Sample Midterm

1. Exactly one of the given functions is in $O(n^2)$.

   $$\log(n^2) + \tfrac{n(n-1)(n-2)}{2010} \qquad 2020n\log n + \frac{2^n \cdot n(n-1)}{3^n} \qquad 20n^2(\log n)^2 + 8$$

   (a) Determine the function in $O(n^2)$ and prove your claim by providing constants c and n_0.
   (b) Of the remaining functions, show for one of them that it is not in $O(n^2)$ (you can pick any one of the two remaining functions).

2. The array $A[1:n]$ contains positive numbers (the numbers are not necessarily integers and can have values less than 1). We want to find the subarray $A[i:j]$ ($1 \le i \le j \le n$) where the product of the numbers is as large as possible. Give a dynamic programming algorithm with running time $O(n)$ to find the largest possible product by solving the following subproblems: $M[j]$ gives the largest product of a subarray of $A[1:j]$ with the additional condition that that subarray ends in j.

3. We are given an **undirected** graph on vertices a, b, c, d, e, f, g. Running a depth-first search (DFS) starting from vertex d, the tree edges discovered are da, ab, ac, de, ef, fg in this order.
   (a) Is it possible to have the edge ae in the graph?
   (b) Is it possible that the graph contains the edge eg?

4. We are given an undirected graph on n vertices with two selected vertices a and b. Each vertex of the graph is coloured either red or blue, and this information is given in an array S, indexed by vertices, where $S[v]$ is the colour of the vertex v. An edge of the graph is called colourful if one of its endpoints is red, while the other is blue. Give an algorithm with running time $O(n+m)$ that determines if there is a path from a to vertex b that consists of only colourful edges. If there is such a path, then the algorithm should also determine the length of the shortest such path.

5. There are many festivals of interest to us next summer, but unfortunately there are some overlaps in the dates. When we go to a festival, we want to be there from the first day to the last, but the next day we can go to another one. For each of the f festivals we are considering, we know on which day they start and on which day they end, and our aim is to spend as many days at festivals as possible. Formulate the problem as a graph theoretic problem and give an algorithm with running time $O(f^2)$ for selecting one of these festivals.

6. Dijkstra's algorithm is run on an undirected graph consisting of vertices a, b, c, d, e, f. The table below shows the status of the array Q of unprocessed vertices.
   Which edges are definitely included in the graph and what is their weight?

a	b	c	d	e	f
$*$	∞	∞	1	3	∞
$*$	∞	5	$*$	2	4
$*$	7	4	$*$	$*$	4
$*$	6	$*$	$*$	$*$	4
$*$	5	$*$	$*$	$*$	$*$

7. We are given a sum of n positive integers $a_1 + a_2 + \ldots + a_n$. We can replace any of the addition signs with multiplication, provided that there aren't two consecutive multiplication signs (i.e., each number can be in at most one multiplication). Give an algorithm with running time $O(n)$ that determines the maximum result we can obtain from such replacements. For example, the maximum number that can be obtained from $1+4+3+2+3+4+2$ is $29 = 1 + 4 \cdot 3 + 2 + 3 \cdot 4 + 2$.