Problem set 11.

P, NP, co-NP, Karp Reductions.

1. Prove that the following two languages are in the class NP. Which of these is in P? Which of them is in coNP?
   $P_1 = \{G | G \text{ is an undirected graph with a cycle on exactly 100 vertices. }\}$
   $P_2 = \{(G, k) | k \text{ is a positive integer and } G \text{ is an undirected graph with a cycle on exactly } k \text{ vertices. }\}$

2. Give a Karp-reduction from the problem 3COL to the problem 2023COL.

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3. For the following languages (of decision problems), prove that the first seven are in NP, the last two in coNP. Which of them are in P?

   $P_1 = \{(G, k) | G$ is a bipartite graph and k a positive integer, and G has a matching of size $k\}$
   $P_2 = \{G | G$ is an undirected graph with an Eulerian circuit. $\}$
   $P_3 = \{G | G$ is an undirected graph with an independent vertex set of size 2023$\}$
   $P_4 = \{(G, k) | G$ is an undirected graph, k a positive integer, and G has an independent vertex set of size $k\}$
   $P_5 = \{(s_1, \ldots, s_n, b) | b$ and all the s_i's are positive integers, and there is a subset of $(s_1, \ldots, s_n)$ such that their sum is $b\}$
   $P_6 = \{G | G$ is an undirected graph with chromatic number 2$\}$
   $P_7 = \{G | G$ is an undirected graph with a proper coloring with 2023 colors$\}$
   $P_8 = \{m | m$ is a positive integer and all its divisors are smaller than $\frac{m}{10}\}$
   $P_9 = \{m | m$ is a positive integer and is a prime number $\}$

4. Let MAXCLIQUE be the language of the decision problem in which we are given a graph G and a positive integer k and we want to decide if G has a clique of size k. Give a Karp reduction from P_4 in the previous problem to MAXCLIQUE.

5. Show that HAMCYCLE $\leq_P$ st-HAMPATH, where HAMCYCLE is the language of the decision problem that decides if a given graph G has a Hamiltonian cycle, while st-HAMPATH is the language of the decision problem which, given an undirected graph G and two designated vertices s and t, decides if there is a Hamiltonian path in G starting from s and ending in t.

6. Show that HAMPATH $\leq_P$ MAXDEG-SPANNINGTREE, where MAXDEG-SPANNINGTREE is the language of the decision problem where given a graph G and a positive integer k, it decides if G has a spanning tree with maximum degree k.

7. Show that HAMPATH $\leq_P$ MAXLEAF-SPANNINGTREE, where MAXLEAF-SPANNINGTREE is the language of the decision problem where given a graph G and a positive integer k, it decides if G has a spanning tree with at most k leaves.

8. Let FACTORIZATION be the language of the following decision problem:

   Input: Two positive integers m and t

   Question: does m have a divisor d such that, $1 < d \leq t$?

   (a) Show that FACTORIZATION is in NP.
   (b) Show that FACTORIZATION is in coNP. (You can use the fact that PRIMES is in P.)
   (c) Show that if $P = NP \cap coNP$, then show that RSA can be decrypted fast, because then prime factorization can be done in polynomial time.

9. Assume that we have an algorithm X that can, given a graph G and a positive integer k, tell in polynomial time if G has an independent vertex set of size at least k.
   (a) Design an algorithm (using X), that can determine $\alpha(G)$, the size of the maximum independent vertex set of G in polynomial time.

(b) Design an algorithm (using X), that will find an independent vertex set of G of size $\alpha(G)$ in polynomial time.

10. Assume that we have an algorithm X that can determine in polynomial time if a given graph G is 3-colorable. Provide an algorithm, using X, that finds a 3-coloring of a graph G (if it exists) in polynomial time.