13. lecture

Mittwoch, 15. November 2023 10:06

Correlation based data compression for image compression (PCA algo)
principle component analysis

each pixel contains info → read out pixel by pixel → $\bar{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix}$ image vector $\xrightarrow[\text{KLT ??}]{\text{Transformation}}$ $\hat{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix}$ $\xrightarrow{\text{Loss of information}}$ $\bar{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_M \end{pmatrix}$ $\xrightarrow{\text{decompr.}}$ $\begin{pmatrix} \hat{x}_1 \\ \vdots \\ \hat{x}_N \end{pmatrix} = \tilde{x}$

$N \sim 10^6$
$\gg M \div 4;5$

z. Bsp. letzten einfach weglassen

→ very aggressive compression under controlled loss of info (QoS-feature)!

$\bar{\bar{R}}_{N\times N}: R_{ij} := \mathbb{E}(x_i x_j)$ — how pixel related to each other

$\bar{\bar{R}} = \mathbb{E}(\underbrace{\bar{x}\bar{x}^T}_{\text{outer product}})$

1) symm. $\bar{\bar{R}} = \bar{\bar{R}}^T$
2) $\bar{a}^T \bar{\bar{R}} \bar{b} = \bar{b}^T \bar{\bar{R}} \bar{a}$
3) $\bar{a}^T \bar{\bar{R}} \bar{a} \geq \emptyset$
4) $\bar{\bar{R}} \bar{s}_i = \lambda_i \bar{s}_i$ $\quad \lambda_i \geq \emptyset, \; \bar{s}_i^T \bar{s}_j = \delta_{ij}$ Eigenvector (orthogonal to eachother) $= \begin{cases} 1 & \text{if } i \neq j \\ \emptyset & \text{otherwise} \end{cases}$
5) $\text{Trace}(\bar{\bar{R}}) = \sum_{i=1}^{N} R_{ii} = \sum_{i=1}^{N} \lambda_i$

$\bar{x} = \sum_{i=1}^{N} y_i \bar{s}_i$ (Eigenvalues) $\quad y_i = \bar{s}_i^T \bar{x} \quad \bar{x} \to \tilde{\bar{y}} = (y_1 \dots y_M \cancel{y_{M+1} \dots y_N})$

image vector

$\lambda_1 > \lambda_2 > \lambda_3 > \dots > \lambda_M > \lambda_{M+1} > \dots > \lambda_N$

Compression: $\bar{y} = (y_1 \dots y_M) \quad M \div 4;5$

Decompressed vector $\tilde{\bar{x}} := \sum_{i=1}^{M} y_i \bar{s}_i$

$\mathbb{E}\|\bar{x} - \tilde{\bar{x}}\|^2 \leq \mathcal{E}$ — QoS criteria

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$\mathbb{E}(\bar{x} y_i) = \mathbb{E}(\bar{x}\bar{x}^T \bar{s}_i) := \mathbb{E}(\bar{x}\bar{x}^T)\bar{s}_i = \bar{\bar{R}}\bar{s}_i = \lambda_i \bar{s}_i$

$\mathbb{E}(y_i y_j) = \mathbb{E}(\bar{s}_i^T x \bar{x}^T \bar{s}_j) = \bar{s}_i^T \mathbb{E}(\bar{x}\bar{x}^T)\bar{s}_j = \bar{s}_i^T \bar{\bar{R}} \bar{s}_j = \lambda_j \underbrace{(\bar{s}_i^T \bar{s}_j)}_{\text{orthog.}} = \lambda_i \delta_{ij} = \begin{cases} \lambda & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$

$$\begin{aligned}
\mathbb{E}\|\bar{x} - \tilde{\bar{x}}\|^2 = \mathbb{E}\|\bar{x} - \sum_{i=1}^{M} y_i \bar{s}_i\|^2 &= \mathbb{E}\|x\|^2 - 2\sum_{i=1}^{M} \mathbb{E}(y_i \bar{x})\bar{s}_i + \sum_{i=1}^{M}\sum_{j=1}^{M} \mathbb{E}(y_i y_j)\bar{s}_i^T \bar{s}_j \\
&= \mathbb{E}\left(\sum_{i=1}^{N} x^2\right) - 2\sum_{i=1}^{M} \lambda_i \bar{s}_i^T \bar{s}_i + \sum_{i=1}^{M}\sum_{j=1}^{M} \lambda_i \delta_{ij} s_i^T s_j^T \\
&= \sum_{i=1}^{N} \lambda_i - 2\sum_{i=1}^{M} \lambda_i + \sum_{i=1}^{M} \lambda_i \\
&= \sum_{i=1}^{N} \lambda_i - \sum_{i=1}^{M} \lambda_i = \sum_{i=M+1}^{N} \lambda_i
\end{aligned}$$

da $\lambda_1 > \lambda_2 > \dots > \lambda_M > \underbrace{\lambda_{M+1} > \dots > \lambda_N}_{\sum \text{ minimal} \;\Rightarrow\; \text{optimal}}$

ALGO:

Given $\bar{\bar{R}}$, $\mathcal{E} > \emptyset$ quality of service criteria → $\mathbb{E}\|\bar{x} - \tilde{\bar{x}}\|^2 \leq \mathcal{E}$

1) $\bar{\bar{R}} \bar{s}_i = \lambda_i \bar{s}_i$ + $\lambda_1 > \lambda_2 > \dots > \lambda_M > \lambda_{M+1} > \dots > \lambda_N$
$\quad \bar{s}_1 \quad \bar{s}_2 \quad \bar{s}_M \quad \bar{s}_{M+1} \quad \bar{s}_N$

2) $M: \sum_{i=M+1}^{N} \lambda_i \leq \mathcal{E}$ (suchen M sodass condition ✓)

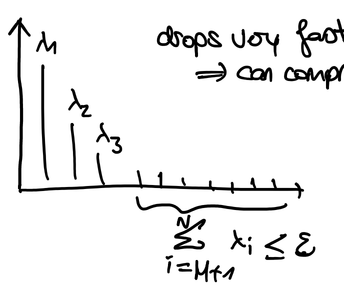
3) $\bar{\bar{S}}_{M\times N} = \begin{pmatrix} — & \bar{s}_1 & — \\ & \vdots & \\ — & \bar{s}_M & — \end{pmatrix} \quad \bar{\bar{S}}^T_{N\times M} = \begin{pmatrix} | & & | \\ \bar{s}_1^T & \dots & \bar{s}_M^T \\ | & & | \end{pmatrix}$

OFF LINE CALCULATION

4) $\bar{x}$ (image) $\to \boxed{\bar{\bar{S}}}$ (DSP) $\to \bar{y}$ (compressed vector) $\dots\; \bar{y} \to \boxed{\bar{\bar{S}}^T}$ (DSP – Digital signal processor) $\to \tilde{\bar{x}}$ $\quad \mathbb{E}\|\bar{x} - \tilde{\bar{x}}\|^2 \leq \mathcal{E}$

ON LINE

$\lambda_i \sim O(\exp(-i^2))$



drops very fast
⇒ can compress very agressively

PROBLEMS:
- $\bar{\bar{R}}$ is unknown $\xrightarrow{??}$ $\bar{\bar{R}} \bar{s}_i = \lambda_i \bar{s}_i$ (unknown) (movie: $\bar{\bar{R}}$ will change in time)
- numerical complexity determine $(\bar{\bar{R}} - \lambda \bar{\bar{I}}) = \emptyset$ ⟶ to much time

Kernel based PCA (in practice used)

$\tau^{(K)} := \{\bar{x}(k), k = 1, \dots, K\}$

$\bar{\bar{R}} \approx \frac{1}{K} \sum_{k=1}^{K} \underbrace{\bar{x}(k)\bar{x}^T(k)}_{\text{outer product}}$ T diadic

$\bar{s}_i = \sum_{j=1}^{N} a_j^{(i)} \bar{x}(j)$

$\underbrace{\bar{\bar{R}} \bar{s}_i = \lambda_i \bar{s}_i}$

$\frac{1}{K} \sum_{k=1}^{K} x(k) x^T(k) \sum_{j=1}^{N} a_j^{(i)} \bar{x}(j) = \lambda_i \sum_{j=1}^{N} a_j^{(i)} \bar{x}(j)$

$\sum_{j=1}^{?} a_j^{(i)} \sum_{k=1}^{K} \underbrace{\bar{x}^T(l)\bar{x}(k)}_{G_{lk}} \underbrace{\bar{x}^T(k)\bar{x}(j)}_{G_{kj}} = \lambda_i \sum_{j=1}^{K} a_j^{(i)} \underbrace{\bar{x}^T(l)\bar{x}(j)}_{G_{lj}}$

$\bar{\bar{G}}: \; G_{lk} = \bar{x}^T(l)\bar{x}(k)$

$\sum_{j=1}^{?} a_j^{(i)} \sum_{k=1}^{K} G_{lk} G_{kj} = \lambda_i \sum_{j=1}^{K} a_j G_{lj}$

$\bar{\bar{G}} \bar{\bar{G}} \bar{a}^{(i)} = K \cdot \lambda_i \bar{\bar{G}} \bar{a}^{(i)}$

$\bar{\bar{G}} \bar{a}^{(i)} = \underbrace{\lambda_i \cdot K}_{S_i} \cdot \bar{a}^{(i)}$ $\quad$ type$(\bar{\bar{G}}) = K \times K$

$K << N$

← λ - Eigenvector problem can be gelöst werden easy

Algo (kernel based PCA) (realtime)

Given $\tau^{(K)} = \{\bar{x}(k), k = 1, \dots, K\}$ training set (observe a couple of frames at beginning of movie)

1) $\bar{\bar{G}}_{K\times K}: \; G_{kl} = \bar{x}^T(l)\bar{x}(k)$

2) $\bar{\bar{G}} \bar{a}^{(i)} = S_i a^{(i)} \qquad i = 1, \dots, K$

3) $\lambda_i = \frac{S_i}{K} \qquad \tilde{\bar{s}}_i = \sum_{j=1}^{K} a_j^{(i)} \bar{x}(j)$

⇓

PCA algo