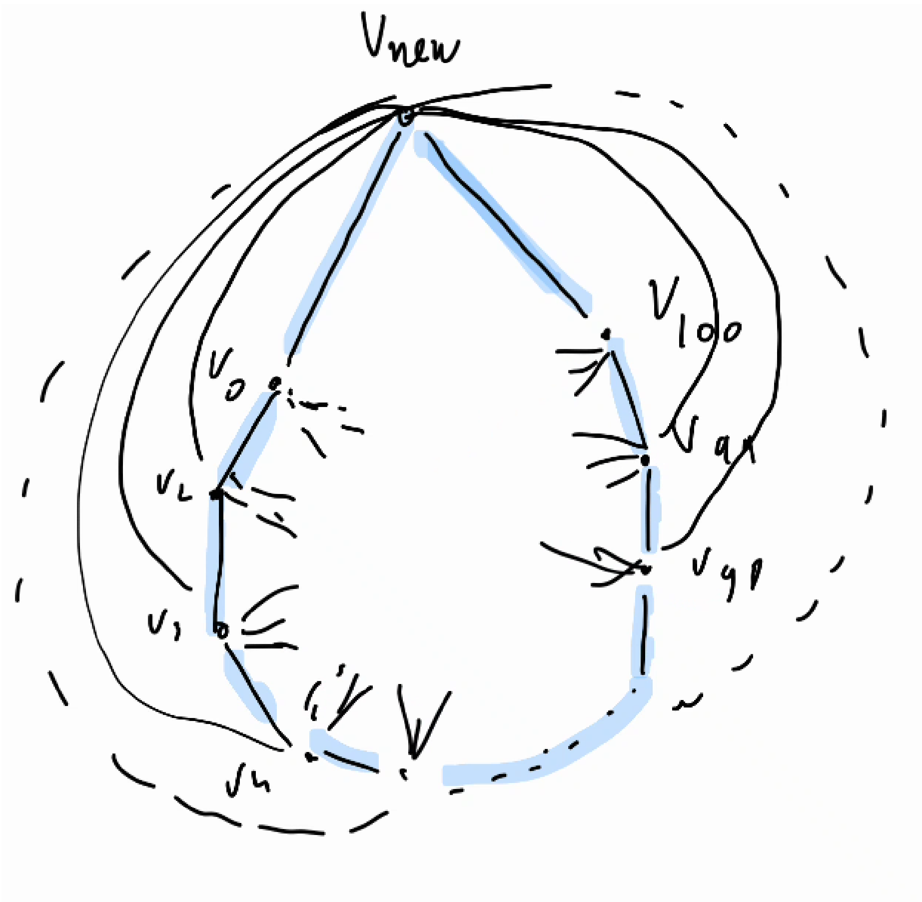
Solution2

2. (KD'2020, Problem 3) Prove that every simple 50-regular graph on 101 vertices has a Hamiltonian cycle. (Out of the 28 submissions, 5 solved this correctly).

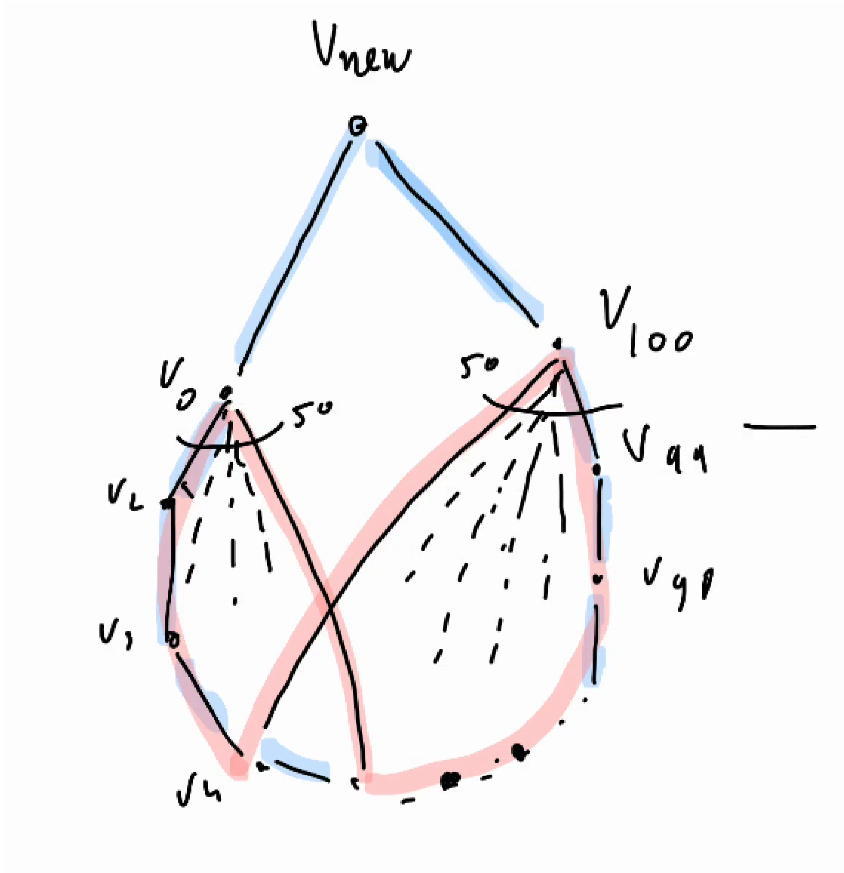
Proof:
Add vertex and connect it to every other vertex so you get G' with n=102 and all the old vertices have deg=51 and the new one 102. Now the ores condition holds (51+51=102=n) => G' is H.



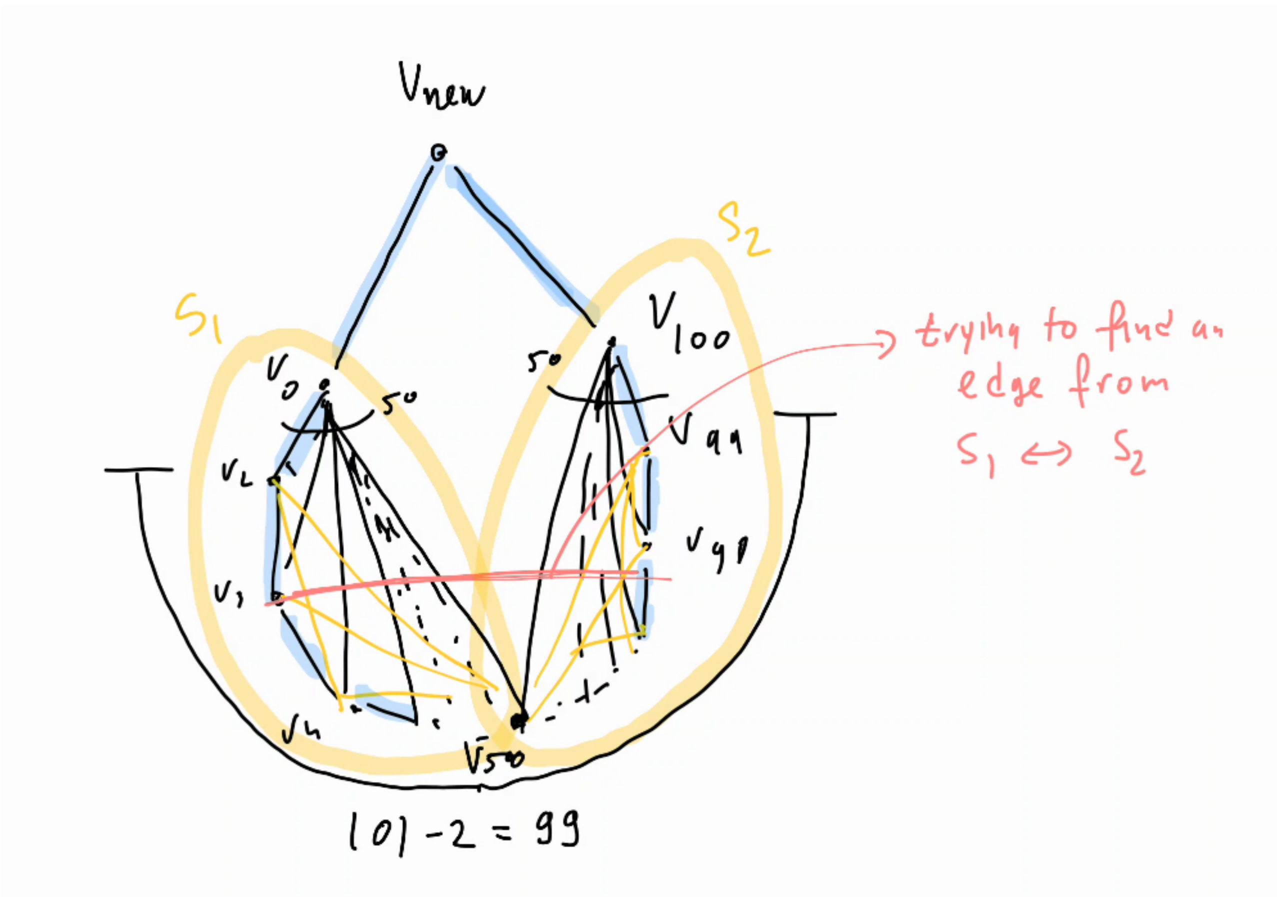
Now we have two option how the edges of v_0 and v_{100} are lays out:

1. They cross each other => immediately H_c.

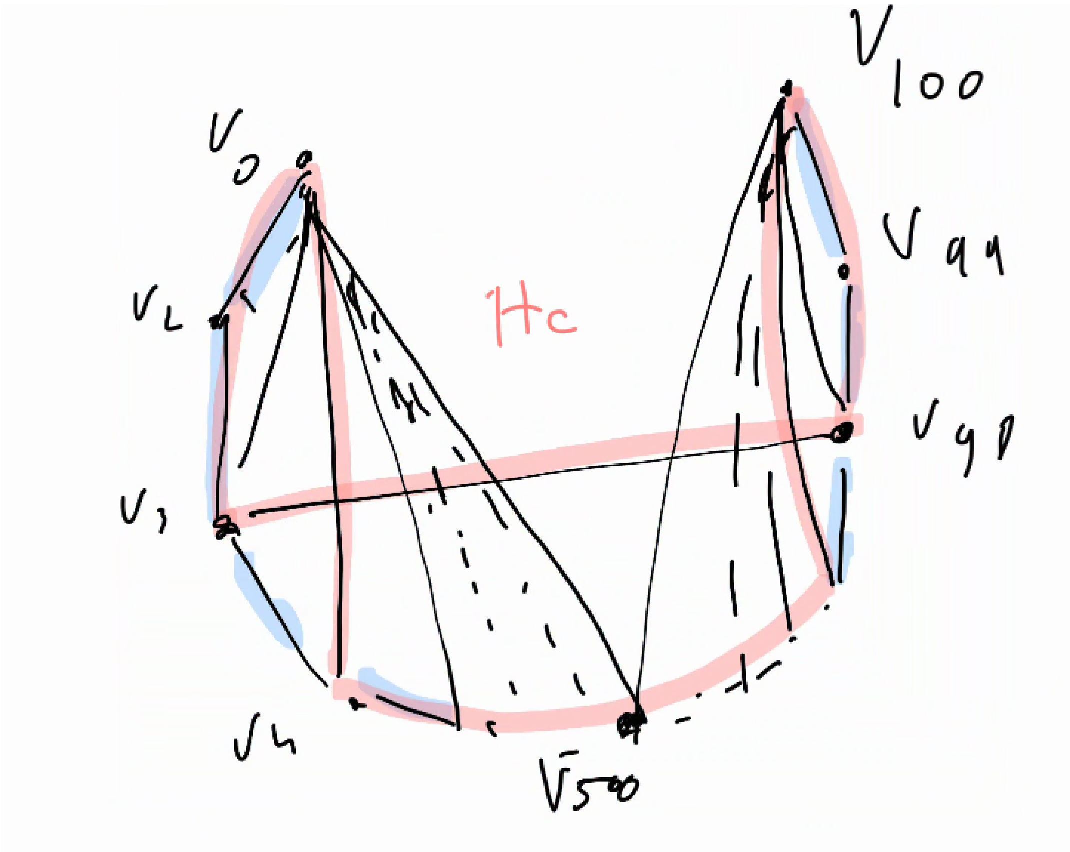
> As mentioned in solution of 1 you if one edge crosses you will always find the cut where the 2 sets meet so you can from the H_c.



2. The edges are very packed together so they don't cross each other. (50+50=100-2+1)
   The edges will met at v_{50} but they wont cross.
   Now we use the 50-regular property. The idea we find and edge from the first subgraph to the second as shown in picture.



We know such an edge should exits because the only way S_1 has no edges going out of it is if you have a K_{50}, but the vertex v_{50} connects to S_2 which implies that S1 can not form a K_{50} => there is an edge between v_i and v_j (0<i<50, 50<j<100). Using that edge you can form the H_c as shown below (You can also remove the new vertex we added cause we only needed it to prove that the G has a H_p):



Note:

> This is just the Nash-Williams theorem for k=50. What we did works generally for any k.
> Every k-regular graph on 2k+1 vertices is Hamiltonian.