Lecture notes ITC2

Binds

>

G_s - simple graph

What is a graph

Def'n: simple grapth

> a simple grapth G=(V,E) is a pair of sets.
>
> 1. V = sets of vertices of G (points)
> 2. E = set of edges, which are unordered pairs (2-elements subsets) of V

Remark:
This definition doesn't allow:

1. parrallel or multiple edges $e_1 = \{u, v\}, \ e_2 = \{v, u\}$ (the same)

2. loop $e = \{u, u\}$ (not an unordered pair)

If we want to contain multiple edges and loops we use multiple sets.

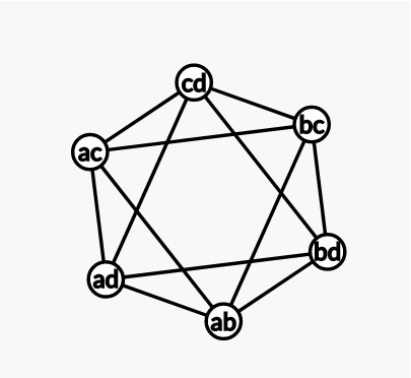
Terminology and notation

> $e = \{u, v\}$:
>
> - e is incidnet tu u, v
> - endpoints of e are u,v
> - u,v are adjacent
>
> G = (V,E) $V \neq \emptyset$ V is finite
> V = V(G), E=E(G) - emphsising G
> |V|, |E| - cardinlity

Example

> 4 people, pairs of people. Adjecent if ∃ a common person.
> V = pairs
> E = common person

v = {ab,ac,ad,bc,bd,cd} |V| = 6



Def'n: Degree of vertex v of G

> = # of edges incident to v (!loops are counted twice)
> $d_G(v)$

Handshake lemma

> $$\sum_{v \in V(G)} d(v) = 2|E(G)|$$

Proof:
Each handshake (=edge) is counted twice for each person(=vertices)

Corollary

> The # of odd degree vertice in any graph should be even

Proof:
Otherwise $\sum d(v) = odd$ but $\sum d(v) = even = 2n$ contradiction

Def'n: Directed Grapths

> G=(V,E) is a directed grapth, if the edges of G are ordered pair of verteces
> $e = (u, v) \neq (v, u)$
>
> - u-starting | inicial point fo e
> - v-endpoint of e

> ind(v) = # edges with the v as inicial point
> outd(v) = # number of edges with v as endpoint

$$\sum_{v \in V(G)} ind(v) + \sum_{v \in V(G)} outd(v) = |E| + |E| = 2|E|$$

Def'n: Compliment

Compliment of $G = (V, E)$ is the grapth $\overline{G} = (V', E')$

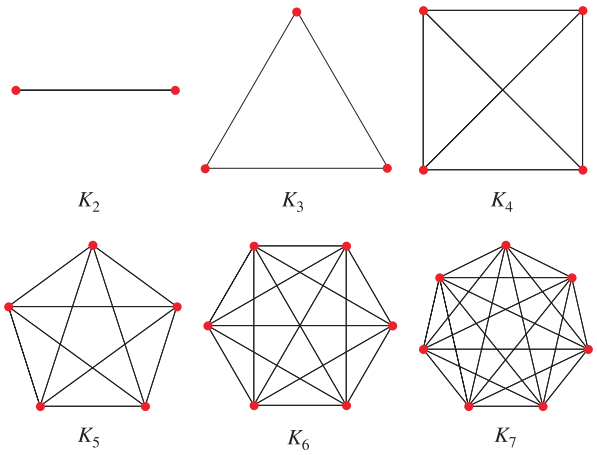
- $V' = V$
- $E' = \{v, u\} \notin E(G) \quad \forall u, v \in V$

Def'n: Complete grapth

The complete grapth $K_n = (V, E)$:

- $|V| = n$
- $E(K_N)$ = all the possible pairs of V

$$E(G) \cup E(\overline{G}) = E(K_{|V|})$$



Proposition

$d_{K_n}(v) = n - 1 \quad \forall v \in V(K_n)$

$$|E(K_n)| = \frac{\sum d(v)}{2} = \frac{n(n-1)}{2}$$

Terminology:

$d_G(v) = k \quad \forall v \in V(G) \iff$ G is k-regular grapth

Corollary

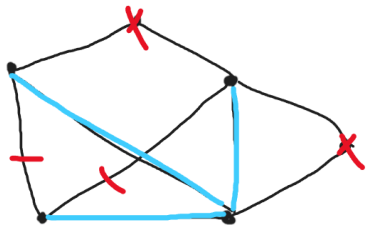
$d_{\overline{G}}(v) + d_G(v) = d_{K_n}(v) = n - 1$
$|E(\overline{G})| + |E(G)| = |E(K_n)| = \frac{n(n-1)}{2}$

Def'n: Subgrapth

> Subgrah of grapth $G = (V, E)$ is $G' = (V', E')$
>
> - $V' \subseteq V$
> - $E' \subseteq E$
> - $e = \{u, v\} \in E' \implies u, v \in V'$

Def'n: Deletion

> deletion of vertex from G $\implies$ delete all the edges incident to it
> deletion of an edge from G $\implies$ delete the edge keep the endpoints
> Than a subgrapth can be obtained by a sequence of deletion of vertexes and edges form G.



Def'n: Induced Subgrapth

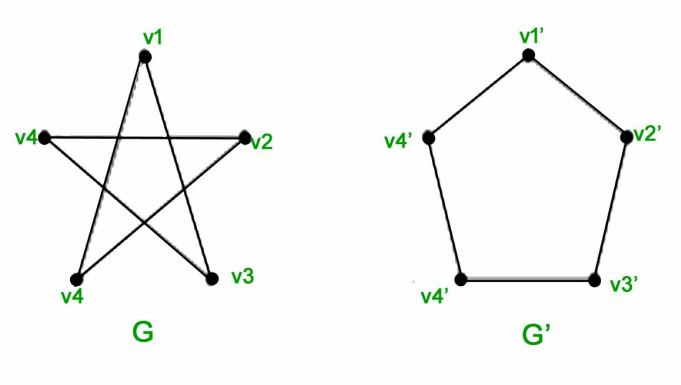
> if we only delete verteces from G

Def'n: Spaning Subgrapth

> if we only delete edges from G

Def'n: Isomorpthism

> $G = (V, E) \cong G' = (V', E')$
> $\exists$ 1-to-1 mapping $f : V \longleftrightarrow V'$
> $\{u, v\} \in E(G) \iff \{f(u), f(v)\} \in E'(G')$



Connectedness of graph

Def'n Walk

> Walk $\in$ G=(V,E) = $(v_0, e_1, v_1, e_2, v_2 \ldots e_k, v_k)$
> A closed walk $v_k = v_0$

> length(walk)=#of edges

Def'n Trail

> Trail $\in$ G=(V,E) = a walk without repeted edges
> A closed trail = circuit

Def'n Path

> Path $\in$ G=(V,E) = a walk without repedted verteces ($\implies \nexists$ repeted edges)
> A closed path = cycle $v_k = v_0$

$C_0 \iff$ vertex
$C_1 \iff$ loop
$C_2 \iff$ parrallel edges
$C_1 \iff$ triangle
G is simple $\iff C_n \quad n \geq 3$

Walk has Path Theorem

> $\exists W_{u \to w} \implies \exists P_{u \to w}$

Proof:
let $\exists W_{u \to w} \implies$ repeted verteces $\implies$ May $\exists$ repeted edges $\in v_i \leftrightarrow v_i$
If we remove the right repeted edge and the edges in between we get a shorted walk $\implies$ termination.

Corollary

> $(a,b,c) \in V(G)$
> $\exists \ P_{a \to b}, P_{b \to c} \implies \exists P_{a \to c}$

Proof:
$\exists \ P_{a \to b}, P_{b \to c} \implies P_{a \to b} \cup P_{b \to c} = W_{a \to c} \implies \exists P_{a \to c}$

Def'n Connected G

> $G = (V,E) = connected \iff \forall v_i, v_j \in V(G) \ \ \exists P_{v_i \to v_j}$

Remark - we can use Walk's

Def'n Component

> Component $G = (V,E) =$ maximal connected induced subgrapth

Remark: G = connected $\iff$ 1 component

Different aproach of Component

> $v \in V(G) \implies V_I(G_I) = \forall u \in V(G) \ \ s.t \ \ \exists P_{v \to u}$

Preposition

> C comp of v $\iff$ C comp of G`

Proof:

1. Maximal
   $\exists u \in V(G),\ \exists P_{v \to u} \implies u \in V_I(G_I)$ so you can not extend $\implies$ maximal
2. Connected
   $\forall u_i, u_j \in V_I \quad \exists P_{u_i \to v}, P_{v \to u_j} \implies \exists P_{v_i \to v_j}$ Walk has Path theorem

Corollary

> Components of G partition V(G) and E(G)

Adding new edge Theorem

> $G = (V, E) + e\{u, v\} \implies$
>
> 1. $u, v \in C_n^o \implies new\ C$
> 2. $u, v \notin C_n^o \implies C_{n_i}^o \cup C_{n_j}^o$

Proof:

1. $u, v \in C_n^o \implies \exists P_{u \to v} + e\{u, v\} \implies new\ C$
2. $u, v \notin C_n^o \implies u \in C_i^o\ v \in C_j^o + e\{u, v\} \implies \exists P_{u \to v}$
   $\forall a \in C_i^o\ \forall b \in C_j^o \implies \exists P_{a \to u}\ \exists P_{u \to v}\ \exists P_{u \to b} \implies \exists P_{a \to b}$
   $\implies C_{n_i}^o \cup C_{n_j}^o$

Def'n Tree T_n

> $Tree$ is a $G(V, E)$:
>
> 1. Connected
> 2. No Cycles

Remark:
Tree := No Cycles $\iff$ simple grapth

Number of Leaves ≥ 2 Preosition

> Leaves = vertex s.t d(v)=1
> Tree $|V(T)| \geq 2 \implies$ # leaves ≥ 2

Proof:
Consider maximal path $\implies$ 2 endpoints are leaves
if the weren't $\implies$ another edge incidnet to endpoins:

1. The endpoint is not in the path $\implies$ longer paths $\Rightarrow\!\!\!\!\!\!\not\Leftarrow$
2. The endpoint is on the path $\implies$ Cycle $\Rightarrow\!\!\!\!\!\!\not\Leftarrow$

$\implies$ the d(endpoints)=1 $\implies$ leaves

A T_n has n-1 edges

> $|V(T_n)| = n - 1$

Proof:
n isolated verteces = n coponents + edge eges $\implies$ Adding new edge Theorem + def'n tree (no cycle + connected) $\implies$ n-1 edges

Def'n Forest

> $F := G(V, E) : \not\subset$

F_n with k componets had n-k edges Proposition

> $C_1^o, \dots C_k^o \in F_n(V_f, E_f) \implies |V_f| = n - k$

remark:
$n = 1 \implies |E(G)| = n - 1 \implies F_n = T_n$

Proof:
Start with isolated edges and edd edges 1 by 1 between components by adding a new edge theorem we get:
0 edge -> n components
1 edge -> n-1 components
....
n-k edge -> k componets

The maxima cycle free (F) grapth = T

> $max|E(F)| = n - 1$ when F=T

Def'n Spanning Tree

> $T_S \ of \ G := G_s \ of \ G : G_s = T, \ |V(G_s)| = |V(G)|$

remark:

1. only connected grapths can have a spanning tree
2. if G is not connected $\implies$ spanning forest

Theorm Containing a Spaning Tree and Connectivity

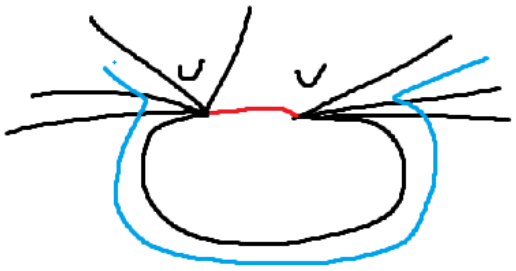
> $T_S \in G = (V, E) \iff G = connected$

Proof:
$\rightarrow$
$T_S \in G = (V, E) \ \ T_s = connected, spanning \implies G = connected$
$\leftarrow$

1. add edges 1 by 1 from E(G)
2. braking cycles by removing edges until we get a cycle free grapth

Breaking the cycle doesn't lose conectivity Lemma

let u, w be part of C
ifven if you remove $e = \{u, w\}$ there still exits a $P_{u \to w}$. So all the paths that went through this edge can go from the other path.



You repeted braking cycles untill you stop.

Corollary Nuber of edges of the connected grapth

> $|E_c(G_c)| \geq n - 1$
> $|E_c(G_c)| = n - 1 \implies G = T$

Proof
If G connected it contains a spanning tree. $\leftarrow$ of the theorem.
Break the cycles.
spanning tree "minimally connected"
Spanning Tree the optimal way to connect all the edges.

Theorm Equivalence G = T

> 1. $G = T \;\; cycle - free, connected$
> 2. $G = connected, |E(G)| = n - 1$ Corollary Nuber of edges of C_c
> 3. $G = cycle - free, |E(G)| = n - 1$ Forest with n-1 edges = Tree
> 4. $C = connected \;\; \forall e = bridge$ (if we delete we get two componets)

proof:
4. The same as $|E(T)| = n - 1$

Eulerian Trail

Def'n Eulerian Trail and Circuit

> $|E(G)| \in T_{rail} \iff$ Eularian Trail
> $|E(G)| \in C_{ircuit} \iff$ Eularian Circut

Reamark:
Eularian Circuit is an Eulerial Trail

Eulerican circut conditions

> Eulerian Circuit $\iff$ G is connected, $\forall v \in V(G) \;\; \deg(v) = even$

Proof
$\rightarrow$ nessecity
G is not connected we cant get from one compoenet to the other $\implies$ our tail can contain all the edges
Eulerian Circiut $\implies$ you go in and out in each vertex where the trail passes $\implies$ $\deg(v) = even$

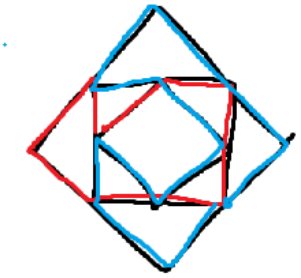
$\leftarrow$ sufficiency
We need an algorithm.

Lemma

> $\forall v \in V(G) \;\; \deg(v) = even \implies$ maximal trail can only end in the starting vertex

Proof:
Degree even means you can alwasy leave from the vertexes accept the first if you use all the edges in it.

After tranversing a mazimal path still all the uncompleted verteces left have even degrees.



Eulerican Trail conditions

> Eulerian Trail $\iff$ G is connected (apart form isolatex vertex), $\exists$ 0 , 2 v $\deg(v) = odd$

Proof:
$\rightarrow$ nessecity
Same but strating and end degress are even\

$\leftarrow$ sufficiency
$\forall v \in V(G) \;\; \deg(v) = even \implies$ Eulerian Circuit which is an eulerian trail.
$\exists$ 0 , 2 v $\deg(v) = odd$
Connect them and use the previous algorithm. Strat the new connected edge $e = \{v, w\}$

Condition for directed grapth

> A directed grapth G contains a directed Euler Circuit.
>
> 1. G is connected (execpt exolated vertex)
> 2. indeg(v) = outdeg(v) $\forall v \in v(G)$

deg(v) = indeg(v)+outdeg(V) = 2n = even

Hamiltonian Cycles and Trails

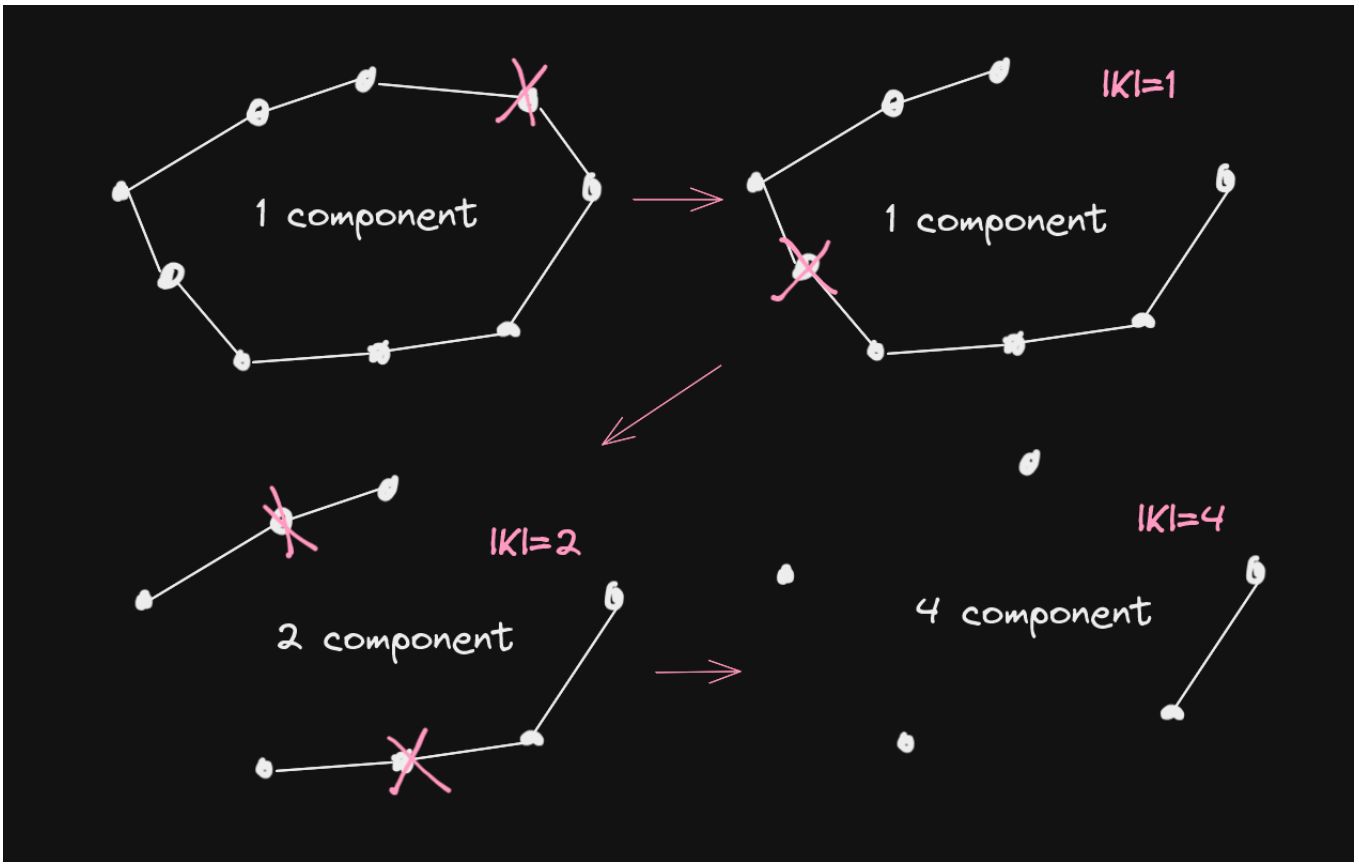
Def'n Hamilton Cycle

> $H_c \subseteq G = (V, E) := C \;\; s.t \;\; \forall v \in V(G) \;\; v \in C$ exacly once
> $H_p \subseteq G = (V, E) := P \;\; s.t \;\; \forall v \in V(G) \;\; v \in P$ exacly once

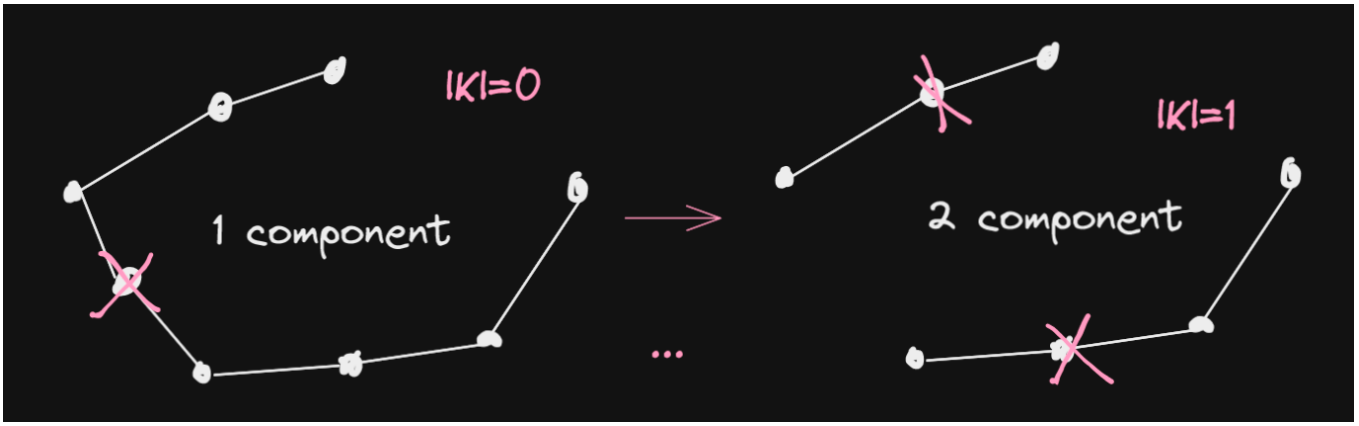
Nessesary conditions

> $H_c \subseteq G \quad K = \{v_1, \ldots, v_k\} \;\; \forall v \in V(G) \implies \#comp(G - K) \leq |K|$
> $H_p \subseteq G \quad K = \{v_1, \ldots, v_k\} \;\; \forall v \in V(G) \implies \#comp(G - K) \leq |K| + 1$

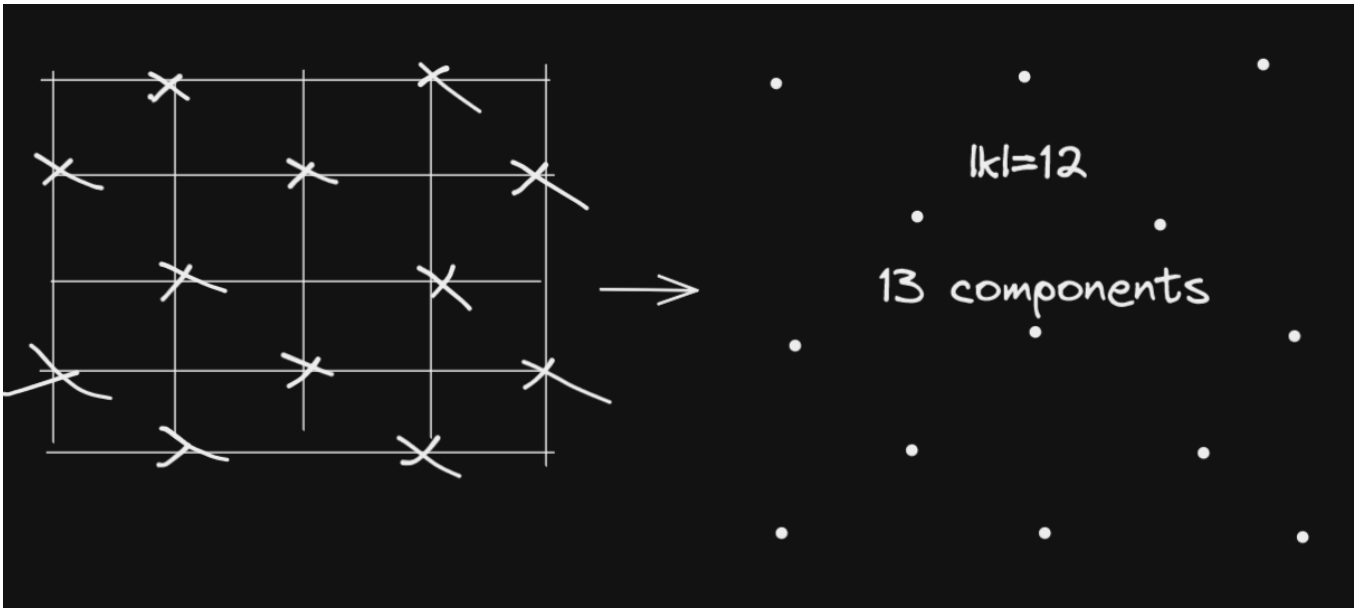
Proof 1:



Proo2:



Example



> You can only disprove with nessesary conditions

Sufficient Conditions

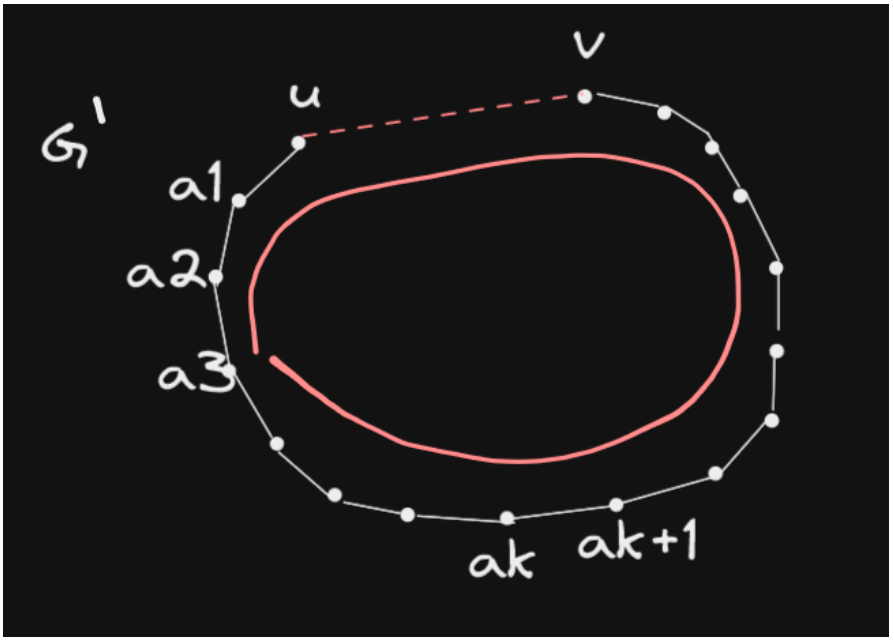
The Ore Theorem

> $G_s \quad V(G_s) \geq 3 \qquad \forall u, v \in V(G_s) \mid u \nleftrightarrow v \quad \deg(v) + \deg(u) \geq n \implies H_c \in G_s$

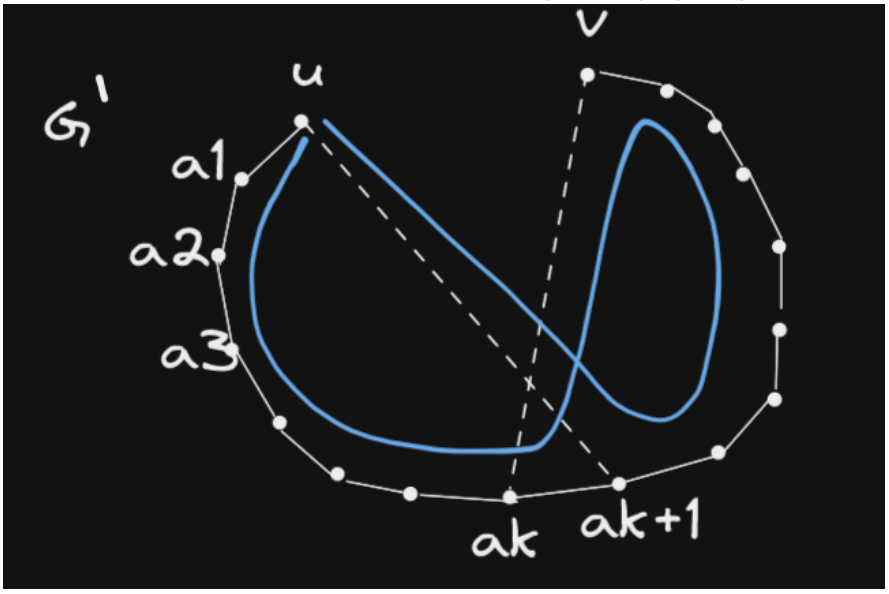
Def'n Saturation of G

> $G = (V, E) + eges \mid$
>
> 1. still has no H-cycle
> 2. but by ading a new one it would
>
> $\implies G` := staturation$

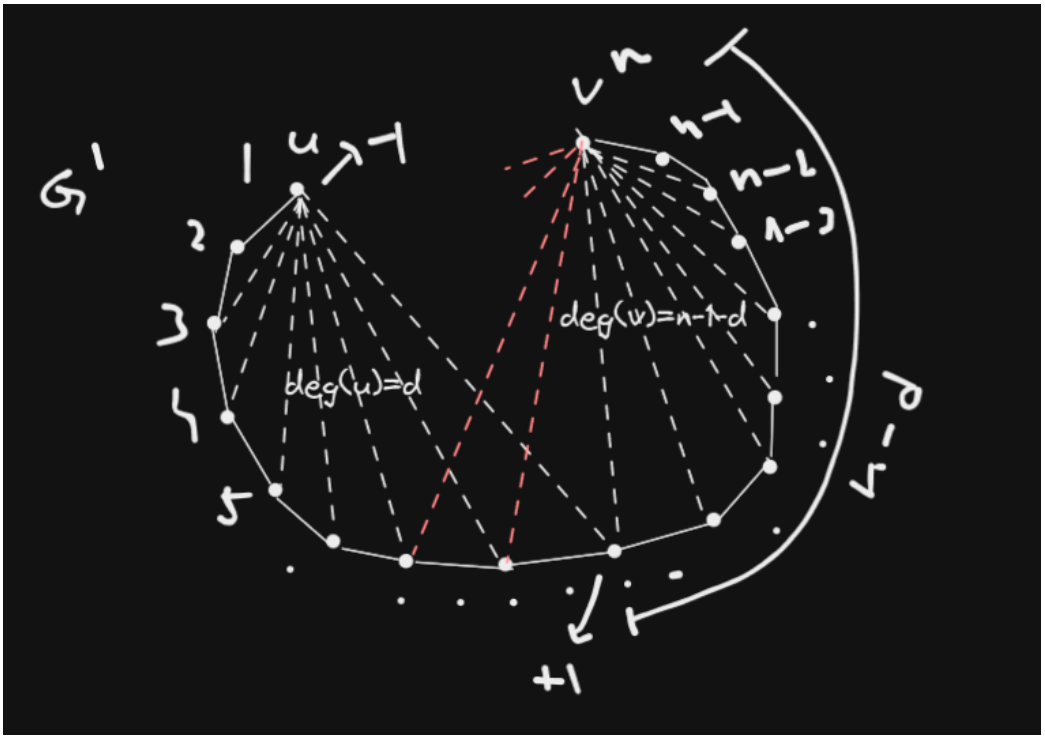
Proof:
Contradiction: $\deg(v) + \deg(u) \geq n$ but there $\nexists H_c \subseteq V$.
We take G and create the staturation of it $G`$
if $u \nleftrightarrow v$ in $G` \implies H_c \subseteq \ G` + \{u, v\}$ by defn of Saturated grapth.



If you have any edge og the from below $\{u, a_{k+1}\}, \{v, a_k\}$ you hace a H_c



This implies:



$\deg(v) + \deg(u) \leq n - 1$ Contradiction.

Dirac Theorem

> $G_s \quad V(G_s) \geq 3 \qquad \forall u \in V \quad \deg(u) \geq \frac{n}{2} \implies H_c \in G_S$

Proof:
Corollary of Ore's Thorem
$\forall u \in V \quad \deg(u) \geq \frac{n}{2} \implies \deg(v) + \deg(u) = \frac{n}{2} + \frac{n}{2} = n \ \ \forall u, v \in V + Ores \ \ Th.$
$\implies H_c \in G_s$

Remark:
Only used to prove if there is. Sufficient Not Necessary.
Cycles. $H_c \subseteq C_5 \quad \deg(v) = 2 \quad \forall v \in V(C_5) \ \ but \ \ \deg(v_1) + \deg(v_2) = 4 < n.$

Graph Coloring

Def'n Proper Coloring

> A proper corloring of verteces := assigment of color to $\forall v \in V(G)$ s.t
> if $v_1 \leftrightarrow v_2$ then $v_1.color() \neq v_2.color()$

remark:

1. loops -> can not color vertex
2. multiple edge -> we can color them

Def'n chromatic number

> $\chi(G) := \text{minOfColors}(\text{properlyColor}(\text{G}))$

$\chi(G) = k \implies$ properlyColor(G) $\geq$ k

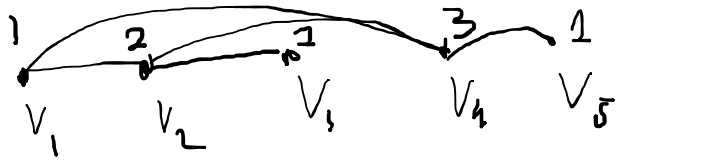
K_n coloring

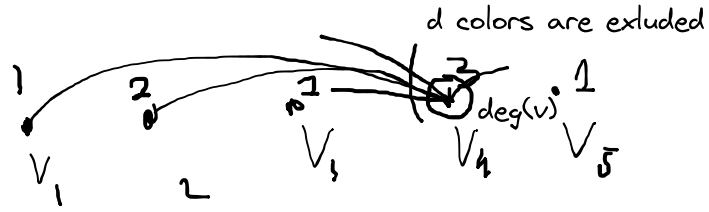
1. color(K_n)=n possible each vertex has a color $\implies \chi(X) \leq n$
2. color(K_n)=n-1 not possible because $\forall v \in V(K_n)\ \ v \leftrightarrow v_n$ threfore you need at lest n colors
   $\implies \chi(X) \geq n$
   $\implies \chi(K_n) = n$

C_n coloring

$$\chi(C_N) = \begin{cases} 2, n\ \ even \\ 3, n\ \ odd \end{cases}$$

Greedy coloring





$$\leq d \leq deg(v_n) \leq \Delta(G)$$
$$\Rightarrow \chi(G) \leq \Delta(G) + 1$$

Bounds

> $\chi(G) \geq w(G)$:= size(max(K_nSubgrapth(G)))
> $x(G) \leq \Delta(G) + 1$ greedy coloring

Sharpness

> $\chi(G) = w(G) := sharp$
> $\chi(G) = \Delta(G) + 1 := sharp$

example:
$\chi(K_n) = w(K_n)$
$\chi(B) = w(B)$

$\chi(K_n) = \Delta(K_n) + 1$
$\chi(C_{2n+1}) = \Delta(C_{2n+1}) + 1$

But it can be very bad:

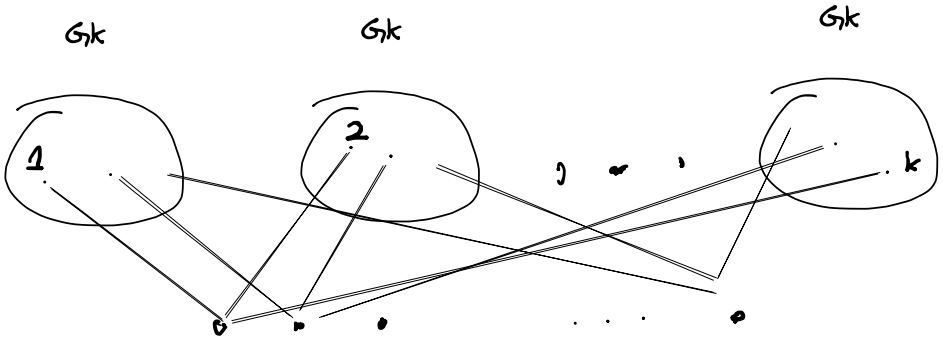
> $\chi(G) - w(G) = \infty$
> $\chi(G) - \Delta(G) - 1 = \infty$

Theorem

> $\exists\{G_n\}$ s.t $\chi(G_k) = k \;\; w(G_k) = 2$

Proof:
Zykov's construction

1. The starting graph $G_0 = C_5$: $\chi(C_5) = 3 \;\; w(C_5) = 2$
2. k new G_k



You can just work with one new vertex.
No edges between new vertexes $\implies w(G_{k+1}) = 2 \;\; \chi(G_{k+1}) \leq k+1$
$deg(new(v)) = k \implies \chi(G_{k+1}) \geq k+1$
$\implies \chi(G_{k+1}) = k+1$

For the greedy algorithm:
If you have $comp(B) = (A \cup B, E)$
order: a_1 b_1 a_2 b_2 a_3 b_3 ...
1 1 2 2 3 3
but $\chi(G) = 2$
order: a_1 a_2 a_3... b_1 b_2 b_3
1 1 1 2 2 2

Special cases

> $\chi(G) = 1 \iff E(G) = \emptyset$ (empty grapth - isolated verteces)
> $\chi(G) = 2 \iff B$:=bipartate grapth (and not empty)

Proof:
$\leftarrow$
$B \implies V(G) = A \cup B \quad A \cap B = \emptyset \implies a_i \nleftrightarrow a_j \;\; b_i \nleftrightarrow b_j \implies$
$color1 \to A \quad color2 \to B$
$\rightarrow$
The algorithm for 2 coloring

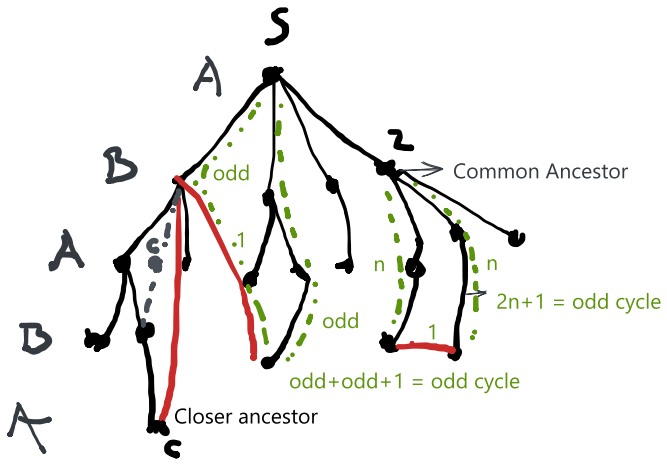
Def'n Bipartite graphs

> $G = (V, E) = B \iff V(G) = A \cup B \quad A \cap B = \emptyset$

Thm: $B = (A \cup B, E) \iff C_{2n+1} \nsubseteq B$

proof:
$\rightarrow$

$C_n = A - B - A - B - A \ldots = C_{2k} \implies C_{2n+1} \nsubseteq B$
$\leftarrow$



Remark:

1. Algorithm for bipartite coloring
2. Trees are bipartite (no cycles at all)

Def'n Interval Graphs

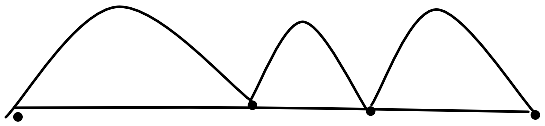
> $G = (V, E) :=$ interval graph:
>
> 1. vertices := finite closed intervals on a real time
> 2. edges := if the intervals overlap

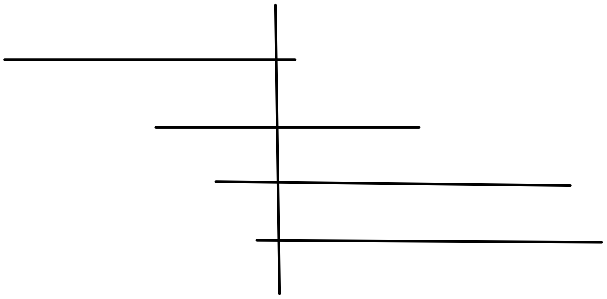
Thm: $I = (V, E) \implies \chi(G) = w(G)$

> $G := I = (V, E) \implies \chi(G) = w(G)$
> $\chi(G)$ = greedy coloring.

Proof:
Order the vertices s.t the the starting time is ordered.
$v_1, v_2, v_3, \ldots, v_k$ s.t $\quad t_1 \leq t_2 \leq t_3 \leq \ldots \leq t_k$

This way they will use $w(G)$ coloring.





All the vertices preceding v_i

- they all intersect v_i
- they intersect each other as well

  $\implies$ c+1 colors to color the clique $\leq w(G)$

  $\implies \chi(G) \leq w(G)$

  $w(G) \leq \chi(G) \quad \forall G$

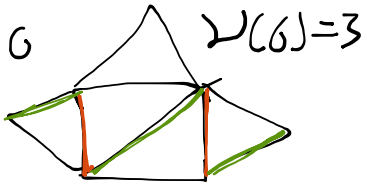
  $\implies \chi(G) = w(G)$

Breaths First Search

Graph Parameters

Def'n Matching

> Matching := set of independent edges
> $\nu(G) :=$ #edges (maxMatch(G))



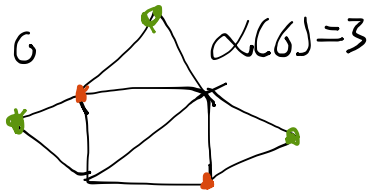
Lemma

> $\nu(G) \leq \lfloor \frac{|V(G)|}{2} \rfloor$

Perfect matching

$|M| = \frac{|V(G)|}{2}$

Def'n Stable set and $\alpha(G)$

> Stable set (independent set):= a set of pairwise non-adjacent vertices
> $\alpha(G) := max\#(independent(\forall v \in V(G))$
>
> 
>

Remark:

- Independent set of vertices $\implies$ Induced subgraph empty
- $\alpha(G) = \omega(\overline{G})$

Minimum weight Spanning tree

Kruskal's algorithm:

> $w(E(G)) \in R^+$
> ...

Proof: Contradiction
$T : w(e_1) \leq w(e_2) \leq \cdots \leq w(e_{n-1})$
$w(T_k) \neq min \implies \exists H$ s.t $w(H) = min.$
(we choose the one that has the most common edges)
$e_i \in E(H), e_i \notin E(T) \implies$ during the algo for T we didn't pick e_0
$\implies w(e_k) \leq w(e_i)$
$C \in T + e_i \quad k(H - e_i) = 2 \implies \exists e_k$ s.t $H - e_i + e_k = T_s$
$w(e_k) \leq w(e_i) \implies H - e_i + e_k \leq H$

- $w(e_k) < w(e_i) \implies H \neq mws$!!!
- $w(e_k) = w(e_i) \implies T_s = msw+$ one more edge in common !!!

$\alpha(G) + \chi(G) \leq n - 1$