1

$$S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2$$

$$E(S_c^2) = E\left(\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2\right)$$

$$= \frac{1}{n-1}E\left(\sum_{i=1}^{n}(X_i - \bar{X}_n)^2\right)$$

$$= \frac{1}{n-1}E\left(\sum_{i=1}^{n}(X_i^2 - 2X_i\bar{X}_n + \bar{X}_n^2)\right)$$

$$= \frac{1}{n-1}\sum_{i=1}^{n}\left(E(X_i^2) - 2E(X_i\bar{X}_n) + E(\bar{X_n}^2)\right)$$

$$=^{1.} \frac{1}{n-1}\sum_{i=1}^{n}\left(E(X_i^2) - E(\bar{X_n}^2)\right)$$

$$=^{2.} \frac{1}{n-1}\sum_{i=1}^{n}(\sigma^2 + \mu^2 - \frac{\sigma^2}{n} - \mu^2)$$

$$= \frac{1}{n-1}\sum_{i=1}^{n}(\frac{n-1}{n}\sigma^2)$$

$$= \sigma^2$$

$$E(\bar{X_n}^2) = \frac{1}{n}E((X_1 + \cdots + X_n)\bar{X}_n)$$

$$= \frac{1}{n}nE(X_i\bar{X}_n)$$

$$= E(X_i\bar{X}_n)$$

$$1. \implies E(\bar{X_n}^2) = E(X_i\bar{X}_n) \quad 0 < i \le n$$

$$2. \; E(\bar{X_n}^2) = Var(\bar{X}_n) + E(\bar{X}_n) =^{E(\bar{X}_n)=E(X_i)}= \frac{\sigma^2}{n} + \mu^2$$

$$Var(\bar{X}_n) = \frac{1}{n^2}Var(X_1 + \cdots + X_n) =^{i.i.d}= \frac{n}{n^2}Var(X_1) = \frac{\sigma^2}{n}$$

2

$$S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2$$

$$E(S_c^2) = E\left(\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2\right)$$

$$= \frac{1}{n-1}E\left(\sum_{i=1}^{n}(X_i - \bar{X}_n)^2\right)$$

$$= \frac{1}{n-1}E\left(\sum_{i=1}^{n}(X_i^2 - 2X_i\bar{X}_n + \bar{X}_n^2)\right)$$

$$= \frac{1}{n-1}\sum_{i=1}^{n}\left(E(X_i^2) - 2E(X_i\bar{X}_n) + E(\bar{X}_n{}^2)\right)$$

$$= \frac{1}{n-1}\sum_{i=1}^{n}(\sigma^2 + \mu^2 - \frac{\sigma^2}{n} - \frac{n-1}{n}\mu^2)$$

$$= \frac{1}{n-1}\sum_{i=1}^{n}\left(\frac{n-1}{n}\sigma^2 + \frac{1}{n}\mu^2\right)$$

$$E(X_i\bar{X}_n) = \frac{1}{n}E\left(X_i^2 + \sum_{j\neq i}^{n}X_iX_j\right)$$

$$= \frac{1}{n}(E(X_i^2) + \sum_{j\neq i}^{n}E(X_i)E(X_j))$$

$$= \frac{\sigma^2}{n} + \frac{n-1}{n}\mu^2 \quad i.i.d$$

$$E(\bar{X}_n{}^2) = \frac{1}{n}E((X_1 + \cdots + X_n)\bar{X}_n)$$

$$= \frac{1}{n}nE(X_i\bar{X}_n)$$

$$= E(X_i\bar{X}_n)$$