Problems of 2022. Duration of the exam: 150 minutes.

1. We are given a graph $G(V, E)$ and a cost function $k : E \to \mathbb{R}^+$.In one step, we can construct along any one chosen edge of the graph, but as a consequence of this, the cost of all the unconstructed edges doubles. Determine an efficient algorithm that constructs a spanning tree for G with the least total cost.

2. Assume that in the network (G, s, t, c), the maximum st-flow has capacity 24. If the capacity of both the edges u_1v_1 and u_2v_2 increases by 32, then the maximum flow increases to 88. Is it possible to determine, with this information, the maximum flow in the original network, if the capacity of the edge u_1v_2 is decreased by 13, while the capacity of the edge u_2v_1 is increased by 31? If yes, then how much is this maximum flow? (Assume that the capacity of the edge u_1v_2 in the original network is at least 13).

3. Can you color the vertices **and** the edges of $K_{100,100}$ complete bipartite graph with 101 colors, such that no two vertices with the same color are adjacent, no two edges with the same color are adjacent, and no edge has the same color as either of its endpoints?

4. Is it true that simple graph on 10 vertices with degree sequence $3, 3, 4, 4, 4, 5, 5, 5, 5, 6$ surely has a Hamiltonian path?

5. Let the vertices of a graph G be $(a, b) \in \mathbb{Z}^2$, and the neighbors of it are the vertices $(a - 1, b)$,$(a + 1, b)$,$(a, b - 1)$ and $(a, b + 1)$. For an arbitrary finite subset $H(1) \subset V(G)$ and for index $i = 1, 2, ...$, let, $H(i + 1) = H(i) \cup f(H(i))$, where $f(H(i))$ comprises of the vertices of G which have at least two neighbors in $H(i)$. Show that for all positive integers n, $|H(n)| \leq 4|H(1)|^2$. Is it also always true that $|H(n)| \leq |H(1)|^2$?

There were 14 submissions and every problem was solved by at least one student. The second was the easiest, while the fourth was the hardest (based on submissions/number of correct solutions).