ |y-b| < \delta \end{cases} \Rightarrow |f(x,y)-l| < \varepsilon$

$$f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^2} & (x,y) \ne (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

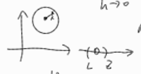
$\lim_{(x,y)\to(a,b)} f(x) = f(a,b)$ if f is continuous at (a,b)

$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$ if f is continuous at (a,b)

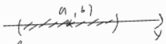


$\lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h} = \frac{\partial f}{\partial x} = f_x$ given that the limit exists.

partial derivative of f with respect to x



$|f(x,y)-2| < \varepsilon$



$\lim_{k\to 0} \frac{f(x,y+k)-f(x,y)}{k} = \frac{\partial f}{\partial y} = f_y$ given that the limit exists.

$\lim_{x\to 0} \frac{|x|}{x} = L.H.L \ne R.H.L$

does not exist D.N.E

$$f(x) = \begin{cases} 2-x^3 & x < 0 \\ x^2+2 & x > 0 \end{cases}$$

$R.H.L = L.H.L = 2 \Rightarrow \lim f(x) = 2$

$\lim_{(x,y)\to(a,b)} \frac{x-y}{x+y}$

$\lim_{(x,y)\to(0,0)} \frac{x-y}{x+y} = \lim_{x\to 0} \frac{x-2x}{x+2x} = -\frac{1}{3}$

$y = 2x$

$\lim_{(x,y)\to(0,0)} \frac{x-y}{x+y} = \lim_{y\to 0} \frac{y^2-y}{y^2+y} = \lim \frac{y(y-1)}{y(y+1)} = -1$

$x = y^2$

limit depends on the path $\Rightarrow$ l. D.N.E

$\lim_{(x,y)\to(0,0)} \frac{x-y}{x+y} = \lim \frac{x-kx}{x+kx} = \frac{1-k}{1+k}$

$y = kx$

$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x+y} = ?$

$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x+y} = \lim_{y\to 0} \frac{kx^3}{x+kx} = \lim_{x\to 0} \frac{kx^2}{1+k} = 0$

$y = kx$

$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x+y} = \lim_{x\to 0} \frac{ky^4}{k^2y^4+y^4} = \frac{k}{k^2+1}$ D.N.E

$x = ky^2$

$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \lim_{x\to 0} \frac{x^2-k^2x^2}{x^2+k^2x^2} = \frac{1-k^2}{1+k^2}$

$y = kx$

limit depends on path $\Rightarrow$ limit D.N.E

$\lim_{y\to 0}\left(\lim_{x\to 0} \frac{x^2-y^2}{x^2+y^2}\right) = \lim_{y\to 0} \frac{-y^2}{y^2} = -1$

$\lim_{x\to 0}\lim_{y\to 0} \frac{x^2-y^2}{x^2+y^2} = \lim_{x\to 0} \frac{x^2}{x^2} = 1$

$(x,y)\to(a,b)$

$x-a = t$

$y-b = s$

$(t,s)\to(0,0)$

$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \lim_{r\to 0} \frac{r^2\cos^2\theta - r^2\sin^2\theta}{r^2}$ $\quad 0 \le \theta < 2\pi$

$= \cos 2\theta$

$\Rightarrow$ depends on θ D.N.E.

$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2} = \lim_{r\to 0} \frac{r\cos\theta\, r^2\sin^2\theta}{r^2}$ $\quad 0 \le \theta < 2\pi$

$= \lim_{r\to 0} r(\cos\theta\sin^2\theta) = 0$

$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2} = \lim_{r\to 0} r$ $\quad 0 \le \theta \le 2\pi$

Step 1. find the limit on a path.

$\lim_{y\to 0}\left(\lim_{x\to 0} \frac{xy^2}{x^2+y^2}\right) = \lim_{y\to 0} 0 = 0$

Step 2. prove it by using definitions

$\forall\varepsilon>0\ \exists\delta>0: \sqrt{x^2+y^2} < \delta \Rightarrow \left|\frac{xy^2}{x^2+y^2} - 0\right| < \varepsilon$

$\left|\frac{xy^2}{x^2+y^2} - 0\right| < \frac{|x||y^2|}{x^2+y^2} \le \frac{|x|(x^2+y^2)}{(x^2+y^2)} = |x| \le \sqrt{x^2+y^2}$ $\quad \delta \le \varepsilon$

$f(x,y) = x^2+y^3+4x^4y^5 - e^{x^2+y^3}+8$

$f_x = 2x+0+16x^3y^5 - 2xe^{x^2+y^3}+0$

$f_y = 0+3y^2+20x^4y^4 - 3y^2e^{x^2+y^3}+0$

$$f(x,y) = \begin{cases} \frac{x^3-y^3}{x^2+y^2} & (x,y) \ne (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$f_x = \frac{3x^2(x^2+y^2)-(x^3-y^3)(x^2-y^2)}{(x^2+y^2)^2}$

$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h}$

$= \lim_{h\to 0} \frac{\frac{h^3}{h^2}-0}{h} = 1$

$f_y(x,y) = \frac{-3y(x^2+y^2)-(x^3-y^3)(2y)}{(x^2+y^2)^2}$

$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h}$

$= \lim_{h\to 0} \frac{-h-0}{h} = -1$

$y = f(x)$

$\frac{df}{dx} = f'(x)$

$df = f'(x)\,dx$

$\frac{f(x+\Delta x)-f(x)}{\Delta x} \approx f'(x)\,\Delta x$ as $\to 0$

$\frac{\Delta f}{\Delta x} - f'(x) = \varepsilon(\Delta x)$

$\Delta f = f'(x)\Delta x + \varepsilon(\Delta x)\cdot\Delta x$

$\Delta f = f(x+\Delta x, y+\Delta y) - f(x,y)$

$= f_x\cdot\Delta x + f_y\cdot\Delta y + \varepsilon_1(\Delta x,\Delta y)\Delta x + \varepsilon_2(\Delta x,\Delta y)\Delta y$

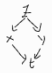
$\lim_{(\Delta x,\Delta y)\to(0,0)} \varepsilon_1(\Delta x,\Delta y) = 0 \quad \lim_{(\Delta x,\Delta y)\to(0,0)} \varepsilon_2(\Delta x,\Delta y) = 0$

$\Rightarrow f(x,y)$ is differentiable at (x,y)

$df = f_x\,dx + f_y\,dy$

chain rule

$z=f(x,y)$, $x=x(t)$, $y=y(t)$ $\quad \frac{dz}{dt}=\frac{\partial z}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}$



$z=x^2+e^{xy}$

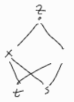
$x=t^2+1,\ y=3t+1 \Rightarrow \frac{y-1}{3}=t$

$\frac{dz}{dt}=(t^2+1)^2+e^{(t^2+1)(3t+1)}$

$=4t(t^2+1)+[3(t^2+1)+2t(3t+1)]e^{(t^2+1)(3t+1)}$

$\frac{dz}{dt}=z_x\frac{dx}{dt}+z_y\frac{dy}{dt}=(2x+ye^{xy})2t+xe^{xy}3$

$=(2x+ye^{xy})\frac{2}{3}(y-1)+3e^{xy}$



$z_t=z_x x_t+z_y y_t$

$z_s=z_x x_s+z_y y_s$

Ex $w=\ln(x^2+y^2+z^2+1)$

$x=3t+4s$

$y=5t-2s$

$z=t-1$

$w_t=w_x x_t+w_y y_t+w_z z_t$

$=\left(\frac{2x}{x^2+y^2+z^2+1}\right)(3)+\left(\frac{2y}{x^2+y^2+z^2+1}\right)(5)$

$+\left(\frac{2z}{x^2+y^2+z^2+1}\right)(1)$

$w_s=w_x x_s+w_y y_s+w_z z_s^{\,0}$

$=\left(\frac{2x}{x^2+y^2+z^2+1}\right)(4)+\left(\frac{2y}{x^2+y^2+z^2+1}\right)(-2)$

Implicit differentiation

$y=f(x) \Rightarrow y'=f'(x)$ explicit

$F(x,y)=0$ implicit

"y as a function of x"

$F=0 \Rightarrow dF=0$

$\Rightarrow F_x dx+F_y dy=0 \Rightarrow \frac{dy}{dx}=-\frac{F_x}{F_y}$

$F(x,y)=xe^y+yx-3$

$y'=-\frac{F_x}{F_y}=-\frac{e^y+y}{xe^y+x}\Big|_{(1,0)}=-\frac{1+0}{1+1}=-\frac{1}{2}$

$F(x,y,z)=0$ "z as a function of x, y"

$z=z(x,y)$

$F=0 \Rightarrow dF=0$

$dF=F_x dx+F_y dy+F_z dz=0$

$=(F_x+F_z z_x)dx+(F_y+F_z\cdot z_y)dy=0$

$\begin{cases}F_x+F_z z_x=0 \longrightarrow z_x=-\frac{F_x}{F_z}\\ F_y+F_z z_y=0 \rightarrow z_y=-\frac{F_y}{F_z}\end{cases}$

$x^2y+tzy^3+4x=8$

$y_z=?$

$y=y(x,t,z)$

$y_z=-\frac{F_z}{F_y}=\frac{ty^3}{x^2+3tzy^2}$

$\begin{cases}F(x,y,u,v)=0\\ G(x,y,u,v)=0\end{cases}$ $\quad u_x,u_y,v_x,v_y=?$ $\quad u=u(x,y)\ v=v(x,y)$

$\begin{cases}F=0\\ G=0\end{cases} \Rightarrow \begin{cases}dF=0\\ dG=0\end{cases} \Rightarrow$

$\begin{cases}F_x dx+F_y dy+F_u du+F_v dv=0\\ G_x dx+G_y dy+G_u du+G_v dv=0\end{cases}$

$du=u_x dx+u_y dy$

$dv=v_x dx+v_y dy$

$\begin{cases}F_x dx+F_y dy+F_u(u_x dx+u_y dy)+F_v(v_x dx+v_y dy)=0\\ G_x dx+G_x dy+G_u(u_x dx+u_y dy)+G_v(v_x dx+v_y dy)=0\end{cases}$

$\begin{cases}(F_x+F_u u_x+F_v v_x)dx+(F_y+F_u u_y+F_v v_y)dy\\ (G_x+G_u u_x+G_v v_x)dx+(G_y+G_u u_y+G_v v_y)dy=0\end{cases}$

$\begin{cases}F_u u_x+F_v v_x=-F_x\\ F_u u_y+F_v v_y=-F_y\\ G_u u_x+G_v v_x=-G_x\\ G_y u_y+G_v v_y=-G_y\end{cases}$

$u_x=\frac{\begin{vmatrix}-F_x & F_v\\ -G_x & G_v\end{vmatrix}}{\begin{vmatrix}F_u & F_v\\ G_u & G_v\end{vmatrix}}=-\frac{\begin{vmatrix}F_x & F_v\\ G_x & G_v\end{vmatrix}}{\begin{vmatrix}F_u & F_v\\ G_u & G_v\end{vmatrix}}=-\frac{\frac{\partial(F,G)}{\partial(x,v)}}{\frac{\partial(F,G)}{\partial(u,v)}}$

$v_x=-\frac{\frac{\partial(F,G)}{\partial(u,x)}}{\frac{\partial(F,G)}{\partial(u,v)}}$

$u_y=-\frac{\frac{\partial(F,G)}{\partial(y,v)}}{\frac{\partial(F,G)}{\partial(u,v)}}$

$v_y=-\frac{\frac{\partial(F,G)}{\partial(u,y)}}{\frac{\partial(F,G)}{\partial(u,v)}}$

$\begin{cases}2t+3s-x+u=0 \rightarrow F\\ 5t+4s-3x+5v=0 \rightarrow G\end{cases}$

$t_x=?\ \Rightarrow t=t(x,v)$

$s_v=?\ \Rightarrow s=s(x,v)$

$t_x=-\frac{\frac{\partial(F,G)}{\partial(s,x)}}{\frac{\partial(F,G)}{\partial(s,t)}}=-\frac{\begin{vmatrix}F_s & F_x\\ G_s & G_x\end{vmatrix}}{\begin{vmatrix}F_s & F_t\\ G_s & G_t\end{vmatrix}}=-\frac{\begin{vmatrix}3 & -1\\ 4 & -3\end{vmatrix}}{\begin{vmatrix}3 & 2\\ 4 & 5\end{vmatrix}}=\frac{5}{7}$

$s_v=-\frac{\frac{\partial(F,G)}{\partial(v,t)}}{\frac{\partial(F,G)}{\partial(s,t)}}=-\frac{\begin{vmatrix}F_v & F_t\\ G_v & G_t\end{vmatrix}}{\begin{vmatrix}F_s & F_t\\ G_s & G_t\end{vmatrix}}=-\frac{\begin{vmatrix}1 & 2\\ 5 & 5\end{vmatrix}}{\begin{vmatrix}3 & 2\\ 4 & 5\end{vmatrix}}=\frac{5}{7}$

$x=r\cos\theta$

$y=r\sin\theta$

$J\left(\frac{x,y}{r,\theta}\right)=\frac{\partial(x,y)}{\partial(r,\theta)}=\begin{vmatrix}x_r & x_\theta\\ y_r & y_\theta\end{vmatrix}=\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}$

$=r\cos^2\theta+r\sin^2\theta$

$=r$

curve passing through $P_0(x_0,y_0,z_0)$ surface

$r(t)=x(t)i+y(t)j+z(t)k$ curve

$v=\frac{dx}{dt}i+\frac{dy}{dt}j+\frac{dz}{dt}k$ tangent to the curve

$T=\frac{v}{|v|}$: tangent to the curve.

$T\cdot N=0$: normal to the curve

$f=f_1i+f_2j+f_3k$

$|f|=C \Rightarrow f\perp f'$

$f\cdot f=|f||f|\cos(0)=|f|^2=C^2$

$f'\cdot f+f\cdot f'=0$

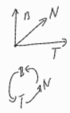
$2f\cdot f'=0$

$f\,f'=0$

$|f|=C \Rightarrow f\perp\frac{df}{dt}$

$N=\frac{\frac{dT}{dt}}{\left|\frac{dT}{dt}\right|}$

$B=T\times N$



Normal vector to the surface is a vector normal to any curve passing through P_0.

$F(x,y,z)=0$

$\frac{dF}{dt}=F_x\frac{dx}{dt}+F_y\frac{dy}{dt}+F_z\frac{dz}{dt}=0$

$(F_x,F_y,F_z)\left(\frac{dx}{dt},\frac{dy}{dt},\frac{dz}{dt}\right)=0$

$\nabla F\cdot V=0 \Rightarrow \nabla F\perp V \Rightarrow \nabla F$ is normal to the surface

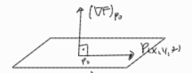
$grad(F)=(F_x,F_y,F_z)$

$\nabla f=grad(f)$

$\nabla=\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)$

Tangent plane to the surface $F(x,y,z)=0$ at $P_0(x_0,y_0,z_0)$;

$(\nabla F)_{P_0}$ is normal to the surface



$(\nabla F)_{P_0}\cdot\overrightarrow{P_0P}=0$

$F_x(P_0)(x-x_0)+F_y(P_0)(y-y_0)+F_z(P_0)(z-z_0)=0$

$z=-x^2-2y^2 \qquad P_0(2,1,-6)$

$F(x,y,z)=z+x^2+2y^2=0$

$\nabla F=(F_x,F_y,F_z)(2x,4y,z)\Big|_{P_0}=(4,4,1)$

$4(x-2)+4(y-1)+(z+6)=0$

$F(P)\simeq F(P_0)+dF$

$dF=F_x dx+F_y dy+F_z dz$

$L(x,y,z)=F(P_0)+F_x(P_0)(x-x_0)+F_y(P_0)(y-y_0)+F_z(P_0)(z-z_0)$

if P is close to $P_0 \Rightarrow f(x,y,z)\simeq L(x,y,z)$ (Linearization)

$f(x,y)\simeq L(x,y)=f_x(P_0)(x-x_0)+f_y(P_0)(y-y_0)$

$|E(x,y)|<\frac{M}{2}((x-x_0)-(y-y_0))^2$

M is a common upper bound for $|f_{xx}|$ $|f_{xy}|$ and $|f_{yy}|$

$f(x,y)=x^2+2xy+\sin x$

$L(x,y)=f(P_0)+f_x(P_0)(x-x_0)-f_y(P_0)(y-y_0)$

$=0+3(x-0)+0(y-y_0)=3x$

$f(0.1,1.1)\simeq L(0.1,1.1)=3(0.1)=0.3$

$f(0.1,1.1)=0.1^2+2(0.1)(1.1)+\sin(0.1)=0.329$ $\quad |E(0.1,1.1)|=0.029$

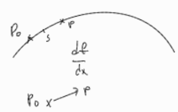
$f_{xx}=2-\sin x \qquad |f_{xx}|\le 3$

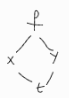
$|f_{xy}|=2 \qquad |f_{xy}|\le 2$

$|f_{yy}|=0 \qquad |f_{yy}|\le 0$ $\quad \boxed{M=3}$

$|E(x,y)|<\frac{3}{2}((x-0)+(y-1.1))^2$

$|E(0.1,1.1)|\le\frac{3}{2}(0.1+0.1)^2=0.06$





$r(t)=x(t)i+y(t)j+z(t)k$

$\frac{dr}{dt}=v, \quad \frac{dr}{ds}=T, \quad \frac{ds}{dt}=|v|$

$\frac{dr}{ds}=\frac{dr}{dt}\cdot\frac{dt}{ds}=\frac{\frac{dr}{dt}}{\frac{ds}{dt}}=\frac{v}{|v|}=T$

$u=\overrightarrow{P_0P}$ $\quad P_0 \rightarrow P$

$\left(\frac{df}{ds}\right)_{P_0}=f_x\frac{dx}{ds}+f_y\frac{dy}{ds}+f_z\frac{dz}{ds}$

$=(f_x,f_y,f_z)\left(\frac{dx}{ds},\frac{dy}{ds},\frac{dz}{ds}\right)$

$=(\nabla f)_{P_0}\cdot T=(\nabla f)_{P_0}\cdot\frac{\overrightarrow{P_0P}}{|\overrightarrow{P_0P}|}=(D_{u_A}f)_{P_0}$

directive of f at P_0 in the direction of vector $\vec{A}$

Ex Find the derivative of $f(x,y)=x^2+3y+y^3$ at $P_0(2,1)$ in the direction of $A=3i-j$

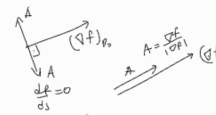
$A=(3,-1) \Rightarrow |A|=\sqrt{10}$

$f_x=2x|_{P_0}=4$

$f_y=3+3y^2|_{P_0}=6$

$(D_{u_A}f)=(\nabla f)_{P_0}\cdot\frac{A}{|A|}$

$=(4,6)\cdot\frac{(3,-1)}{\sqrt{10}}=\frac{12-6}{\sqrt{10}}=\frac{6}{\sqrt{10}}$

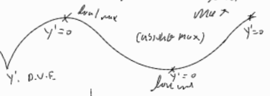


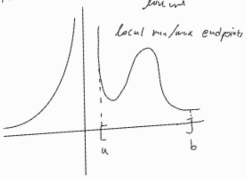
$\left(\frac{df}{ds}\right)_{P_0}=\nabla f\cdot\frac{\nabla f}{|\nabla f|}=\frac{|\nabla f||\nabla f|\cos 0}{|\nabla f|}=|\nabla f|$ max

$\left(\frac{df}{ds}\right)_{P_0}=-|\nabla f|=$ min

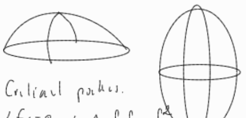
$(4,6)$

LOCAL MAX / MIN





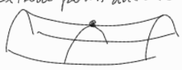
$y=f(x)$



Critical points.

$\begin{cases} f_x=0 \\ f_y=0 \end{cases}$ at $\Rightarrow \Delta=f_{xx}f_{yy}-f_{xy}^2$ also

Extreme points and saddle points



$$\Delta=\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy}\end{vmatrix}=f_{xx}f_{yy}-f_{xy}^2$$

if $f, f_x, f_y, f_{xy}, f_{yx}$ are all continuous then $f_{xy}=f_{yx}$

$f, f_x, f_y, f_{xy}, f_{yx}$ all are defined at (a,b) and in a open region containing (a,b) then continuous

$f_{xy}(a,b)=f_{yx}(a,b)$

$\exists\delta>0: \sqrt{(x-a)^2+(y-b)^2}<\delta$

$\Delta<0 \Rightarrow$ there is a saddle point (a,b)

$\Delta>0 \Rightarrow \begin{cases} f_{xx}<0 \to \text{local max} \\ f_{xx}>0 \to \text{local min}\end{cases}$

$D=0 \Rightarrow$ test is inconclusive.

$z=-x^2+y^2$

$z=0 \quad x^2=y^2 \quad z=-x^2$

$f(x,y)=-x^2+y^2+5$

$\begin{cases} f_x=0 \\ f_y=0\end{cases} \Rightarrow \begin{cases} 2x=0 \\ 2y=0\end{cases} \Rightarrow (x,y)=(0,0)$

$f_{xx}=-2 \quad f_{xy}=0 \quad f_{yy}=2$

$\Delta=-4-0=-4<0$

$\to \Delta<0 \Rightarrow$ there is a saddle point at $(a,b)=0$

Ex. $f(x,y)=x^2-xy+y^2+2x+2y-4$

$\begin{cases} f_x=0 \\ f_y=0\end{cases} \Rightarrow \begin{cases} 2x-y=-2 \\ -x+2y=-2\end{cases}$

$\begin{cases} 2x-y=-2 \\ -2x+4y=-4\end{cases} \Rightarrow 3y=-6 \Rightarrow y=-2 \Rightarrow x=-2$

$\Rightarrow (x,y)=(-2,-2)$

$f_{xx}=2 \quad f_{xy}=-1 \quad f_{yy}=2$

$\Delta=4-1=3>0$

$\begin{cases} \Delta>0 \\ f_{xx}>0 \text{ then min at } (x,y)=(-2,-2)\end{cases}$

$f(x,y)=2x^3+3xy+2y^3$

$\begin{cases} f_x=0 \\ f_y=0\end{cases} \Rightarrow \begin{cases} 6x^2+3y=0 \\ 3x+6y^2=0\end{cases}$

$\Rightarrow y=-2x^2 \Rightarrow 8x^4-x=0$

$\Rightarrow x=0 \Rightarrow y=0 \Rightarrow (x,y)=(0,0)$

$x=-\frac{1}{2} \Rightarrow y=-\frac{1}{2} \Rightarrow (x,y)=(-\frac{1}{2},-\frac{1}{2})$

$f_{xx}=12x$

$f_{yy}=12y$

$f_{xy}=3$

$\Delta=144xy-9$

$\Delta_{(0,0)}=-9<0$ saddle point

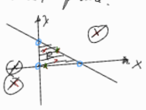
$\Delta(-\frac{1}{2},-\frac{1}{2})=144(\frac{1}{4})-9>0$

$f_{xx}(-\frac{1}{2},-\frac{1}{2})=12(-\frac{1}{2})=-6<0$

$=\begin{cases} \Delta>0 \\ f_{xx}<0\end{cases} \Rightarrow$ local max

Absolute extrema

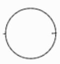
Absolute max and absolute min of function $f(x,y)$ on the Region R in xy-plane.



$\begin{cases} f_x=0 \\ f_y=0\end{cases}$ in the boundary $\Rightarrow y=y(x)$, $a_1<x<b_1$

or $\begin{cases} x=x(y) \\ c<y<d_1\end{cases}$ $g(x)$ study of function of single variable

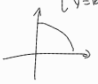
$\begin{cases} x=x(t) \\ y=y(t) \\ \alpha\le t\le\beta\end{cases}$



$y=\pm\sqrt{r^2-x^2}$

$x=\pm\sqrt{r^2-y^2}$

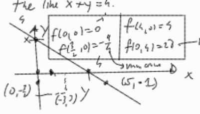
$\begin{cases} x=r\cos t \\ y=r\sin t\end{cases}$ $0\le t\le 2\pi$



$x=3\cos t$, $y=3\sin t$ $\quad 0\le t\le\frac{\pi}{2}$

$f(x,y)=x^2+xy+y^2-3x+3y$

R: a triangular region cut from the first quadrant by the line $x+y=4$.



$\begin{cases} f_x=0 \\ f_y=0\end{cases} \Rightarrow \begin{cases} 2x+y-3=0 \\ x+2y+3=0\end{cases}$

$-3y-9=0 \Rightarrow y=-3$

$x_1=3$

$(x_1,y_1)=(3,-3)\notin R$

boundary ①: $\begin{cases} y=0 \\ 0\le x\le 4\end{cases}$

$\Rightarrow g_1(x)=x^2-3x$

$g_1'(x)=0 \Rightarrow 2x-3=0 \Rightarrow x_2=\frac{3}{2}$

$(x,y)=(\frac{3}{2},0)$

$f(\frac{3}{2},0)=(\frac{3}{2})^2-3(\frac{3}{2})=\frac{9}{4}-\frac{9}{2}=-\frac{9}{4}$

$(x_3,y_3)=(0,0) \Rightarrow f(0,0)=0$

$(4,0) \quad f(4,0)=4$

$b(2)=\begin{cases} x=0 \\ 0\le y\le 4\end{cases}$

$h_2(y)=y^2+3y$

$2y+3=0 \quad y=-\frac{3}{2}$

b) $\begin{cases} y=4-x \\ 0\le x\le 4\end{cases}$

$g_3(x)=x^2+x(4-x)+(4-x)^2-3x+3(4-x)$

$=\cancel{x^2}x-\cancel{x^2}+(4-x)^2+12-3x$

$=-2x+(4-x)^2+12$

$=-2x+16-8x+x^2+12$

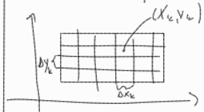
$=-10x+28+x^2$

$g_3'=0 \quad 2x-10=0$

$x=5$

$(5,-1)\notin R$

Double-integral



$f(x,y)$ is defined over the region R $R\subseteq D_f$

$\|P\|=\max\{\Delta x_k,\Delta y_k\}$

$\|P\|\to 0\Rightarrow \Delta x_k\to 0, \Delta y_k\to 0\Rightarrow n\to\infty$

$S_n=\sum_{k=1}^{n} f(x_k,y_k)\cdot\Delta A_k$ $\qquad dA=dy\,dx=dx\,dy$

$\Delta A_k=\Delta x_k\Delta y_k$

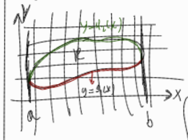
$$\lim_{\|P\|\to 0} S_n=\iint_R f(x,y)\,dA$$

given that the limit exists and is independent from the choice of (x_k,y_k) and the partition $p\Rightarrow$ the f is integrable over the region R

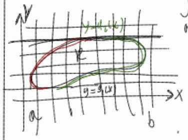
Fubinis Theorem:

$dA=dy\,dx=dx\,dy$

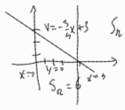
$$\int_a^b\left(\int_c^d f(x,y)\,dy\right)dx=\int_c^d\left(\int_a^b f(x,y)\,dx\right)dy$$



$$\int_a^b\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx=\int_c^d\int_{x=h_1(y)}^{x=h_2(y)} f(x,y)\,dx\,dy$$



$\iint_R dA=S_R$ (Area of the region R)

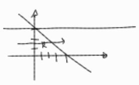


$$S_R=\iint_R dA=\int_0^4\int_0^{-\frac{3}{4}x+3} dy\,dx$$

$$=\int_0^4\left(y\Big|_0^{-\frac{3}{4}x+3}\right)dx=\int_0^4\left(-\frac{3}{4}x+3-0\right)dx$$

$$=-\frac{3}{8}x^2+3x\Big|_0^4=-6+12=6$$

$S_R=\iint dA=\iint dx\,dy$ $\qquad y=-\frac{3}{4}x+3$



$4y=-3x+12$

$-\frac{4}{3}y=x-4$

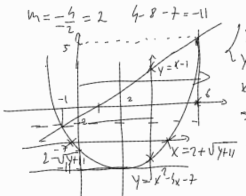
$-\frac{4}{3}y+4=x$

$$S_R=\int_R dA=\int_0^3\int_0^{-\frac{4}{3}y+4} dx\,dy=\int_0^3 x\Big|_0^{-\frac{4}{3}y+4} dy$$

$$=\int_0^3 -\frac{4}{3}y+4\,dy=-\frac{2}{3}y^2+4y\Big|_0^3=-6+12=6$$

Ex. find the area of the region R enclosed by the line $y=x-1$ and the parabola $y=x^2-4x-7$

$m=\frac{-4}{-2}=2 \qquad 4-8-7=-11$



$\begin{cases} y=x-1 \\ y=x^2-4x-7\end{cases}$

$x^2-4x-7=x-1$

$\Rightarrow (x-6)(x+1)=0$

$x=-1 \quad x=6$

$$S_R=\int_R dA=\int_{-1}^{6}\int_{x^2-4x-7}^{x-1} dy\,dx=\int_{-1}^{6} y\Big|_{x^2-4x-7}^{x-1} dx$$

$$=\int_{-1}^{6} x-1-x^2+4x+7\,dx$$

$$=\int_{-1}^{6} -x^2+5x+6\,dx=-\frac{x^3}{3}+\frac{5x^2}{2}+6x\Big|_{-1}^{6}=-\frac{216}{3}+\frac{5(36)}{2}+36+\frac{1}{3}+\frac{5}{2}-6$$

$$S_R=\iint_R dA=\int_{-11}^{-2}\int_{2-\sqrt{y+11}}^{2+\sqrt{y+11}} dx\,dy+\int_{-2}^{5}\int_{y+1}^{2+\sqrt{y+11}} dx\,dy=-\frac{12}{3}+\frac{5(37)}{2}+30$$

$=57.666$

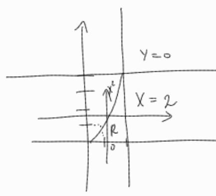
$x^2-4x-7-y=0$

$m=2 \quad c=+7$

$x=2\pm\sqrt{4+7+y}$

$x=2\pm\sqrt{11+y}$

$$\int_{y=0}^{y=4}\int_{x=\sqrt{y}}^{x=2}\frac{e^x}{x}\,dx\,dy=\int_0^2\int_0^{x^2}\frac{e^x}{x}\,dy\,dx$$



$$y=4\int_0^2\frac{e^x}{x}\cdot y\Big|_0^{x^2}$$

$$y=4\int_0^2 x e^x\,dx$$

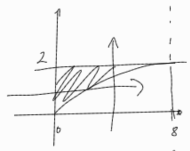
$y=0$ $\quad =xe^x-e^x\Big|_0^2$

$=e^2+1$

$$\int_{x=0}^{x=8}\int_{y=\sqrt[3]{x}}^{y=2}\frac{dy\,dx}{y^4+1}$$

$y^4+1=(y^2+\sqrt{2}y+1)(y^2-\sqrt{2}y+1)$

$y=x^{\frac{1}{3}}$



$$\int_{y=0}^{y=2}\int_{x=0}^{x=y^3}\frac{1}{y^4+1}\,dx\,dy$$

$$=\int_0^2\left(\frac{x}{y^4+1}\Big|_0^{y^3}\right)dy$$

$v=y^4+1 \quad du=3y\,dy$

$$=\int_0^2\frac{y^3}{y^4+1}dy=\int_0^2\frac{1}{u}du$$

$$=\ln|y^4+1|\Big|_0^2=\ln|5|-\ln|1|=\ln|5|$$

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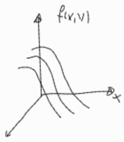


$a=(1,1)$
$v=\frac{u}{|u|}$

$\partial_{\underline{u}} f \leftarrow \Delta f \cdot \underline{v}$

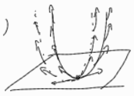
$f(x,y)=2xy-3y^2 \quad p_0=\begin{bmatrix}5\\5\end{bmatrix} \quad a=\begin{bmatrix}4\\3\end{bmatrix}$

$\nabla f_{\left(\substack{5\\5}\right)}=\begin{bmatrix}2x\\2x-6y\end{bmatrix}_{\left(\substack{5\\5}\right)}=\begin{bmatrix}10\\-20\end{bmatrix}$

$\underline{v}=\frac{a}{|a|}=\begin{bmatrix}4\\3\end{bmatrix}\cdot\frac{1}{5}=\begin{bmatrix}\frac{4}{5}\\\frac{3}{5}\end{bmatrix}$

$\nabla f\cdot\underline{v}=\begin{bmatrix}10\\-20\end{bmatrix}\begin{bmatrix}\frac{4}{5}\\\frac{3}{5}\end{bmatrix}=8-12=-4$

$f(x,y)=2x^2+y^2 \quad P_0(-1,1)$
$\underline{a}=(3,-4)$



$\underline{u}=\begin{bmatrix}\frac{3}{5}\\-\frac{4}{5}\end{bmatrix}$

$\Delta f_{(-1,1)}=\begin{bmatrix}4x\\2y\end{bmatrix}_{[-1,1]}=\begin{bmatrix}-4\\2\end{bmatrix}$

$\Delta f_{(-1,1)}\cdot\underline{u}=\begin{bmatrix}-4\\2\end{bmatrix}\begin{bmatrix}\frac{3}{5}\\-\frac{4}{5}\end{bmatrix}=-4\cdot\frac{3}{5}-2\cdot\frac{4}{5}=-\frac{12}{5}-\frac{8}{5}=\frac{-20}{5}$
$=-4$

3) $f(x,y)=x^2y+e^{xy}\sin y$
$P_0(1,0)$

$\nabla f=\begin{bmatrix}2xy+ye^{xy}\sin y\\x^2+xe^{xy}\sin y+e^{xy}\cos y\end{bmatrix}$

$\nabla f(1,0)=\begin{bmatrix}0\\2\end{bmatrix}$

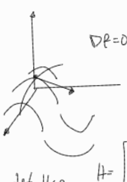
$u=\frac{\nabla f}{|\nabla f|}$

$\nabla f\cdot\underline{u}=\nabla f\frac{\nabla f}{|\nabla f|}=|\nabla f|=2$

Extremal values

$\nabla f\cdot\underline{u}=0$

$\nabla f=0$



$f_{xx}<0$

$f_{xx}>0$

det $H<0$
det $H>0$

$H=\begin{bmatrix}f_{xx}&f_{xy}\\f_{xy}&f_{yy}\end{bmatrix}$

det $H<0$

det $H>0$ $\cup$ $\cap$

$f(x,y)=x^2+xy+y^2+3x-3y+5$

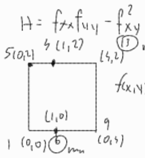
$\nabla f=\begin{bmatrix}2x+y+3\\x+2y-3\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

$\begin{cases}2x+y=-3\\x+2y=3\end{cases}$

$\begin{cases}2x+y=-3\\2x+4y=6\end{cases}$ $2x=-6 \Rightarrow x=-3$

$3y=9 \quad y=3$

$H=f_{xx}f_{yy}-f_{xy}^2=4-1=3$



$f(x,y)=x^2-y^2-2x+4y+1$

$(1,2)$

$\Delta f=\begin{bmatrix}2x-2\\-2y+4\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \quad \begin{matrix}x=1\\y=2\end{matrix}$

$(x,y)=(1,2)$

$f(x,0)=x^2-2x+1$
$f'(x,0)=2x-2=0 \quad x=1$
$f'(1,0)=$

$f(0,y)=-y^2+4y+1$
$f'(0,y)=-2y+4=0$
$y=2$

$f(x,2)=x^2-2x+5$
$f'(x,2)=2x-2=0$
$x=1$
$f(1,2)=1-2+5=4$

$f(4,y)=16-y^2-8+4y+1$
$f'(4,y)=-2y-4 \quad y=2$
$(4,2)$

6) $\int_0^1\int_1^2 xy\,dx\,dy=\int_0^1 y\frac{x^2}{2}\Big|_1^2 dy$

$=\int_0^1\frac{y}{2}(4-1)\,dy=\int_0^1 y\,dy=\frac{3}{2}\frac{y^2}{2}\Big|_0^1$

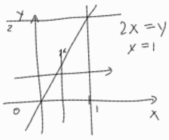
$=\frac{3}{4}$

$\int_{y=1}^{y=10}\int_{x=0}^{x=\frac{1}{y}} ye^{xy}\,dx\,dy$

$=\int_2^{10} e^{xy}\Big|_0^{\frac{1}{y}} dy$

$=\int_1^{10} e-1\,dy=9(e-1)$

7) $\int_0^2\int_{x=\frac{y}{2}}^{x=1} e^{x^2}\,dx\,dy$



$\int_0^1\int_0^{2x} e^{x^2}\,dy\,dx$

$=\int_0^1 e^{x^2}2x\,dx$

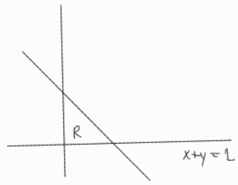
$=e^{x^2}\Big|_0^1=e-1$

Substitution on double Integrals

$\int_R\int f(x,y)\,dy=\int_{R'}\int f(u,v)\,|J|\,du\,dv \quad \frac{R}{\text{boundary ①}} \xrightarrow[y=y(u,v)]{x=x(u,v)} \frac{R'}{\text{boundary ①'}}$

$\begin{cases}x=x(u,v)\\y=y(u,v)\end{cases} \Rightarrow dx\,dy=|J|\,du\,dv$

$J=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}x_u&x_v\\y_u&y_v\end{vmatrix}$



Ex $\int_0^1\int_0^{1-y} e^{\frac{x-y}{x+y}}\,dy\,dx$

$\begin{cases}x-y=u\\x+y=v\end{cases} \rightarrow \begin{cases}x=\frac{1}{2}u+\frac{1}{2}v\\y=\frac{1}{2}v-\frac{1}{2}u\end{cases}$

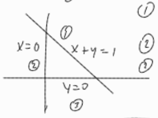
$J=\begin{vmatrix}x_u&x_v\\y_u&y_v\end{vmatrix}=\begin{vmatrix}\frac{1}{2}&\frac{1}{2}\\-\frac{1}{2}&\frac{1}{2}\end{vmatrix}=\frac{1}{2}$

$\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}1&-1\\1&1\end{vmatrix}=2$

reprocical

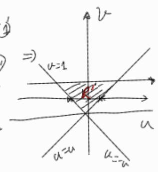
$|J|=\left|\frac{\partial(x,y)}{\partial(u,v)}\right|=\frac{1}{\left|\frac{\partial(u,v)}{\partial(x,y)}\right|}=\frac{1}{2}$

$\Rightarrow \quad dx\,dy=\frac{1}{2}du\,dv$



① $x+y=1 \Rightarrow v=1$ ①'
② $x=0 \Rightarrow v=-u$ ②'
③ $y=0 \Rightarrow v=u$ ③'

$\Rightarrow$



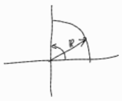
$\int_0^1\int_{-v}^{v} e^{\frac{u}{v}}\frac{1}{2}\,du\,dv=\int_0^1\frac{1}{2}ve^{\frac{u}{v}}\Big|_{-v}^{v}dv$

$=\int_0^1\frac{1}{2}v(e^1-e^{-1})\,dv$

$=\sinh(1)\frac{v^2}{2}\Big|_0^1$

$=\frac{\sinh(1)}{2}$

$\int_0^1\int_0^{\sqrt{1-x^2}}(x+y)\,dy\,dx$



$\begin{cases}x=r\cos\theta\\y=r\sin\theta\end{cases}$

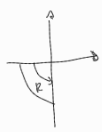
$J=\frac{\partial(x,y)}{\partial(r,\theta)}=\begin{vmatrix}x_r&x_\theta\\y_r&y_\theta\end{vmatrix}=\begin{vmatrix}\cos\theta&-r\sin\theta\\\sin\theta&r\cos\theta\end{vmatrix}=r$

$\int_0^{\frac{\pi}{2}}\int_0^1(r\cos\theta+r\sin\theta)\,r\,dy\,d\theta$

$=\int_0^{\frac{\pi}{2}}(\cos\theta+\sin\theta)\frac{r^3}{3}\Big|_0^1 dy\,d\theta$

$=\frac{1}{3}\int_0^{\frac{\pi}{2}}\cos\theta+\sin\theta\,d\theta=\frac{1}{3}(\sin\theta-\cos\theta)\Big|_0^{\frac{\pi}{2}}=\frac{2}{3}$

$\int_{-1}^0\int_{-\sqrt{1-x^2}}^0\frac{2}{1+\sqrt{x^2+y^2}}\,dy\,dx$



$=\int_\pi^{\frac{3\pi}{2}}\int_0^1\frac{2r}{1+r}\,dr\,d\theta$

$=\int_\pi^{\frac{3\pi}{2}}2\int_0^1 1-\frac{1}{r+1}\,dr\,d\theta$

$=\int_\pi^{\frac{3\pi}{2}}2(r-\ln(r+1))\Big|_0^1 d\theta$

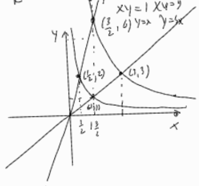
$=\int_\pi^{\frac{3\pi}{2}}2(1-\ln(2))\,d\theta=2(1-\ln(2))\theta\Big|_\pi^{\frac{3\pi}{2}}$

$=3(1-\ln(2))\pi-2(1-\ln(2))\pi$

$=(1-\ln(2))\pi$

$\int_R\int\left(\sqrt{\frac{y}{x}}+\sqrt{xy}\right)dy\,dx$

R: in first quadrant bounded by;



$u=xy$
$v=\frac{y}{x}$

$v=1 \quad u=9 \rightarrow 1\le u\le 9$
$v=1 \quad v=4 \rightarrow 1\le v\le 4$

$\int_1^4\int_1^9\left(\sqrt{v}+\sqrt{u}\right)\frac{1}{2v}\,du\,dv$

$=\int_1^4\frac{\sqrt{v}}{2v}u-\frac{2}{3}\frac{u^{\frac{3}{2}}}{v}\Big|_1^9 dv$

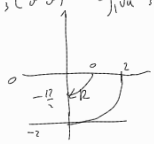
$J=\frac{\partial(x,y)}{\partial(u,v)}=\frac{1}{\frac{\partial(u,v)}{\partial(x,y)}}=\frac{1}{\begin{vmatrix}u_x&u_y\\u_x&u_y\end{vmatrix}}$

$=\frac{1}{\begin{vmatrix}y&x\\-\frac{y}{x^2}&-\frac{1}{x}\end{vmatrix}}=\frac{1}{\left|\frac{y}{x}-\frac{y}{x}\right|}=\frac{1}{2\frac{y}{x}}=\frac{1}{2v}$

$=\frac{1}{2}\int_1^4\frac{9}{\sqrt{v}}+\frac{2}{3}\frac{9^{\frac{3}{2}}}{v}-\left(\frac{1}{\sqrt{v}}+\frac{2}{3v}\right)dv$

$\Rightarrow \frac{1}{2}\int_1^4\frac{8}{\sqrt{u}}+\frac{2}{3}\left(\frac{27}{v}+\frac{1}{v}\right)dv=\frac{1}{2}\int_1^4\frac{8}{\sqrt{u}}+\frac{2}{3}\left(\frac{26}{v}\right)dv$

$\int_{-2}^0\int_0^{\sqrt{4-y^2}}\frac{\sqrt{x^2+y^2}}{1+x^2+y^2}\,dx\,dy$



$=\int_{-\frac{\pi}{2}}^0\int_0^2\frac{r^2}{1+r^2}\,dr\,d\theta$

$=\int_{-\frac{\pi}{2}}^0\int_0^2 1-\frac{1}{1+r^2}\,dr\,d\theta=\int_{-\frac{\pi}{2}}^0 r-\tan^{-1}(r)\Big|_0^2 d\theta$

$=\int_0^{-\frac{\pi}{2}}(2-\tan(2))\,d\theta=(2-\tan^{-1}(2))\theta\Big|_{-\frac{\pi}{2}}^0$

$=\frac{\pi}{2}(2-\tan^{-1}(2))$

Let $f(x,y,z)$ be defined in D.

Partition divides D into n cubes with

(the limit does not depend on the partition P)

$$\Delta V_k = \Delta x_k \Delta y_k \Delta z_k$$

if we select the point $p(x_k, y_k, z_k)$ from k^{th} cube

(if the sum does not depend on the choice of point p)

$\|P\| = \min\{\Delta x_i, \Delta y_i, \Delta z_i\}$

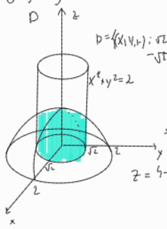
$$\lim_{\|P\|\to 0} S_n = \iiint_D f(x,y,z)\,dv$$

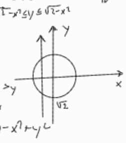
if the limit is independent from partition P and the choice of (x_k, y_k, z_k)

$$\iiint_D f(x,y,z)\,dv = \iint \int_{z=u_1(x,y)}^{z=u_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$

$D = \{(x,y,z) : -\sqrt{2} \le x \le \sqrt{2},\ -\sqrt{2-x^2} \le y \le \sqrt{2-x^2}$

$$\Rightarrow V_D = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_0^{4-x^2-y^2} dz\,dy\,dx$$





$$V_D = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \left(\int_0^{4-x^2-y^2} dz \right) dy\,dx = \int_{-\sqrt{2}}^{\sqrt{2}} \left(\int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} 4-x^2-y^2 \right) dy\,dx$$

$\Downarrow$

$$= \int_0^{2\pi} \int_0^{\sqrt{2}} \left(\int_0^{4-r^2} r\,dz \right) dr\,d\theta = \int_0^{2\pi} \int_0^{\sqrt{2}} 4r - r^3\,dr\,d\theta$$

$$= 2\pi \left(2r^2 - \frac{r^4}{4} \Big|_0^{\sqrt{2}} \right) = (2\pi)(4-1) = 6\pi$$

$$V = \int_0^{2\pi} \int_0^{\pi} \int_0^{a} \rho^2 \sin\varphi\,d\rho\,d\varphi\,d\theta$$

$$= (2\pi)\left(-\cos\varphi \Big|_0^{\pi}\right)\left(\frac{\rho^3}{3}\Big|_0^{a}\right) = 2\pi\,((-1)(-1))\,\frac{a^3}{3}$$

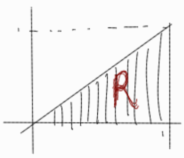
$$= \frac{4\pi}{3} a^3$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$V_D = \iiint_{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \le 1} dz\,dy\,dx = \iiint_{u^2+v^2+w^2 \le 1} (a,b,c)\,dv\,dv\,dw$$

$$= \frac{4\pi}{3}(1)^3(abc)$$

$$= \frac{4\pi}{3}(abc)$$



$$\iint_R \frac{y}{\sqrt{x^2+y^2}}\,dy\,dx$$

$$= \int_0^{\frac{\pi}{4}} \int_0^{\frac{1}{\cos\theta}} \frac{\cancel{r}\sin\theta}{\cancel{r}}\,r\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{r^2 \sin\theta}{2} \Big|_0^{\frac{1}{\cos\theta}} = \int_0^{\frac{\pi}{4}} \sec\theta \tan\theta\,d\theta$$

$$= \frac{1}{2}\sec\theta \Big|_0^{\frac{\pi}{4}} = \frac{1}{2}(\sqrt{2}-1)$$

$(x,y,z) \to (\rho, \varphi, \theta)$

$$J = \frac{\partial(x,y,z)}{\partial(\rho,\varphi,\theta)} = \begin{vmatrix} x_\rho & x_\varphi & x_\theta \\ y_\rho & y_\varphi & y_\theta \\ z_\rho & z_\varphi & z_\theta \end{vmatrix} = \begin{vmatrix} \sin\varphi\cos\theta & \rho\cos\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta \\ \cos\varphi & -\rho\sin\varphi & 0 \end{vmatrix}$$

$x = \rho\sin\varphi\cos\theta$
$y = \rho\sin\varphi\sin\theta$
$z = \rho\cos\varphi$

$$J = \cos\varphi\,(\rho^2\cos\varphi\sin\varphi\cos^2\theta + \rho^2\cos\varphi\sin\varphi\sin^2\theta)$$
$$+$$
$$\rho\sin\varphi\,(\rho\sin\varphi^2\cos^2\theta + \rho\sin\varphi^2\sin\theta^2)$$
$$= \rho^2\sin\varphi\cos^2\varphi + \rho^2\sin\varphi^3 = \rho^2\sin\rho.$$

$$\frac{64}{3}\int_0^{2\pi}$$

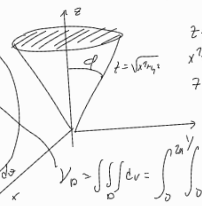
$$V_D = \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \frac{\rho^3}{3}\Big|_0^{\frac{4}{\cos\varphi}} d\varphi\,d\theta \quad = \frac{32}{3}(2\pi)$$

$$= \frac{64}{3} \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \frac{\sin\varphi}{\cos^3\varphi}\,d\varphi\,d\theta$$

$t = \cos\varphi \Rightarrow dt = -\sin\varphi\,d\varphi$

$$\int -\frac{1}{t^3}\,dt \quad \int \frac{1}{2t^2}\,dt$$

$$= \frac{64}{3} \int_0^{2\pi} \frac{1}{2\cos^2\varphi}\Big|_0^{\frac{\pi}{4}} d\theta$$



$z = \rho\cos\varphi$
$x^2+y^2 = \rho^2\sin^2\varphi$
$z = 4 \Rightarrow \rho = \frac{4}{\cos\varphi}$

$$V_D = \iiint_D dv = \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{\frac{4}{\cos\varphi}} (\rho^2\sin\varphi)\,d\rho\,d\varphi\,d\theta$$