Exercise-set 4.
Solutions

1. a) There is no Hamilton cycle: if we delete 4 vertices we get 5 components.
   There is a Hamilton path: draw.
   b) There is no Hamilton cycle: if we delete 2 vertices we get 3 components.
   There is a Hamilton path: draw.
   c), d) There is no Hamilton cycle: if we delete 4 vertices we get 5 components.
   There is a Hamilton path: draw.

2. a) Yes (draw); yes.
   b) No (delete 11 vertices); yes (draw).

3. a) No (delete the 9 vertices divisible by 3 or 5).
   b) No as well.

4. a) If we delete 2 vertices we get 3 components $\Longrightarrow$ need to add at least 1 edge. That is enough (draw).
   b) If we delete 2 vertices we get 4 components $\Longrightarrow$ need to add at least 2 edges. That is enough (draw).

5. If we delete 1 vertex (the center) we get 100 components $\Longrightarrow$ need to add at least 99 edges. That is enough (if we add a path).

6. a) Construct a graph G: $V(G)$ = squares, and the edges are the possible moves of the horse. This graph contains no Hamilton path: if we delete the 4 middle vertices we get 6 components.
   b) The graph contains no Hamilton cycle: if we delete 12 vertices we get 13 components.
   c) No: if we delete 5 vertices we get 7 components.

7. Only one (up to isomorphism), with 50 vertices in both classes.

8. Construct a graph G: $V(G)$ = people, and the edges are the acquaintances. Then $\deg(v) \geq 6 = 12/2$ $\Longrightarrow$ by Dirac's theorem $\exists$ a Hamilton cycle.

9. a) The condition in Ore's theorem holds for $G \Longrightarrow \exists$ a Hamilton cycle.
   b) If the two vertices of degree 50 are adjacent, then use Ore's theorem. If they are not adjacent, then add the edge between them.

10. Construct a graph G: $V(G)$ = people, and the edges are the acquaintances. G is k-regular for some k. If $k \geq 10 \Longrightarrow G$ contains a Hamilton cycle, if $k \leq 9 \Longrightarrow \overline{G}$ contains a Hamilton cycle.

11. Construct a graph G: $V(G)$ = people, and the edges are the acquaintances. We get G' by adding the edges between the second neighbors to G. In G' the degree of each vertex is at least $5 + 5 \cdot 4 = 25 \Longrightarrow G'$ contains a Hamilton cycle.

12. Add a new vertex to G, and connect it to all the old vertices. Then the new graph contains a Hamilton cycle from which we can get a Hamilton path of G.

13. Add two new non-adjacent vertices to G, and connect them to all the old vertices. Then the new graph contains a Hamilton cycle from which we can get a Hamilton path of G by adding an edge if necessary.

14. Delete v from G. Then the new graph contains a Hamilton cycle from which we can get a Hamilton path of G.

15. The 8 edges must have pairwise no common endpoints (i.e. be independent). Every second edge of a Hamilton cycle will do (which exists because $\deg(v) \geq n/2\ \forall v$).

16. Need to add k pairwise non-adjacent edges (from $\overline{G}$). $\overline{G}$ contains a Hamilton cycle ($\deg_{\overline{G}}(v) = k,\ \forall v \in V(G)$). Every second edge of it will do.

17. The first graph is not bipartite (contains 5-cycles), but the second graph is.

18. Deleting 2 edges are enough, but less is not, since $\exists$ 2 edge-disjoint odd cycles in G.

19. The graph determined by the knights and attacks is bipartite (the two classes are to the white and black squares), and each of its degrees is at least 2 $\Longrightarrow$ $\exists$ a degree ≥ 3.

20. Yes (the two classes of vertices are sequences with an even or odd number of 1's, resp.).

21. No (the complement contains a triangle).

22. The vertices cannot be divided into two classes (count the degrees).

23. Complete bipartite graphs are like that.

24. The graphs are exactly the odd cycles (so in particular n must be odd). G must contain an odd cycle (otherwise $\chi(G') = 2$), and cannot contain more vertices or edges.

25. $\omega(G) \geq 3 \implies \chi(G) \geq 3$, and G can be colored with 3 colors $\implies \chi(G) \leq 3$, so $\chi(G) = 3$.

26. $\omega(G) \geq 8$ (each row and column is a clique) $\implies \chi(G) \geq 8$, and G can be colored with 8 colors (colors are diagonal) $\implies \chi(G) \leq 8$, so $\chi(G) = 8$.

27. G is bipartite (the two classes of vertices are the even and odd numbers, resp.) $\implies \chi(G) = 2$

28. $\omega(G) \geq 4 \implies \chi(G) \geq 4$, and G can be colored with 4 colors $\implies \chi(G) \leq 4$.

29. a), b) $\omega(G) \geq 3 \implies \chi(G) \geq 3$, but G cannot be colored with 3 colors (proof!) $\implies \chi(G) \geq 4$. G can be colored with 4 colors $\implies \chi(G) \leq 4$.

30. $\omega(G) \geq 3 \implies \chi(G) \geq 3$, but G cannot be colored with 3 colors (proof!) $\implies \chi(G) \geq 4$. G can be colored with 4 colors $\implies \chi(G) \leq 4$.