Problem set 12.
NP-completeness.

1. Prove that the following language is NP-complete.
   ALMOST-HAMCYCLE = $\{G|G$ is an undirected graph with a cycle on all but one vertex. $\}$
   (So if the graph has n vertices and a cycle on $n-1$ vertices then it belongs to ALMOST-HAMCYCLE).

2. Assume $P \neq NP$, and that X is a language such that $X \in P$. Is it possible that,
   (a) X has a Karp-reduction to a NP-complete problem Y (i.e. $X \leq_P Y$)?
   (b) there is a Karp-reduction of a NP-complete problem Y to X (i.e. $Y \leq_P X$)?
   (c) X is a language in the class NP?

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3. Show that the following language is NP-complete: 3EQUAL-COLOR = $\{G|G$ is an undirected graph with a proper 3-coloring such that each color class has the same number of vertices. $\}$

4. Prove that TSP (travelling salesman problem) is NP-complete.
   TSP = $\{(G,k)|G$ is an undirected weighted graph, k a positive integer, and G has a hamiltonian cycle with total weight at most k. $\}$

5. Show the following Karp-reductions:
   (a) SUBSETSUM $\leq_P$ 3-COLOR
   (b) CONNECTED $\leq_P$ HAMPATH
   (r) CONNECTED $\leq_P$ BIPARTITE
   where, CONNECTED = $\{G|G$ is an undirected connected graph $\}$ and BIPARTITE = $\{G|G$ is an undirected bipartite graph $\}$.

6. An organisation of n people has b different committees. We want to find a schedule for the committee meetings. Two different committees can meet on the same day if they don't have a common member. We are given a positive number k and a list of names of members of each committee. We want to decide if the b committee meetings can be held in at most k days. Provide a polynomial time algorithm for this problem, or prove that it is NP-complete.

7. Decide if the following problems are in P or are NP-complete:
   (a) ATLEAST-kPATH = $\{(G,k)|G$ is an undirected graph with a path of length at least $k\}$
   (b) kPATH = $\{(G,k)|G$ is an undirected graph with a path of length exactly $k\}$
   (c) ATMOST-kPATH = $\{(G,k)|G$ is an undirected graph with a path of length at most $k\}$

8. We know that $X_1 \leq_P X_2$. Assume further that we have a Karp-reduction of the complement of X_2 to PARTITION. Prove that $X_1 \in$ coNP.

9. Prove that SUBSET-SUM is NP-complete using the fact that PARTITION is NP-complete.
   SUBSET-SUM = $\{(s_1, s_2, ..., s_n, b)|s_i$'s and b are positive integers and there is a subset of the s_i's whose sum is exactly $b\}$
   PARTITION = $\{(s_1, s_2, ..., s_n)|s_i$'s are positive integers and there is a partition of the s_i's into two subsets of equal sum $\}$

10. Prove that KNAPSACK is NP-complete using the fact that SUBSET-SUM is NP-complete.

11. Is it possible that CONNECTED $\leq_P$ HAMCYCLE? Is it possible that HAMCYCLE $\leq_P$ CONNECTED?

12. Determine if the following language is in P or NP-complete: 12-4COLOR = $\{G|G$ is an undirected graph with a proper 4-coloring such that two color classes have exactly 1 and 2 vertices in them respectively $\}$.