Solution1

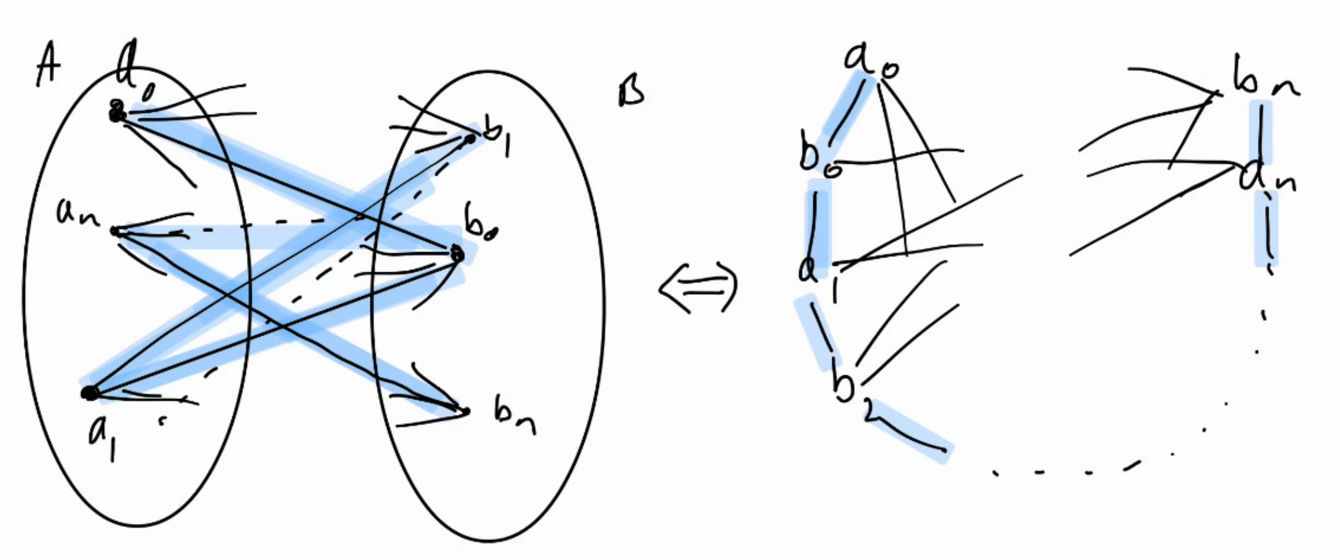
1. (KD'2018, Problem 5) Let G be a simple bipartite graph with n vertices in both of its partitions A, B. Further, for any $u \in A, v \in B$, if $uv \notin E$, then $d(u)+d(v) \geq n+1$. Prove that G contains a Hamiltonian cycle. (Note: In the notes they said that in 2018, of the 46 participants, 43 participants chose to hand in a submission. There were many correct submissions for the first and second problems. The fourth and fifth problem each had two almost correct submissions. Nobody could solve problem 3. The first prize of 25000 Ft was awarded to two students, both who had three full correct solutions.)

Proof:

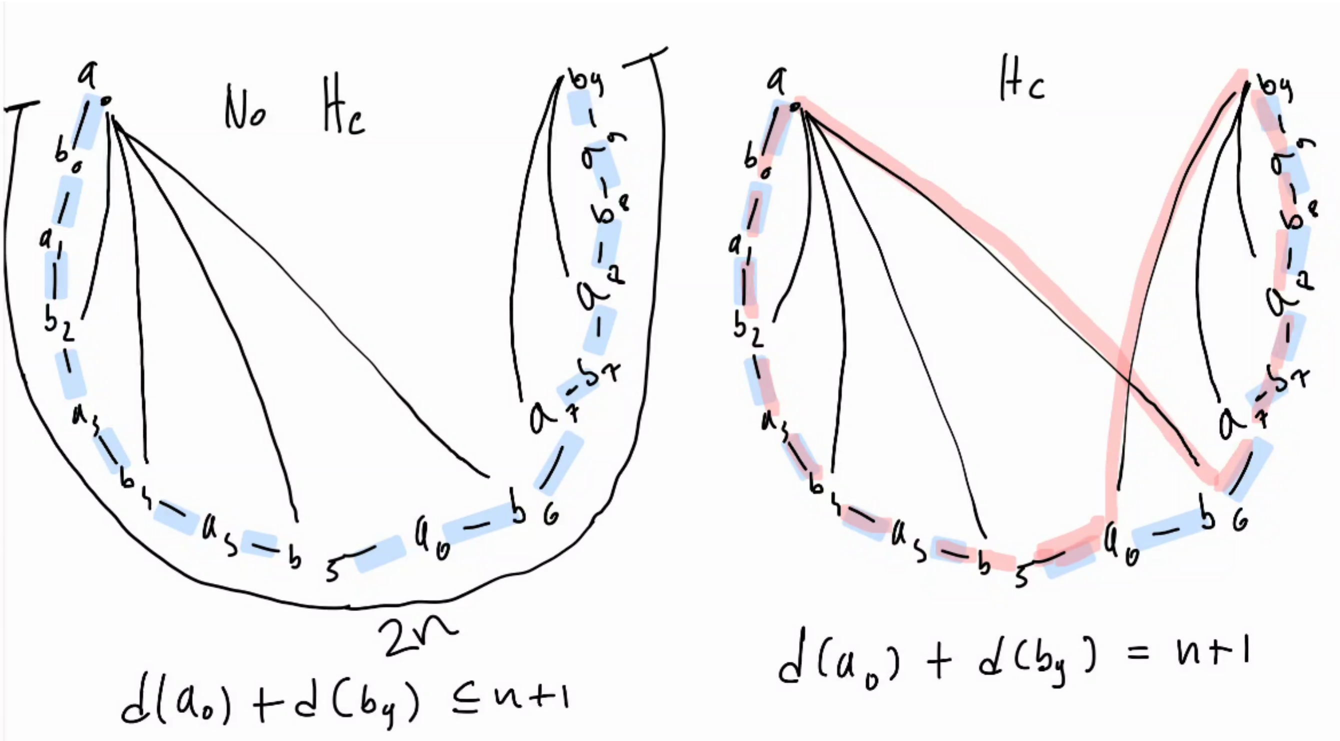
Let G=G(A,B,E)=B(n,n). Vertices a are in A and b are in B.

Add edges $a_i b_j$ to G (keeping it bipartite) until you get a Hamiltonian path. Because the graph G' is bipartite this implies you have an alternating H_p: $a_0 b_0 a_1 b_1 \ldots a_n b_n$.

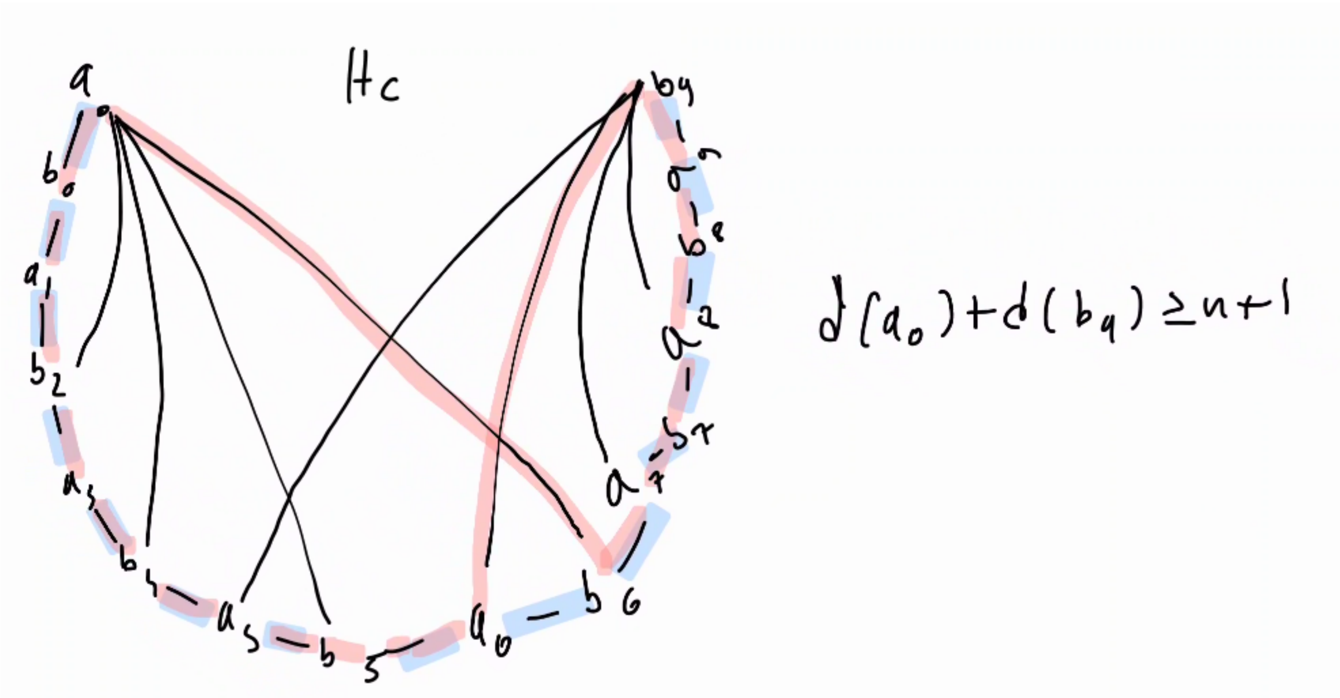
> G' is a supergraph of G.



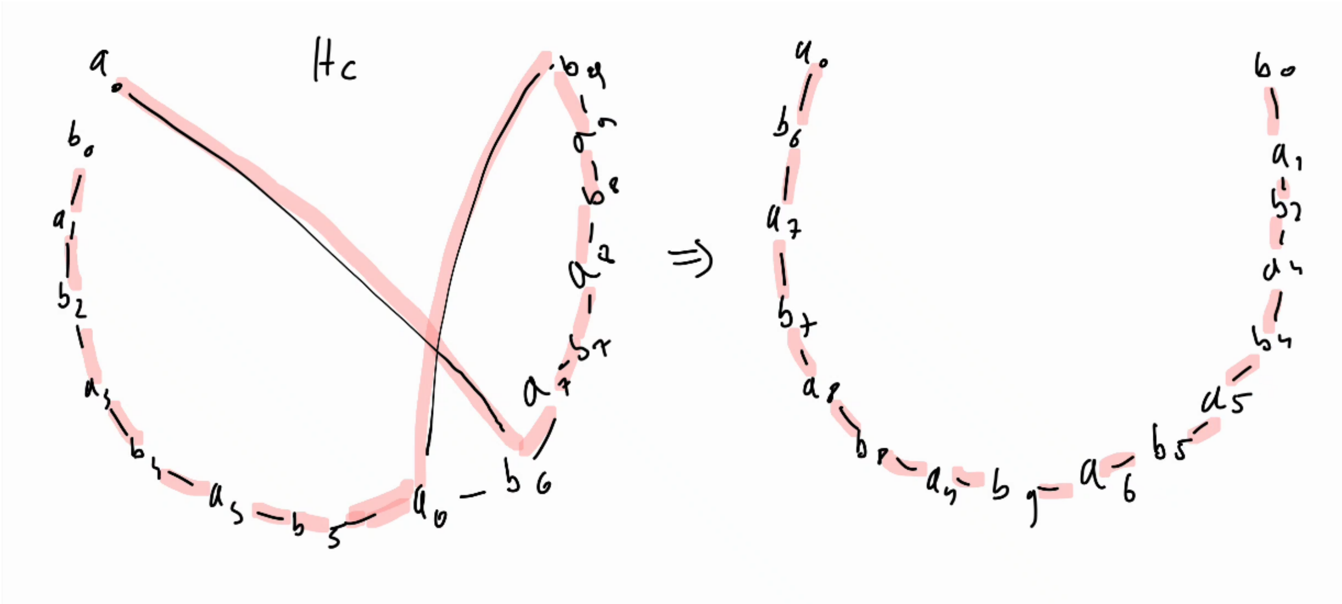
Now we can see if $d(a_0) + d(b_n) \geq n + 1$ => we get the H_c. Because the edges will cross and you can form the H_c.



> Every time the edges cross and are one after the other you can form the H_c. So Here if one edge has crossed you will always find an edge cross after the other because there will always be a cut between the two set of edges.



We added the edges to form the H_p now we should remove them.
We know that all edges which are not connected hold the $d(a_i) + d(b_j) \geq n + 1$. So our 'red' H_c will still be there. We remove $a_0 b_0$. You can rearrange:



You are at the same step as previously. Now you do it recursively until you have removed all the edged edges (the 'blue' H_p). You are left with a H_c if all the vertices not between sets have $d(a_i) + d(b_j) \geq n + 1$.

> You can also use the closure of G instead of recursion

Note:

> All the theorems for proving sufficiency (if condition hold => H_c) have an equivalent in the B(n,n) graph. The one we proved was 'Ore's Theorem for bibartite graphs'. You can easely derive the Dira's from Ore's (just take $\forall a, b \;\; d(a) = \frac{n}{2} \; d(b) = \frac{n}{2}$)