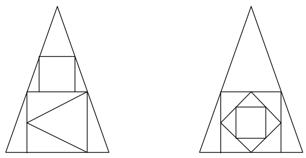
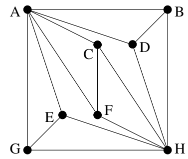
Problem set 4.

Eulerian trails and circuits, Hamiltonian cycles.

1. If it is possible, draw the figures below without lifting the pen.



2. (MT++'11) At least how many edges must be added to the graph below so that the graphs obtained contain a Hamiltonian cycle?



3. ($\sim$ MT'03) A simple graph G has 101 vertices. One of its vertices has degree 50, while all the other vertices have degree at least 51. Prove that G contains a Hamiltonian cycle.

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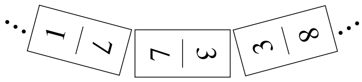
4. (MT'19) For which values of $r = 1, 2, ..., 9$ is it true that every r-regular simple graph on 10 vertices contains an Eulerian circuit? (A graph is said to be r-regular if the degree of every vertex is r).

5. Can a horse travel on a 4×4 chessboard in such a way that it steps on each square exactly once?

6. (MT'07) A domino game set is such that for any pair of distinct numbers between 1 and n, there is exactly one domino in it which contains exactly these two numbers on its two halves and these are all the dominos in the set. We want to arrange all the dominos in the set in a circle such that adjacent numbers on two adjacent dominos are the same (see figure below). For what values of n is this possible?



7. In a group of 20 people, everybody has the same number of acquaintances. Show that they can be seated around a round table such that, either everybody knows their neighbors or nobody knows their neighbors.

8. (MT++'08) Show that if G is a simple 9-regular graph on 16 vertices, then we can delete 8 edges of G in such a way that the remaining graph contains an Eulerian circuit.

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9. (MT'13) Let the vertices of the graph G be the squares of a 5×5 chessboard, and two vertices be adjacent if and only if the corresponding squares have a common edge. The graph G_1 is obtained from G by deleting a vertex corresponding to one of the corners of the chessboard from it (so G_1 has 24 vertices). The graph G_2 is obtained from G by deleting two vertices corresponding to opposite corners of the chessboard from it (so G_2 has 23 vertices).
(a) Does G_1 contain a Hamilton cycle? And a Hamilton path?
(b) Does G_2 contain a Hamilton cycle? And a Hamilton path?

10. Is there an Eulerian trail or an Eulerian circuit in the following graphs?
(a) Graph G where the vertices of G are three element subsets of $\{1, 2, 3, 4, 5, 6\}$ such that two vertices are adjacent if their corresponsing three element subsets have at most one element in common.
(b) Graph H where the vertices of H are 100 long $0 - 1$ sequences such that, two vertices are adjacent if their corresponding sequences differ in exactly two places.

11. A simple graph G has 101 vertices. **Two** of its vertices have degree exactly 50, while all the other vertices have degree at least 51. Prove that G contains a Hamiltonian cycle.

12. (MT'19) Can a horse travel on a 3×5 chessboard in such a way that it steps on each square exactly once?

13. (MT'12) There are 10 vowels and 21 consonants in an imaginary language. Words in this language don't have double letters (i.e. no letter can stand next to itself) and two different consonants cannot stand next to each other (either). But except for these anything else is possible, that is, any two

different letters can stand next to each other if at least one of them is a vowel. What is the length of the longest sequence of letters in this language under the conditions that every letter can be used several times, but any two different letters can stand next to each other at most once in it?

14. (MT+'11) There are 50 guests at a banquet, each of them knows at least 5 people from the others. (Acquaintances are mutual.) No matter how we choose 3 or 4 from the guests they cannot sit down to a round table in such a way that everybody knows both of his/her neighbors. Show that in this case all the guests can be seated around a round table for 50 persons in such a way that any two people who sit next to each other and don't know each other have a common friend among the guests.

15. (MT'09) Let G be a simple graph on 101 vertices in which the degree of one vertex is 50, and the degree of all the other vertices is 49. Show that we can add 50 edges to G in such a way that the graph obtained is still simple and contains an Euler circuit.

16. In a simple graph G on $2k+1$ vertices, the degree of every vertex is at least k. Show that it has a Hamiltonian path.