2. Practice

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BINARY LINEAR CODES

$\uparrow \bar{G}, \bar{H}$ linear map

u: $01 + 10 = 11$ c: $001 + 010 \neq 111$ $\Rightarrow$ no linear map ↯

Systematic code $\underline{01}$, $\underline{10}$ $\underline{01}111$, $\underline{10}110$

identity

① $G_{2\times5} = \begin{pmatrix} 10 & 110 \\ 01 & 111 \end{pmatrix}$ ($B = \begin{pmatrix}110\\111\end{pmatrix}$) $H_{3\times5} = \begin{pmatrix} 111 & 00 \\ 110 & 10 \\ 01 & 001 \end{pmatrix}$ (B^T) $GH^T = 0$ $C(5,2)$

$s^T = Hv^T = He^T$
→ s only depends on error vector not the original codeword
→ based on s try to find e Error group

a) code vectors

$$c_0 = u_0 G_{2\times5} = (00)\cdot\begin{pmatrix}10110\\01111\end{pmatrix} = (00000)$$
$$c_1 = u_1 G_{2\times5} = (01)\cdot\begin{pmatrix}10110\\01111\end{pmatrix} = (01111)$$
$$(10) \qquad = (10110)$$
$$(11) \qquad = (11001)$$

b) error detecting & correcting capability

$d_{min} = 3 \rightarrow d_{min} - 1 = 2$ detecting

$\rightarrow \left\lfloor \frac{d_{min}-1}{2} \right\rfloor = 1$ correcting

c) $e = (01110)$

$s^T = He^T$

$$= \begin{pmatrix} 1&1&1&0&0 \\ 1&1&0&1&0 \\ 0&1&0&0&1 \end{pmatrix} \cdot \begin{pmatrix}0\\1\\1\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

How error groups?
1) → für alle e mit H s ausrechnen
2) → $e, e+c_1, \dots, e+c_{(2^k-1)}$

$\Rightarrow E_{(001)} = \{(01110), (01110)+(01111), (01110)+(10110), (01110)+(11001)\}$

$= \{\underset{w=3}{(01110)}, \underset{w=1}{(00001)}, \underset{w=2}{(11000)}, \underset{w=4}{(10111)}\}$ $P_b = 0.01$

$9.3\cdot10^{-7}$ $9.6\cdot10^{-3}$ $9.7\cdot10^{-5}$ $9.9\cdot10^{-9}$

$\Rightarrow$ group leader (minimal weight)

$\Rightarrow$ for each syndrom s calculate E

- $e = (00000) \rightarrow s = (000)$
- $E_{(000)} = \{(00001), (00001)+(01111); \dots\}$

... Syndrom decoding table

if several errors same minimal weight in E → pick one

Code can correct t errors $\Leftrightarrow$ in each group with minimal weight t or less the group leader is unique!

d) $u = (01)$ $e = (00100)$ $G = \begin{pmatrix}10110\\01111\end{pmatrix}$

$c = u\cdot G$
$= (01)\cdot\begin{pmatrix}10110\\01111\end{pmatrix}$
$= (01111)$

$v = c \oplus e$
$= (01111) \oplus (00100)$
$= (01011)$

$H = \begin{pmatrix}11100\\11010\\01001\end{pmatrix}$

$s^T = Hv^T$
$= \begin{pmatrix}11100\\11010\\01001\end{pmatrix}\cdot\begin{pmatrix}0\\1\\0\\1\\1\end{pmatrix}$
$= \begin{pmatrix}1\\0\\0\end{pmatrix}$

→ syndrom decoding table
$s = (100) \rightarrow e' = (00100)$

$\Rightarrow c' = v \oplus e'$
$= (01011) \oplus (00100)$
$= (01111)$
systematic code
$\Rightarrow u' = (01)$

e) $u = (01)$ $e = (00111)$

$c = u\cdot G$
$= (01)\cdot\begin{pmatrix}10110\\01111\end{pmatrix}$
$= (01111)$

$v = c \oplus e$
$= (01111) \oplus (00111)$
$= (01000)$

$H = \begin{pmatrix}11100\\11010\\01001\end{pmatrix}$

$s = Hv^T$
$= \begin{pmatrix}11100\\11010\\01001\end{pmatrix}\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}$
$= \begin{pmatrix}1\\1\\1\end{pmatrix} \rightarrow e' = (01000)$ $\rightarrow c' = (00000)$ $\rightarrow u' = (00)$ ↯ kann nicht mehr als 1 Fehler korrigieren

f) $P_b = 0.1$

$$P_e = \left(\binom{5}{2} - 2\right) P_b^2 (1-P_b)^3 + \binom{5}{3} P_b^3 (1-P_b)^2 + 5 P_b^4 (1-P_b) + P_b^5$$

2 davon group leader block error probability

g) $c_0 = (000000)$ $c_1 = (010101)$ $c_2 = (101010)$ $c_3 = (111111)$ Systematic code

(1) $C(6,2)$ $4 = 2^k$ $k = 2$ $n = 6$

(2) $d_{min} = 3$

(3) generator matrix

$G = \begin{pmatrix}10\,1010\\01\,0101\end{pmatrix}$ → c so wählen das vorne Identitymatrix

$H = \begin{pmatrix}101000\\010100\\100010\\010001\end{pmatrix}$

(4) $s = He^T$ $e = (010100)$ $E_{(0001)} = \{(010100), (000001) \dots\}$ group leader

$s = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$

(5) how many rows: $2^4 - 1$
all (0000) → kein Fehler od. so viele das nicht sehen

HAMMING CODES

- correct any single error

$n = 2^{n-k} - 1 \Leftrightarrow \sum_{i=0}^{1} \binom{n}{i} = 2^{n-k} \Rightarrow$ perfect codes

(3,1)
(7,4)
(15,11)

$(7,4)$: $\cancel{\begin{pmatrix}0\\0\\0\end{pmatrix}}\begin{pmatrix}1\\0\\0\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}\begin{pmatrix}1\\1\\0\end{pmatrix}\begin{pmatrix}1\\0\\1\end{pmatrix}\begin{pmatrix}0\\1\\1\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}$

$$H = \begin{pmatrix} 0&1&1&1&1&0&0 \\ 1&0&1&1&0&1&0 \\ 1&1&0&1&0&0&1 \end{pmatrix}$$

$$G = \begin{pmatrix} 1&0&0&0&0&1&1 \\ 0&1&0&0&1&0&1 \\ 0&0&1&0&1&1&0 \\ 0&0&0&1&1&1&1 \end{pmatrix}$$

$\Rightarrow$ AND-gate

① $P_b = 0.001$

(1) length 57
block error propability? $(1-P_b)^{57} \approx 0.9446$ → all correct
↳ $1 - (1-P_b)^{57}$

(2) $C(63,57)$ Hamm. message 57 length
block error

$\underbrace{(1-P_b)^{63} + 63(1-P_b)^{62} P_b}$ ≈ 0.99812 nichts / 1 falsch

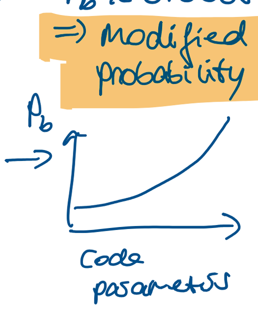
$\Rightarrow 1-$ $\hat{=}$ block error pr. ≈ 0.00188
$\Rightarrow$ smaller aber auch channel capacity ↯ $\frac{57}{63}$

b) (1) $P_b = 0.01$ $C(7,4)$ Hamming code $1 - (1-P_b)^7 - 7(1-P_b)^6 P_b \approx 0.00203$

(2) P_b' without coding, message length = 4 $1 - (1-P_b')^4 = 0.00203 \rightarrow P_b' \approx 0.000508$
$\Rightarrow$ modified bit-error probability

3) $P_b = 0.001$ $\overset{\text{wollen}}{P_b'} = 0.00001$ wie Hamming code?

$\Rightarrow n = 7$ $k = 4$



aber channel Kapazität leidet darunter

4) $G_{3\times5} = \begin{pmatrix}10001\\01010\\00111\end{pmatrix}$ $\rightarrow n = 5$ $k = 3$ ↯ kein Hamming code

$n = 5 \neq 2^{n-k} - 1$

5) $H_{3\times7} = \begin{pmatrix}0110100\\1011010\\1101001\end{pmatrix}$ ↯ identy matrix ✓