Exercise-set 3.
Solutions

1. Not possible; possible.

2. a) $|V(G)| = \binom{8}{2} = 28$, $\deg(v) = \binom{6}{2} = 15\ \forall v \in V(G) \implies$ no Euler-circuit.
   b) $|V(G)| = \binom{6}{3} = 20$, $\deg(v) = 1+\binom{3}{2}\cdot\binom{3}{1} = 10\ \forall v \in V(G)$, and G is connected $\implies \exists$ Euler-circuit.

3. $|V(G)| = 2^4 = 16$, $\deg(v) = \binom{4}{2} = 6\ \forall v \in V(G)$, but G is not connected $\implies$ no Euler-circuit.

4. Construct a graph G: $V(G)$ = children, and u and v are adjacent $\iff$ not next to each other in the circle. This graph contains an Euler-circuit ($\deg(v) = 8\ \forall v \in V(G)$, connected). Most number of passes = length of an Euler-circuit = $|E(G)| = 44$.

5. Construct a graph G: $V(G)$ = digits = $\{0, 1, \ldots, 9\}$, and u and v are adjacent $\iff u+v \neq 9$. This graph contains an Euler-circuit ($\deg(v) = 8\ \forall v \in V(G)$, connected) $\iff \exists n$.

6. Construct a graph G: $V(G)$ = letters, and u and v are adjacent $\iff$ can stand next to each other. This graph contains an Euler-circuit ($\deg(v) = 30$ for vowels and $\deg(v) = 10$ for consonants, connected). Length of the longest sequence of letters = length of an Euler-circuit + 1 = $|E(G)|+1 = \binom{10}{2} + 10 \cdot 21 + 1 = 256$.

7. a) Add k new edges ($\implies \exists$ Euler-circuit), then delete them.
   b) No: each trail eliminates ≤ 2 odd degrees from G.

8. Equivalently: at least how many edges have to be deleted, s.t. the remaining graph contains an Euler-trail? At least 2 ($\exists$ 6 vertices of odd degree in G) $\implies$ length of a trail = $|E(G)| - 2 = 2 \cdot 4 + 5 \cdot 5 - 2 = 31$.

9. a), b) There can be at most 2 components $\implies$ adding one edge can make it connected, and the degrees will be OK.

10. r = 1,2,3,4,5,7,9 NO; r = 6,8 YES (each degree is even + connected).

11. YES: both components contain 2 vertices of odd degree.

12. YES, but only one (up to isomorphism), with 2 and 99 vertices in the classes.

13. Only for $\{D, H\}$.