Problem set 0.

Recurrences

For all the problems below, try to devise a method to solve it. If possible, try to write a recurrence for your solution of the problem.

1. Given a $2 \times n$ chessboard, we want to tile it with dominos. How many ways are there of doing it?
2. (Tower of Hanoi) There are three vertical columns in Hanoi, the first of them contains disks numbered $1, 2, ..., n$ from top to bottom. The second and the third column are empty. You have to transfer all the disks to the second column, given that you can only move one disk at a time and that the disks on any column at any time must be in increasing order of their numbers from top to bottom.
3. Given a $2 \times n$ chessboard, we cover it with n red squares and $n - 1$ blue squares in a random order. (So exactly one square will be left uncovered). We want to now rearrange the squares so that all the red squares are on the top row. The only legal move allowed to achieve this is to move an adjacent colored square to the empty square. Can we do this?
4. How many correctly parenthesized expressions are there on n pairs of parenrhesis?
5. How many regions do n lines divide the plane into? (A region is a connected portion of the plane left after removing all the lines)
6. (Josephus problem) You are given numbers $1, 2, ..., n$ written around a circle in clockwise order. Starting from 1 we eliminate every second number that has not been eliminated so far. (So, first 2 is eliminated, then 4 and so on). What is the number that survives this elimination process?
7. Balls are distributed in n boxes, such that in the k^{th} box there are k balls. In one step we can choose an arbitrary subset of the boxes and take out the same number of balls from each. How many steps are needed to empty out all the boxes.

Induction Practice

For all the problems below, guess a solution to the summation and prove it by Induction. Remember to check the Base case, state the Induction Hypothesis and then check the Induction step. Assume that all problems below are stated for $\mathbb{N}$.

8. $\sum_{i=1}^{n} i$
9. Show that $\sum_{i=1}^{n}(-1)^i i^2 = \frac{(-1)^n n(n+1)}{2}$
10. $\sum_{i=1}^{n} \frac{1}{i(i+1)}$
11. $\sum_{i=1}^{n}(2i - 1)$
12. $\Pi_{i=2}^{n}(1 - \frac{1}{i^2})$