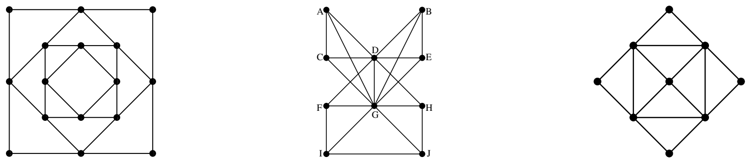
Exercise-set 4.

1. Do the following graphs contain a Hamilton cycle? And a Hamilton path?



2. Let the vertices of the graph G be the squares of a 5×5 chessboard, and two vertices be adjacent if and only if the corresponding squares have a common edge. The graph G_1 is obtained from G by deleting a vertex corresponding to one of the corners of the chessboard from it (so G_1 has 24 vertices). The graph G_2 is obtained from G by deleting two vertices corresponding to opposite corners of the chessboard from it (so G_2 has 23 vertices).
   a) Does G_1 contain a Hamilton cycle? And a Hamilton path?
   b) Does G_2 contain a Hamilton cycle? And a Hamilton path?

3. Let the vertex set of the graph G be $V(G) = \{1, 2, \ldots, 20\}$. Let the vertices $x, y \in V(G)$ be adjacent in G if $x \neq y$ and $x \cdot y$ is divisible by 3 or 5 (or both).
   a) Does G contain a Hamilton path?
   b) Does G contain a Hamilton cycle?

4. At least how many edges must be added to the graphs below so that the graphs obtained contain a Hamilton cycle?

5. The graph G is a star on 101 vertices (i.e. G has one vertex of degree 100 and hundred vertices of degree 1). At least how many edges must be added to G so that the graph obtained contains a Hamilton cycle?

6. a) Show that it is impossible to visit each square of a 4×4 chessboard (exactly once) with a horse.
   b) Show that it is impossible to visit each square of a 5×5 chessboard (exactly once) with a horse such that in the 25th move we arrive back to the starting square.
   c) $*$ (MT+'19) Can we visit each square of a 3×5 chessboard exactly once with a horse?

7. (MT+'21) The vertex set of a graph on 100 vertices can be divided into two classes and all the vertices in one class are connected to all the vertices in the other class. How many non-isomorphic such a graphs are there which contain a Hamilton cycle?

8. In a company of 12 everybody knows at least 6 others (acquaintances are mutual). Show that this company can be seated around a round table in such a way that everybody knows his/her neighbors.

9. a) (MT'03) The simple graph G has 101 vertices. One of its vertices has degree 50, and all the other vertices have degree at least 51. Prove that G contains a Hamilton cycle.
   b) The simple graph G has 101 vertices. Two of its vertices have degree 50, and all the other vertices have degree at least 51. Prove that G contains a Hamilton path.

10. In a company of 20 everybody knows the same number of people (acquaintances are mutual). Show that this company can be seated around a round table in such a way that either everybody knows his/her neighbors or nobody knows his or her neighbors.

11. $*$ There are 50 guests at a banquet, each of them knows at least 5 people from the others. (Acquaintances are mutual.) No matter how we choose 3 or 4 from the guests they cannot sit down to a round table in such a way that everybody knows both of his/her neighbors. Show that in this case all the guests can be seated around a round table for 50 persons in such a way that any two people who sit next to each other, but don't know each other have a common friend among the guests.

12. $*$ In the simple graph G on $2k+1$ vertices each vertex has degree at least k. Prove that G contains a Hamilton path.

13. * In a simple graph on 20 vertices the degree of each vertex is at least 9. Prove that we can add one new edge to the graph in such a way that the resulting graph contains a Hamilton path.

14. * In the simple graph G on 201 vertices the degree of each vertex, except for v, is at least 101. About v we only know that it is not an isolated vertex. Show that G contains a Hamilton path.

15. * Show that if G is a simple 9-regular graph on 16 vertices, then we can delete 8 edges of G in such a way that the remaining graph contains an Euler circuit.

16. * Let G be a simple graph on $2k$ vertices in which the degree of each vertex is $k-1$, where $k > 1$ is an integer. Prove that we can add k new edges to G in such a way that the resulting graph contains a Hamilton cycle.

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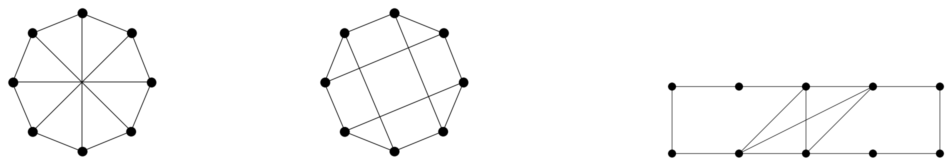
17. Determine whether the first two graphs below are bipartite or not:



18. (MT'09) At least how many edges must be deleted from the third graph above to get a bipartite graph?

19. (MT'10) 7 knights are put on a chessboard in such a way that each of them attacks at least two others. Show that there is such a knight among them which attacks three others.

20. (MT+'16) Let the vertices of the graph G be the all the 0-1 sequences of length 5, and two sequences be adjacent if they differ in eactly one position. Is G a bipartite graph?

21. (MT++'16) Is there a simple bipartite graph on at least 5 vertices whose complement is also a bipartite graph?

22. (MT+'17) In a graph on 99 vertices two vertices have degree 3, and the degree of the other vertices is 4. Show that the graph contains an odd cycle.

23. * Determine all the nonisomorphic simple graphs G on 8 vertices for which $\chi(G) = 2$ but if we add any edge to G (between two nonadjacent vertices) then for the graph G' obtained this way $\chi(G') = 3$ holds.

24. * (MT+'03) Determine all the nonisomorphic simple graphs G on n vertices for which $\chi(G) = 3$ but if we delete any vertex from G (together with the edges adjacent to it) then for the graph G' obtained $\chi(G') = 2$ holds.

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25. Determine the chromatic number of the graph of the regular octahedron. (The octahedron has 6 vertices and 8 triangular faces.)

26. Let the vertices of the graph G be the squares of the chessboard, and two vertices be adjacent if and only if the corresponding squares can be reached from each other by one move of a rook. Determine $\chi(G)$, the chromatic number of G. (A rook in chess can move either horizontally or vertically, and in one move it can go to any square along the selected line.)

27. Let the vertices of the graph G be the integers 1,2,...,100, and two vertices, m and n be adjacent if and only if $m+n$ is odd. Determine $\chi(G)$, the chromatic number of G.

28. (MT+'18) We add two non-adjancent edges to the complete bipartite graph $K_{3,3}$ in such a way that the resulting graph G is simple. Determine $\chi(G)$, the chromatic number of G.

29. (MT'05, MT+'15) Determine the chromatic number of the graphs below:

30. (MT'14) Let G be the graph obtained from a regular 8-sided polygon by adding all the shortest diagonals to it (i.e. G has 8 vertices and 16 edges). Determine $\chi(G)$ and $\omega(G)$.