Exercise-set 8.

1. Determine the edge-chromatic number of the graph of the regular octahedron.

2. Show that in a graph on n vertices with e edges the following hold:
a) $\chi(G) \geq n/\alpha(G)$,
b) $\chi_e(G) \geq e/\nu(G)$, if G is loop-free.

3. Determine $\chi_e(K_5)$ and $\chi_e(K_6)$. (In general, determine $\chi_e(K_{2n+1})$ and $\chi_e(K_{2n})$.)

4. Show that a championship with 20 participants can be arranged in 19 rounds. (Everybody plays with everybody else once, and in one round everybody can play at most once.)

5. Let G be the following graph: $V(G) = \{1, 2, \ldots, 10\}$ és $E(G) = E_1 \cup E_2 \cup E_3$, ahol $E_1 = \{\{i, j\} : 1 \leq i < j \leq 5\}, E_2 = \{\{i, j\} : 6 \leq i < j \leq 10\}, E_3 = \{\{i, j\} : j = i+5\}$. (In words: G consists of two vertex-disjoint K_5 graphs, connected by a perfect matching.) Determine $\chi_e(G)$, the edge-chromatic number of the graph G.

6. a) (MT++'17) We double the edges of a cycle of length 5 of the complete graph on 5 vertices (i.e. we substitute each edge by two parallel edges). Determine the edge-chromatic number of the graph obtained.
b) Let G be the graph obtained from a 5-cycle by substituting all of its edges by three parallel edges. Determine the edge-chromatic number of G.

7. Let G be a 10-regular simple graph on 1999 vertices. Determine $\chi_e(G)$, the edge-chromatic number of G.

8. (MT'16) The vertex v of the simple graph G has degree 2, but all the other vertices of G have degree 3. Determine $\chi_e(G)$, the edge-chromatic number of G.

9. (MT+'15) Show that if G is a simple k-regular graph on 9 vertices then $\chi_e(G) + \chi_e(\overline{G}) \geq 10$.

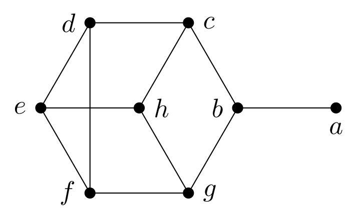
10. (MT+'09) Let G be a k-regular graph with $\chi_e(G) = k$. Show that G contains a perfect matching.

11. In the simple graph on 9 vertices five vertices have degree 4, and the other four vertices have degree 3. Show that $\nu(G) = 4$.

12. a) Show that if G is a 3-regular graph which contains a Hamilton cycle, then the edge-chromatic number of G is 3.
b) Show that the Petersen graph doesn't contain a Hamilton cycle.

13. (MT'09) Let $G = (V, E)$ be a simple graph on 20 vertices in which the degree of each vertex is 8. Let v be an arbitrary vertex of G and denote by $G - v$ the graph obtained from G by deleting v (and all the edges incident to it). Show that $\chi_e(G - v) = \chi_e(G)$, where χ_e denotes the edge-chromatic number of the graphs.

14. (MT'19) Determine the edge-chromatic number of the graph below.



15. ($*$) (MT++'19) We obtained the graph G from K_9, the complete graph on 9 vertices by deleting 4 edges from it. Is it possible that the edge-chromatic number of G is 8?

16. (MT'20) From a graph on $2n$ vertices we delete the edges of a Hamilton cycle. Show that the edge-chromatic number of the graph cannot decrease by more than 2.

17. ($*$) (MT++'20) Does there exist a simple graph whose edge-chromatic number is 5, but if we delete the edges of a Hamilton cycle from it then the edge-chromatic number of the graph obtained is only 2?

18. (MT+'21) Let the graph G be the complement of a path on 6 vertices (with 5 edges). Determine $\chi_e(G)$, the edge-chromatic number of the graph G.

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19. Somebody selected 30 squares on a 10×10 chessboard in such a way that each row and each column contains exactly three selected squares. We want to place 10 white, 10 black and 10 red stones on the 30 selected squares in such a way that each row and each column contains exactly one white, one black and one red stone. Prove that it is always possible with the given conditions.

20. (MT'11) We build a $4 \times 4 \times 4$ cube from 64 small cubes (so the length of an edge in the large cube is four times that of in the small cube). Let the vertices of the graph G be the small cubes, and two different vertices be adjacent if and only if the corresponding small cubes have a common face in the large cube. Determine $\chi_e(G)$, i.e. the edge-chromatic number of G.

21. (MT'18) Let the vertex set of the simple graph be $V(G) = \{1, 2, \ldots, 10\}$. Let the vertices $x, y \in V(G)$ be adjacent if and only if $|x - y| = 3$ or $|x - y| = 5$. Determine $\chi_e(G)$, the edge-chromatic number of G.

22. (MT+'18) In a tree T on 20 vertices 11 vertices have degree 1, and the degree of the remaining 9 vertices are the same as well. Determine the $\chi_e(T)$, edge-chromatic number of the tree.

23. (MT++'18) The graph G on 200 vertices is constructed from two (vertex-disjoint) cycles on 100 vertices each in such a way that we connect each vertex of one cycle with every vertex of the other cycle. Determine $\chi_e(G)$, the edge-chromatic number of the graph G.

24. (MT+'19) We only know of the graph G that it was obtained from $K_{9,9}$ by deleting 8 edges from it. Determine $\chi_e(G)$, the edge-chromatic number of G.

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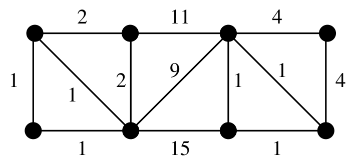
25. Determine a minimum weight spanning tree in the weighted graph below. How many such trees are there?



26. Let G be the complete graph on the vertex set $V(G) = \{1, 2, \ldots, 100\}$. For every $1 \leq i, j \leq 100$, $i \neq j$ let the weight of the edge $\{i, j\}$ be the larger of the values of i and j. What is the weight of a minimum weight spanning tree in G? Determine such a tree. How many minimum weight spanning trees are there?

27. Let G be the complete graph on the vertex set $V(G) = \{1, 2, \ldots, 100\}$. For every $1 \leq i, j \leq 100$, $i \neq j$ let the weight of the edge $\{i, j\}$ be 1, if $i, j \leq 50$, 2, if $i, j \geq 51$, and 3 for all the other edges. What is the weight of a minimum weight spanning tree in G? Determine such a tree.

28. (MT+'21) Let G be the complete graph on the vertex set $V(G) = \{1, 2, \ldots, 10\}$. For every $1 \leq i < j \leq 10$ let the weight of the edge $\{i, j\}$ be $\lfloor \frac{2j-i}{3} \rfloor$ (where $\lfloor\ \rfloor$ denotes the lower integer part). Determine a minimum weight spanning tree in G.

29. (MT'15) Let G be a connected graph and $w : E(G) \to \mathbf{R}$ be a weight function on the edges of G. Suppose that one of the endpoints of the edge e of G is v and for all the edges f which are incident to v the inequality $w(e) \leq w(f)$ holds. Show that G has a minimum weight spanning tree which contains e.

30. (MT+'15) Let G be a connected graph and $w : E(G) \to \mathbf{R}$ be a weight function on the edges of G. Furthermore, let C be a cycle in G and e an edge of C. Suppose that $w(e) \geq w(f)$ holds for all the edges f of the cycle C. Show that G has a minimum weight spanning tree which doesn't contain e.

31. In a connected weighted graph G the weight of every edge is at most 100. We know that G has a minimum weight spanning tree which contains an edge of weight 100. Show that in this case all the (not necessarily minimum weight) spanning trees of G contain an edge of weight 100.

32. * Let G be a connected weighted graph. Show that all the minimum weight spanning trees of G can be obtained by Kruskal's algorithm.