Problem set 6.

Exponential distribution, Expected value, Transforms.

1. Let X be a random variable with exponential distribution. We further know that $P(X > 3) = e^{-6}$. Find the following:
(a) λ (b) $P(X < 2)$ (c) $E(X)$

2. On a clear evening in August, Amber is out star-gazing, and in particular, is waiting to see a shooting star. If we know that the probability that she would spot a shooting star in the first 20 minutes of star-gazing is $1 - e^{-\frac{2}{3}}$, what is the probability of spotting a shooting star in the first one hour?

3. Suppose that a certain kind of washing machine, on an average, runs for 2 years until its first repair. If we know that one particular machine worked for 2 years without requiring a repair, what is the probability that it would not require repairs in its first 3 years?

4. A badly written piece of software has a memory leak that causes the program to occasionally crash and need to be restarted. The runtime between two crashes, measured in hours, has exponential distribution with parameter $\frac{1}{10}$. Dominik starts the software at 4 pm, before leaving work. Assuming that when he returns to work the next morning at 8am he finds that the software has not stopped even once, what is the probability that it will not stop until the lunch break at noon?

5. Let X and Y be random variables with exponential distribution. We know that the expected value of X is twice that of Y. Further, we also know that $3P(X > 1) = 2P(Y < 1)$. What is the expected value of X?

6. On the M3 metro, a sleepy passenger falls asleep and wakes up at a point chosen according to a uniform distribution along the 16.4 km metro line. How many kilometres is the standard deviation of the random variable describing the location of his wake-up? How would the answer change if we asked the same question for another metro line of the same length instead of the M3?

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7. We roll a fair dice and let X be a random variable denoting its outcome. Further, let $Y = |X - 3|$. What is the probability density function of Y? Determine $\int_0^\infty (1 - F_Y(y))\mathrm{d}y$ and $E(Y)$.

8. We pick two random points from two non-adjacent sides of a unit square. Let X denote the square of the distance between them. What is the average value of X?

9. Three numbers are picked uniformly at random from the $[0, 1]$ interval. Let Y denote the middle of these numbers. What is the average value of Y?

10. The fuel tank at a petrol station is filled up every week. Let X denote the weekly consumption (in hundreds of thousands of litres), with the following probability density function:
$$f_X : x \mapsto \begin{cases} 5\,(1-x)^4 & \text{if } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$
What should be the capacity of the tank so that the probability of running out of fuel in a week is less than 0.05? What is the average weekly consumption?

11. The MGF of a random variable X is given by $e^{5(e^t-1)}$. Find $P(X \leq 1)$.

12. The MGF of a random variable X is given by $\frac{1}{1-2t}, t < \frac{1}{2}$. Find $P(X > 3|X > 1)$.

13. If $X \sim Exp(1)$ and $Y \sim Exp(2)$ are independent, does it follow that $X+Y \sim Exp(3)$. Prove your claim.

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14. Let the probability density function of a random variable X be $f_X(x) = \frac{1}{2\sqrt{x}}$ if $0 < x < 1$ and 0 otherwise. Let $Y = X\sqrt{X}$.
    (a) Determine the distribution function of X.
    (b) Determine the distribution function of Y.
    (c) Determine the density function of Y.
    (d) Determine $E(Y)$ using the density function of Y.
    (e) Determine $E\big(X\sqrt{X}\big)$ using the density function of X.

15. Let the cumulative distribution function of a random variable X be given by $F_X(x)$. Determine the cumulative distribution function of the following random variables in terms of F_X:
    (a) $Y = \max\{0, X\}$ (b) $Z = -X$ (c) $V = |X|$ (d) $W = \min\{0, -X\}$.

16. Let $X \sim \text{Exp}(\lambda)$ and $Y = X^2$. Determine the density function and expected value of Y.

17. Let $X \sim U(0,1)$, and let $Y = \sqrt{2X}$, $V = \ln \frac{1}{X}$ and $Z = \text{arctg}(X)$. Determine the density functions of Y, V and Z.

18. Mileage of cars in America is expressed in miles per gallon (mpg), which is the number of miles a car travels on a gallon of fuel. In Europe, mileage is expressed in litres per 100 km. For a car, we know that the mpg mileage is given by the random variable X, with density function f_X. How should we transform f_X if we wish to switch to the litre/100km scale? (1 mile = a km, 1 gallon = b litres, where $a = 1.609$ and $b = 3.785$).

19. * The density function of the *standard Cauchy distribution* is as follows:

$$f(x) = \frac{1}{\pi}\frac{1}{1+x^2}, \qquad x \in \mathbb{R}.$$

    (a) Let X have the standard Cauchy distribution. Show that $\mathbb{E}(|X|) = \infty$.
    Note: The distribution of X is symmetric around 0, but since $\mathbb{E}(|X|) = \infty$, $\mathbb{E}(X)$ *does not exist* (not just as a real number, it cannot be defined as $+\infty$ or $-\infty$).
    (b) Let the following 10 numbers be (pseudo) random numbers generated from the interval $(0,1)$:
    0,966273477967348 0,394713268595531 0,899735231113891 0,474621461118444
    0,728969725975848 0,0146911108455452 0,907138254095836 0,690170289037809
    0,446156897231753 0,374033711751211
    Of the standard Cauchy-distributed random numbers obtained by the appropriate transformation of these random numbers, what will be the smallest and the largest numerical value? Also describe how the transformation is calculated, but the transform of the other 8 numbers need not be given.
    Note: We get results with relatively large absolute values – this also shows that the expected value does not exist.

Final Answers

1. (a) 2 (b) 0.9817 (c) $\frac{1}{2}$
2. 0.8647, $\frac{1}{4}$
3. 0.6065
4. 0.6703
5. 1.443
6. 4.7343; does not change

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7. $F_Y(y) = \begin{cases} 0 & y < 0 \\ \frac{1}{6} & 0 \leq y < 1 \\ \frac{3}{6} & 1 \leq y < 2 \\ \frac{5}{6} & 2 \leq y < 3 \\ 1 & y \geq 3 \end{cases} \qquad E(Y) = \int_0^\infty (1 - F_Y(y))\mathrm{d}y = 1.5$

8. $\dfrac{7}{6}$

9. $\dfrac{1}{2}$

10. 45.072, 16.667 (both numbers are in units of thousands of liters)

11. 0.0404

12. 0.3679

13. No

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14. a) $F_X(x) = \begin{cases} 0 & x < 0 \\ \sqrt{x} & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$ b) $F_Y(y) = \begin{cases} 0 & y < 0 \\ \sqrt[3]{y} & 0 \leq y < 1 \\ 1 & y \geq 1 \end{cases}$ c) $f_Y(y) = \begin{cases} 0 & y \leq 0 \\ \frac{1}{3y^{\frac{2}{3}}} & 0 < y < 1 \\ 1 & y \geq 1 \end{cases}$

    d) $\dfrac{1}{4}$ e) $\dfrac{1}{4}$

15. a) $F_Y(x) = \begin{cases} 0 & x < 0 \\ F_X(x) & x \geq 0 \end{cases}$ b) $F_Z(x) = 1 - F_X(-x-0)$

    c) $F_V(x) = \begin{cases} 0 & x < 0 \\ F_X(x) - F_X(-x-0) & x \geq 0 \end{cases}$ d) $F_W(x) = \begin{cases} 1 - F_X(-x-0) & x < 0 \\ 1 & x \geq 0 \end{cases}$

16. $f_Y(x) = \begin{cases} \frac{\lambda}{2\sqrt{x}} e^{-\lambda\sqrt{x}} & 0 < x \\ 0 & \text{otherwise} \end{cases} \qquad E(Y) = \frac{2}{\lambda^2}$

17. $f_Y(x) = \begin{cases} x & 0 < x < \sqrt{2} \\ 0 & \text{otherwise} \end{cases}, \qquad f_V(x) = \begin{cases} e^{-x} & 0 < x \\ 0 & \text{otherwise} \end{cases}, \qquad f_Z(x) = \begin{cases} \frac{1}{\cos^2(x)} & 0 < x < \frac{\pi}{4} \\ 0 & \text{otherwise} \end{cases}$

18. $x \mapsto \frac{100b}{ax^2} f_X\left(\frac{100b}{ax}\right)$