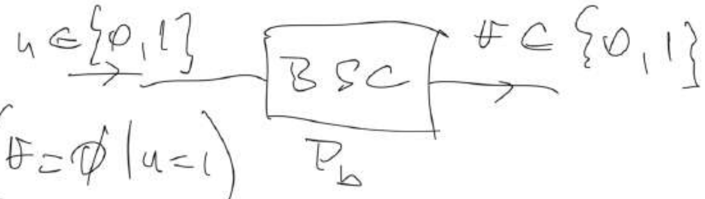
Error Control Codes

How to communicate reliably over an unreliable channel? – in QoS sense,

$$P(\text{error}) \le 10^{-\gamma}; \quad \gamma \text{ QoS parameter}$$

Error: random
- independent (no memory)
- it depends on the signal energy.

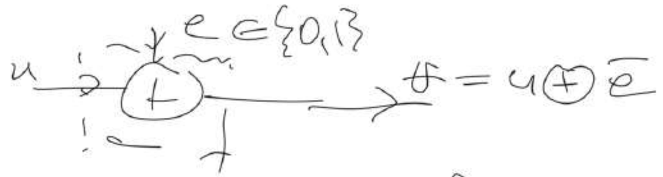
Binary model:



$$P_b := P(v=1 \mid u=0) = P(v=0 \mid u=1)$$

$\equiv$

Additive model

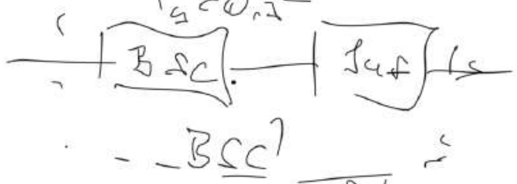


u	e	v		
0	0	0	OK	$1-P_b$
0	1	1	Error	P_b
1	0	1	OK	$1-P_b$
1	1	0	Error	P_b

RNG $\quad P_b, 1-P_b$

$0 \le P_b < 0.5$

$P_b \ge 0.5$



$P_b' < 0.5$

Vectorial channel

$$\bar{u}=(10101) \longrightarrow \oplus \longleftarrow \bar{e}=(01010) \longrightarrow \bar{r}=\bar{u}\oplus\bar{e}=(11111)$$

$$d(\bar{u},\bar{r})=2 \quad \rightarrow \quad d(\bar{u},\bar{r})=w(\bar{u}\oplus\bar{r})=w(\bar{e})$$

$$\begin{array}{r} 10101 \\ 11111 \\ \hline w(01010)=2 \end{array}$$

$$P\left(\bar{r}=(\overset{P_b}{1}1\overset{P_b}{1}11) \mid \bar{u}=(10101)\right) = P_b^2(1-P_b)^3 = P_b^2(1-P_b)^{5-2}$$

$$P(\bar{r}\mid\bar{u}) = P_b^{d(\bar{u},\bar{r})}(1-P_b)^{n-d(\bar{u},\bar{r})} = P_b^{w(\bar{e})}(1-P_b)^{n-w(\bar{e})}$$

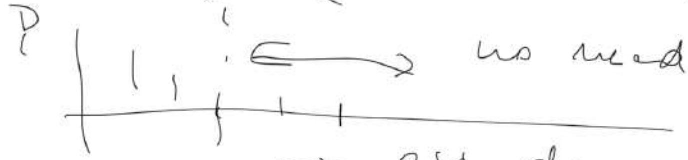
binary vectors of length "n"

$$\left(\frac{P_b}{1-P_b}\right)^{w(\bar{e})}(1-P_b)^n$$

→ exp. decrease

$$0\le P_b<0.5 \rightarrow \frac{P_b}{1-P_b}<1$$

$$P\left(w(\bar{e}) = i\right) = \binom{n}{i}\left(\frac{p_b}{1-p_b}\right)^i (1-p_b)^n$$



no need to pay attention

→ exp decay

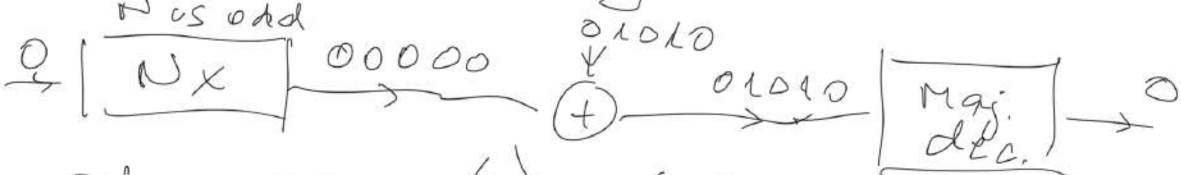
Correcting "t" # of errors

$$P(\text{error}) = \sum_{i=t+1}^{n} \binom{n}{i}\left(\frac{p_b}{1-p_b}\right)^i (1-p_b)^n \leq 10^{-\gamma}$$

Given: $p_b, \gamma \rightarrow n, t$

How to design a code of correcting "t" errors?

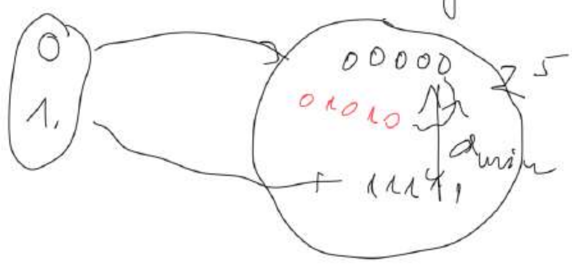
Error control coding



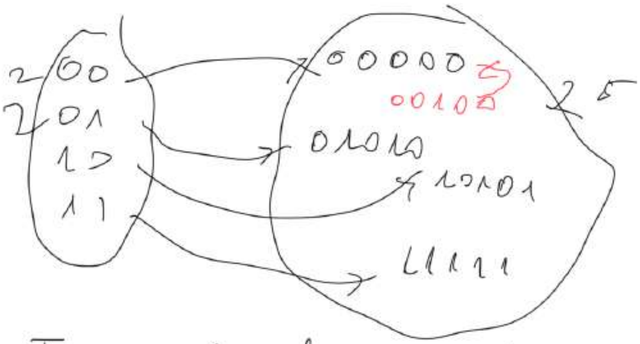
$$P_e' = \sum_{i=\lceil \frac{n}{2} \rceil} \binom{n}{i} p_e^{\,i} (1-p_e)^{n-i} \le 10^{-\gamma}$$

Loss $\frac{1}{n}$

Geometric interpretation - space extension



Detect and correct errors



$\frac{2}{5}$

Formal description

Messages $\bar{u} \in \{0,1\}^k$; $\dim(\bar{u}) = k$ $\quad \bar{u} = (\underbrace{10\ldots1}_{k})$; $M = 2^k$

Code $C = \{\bar{c}_1, \bar{c}_2 \ldots \bar{c}_M\}$ $\quad \dim(\bar{c}) = n > k = \dim(\bar{u})$

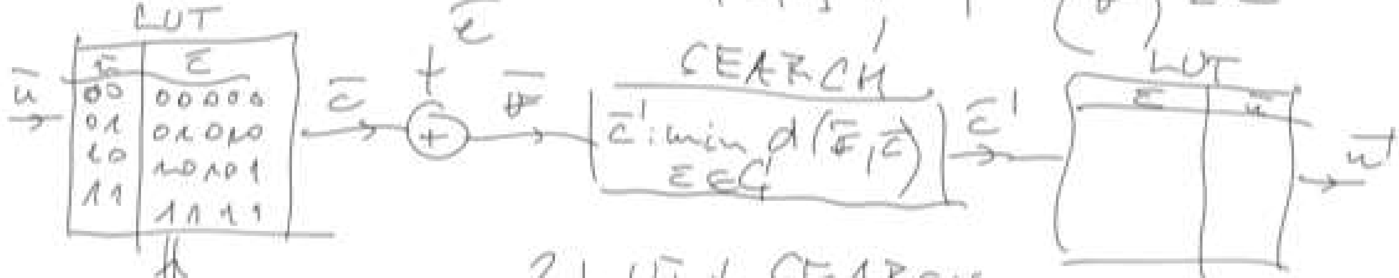
$C(n,k)$, $\frac{k}{n}$, $n-k$

Coding: $\Psi : \{0,1\}^k \to C$; $\Psi(\bar{u}) = \bar{c}$

$\bar c \xrightarrow{} \oplus \xrightarrow{} \bar r = \bar c + \bar e$ Received vector $\bar r \in \{0,1\}^n$

Detection : $\varphi : \{0,1\}^n \to C$; $\varphi(\bar r) = \bar c'$

Decoding : $\psi^{-1} : C \to \{0,1\}^k$; $\psi^{-1}(\bar r) = \bar c'$



maxdmin

2 LUT + SEARCH

Complexity analysis

Off-line complexity $\binom{2^n}{2^k}\binom{2^k}{2}$

NON REALTIME On-line compl.

$3 \cdot O(2^k)$