Physics II.

Introduction

Present work is the summary of the lectures held by the author at Budapest University of Technology and Economics. Long verbal explanations are not involved in the text, only some hints which make the reader to recall the lecture. Refer here the book: Alonso/Finn Fundamental University Physics, Volume II where more details can be found.

Physical quantities are the product of a measuring number and the physical unit (dimension). In contrast to mathematics, the accuracy or in other words the precision is always an essential parameter of each physical quantity. Accuracy is determined by the number of valuable digits of the measuring number. Because of this 1500 V and 1.5 kV are not equivalent in terms of accuracy. They have 1 V and 100 V absolute errors respectively. The term relative error is the ratio of the absolute error over the nominal value. The smaller the relative error the higher the accuracy of the measurement. When making operations with physical quantities, remember that the result may not be more accurate than the worst of the factors involved. For instance, when dividing 3.2165 V with 2.1 A to find the resistance of some conductor, the result 1.5316667 ohm is physically incorrect. Correctly it may contain only two valuable digits, just like the current data, so the correct result is 1.5 ohm.

The physical quantities are classified as fundamental quantities and derived quantities. The fundamental quantities and their units are defined by standard or in other words by etalon. The etalons are stored in relevant institute in Paris. The fundamental quantities are the length, the time and the mass. The corresponding units are meter (m), second (s) and kilogram (kg), respectively. These three fundamental quantities are enough to build up the mechanics. The derived quantities are all other quantities, which are the result of mathematical operations. To describe electric phenomena the fourth fundamental quantity was introduced. This is Ampere (A) the unit of electric current. This is used extensively in Physics II, when dealing with electricity.

Chapter 1.

Electrostatic phenomena - György Hárs

1.1./ Fundamental experimental phenomena

Electrostatics deals with the phenomena of electric charges at rest. Electric charges can be generated by rubbing different insulating materials with cloth or fur. The device called electroscope is used to detect and roughly measure the electric charge. By rubbing a glass rod and connecting it to the electroscope the device will indicate that charge has been transferred to it. By doing so second time the electroscope will indicate even more charges. Accordingly, the same polarity charges are added together and are accumulating on the electroscope. Now replace the glass rod with a plastic rod. If the plastic rod is rubbed and connected to the charged electroscope, the excursion of the electroscope will decrease. This proves that there are two opposite polarity charges in the nature, therefore they neutralize each other. The generated electricity by glass and plastic are considered positive and negative respectively. The unit of the charge is called "Coulomb" which is not fundamental quantity in System International (SI) so "Ampere second" (As) is used mostly.

Now take a little (roughly 5 mm in diameter) ball of a very light material and hang it on a thread. This test device can detect forces by being deflected from the vertical. Charge up the ball to positive and approach it with a charged rod. If the rod is positive or negative the force is repulsive or attractive, respectively. This experiment demonstrates that the opposite charges attract the same polarity charges repel each other.

Now use neutral test device in the next experiment. Put the ball to proximity of the rod with charge on it. The originally neutral ball will be attracted. By approaching the rod with the ball even more the ball will suddenly be repelled once mechanically connected. The explanation of this experiment is based on the phenomenon of electrostatic induction (or some say electrostatic influence). By the effect of the external charge the neutral ball became a dipole. For the sake of simplicity assume positive charge on the rod. The surface closer to the rod is turned to negative, while the opposite side became positive. The attractive force of the opposite charges is higher (due to the smaller distance) than the repelling force of the other side. So altogether the ball will experience a net attractive force. When the rod connected to the ball it became positively charged and was immediately repelled.

1.2./ The electric field

In the proximity of the charged objects forces are exerted to other charges. The charge under investigation is called the "source-charge". To map the forces around the source-charge a hypothetic positive point-like charge is used which is called the "test-charge" denoted with q. By means of the test-charge, the force versus position function can be recorded. In terms of mathematics this is a vector-vector function or in other words force field $\mathbf{F}(\mathbf{r})$. Experience shows that the intensity of force is linearly proportional with the test charge. By dividing the force field with the amount of the test charge one recovers a normalized parameter. This parameter is the electric field $\mathbf{E}(\mathbf{r})$ which is characteristic to electrification state of the space generated solely by the source charge. The unit of electric field is *N/As* or much rather *V/m*. In Cartesian coordinates the vector field consists of three pieces of three variable functions.

$$\frac{\mathbf{F}(\mathbf{r})}{q} = \mathbf{E}(\mathbf{r}) = E_x(x,y,z)\mathbf{i} + E_y(x,y,z)\mathbf{j} + E_z(x,y,z)\mathbf{k}$$

One variable scalar functions *y=f(x)* are easy to display in Cartesian system as curve. In case of two or three independent variables a scalar field is generated. This can be displayed like level curves or level surfaces. To display the vector field requires the concept of force line. Force lines are hypothetic lines with the following criteria:

- Tangent of the force line is the direction of the force vector
- Density of the force lines is proportional with the magnitude (intensity) of the vector.

The positive test charge is repelled by the positive source charge therefore the electric field **E(r)** lines are virtually coming out from the positive source charge. One might say that the positive charge is the source of the electric field lines. (The outcome would be the same by assuming negative test charge, this time the force would be opposite but after division with the negative test charge the direction of the electric field would revert.) The negative source charge is the drain of the electric field lines due to symmetry reasons. So, the electric field lines start on the positive charge and end on the negative charge. When both positive and negative charges are present in the space the electric field lines leaving the positive charges are drained fully or partially by the negative charges. The electric field lines of the uncompensated positive or negative charge will end or start in the infinity, respectively.

In case when more source charges are present in the empty space the principle of superposition is valid. Accordingly, the electric field vectors are added together as usual vector addition in physics.

1.3./ The flux

To understand the concept of flux we start with a simple example and proceed to the general arrangement.

Assume we have a tube with stationary flow of water in which the velocity versus position vector field **v(r)** is homogeneous, in other words the velocity vector is constant everywhere. Now take a plane-like frame made of a very thin wire with the area vector **A**. The area vector by definition is normal to the surface and the magnitude of the vector is the area of the surface. Let us submerge the frame into the flowing water. The task is to find a formula for the amount of water going through the frame.

If the area vector is parallel with the velocity (this means that the velocity vector is normal to the surface) the flow rate (Φ) through the frame (m^3/s) is simply the product of the area and the velocity. If the angle between the area vector and the velocity is not zero but some other φ angle, the area vector should be projected to the direction of the velocity. The projection can be carried out by multiplying with the cosine of the angle. So ultimately it can be stated that the flow rate is the dot product of the velocity vector and the area vector.

$$\Phi = \mathbf{v} \cdot \mathbf{A}.$$

Remember that the above simple formula is valid in case of homogeneous vector field and plane-like frame alone. The question is how the above argument can be implemented to the general case where the vector field is not homogeneous, and the frame has a curvy shape. The solution requires subdividing the area to very small mosaics which represent the surface like tiles on a curvy wall. If the mosaics are sufficiently small (math says they are infinitesimal) then the vector field can be considered homogeneous within the mosaic, and the mosaic itself can be considered plain. So ultimately the above simple dot product can be readily used for the little mosaic. At each mosaic one must choose a representing value of the velocity vector since the velocity vector changes from place to place. The surface vector also changes from point to point, since the surface is not plane-like anymore. Finally, the contribution of each mosaic must be summarized. If the process of subdivision goes to the infinity than the summarized value tends to a limit which is called flux, or in terms of mathematics it is called the scalar value surface integral (denoted as below).

$$\Phi = \int_S \mathbf{v}(\mathbf{r}) \cdot d\mathbf{A}$$

Here S indicates an open surface on which the integration should be carried out. The open surface has a rim and has two sides just like a sheet of paper. In contrast to it, the closed surface does not have any rim and divides the 3D space to internal and external domains just like a ball. In case of open surface, the circulation of the rim determines the direction of the area vector like a right-hand screw turning. Since at closed surface there is no rim a convention states that the area vector is directed outside direction.
Let us find out how much the above integral would be if a closed surface would be submerged into the flow of water, of course with a penetrable surface. In physical context the fact is clear that on one side of the surface the water flows in and on the other side if flows out. After some consideration one can readily conclude that the overall flux on a closed surface is zero. This statement is true if the closed surface does not contain source or drain of the water. If the closed surface contains source then the velocity vectors all point away from the surface, thus the flux will be a positive value equal to the intensity of the source. Plausibly negative result comes out when the drain is contained by the surface. This time the negative value is the intensity of the drain enclosed.

1.4./ Gauss's law

In section 1.2. the fact has been stated that positive and negative charge are the source and the drain of the electric field lines, respectively. Combining this, with the features flux on a closed surface the conclusion is clear: The flux of the electric field to a close surface is zero if the surface does not contain charge. When it does contain charge the flux will be proportional with the amount of charge enclosed. If the charge is positive or negative the flux will be the same sign value. In terms of formula this is the Gauss's law:

$$\oint_S \mathbf{E}(\mathbf{r}) \cdot d\mathbf{A} = \frac{Q}{\varepsilon_0}$$

On the right-hand side Q denotes the total charge contained by the surface, vacuum permittivity ε_0 is a universal constant in nature. ($\varepsilon_0 = 8.86\ 10^{-12}$As/Vm)
We want to use Gauss's law for solving problems in which the charge arrangement is given, and the distribution of the electric field is to be found. This law is an integral type law. In general case information is lost by integration. The only case when information is preserved is when the function to be integrated is constant. Therefore, there will be three distinct classes of charge arrangements when the Gauss's law can be effectively used. These are as follows:

- Spherically symmetric
- Cylindrically symmetric, infinite long
- Plane parallel, infinite large

In all other cases the Gauss's law is also true in terms of integral, but the local electric field is impossible to determine.
To use the law, one needs a closed surface with the same symmetry as that of the charge arrangement. On this surface the angle of electric field vector is necessarily normal, and its intensity is constant. This way the vector integral of flux is majorly simplified to the product of the area and the electric field magnitude.

1.5./ Point charges and the Coulomb's law

Let us use the Gauss's law for the case of point charge. Point charge is a model with zero extension and finite (non-infinitesimal) charge. Accordingly, the charge density and the electrostatic energy are infinite. Even though, this is a useful model for many charge arrangements which are much larger than the distinct charges themselves.

The electric field is perfectly spherical around the point charge. So, surface to be used is obviously sphere. The surface of the sphere is $4r^2\pi$. Accordingly, Gauss's law can be written as follows:

$$4r^2\pi \cdot E = \frac{Q}{\varepsilon_0}$$

The electric field can readily be expressed:

$$E = \frac{Q}{4\pi\varepsilon_0} \cdot \frac{1}{r^2}$$

The above formula is the electric field of the point charge which will be used extensively later in this chapter.
The exerted force to a q charge can be written:

$$F = q \cdot E$$

$$F = \frac{1}{4\pi\varepsilon_0} \cdot \frac{Qq}{r^2}$$

This is the Coulomb's law which describes the force between point charges. For practical reason it is worth remembering that the value of the constant in Coulomb's law is the following:

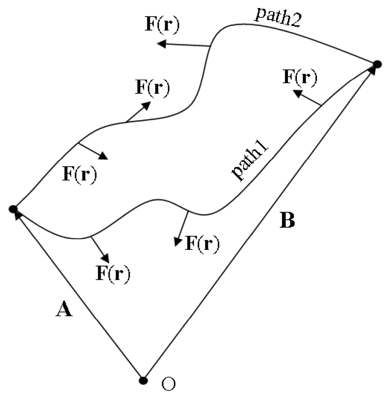
$$\frac{1}{4\pi\varepsilon_0} = 9 \cdot 10^9 \frac{Vm}{As}$$

1.6./ Conservative force field

Force field is a vector-vector function in which the force vector **F** depends on the position vector **r**. In terms of mathematics the force field **F(r)** is described as follows:

$$\mathbf{F}(\mathbf{r}) = X(x,y,z)\mathbf{i} + Y(x,y,z)\mathbf{j} + Z(x,y,z)\mathbf{k}$$

where **i, j, k** are the unit vectors of the coordinate system.
Take a test charge and move it slowly in the **F(r)** force field from position **A** to position **B** on two alternative paths.



Integration on two paths
Fig. 1.1

Let us calculate the amount of work done on each path. The force exerted to the test charge by my hand is just opposite of the force field. **-F(r)** If it was not the case, the charge would accelerate. The moving is thought to happen quasi-statically without any acceleration.
Let us calculate my work for the two alternate paths:

$$W_1 = \left(\int_{\mathbf{A}}^{\mathbf{B}} (-\mathbf{F}) d\mathbf{r} \right)_{path1} \qquad W_2 = \left(\int_{\mathbf{A}}^{\mathbf{B}} (-\mathbf{F}) d\mathbf{r} \right)_{path2}$$

In general case W_1 and W_2 are different. However, in some special cases they may be equal for any two paths. Imagine that our force field is such, that W_1 and W_2 are equal. In this case a closed loop path can be made which starts with path 1 and returns to the starting point on path 2. Since the opposite direction passage turns W_2 to its negative, ultimately the closed loop path will result in zero. That special force field where the integral is zero for any closed loop is considered CONSERVATIVE force field. In formula:

$$\oint \mathbf{F}(\mathbf{r})d\mathbf{r} = 0$$

Using the concept of electric field with the formula $\mathbf{F} = q\mathbf{E}$ above equation is transformed:

$$\oint \mathbf{E}(\mathbf{r})d\mathbf{r} = 0$$

According to the experience the electric field obeys the law of conservative field. The integral on any closed loop results zero. That also means that curve integral between any two points is independent of the path and solely depends on the starting and final point.

1.7./ Voltage and potential

The work done against the force of the electric field is as follows:

$$W_{\mathbf{A}}^{\mathbf{B}} = \int_{\mathbf{A}}^{\mathbf{B}} (-\mathbf{F})d\mathbf{r} = q\int_{\mathbf{A}}^{\mathbf{B}} (-\mathbf{E})d\mathbf{r}$$

Let us rearrange and divide with q.

$$U_{\mathbf{A}}^{\mathbf{B}} = \frac{W_{\mathbf{A}}^{\mathbf{B}}}{q} \qquad\qquad U_{\mathbf{A}}^{\mathbf{B}} = -\int_{\mathbf{A}}^{\mathbf{B}} \mathbf{E}(\mathbf{r})d\mathbf{r}$$

The voltage of point **B** relative to **A** is given by the formula above. The fact is clear that the voltage is dependent on two points. If the starting point is considered as a reference point for all the integrals, that specific voltage will be dependent on the final point only. This voltage with only one position parameter is called the potential.

$$U_{\mathbf{B}} = -\int_{\mathbf{ref}}^{\mathbf{B}} \mathbf{E}(\mathbf{r})d\mathbf{r}$$

Voltages can be expressed as the difference of potentials proven below:

$$U_{\mathbf{A}}^{\mathbf{B}} = \left(-\int_{\mathbf{A}}^{\mathbf{ref}} \mathbf{E}(\mathbf{r})d\mathbf{r}\right) + \left(-\int_{\mathbf{ref}}^{\mathbf{B}} \mathbf{E}(\mathbf{r})d\mathbf{r}\right) = \left(-\int_{\mathbf{ref}}^{\mathbf{B}} \mathbf{E}(\mathbf{r})d\mathbf{r}\right) - \left(-\int_{\mathbf{ref}}^{\mathbf{A}} \mathbf{E}(\mathbf{r})d\mathbf{r}\right) = U_{\mathbf{B}} - U_{\mathbf{A}}$$

The potential of any point can also be written as follows:

$$U(\mathbf{r}) = -\int_{\mathbf{ref}}^{\mathbf{r}} \mathbf{E}(\mathbf{r}')d\mathbf{r}'$$

The concept of voltage exists always, and its value is definite and depends on the starting and the final point of integration. The value of potential requires a definite reference point. The physically real object, by definition, is such an object which virtually shrinks to a point if one departs infinite far away. The reference point of the integral should be placed to the infinity. This can be done in case of physically real objects only, when the corresponding improper integral is convergent. The spherical charge arrangement is the only physically real object among those three symmetry classes mentioned above. An infinite long cylinder or an infinite plan-parallel plate are not physically real, since viewed from infinite far, they still look infinite instead of shrinking to a point. This case, the reference point is the center of the charge structure mostly.

1.8./ Gradient

The gradient is an operation in vector calculus which generates the electric field vector from the potential scalar field. In general case the formula is as follows:

$$gradU(\mathbf{r}) = \frac{\partial U(\mathbf{r})}{\partial x}\mathbf{i} + \frac{\partial U(\mathbf{r})}{\partial y}\mathbf{j} + \frac{\partial U(\mathbf{r})}{\partial z}\mathbf{k} = -\mathbf{E}(\mathbf{r})$$

Therefore:

$$\mathbf{E}(\mathbf{r}) = -gradU(\mathbf{r})$$

In the special case of spherical cylindrical and plane parallel structures the gradient operation is merely a derivation according to the position variable.

$$E(r) = -\frac{dU(r)}{dr}$$

1.9./ Spherical structures

1.9.1./ Metal sphere

Metal sphere with radius $R=0.1m$ contains $Q=10^{-8}As$ charge. Find the function of the electric field and the potential as the function of distance from the center and sketch the result. Calculate the values of the electric field and the potential on the surface of the metal sphere. Determine the capacitance of the metal sphere.

Metal contains free electrons therefore electric field may not exist inside the bulk of the metal. If there was electric field in the metal the free electrons would move to compensate it to zero very fast. Since there is no electric field inside the metal the total volume of the metal is equipotential. The vector of the electric field is always normal (perpendicular) to the metal (equipotential) surface. The proof of this as follows: If there was an angle different of ninety degrees, then this electric field vector could be decomposed to normal and tangential components. The tangential component would readily move the electrons until this component gets compensated. In stationary case all the excess charge resides on the surface of the metal. Therefore, a hollow metal is equivalent with a bulky metal in terms of electrostatics. The surface charge density and the surface electric field are proportional to the reciprocal of the curvature radius.

$$\oint_S \mathbf{E}(\mathbf{r}) \cdot d\mathbf{A} = \frac{Q}{\varepsilon_0}$$

Gauss's law is used to solve the problem. We pick a virtual point-like balloon and inflate it from zero to the infinity radius. Inside the metal sphere there is no contained charge in the balloon.

$$4r^2\pi \cdot E = 0$$
$$E = 0$$

So, the electric field inside the metal sphere is zero.
Out of the metal sphere however, the contained charge is the amount given in this problem.

$$4r^2\pi \cdot E = \frac{Q}{\varepsilon_0}$$

The electric field can be expressed:

$$E(r) = \frac{Q}{4\pi\varepsilon_0} \cdot \frac{1}{r^2}$$

On the surface of the metal sphere the electric field comes out if $r=R$ is substituted to the above function

$$E(R) = \frac{Q}{4\pi\varepsilon_0} \cdot \frac{1}{R^2} = 9 \cdot 10^9 \frac{10^{-8}}{0.1^2} = 9000\frac{V}{m}$$

The potential function can be determined by integrating the electric field:

$$U(r)=-\int_{\infty}^{r}E(r')dr'=-\int_{\infty}^{r}\frac{Q}{4\pi\varepsilon_0}\cdot\frac{1}{r'^2}dr'=\frac{Q}{4\pi\varepsilon_0}\cdot\int_{\infty}^{r}(-\frac{1}{r'^2})dr'=\frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r'}\right]_{r'=\infty}^{r'=r}=\frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r}-\frac{1}{\infty}\right)=\frac{Q}{4\pi\varepsilon_0}\frac{1}{r}$$

So briefly the potential function out of the metal sphere is as follows:

$$U(r)=\frac{Q}{4\pi\varepsilon_0}\frac{1}{r}$$

On the surface of the metal sphere the potential comes out if $r=R$ is substituted to the above function

$$U(R)=\frac{Q}{4\pi\varepsilon_0}\frac{1}{R}=9\cdot10^9\frac{10^{-8}}{0.1}=900V$$

Inside the metal sphere the potential is constant, because the electric field is zero.

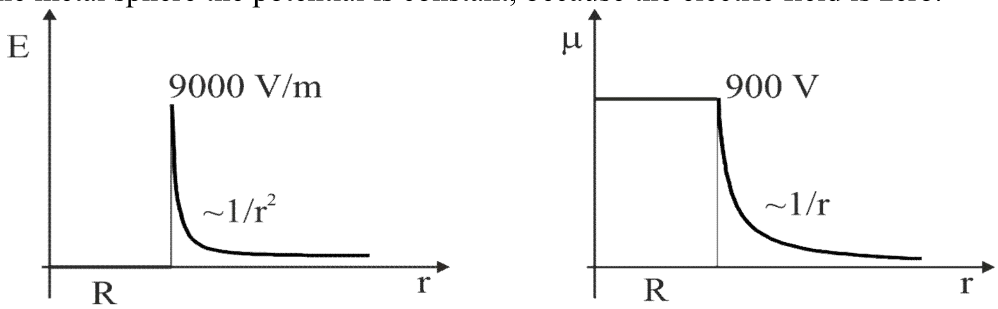


Fig. 1.2 Metal sphere

Electric field vs. radial position function　　Potential vs. radial position function

An interesting result can be concluded. Let us divide the formula of the potential and the electric field on the surface.

$$\frac{U(R)}{E(R)}=\frac{Q}{4\pi\varepsilon_0}\frac{1}{R}\cdot\frac{4\pi\varepsilon_0}{Q}R^2=R$$

The electric field on the surface is the ratio of the potential and the radius.

$$E(R)=\frac{U(R)}{R}$$

The result is in perfect agreement with the numerical values.

This result is useful when high electric field is desired. This time ultra-sharp needle is used, and the needle is hooked up to high potential. By means of this device corona discharge can be generated in air.

Capacitance is a general term in physics which means a kind of storage capability. More precisely, this is the ratio of extensive parameter over the corresponding intensive parameter. For instance, the heat capacitance is the ratio of the heat energy over the temperature. Similarly, the electric capacitance is the ratio of the charge over the generated potential. The unit of the capacitance is *As/V* which is called Farad (*F*) to commemorate the famous scientist Faraday. Farad as a unit is very large therefore *pF* or μF is used mostly. Capacitance denoted with *C* is a feature of all physically real conductive objects. In contrast to this, the capacitor is a device used in the electronics with intentionally high capacitance.

$$U=\frac{Q}{4\pi\varepsilon_0}\frac{1}{R}$$

$$C=\frac{Q}{U}=4\pi\varepsilon_0 R=\frac{1}{9\cdot10^9}\cdot 1Farad=110pF$$

This is an important formula, worth remembering: $C=4\pi\varepsilon_0 R$

So, the capacitance of the metal sphere is proportional to the radius. A big sphere of 1 meter radius has a capacitance of 110 *pF*. The capacitance of the human body is in the range of some tens of *pF*.

1.9.2./ Sphere with uniform space charge density

Uniform space charge density (ρ = 10^{-6} As/m^3) is contained by a sphere with radius R = 0.1 meter. (The charge density is immobile. Imagine this in the way, that wax is melted, charged up and let it cool down. The charges are effectively trapped in the wax.) Find the function of the electric field and the potential as the function of distance from the center and sketch the result. Calculate the value of the electric field on the surface of the sphere and find the value of the potential on the surface and in the center.

$$\oint_S \mathbf{E}(\mathbf{r}) \cdot d\mathbf{A} = \frac{Q}{\varepsilon_0}$$

We pick a virtual point-like balloon and inflate it from zero to the infinity radius. Inside the charged sphere, the Gauss's law is as follows:

$$4r^2\pi \cdot E = \frac{4r^3\pi}{3}\frac{\rho}{\varepsilon_0}$$

On the left-hand side there is the flux, on the right-hand side there is the volume of the sphere multiplied with the charge density. Many terms cancel out.

$$E(r) = \frac{\rho}{3\varepsilon_0} r$$

The result is not surprising. By increasing the radius inside the sphere, the charge contained grows with the third power, the surface area increases with the second power so the ratio will be linear.

Outside the charged sphere the amount of the charge contained does not grow any more only the surface of the sphere continues to grow with the second power.

$$4r^2\pi \cdot E = \frac{4R^3\pi}{3}\frac{\rho}{\varepsilon_0}$$

$$E(r) = \frac{\rho}{3\varepsilon_0}\frac{R^3}{r^2}$$

The two above equations show that the function of the electric field is continuous, since on the surface of the charged sphere r=R substitution produces the same result.

On the surface of the sphere the numerical value of the electric field can readily be calculated:

$$E(R) = \frac{\rho}{3\varepsilon_0} R = \frac{10^{-6}}{3 \cdot 8{,}86 \cdot 10^{-12}} \cdot 0.1 = 3762\frac{V}{m}$$

The potential function can be determined by integrating the electric field. First the external region is integrated:

$$U_{out}(r) = -\int_{\infty}^{r} E(r')dr' = -\int_{\infty}^{r} \frac{\rho}{3\varepsilon_0}\frac{R^3}{r'^2}dr' = \frac{\rho R^3}{3\varepsilon_0} \cdot \int_{\infty}^{r} (-\frac{1}{r'^2})dr' = \frac{\rho R^3}{3\varepsilon_0}\left[\frac{1}{r'}\right]_{r'=\infty}^{r'=r} = \frac{\rho R^3}{3\varepsilon_0}\left(\frac{1}{r} - \frac{1}{\infty}\right) = \frac{\rho R^3}{3\varepsilon_0}\frac{1}{r}$$

So briefly the potential function out of the charged sphere is as follows:

$$U_{out}(r) = \frac{\rho R^3}{3\varepsilon_0}\frac{1}{r}$$

The surface potential of the sphere is the above function with r=R substitution:

$$U(R) = \frac{\rho R^2}{3\varepsilon_0} = \frac{10^{-6}\cdot 10^{-2}}{3\cdot 8{,}86\cdot 10^{-12}} = 376V$$

Remember that this value should be added to the integral calculated next.
Inside the charged sphere the integral is different:

$$U_{in}(r) = U(R) + U_R^r = U(R) + \left(-\int_R^r E(r')dr'\right)$$

For simplicity reason only the integral in the parenthesis is transformed first:

$$U_R^r = -\int_R^r \frac{\rho}{3\varepsilon_0} r' dr' = -\frac{\rho}{3\varepsilon_0}\int_R^r r' dr' = -\frac{\rho}{3\varepsilon_0}\left[\frac{r'^2}{2}\right]_{r'=R}^{r'=r} = -\frac{\rho}{3\varepsilon_0}\left(\frac{r^2}{2} - \frac{R^2}{2}\right)$$

Altogether:

$$U_{in}(r) = U(R) + U_R^r = \frac{\rho R^2}{3\varepsilon_0} + \left(-\frac{\rho}{3\varepsilon_0}\left(\frac{r^2}{2} - \frac{R^2}{2}\right)\right) = \frac{\rho}{3\varepsilon_0}\left(\frac{3R^2}{2} - \frac{r^2}{2}\right) = \frac{\rho}{6\varepsilon_0}\left(3R^2 - r^2\right)$$

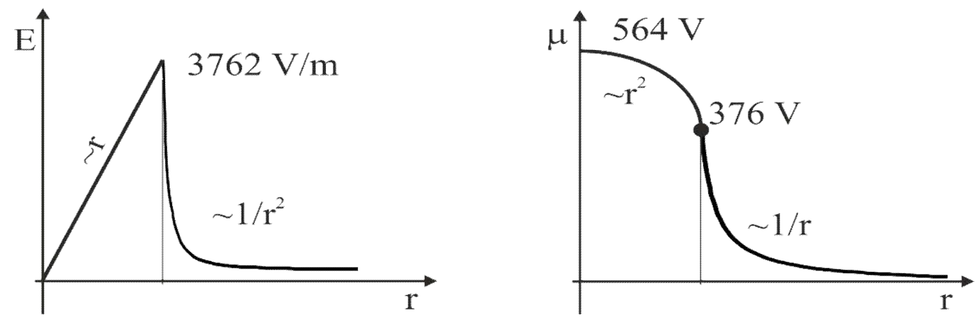


Fig. 1.3 Sphere with uniform charge density
Electric field vs. radial position function Potential vs. radial position function

The final result is:

$$U_{in}(r) = \frac{\rho}{6\varepsilon_0}\left(3R^2 - r^2\right)$$

The numerical value of the central potential is given by the above equation at $r = 0$ substitution.

$$U_{in}(0) = \frac{\rho}{6\varepsilon_0}\left(3R^2\right) = \frac{\rho R^2}{2\varepsilon_0} = \frac{10^{-6}\cdot 10^{-2}}{2\cdot 8{,}86\cdot 10^{-12}} = 564V$$

1.10./ Cylindrical structures

1.10.1./ Infinite metal cylinder

Infinite metal cylinder (tube) with radius $R=0.1m$ contains $\omega = 10^{-8} As/m^2$ surface charge density. Find the function of the electric field and the potential as the function of distance from the center and sketch the result. The reference point of the potential should be the center. Calculate the value of the electric field and of the potential on the surface of the metal cylinder.

$$\oint_S \mathbf{E}(\mathbf{r})\cdot d\mathbf{A} = \frac{Q}{\varepsilon_0}$$

Gauss's law is used to solve the problem. We pick a virtual tube and inflate it from zero to the infinity radius. Inside the metal cylinder there is no contained charge in the virtual tube.

$$2r\pi \cdot l \cdot E = 0$$
$$E = 0$$

So, the electric field inside the metal cylinder is zero.
Out of the metal cylinder however the contained charge is as follows.

$$2r\pi \cdot l \cdot E = \frac{2R\pi \cdot l \cdot \omega}{\varepsilon_0}$$

The electric field can be expressed:

$$E(r) = \frac{\omega R}{\varepsilon_0} \cdot \frac{1}{r}$$

On the surface of the metal cylinder the electric field comes out if $r=R$ is substituted to the above function

$$E(R) = \frac{\omega R}{\varepsilon_0} \cdot \frac{1}{R} = \frac{\omega}{\varepsilon_0} = \frac{10^{-8}}{8.86 \cdot 10^{-12}} = 1129 \frac{V}{m}$$

The potential function can be determined by integrating the electric field:

$$U(r) = -\int_R^r E(r')dr' = -\int_R^r \frac{\omega R}{\varepsilon_0} \cdot \frac{1}{r'} dr' = -\frac{\omega R}{\varepsilon_0} \int_R^r \frac{1}{r'} dr' = -\frac{\omega R}{\varepsilon_0} \left[\ln r'\right]_{r'=R}^{r'=r} = -\frac{\omega R}{\varepsilon_0} \ln\left(\frac{r}{R}\right)$$

So briefly the potential function out of the metal cylinder is as follows:

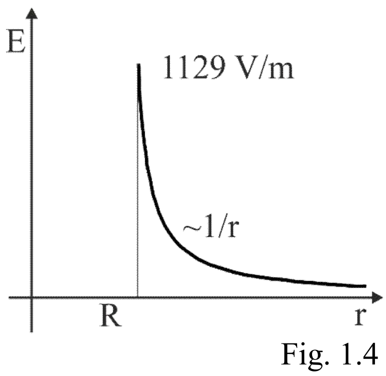
$$U(r) = -\frac{\omega R}{\varepsilon_0} \ln\left(\frac{r}{R}\right)$$

On the surface of the metal cylinder the potential comes out if $r=R$ is substituted to the above function

$$U(R) = -\frac{\omega R}{\varepsilon_0} \ln\left(\frac{R}{R}\right) = 0V$$

The result is obvious, since inside the metal cylinder the potential is constant, since electric field is zero.

Note that the reference point of the potential could not be placed to the infinity because the infinite long cylinder is not physically real object. Therefore, the improper integral is not convergent.



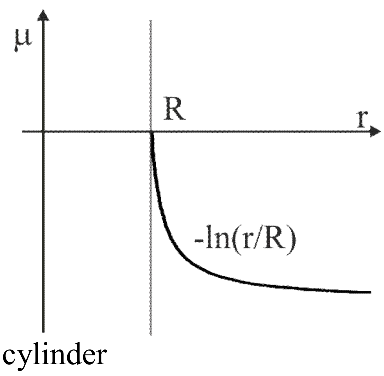


Fig. 1.4 Metal cylinder
Electric field vs. radial position function Potential vs. radial position function

1.10.2./ Infinite cylinder with uniform space charge density

Uniform space charge density ($\rho = 10^{-6}$ As/m^3) is contained by an infinite cylinder with radius R = 0.1 meter. (The charge density is immobile. Imagine this in the way that wax is melted charged up and let it cool down. The charges are effectively trapped in the wax.) Find the function of the electric field and the potential as the function of distance from the center and sketch the result. Calculate the value of the electric field and the potential on the surface of the cylinder. The reference point of the potential should be the central line.

$$\oint_S \mathbf{E}(\mathbf{r}) \cdot d\mathbf{A} = \frac{Q}{\varepsilon_0}$$

We pick a virtual tube and inflate the radius from zero to the infinity. Inside the charged sphere the Gauss's law is as follows:

$$2r\pi \cdot l \cdot E = r^2\pi \cdot l \frac{\rho}{\varepsilon_0}$$

On the left-hand side there is the flux on the right-hand side there is the volume of the cylinder multiplied with the charge density. Many terms cancel out.

$$E(r) = \frac{\rho}{2\varepsilon_0} r$$

The result is not surprising. By increasing the radius, the charge contained grows with the second power, the surface area increases linearly so the ratio will be linear.
Outside the charged cylinder, the amount of the charge contained does not grow anymore, only the surface continues to grow linearly.

$$2r\pi \cdot l \cdot E = R^2\pi \cdot l \frac{\rho}{\varepsilon_0}$$

$$E(r) = \frac{\rho}{2\varepsilon_0} \frac{R^2}{r}$$

The two above equations show that the function of the electric field is continuous, since on the surface of the cylinder $r=R$ substitution produces the same result.
On the surface of the sphere the numerical value of the electric field can readily be calculated:

$$E(R) = \frac{\rho}{2\varepsilon_0} R = \frac{10^{-6}}{2 \cdot 8{,}86 \cdot 10^{-12}} \cdot 0.1 = 5643 \frac{V}{m}$$

The potential function can be determined by integrating the electric field. First the internal region is integrated: The reference point of the potential will be the center.

$$U_{in}(r) = -\int_0^r E(r')dr' = -\int_0^r \frac{\rho}{2\varepsilon_0} r'dr' = -\frac{\rho}{2\varepsilon_0} \int_0^r r'dr' = -\frac{\rho}{2\varepsilon_0} \left[\frac{r'^2}{2} \right]_{r'=0}^{r'=r} = -\frac{\rho}{2\varepsilon_0} \left(\frac{r^2}{2} \right) = -\frac{\rho}{4\varepsilon_0} r^2$$

$$U_{in}(r) = -\frac{\rho}{4\varepsilon_0} r^2$$

The surface potential of the cylinder is the above function with $r=R$ substitution:

$$U_{in}(R) = -\frac{\rho R^2}{4\varepsilon_0} = -\frac{10^{-6} \cdot 10^{-2}}{4 \cdot 8{,}86 \cdot 10^{-12}} = -282V$$

Remember that this value should be added to the integral calculated next.

$$U_{out}(r) = U_{in}(R) + U_R^r = U_{in}(R) + \left(-\int_R^r E(r')dr' \right)$$

For simplicity reason only the integral in the parenthesis is transformed first:

$$U_R^r = -\int_R^r \frac{\rho}{2\varepsilon_0} \frac{R^2}{r} dr' = -\frac{\rho R^2}{2\varepsilon_0} \int_R^r \frac{dr'}{r'} = -\frac{\rho R^2}{2\varepsilon_0} \left[\ln r' \right]_{r'=R}^{r'=r} = -\frac{\rho R^2}{2\varepsilon_0} \ln\left(\frac{r}{R} \right)$$

Altogether:

$$U_{out}(r) = U_{in}(R) + U_R^r = -\frac{\rho R^2}{4\varepsilon_0} + \left(-\frac{\rho R^2}{2\varepsilon_0}\ln(\frac{r}{R})\right) = -\frac{\rho R^2}{4\varepsilon_0}\left(1 + 2\ln(\frac{r}{R})\right)$$

The final result is:

$$U_{out}(r) = -\frac{\rho R^2}{4\varepsilon_0}\left(1 + 2\ln(\frac{r}{R})\right)$$

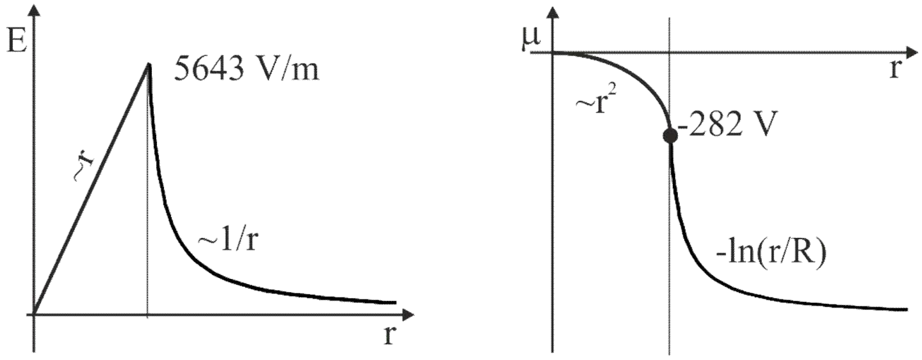


Fig. 1.5 Cylinder with uniform charge density
Electric field vs. radial position function Potential vs. radial position function

1.11./ Infinite parallel plate with uniform surface charge density

Infinite metal plate contains $\omega = 10^{-8} As/m^2$ surface charge density. Find the function of the electric field and the potential as the function of distance from the plate and sketch the result. The reference point of the potential should be the center. Calculate the value of the electric field on the surface of the metal plate.

$$\oint_S \mathbf{E}(\mathbf{r}) \cdot d\mathbf{A} = \frac{Q}{\varepsilon_0}$$

Pick a virtual drum with base plate area A. The drum should be cut to two identical halves by charged plate, parallel with the base plates. Place the half-drums symmetrically on the two sides of the charged plate. Gauss's law is used to solve the problem.

$$E \cdot 2A = \frac{\omega A}{\varepsilon_0}$$

The magnitude of the electric field can be expressed:

$$E = \frac{\omega}{2\varepsilon_0} = \frac{10^{-8}}{2 \cdot 8.86 \cdot 10^{-12}} = 564\frac{V}{m}$$

The result shows that the magnitude of the electric field is constant in each half space. The direction of the electric field is opposite in the two half spaces. In contrast to the spherical and cylindrical structures where the radial distance is the position parameter, here a reference direction line will be used.

The potential function can be determined by integrating the electric field in the positive half space:

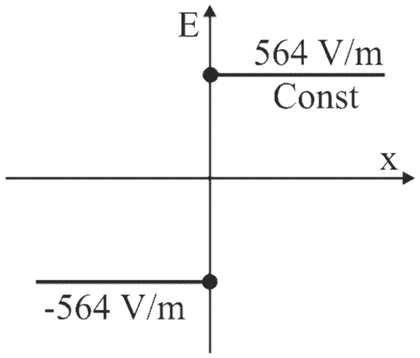
$$U(x) = -\int_0^x E(x')dx' = -\int_0^x \frac{\omega}{2\varepsilon_0}dx' = -\frac{\omega}{2\varepsilon_0}\int_0^x dx' = -\frac{\omega}{2\varepsilon_0}\left[x'\right]_{x'=0}^{x'=x} = -\frac{\omega}{2\varepsilon_0}x$$

So briefly the potential function is as follows:

$$U(x) = -\frac{\omega}{2\varepsilon_0}x$$

Obviously, the potential function turns to its negative in the negative half space. This can be taken account by using the absolute value function.

$$U(x) = -\frac{\omega}{2\varepsilon_0}|x|$$



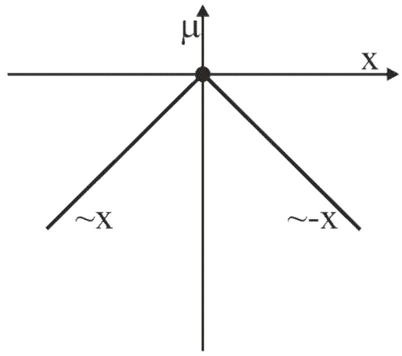


Fig. 1.6 Infinite parallel plate with uniform surface charge density
Electric field vs. radial position function Potential vs. radial position function

Note that the reference point of the potential could not be placed to the infinity because the infinite plate is not physically real object.

1.12./ Capacitors

Capacitors consist of two plates to store charge. The overall contained charge is zero since the charges on the plates are opposite therefore the electric field is confined to the inner volume of the capacitor. The capacitance is the ratio of the charge over the voltage generated between the plates. $C = \frac{Q}{U}$ Three different geometries will be treated below.

1.12.1/ Parallel plate capacitor

The parallel plate capacitor is made of two parallel metal plates facing each other with the active surface area *A*. The distance between the plates and the charge are denoted by *d* and Q, respectively.

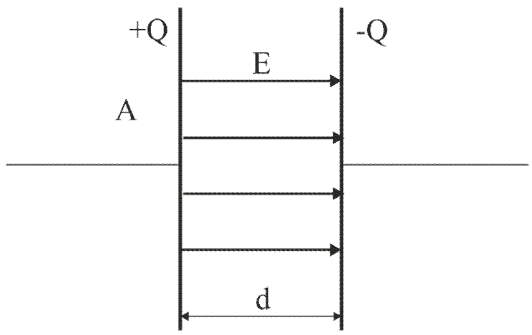


Fig 1.7
Parallel plate capacitor

There is homogeneous electric field between the plates, while out of the capacitor there is no electric field. Use the Gauss's law for a drum-like surface which surrounds one of the plates.

$$EA = \frac{Q}{\varepsilon_0} \qquad\qquad E = \frac{Q}{A\varepsilon_0}$$

To find out the voltage between the plates, does not need integration due to the homogenous field. $U = d \cdot E$

$$U = \frac{d \cdot Q}{A\varepsilon_0}$$

And capacitance can be expressed from here.

$$C = \varepsilon_0 \frac{A}{d}$$

1.12.2/ Cylindrical capacitor

The cylindrical capacitor is made of two coaxial metal cylinders. The inner and the outer radii as well as the length are denoted R_1, R_2 and l, respectively. The coaxial cable is the only practically used cylindrical capacitor.

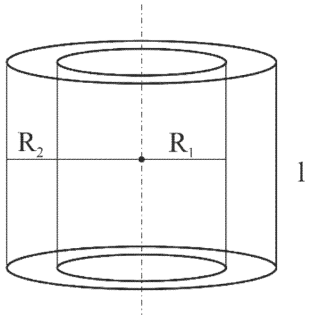


Fig. 1.8
Cylindrical capacitor

Let us use the Gauss's law. A coaxial cylinder should be inflated from R_1 to R_2.

$$\oint_S \mathbf{E}(\mathbf{r}) \cdot d\mathbf{A} = \frac{Q}{\varepsilon_0}$$

$$E \cdot 2r\pi l = \frac{Q}{\varepsilon_0}$$

$$E = \frac{Q}{2\pi\varepsilon_0 l} \cdot \frac{1}{r}$$

To find out the voltage the following integral should be evaluated:

$$U = -\int_{R_2}^{R_1} \frac{Q}{2\pi\varepsilon_0 l} \cdot \frac{1}{r} dr = \frac{Q}{2\pi\varepsilon_0 l} \int_{R_1}^{R_2} \frac{1}{r} dr = \frac{Q}{2\pi\varepsilon_0 l} \ln(\frac{R_2}{R_1})$$

The capacitance can be expressed from here.

$$C = \frac{Q}{U} = \frac{2\pi\varepsilon_0 l}{\ln(R_2 / R_1)}$$

The above formula shows the obvious fact that the capacitance is proportional to the length of the structure. Because of this, the capacitance of one-meter coaxial cable is used mostly. This is denoted with c and measured in *F/m* units. Most coaxial cables represent some 10 *pF/m* value.

$$c = \frac{C}{l} = \frac{2\pi\varepsilon_0}{\ln(R_2 / R_1)}$$

1.12.3/ Spherical capacitor

The spherical capacitor is made of two concentric metal spheres. The inner and the outer radii as well as the charge on the inner sphere are denoted R_1, R_2 and Q, respectively.

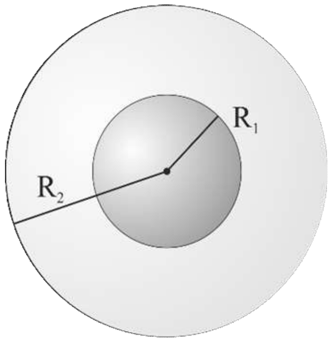


Fig. 1.9
Spherical capacitor

Let us use the Gauss's law. A concentric sphere should be inflated from R_1 to R_2.

$$\oint_S \mathbf{E}(\mathbf{r}) \cdot d\mathbf{A} = \frac{Q}{\varepsilon_0}$$

$$E \cdot 4r^2\pi = \frac{Q}{\varepsilon_0}$$

$$E = \frac{Q}{4\pi\varepsilon_0} \cdot \frac{1}{r^2}$$

To find out the voltage, the following integral should be evaluated:

$$U = -\int_{R_2}^{R_1} \frac{Q}{4\pi\varepsilon_0} \cdot \frac{1}{r^2} dr = \frac{Q}{4\pi\varepsilon_0} \int_{R_2}^{R_1} (-\frac{1}{r^2}) dr = \frac{Q}{4\pi\varepsilon_0} \left[\frac{1}{r} \right]_{r=R_2}^{r=R_1} = \frac{Q}{4\pi\varepsilon_0} \left(\frac{1}{R_1} - \frac{1}{R_2} \right)$$

The capacitance can be expressed from here.

$$C = \frac{Q}{U} = \frac{4\pi\varepsilon_0}{\frac{1}{R_1} - \frac{1}{R_2}}$$

One can see the correspondence between this formula and the general formula for the capacitance of sphere. If R_2 goes to infinity, the above formula turns into $C = 4\pi\varepsilon_0 R_1$.

1.13./ Principle of superposition

The Gauss's law can only be used effectively in the three symmetry classes mentioned earlier. If the charge arrangement does not belong to any of those classes, the principle of superposition is the only choice. This time the charge arrangement is virtually broken to little pieces and the electric fields of these little pieces are superimposed (added together) like point charges.

- Find the electric field of a finite long charged filament in the equatorial plane as the function of distance from the filament. The linear charge density is denoted σ.

Since the filament is not infinite long, Gauss's law cannot be used for solution. The charged filament is divided to little infinitesimal pieces and the electric fields of such pieces are added together. Only the normal components of the electric field are integrated, since the parallel components cancel out by pairs due to symmetry reasons. The mathematical deduction of the final formula follows below, without close commenting to the transformations. For the definition of the notations refer the figure below:

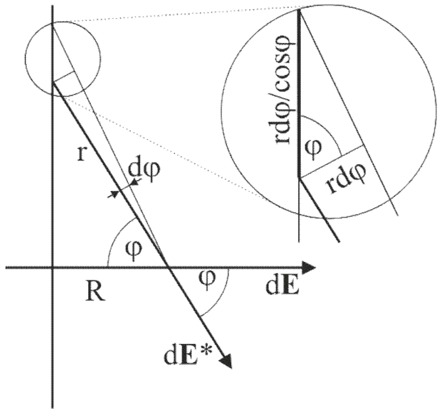


Fig. 1.10
Principle of superposition

The infinitesimal contribution of the electric field is calculated as a point charge. The infinitesimal charge is contained by the infinitesimal angle.

$$dE^* = \frac{dQ}{4\pi\varepsilon_0} \cdot \frac{1}{r^2} \qquad dQ = \frac{rd\varphi}{\cos\varphi}\sigma \qquad r = \frac{R}{\cos\varphi}$$

After some ordering:

$$dE^* = \frac{rd\varphi}{\cos\phi}\sigma \frac{1}{4\pi\varepsilon_0} \cdot \frac{1}{r^2} = \frac{\sigma}{4\pi\varepsilon_0} \frac{1}{r\cos\varphi} d\varphi = \frac{\sigma}{4\pi\varepsilon_0} \frac{1}{R} d\varphi$$

The electric field of the point charge is projected to the perpendicular direction. The parallel direction components cancel out by symmetric pairs.

$$dE = dE^* \cos\varphi$$

$$dE = \frac{\sigma}{4\pi\varepsilon_0} \frac{1}{R} \cos\varphi \cdot d\varphi$$

The integration is carried out in α half visual angle.

$$E = \int_{-\alpha}^{\alpha} dE = \int_{-\alpha}^{\alpha} \frac{\sigma}{4\pi\varepsilon_0} \frac{1}{R} \cos\varphi \cdot d\varphi = \frac{\sigma}{4\pi\varepsilon_0} \frac{1}{R} \int_{-\alpha}^{\alpha} \cos\varphi \cdot d\varphi = \frac{\sigma}{4\pi\varepsilon_0} \frac{1}{R} [\sin\varphi]_{\varphi=-\alpha}^{\varphi=\alpha} = \frac{\sigma}{2R\pi\varepsilon_0} \sin\alpha$$

The result of the superposition is the formula below which could not have been attained with Gauss's law.

$$E = \frac{\sigma}{2R\pi\varepsilon_0} \sin\alpha$$

If α approaches ninety degrees, the filament tends to the infinity.

$$E_\infty = \lim_{\alpha \to \frac{\pi}{2}} E = \frac{\sigma}{2R\pi\varepsilon_0}$$

Using Gauss's law, the above result can be reached far easier for the infinite long filament.

$$E_\infty \cdot 2R\pi \cdot l = \frac{\sigma l}{\varepsilon_0}$$

$$E_\infty = \frac{\sigma}{2R\pi\varepsilon_0}$$

The results are in perfect match. However, superposition principle can be used in full generality, but it is far more meticulous and tedious than using Gauss's law if that is possible.

Chapter 2.

Dielectric materials - György Hárs

Insulators or in other words dielectric materials will be discussed in this chapter. In chapter 1 only metal electrodes and immobile charges are the sources of the electric field. In contrast to metals, insulating materials are lacking mobile electron plasma therefore the electric field can penetrate the insulators. In terms of phenomenology, the insulating material is polarized, which means that whole material will become an electric dipole. In terms of microphysical explanation, the overall effects of huge number of elementary dipoles will create the external dipole effect. The elementary dipoles are either generated or oriented by the external electric field.

2.1./ The electric dipole

Consider a pair of opposite point charges ($+q, -q$). Initiate the vector of separation (**s**) from the negative to the positive point charge. The following product defines the electric dipole moment:

$$\mathbf{p} = q\mathbf{s} \qquad [Asm]$$

In order to generate a point-like dipole the definition is completed with a limit transition. Accordingly, the magnitude of the displacement vector shrinks to zero while the charge tends to the infinity such a way, that the product is a constant vector. The point-like dipole is a useful model when the distance of the charges is far smaller than the corresponding geometry, for example if a dipole molecule is in the proximity of centimeter size electrodes.

Force couple is exerted to the dipole by homogeneous electric field. The generated torque turns the dipole parallel to the electric field.

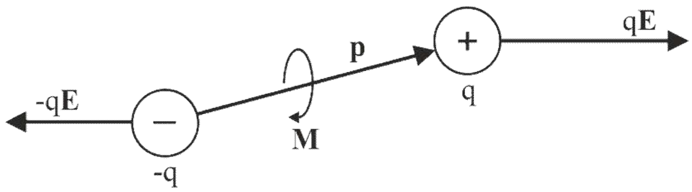


Fig. 2.1
Dipole turns parallel to the **E** filed

$$\mathbf{M} = \mathbf{p} \times \mathbf{E} \qquad \left[Asm \cdot \frac{V}{m} = Nm \right]$$

The dipole moment turns into the direction of the electric field spontaneously and stays there. Having reached this position, the least amount of potential energy is stored by the dipole. Obviously, the most amount of potential energy is stored, just in the opposite position. Let us find out the work needed to turn the dipole from the deepest position to the highest energy.

$$W = \int_0^\pi M d\varphi = \int_0^\pi pE \sin\varphi \cdot d\varphi = pE \int_0^\pi \sin\varphi \cdot d\varphi = -pE[\cos\varphi]_0^\pi = 2pE$$

According to this result the potential energy of the dipole is as follows:

$$E_{pot} = -\mathbf{p} \cdot \mathbf{E}$$

This formula provides the deepest energy at parallel spontaneous position and the highest at anti-parallel position. The zero potential energy is at ninety degrees. The difference between the highest and lowest is just the work needed to turn it around.

2.2./ Polarization

Take a plate capacitor and fill its volume with a dielectric material. The experiment shows that the capacitance increased relative to the empty case. The explanation behind is the polarization of the dielectric material.

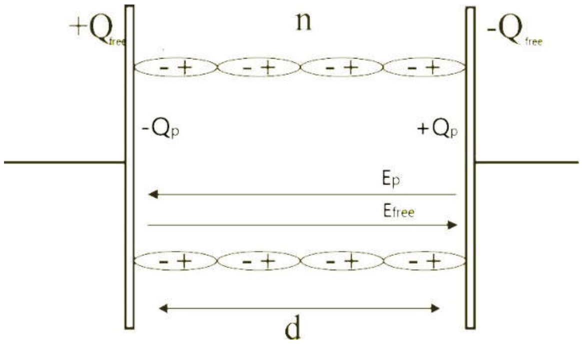


Fig. 2.2
Plate capacitor with dipole chain

In the figure above, the formation of dipole chains can be seen. Each piece of macroscopic volume inside the dielectric material contains equal number of positive and negative charges, thus they do not affect the surrounding materials. In contrast to this, the uncompensated endings of the dipole chains show up like a polarization surface charge layer with a polarity, opposite to that of the metal plate adjacent to it.

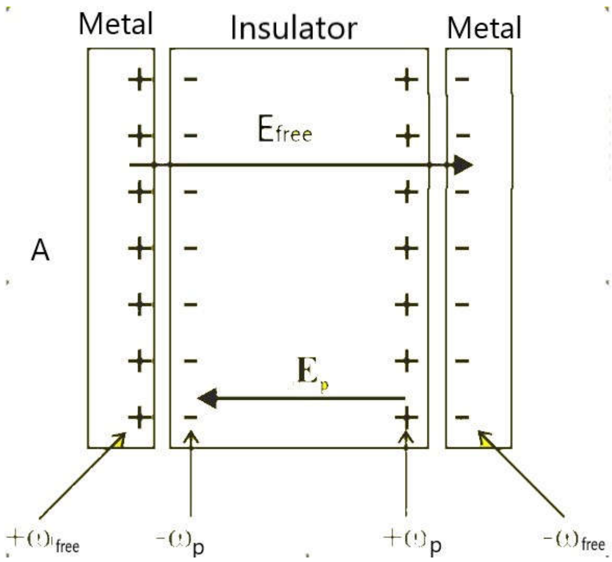


Fig. 2.3
Plate capacitor with polarized dielectric material

The metal plates of the capacitor contain what are called the “free charges”. The free charges are mobile and can be conducted away by means of a wire. In contrast to this, the polarization charges are immobile, so they can not move.

In chapter 1 the homogeneous electric field in plate capacitor has been expressed, provided the free surface charge densities ($+\omega_{free}$, $-\omega_{free}$) are located on the metal plates.

$$E_{free} = \frac{\omega_{free}}{\varepsilon_0} \qquad\qquad \varepsilon_0 E_{free} = \omega_{free}$$

The free electric field (E_{free}) is diminished by the effect of the polarization electric field (E_p), since the polarization electric field is always in opposite direction to the free electric field. So, the total electric field (E) results as below:

$$E = E_{free} - E_p$$

The last formula has been deduced for one dimensional case. The parameters show up here as they were real numbers. However, the result is true in full generality in three dimensions with vectors as well.

$$\mathbf{E} = \mathbf{E_{free}} - \mathbf{E_p}$$

$$\mathbf{E_{free}} = \mathbf{E} + \mathbf{E_p}$$

Note the important fact that the **E** electric field and so the intensity of forces are always reduced in presence of dielectric material.

2.4./ Electric permittivity (dielectric constant)

Experiments show that the polarization of some isotropic material is the monotonous function of the external electric field. At external fields of higher intensity, the insulating material is gradually saturated. At relatively low levels, the function can be considered linear. Our next discussion is confined to the linear range. This case the proportionality is valid between the polarization and the external electric field. In order to make an equation from the proportionality a coefficient (χ) is introduced.

$$\mathbf{E}_p = \chi \mathbf{E}$$

The coefficient is the electric susceptibility (χ).

Substitute this equation to the former expression of $\mathbf{E_{free}}$ vector:

$$\mathbf{E_{free}} = \mathbf{E} + \mathbf{E_p}$$

$$\mathbf{E_{free}} = \mathbf{E} + \chi\mathbf{E} = (1+\chi)\mathbf{E} = \varepsilon_r \mathbf{E}$$

Here the relative permittivity (ε_r) has been introduced:

$$1 + \chi = \varepsilon_r$$

Typical values of the relative permittivity are up to five or so. Very high numbers are technically impossible.

Finally, the results to be remembered is as follows:

$$\mathbf{E_{free}} = \varepsilon_r \mathbf{E} \qquad \left[\frac{V}{m}\right]$$

Here we must mention a historically introduced physical quantity, which is still being used. This is the vector of "dielectric displacement" $\mathbf{D}$ with the following definition:

$$\mathbf{D} = \varepsilon_0 \mathbf{E_{free}} = \varepsilon_0 \varepsilon_r \mathbf{E} \qquad \left[\frac{As}{m^2}\right]$$

2.6./ Inhomogeneous dielectric materials

Consider two different dielectric materials with plane surface. The plane surfaces are connected, thus creating an interface between the insulators. This structure is subjected to the experimentation.

First the $\mathbf{E_{free}}$ field is studied.

The interface is contained by a symmetrical disc-like drum with the base area A. The upper and lower surface vectors are $\mathbf{A_1}$ and $\mathbf{A_2}$ respectively.

$$\mathbf{A}_1 = -\mathbf{A}_2 \qquad\qquad |\mathbf{A}_1| = |\mathbf{A}_2| = A$$

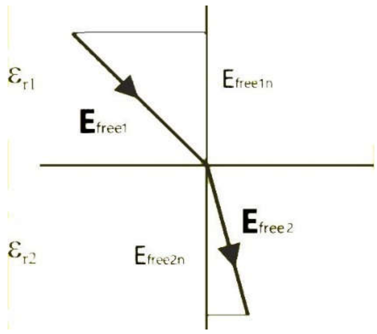


Fig. 2.4
$\mathbf{E_{free}}$ field at the interface of different dielectric materials

The volume does not contain free charges therefore the flux of the $\mathbf{E_{free}}$ vector is zero.

$$\oint_S \mathbf{E_{free}} d\mathbf{A} = \mathbf{E_{free1}} \cdot \mathbf{A}_1 + \mathbf{E_{free2}} \cdot \mathbf{A}_2 = 0$$

$$\mathbf{E_{free1}A_2} = \mathbf{E_{free2}A_2}$$

The operation of dot product contains the projection of the $\mathbf{E_{free}}$ vectors to the direction of $\mathbf{A_2}$ vector which is the normal direction to the surface. The subscript n means the magnitude of the normal direction component.

$$E_{free1n}A_2 = E_{free2n}A_2$$

Once we are among real numbers, the surface area cancels out readily.

$$E_{free1n} = E_{free2n}$$

According to this result the normal component of $\mathbf{E_{free}}$ vector is continuous on the interface of dielectric materials.

Secondly the electric field $\mathbf{E}$ is the subject of analysis.

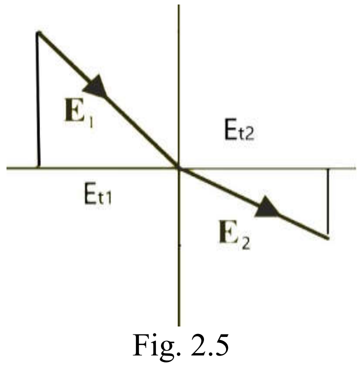


Fig. 2.5
$\mathbf{E}$ field at the interface of different dielectric materials

The interface is surrounded by a very narrow rectangle-like loop with sections parallel and normal to the surface. The parallel sections of the loop are **s** and **–s** vectors. The normal direction sections are ignored due to the infinitesimal size. The closed loop integral of the **E** in static electric field is zero.

$$\oint_g \mathbf{E} d\mathbf{r} = \mathbf{sE_1} + (-\mathbf{s})\mathbf{E_2} = 0$$

$$\mathbf{sE_1} = \mathbf{sE_2}$$

The operation of dot product contains the projection of the **E** vectors to the direction of **s** vector which is the tangential direction to the surface. The subscript t means the magnitude of the tangential direction component.

$$sE_{1t} = sE_{2t}$$

Once we are among real numbers the length of the tangential section cancels out readily.

$$E_{1t} = E_{2t}$$

According to this result, the tangential component of the **E** vector is continuous on the interface of dielectric materials.

2.7./ Demonstration example

A metal sphere with radius (R_1 = 10cm) contains free charges (Q_{free} = 10^{-8}As). The metal sphere is surrounded by an insulating layer (ε_r = 3) up to the radius (R_2 = 15cm). Find and sketch the radial dependence of $E(r), E_{free}(r)$ and $E_p(r)$ vectors. Determine the numerical peak values in the break points.

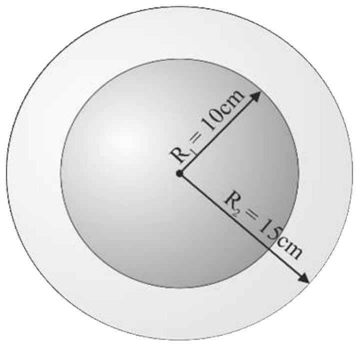


Fig. 2.6
Metal sphere surrounded by insulating layer

The first parameter to deal with is the $\mathbf{E_{free}}$ vector, because the normal component is continuous on the interface of dielectric materials. Let us use the Gauss's law.

$$\oint_S \mathbf{E_{free}} d\mathbf{A} = \frac{Q_{free}}{\varepsilon_0}$$

Inside the metal sphere all the parameters are zero, only out of the metal sphere is of interest.

$$4r^2\pi \cdot E_{free} = \frac{Q_{free}}{\varepsilon_0}$$

$$E_{free} = \frac{Q_{free}}{4\pi\varepsilon_0} \cdot \frac{1}{r^2}$$

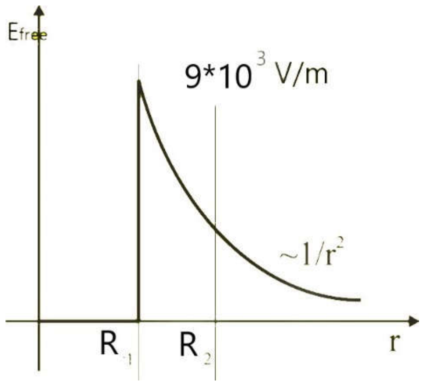


Fig. 2.7
The magnitude of E_{free} vs. radial position function

The peak value at the brake point results once $r = R_1$ is substituted.

$$E_{free}(R_1) = \frac{Q_{free}}{4\pi\varepsilon_0} \cdot \frac{1}{R_1^2} = \frac{10^{-8}}{4\pi\varepsilon_0} 10^2 = 9 \cdot 10^9 \cdot 10^{-8} \cdot 10^2 = 9 \cdot 10^3 \frac{V}{m}$$

Here we used a useful shortcut in substituting numerical values.

$$\frac{1}{4\pi\varepsilon_0} = 9 \cdot 10^9 \frac{Vm}{As}$$

It is worth remembering since it is frequently used:
The E field is equal with the E_{free} function, outside of the insulator. In the insulator however, the E function is reduced to its one third, according to $\varepsilon_r = 3$ value.

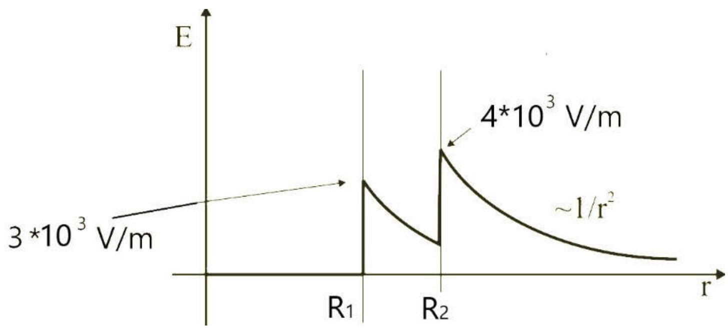


Fig. 2.8
The magnitude of E vs. radial position function

The E functions in the insulator and out of the structure are as follows:

$$E_{in} = \frac{Q_{free}}{4\pi\varepsilon_0\varepsilon_r} \cdot \frac{1}{r^2} \qquad E_{out} = \frac{Q_{free}}{4\pi\varepsilon_0} \cdot \frac{1}{r^2}$$

The brake point peak values of E function are as follows:

$$E_{in}(R_1) = \frac{Q_{free}}{4\pi\varepsilon_0\varepsilon_r} \cdot \frac{1}{R_1^2} = 9 \cdot 10^9 \cdot \frac{10^{-8}}{3} \cdot 10^2 = 3 \cdot 10^3 \frac{V}{m}$$

$$E_{out}(R_2) = \frac{Q_{free}}{4\pi\varepsilon_0} \cdot \frac{1}{R_2^2} = 9 \cdot 10^9 \cdot 10^{-8} \cdot \frac{1}{0.15^2} = 4 \cdot 10^3 \frac{V}{m}$$

The magnitude of the polarization electric field (E_p) is generally zero, except for the inside of the insulating material. In the insulating material this is as follows:

$$E = E_{free} - E_p$$

$$E_p = E_{free} - E$$

$$E_p = \frac{Q_{free}}{4\pi\varepsilon_0} \cdot \frac{1}{r^2} - \frac{Q_{free}}{4\pi\varepsilon_0\varepsilon_r} \cdot \frac{1}{r^2}$$

$$E_p = \left(1 - \frac{1}{\varepsilon_r}\right) \cdot \frac{Q_{free}}{4\pi\varepsilon_0} \cdot \frac{1}{r^2}$$

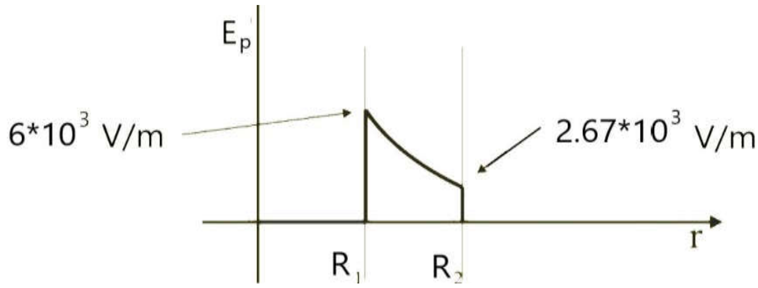


Fig. 2.9
The magnitude of E_p vs. radial position function

Let us determine the peak values in the break points:

$$E_p(R_1) = \left(1 - \frac{1}{\varepsilon_r}\right) \frac{Q_{free}}{4\pi\varepsilon_0} \cdot \frac{1}{R_1^2} = \left(1 - \frac{1}{3}\right) \cdot 9 \cdot 10^9 \cdot 10^{-8} \frac{1}{0.1^2} = 6 \cdot 10^3 \frac{V}{m}$$

$$E_p(R_2) = \left(1 - \frac{1}{\varepsilon_r}\right) \frac{Q_{free}}{4\pi\varepsilon_0} \cdot \frac{1}{R_2^2} = \left(1 - \frac{1}{3}\right) \cdot 9 \cdot 10^9 \cdot 10^{-8} \frac{1}{0.15^2} = 2.67 \cdot 10^3 \frac{V}{m}$$

2.8./ Energy relations

Any electrostatic charge arrangement represents potential energy. This energy equals the amount of work needed to create the arrangement.

2.8.1/ Energy stored in the capacitor

Consider a capacitor without charges initially. Carry an infinitesimal amount of dQ charge from one plate to the other. Therefore voltage (dQ/C) will appear between the plates. The next packet of dQ charge needs to be carried against the electric field generated by the previous packets. This way the voltage on the capacitor and the infinitesimal amounts of works will increase linearly. The triangle under the graph can represent the work done.
The total amount of work can be calculated by integration of those infinitesimal contributions.

$$E_{pot} = W = \int_0^Q U(Q')dQ' = \int_0^Q \frac{Q'}{C} dQ' = \frac{1}{C}\int_0^Q Q' dQ' = \frac{1}{C}\left[\frac{Q'^2}{2}\right]_{Q'=0}^{Q'=Q} = \frac{1}{2}\frac{Q^2}{C}$$

The fundamental formula $Q = CU$ can be combined into the result.

$$E_{pot} = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}\frac{(CU)^2}{C} = \frac{1}{2}CU^2$$

In practical cases the last formula is used, since the voltage is the known parameter mostly.

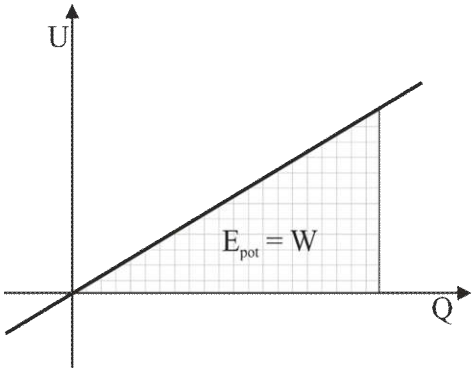


Fig. 2.16
Voltage vs. charge function

2.8.2 Electrostatic energy density

A plate capacitor is studied. The following pieces of information are at disposal:

$$U = Ed \qquad C = \varepsilon_0 \varepsilon_r \frac{A}{d} \qquad E_{pot} = \frac{1}{2} C U^2$$

The notations are as defined earlier. Let us substitute to the final formula:

$$E_{pot} = \frac{1}{2} C U^2 = \frac{1}{2} \varepsilon_0 \varepsilon_r \frac{A}{d} (Ed)^2 = \frac{1}{2} \varepsilon_0 \varepsilon_r E^2 (Ad)$$

In the last formula, the volume of the plate capacitor emerges. The energy density (e_{pot}) can be calculated as follows:

$$e_{pot} = \frac{E_{pot}}{Ad} = \frac{1}{2} \varepsilon_0 \varepsilon_r E^2 \qquad \left[\frac{J}{m^3} \right]$$

This result is also true in full generality in isotropic insulators.

$$e_{pot} = \frac{1}{2} \varepsilon_0 \varepsilon_r \mathbf{E}^2 \qquad \left[\frac{J}{m^3} \right]$$

2.8.3 Principle of the virtual work

Electrostatic forces can be determined with the principle of the virtual work, provided the potential energy of a charge arrangement can be expressed as the function of a position coordinate. This time, the derivative of the potential energy results the magnitude of force. The tedious integration of Coulomb's law can be replaced with the calculation of the potential energy, which is far easier task in most cases.
Demonstration example:
Find the pressure, exerted to the dielectric material between the two plates of a charged and disconnected plate capacitor.

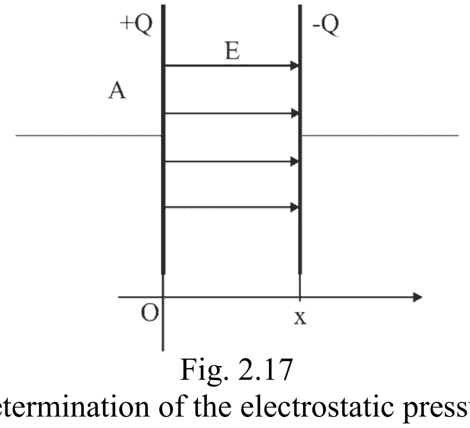


Fig. 2.17
Determination of the electrostatic pressure

The following pieces of information are at disposal:

$$C(x) = \varepsilon_0 \varepsilon_r \frac{A}{x} \qquad\qquad E_{pot} = \frac{1}{2}\frac{Q^2}{C}$$

The notations are defined as earlier. Let us substitute the formula of the capacitance to the final formula:

$$E_{pot} = \frac{1}{2}\frac{Q^2}{C} = \frac{Q^2}{2}\frac{1}{C} = \frac{Q^2}{2}\frac{x}{\varepsilon_0 \varepsilon_r A}$$

Let us make the derivation.

$$F = \frac{dE_{pot}}{dx}$$

The magnitude of the force is as follows:

$$F = \frac{Q^2}{2A\varepsilon_0 \varepsilon_r}$$

Thus, between the plates of the capacitor, a constant attractive force emerges. This is not surprising, since opposite charges are facing each other at a little distance.
The following pieces of additional information are at disposal:

$$Q = CU \qquad E = \frac{U}{d} \qquad C = \varepsilon_0 \varepsilon_r \frac{A}{d} \qquad F = \frac{Q^2}{2A\varepsilon_0 \varepsilon_r}$$

The notations are defined as earlier. The attractive pulling force density is a pressure over the surface. This pressure compresses the dielectric material in the capacitor. The pressure is denoted p. Let us substitute to the final formula:

$$pA = F = \frac{Q^2}{2A\varepsilon_0 \varepsilon_r} = \left(\varepsilon_0 \varepsilon_r \frac{A}{d}\right)^2 U^2 \frac{1}{2A\varepsilon_0 \varepsilon_r} = \frac{A}{2}\varepsilon_0 \varepsilon_r \left(\frac{U}{d}\right)^2 = \frac{A}{2}\varepsilon_0 \varepsilon_r E^2$$

The pressure can readily be expressed:

$$p = \frac{1}{2}\varepsilon_0 \varepsilon_r E^2 \qquad\qquad \left[Pa = \frac{J}{m^3}\right]$$

This formula showed up earlier in this chapter. The energy density, and the pressure to the dielectric material between the plates are expressed by the same formula.
The critical electric field (E_{cr}) is the limit at which electric discharge occurs. The manufacturers of the capacitors carefully approach this limit by using tough materials. So, the maximum pressure is determined approximately by the above formula at critical electric field intensity. For estimation purposes let us choose the following values: ε_r=3 and $E_{cr} = 10^6$ V/m.

$$p_{\max} = \frac{1}{2}\varepsilon_0 \varepsilon_r E_{cr}{}^2 = \frac{1}{2} 8.86 \cdot 10^{-12} \cdot 3 \cdot 10^{12} = 13.3 Pa$$

This pressure is an insignificant mechanical load on the dielectric material between the plates.

Chapter 3.

Stationary electric current - György Hárs

3.1./ Definition of Ampere

Consider two pieces of metal electrodes on different potentials. The voltage between them is the difference of the potentials. Now connect the electrodes by means of a wire. The experiment proves that electric current flows on the wire if the voltage is sustained. The value of the current is the time derivative of the charge transferred. The unit of electric current is Ampere [A] which is a fundamental quantity in the SI system. Therefore, the electric charge is a derived quantity and its unit is Ampere second [As], which is called Coulomb.

$$I = \frac{dQ}{dt}$$

Currents in proximity exert forces to each other. The definition of Ampere is based on the force interaction between two parallel wires which carry the same current. It is worth emphasizing here the important fact, that parallel direction currents attract while the opposite direction currents repel each other. This is somewhat in contrary to the anticipation, which might prompt otherwise.

It is also important to note, that the direction of electric current is downhill the potential field from the plus to the minus electrode. And this is always true, no matter what kind of charge carrier is involved. If the charge carrier is negative (mostly electron) then the direction of mechanical flow is just opposite to the current direction.

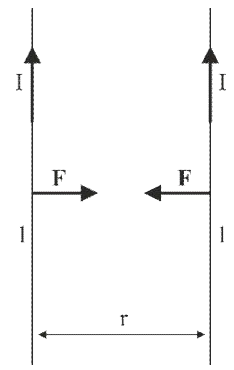


Fig. 3.1
Attractive force between parallel currents

Experiments show that the intensity of force (*F*) is proportional to the currents (*I*) and to the length (*l*) of the wire, while it is reversely proportional to the separation (*r*) of the parallel wires. To create equation from the proportionalities, a coefficient is introduced ($\mu_0/2\pi$).

$$F = \frac{\mu_0}{2\pi}\frac{I^2}{r}l$$

The parameter μ_0 is a universal constant in nature and this is called the permeability of vacuum. The numerical value is $4\pi 10^{-7}\ Vs/Am$.

Based on the formula, the definition of Ampere is as follows: The values are *1A* of two identical parallel currents if the attractive force is $2\ 10^{-7}\ N$ between them provided both the length of the wire and the separation are one meter. The force to be measured is obviously very small, so much higher current and far smaller separation are used in the real measurements.

3.2./ Current density (j)

The current density is a more essential physical quantity than the current itself. The current density is a vector. Its direction shows the local direction of the current. The flux of the current density on an open (*S*) surface results the actual current flowing through the rim of the surface. The measuring unit of the current density is A/m^2.

$$I = \int_S \mathbf{j} d\mathbf{A}$$

At homogeneous current density and plane surface the above integral can be replaced by the dot product of the current density and the corresponding area vector.

$$I = \mathbf{j} \cdot \mathbf{A}$$

If the current density vector and the area vector are parallel, then the magnitude of the current density can be expressed as below with simple division, where the current and the area are the magnitudes of the relevant quantities.

$$j = \frac{I}{A}$$

3.3./ Ohm's law

Experiments show that the current (I) is proportional to the voltage (U) applied. The coefficient between them is the conductance (G) of the conductor. The unit of the conductance is A/V called Siemens.

$$I = GU$$

The reciprocal value of the conductance is called the resistance (R). The unit of the resistance is V/A called Ohm (Ω).

$$R = \frac{1}{G} = \frac{U}{I}$$

The resistance of a cylindrical conductor is proportional to the length (l) and reversely proportional to the cross-sectional area (A). The coefficient is characteristic to the material of the conductor which is called the resistivity (ρ). Its measuring unit is ohm meter (*Vm/A*).

$$R = \rho \frac{l}{A}$$

The reciprocal of the resistivity is called the conductivity (σ):

$$\sigma = \frac{1}{\rho}$$

Let us substitute the expression of the resistance into the Ohm's law:

$$U = RI = \rho \frac{l}{A} I$$

Divide it with the length:

$$\frac{U}{l} = \rho \frac{I}{A}$$

The left-hand side is the electric field (E) in the conductor while the right-hand side is the current density (j).

$$E = \rho \cdot j$$

This equation is the **differential ohm's law**. The above formula is valid in full generality in vector quantities as well: By means of conductivity the formula is as follows:

$$\mathbf{E} = \rho \cdot \mathbf{j} \qquad\qquad \mathbf{j} = \sigma \cdot \mathbf{E}$$

3.4./ Joule's law

The power dissipated by the conductor is the time derivative of the work done by the electric field. The measuring unit is Watt (J/s = W).

$$P = \frac{dW}{dt} = U\frac{dQ}{dt} = UI$$

This formula is the Joule's law. Combining it with the Ohm's law the following formulas can be concluded.

$$P = UI = RI^2 = \frac{U^2}{R}$$

Let us use the formulas of U and I and substitute them to the above equation.

$$P = UI = (El)(jA) = Ej(Al)$$

The Al product is the volume of the conductor. After dividing with the volume, the power density (p) can be expressed: The measuring unit is watt per cubic meter (W/m^3).

$$p = E \cdot j$$

This formula is the **differential Joule's law**, which is valid between vector quantities as dot-products: $p = \mathbf{E} \cdot \mathbf{j}$. Involving the differential Ohm's law two more expressions can be found:

$$p = \sigma \cdot \mathbf{E}^2 \qquad\qquad p = \rho \cdot \mathbf{j}^2$$

3.5./ Microphysical interpretation

The charge carriers collide frequently with the ionic lattice in the conductive material. Between collisions they are accelerated by the electric field. So, the motion consists of short acceleration periods and sudden stops. The resulting motion can be characterized by the average speed which is called the drift velocity (v_{drift}). Surprisingly, this value is very small, roughly one meter per hour. Experiments show that the drift velocity is proportional to the electric field affecting the conductor. The coefficient is called the mobility (μ).

$$v_{drift} = \mu \cdot E$$

Consider a piece of conducting material with cross sectional area A. The material contains charge carriers with the density n and with charge q. The infinitesimal amount of charge, transferred by the material in infinitesimal time period is as follows:

$$dQ = v_{drift} dt \cdot A \cdot nq$$

The current can be expressed:

$$\frac{dQ}{dt} = I = v_{drift} \cdot A \cdot nq$$

The current density can also be calculated:

$$j = v_{drift} \cdot nq$$

Now we substitute the mobility:

$$j = \mu nq \cdot E$$

Compare this result with the differential Ohm's law. This provides a microphysical substantiation to the conductivity, which was introduced earlier as a phenomenological material parameter.

$$\sigma = \mu nq$$

Accordingly, the conductivity of some material depends on two major factors such as the mobility and the density of the charge carriers. If several different types of charge carriers are involved in the current, the individual conductivities and mobilities should all be counted.

If the temperature of the material is increased the conductivity can either increase or decrease. The conductivity decreases by the reduction of the mobility, due to the more frequent collisions with the even more oscillating ionic lattice (mostly metals). Increasing in the conductivity occurs if the generation of the charge carriers overcompensate the former effect and becomes dominant (mostly semiconductors).

Chapter 4.

Magnetic phenomena in space György-Hárs

4.1./ The vector of magnetic induction (B)

In chapter 3 the definition of Ampere is based on the attractive force between two identical parallel currents. In this chapter the current values can be different. One of the currents is considered to be the source-current (*I*), while the other one is the test-current (*i*). Accordingly, the intensity of force (*F*) is proportional to the currents (*I, i*) and to the length (*l*) of the wire, while it is reversely proportional to the separation (*r*) of the parallel wires. To create equation from the proportionalities a coefficient is introduced ($\mu_0/2\pi$).

$$F = \frac{\mu_0}{2\pi}\frac{Ii}{r}l$$

The parameter μ_0 is a universal constant in nature and this is called the permeability of vacuum. The numerical value is $4\pi 10^{-7}$ Vs/Am.

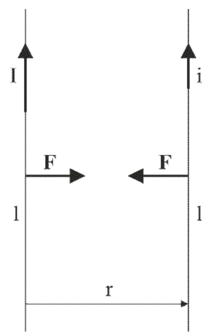


Fig. 4.1
Parallel currents (side view)

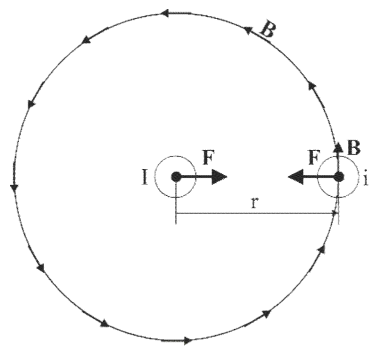


Fig. 4.2
Parallel currents (upper view)

Experiments showed that the test current (*i*) and the length (*l*) of the test wire are proportional to the force. Accordingly, the force can be written as follows:

$$F = \frac{\mu_0}{2\pi}\frac{Ii}{r}l = Bil$$

Here *B* is a coefficient which is determined solely by the source current (*I*), and somehow characteristic to the magnetization level of the space (magnetic induction field) generated by the source current.

$$\frac{\mu_0}{2\pi}\frac{I}{r} = B \qquad \left[\frac{Vs}{m^2} = Tesla\right]$$

Up to this point, this almost looks like if *B* were a scalar. This is not the case. The *B* will be the vector of the magnetic induction (**B**) with the definition below:

Let us study the upper view of the currents (Fig 4.2). Due to the cylindrical symmetry of the infinite straight current the magnetic induction lines are circles around the current. Let us attribute right hand screw turning direction to the lines relative to the current direction. Accordingly, when current (*I*) flows out of the sheet of the paper, the magnetic induction (**B**) lines go around current in counterclockwise (CCW) direction.

4.2./ The Lorentz force

On the other hand, the attractive force vector points toward the source current. This direction is perpendicular both to the direction of **B** vector and the test current. This relation implies the application of vector product as a mathematical means.

$$\mathbf{F} = i\mathbf{l} \times \mathbf{B}$$

The above formula describes accurately both the direction and the magnitude of the force. The direction of the current is turned into the direction of the magnetic induction, and the right-hand screw turning will determine the direction of the force.

The test wire is not necessarily straight, this can be any curve. Very small (infinitesimal) section of a curve can be considered straight, so the above formula is valid for the infinitesimal contribution to the force.

$$d\mathbf{F} = id\mathbf{l} \times \mathbf{B}$$

The total force results as the curve integral of the contributions.

$$\mathbf{F} = \int_g id\mathbf{l} \times \mathbf{B}$$

The validity of the earlier formula can be extended to point charges traveling in the space. If a point charge moves, this is equivalent with a certain current. This relation is summarized below:

$$id\mathbf{l} = \mathbf{j}A \cdot dl = \mathbf{v}\rho \cdot Adl = \mathbf{v} \cdot \rho Adl = \mathbf{v}dQ \qquad [Am]$$

Here we used the expression of current density by means of the charge density and the velocity: $\mathbf{j} = \rho\mathbf{v}$

The above expression can be substituted to expression of force:

$$d\mathbf{F} = dQ(\mathbf{v} \times \mathbf{B})$$

In case of a definite point charge there is a definite amount of charge and so the force is a definite vector too.

$$\mathbf{F} = Q(\mathbf{v} \times \mathbf{B})$$

This formula describes the Lorentz force. Accordingly, the magnetic field may affect a charged particle only when it moves. Standstill particle does not "feel" the magnetic field. If the above formula is divided with the charge, the Lorentz electric field ($\mathbf{E_L}$) is the result.

$$\mathbf{E}_\mathrm{L} = \mathbf{v} \times \mathbf{B}$$

This quantity will be used extensively in connection with the motion related electromagnetic induction phenomena.

4.2.1./ Cyclotron frequency

Consider homogeneous magnetic field (B = 0.1 *Tesla*). Inject a proton (m = 1.67 10^{-27}kg, q=1.6 10^{-19} As) normal to the magnetic field with initial kinetic energy (U_0 = 1 keV). Determine how the particle moves in the field.

The Lorentz force is always normal to the velocity, therefore the magnitude of the velocity (speed) and so the kinetic energy of the particle is constant. The Lorentz force generates only centripetal acceleration, and this way the particle goes around a circular trajectory. The parameters of the motion can be determined by means of the equation of motion (with the usual notations).

$$qvB = m\frac{v^2}{r}$$

The equation above is written in the radial direction of the circle. The left-hand side is the magnitude of the Lorentz force, while the right-hand side is the mass multiplied with the centripetal acceleration. After some ordering, the angular cyclotron frequency results.

$$\frac{qB}{m}=\frac{v}{r}=\omega_c=\frac{1.6\cdot 10^{-19}\cdot 0.1}{1.67\cdot 10^{-27}}=9.58\cdot 10^{6}\frac{rad}{s}=1.53MHz$$

The velocity can be calculated from the initial kinetic energy:

$$\frac{1}{2}mv^2=qU_0$$

$$v=\sqrt{\frac{2qU_0}{m}}=\sqrt{\frac{2\cdot 1.6\cdot 10^{-19}\cdot 10^{3}}{1.67\cdot 10^{-27}}}=4.38\cdot 10^{5}\frac{m}{s}$$

The radius of the circulation can be expressed:

$$r=\frac{v}{\omega_c}=\frac{4.38\cdot 10^{5}}{9.58\cdot 10^{6}}=4.57\cdot 10^{-2}m=4.57cm$$

Note the cyclotron frequency is independent of the energy of the particle. This feature made possible to construct the first particle accelerator (1932 Ernest Lawrence). The charged particles are forced to circulate by means of homogeneous magnetic field. They are accelerated with a high frequency electric field, which is in resonance with the cyclotron frequency. As the energy of the particles grew the radius of the circulation increased, but the cyclotron frequency did not change, so the resonance stayed. The particles could be accelerated to high kinetic energies in classical range. At even higher energies, the relativistic description is necessary.

If the injection of the particle is not fully perpendicular to the **B** vectors, then the initial velocity should be decomposed to parallel and normal components. The parallel component is unaffected by the magnetic field while the normal component generates uniform circulation with the cyclotron frequency. Ultimately, the trajectory of the particle is twisted around the magnetic induction lines. This feature is used extensively in plasma generation techniques, when additional external magnetic field is used to increase the efficiency of the ionization, by increasing the path length of the charged particles.

4.2.2./ The Hall effect

The effect was discovered by Edwin Hall in 1879. Consider a layer of a conducting material in the form of a stripe. The direct current (*I*) flows parallel with the longer dimension. Homogeneous magnetic field (*B*) crosses the material normal to the surface. The Hall voltage is measured between the two sides of the stripe.

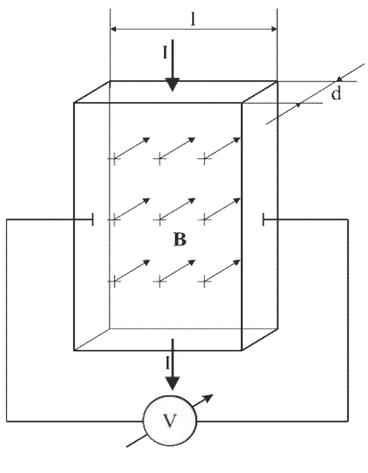


Fig. 4.3 Hall effect

The following pieces of information are at disposal:

$$I = j \cdot l \cdot d \qquad [A] \qquad\qquad j = v_{drift} \cdot nq \qquad \left[\frac{A}{m^2}\right]$$

Here *l* and *d* are the width and the thickness of the stripe, respectively.

$$I = v_{drift} \cdot nq \cdot l \cdot d$$

The drift velocity can be expressed:

$$v_{drift} = \frac{I}{nq \cdot l \cdot d}$$

The magnitude of the Lorentz electric field is merely the product of the factors due to the perpendicular arrangement.

$$E_L = v_{drift} B$$

The generated Hall voltage is the product of the width (*l*) and the Lorentz field intensity. Integration can be omitted because the field is homogeneous.

$$U_L = v_{drift} B \cdot l = \frac{BI}{nq \cdot d} = \frac{1}{nq}\frac{BI}{d} = R_H \frac{BI}{d}$$

Here R_H is the Hall coefficient as follows:

$$R_H = \frac{1}{nq} \qquad \left[\frac{m^3}{As}\right]$$

The formula shows that the polarity of the Hall coefficient depends on the charge carrier polarity. In all other electrodynamic experiments, a current when electrons move from left to right or positive particles move from the right to left generates the same physical effect. So, one never knows just from the current, which is the case from these two above. Hall effect is the only experiment in which the polarity of the charge carrier makes a qualitative difference.
In modern electronic devices Hall detector is used mostly for measuring magnetic field. It is also used as commutators in electric motors and in the ignition system of the cars.

4.3./ Magnetic dipole

Consider a circular loop current (*I*) with given radius (*r*). The center of the circle is the origin of the Cartesian coordinate system, and the circle is in the x, y plane. The loop current is surrounded by homogeneous magnetic (**B**) field in arbitrary direction. The infinitesimal force vector affecting an infinitesimal section of the loop is as follows:

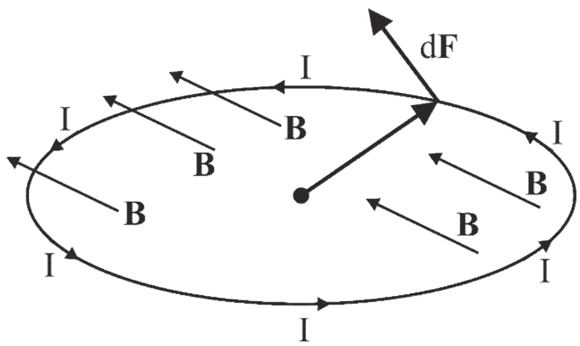


Fig. 4.4
Determination of the torque affecting a current loop

$$d\mathbf{F} = Id\mathbf{r} \times \mathbf{B}$$

The infinitesimal contribution of torque is based on the definition:

$$d\mathbf{M} = \mathbf{r} \times d\mathbf{F} = \mathbf{r} \times (Id\mathbf{r} \times \mathbf{B})$$

Now we use the formula for triple product: $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{ac}) - \mathbf{c}(\mathbf{ab})$

Accordingly:

$$d\mathbf{M} = Id\mathbf{r}(\mathbf{B} \cdot \mathbf{r}) - \mathbf{B}(\mathbf{r} \cdot d\mathbf{r})$$

The second term is zero because the **r** and $d\mathbf{r}$ vectors are perpendicular due to the circle. So ultimately the torque to be integrated is the following: $d\mathbf{M} = Id\mathbf{r}(\mathbf{B} \cdot \mathbf{r})$

$$\mathbf{M} = \int_0^{2\pi} d\mathbf{M} = I \int_0^{2\pi} (\mathbf{B} \cdot \mathbf{r}) d\mathbf{r}$$

Next the formulas to be substituted:

$$\mathbf{B} = B_x \mathbf{i} + B_y \mathbf{j} + B_z \mathbf{k}$$

$$\mathbf{r} = \mathbf{i} \cdot r\cos\varphi + \mathbf{j} \cdot r\sin\varphi + 0 \cdot \mathbf{k}$$

$$d\mathbf{r} = \frac{d\mathbf{r}}{d\varphi} d\varphi = r(-\mathbf{i} \cdot \sin\varphi + \mathbf{j} \cdot \cos\varphi) d\varphi$$

The formula is simplified by the following rule: **ij = jk = ki =** 0 and $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = 1$

$$\mathbf{M} = Ir^2 \int_0^{2\pi} (B_x \cos\varphi + B_y \sin\varphi)(-\mathbf{i} \cdot \sin\varphi + \mathbf{j} \cdot \cos\varphi) d\varphi$$

$$\frac{\mathbf{M}}{Ir^2} = \mathbf{j} \int_0^{2\pi} (B_x \cos\varphi + B_y \sin\varphi) \cos\varphi \cdot d\varphi - \mathbf{i} \int_0^{2\pi} (B_x \cos\varphi + B_y \sin\varphi) \sin\varphi \cdot d\varphi$$

$$\frac{\mathbf{M}}{Ir^2} = \mathbf{j} \int_0^{2\pi} (B_x \cos^2\varphi + B_y \sin\varphi\cos\varphi) \cdot d\varphi - \mathbf{i} \int_0^{2\pi} (B_x \sin\varphi\cos\varphi + B_y \sin^2\varphi) \cdot d\varphi$$

$$\frac{\mathbf{M}}{Ir^2} = \mathbf{j}B_x \int_0^{2\pi} \cos^2\varphi \cdot d\varphi + \mathbf{j}B_y \int_0^{2\pi} \sin\varphi\cos\varphi \cdot d\varphi - \mathbf{i}B_y \int_0^{2\pi} \sin^2\varphi \cdot d\varphi - -\mathbf{i}B_x \int_0^{2\pi} \sin\varphi\cos\varphi \cdot d\varphi$$

The sine and cosine functions are orthogonal functions, therefore the integral below is zero:

$$\int_0^{2\pi} \sin\varphi\cos\varphi \cdot d\varphi = 0$$

The only remaining terms are either pure sine or pure cosine functions.

$$\mathbf{M} = Ir^2 \left(\mathbf{j}B_x \int_0^{2\pi} \cos^2\varphi \cdot d\varphi - \mathbf{i}B_y \int_0^{2\pi} \sin^2\varphi \cdot d\varphi \right)$$

$$\int_0^{2\pi} \sin^2\varphi \cdot d\varphi = \int_0^{2\pi} \cos^2\varphi \cdot d\varphi = \pi$$

The values of the integral can be calculated:

$$\mathbf{M} = Ir^2 \left(B_x\pi \cdot \mathbf{j} - B_y\pi \cdot \mathbf{i} \right) = Ir^2\pi \left(-B_y \cdot \mathbf{i} + B_x \cdot \mathbf{j} \right)$$

Here one can find the area of the circle. $A = r^2\pi$

Finally, the formula of torque emerges.

$$\mathbf{M} = IA \cdot \left(-B_y \cdot \mathbf{i} + B_x \cdot \mathbf{j} \right)$$

This formula is not easy to handle let alone to remember that. Therefore, the following argument gives a far more memorable description of the result.

If one attributes vector character to the area as it was done earlier, the torque above can be interpreted much easier.

$$\mathbf{M} = I\mathbf{A} \times \mathbf{B}$$

Let us verify this cross product by calculating the components.

$$\mathbf{M} = I\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & A \\ B_x & B_y & B_z \end{vmatrix} = IA\cdot\left(-B_y\cdot\mathbf{i} + B_x\cdot\mathbf{j}\right)$$

This is a perfect match with the formula above.
The first factor is the magnetic dipole moment (**m**) of the current loop.

$$\mathbf{m} = I\mathbf{A} \qquad \left[Am^2\right]$$

By means of this physical quantity, the final formula of the torque results:

$$\mathbf{M} = \mathbf{m}\times\mathbf{B} \qquad \left[Am^2\cdot\frac{Vs}{m^2} = Nm\right]$$

The magnetic dipole moment is turned into the direction of the external magnetic field spontaneously and stays there. Having reached this position, the least amount of potential energy is stored in the magnetic dipole. Obviously, the most amount of potential energy stored is just in the opposite position. Let us find out the work needed to turn the dipole from the deepest position to the highest energy.

$$W = \int_0^\pi Md\varphi = \int_0^\pi mB\sin\varphi\cdot d\varphi = mB\int_0^\pi \sin\varphi\cdot d\varphi = -mB\left[\cos\varphi\right]_0^\pi = 2mB$$

According to this result the potential energy of the magnetic dipole is as follows:

$$E_{pot} = -\mathbf{m}\cdot\mathbf{B} \qquad \left[Nm = J\right]$$

This formula provides the deepest energy at parallel, spontaneous position and the highest at anti-parallel position. The zero potential energy is at ninety degrees. The difference between the highest and lowest is just the work needed to turn it around.
Note:
Newton meter is not always Joule. If there is dot product it is *J*, but if it is cross product to make torque, the result is *Nm* only.

4.4./ Earth as a magnetic dipole

Magnetic dipole moment of a solenoid is directed according to the right-hand screw rule. This means that circulating parallel with the circulation of the current, the progress of the right-hand screw defines the direction of the magnetic dipole moment. Let us suspend the solenoid in its center of gravity on a thin thread which provides free turning in the horizontal plane. The solenoid slowly turns parallel to the Earth's magnetic field such a way, that the magnetic dipole moment points to the geographic North Pole. If a permanent magnet rod is suspended in the same way as the solenoid, this will also turn parallel to the Earth's magnetic field.
In case of an ordinary dipoles such as a solenoid or a bar magnet the magnetic dipole moment is directed from the south end to the north end. The magnetic induction lines are virtually exiting from the north end, and entering the south end.

Planet Earth is an exceptional magnetic dipole because the geographic North Pole is in fact a magnetic South Pole. This weird-looking switch is required to dissolve the contradiction of naming the poles of an ordinary dipole. That end of an ordinary dipole is called north-end which is closer to the north geographic pole of the Earth. However, the same kinds of poles repel each other so the mentioned switch clears the situation.

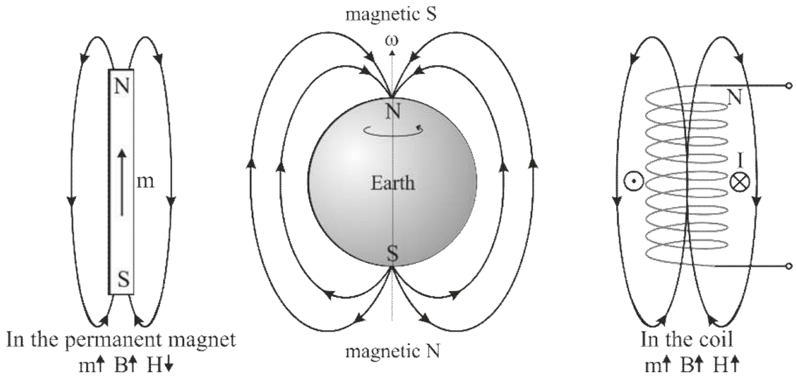


Fig. 4.5
The geographic North Pole of the Earth is in fact a magnetic South Pole.

4.5./ Biot-Savart law

The magnetic field has been characterized by the “magnetic induction” (**B** vector) field so far. This type of description is based on the generated forces and torques.
In order to describe the magnetic field, initiated by the electric current which generated it, the “magnetic field” **(H** vector) will be used with the simple definition below:

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} \qquad \left[\frac{A}{m}\right]$$

$$\mathbf{B} = \mu_0 \mathbf{H} \qquad \mu_0 = 4\pi \cdot 10^{-7} \frac{Vs}{Am}$$

The importance of these two fields will be plausible in the next chapter, when dealing with role of the magnetic materials. In this chapter their physical meaning is the same, apart from a constant multiplier.
The magnetic field of an infinitesimal current element is described by the Biot-Savart law.

$$d\mathbf{H} = \frac{I}{4\pi} \frac{d\mathbf{l} \times \mathbf{r}_0}{r^2}$$

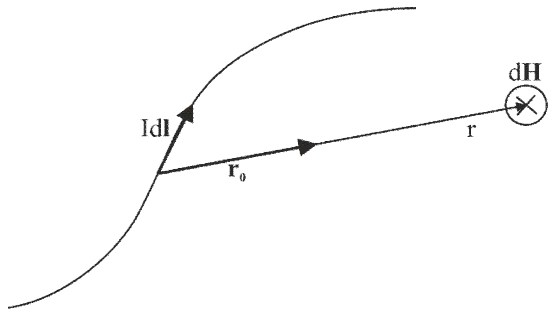


Fig 4.6 Biot-Savart law

The infinitesimal current element is surrounded by circular magnetic field. The rotation of the magnetic field is in accordance with the right-hand screw rule. In the equatorial plane of the current element, the magnetic field diminishes with the negative second power (just like Coulomb’s law). Below and above the equatorial plane the magnetic field diminishes with the increasing angle and vanishes on the line of the current element. The infinitesimal magnetic contributions can be superimposed, and the overall magnetic effect of any extended current can be calculated by integration. The Biot–Savart law is used typically for currents in a thin wire where the integration by line provides a fair result.

4.5.1./ Magnetic field of the finite long straight current

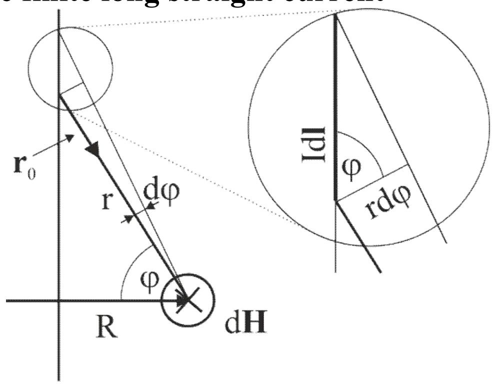


Fig. 4.7 Biot-Savart law for a straight wire

The following pieces of information are at disposal:

$$d\mathbf{H} = \frac{I}{4\pi}\frac{d\mathbf{l}\times\mathbf{r}_0}{r^2} \qquad |d\mathbf{l}| = \frac{rd\varphi}{\cos\varphi} \qquad r = \frac{R}{\cos\varphi}$$

All contributions of the magnetic field point to the same direction therefore the integration of the magnitude is satisfactory.

$$|d\mathbf{H}| = \frac{I}{4\pi}\frac{|d\mathbf{l}|\cdot|\mathbf{r_0}|\sin(\varphi+90^0)}{r^2} = \frac{I}{4\pi}\frac{rd\varphi}{\cos\varphi}\frac{\cos\varphi}{r^2} = \frac{I}{4\pi}\frac{d\varphi}{r} = \frac{I}{4\pi}\frac{\cos\varphi}{R}d\varphi$$

After some ordering the infinitesimal contribution results:

$$dH = \frac{I}{4R\pi}\cos\varphi\cdot d\varphi$$

The integration will be carried out for symmetrical α half visual angle domain.

$$H = \int_{-\alpha}^{\alpha}\frac{I}{4R\pi}\cos\varphi\cdot d\varphi = \frac{I}{4R\pi}[\sin\varphi]_{-\alpha}^{\alpha} = \frac{I}{2R\pi}\sin\alpha$$

The magnetic field of a finite current section under symmetrical α half visual angle domain can finally be expressed.

$$H = \frac{I}{2R\pi}\sin\alpha$$

If the current tends to the infinity, then α tends to ninety degrees so sinα equals unit. This way for infinite long filament the result is as follows:

$$H = \frac{I}{2R\pi}$$

4.5.2./ Central magnetic field of the polygon and of the circle

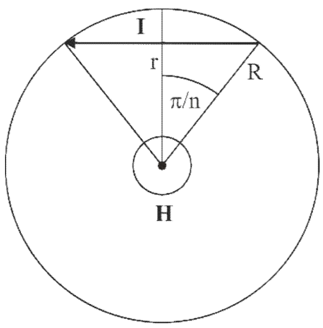


Fig. 4.8
Regular polygon

Consider a part of a regular polygon with sides *n* which carries a current *I*. The containing circle has the radius *R*. The half central angle is *π/n*. The magnetic effect of one side is denoted H^*. For other notations see the figure above.
The following pieces of information are at disposal:

$$H^* = \frac{I}{2r\pi}\sin(\frac{\pi}{n}) \qquad r = R\cos(\frac{\pi}{n}) \qquad H_n = n \cdot H^*$$

After substitution:

$$H_n = n\frac{I}{2\pi}\frac{1}{R\cos(\frac{\pi}{n})}\sin(\frac{\pi}{n}) = \frac{I}{2R}\left(\frac{n}{\pi}\right)\frac{\sin(\frac{\pi}{n})}{\cos(\frac{\pi}{n})} = \frac{I}{2R}\left(\frac{n}{\pi}\right)tg(\frac{\pi}{n})$$

So altogether the magnetic field in the center of the n sided regular polygon is as follows:

$$H_n = \frac{I}{2R}\left(\frac{n}{\pi}\right)tg(\frac{\pi}{n})$$

Please find some numerical values:

$$H_3 = \frac{I}{2R}\left(\frac{3}{\pi}\right)tg60^0 = 1.65\frac{I}{2R}$$

$$H_4 = \frac{I}{2R}\left(\frac{4}{\pi}\right)tg45^0 = 1.27\frac{I}{2R}$$

$$H_5 = \frac{I}{2R}\left(\frac{5}{\pi}\right)tg36^0 = 1.15\frac{I}{2R}$$

$$H_6 = \frac{I}{2R}\left(\frac{6}{\pi}\right)tg30^0 = 1.10\frac{I}{2R}$$

Let us make a transformation on the original result:

$$H_n = \frac{I}{2R}\left(\frac{n}{\pi}\right)tg(\frac{\pi}{n}) = \frac{I}{2R}\left(\frac{tg(\frac{\pi}{n})}{(\frac{\pi}{n})}\right)$$

If the number of sides tends to the infinity, then the polygon will tend to the circle. The limit value of the big parenthesis is unit. Ultimately the magnetic field in the center of the circle is equal with the current over the diameter (worth remembering).

$$H_{circle} = H_\infty = \frac{I}{2R}$$

4.6./ Ampere's law

The magnetic field of the infinite long filament was reached in this chapter above.

$$H = \frac{I}{2r\pi}$$

One can transform this formula in the following way:

$$2r\pi \cdot H(r) = I$$

If the circumference of a circle is multiplied with the actual magnetic field, the result will be independent of the radius and will be equal with the current. Since the magnetic field is constant on a radius one may write the above equation also by means of curve integral on the circle.

$$\oint_{circle} \mathbf{H}(\mathbf{r})d\mathbf{r} = I$$

This equation prompts the following hypothesis. May be the integration path needs not to be a circle, but this could be any closed loop around the current. This hypothesis is proven, and it is called the Ampere's law. The current may even be the sum of several currents. The direction of integration determines the direction of positive currents based on the right-hand screw rule.

$$\oint_{loop} \mathbf{H}(\mathbf{r})d\mathbf{r} = \sum I$$

The proof the Ampere's law follows next:
The figure below shows an infinite straight current normal to the sheet of paper pointing up just in front of our eyes. An arbitrary closed loop surrounds the current which is the path of the line integral. The current is also surrounded by several concentric circles in equidistant steps. The path of the integration can be approximated by very small (infinitesimal) sections which are either in tangential or in radial steps relative to the concentric circles. This way the integration can be carried out by moving on any of the circles or by moving radial direction. No contribution is generated in radial direction since the dot product vanishes at perpendicular step. On the circles however the contribution is the product of the magnetic field and the length of the arc. The corresponding central angle is donated φ.

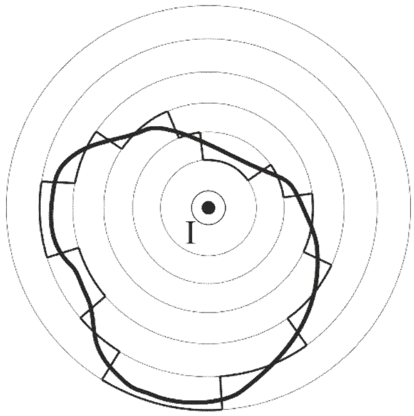


Fig. 4.9 Ampere's law proof

$$\oint_{loop} \mathbf{H}(\mathbf{r})d\mathbf{r} = (\varphi_1 r_1)\cdot\frac{I}{2r_1\pi} + (\varphi_2 r_2)\cdot\frac{I}{2r_2\pi} + (\varphi_3 r_3)\cdot\frac{I}{2r_3\pi} + ...(\varphi_n r_n)\cdot\frac{I}{2r_n\pi}$$

The radii all cancel out.

$$\oint_{loop} \mathbf{H}(\mathbf{r})d\mathbf{r} = \frac{I}{2\pi}\left(\varphi_1 + \varphi_2 + \varphi_3 + ... + \varphi_n\right) = I$$

The sum of the central angles stacks up to 2π due to the closed loop. So altogether the angles cancel out. Ultimately, the statement to be proven is the result. Q.E.D.

Application of Ampere's law for solving problems requires similar considerations to that of the Gauss's law. Ampere's law is an integral law. If one wants to use it for finding out local magnetic field, the symmetries or regularities of the magnetic field must be known prior to the application. If so, one must choose the path of the integration in which the magnetic field is constant, thus the integral converts to a simple product. The actual magnetic field results after dividing this product with the length of the integration path.

4.6.1./ Thick rod with uniform current density

Consider a thick metal rod wit radius R, which conducts current with uniform current density denoted j. Find the intensity of the magnetic field as the function of radius.

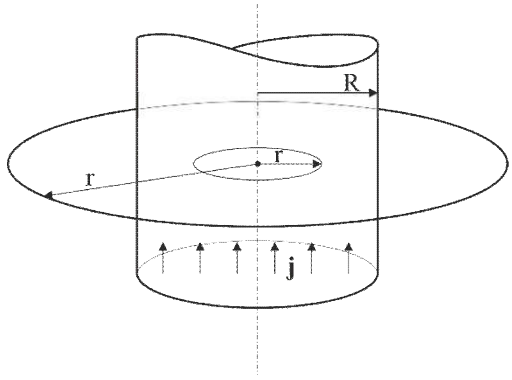


Fig. 4.10 Current in the thick rod

The solution uses Ampere's law. A circle is expanded in radius from zero to the infinity.

$$\oint_{loop} \mathbf{H}(\mathbf{r})d\mathbf{r} = \sum I$$

In the rod:

$$2r\pi \cdot H = r^2\pi \cdot j$$

$$H = \frac{j}{2}r$$

Out of the rod

$$2r\pi \cdot H = R^2\pi \cdot j$$

$$H = \frac{j}{2}\frac{R^2}{r}$$

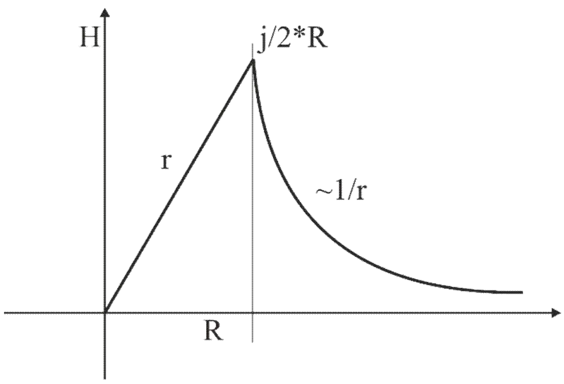


Fig. 4.11 *H(r)* function

Inside there is a linear slope of the magnetic field intensity. Outside, the intensity decays like a hyperbola. The function is continuous on the surface, and the maximum value is $H = \frac{j}{2}R$.

4.6.2./ Solenoid

This is a straight rod coil. The physical model of solenoid requires the length to be roughly ten times longer then the diameter.

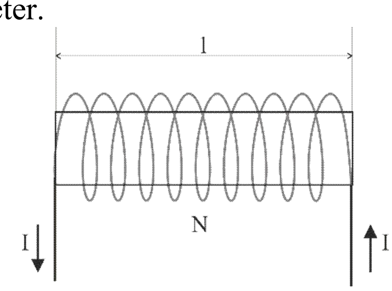


Fig. 4.12
The solenoid coil

Let us use Ampere's law. The magnetic field is homogeneous in the cavity of the solenoid. A closed path is chosen, which is parallel with the coil and located in the cavity on one side. The front side of the path is outside the coil where there is no magnetic field. The remaining two little sides of the rectangle are normal to the magnetic field thus can be ignored. The length is denoted l and the number of turns is N. So ultimately the Ampere's law emerges in a simple form:

$$\oint_{loop} \mathbf{H}(\mathbf{r})d\mathbf{r} = \sum I$$
$$Hl = NI$$
$$H = \frac{NI}{l}$$

4.6.3./ Toroid coil

This is a doughnut shape coil. The physical model requires the circumference to be roughly ten times longer then the diameter.

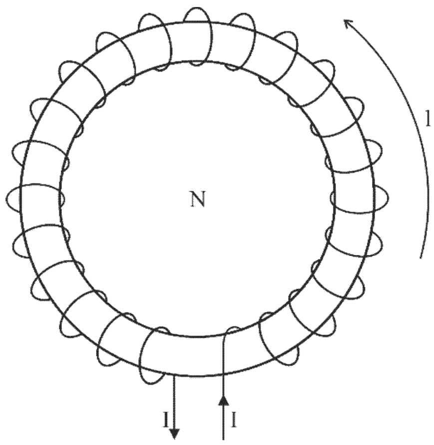


Fig. 4.13
The toroidal coil

Let us use Ampere's law again. The magnetic field is homogeneous in the cavity of the coil. A closed circular path is chosen, which is running in the central of the cavity. The circumference is denoted l and the number of turns is N. So ultimately the Ampere's law emerges in a simple form:

$$\oint_{loop} \mathbf{H}(\mathbf{r})d\mathbf{r} = \sum I$$
$$Hl = NI$$
$$H = \frac{NI}{l}$$

4.7./ Magnetic flux

The general concept of flux has been treated earlier. This is a scalar value surface integral of some vector field. The vector field in present case is the field of magnetic induction **B(r)**. The magnetic flux as follows:

$$\Phi_m = \int_g \mathbf{B}d\mathbf{A} \qquad [Vs]$$

In case of both solenoid and toroid, the formula of the magnetic turn flux is as follows:

$$B = \frac{\mu_0 NI}{l} \qquad \Phi_{turn} = BA = \frac{\mu_0 NIA}{l} \qquad \Phi_{coil} = \frac{\mu_0 N^2 IA}{l}$$

Later coil flux will also be used in conjunction with the induced voltage of the coil.

Chapter 5.

Magnetic field and the materials - György Hárs

In the former chapter magnetic phenomena without any magnetic material in the surrounding space have been treated. In technology however, the magnetic field is used in conjunction with magnetic materials, which enhance the intensity of the forces and the related interactions.

5.1./ Three basic types of magnetic behavior

Non-physicist public opinion divides the materials such as magnetic and non-magnetic materials. The former one is mostly iron and some alloys which are attracted by magnet, and the latter ones are the rest of the materials which are seemingly unaffected by magnet. However, all materials are affected by the magnetic field though the intensity of the attraction varies several orders of magnitude. The commonly mentioned "magnetic materials" are in fact the ferromagnetic materials in technical terms. The nonmagnetic materials can be classified to two distinct groups such as paramagnetic and diamagnetic materials in which the intensity of the interaction is so low that it is simply overlooked by the easy observer.

The following experiment makes it possible to distinguish between the sorts of the magnetic behavior: Make little samples of the materials to be tested. The sample geometry is roughly fifty-millimeter-long and five millimeter in diameter cylindrical rod. In the symmetry axis there is an indentation normal to the rotational axis, which contains a needle on which the rod can be rotated freely. The depth of the indentation is roughly eighty percent of the rod diameter. The rotatable sample is placed into the airgap of the unexcited toroidal electromagnet, which can create high intensity homogeneous magnetic field. Now switch the electromagnet which creates a magnetic induction field (B) in the order of magnitude 0.01 Tesla at least. The samples of different materials will behave as shown in the figures below:

Diamagnetism

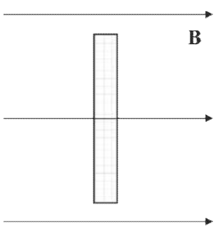


Fig.5.1
copper, mercury, gold, bismuth

Paramagnetism

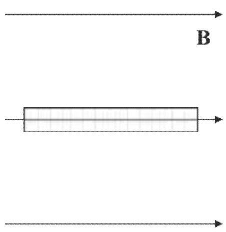


Fig.5.2
aluminum, chromium, platinum, tungsten

Ferromagnetism

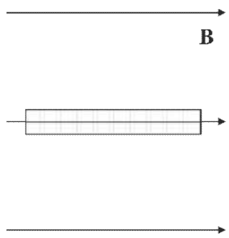


Fig.5.3
iron, nickel, cobalt and their alloys

Some samples orient themselves diagonally (perpendicularly) to the direction of the magnetic field. These materials are called diamagnetic materials (Fig 5.1). Some other materials orient themselves parallel to the magnetic field. These are the paramagnetic (Fig 5.2) and ferromagnetic (Fig 5.3) materials. The intensity of the interaction can be estimated based on the dynamics of the turning. The most sluggish turning happened at the diamagnetic materials. The paramagnetic material turned somewhat more agile but still slow. The turning reaction of ferromagnetic material was instantaneous relative to the others. The estimated intensities of the torques are roughly one, ten and several thousand, respectively.

So far, the experimental distinction has been carried out. The microphysical interpretation of the experimental results should provide the understanding of the phenomena. The roots of the interpretation come from the atomic structure of the material. Some materials contain atoms

without any magnetic dipole moment. Upon placing these materials into the magnetic field, the atoms become weak dipoles. Due to quantum mechanical reasons, the direction of the dipole moment will be just opposite of the external magnetic field. This way the sample becomes a dipole of opposite direction to the external magnetic field. Now the external field wants to turn the dipole parallel with itself. As the turning goes on, the atomic dipoles also turn their orientation again, against the external field. So, the only position where the sample gets rest is the perpendicular position, where virtually no torque affects the sample. This behavior is characteristic of the diamagnetic materials.
Atomic dipoles are originally located in the paramagnetic materials. Once external magnetic field emerges, the atomic dipoles in the material orient themselves into the direction of the field, thus the sample becomes a magnetic dipole. In the former chapter, the fact has been presented that the magnetic dipoles turn to the parallel direction to the external field. This way the sample is positioned as shown in the figure 2. The diamagnetic effect also shows up in paramagnetic materials, but it is overcompensated by the paramagnetic effect.
From technical point of view the most important type, the ferromagnetic material is treated finally. There are atomic dipoles in the material similarly the paramagnetic materials, but these atomic dipoles interact with each other in contrast to the simple paramagnetic case, where the dipoles are affected by the external field alone. Due to the interaction, the dipoles orient themselves parallel with each other and create the magnetic domains. The size of such domains is roughly in the order of micrometers, which in terms of atomic dimensions is large, though in terms of macroscopic dimensions is still rather small. Such material is magnetically neutral, since the domains are oriented randomly, this way the effects of domains average out to zero. When the external magnetic field is switched on, the domains get oriented, and very high magnetic field is generated. The magnetization of the material is proportional to the relatively low external magnetic fields. At high external fields however the magnetization gets saturated, since the domains have all been oriented. This phenomenon called the hysteresis. Another interesting fact is related to the Curie temperature. Above this temperature the material loses the ferromagnetic behavior and reverts to be paramagnetic. In the case of iron, the Curie temperature is 770 degrees Celsius. The relatively high temperature breaks the bonds of the interaction between the dipoles, and domains are disintegrated.
Later in this work the discussion of magnetic phenomena is limited to ferromagnetic materials in the linear magnetization range, when the magnetization and the external field are proportional. Hysteresis phenomena will not be treated in detail.

5.2./ Solenoid coil with iron core
Take a solenoid coil with empty cavity. Switch on the DC current and feel how strong the magnetic force is, by approaching it with an iron screwdriver. Now place the iron core into the cavity and check the force again. The experiment confirms that the force is much stronger than previously. The conclusion can be drawn readily, that the presence of iron core increased the intensity of the force, in contrast to the electrostatic phenomena where the presence of dielectric material diminished the intensity of the force. This anti-symmetry is rooted in the fact that the parallel currents and the opposite charges attract each other.

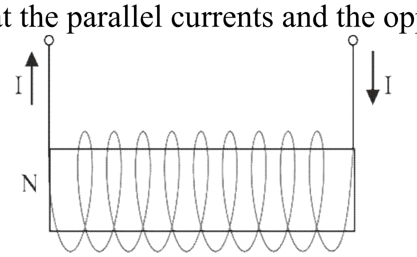


Fig.5.4
Solenoid from side view

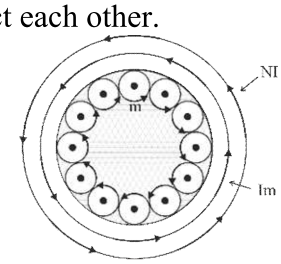


Fig.5.5
Solenoid from top view

A simple model will be discussed here suggested by Ampere:
The atomic dipole is equivalent with a tiny loop current. So, the magnetic material is assumed to be filled with such loop currents in random orientation, so their total magnetic moment is zero. By switching the external DC field on, they get oriented and the loop currents will all circulate in the same direction. Let us consider a point within the iron core. Due to symmetry reasons, same amount of current goes through this point from right to left and opposite. Inside the volume of the core, the loop currents neutralize each other. The situation is very much different on the surface of the core, where the direction of the loop currents is parallel with the coil current. Because of this, the core will behave as it was a coil in which the so-called magnetizing current (I_m) flows. The direction of the coil current and the direction of magnetizing current are the same.
The magnetic field of both the coil current, and that of magnetizing current are as follows:

$$H = \frac{NI}{l} \qquad\qquad H_m = \frac{I_m}{l}$$

Here N is the number of turns, I and I_m are the coil current in the wire, and the magnetizing current on the surface of the iron core, respectively.
Due to the same direction, the total magnetic field is the sum of these two:

$$H_{tot} = H + H_m$$

Let us multiply the above equation with μ_0.

$$\mu_0 H_{tot} = \mu_0 H + \mu_0 H_m$$

The left-hand side of the above equation is the field of the magnetic induction (B), which is characteristic to the total forces and torques. The first term on the right-hand side is associated with the forces and torques generated by the current in the wire of the coil alone. The second term on the right-hand side is the field of magnetization M, which is characteristic to the forces and torques, generated by the magnetizing current on the surface of the iron core. Accordingly, this can be written:

$$B = \mu_0 H_{tot} \qquad\qquad M = \mu_0 H_m$$

$$B = \mu_0 H + M$$

The deduction of this formula was made in a special geometry for the sake of simplicity. However, the result is true in full generality for vectors as well.

$$\mathbf{B} = \mu_0 \mathbf{H} + \mathbf{M} \qquad\qquad \left[\frac{Vs}{m^2}\right]$$

The meaning of the above formula can be summarized as follows:
The magnetic field (**H**) is generated by the coil current alone, while the vector of magnetization (**M**) is generated solely by the magnetizing current. The vector of magnetic induction (**B**) contains the effects of both the coil current and the magnetizing current. The emerging torques and forces are determined by the **B** field.
The experiment showed that the created magnetization (**M**) is proportional to the external field (**H**) provided no saturation happens. Proportionality can be transformed to equation by introducing a coefficient, which is called the magnetic susceptibility (χ_m).

$$\mathbf{M} = \mu_0 \chi_m \mathbf{H}$$

Let us substitute this into the former one:

$$\mathbf{B} = \mu_0 \mathbf{H} + \mathbf{M} = \mu_0 \mathbf{H} + \mu_0 \chi_m \mathbf{H} = \mu_0 (1 + \chi_m) \mathbf{H}$$

Here we introduce the concept of relative permeability (μ_r).

$$\mu_r = 1 + \chi_m$$

By means of this, the final equation can be written:

$$\mathbf{B} = \mu_0 \mu_r \mathbf{H}$$

5.3./ Ampere's law and the magnetic material

In this section the vector calculus will be used at somewhat higher level.

The rotation operation (*rot*) generates a vector field which represents the vortexes of some vector field.

$$\mathbf{V}(\mathbf{r}) = V_x(x,y,z)\mathbf{i} + V_y(x,y,z)\mathbf{j} + V_z(x,y,z)\mathbf{k}$$

$$rot\mathbf{V}(\mathbf{r}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

The Stoke's theorem integrates the rotation to a surface as follows:

$$\oint_g \mathbf{V}(\mathbf{r})d\mathbf{r} = \oint_S (rot\mathbf{V})d\mathbf{A}$$

Let us divide the following equation with μ_0:

$$\mathbf{B} = \mu_0\mathbf{H} + \mathbf{M}$$

$$\frac{\mathbf{B}}{\mu_0} = \mathbf{H} + \frac{\mathbf{M}}{\mu_0}$$

Generate the rotation of the equation above:

$$rot(\frac{\mathbf{B}}{\mu_0}) = rot\mathbf{H} + rot(\frac{\mathbf{M}}{\mu_0})$$

Each term in the above equation can be interpreted separately based on the first Maxwell equation. The current densities are related both to the conductive current in the coil, and to the magnetizing current on the surface of the iron core.

$$rot(\frac{\mathbf{B}}{\mu_0}) = \mathbf{j_{tot}} \qquad rot\mathbf{H} = \mathbf{j_{coil}} \qquad rot(\frac{\mathbf{M}}{\mu_0}) = \mathbf{j_{magn}}$$

$$\mathbf{j_{tot}} = \mathbf{j_{coil}} + \mathbf{j_{magn}}$$

Stokes theorem generates integral form from the relations above:

$$\oint_g (\frac{\mathbf{B}}{\mu_0})d\mathbf{r} = I_{tot} \qquad \oint_g \mathbf{H}d\mathbf{r} = I_{coil} \qquad \oint_g (\frac{\mathbf{M}}{\mu_0})d\mathbf{r} = I_m$$

The central integral above is the well-known form of Ampere's law, which shows that the curve integral of the **H** vector on a closed path (*g*) equals the amount of the conductive current (current in a wire) surrounded by the *g* path. The integral on the right expresses, that the curve integral the magnetization vector (**M**/ μ_0) equals the total magnetizing current surrounded by the *g* path. Finally, the left-hand side states, that the curve integral of the magnetic induction vector (**B**/ μ_0) is equal to the total current (conductive and magnetizing) surrounded by the *g* path.

5.4./ Inhomogeneous magnetic material

Consider two different magnetic materials with plane surface. The plane surfaces are connected thus creating an interface between the materials. This structure is subjected to the experimentation.

First the **B** field is studied.

The interface is contained by a symmetrical disc-like drum with the base area *A*. The upper and lower surface vectors are $\mathbf{A_1}$ and $\mathbf{A_2}$ respectively.

$$\mathbf{A_1} = -\mathbf{A_2} \qquad |\mathbf{A_1}| = |\mathbf{A_2}| = A$$

Since magnetic monopoles do not exist, the **B** field does not have sources. So, the flux of the **B** field to a close surface is necessarily zero (Maxwell equation 3.)

$$\oint_S \mathbf{B}d\mathbf{A} = \mathbf{B}_1\mathbf{A}_1 + \mathbf{B}_2\mathbf{A}_2 = 0$$

$$\mathbf{B}_1\mathbf{A}_2 = \mathbf{B}_2\mathbf{A}_2$$

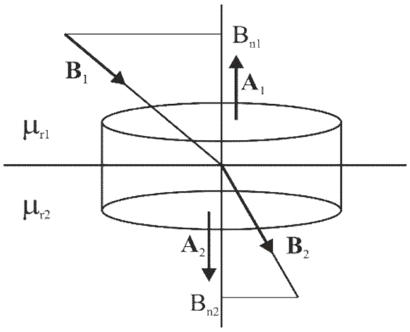


Fig 5.6
The **B** field on the interface of different magnetic materials

The operation of dot product contains the projection of the **B** vectors to the direction of $\mathbf{A_2}$ vector, which is the normal direction to the surface. The subscript n means the magnitude of the normal direction component.

$$B_{1n}A_2 = B_{2n}A_2$$

Once we are among real numbers the surface area cancels out readily.

$$B_{1n} = B_{2n}$$

According to this result, the normal component of **B** vector is continuous on the interface of magnetic materials.

Secondly, the magnetic field (**H)** is the subject of the analysis.

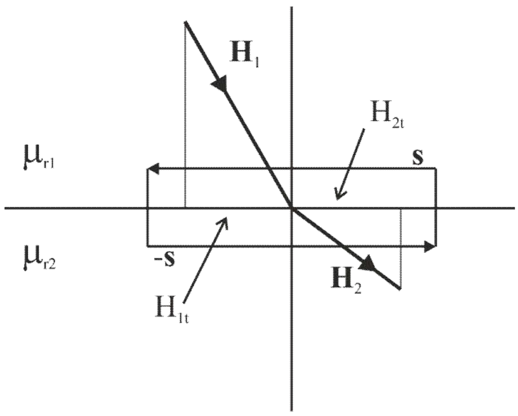


Fig 5.7
The **H** field on the interface of different magnetic materials

The interface is surrounded by a very narrow rectangle-like loop with sections parallel and normal to the surface. The parallel sections of the loop are **s** and **–s** vectors. The normal direction sections are ignored due to the infinitesimal size. The closed loop integral of the **H** field equals the total conductive current through the loop, according to Ampere's law. Here there is no such current, so the right-hind side of the equation will be zero.

$$\oint_g \mathbf{H}d\mathbf{r} = \mathbf{sH}_1 + (-\mathbf{s})\mathbf{H}_2 = 0$$

$$\mathbf{sH}_1 = \mathbf{sH}_2$$

The operation of dot product contains the projection of the **H** vectors to the direction of **s** vector, which is the tangential direction to the surface. The subscript *t* means the magnitude of the tangential direction component.

$$sH_{1t} = sH_{2t}$$

Once we are among real numbers, the length of the tangential section cancels out readily.

$$H_{1t} = H_{2t}$$

According to this result, the tangential component of the **H** vector is continuous on the interface of magnetic materials.

5.5./ Demonstration example

A conductive rod with a radius (R_1=*10cm*) made of copper carries a uniform current density (j=10^5 A/m^2). The rod is surrounded by magnetic coating (μ_r=10^3) up to the radius (R_2=15cm). Sketch the radial dependence of *H, B* and *M* magnitudes. Determine the numerical peak values in the break points and find the magnitudes of the magnetizing current at R_1 and R_2 radii.

Solution:

First parameter to be calculated is the magnetic field (*H*). The tangential component of the *H* field is continuous on the interface of the magnetic materials. In our case the circular magnetic field is tangential to the surface of the magnetic coating, therefore the *H* field is unaffected by the presence of the magnetic coating.

In the rod

$$2r\pi \cdot H(r) = r^2\pi \cdot j$$

$$H(r) = \frac{j}{2}r$$

$$H(r = R_1) = \frac{j}{2}R_1 = \frac{10^5}{2}0.1 = 5 \cdot 10^3 \frac{A}{m}$$

Out of the rod

$$2r\pi \cdot H(r) = R_1^2\pi \cdot j$$

$$H(r) = \frac{j}{2}\frac{R_1^2}{r}$$

$$H(r = R_1) = \frac{j}{2}R_1 = \frac{10^5}{2}0.1 = 5 \cdot 10^3 \frac{A}{m}$$

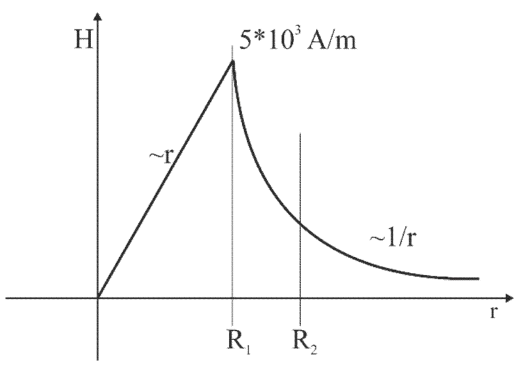


Fig. 5.8

The magnetic field (*H*) as the function of distance

The peak value of the magnetic field (*H*) can be calculated as above. It shows that the function is continuous. The maximum value is:

$$H(R_1) = \frac{j}{2}R_1 = 5 \cdot 10^3 \frac{A}{m}$$

Concerning the *B* field, the following argument is presented. Earlier result was:

$$B(r) = \mu_0\mu_r H(r)$$

In order to display the same physical quantity in all figures, the above equation is transformed.

$$\frac{B(r)}{\mu_0} = \mu_r \cdot H(r) \qquad \left[\frac{A}{m}\right]$$

Where there is no magnetic material $(\mu_r = 1)$, the above function is qualitatively identical with the *H(r)* function, apart from a universal constant μ_0. On the other hand, where the magnetic material is present, the function is enlarged to μ_r times higher value. Difficulty lies in the drawing of such figure, due to the large (μ_r=10^3) dynamic range. Domains without magnetic material are so low in magnitude that they can not be shown in the same graph. In measuring techniques, this is a frequently arising problem. The general solution is using the logarithmic scale and increasing the displayable dynamic range by this way. In present case however, the low values fall on the horizontal axis, thus they can not be displayed. In technology the three order of magnitude lower values can be ignored generally.

The peak values in the magnetic material and in the copper are $\frac{B(R_1)}{\mu_0} = 5 \cdot 10^6 \frac{A}{m}$ and $\frac{B(R_1)}{\mu_0 \mu_r} = 5 \cdot 10^3 \frac{A}{m}$ respectively. The corresponding *B* values are $B_{magn}(R_1) = 6{,}28T$ and $B_{air}(R_1) = 6{,}28 \cdot 10^{-3} T$ respectively. Thus $B(r)$ function is not continuous in R_1 and R_2.

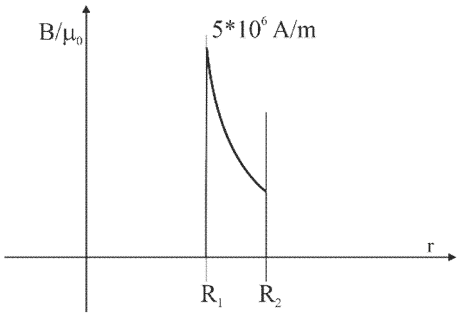


Fig. 5.9 The magnetic induction (B/μ_0) as the function of distance

Magnetization results as the difference of the above functions:

$$\frac{M(r)}{\mu_0} = \frac{B(r)}{\mu_0} - H(r)$$

$$\frac{M(R_1)}{\mu_0} = \frac{B(R_1)}{\mu_0} - H(R_1) = 5 \cdot 10^6 \frac{A}{m} - 5 \cdot 10^3 \frac{A}{m} = 4{,}995 \cdot 10^6 \frac{A}{m}$$

The peak value is $\frac{M(R_1)}{\mu_0} = 4{,}995 \cdot 10^6 \frac{A}{m}$. The corresponding *M* value is 6.277 *Tesla*.

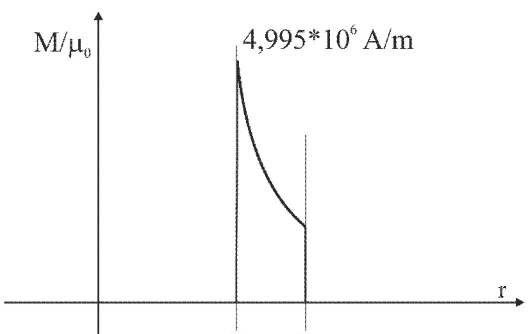


Fig. 5.10 The magnetization (M/μ_0) as the function of distance

The amount of the magnetizing current can be calculated as follows:

$$\oint_g \left(\frac{\mathbf{M}}{\mu_0}\right) d\mathbf{r} = I_{magn}$$

The g curve of the integration is the circle with R_1 radius. The magnetizing current can be expressed as follows:

$$I_m = 2R_1\pi \frac{M(R_1)}{\mu_0} = 2R_1\pi\left(\frac{B(R_1)}{\mu_0} - H(R_1)\right) = 2R_1\pi\left(\frac{1}{\mu_0}\mu_0\mu_r \frac{j}{2}R_1 - \frac{j}{2}R_1\right) = R_1^2\pi \cdot j(\mu_r - 1)$$

The cross-sectional area is multiplied with the current density. This is the conductive current (I_{cond}) in the rod.

$$I_{Cond} = R_1^2\pi \cdot j = 10^{-2} \cdot 3{,}14 \cdot 10^5 = 3140A$$

The magnetizing currents are as follows:

$$I_m(R_1) = I_{Cond} \cdot (\mu_r - 1) = 3140 \cdot 999 = 3{,}14 \cdot 10^6 A$$

$$I_m(R_2) = I_m(R_1)\frac{R_1}{R_2} = 3.14 \cdot 10^6 \frac{10}{15} = 2{,}093 \cdot 10^6 A$$

The magnetizing current is virtual current and shows up exclusively on the surface of the magnetic material. At R_1 radius the magnetizing current is in parallel direction with the conductive current, while at R_2 radius the magnetizing current flows in opposite direction. At bigger radii than R_2, the effects of two opposite direction magnetizing currents compensate each other so the magnetization intensity drops to zero.

5.6./ Solenoid with iron core

In the former chapter at section 5.2, the empty solenoid coil has already been treated. Now the cavity contains the iron core which is characterized by μ_r value. The physical model of solenoid requires the length to be roughly ten times longer then the diameter.

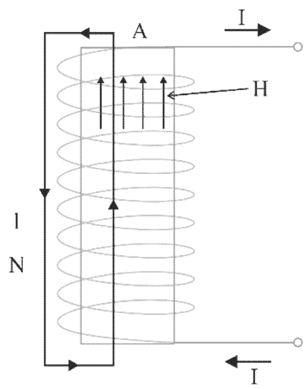


Fig. 5.11
Solenoid with core

Let us use Ampere's law. The magnetic field is homogeneous in the solenoid. A closed path is chosen, which is parallel with the coil and located in the cavity on one side. The front side of the path is outside the coil where there is no magnetic field. The remaining two little sides of the rectangle are normal to the magnetic field, thus can be ignored. The length is denoted l and the number of turns is N. So ultimately the Ampere's law emerges in a simple form:

$$\oint_{loop} \mathbf{H}(\mathbf{r}) d\mathbf{r} = \sum I$$

$$Hl = NI$$

$$H = \frac{NI}{l}$$

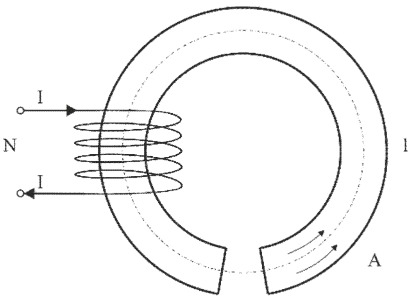
The generated magnetic induction is as follows:

$$B = \mu_0 \mu_r \frac{NI}{l}$$

The generated turn flux and coil flux values can be expressed:

$$\Phi_{turn} = BA = \frac{\mu_0 \mu_r NIA}{l} \qquad \Phi_{coil} = BA \cdot N = \frac{\mu_0 \mu_r N^2 IA}{l}$$

5.7./ Toroid coil with air gap in the iron core



Fig, 5.12
Toroid with air gap

The model of the toroid coil requires, that the coil should resemble to a bicycle wheel much rather than a car wheel. So, the perimeter of the coil should be about ten times larger than the diameter of the cavity. On the other hand, the airgap should be ten times smaller than the diameter of the cavity, and it must be perpendicular to the axis of the coil. The magnetic induction is continuous on the surfaces of the airgap, since the normal component of the *B* field is always continuous on the surface of magnetic material. Due to the small size of the airgap, the *B* magnetic induction field can be considered constant all around the coil and in the gap.

$$B_{iron} = B_{air} = B \qquad H_{air} = \frac{B}{\mu_0} \qquad H_{iron} = \frac{B}{\mu_0 \mu_r}$$

Let us use the Ampere's law on circle which goes around the central line of the toroid:

$$H_{iron} l + H_{air} \delta = NI$$

The letters *l* and δ are the circumference of the iron and the width of the air gap, respectively. After substitution:

$$\frac{B}{\mu_0 \mu_r} l + \frac{B}{\mu_0} \delta = NI$$

From here *B* can readily be expressed:

$$B = \frac{\mu_0 NI}{\frac{l}{\mu_r} + \delta}$$

The generated turn flux and coil flux values can be expressed, for use in the next chapter:

$$\Phi_{turn} = BA = \frac{\mu_0 NIA}{\frac{l}{\mu_r} + \delta} \qquad \Phi_{coil} = BA \cdot N \frac{\mu_0 N^2 IA}{\frac{l}{\mu_r} + \delta}$$

Chapter 6.

Time dependent electromagnetic field – György Hárs

The phenomena of electromagnetic induction are majorly important in the applications. The production and the transformation of electric energy are carried out this way.

6.1./ Motion related electromagnetic induction
6.1.1./ Plane generator (DC voltage)

The plane generator is a hypothetical device which is unpractical to use in its original form, however this can demonstrate the operation some practically used generators. The physical principles of operation are clearly apparent without the disturbing technical details.

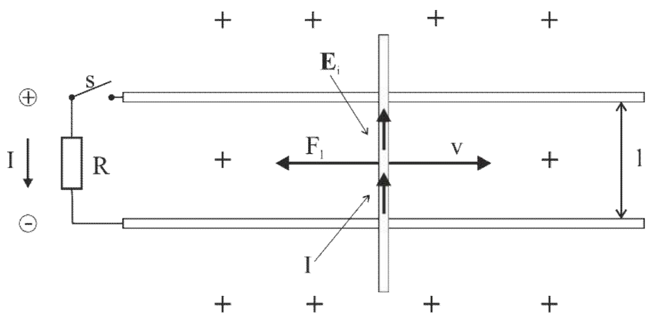


Fig.6.1
Plane generator

The plane generator consists of two parallel conductive rails and a similarly conductive crossbar perpendicular to the rails. The crossbar can travel freely on the rails, while staying in galvanic contact with both. A resistor *R* is also connected between the rails. The whole setup is placed into homogeneous ***B*** magnetic field which points into the plane of the paper. Let us move the crossbar with uniform *v* velocity parallel with the rails. Let the velocity vector point to the right-hand side direction.
In the first part of the experiment the S switch, which connects the resistor to the setup is open.
The generated Lorentz electric field is as follows:

$$\mathbf{E_L} = \mathbf{v} \times \mathbf{B}$$

According to the vector product, the Lorentz electric field points upside direction. The Lorentz electric field pushes the positive charge carriers to upside direction, so the upside terminal is the positive one. The result was the same if electrons were considered as charge carriers. This time the electrons were pushed downside direction making the downside terminal negative, which matches the earlier result.
The **v** and **B** vectors are normal to each other, so the magnitude of the result is merely the product of the magnitudes:

$$E_L = v \cdot B$$

The magnitude of the induced voltage can be calculated without any integration by a simple product.

$$U_{ind} = E_L \cdot l = B \cdot l \cdot v$$

Here the distance between the rails is denoted *l*.

The magnitude of the induced voltage can also be calculated in the following way: The magnetic flux affecting the setup is the product of the magnetic induction (B) and the active area (A).

$$\Phi = BA = Blx$$

The time derivative of the above formula is as follows:

$$\frac{d\Phi}{dt} = B\frac{dA}{dt} = Bl\frac{dx}{dt} = B \cdot l \cdot v$$

The result matches the magnitude of the induced voltage. The polarity of the result should be considered separately. If the velocity vector points to the right, then the active flux increases due to the increasing area. The increasing flux points into the paper, so the generated electric field rotates counterclockwise as it had been shown in the first part of this argument. Altogether one can summarize the conclusion in the following formula:

$$U_{ind} = -\frac{d\Phi}{dt}$$

This formula is the Faraday induction law. This is true for all kinds of induction process.
Now let us return to the discussion of the plane generator by closing the S switch, this way applying a load to the generator. Current flowing through the resistor is denoted i.

$$i = \frac{U_{ind}}{R} = \frac{Blv}{R}$$

The electric power generated P_{el} is the following:

$$P_{el} = U_{ind} i = \frac{(Blv)^2}{R}$$

The induced current flows through the crossbar. The current and magnetic field interact according to Lorentz law.

$$\mathbf{F_L} = i\mathbf{l} \times \mathbf{B}$$

The direction of the Lorentz force is just the opposite of the velocity. If I move the crossbar with my hand, I must overcome the Lorentz force.
This is the point to mention Lenz's law which states that following: The direction of the induced current is determined accordingly, that by means of its magnetic field, the induced current always opposes the original change in the magnetic flux. So, if the flux is increased by my hand, the induced current opposes my hand's motion.
My force will be parallel direction with the velocity. This way I make positive power on the system.

$$F_L = il \cdot B = \frac{Blv}{R} l \cdot B$$

The positive power exerted to the system is the product of the force and the velocity:

$$P_{mech} = F_L \cdot v = \frac{(Blv)^2}{R}$$

The amount of the electrical power matches the formula of the mechanical power. This means that the mechanical power required for moving the crossbar against the force of the magnetic field is equal to the electrical power, which heats up the resistor.
Based on the principle of the plane generator there are practically usable generator types such as the "Drum generator" and the "Unipolar generator" both generating DC voltage.

6.1.2./ Rotating frame generator (AC voltage)

The rotating frame generator is a hypothetical device, which is unpractical to use in its original form, however this can demonstrate the operation some practically used generators.

The physical principles of operation are clearly apparent without the disturbing technical details.

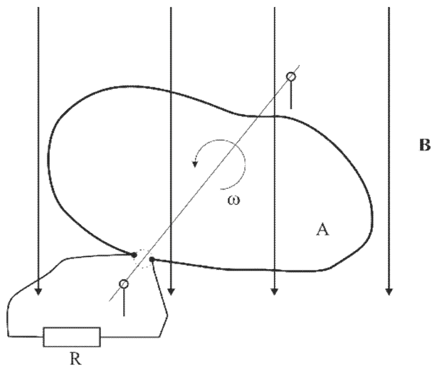


Fig. 6.2
Rotating frame generator

The rotating frame generator consists of a wireframe with the surface area A without any special condition for the shape of the frame. The frame is rotating around an axis which is expanded between two diagonal points of the frame. The axis is positioned perpendicular to a homogeneous magnetic field. On the rotational axis there is a pair of sliding rings, which is solidly connected to the two ends of the wire frame. The sliding connectors are hooked up to a resistor of resistance R through a switch, which is open during the first part of the experiment. The flux in the frame as the function of time can be written easily:

$$\Phi(t) = BA\cos(\omega t)$$

Let us use the Faraday induction law:

$$U_{ind} = -\frac{d\Phi}{dt} = BA\omega\sin(\omega t)$$

The induced current is expressed by Ohm's law:

$$i = \frac{U_{ind}}{R} = \frac{BA\omega}{R}\sin(\omega t)$$

The electrical power generated is as follows:

$$P_{el} = U_{ind} i = \frac{(BA\omega)^2}{R}\sin^2(\omega t)$$

Let us check out the mechanical power required to rotate the generator.
The torque **M** is affecting a magnetic dipole in a **B** magnetic field. It has been discussed in chapter 4.

$$\mathbf{M} = i\mathbf{A}\times\mathbf{B}$$

The magnitude of the torque is as follows:

$$M = iAB\sin(\omega t)$$

The current is substituted:

$$M = \frac{BA\omega}{R}\sin(\omega t)\cdot AB\sin(\omega t)$$

The mechanical power exerted to the system is the product of the torque and the angular velocity of the rotation.

$$P_{mech} = M\omega = \frac{BA\omega}{R}\sin(\omega t)\cdot AB\sin(\omega t)\cdot\omega = \frac{(BA\omega)^2}{R}\sin^2(\omega t)$$

The final formula of the mechanical power completely matches the formula of the electrical power. So altogether the situation is clear. The torque of my hand which rotates the frame

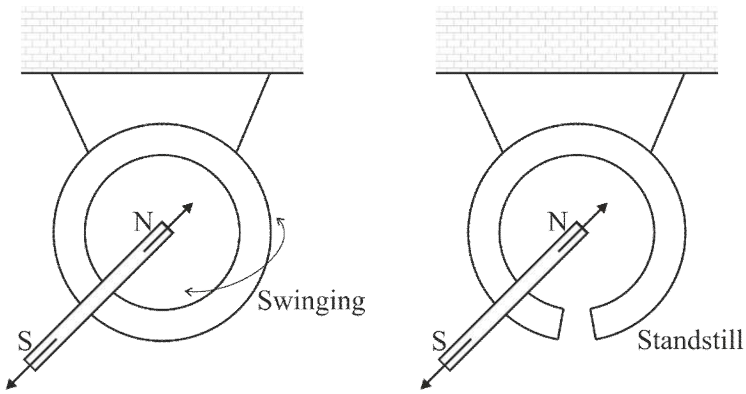
generator overcomes the opposition of the induced current according to Lenz's law. The generated power has been consumed in the resistor by warming it up.
Generators which supply AC voltage into the electrical energy systems operate on the principle of the rotating frame generator.

6.1.3./ Eddy currents

If the magnetic field changes over time inside of a conductive medium, the generated electric field gives rise to loop currents which circulate in the medium. These are the eddy currents, which cause energy dissipation in the medium. The direction of the current is determined by Lenz's law.

6.1.3.1./ Swinging rings experiment

The rings are made of aluminum with an approximate diameter of twenty centimeters. One of them has a thin cutting so this ring is not continuous all around. The rings are suspended according to the figure. A bar magnet is pushed back and forth into the ring. The ring with the cutting is unaffected by the periodic motion of the magnet. However, the intact ring gradually starts to swing if the operator moves the magnet in synchronism with the oscillation frequency of the pendulum.



Continuous aluminum ring   Aluminum ring with cutting

Fig. 6.3 Swinging rings experiment

Explanation: The motion of the bar magnet causes the variation of the flux in the ring. The induced electric field generates the induced loop current (eddy current). The magnetic field of the induced loop-current, opposes the original effect according to Lenz's law. The original effect is the motion of the bar magnet which cannot be stopped; therefore, the ring starts to swing by the periodic effect of the braking force. In case of the ring without cutting, no effect will show up, since the cutting prevents the formation of the eddy current.

6.1.3.2./ Waltenhofen pendulum

A pendulum is made with a copper workpiece on its end according to the figure. One of the workpieces is intact, the other one is having comb-like cuttings. The intact workpiece is swinging between the jaws of the electromagnet which is initially inactive. Let the pendulum swing and observe, that the attenuation of the motion is insignificant. Now switch DC voltage to the electromagnet. The swinging will stop completely in three oscillations. Now replace the workpiece with cuttings in it. By repeating the experiment, the attenuation does not appear upon switching the magnet.

Pendulum with intact workpiece Pendulum with comb-like cuttings

Fig.6.4 Waltenhofen pendulum experiment

Explanation: When the intact piece was moving through the magnetic field, eddy current was generated in the workpiece by the effect of the flux variation. The magnetic effect of the eddy current attenuated the swing, according to Lenz's law. Once the workpiece with cuttings has been placed, the attenuation seized since eddy currents have been prevented.

6.2./ Electromagnetic induction at rest

Electromagnetic induction can also occur without mechanical motion. The primary cause of the induction process is the variation of the electric current, which in turn generates time variant magnetic field.

6.2.1./ The mutual and the self induction

Consider *n* pieces of current loops. Each of them carries a current i_i and each of them contains a flux Φ_i. The flux is originated partly from its own current, and partly by all other current loops.

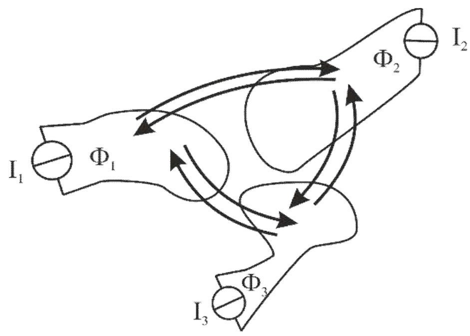


Fig. 6.5
Current loops

Experience shows that the coupled fluxes and the self fluxes are proportional with the corresponding currents, so the total flux in a loop can be expressed by a linear relation.

$$L_1 i_1 + M_{12} i_2 + M_{13} i_3 = \Phi_1$$
$$M_{21} i_1 + L_2 i_2 + M_{23} i_3 = \Phi_2$$
$$M_{31} i_1 + M_{32} i_2 + L_3 i_3 = \Phi_3$$

This relation is presented in the best way by using matrix formalism.

$$\begin{bmatrix} L_1 & M_{12} & M_{13} \\ M_{21} & L_2 & M_{23} \\ M_{31} & M_{32} & L_3 \end{bmatrix} \cdot \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{bmatrix}$$

Here the self induction coefficients and the mutual induction coefficients are denoted with letters L and M respectively. In the M coefficients, the first subscripts indicate the current loop that received the external flux, while the second subscript shows the current loop that generated the magnetic field. Obviously, the self induction coefficients do not need double subscripts.

The major physical point of the above description is the fact, that the induction matrix is symmetrical. That means for instance M_{12} and M_{21} elements are equal. This contains the important fact that by applying current to loop 1 and measuring the flux in loop 2 the mutual induction coefficient turns out to be the same when the current conducting loop and the measuring loop are swapped. This symmetry provides the possibility to determine the mutual induction coefficient in the most convenient way.

6.2.2./ Induced voltage of a current loop

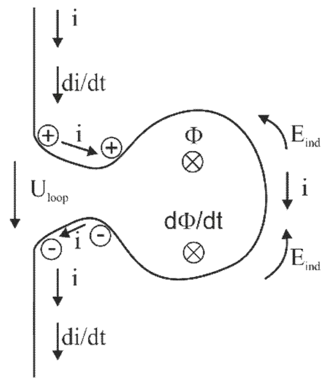


Fig. 6.6
Induced voltage on current loop

Consider a current loop according to the figure above. Let us assume an electric current flowing down-side direction and increasing in magnitude. Therefore, the time derivative of the current also points down-side direction. In the loop, both the generated flux and the time derivative of the flux point into the sheet of the paper. The induced electric field (some earlier books call it as electromotive force) in the loop performs a counter-clockwise rotation due to the negative sign in the Faraday induction law.

$$U_{ind} = -\frac{d\Phi}{dt}$$

Accordingly, the induced electric field pushes the positive charge carriers counter-clockwise direction. This means, that the positive pole of the induced voltage will be the upper pole and the negative is the lower one. The voltage drop of a two-pole component is directed from the positive to the negative pole. So, the measurable loop voltage is pointing down-side direction, similarly to the direction of the time derivative of the current. Thus, the measured loop voltage on the two-pole component, will be the time derivative of the loop current multiplied with the self induction coefficient without the negative sign.

$$U_{loop} = L\frac{di}{dt}$$

If there are more loops in the setup, the loop voltages can be expressed as the time derivative of the above matrix equation:

$$\begin{bmatrix} L_1 & M_{12} & M_{13} \\ M_{21} & L_2 & M_{23} \\ M_{31} & M_{32} & L_3 \end{bmatrix} \cdot \begin{bmatrix} di_1/dt \\ di_2/dt \\ di_3/dt \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}$$

6.2.3./ The transformer in easer version

The transformer consists of at least two coils on a common iron core. The coil which receives the excitation is called the primary coil, while the coil which provides the output is called the secondary coil. Two geometrical arrangements will be discussed, the solenoid and the toroid.

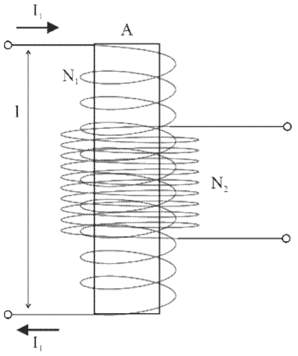


Fig. 6.7 The solenoid transformer

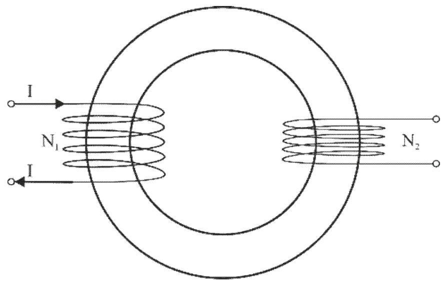


Fig. 6.8 The toroid transformer

These two types of geometry have already been discussed in chapter 5 in some extent. The discussion of these types will be carried out in a uniform way with the following notations: The cross-sectional area of the coil is A. The length of the solenoid and the circumference of the toroid both are denoted l. The number of turns is denoted N_1 *and* N_2. In both cases the Ampere's law emerges in a simple form:

$$\oint_{loop} \mathbf{H}(\mathbf{r})d\mathbf{r} = \sum I$$

$$Hl = N_1 I$$

$$H = \frac{N_1 I}{l}$$

The generated magnetic induction is as follows:

$$B = \mu_0 \mu_r \frac{N_1 I}{l}$$

The generated turn flux can be expressed:

$$\Phi_{turn} = BA = \frac{\mu_0 \mu_r N_1 I A}{l}$$

The voltage of one turn of the coil is called the turn voltage. This is the time derivative of the turn flux.

$$U_{turn} = \frac{dB}{dt} A = \frac{\mu_0 \mu_r N_1 A}{l} \frac{dI}{dt}$$

The coil voltage on the primary coil is as follows:

$$U_1 = N_1 \cdot U_{turn} = \frac{\mu_0 \mu_r N_1^2 A}{l} \frac{dI}{dt} = L_1 \frac{dI}{dt}$$

The formula of the self induction coefficient can be identified:

$$L_1 = \frac{\mu_0 \mu_r N_1^2 A}{l}$$

Due to symmetry, the secondary coil the self induction coefficient is as follows:

$$L_2 = \frac{\mu_0 \mu_r N_2^2 A}{l}$$

The secondary voltage also can be determined by means of the turn voltage:

$$U_2 = N_2 \cdot U_{turn} = \frac{\mu_0 \mu_r N_1 N_2 A}{l} \frac{dI}{dt} = M \frac{dI}{dt}$$

The formula of the mutual induction coefficient can be identified:

$$M = \frac{\mu_0 \mu_r N_1 N_2 A}{l}$$

An important relation can be seen easily:

$$M^2 = L_1 L_2$$

Let us compare the primary and the secondary voltages.

$$\frac{U_2}{U_1} = \frac{N_2 U_{turn}}{N_1 U_{turn}} = \frac{N_2}{N_1} = T$$

The ratio of number of turns is called the transmission (*T*) of the transformer.

The fact is known that the transformer is a passive element. Accordingly, the transformer input and output power values are the same.

$$U_1 \cdot I_1 = U_2 \cdot I_2$$

$$\frac{U_2}{U_1} = \frac{I_1}{I_2} = T$$

Let us find out the input resistance of the transformer, provided a load resistor is hooked up on the secondary coil with the resistance *R*. The input resistance of the transformer is the ratio of the primary voltage over the primary current.

$$R_{in} = \frac{U_1}{I_1}$$

From the equation above, U_1 and I_1 can be easily expressed:

$$U_1 = \frac{U_2}{T} \qquad\qquad I_1 = T \cdot I_2$$

Substitute this, into the formula of the input resistance:

$$R_{in} = \frac{U_1}{I_1} = \frac{U_2}{T} \cdot \frac{1}{T \cdot I_2} = \frac{U_2}{I_2} \cdot \frac{1}{T^2}$$

Here $\frac{U_2}{I_2} = R$ is the load resistance.

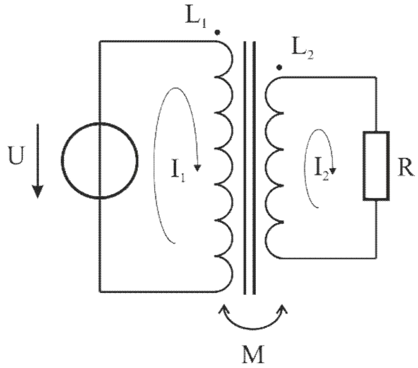
$$R_{in} = R \cdot \frac{1}{T^2}$$

The conclusion is that the load resistance transformed to the input with the square of the transmission. Take a numerical example. Let the values be: $T := 3$ and $R := 900 \cdot \Omega$.

This time the secondary voltage is three times greater than the primary voltage, and the secondary current is one third of the primary current. So, the power on both coils is the same. In addition, the input resistance on the primary coil is only one ninth of the load on the secondary coil, which is a resistance of one hundred ohms only.

6.2.4./ The transformer more in detail

Let us analyze the transformer, when load is applied to the secondary coil. The analysis is carried out by means of the complex amplitudes of the sinusoidal quantities. Here *j* denotes the imaginary unit



Transformer with load

The Kirchhoff loop equations are used.

$$j\omega L_1 I_1 - j\omega M I_2 - U = 0$$
$$R I_2 + j\omega L_2 I_2 - j\omega M I_1 = 0$$

Regroup the equation.

$$j\omega L_1 I_1 - j\omega M I_2 = U$$
$$-j\omega M I_1 + j\omega L_2 I_2 = -R I_2$$

The first and the second equations are multiplied with M and L_1 respectively.

$$j\omega L_1 M I_1 - j\omega M^2 I_2 = UM$$
$$-j\omega L_1 M I_1 + j\omega L_1 L_2 I_2 = -R L_1 I_2$$

Add the equations together. The first term cancels out.

$$j\omega I_2 (L_1 L_2 - M^2) = UM - R L_1 I_2$$
$$I_2 \left(j\omega (L_1 L_2 - M^2) + R L_1 \right) = UM$$

From here I_2 can be expressed:

$$I_2 = \frac{UM}{j\omega (L_1 L_2 - M^2) + R L_1}$$

$$I_2 = \frac{U}{j\omega \left(\frac{L_1 L_2}{M} - M \right) + \frac{R L_1}{M}} = \frac{U}{j\omega (M - M) + \frac{R_1}{T}} = \frac{UT}{R}$$

Here $L_1 L_2 = M^2$ and *the* $L_1/M = 1/T$ relations have been used.
The generated secondary voltage is simply:

$$U_2 = R I_2 = UT$$

This equation proves that the secondary voltage matches the value of the case without load. This result is the conclusion of the model that we are using, namely the coils are free of any serial resistance. The output effective power is the half value of the product between the voltage and the current.

$$P_{out} = \frac{1}{2} U_2 I_2 = \frac{(UT)^2}{2R}$$

Now let us find the relations on the primary side. For this purpose the value of I_1 should be determined from the initial equations.

$$j\omega L_1 I_1 - j\omega M I_2 = U$$

The value of I_2 is substituted:

$$I_2 = \frac{UT}{R}$$

$$j\omega L_1 I_1 - j\omega M\frac{UT}{R} = U$$

$$j\omega L_1 I_1 = U\left(1 + j\omega M\frac{T}{R}\right)$$

I_1 can be expressed readily.

$$I_1 = U\frac{1 + j\omega\frac{M}{R}T}{j\omega L_1}{}_1 = \frac{U}{R}\frac{R + j\omega MT}{j\omega L_1} = \frac{U}{R}\left(\frac{R}{j\omega L_1} + \frac{MT}{L_1}\right) = \frac{U}{R}\left(\frac{R}{j\omega L_1} + T^2\right)$$

$$I_1 = \frac{U}{j\omega L_1} + \frac{U}{R}T^2$$

Further conclusions follow:

a./ Let us calculate the input impedance Z_{in} of the transformer, which is the ratio of the input voltage over the primary current.

$$Z_{in} = \frac{U}{I_1} = \frac{U}{\frac{U}{j\omega L_1} + \frac{U}{R}T^2} = \frac{1}{\frac{1}{j\omega L_1} + \frac{1}{R}T^2}$$

$$Z_{in} = \frac{R}{\frac{R}{j\omega L_1} + T^2}$$

In general case the frequency is high enough, so the magnitude of the primary reactance ($j\omega L_1$) is much higher than the load resistance R. This time the first term in the denominator can be ignored next to the square of the transmission. So ultimately:

$$Z_{in} = \frac{R}{T^2}$$

The well-known result came out, namely the load resistance on the secondary coil is transformed to the input of the transformer, with the reciprocal of the square of the transmission.

b./ A frequently occurring use case, when the load resistor is switched off from the secondary coil. This can be calculated, by tending the value of load resistance (*R)* to the infinity.

$$\lim_{R\to\infty} Z_{in} = \lim_{R\to\infty}\frac{R}{\frac{R}{j\omega L_1} + T^2} = j\omega L_1$$

This time the transformer shows its imaginary primary impedance on the input.
The input primary current is as follows:

$$\lim_{R\to\infty} I_1 = \lim_{R\to\infty}\left(\frac{U}{j\omega L_1} + \frac{U}{R}T^2\right)$$

$$I_1 = \frac{U}{j\omega L_1}$$

Note, that this current is not zero. This seems to contradict to an earlier statement in here, which was exposed in the framework of point 6.2.3 in this chapter.
The earlier exposed statement was as follows, in quotes:

"The fact is known that the transformer is a passive element.

$$U_1 \cdot I_1 = U_2 \cdot I_2$$

Accordingly, the transformer input and output power values are the same."
From the above equation $I_1 = 0$ would follow, if $I_2 = 0$ is the case, due to the disconnected load resistor.
The resolution of the virtual contradiction is as follows: The continuity of effective power is always true. The above formula should be amended, by taking the phase situation into account with the effective power factors.

$$U_1 \cdot I_1 \cdot \cos\varphi_1 = U_2 \cdot I_2 \cdot \cos\varphi_2$$

Here φ_1 and φ_2 are the phase differences between the voltage and the current functions in the primary and secondary coils, respectively. In the concrete case, if $I_2 = 0$, then the secondary power is zero. The primary effective power will be zero too, since the phase difference in the primary coil is $\varphi_1 = 90^o$, so the effective primary power is also zero, even though the primary current is not zero.

c./ The input current above consists of two terms. The first one is a reactant current, which is in ninety-degree lag relative to the voltage. This term does not produce effective power. The second term is in phase with the voltage, so the effective power will be generated accordingly.

$$I_1 = \frac{U}{j\omega L_1} + \frac{U}{R}T^2$$

$$I_1 = \frac{U}{R}T^2$$

$$P_{in} = \frac{1}{2}UI_1 = \frac{(UT)^2}{2R}$$

The result matches perfectly the form of the output power calculated earlier here.

6.2.5./ Energy stored in the coil

Let us increase the current in a coil from zero to some I value gradually in time. The coil reacts with an opposition to the increasing current, according to Lenz's law. The induced voltage of the coil needs to be overcome in order to press through the growing current. This way we must carry out positive work, and because of this the coil will contain magnetic energy. The induced voltage is expressed by the known formula:

$$U = L\frac{dI}{dt}$$

Multiplying it with the current, one recovers to invested power.

$$P(t) = UI = (L\frac{dI}{dt})I = L(I\frac{dI}{dt})$$

On the right hand-side a function and its time derivative are multiplied together. It is known from mathematics, that the following rule is true.

$$f(x)\frac{df(x)}{dx} = \frac{1}{2}\frac{d}{dx}f^2(x)$$

Let us apply this rule to the original case.

$$I\frac{dI}{dt} = \frac{1}{2}\frac{d}{dt}(I^2)$$

This can be readily substituted.

$$P(t) = \frac{1}{2} L \frac{d}{dt}(I^2(t))$$

Let us integrate the two sides of the equation over time with homogeneous initial conditions. This means that in the initial state there was no current and no energy stored in the coil.

$$\int_0^t P(t^{'})dt^{'} = \frac{L}{2}\int_0^t \frac{d}{dt^{'}}\left(I^2(t^{'})\right)dt^{'} = \frac{L}{2}\int_0^t d\left(I^2(t^{'})\right) = \frac{L}{2} I^2(t)$$

The integral of the invested power is the magnetic energy stored. $\int_0^t P(t^{'})dt^{'} = E_m(t)$

The last two equations combined provide the result of the magnetic energy of the coil:

$$E_m = \frac{1}{2} L I^2$$

Let us substitute the formula of the self induction coefficient.

$$L = \frac{\mu_0 \mu_r N^2 A}{l} \qquad E_m = \frac{L}{2} I^2 = \frac{1}{2}\frac{\mu_0 \mu_r N^2 A}{l} I^2$$

The magnetic energy density (ε_m) in the coil is the ratio of the energy over the volume (Al) :

$$\varepsilon_m = \frac{E_m}{Al} = \frac{1}{2}\frac{\mu_0 \mu_r N^2}{l^2} I^2 = \frac{1}{2}\mu_0 \mu_r \left(\frac{NI}{l}\right)^2 = \frac{1}{2}\mu_0 \mu_r H^2$$

The following two formulas are well-known:

$$\frac{NI}{l} = H \qquad \mu_0 \mu_r H = B$$

By means of these, the magnetic energy density in the coil comes out.

$$\varepsilon_m = \frac{1}{2} HB \qquad \varepsilon_m = \frac{1}{2}\mathbf{HB} \qquad \left[\frac{J}{m^3}\right]$$

This result was deduced for the specific condition of a solenoid or toroid coil, the formula for energy density is universally valid for any condition and geometry. In general case the dot product of the magnetic field and the magnetic induction field provides the result.

6.3./ The Maxwell equations

In the table below the Maxwell equations are summarized along with some auxiliary relations

Equation number	Differential form	Integral form
Maxwell 1. Ampere's law	$rot\mathbf{H} = \mathbf{j} + \frac{\partial \mathbf{D}}{\partial t}$ $\frac{\partial \mathbf{D}}{\partial t}$ =displacement current density	$\oint_g \mathbf{H}d\mathbf{r} = I_{cond} + \frac{d\Phi_D}{dt}$ $\quad \Phi_D = \int_S \mathbf{D}d\mathbf{A}$ $I_{cond} = \int_S \mathbf{j}d\mathbf{A}$ $\quad \mathbf{D} = \varepsilon_0\varepsilon_r\mathbf{E}$ $\quad \varepsilon_0\mu_0 = c^2$
Maxwell 2. Faraday induction law Scalar potential U	$rot\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ If $rot\mathbf{E} = 0$ Then $\mathbf{E} = -gradU$	$\oint_g \mathbf{E}d\mathbf{r} = -\frac{d\Phi_B}{dt}$ $\quad \Phi_B = \int_S \mathbf{B}d\mathbf{A}$ $\varepsilon_0\mu_0 = c^2$ $\quad \mathbf{B} = \mu_0\mu_r\mathbf{H}$
Stokes' theorem	Conversion $\oint_g \mathbf{v}d\mathbf{r} = \int_S rot\mathbf{v} \cdot d\mathbf{A}$	
Maxwell 3 Magnetic Gauss law	$div\mathbf{B} = 0$	$\oint_V \mathbf{B}d\mathbf{A} = 0$
Maxwell 4. Gauss law	$div\mathbf{D} = \rho_{free}$	$\oint_V \mathbf{D}d\mathbf{A} = Q_{free}$
Gauss Ostrogradsky theorem	Conversion $\oint_S \mathbf{v}d\mathbf{A} = \oint_V div\mathbf{v} \cdot dV$	

Maxwell 1. Ampere's law

This expresses the fact that the magnetic force lines do not have starting and final points, much rather they are closing into themselves like closed loops. The rotation of the magnetic force lines is determined by the sum of the conductive current and the displacement current. The displacement current alone can generate magnetic field without any real electric conduction. This makes possible the propagation of electromagnetic waves in the vacuum.

Maxwell 2. Faraday induction law

This law states that rotational electric field is generated around the time variant magnetic field. The direction of the rotation is opposite of the right-hand screw direction. In absence of time variant magnetic field, the electric field is irrotational, thus scalar potential can be introduced.

Maxwell 3. Magnetic Gauss law

This equation declares that magnetic force lines are closing to themselves. They do not have starting and ending points. The magnetic flux to any closed surface is zero.

Maxwell 4. Gauss law

This law declares that the electric force lines start on the positive charge and end on the negative charge. The electric flux on a closed surface is proportional to the amount of the contained free charges, and it is zero if the charges are out of the closed volume.

Stokes' and Gauss Ostrogradsky theorems provide the conversion between the differential and integral laws.