Problem set 5.

DFS vs BFS, DAG, Topological ordering, Shortest paths in DAG.

1. We are given the adjacency lists of two directed (edge-)weighted graphs G_1 and G_2.
   G_1: a:b(3),c(8); b:d(-7); c:d(5); d:e(2); e:a(-10);
   G_2: a:g(2),f(10); b:a(-2),g(1); c:-; d:-; e:c(5),d(6); f:e(7); g:f(1), e(8);
   (a) Use the DFS algorithm to determine which of them is a DAG.
   (b) For the graph that is acyclic, find a Topological ordering and use that to find the shortest and longest paths from a to all the other vertices.

2. In a directed graph G given by an adjacency list, some vertices are red, some vertices are blue, and the remaining vertices are colorless. (The coloring is given in an array C indexed by vertices). Which algorithm can be run once in $O(n + m)$ steps to solve the following problems and how?
   (a) We want to decide which blue vertices are reachable from a given red vertex p_1 (via a directed path).
   (b) Of all the blue vertices, we want to determine the ones nearest to a given red vertex p_1.

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3. Let G be an undirected connected graph. Is it true that
   (a) For each edge f of G, it is possible to run the DFS on G from some vertex such that f will be a tree-edge.
   (b) For each edge f of G, it is possible to run the BFS on G from some vertex such that f will be a tree-edge.
   (c) For every spanning tree F of G, it is possible to obtain F as a BFS-tree (that is, all edges of F are tree edges).
   (d) For every spanning tree F of G, it is possible to obtain F as a DFS-tree (that is, all edges of F are tree edges).

4. We are given the adjacency list of a directed graph G with no directed cycles. Given a vertex s of the graph, we want to determine the number of paths from s to v for all vertices v in the graph. Give an algorithm with running time $O(n + m)$ for this task.

5. Continuation of exercise 2:
   (c) We want to decide whether each blue vertex is reachable from at least one red vertex (i.e., there is a directed path to each blue vertex from some red vertex).
   (d) We want to find the shortest path in the graph that leads from a red vertex to a blue vertex. (Note: the starting red vertex, and the ending blue vertex are not fixed).

6. Circus acrobats stand on each other's shoulders and want to build a tower as big as possible (there will be only one acrobat on each level). For aesthetic and practical reasons, only an acrobat who is both lighter and smaller can stand on the shoulder of another acrobat. In the circus, there are n acrobats, and the height and weight of each is given.
   (a) Give an algorithm with running time $O(n^2)$, that gives the maximum number of people that can stand in a tower.
   (b) Give an algorithm with running time $O(n^2)$, that returns a subset of the acrobats that constitute a tallest possible tower.

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7. We are given the adjacency list of a weighted graph, where the edges have weights $1, 2, 3$. Give an algorithm with running time $O(n + m)$ to determine the length of the shortest paths from a given vertex s to all other vertices in the graph.

8. We are given the adjacency list of a directed graph G with weights (negative edges are allowed), in which there are no directed cycles. Some vertices of the graph are colored purple, the other vertices are colorless, and one s (colorless) vertex of the graph is given. For all vertices v of the graph, we wish to determine the length of the shortest path from s to v, with the further restriction that the path may contain at most one purple vertex. Give an algorithm with running time $O(n + m)$ for this task.

9. We have n files, and let h_i denote the length of the i^{th} file. Assume that the numbers h_i are integers. We have two disks of equal size L (L is a positive integer) that we will use to store these files. Our goal is to determine the largest number k, such that we can store the first k files on the two disks (the order of the files is fixed). Files should not be split, each file is written in its entirety to one of the two disks. Given the numbers L and $h_i, 1 \leq i \leq n$, give an algorithm that determines which disk should be used to store the i^{th} file so that the number of files stored (k) is the largest possible. The running time of your algorithm should be $O(L^2)$.

10. We are given a black and white image of $n \times n$ pixels. We would like to draw boundary line from the upper left corner to the lower right corner of the image in such a way that the sum of the number of black pixels above and to the right of the line and the white pixels below and to the left of the line is as small as possible. Determine this boundary in running time $O(n^2)$.

11. We are given a weighted directed graph $G(V, E)$ on n vertices, where the weight of the edges (denoted by $c(e)$) is positive ($c(e) > 0$). From a given vertex $s \in V(G)$, we want to find the shortest path to another given vertex $t \in V(G)$, but the length of the path is computed a little unconventionally as follows: if e is the k-th edge of the path from s, then it adds $k \cdot c(e)$ to the length of the path. Give an algorithm that determines the length of the shortest path (length computed in the above sense) with running time $O(n(n + m))$.