Problems of 2023. Duration of the exam: 150 minutes.

1. Let A be a square matrix of real numbers. We can perform the following operations on it: row swap, multiplication of a row with a non-zero scalar, adding a row to another row, and finally, cyclic permutation, in which all the elements move to an adjacent cell to its right, the last element in any row goes to the first position of the next row, while the last element of the last row goes to the first position in the first row. Prove that with such steps, we can ensure that the elements of the resulting matrix are either 0 or 1, and that 1's are only in the first row.

2. Suppose that each edge e of the tree F is given a non-negative edge length, $l(e)$. We want to find a pairing of the vertices of F such that each vertex of F is contained in at most one pair. Further, we want to maximize the total of the path lengths of all pairs of vertices. Find a fast algorithm to find such a pairing. Given another non-negative edge length function l', can you determine the optimal pairing for it with the knowledge of the optimal pairing in case of l?

3. Show that a simple connected graph G on 10 vertices with the degree sequence $1, 1, 4, 4, 4, 4, 4, 4, 4, 4$, has a spanning tree with at most 4 leaves. (28 students handed in a sumbission on this contest, this was not the easiest or the hardest problem in this test, and multiple solutions were present for this problem.)

4. Hansel and Gretel have three additional boys and three additional girls in their nursery group. On Easter monday, the Easter bunny gave Hansel and Gretel 42 eggs each, while it gave the other six children 41 eggs each. The rule in the kindergarten is that if a child eats an egg, then he or she must give one egg to each member of the opposite sex. The children keep eats eggs until they no longer can while still respecting the mentioned rule. How many eggs does Hansel have left with him?

5. Can the vertices and edges of a complete bipartite graph $K_{n,n-1}$ be coloured with $n+1$ such that adjacent vertices or adjacent edges don't have the same color, and the color of every edge should be different from its endpoints?

There were 28 submissions, a correct solution was received for every problem. The easiest problem was the fifth, while the hardest proved to be the second (based on submissions).