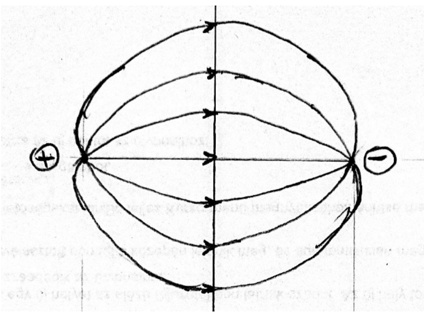
**Problem E8**

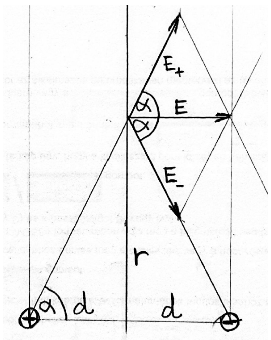
A negative particle with the charge of $Q := -1\mu C$ is placed into close proximity of an infinite large metal sheet. The separation is $d := 1cm$. The distribution of charge in the metal sheet changed, by the effect of the closely located point charge.
a./ Find the surface charge density in the metal sheet, in the circles around the stem point as a function radius.
b./ Sketch the function and find its value at 2cm radial position.
Hint: Use the principle of electrostatic mirror.

Solution:

The principle of electrostatic mirror is based on the fact, that metal is equipotential always. A negative point charge and in close proximity a metal sheet, generate the same electric field in the half space of the negative point charge, as if there were a positive mirror charge in the other half space. Due to the mirror positioning, the metal sheet is equipotential, since the electric field vectors are all perpendicular to it.



Force line distribution




Mirror charges

The individual electric field vectors of $Q_-$ and $Q_+$ charges on the symmetry axis, are **E**₋ and **E**₊ respectively. The magnitude of the individual electric field vectors is as follows:

$$E_\pm = \frac{Q}{4\pi\varepsilon_0} \cdot \frac{1}{d^2 + r^2}$$

Here the electric field of the point charge and the Pythagoras theorem has been used.

Now, the angular position of the actually studied point $\alpha$ is introduced, relative to normal direction to the metal sheet. In the right angle triangle, $\cos\alpha$ can be written:

$$\cos\alpha = \frac{d}{\sqrt{d^2 + r^2}}$$

The sum of the two individual $\mathbf{E}_-$ and $\mathbf{E}_+$ vectors is **E** vector, which is normal to the metal sheet, since the parallel components all cancel out due to symmetry. The magnitude of the **E** vector is as follows: *E*.

$$E = 2E_\pm \cos\alpha$$

$$E = 2\frac{Q}{4\pi\varepsilon_0} \cdot \frac{1}{d^2 + r^2} \cdot \frac{d}{\sqrt{d^2 + r^2}}$$

$$E = \frac{Q}{2\pi\varepsilon_0} \cdot \frac{d}{\left(d^2 + r^2\right)^{3/2}}$$

Let us use an infinitesimally narrow drum volume, with an axis normal to the metal plate. The Gauss law for this volume is as follows:

$$E \cdot dA = \frac{\omega \cdot dA}{\varepsilon_0} \qquad \left[ Vm = \frac{As}{m^2} m^2 \frac{Vm}{As} \right]$$

$$E \cdot \varepsilon_0 = \omega \qquad \left[ \frac{V}{m} \cdot \frac{As}{Vm} = \frac{As}{m^2} \right]$$

So, by multiplying the above result with $\varepsilon_0$, the surface charge density $\omega$ results.

$$\omega(r) = \frac{Q}{2\pi} \frac{d}{\left(d^2 + r^2\right)^{3/2}} = \frac{Q}{2\pi d^2} \frac{1}{\left(1 + \left(\frac{r}{d}\right)^2\right)^{3/2}}$$

Here we introduce $\omega_0 = \frac{Q}{2\pi d^2}$ and the relative radius $0 \le x = \frac{r}{d}$.

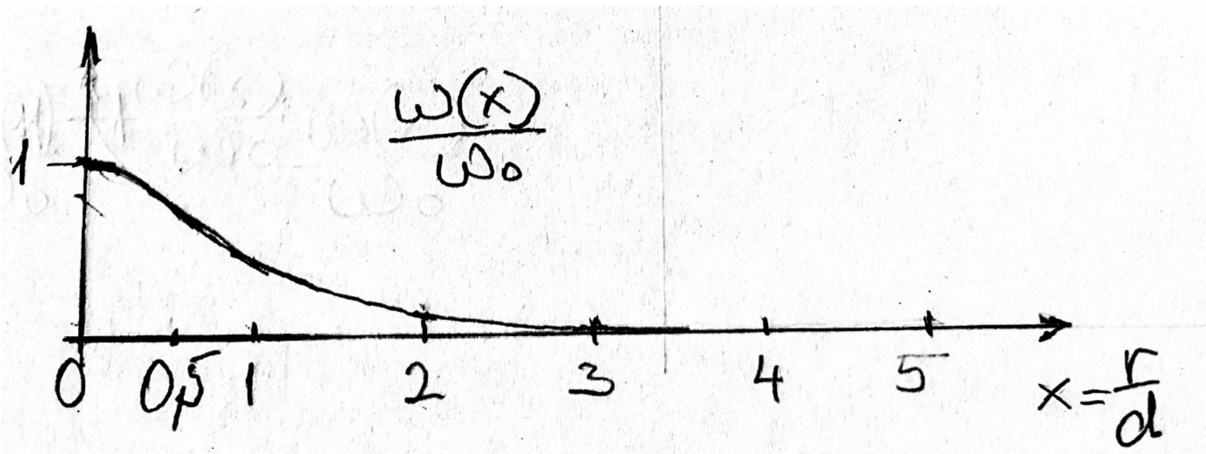
$$\frac{\omega(x)}{\omega_0} = \frac{1}{\left(1 + x^2\right)^{3/2}}$$

After analyzing the function above the following features revealed: At zero it has horizontal tangent, secondly it has an inflection point at $x = 0{,}5$ value. (See appendix below)
Numerically:

| $x = r/d$ | 0 | 0,5 | 1 | 2 | 3 |
|---|---|---|---|---|---|
| $\omega(x)/\omega_0$ | 1 | 0,72 | 0,35 | 0,09 | 0,03 |

$$\omega_0 = \frac{Q}{2\pi d^2} = \frac{10^{-6}}{6{,}28 \cdot 10^{-4}} = \frac{10^{-2}}{6{,}28} = 1{,}59 \cdot 10^{-3} \frac{As}{m^2}$$

$$\omega(2) = 1{,}59 \cdot 10^{-3} \frac{1}{(5)^{3/2}} = 1{,}42 \cdot 10^{-4} \frac{As}{m^2}$$



Appendix:

$$y(x) = \left(1+x^2\right)^{-\frac{3}{2}}$$

$$\frac{dy}{dx} = -\frac{3}{2}\left(1+x^2\right)^{-\frac{5}{2}} \cdot 2x = -\frac{3x}{\left(1+x^2\right)^{\frac{5}{2}}}$$

$$\frac{dy}{dx} = -\frac{3x}{\left(1+x^2\right)^{\frac{5}{2}}}$$

$$\frac{d^2y}{dx^2} = -\frac{3\left(1+x^2\right)^{\frac{5}{2}} - 3x \cdot \frac{5}{2}\left(1+x^2\right)^{\frac{3}{2}} \cdot 2x}{\left(1+x^2\right)^5} = \frac{-3\left(1+x^2\right)^{\frac{5}{2}} + 3 \cdot 5x^2\left(1+x^2\right)^{\frac{3}{2}}}{\left(1+x^2\right)^5} = 3\frac{\left(1+x^2\right)^{\frac{3}{2}}\left[5x^2 - (1+x^2)\right]}{\left(1+x^2\right)^5}$$

$$\frac{d^2y}{dx^2} = 3\frac{\left(1+x^2\right)^{\frac{3}{2}}\left[5x^2 - (1+x^2)\right]}{\left(1+x^2\right)^5} = \frac{3(4x^2-1)}{\left(1+x^2\right)^{\frac{7}{2}}}$$

$$\frac{d^2y}{dx^2} = \frac{3(4x^2-1)}{\left(1+x^2\right)^{\frac{7}{2}}}$$

$$\left.\frac{dy}{dx}\right|_{x=0} = -\frac{3x}{\left(1+x^2\right)^{\frac{5}{2}}} = 0$$ Horizontal tangent is in the center.

$$\frac{d^2y}{dx^2} = \frac{3(4x^2-1)}{\left(1+x^2\right)^{\frac{7}{2}}} = 0 \qquad 4x^2 - 1 = 0 \qquad x = \pm\frac{1}{2}$$ Inflexion point is here.

The function value at the inflexion point is:

$$y\left(x = \frac{1}{2}\right) = \left(\frac{5}{4}\right)^{-\frac{3}{2}} \approx 0{,}72$$

The slope value at the inflexion point is:

$$\left.\frac{dy}{dx}\right|_{x=\frac{1}{2}} = -\frac{3x}{\left(1+x^2\right)^{\frac{5}{2}}} = -\frac{\frac{3}{2}}{\left(\frac{5}{4}\right)^{\frac{5}{2}}} = -\frac{1{,}5}{(1{,}25)^{2{,}5}} \approx -0{,}86$$