Exercise-set 2.

1. Let the vertex set of the simple graph be $V(G) = \{2, 3, 4, \ldots, 21\}$. Let two different vertices be adjacent if and only if the smaller of them divides the larger one.
   a) How many components does G have?
   b) Give a spanning tree in each component of G.
   c) Determine the distances of the pairs of vertices in G.

2. Let the vertices of the graph G be all the 0-1 sequences of length 4, and two vertices be adjacent if and only if the corresponding sequences differ in exactly two digits.
   a) How many components does G have?
   b) Give a spanning tree in each component of G.
   c) Determine the distances of the pairs of vertices in G.

3. a) In a graph on n vertices all the degrees are at least $\frac{n}{2}$. Does it follow that the graph is connected?
   b) And if we suppose that the graph is simple?

4. (MT++'06) In a simple graph on 100 vertices each degree is at least 33. Show that we can add one edge to the graph in such a way that the resulting graph is connected.

5. (MT++'17) In a simple graph on 23 vertices the degree of each vertex is at least 7. Show that no matter how we choose three vertices of the graph, there will be a path between two of them.

6. (MT++'15) In a simple graph G on 20 vertices the degree of 10 vertices is 5, and the degree of the remaining 10 vertices is 14. Is the complement of G connected?

7. Show that for a simple graph G either G or its complement $\overline{G}$ is connected.

8. (MT++'16) Let G be a simple graph and $v \in V(G)$ be a vertex of odd degree. Show that there is a path in G which starts at v and ends in a vertex of odd degree different from v.

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9. Sketch all the non-isomorphic trees on 3, 4, 5 and 6 vertices!

10. a) In a tree the degree of each vertex is 1 or 2 or 3. How many vertices of degree 1 are there if there are 5 vertices of degree 3?
    b) Draw two such trees in which the number of vertices of degree 2 are different.

11. (MT++'10) In a tree only two kinds of degrees occur, one of them 9 times, the other one 92 times. What are the two degrees?

12. (MT+'19) Show that there is no such tree on $n \geq 5$ vertices which has only two kinds of degrees, each of them occurring exactly $\frac{n}{2}$ times (n is even).

13. (MT++'22) A tree on 10 vertices contains two vertices of degree 5. Show that these two vertices are adjacent to each other.

14. Show that if all the degrees in a tree are odd, then the number of edges is also odd.

15. How many vertices does the tree T have if the number of its edges is exactly one-tenth of the number of edges of its complement?

16. (MT+'08) Determine all the trees (on at least two vertices) which are isomorphic to their complement.

17. A cycle-free graph on 100 vertices has 80 edges. How many components can it have?

18. A graph on 20 vertices has 18 edges and 3 components. Show that exactly two of its components are trees.

19. The maximum degree in a tree is Δ. Prove that the tree has at least Δ vertices of degree 1.

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20. (MT'18) Let the vertex set of the simple graph be $V(G) = \{1, 2, \ldots, 10\}$. Let the vertices $x, y \in V(G)$, $x \neq y$ be adjacent if and only if $|x - y| \leq 2$. Does G have a spanning tree, which
    a) contains all the edges $\{x, y\}$ of G for which $x, y \leq 3$ holds;
    b) contains all the edges $\{x, y\}$ of G for which $|x - y| = 2$ holds?

21. (MT'17) A simple connected graph on 100 vertices has 102 edges. Show that the graph contains three pairwise different cycles. (Two cycles are different if their edge sets are not the same.)

22. (MT+'17) A simple connected graph on 100 vertices has 100 edges. Show that the graph contains three pairwise different spanning trees. (Two spanning trees are different if their edge sets are not the same.)

23. Let G be a simple connected graph and e, f be two of its edges. Show that G has a spanning tree which contains both e and f.

24. Show that every connected graph contains a vertex whose deletion doesn't disconnect the graph.