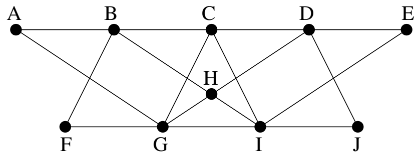
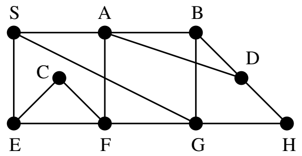
Exercise-set 6.

1. Can the vertices of the graph below be reached in the following order using the BFS algorithm?
a) H, B, D, G, I, C, A, F, J, E
b) F, B, A, G, C, H, I, D, E, J
c) J, D, I, C, E, G, H, A, F, B
d) A, B, G, C, H, F, I, D, E, J



2. (MT'15) The BFS algorithm visited the vertices of the graph below in the following order: $S, \square, \square, \square, H, \square, F, C, \square$.
a) Complete the sequence with the missing vertices (which are denoted by $\square$), and determine the corresponding BFS tree.
b) Can the edge $\{D, H\}$ be contained in an arbitrary BFS spanning tree started from S?



3. In the connected graph G the degree of each vertex is 3. We start a BFS algorithm from vertex s which reaches vertex v in the 13th place (we consider s to be the vertex first reached). Is it possible that the distance of v from s is
a) 2, b) 3, c) 8?

4. a) We want to decide for a given graph G and vertex s whether there is a cycle in G containing s and if yes then we want to find a shortest such cycle. How can we use the BFS algorithm to solve this problem?
b) And if we want to find a shortest cycle containing a given edge e?

5. (MT++'15) We call the spanning tree F of a connected graph G *suitable* for a vertex v of G, if there is a BFS started from v which is exactly F. At most how many edges can a connected graph G on 100 vertices have if it has a spanning tree which is suitable for every vertex of G?

6. (MT+'20) Let G be a simple undirected graph, and let a and b be two different vertices of it. Furthermore, we know that in a BFS started from vertex a the fifth vertex found is b. Is it true then that there is a BFS started from vertex b in which the fifth vertex found is a? (If the answer is yes, prove it, if no, give a counterexample.)

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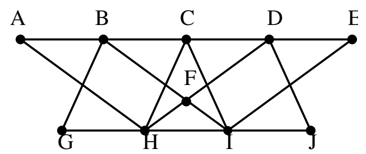
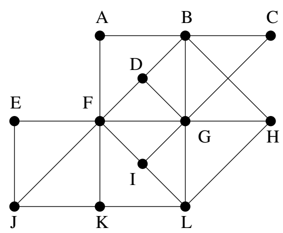
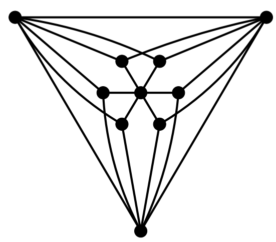
7. Determine the values $\nu(G), \tau(G), \alpha(G)$ and $\rho(G)$ for the graphs below, and determine a maximum matching, a maximum independent set (of vertices), and minimum vertex- and edge covers.







8. The vertices of an 18-vertex graph G can be divided into 3 classes of six vertices each, in such a way that 2 vertices are adjacent if and only if they are in different classes. Determine the values $\chi(G), \nu(G), \tau(G), \alpha(G)$ and $\rho(G)$ for this graph G.

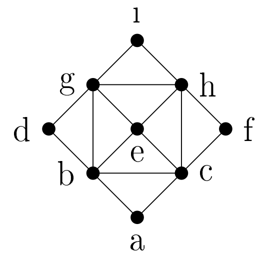
9. Let the vertices of G be the numbers 1,2,...,2005, and two vertices, $i, j \in V$ be adjacent if and only if $i + j$ divided by 3 gives a remainder of 1. Determine the values $\chi(G), \nu(G), \tau(G), \alpha(G)$ and $\rho(G)$ for this graph G.

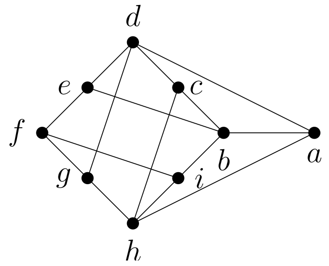
10. (MT'09) Let the vertex set of the graph G be $V(G) = \{1, 2, \ldots, 60\}$. The vertices $x, y \in V(G)$ are adjacent in G if $x \neq y$ and $x \cdot y$ is divisible by 6. Determine $\nu(G)$, i.e. the size of a maximum matching in G.
(Hint: group the vertices according to the remainder when divided by 6.)

11. (MT+'10) The vertices of a graph G are the numbers 1,2,...,100, and two (different) vertices are adjacent if and only if their product is divisible by 7. Determine the the values $\alpha(G), \tau(G), \nu(G)$ and $\rho(G)$ for this graph G.

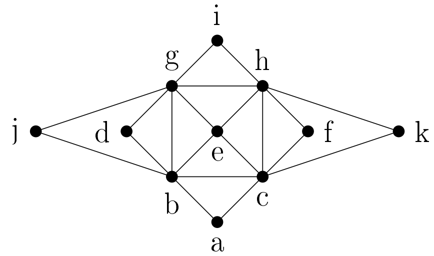
12. (MT+'08) Let the vertices of the graph G be the numbers $\{1, 2, \dots, 100\}$, and two vertices, i and j be adjacent if and only if $i \neq j$ and the g.c.d. of i and j is even but not divisible by 4. Determine the parameters $\nu(G)$ and $\alpha(G)$. ($\nu(G)$ and $\alpha(G)$ denote the maximum number of independent edges and vertices, resp.)

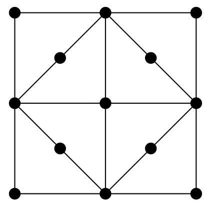
13. (MT+'17, MT+'19) Determine a minimum vertex cover in the graphs below.





14. (MT++'17, MT++'20) Determine a maximum matching in the graphs below.





15. a) Show that in a graph G without loops the endpoints of a maximum matching M form a vertex cover.
b) Show that in a graph G without loops $\tau(G) \leq 2\nu(G)$.
c) Show that in a graph G without loops $\alpha(G) + 2 \cdot \nu(G) \geq |V(G)|$.

16. a) True or false: If a bipartite graph contains a Hamilton cycle then it contains a perfect matching?
b) Is the reverse of the stamement true or false?
c) Does the original statement remain true for non-bipartite graphs?

17. (MT++'03) In the simple graph G on $2k+1$ vertices the degree of each vertex is at least $k+1$. Determine $\nu(G)$, the maximum number of independent edges in G.

18. (MT+'20) In a simple graph on 20 vertices the degree of each vertex is at least 10. Is it true then that the size of a minimum vertex cover is also at least 10?

19. (MT'07) Let G be a simple graph on $2n$ vertices. We know that two nonadjacent vertices, u and v have degree $n-1$, but all the other vertices of G have degree at least n (where $n > 1$ is an integer). Show that G contains a perfect matching!

20. In a simple graph on 20 vertices the degree of each vertex is at least 9. Prove that the graph contains a matching of 9 edges.

21. Let M be an $n \times n$ matrix. Construct a bipartite graph G from M in the following way: let the two vertex classes of G be $A = \{a_1, a_2, \dots, a_n\}$ and $B = \{b_1, b_2, \dots, b_n\}$, and for each $1 \leq i, j \leq n$ let a_i and b_j be adjacent if and only if the entry in the ith row and jth column of M is not zero. Show that if $\det M \neq 0$ then there is a perfect matching in G.

22. At most how many edges can a simple graph G on 100 points have if $\tau(G) \leq 20$? (The parameter $\tau(G)$ denotes the size of a minimum vertex cover in G.)

23. Let the vertices of the graph G be the numbers $1, 2, \dots, 100$, and two vertices, i and j be adjacent if one of them divides the other. Determine the value of $\tau(G)$.

24. (MT++'20) In a simple graph on 50 vertices the maximum degree is 7. Show that we can find 7 independent vertices in the graph.

25. (MT++'21) Let G be a connected graph on 20 vertices. We know that no matter how we choose 8 edges of G there will be a vertex of G which is incident to at least 2 of them. Show that in this case no matter how we choose 12 edges of G there will be a vertex of G which is not incident to any of them.