Exercise-set 7.

1. In a school the students elect several committees. A student can be a member on several committees. Now every committee wants to select a president from its members. Every member of a committee is eligible for presidency, but the committees don't want to share presidents (i.e., one person can be a president of at most one committee). When can this be achieved?

2. a) In an Indian tribe there are 7 girls (A,B,...,G) and 6 boys (H,I,...,M) to be married. The chieftain made the table below about the possible couples. Can he find a wife for each of the boys?
b) G and L don't want to get married anymore. Solve the problem in this case as well.

	A	B	C	D	E	F	G
H		$\star$				$\star$	
I	$\star$	$\star$	$\star$	$\star$	$\star$		$\star$
J		$\star$			$\star$	$\star$	
K	$\star$		$\star$	$\star$		$\star$	$\star$
L					$\star$	$\star$	$\star$
M		$\star$			$\star$		

3. a) Show that in an r-regular bipartite graph $|A| = |B|$.
b) Show that an r-regular bipartite graph satisfies Hall's condition.
c) Show that an r-regular bipartite graph has a perfect matching.

4. There are n couples on a hike. They want to distribute $2n$ different chocolate bars among themselves (so that everybody gets one). We know that everybody likes at least n kinds from the $2n$ types, and each kind of chocolate is liked by at least one person in each couple. Prove that the chocolate bars can be distributed in such a way that everybody gets a type that he/she likes.

5. (MT'08) Suppose that the bipartite graph G on $2n$ vertices has n vertices in both of its classes, and that the degree of each vertex of G is more that $\frac{n}{2}$. Show that G contains a perfect matching.

6. (MT+'10) Each class of a bipartite graph contains exactly 5 vertices, and the degree of each vertex is at least 2. Show that this doesn't imply that the graph contains a perfect matching.

7. (MT+'10) The organizer of a 10-day trip offers exactly 5 programs for each of the ten days, chosen out of 16 possible ones. He never offers the same program for two consecutive days. Show that we can select one program for each day from the programs offered for that day in such a way that we select a different program for each of the 10 days.

8. Let G be a simple, connected bipartite graph with n vertices in both of its vertex classes, and let all the degrees in one class be different. Show that G contains a perfect matching.

9. a) In a bipartite graph on 20 vertices 18 vertices have degree 5, and the degree of the other 2 vertices is 3. Show that the graph contains a perfect matching.
b) In a bipartite graph on 19 vertices 17 vertices have degree 6, and the degree of the other 2 vertices is 3. Show that the graph contains a matching of 9 edges.

10. (MT'14) Let the two vertex classes of the bipartite graph $G(A, B; E)$ be $A = \{a_1, a_2, \ldots, a_8\}$ and $B = \{b_1, b_2, \ldots, b_8\}$. For each $1 \le i, j \le 8$ let a_i and b_j be adjacent if the entry in the ith row and jth column of the matrix below is 1. Determine whether G contains a perfect matching or not.

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

11. Somebody selected 32 squares on a (8×8) chessboard in such a way that each row and each column contains exactly four selected squares. Show that we can select 8 out of the 32 squares in such a way that each row and each column contains exactly one of them.

12. Somebody divided a pack of 52 cards into 13 sets of 4 cards each at random. Prove that we can select one card from each set in such a way that we select exactly one of each of the 13 figures.

13. (**) (MT'19) In a simple bipartite graph on 20 vertices the degree of each vertex is either 5 or 6. Show that the graph contains a perfect matching.

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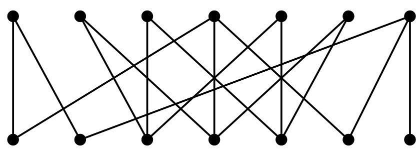
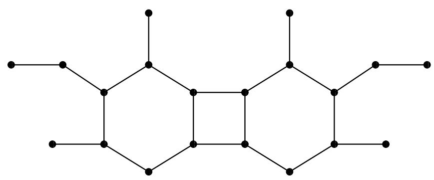
14. (MT'15, MT++'19) Let the two vertex classes of the bipartite graph $G(A, B; E)$ be $A = \{a_1, a_2, \ldots, a_9\}$ and $B = \{b_1, b_2, \ldots, b_9\}$. For each $1 \leq i \leq 9$ and $1 \leq j \leq 9$ let a_i and b_j be adjacent if the entry in the ith row and jth column of the matrix below is 1. Determine a maximum matching and a minimum vertex cover in G.

$$\begin{pmatrix} 1&0&0&1&0&0&1&0&1\\ 1&0&1&0&1&1&1&0&1\\ 1&1&0&1&1&0&0&1&1\\ 1&0&0&1&0&0&1&0&1\\ 1&0&0&0&0&0&1&0&1\\ 0&1&1&0&1&0&0&1&0\\ 0&0&0&1&0&0&1&0&1\\ 0&1&0&0&1&1&0&1&0\\ 1&0&0&1&0&0&1&0&0 \end{pmatrix}, \qquad \begin{pmatrix} 1&1&0&1&1&0&1&1&0\\ 1&0&0&0&0&1&0&1&0\\ 1&0&1&1&0&0&0&1&1\\ 1&0&1&0&0&1&0&1&0\\ 1&0&0&0&0&1&0&1&0\\ 0&1&0&1&0&0&1&0&1\\ 0&0&1&0&0&1&0&1&0\\ 1&0&1&0&0&1&0&0&0\\ 0&0&0&1&1&0&1&0&1 \end{pmatrix}$$

15. (MT'16) Let the two vertex classes of the bipartite graph $G(A, B; E)$ be $A = \{a_1, a_2, \ldots, a_{101}\}$ and $B = \{b_1, b_2, \ldots, b_{101}\}$. For each $1 \leq i \leq 101$ and $1 \leq j \leq 101$ let a_i and b_j be adjacent if $i \cdot j$ is even. Determine $\nu(G)$, the maximum number of independent edges, $\rho(G)$, the minimum number of covering edges, and give a maximum matching and a minimum edge cover in G.

16. Determine a maximum matching in each of the graphs below. Show that they are really maximum!





17. (MT'18) Let the vertex set of the simple graph be $V(G) = \{1, 2, \ldots, 10\}$. Let the vertices $x, y \in V(G)$ be adjacent if and only if $|x - y| = 3$ or $|x - y| = 5$. Delete the edge $\{3, 8\}$ from the graph G, and denote the graph obtained by H.
a) Determine $\nu(H)$, the maximum number of independent edges in H and determine a maximum matching in H.
b) Determine $\alpha(H)$, the maximum number of independent vertices in H and determine a maximum independent set of vertices in H.

18. (MT++'18) Let the two vertex classes of the bipartite graph $G(A, B; E)$ be $A = \{a_1, a_2, \ldots, a_8\}$ and $B = \{b_1, b_2, \ldots, b_8\}$. For each $1 \leq i, j \leq 8$ let a_i and b_j be adjacent if the entry in the ith row and jth column of the matrix below is 1. Determine $\tau(G)$, the minimum number of covering vertices and $\rho(G)$, the minimum number of covering edges, and give a minimum vertex cover and a minimum edge cover in G.

$$\begin{pmatrix} 0&0&1&0&0&1&0&1\\ 1&1&0&1&0&0&1&0\\ 0&0&1&0&0&0&0&1\\ 0&0&0&0&0&1&0&1\\ 0&1&1&1&1&0&0&1\\ 0&0&1&0&0&1&0&1\\ 1&1&0&0&1&1&1&0\\ 0&0&1&0&0&1&0&0 \end{pmatrix}$$

19. a) (MT'15) In a tree on 11 vertices each vertex has degree at most 3. Show that the tree has a matching of 4 edges.
b) (MT++'15) In a simple bipartite graph G on 9 vertices the degree of each vertex is either 2 or 4. Show that G contains a matching of 3 edges.

20. * (MT'21) We select 30 squares on a 100×100 chessboard in such a way that they form a connected area (i.e. we can get from any of the selected squares to any other one by moving through selected squares with a common side only). Show that we can place eight 1×2 domino pieces without overlap on the selected squares in such a way that each domino covers exactly 2 (neighboring) squares.