Analysis of matrices

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Contents

1. Introduction

These notes cover the material of the course *Analysis of Matrices* offered at BME VIK. The notes were written during the COVID-19 pandemic in the Spring semester of 2020. Since then several corrections have been made, in particular we would like to thank the following former participants of the course for several valuable comments András Palkó (2020), Attila Zoltán Jenei (2021), András Fridvalszky (2021), Anett Kenderes (2021), Gábor Révy (2022) and Ónozó Lívia Réka (2024).

2. Notations, preliminary observations

An $n \times m$ rectangular matrix has n rows and m columns. We will denote matrices by capital letters: A, B, C, and so on. For instance:

$$A = (a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{pmatrix}.$$

The entries a_{ij} will be real numbers ($a_{ij} \in \mathbb{R}$) or complex numbers ($a_{ij} \in \mathbb{C}$).

If the number of rows is the same as the number of columns, that is, $n = m$, then we say that the matrix is a square matrix.

If A and B are two matrices, then we write $A = B$ if A and B are of the same type (say, $n \times m$) and $a_{ij} = b_{ij}$ for every $1 \le i \le n$, $1 \le j \le m$.

The transpose of an $n \times m$ matrix A is denoted by A^T, its type is $m \times n$ and $(A^T)_{ij} = (A)_{ji}$.

The matrix $\overline{A}$ has the same type as A and $(\overline{A})_{ij} = \overline{a}_{ij}$, that is, we take the complex conjugate of all of the entries of A.

Note that $A = \overline{A}$ if and only if A is a real matrix, that is, if all of the entries of A are real.

We say that a square matrix A is *symmetric*, if $A^T = A$.

The matrix A satisfies $\overline{A} = -A$ if and only if all the entries of A are purely imaginary.

If a square matrix satisfies that $A^T = -A$, then we say that A is *skew-symmetric*. Note that on the main diagonal of a skew-symmetric matrix all the entries are equal to 0. An example for a skew-symmetric matrix is as follows:

$$\begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{pmatrix}.$$

The adjoint of a matrix A is denoted by A^* and defined to be $A^* = \overline{A^T} = \overline{A}^T$.

If $A^* = A$, then we say that A is *self-adjoint* (or Hermitian).

If $A^* = -A$, then A is said to be *skew-Hermitian.*

Note that for real matrices $A^* = A^T$.

If A is an $n \times m$ matrix and c is a scalar, then cA is also an $n \times m$ matrix and $(cA)_{ij} = c \cdot (A)_{ij}$. If A is self-adjoint and c is real, then cA is also self-adjoint.

Now, we continue with some special matrices. The everywhere-zero matrix is denoted by 0, or $0_{n\times m}$, if we want to specify in the notion its size. For instance:

$$0_{2\times 3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The identity matrix is a square matrix having 1's along its main diagonal and 0s everywhere else and it is denoted by I (or I_n to specify its size). For instance:

$$I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The $n \times m$ matrix in which all the entries are 0s with the exception of the jth entry of the ith row is denoted by E_{ij}. Note that for $n \times n$ matrices we have $I = I_n = E_{11} + E_{22} + \cdots + E_{nn}$.

An $n \times 1$ matrix is a column vector and a $1 \times m$ matrix is a row vector. For instance:

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \end{pmatrix}.$$

The sum $A+B$ is defined if A and B are of the same type and $(A+B)_{ij} = (A)_{ij} + (B)_{ij}$. Note that addition of matrices is

- associative: $(A+B)+C = A+(B+C)$,
- commutative: $A+B = B+A$,
- there is a 0 element, namely the matrix 0: $A+0 = A = 0+A$.

Proposition 1. *Let A be an $n \times n$ matrix. Then there exist $n \times n$ self-adjoint matrices A_1, A_2 such that $A = A_1 + iA_2$.*

Proof. Let $A_1 := \frac{1}{2}(A + A^*)$ and $A_2 := \frac{1}{2i}(A - A^*)$. Note that $A = A_1 + iA_2$ indeed holds. Observe that A_1 is self-adjoint, since

$$A_1^* = \frac{1}{2}(A^* + \underbrace{(A^*)^*}_{A}) = A_1.$$

A_2 is also self-adjoint, since

$$A_2^* = \frac{-1}{2i}(A^* - \underbrace{(A^*)^*}_{A}) = A_2.$$

$\square$

Proposition 2. *Let A be an $n \times n$ real matrix. Then there exist $n \times n$ matrices A_1, A_2, where A_1 is symmetric, A_2 is skew-symmetric, and $A = A_1 + A_2$.*

Proof. Let $A_1 := \frac{1}{2}(A + A^T)$ and $A_2 := \frac{1}{2}(A - A^T)$. Note that $A = A_1 + A_2$ indeed holds. Observe that A_1 is symmetric, since

$$A_1^T = \frac{1}{2}(A^T + \underbrace{(A^T)^T}_{A}) = A_1.$$

A_2 is skew-symmetric, since

$$A_2^T = \frac{1}{2}(A^T - \underbrace{(A^T)^T}_{A}) = -A_2.$$

□

If A is an $n \times k$ matrix and B is a $k \times m$ matrix, then the product AB is defined to be the $n \times m$ matrix for which

$$(AB)_{jt} = \sum_{\ell=1}^{k} a_{j\ell} b_{\ell t}.$$

Note that matrix multiplication is associative:

$$(AB)C = A(BC).$$

(In the sense that the left hand side is defined if and only if so does the right hand side, and in this case the two sides are equal to each other.)

Matrix multiplication is *not* commutative, $AB \neq BA$ in general. If $AB = BA$, then we say that A and B commute, or A and B are interchangeable. If A is an $n \times n$ matrix, then $AI = A = IA$ for the $n \times n$ identity matrix $I = I_n$.

Note that $A \cdot 0 = 0$ for the 0 matrix.

Also, note that for square matrices we have

$$E_{ij}E_{k\ell} = \begin{cases} E_{i\ell} & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}.$$

Let A be an $n \times n$ matrix. Then the p-factor product $\underbrace{A \cdot A \cdot \ldots \cdot A}_{p \text{ copies of } A} =: A^p$ is well-defined, since multiplication is associative. Note that $A^p A^q = A^{p+q}$. Let us define $A^0 := I$.

Definition 1. *If $p(x) = a_n x^n + \cdots + a_1 x + a_0$ is a polynomial and A is a square matrix, then we define $p(A) := a_n A^n + \cdots + a_1 A + a_0 I$.*

Example 1. *If there is a one-variable polynomial identity, say, $(x-1)(x+1) = x^2 - 1$, then the identity also holds for a square matrix:*

$$(A - I)(A + I) = A^2 - I,$$

since the powers of A commute with each other. However, for polynomials with multiple variables this might not be the case. For instance, $(x+y)(x-y) = x^2 - y^2$ is a polynomial identity, but $(A+B)(A-B) \neq A^2 - B^2$ in general. Since, the left hand side is $A^2 - B^2 + BA - AB$, which means that for $AB \neq BA$ the equation fails to hold.

We list some properties of the operations in the following proposition.

Proposition 3. *The following statements hold:*

- $\overline{A \cdot B} = \overline{A} \cdot \overline{B}$,
- $(AB)^T = B^T A^T$, *and in general,*

- $(A_1 A_2 \dots A_m)^T = A_m^T \dots A_2^T A_1^T$,
- $(AB)^* = B^* A^*$, *and in general,*
- $(A_1 A_2 \dots A_m)^* = A_m^* \dots A_2^* A_1^*$.

There are some special products that often appear in our problems. If A is an $n \times k$ matrix and b is a column vector with k entries, then the product Ab is also a column vector, the number of its entries is n. Note that we can think of Ab as a linear combination of the columns of A, where the coefficients are the entries of b. For instance:

$$\begin{pmatrix}1 & 2 & 3 & 4\\ 5 & 6 & 7 & 8\\ 9 & 10 & 11 & 12\end{pmatrix}\begin{pmatrix}1\\1\\0\\-2\end{pmatrix} = 1\begin{pmatrix}1\\5\\9\end{pmatrix} + 1\begin{pmatrix}2\\6\\10\end{pmatrix} + 0\begin{pmatrix}3\\7\\11\end{pmatrix} + (-2)\begin{pmatrix}4\\8\\12\end{pmatrix} = \begin{pmatrix}-5\\-5\\-5\end{pmatrix}.$$

Analogously, for a row vector c^T the product $c^T A$ is also a row vector, which is a linear combination of the rows of A. For instance:

$$\begin{pmatrix}1 & 2 & 0\end{pmatrix}\begin{pmatrix}1 & 2 & 3 & 4\\ 5 & 6 & 7 & 8\\ 9 & 10 & 11 & 12\end{pmatrix} = 1\begin{pmatrix}1 & 2 & 3 & 4\end{pmatrix} + 2\begin{pmatrix}5 & 6 & 7 & 8\end{pmatrix} + 0\begin{pmatrix}9 & 10 & 11 & 12\end{pmatrix} =$$
$$= \begin{pmatrix}11 & 14 & 17 & 20\end{pmatrix}.$$

If c^T is a row vector, b is a column vector and they have the same number of entries, then $c^T b$ is a 1×1 matrix, that is, a scalar. Note that this is the dot product of the two vectors. For instance:

$$\begin{pmatrix}1 & 2 & 3\end{pmatrix}\begin{pmatrix}4\\5\\6\end{pmatrix} = 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 = 32.$$

If we take the product of a column vector b and a row vector c^T (in this order), then we get a matrix. For instance:

$$\begin{pmatrix}1\\2\\3\\4\end{pmatrix}\begin{pmatrix}5 & 6 & 7\end{pmatrix} = \begin{pmatrix}5 & 6 & 7\\ 10 & 12 & 14\\ 15 & 18 & 21\\ 20 & 24 & 28\end{pmatrix}.$$

Note that if both $b \neq 0$ and $c \neq 0$, then bc^T is a so-called rank-one matrix. (See Example 4.)

In dimension n the standard basis vectors are denoted by $e_1, e_2, \dots, e_n$, where in e_i the ith entry is 1 and all the other entries are 0s. Note that

$$e_i^T e_j = \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases},$$

where δ is the Kronecker-symbol.

If we take the product in the other order, $e_i e_j^T = E_{ij}$.

Finally, when the matrix product AB is defined, the number of columns of A is the same as the number of rows of B:

$$A = \begin{pmatrix} a_1 & a_2 & \dots & a_k \end{pmatrix}, \quad B = \begin{pmatrix} b^1 \\ b^2 \\ \vdots \\ b^k \end{pmatrix},$$

where $a_1, \dots a_k$ are the columns of A and $b^1, \dots, b^k$ are the rows of B. The product AB can be expressed as

$$AB = a_1 b^1 + \dots + a_k b^k,$$

that is we can think of the product of two matrices as a sum of rank-one (or 0) matrices.

2.1. **Determinant.** The determinant is defined only for square matrices. Let A be an $n \times n$ matrix. The determinant of A, denoted by $\det A$ or $|A|$ is

$$\det A = \sum_{\pi:\text{ permutation}} (-1)^{I(\pi)} a_{1\pi(1)} a_{2\pi(2)} \dots a_{n\pi(n)},$$

where $I(\pi)$ is the number of inversions in π. For instance, for the permutation

i	1	2	3	4	5
$\pi(i)$	5	3	1	2	4

the number of inversions is $I(\pi) = 6$, because the pairs $53, 51, 52, 54, 31, 32$ are the ones forming inversions.

That is, to get the determinant, for each rook configuration (selection of n entries such that we select one from each row and one from each column) we take the corresponding signed n-factor product, and add these up.

Some basic properties of the determinant:

Proposition 4. *If A is an $n \times n$ matrix, then the following statements hold:*

- $\det(cA) = c^n \det(A)$
- $\det(A^T) = \det(A)$
- $\det A^* = \overline{\det A}$

Remark 1. Specially, $\det(-A) = (-1)^n \det A$. So $\det(-A) = -\det A$ for odd n, while $\det(-A) = \det A$ for even n.

Let A_{ij} denote the matrix obtained from A by deleting its ith row and jth column.

Theorem 5 (Expansion by row)**.**

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}$$

Analogously:

Theorem 6 (Expansion by column)**.**

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}$$

Corollary 7. *If $i \neq k$, then*

$$0 = \sum_{j=1}^{n} (-1)^{i+j} a_{kj} \det A_{ij}.$$

Proof. By changing the ith row to be the same as the kth row we get a matrix having two identical rows, thus its determinant is 0. However, by using expansion by row i we get the sum from the right hand side. □

We can summarize the results of Theorems 5 and Corollary 7 in the form of a matrix equation. In order to this, let us define the adjugate of a matrix as follows.

Definition 2. *The matrix arising in the previous statement is the adjugate of the matrix A:*

$$\operatorname{adj} A = \begin{pmatrix} (-1)^{1+1} \det A_{11} & \dots & (-1)^{n+1} \det A_{n1} \\ \vdots & \ddots & \vdots \\ (-1)^{1+n} \det A_{1n} & \dots & (-1)^{n+n} \det A_{nn} \end{pmatrix}.$$

Now, we can summarize the results of Theorems 5 and Corollary 7 in the form of the following matrix equation.

Corollary 8.

$$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} (-1)^{1+1} \det A_{11} & \dots & (-1)^{n+1} \det A_{n1} \\ \vdots & \ddots & \vdots \\ (-1)^{1+n} \det A_{1n} & \dots & (-1)^{n+n} \det A_{nn} \end{pmatrix} =$$

$$= \begin{pmatrix} \det A & & \\ & \ddots & \\ & & \det A \end{pmatrix} = (\det A) I.$$

That is,

$$A \cdot \operatorname{adj} A = (\det A) I.$$

Analogously:

Corollary 9. *If A is a square matrix, then*

$$(\operatorname{adj} A) \cdot A = (\det A) I.$$

Example 2. *If* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, *then* $\det A = ad - bc$ *and* $\operatorname{adj} A = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ *and*

$$A \cdot \operatorname{adj} A = (\det A) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let us recall the following theorem.

Theorem 10. *If A and B are $n \times n$ matrices, then*

$$\det(AB) = (\det A)(\det B).$$

Definition 3. *Let A be az $n \times m$ matrix. If $1 \leq i_1 < \cdots < i_r \leq n$ and $1 \leq j_1 < \cdots < j_s \leq m$, then the $r \times s$ matrix obtained by deleting the rows with indices different from $i_1, \ldots, i_r$ and columns with indices different from $j_1, \ldots, j_s$ is called a submatrix of A and denoted by*

$$A^{i_1,\ldots,i_r}_{j_1,\ldots,j_s}.$$

Example 3. *Let* $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix}$.

Then $A^{1,3}_{1,2,4} = \begin{pmatrix} 1 & 2 & 4 \\ 9 & 10 & 12 \end{pmatrix}$ *is a 2×3 submatrix of A.*

3. Minimal rank-one decomposition

3.1. **Rank.** Let us start with the definition of the rank of a matrix.

Definition 4. *Let A be a matrix. The row rank of A is the largest k such that k linearly independent rows of A can be chosen. Alternatively, the row rank is the dimension of the row space of A.*

The column rank of A is the largest ℓ such that ℓ linearly independent columns of A can be chosen. Alternatively, the column rank is the dimension of the column space of A.

Finally, the determinant rank of A is the largest r such that A has an $r \times r$ submatrix with nonzero determinant.

Proposition 11. *The row rank, the column rank and the determinant rank of A is always the same.*

Definition 5. *The rank of A, denoted by* $\operatorname{rk} A$*, is the row/column/determinant rank of A.*

Example 4. *If $u \neq 0, v \neq 0$, then the rank of uv^T is 1. Indeed,*

$$\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \begin{pmatrix} v_1 & \dots & v_m \end{pmatrix} = \begin{pmatrix} u_1 v_1 & \dots & u_1 v_m \\ \vdots & \ddots & \vdots \\ u_n v_1 & \dots & u_n v_m \end{pmatrix},$$

for $u_i \neq 0$ the ith row is a nonzero vector, and all the other rows are scalar multiples of the ith row.

3.2. **Minimal rank-one decomposition.** Our aim is to express A as a sum of rank-one matrices in such a way that we use as few as we can.

The decomposition $A = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} e_i e_j^T$ shows that an $m \times n$ matrix can always be expressed as a sum of mn rank-one matrices.

We have seen that we can think of a rank-one decomposition as a matrix product (and vice versa), thus by $AI_n = A$ we obtain the decomposition

$$A = AI_n = a_1 e_1^T + \dots + a_n e_n^T,$$

where $a_1, \dots, a_n$ are the columns of A. So n rank-one matrices are always enough for an $m \times n$ matrix. Analogously, from $I_m A = A$ we also get a decomposition, where the number of summands is m.

Now, we present an algorithm which always determines a rank-one decomposition using a minimum number of rank-one matrices.

Algorithm (for finding a minimal rank-one decomposition). Let us assume that $a_{11} \neq 0$. (Otherwise we may swap rows, columns.) Let us zero out the first row and first column by subtracting a suitably chosen rank-one matrix:

$$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} - \frac{1}{a_{11}} \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix} \begin{pmatrix} a_{11} & \dots & a_{1n} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & a_{22}^{(2)} & \dots & a_{2n}^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{m2}^{(2)} & \dots & a_{mn}^{(2)} \end{pmatrix} =: A^{(2)}.$$

Now, for simplicity let us assume that $a_{22}^{(2)} \neq 0$. (Otherwise we swap rows, columns...) Now, we zero out the second row and second column of $A^{(2)}$:

$$\begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & a_{22}^{(2)} & \dots & a_{2n}^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{m2}^{(2)} & \dots & a_{mn}^{(2)} \end{pmatrix} - \frac{1}{a_{22}^{(2)}} \begin{pmatrix} 0 \\ a_{22}^{(2)} \\ \vdots \\ a_{m2}^{(2)} \end{pmatrix} \begin{pmatrix} 0 & a_{22}^{(2)} & \dots & a_{2n}^{(2)} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & & \end{pmatrix} =: A^{(3)},$$

and so on...

This way in a finite number of steps we obtain a rank-one decomposition:

$$A = \frac{1}{a_{11}} \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix} \begin{pmatrix} a_{11} & \dots & a_{1n} \end{pmatrix} + \frac{1}{a_{22}^{(2)}} \begin{pmatrix} 0 \\ a_{22}^{(2)} \\ \vdots \\ a_{m2}^{(2)} \end{pmatrix} \begin{pmatrix} 0 & a_{22}^{(2)} & \dots & a_{2n}^{(2)} \end{pmatrix} + \dots =$$

$$= u_1 v_1^T + u_2 v_2^T + \dots + u_r v_r^T.$$

That is, $A = \begin{pmatrix} u_1 & \dots & u_r \end{pmatrix} \begin{pmatrix} v_1^T \\ \vdots \\ v_r^T \end{pmatrix} =: UV^T$, where U is an $m \times r$ and V^T is an $r \times n$ matrix that look like below:

$$U = \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ * & a_{22}^{(2)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ * & \dots & * & a_{rr}^{(r)} \\ * & \dots & * & * \\ \vdots & \ddots & \vdots & \vdots \\ * & \dots & * & * \end{pmatrix}, \quad V^T = \begin{pmatrix} 1 & * & \dots & * & * & \dots & * \\ 0 & 1 & \ddots & \vdots & * & \dots & * \\ \vdots & \ddots & \ddots & * & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & * & \dots & * \end{pmatrix}.$$

Observe that since U has exactly r columns and V^T has exactly r rows we have

$$\det A_{1,\dots,r}^{1,\dots,r} = (\det U_{1,\dots,r}^{1,\dots,r})(\det (V^T)_{1,\dots,r}^{1,\dots,r}) = \prod_{k=1}^{r} u_{kk} v_{kk} = \prod_{k=1}^{r} a_{kk}^{(k)} \neq 0,$$

which implies that $\operatorname{rk} A \geq r$ (where r is the number of summands in the above obtained rank-one decomposition).

Now, we show that the obtained rank-one decomposition uses a minimum number of rank-one matrices.

For the sake of contradiction let us assume that $A = \sum_{k=1}^{r-1} \tilde{u}_k \tilde{v}_k^T$, that is, A can be expressed as a sum of (at most) $r-1$ rank-one matrices. Let us consider the matrices

$$\tilde{U} = \begin{pmatrix} \tilde{u}_1 & \tilde{u}_2 & \dots & \tilde{u}_{r-1} & 0 \end{pmatrix}, \quad \tilde{V}^T = \begin{pmatrix} \tilde{v}_1^T \\ \vdots \\ \tilde{v}_{r-1}^T \\ 0 \end{pmatrix}.$$

Observe that

$$A = \sum_{k=1}^{r-1} \tilde{u}_k \tilde{v}_k + 0 \cdot 0 = \tilde{U}\tilde{V}^T.$$

For any choice of indices $1 \le i_1 < \dots < i_r \le m$ and $1 \le j_1 < \dots < j_r \le n$ we have

$$\det A_{j_1,\dots,j_r}^{i_1,\dots,i_r} = (\det \tilde{U}_{1,\dots,r}^{i_1,\dots,i_r})(\det(\tilde{V}^T)_{j_1,\dots,j_r}^{1,\dots,r}) = 0,$$

since $\tilde{U}$ (and $\tilde{V}^T$) clearly have rank at most $r-1$. (In fact $\tilde{U}_{1,\dots,r}^{i_1,\dots,i_r}$ has a full-zero column.)

We obtained that every $r \times r$ submatrix of A has 0 determinant, which contradicts that $\operatorname{rk} A \ge r$.

Hence, the obtained rank-one decomposition is indeed minimal.

Moreover, by a similar reasoning (applied for the rank-one decomposition $\sum_{k=1}^{r} u_k v_k$ in place of $\sum_{k=1}^{r-1} \tilde{u}_k \tilde{v}_k$) we get that every $(r+1) \times (r+1)$ submatrix of A has determinant 0, thus $\operatorname{rk} A = r$. Let us formulate this result in the following statement.

Corollary 12. *The number of rank-one matrices in a minimal rank-one decomposition of A is $\operatorname{rk} A$, the rank of A.*

Example 5. *The task is to find a minimal rank-one decomposition for the matrix*

$$A = \begin{pmatrix} 3 & 3 & 6 & 5 & 5 \\ 7 & 4 & 7 & 2 & 0 \\ -1 & -2 & -3 & -4 & -5 \\ -1 & -3 & -8 & -9 & -10 \end{pmatrix}.$$

First we zero out the first row and first column:

$$\begin{pmatrix} 3 & 3 & 6 & 5 & 5 \\ 7 & 4 & 7 & 2 & 0 \\ -1 & -2 & -3 & -4 & -5 \\ -1 & -3 & -8 & -9 & -10 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} 3 \\ 7 \\ -1 \\ -1 \end{pmatrix}\begin{pmatrix} 3 & 3 & 6 & 5 & 5 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & -7 & -29/3 & -35/3 \\ 0 & -1 & -1 & -7/3 & -10/3 \\ 0 & -2 & -6 & -22/3 & -25/3 \end{pmatrix} = A^{(2)}.$$

Now, we continue with the 2nd row and 2nd column:

$$A^{(2)} - \frac{1}{-3}\begin{pmatrix}0\\-3\\-1\\-2\end{pmatrix}\begin{pmatrix}0 & -3 & -7 & -29/3 & -35/3\end{pmatrix} = \begin{pmatrix}0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0\\0 & 0 & 4/3 & 8/9 & 5/9\\0 & 0 & -4/3 & -8/9 & -5/9\end{pmatrix} = A^{(3)}.$$

Finally,

$$A^{(3)} - \frac{1}{4/3}\begin{pmatrix}0\\0\\4/3\\-4/3\end{pmatrix}\begin{pmatrix}0 & 0 & 4/3 & 8/9 & 5/9\end{pmatrix} = 0.$$

Hence, we obtained that a minimal rank-one decomposition of A is as follows:

$$A = \frac{1}{3}\begin{pmatrix}3\\7\\-1\\-1\end{pmatrix}\begin{pmatrix}3 & 3 & 6 & 5 & 5\end{pmatrix} + \left(-\frac{1}{3}\right)\begin{pmatrix}0\\-3\\-1\\-2\end{pmatrix}\begin{pmatrix}0 & -3 & -7 & -29/3 & -35/3\end{pmatrix} +$$

$$+ \frac{3}{4}\begin{pmatrix}0\\0\\4/3\\-4/3\end{pmatrix}\begin{pmatrix}0 & 0 & 4/3 & 8/9 & 5/9\end{pmatrix} = \begin{pmatrix}3 & 0 & 0\\7 & -3 & 0\\-1 & -1 & 4/3\\-1 & -2 & -4/3\end{pmatrix}\begin{pmatrix}1 & 1 & 2 & 5/3 & 5/3\\0 & 1 & 7/3 & 29/9 & 35/9\\0 & 0 & 1 & 2/3 & 5/12\end{pmatrix}.$$

Example 6. *The task is to find a minimal rank-one decomposition for the matrix*

$$B = \begin{pmatrix}1 & 2 & 3 & 4\\3 & 6 & 7 & 8\\1 & 2 & 1 & 0\end{pmatrix}.$$

After subtracting the first rank-one matrix:

$$B - \begin{pmatrix}1\\3\\1\end{pmatrix}\begin{pmatrix}1 & 2 & 3 & 4\end{pmatrix} = \begin{pmatrix}0 & 0 & 0 & 0\\0 & 0 & -2 & -4\\0 & 0 & -2 & -4\end{pmatrix} = B^{(2)}.$$

That is, $b_{22}^{(2)} = 0$. In such a case we can simply choose a nonzero element, and subtract the product of the corresponding column and row:

$$B^{(2)} - \frac{1}{-2}\begin{pmatrix}0\\-2\\-2\end{pmatrix}\begin{pmatrix}0 & 0 & -2 & -4\end{pmatrix} = 0.$$

Hence,

$$B = \begin{pmatrix}1\\3\\1\end{pmatrix}\begin{pmatrix}1 & 2 & 3 & 4\end{pmatrix} + \frac{1}{-2}\begin{pmatrix}0\\-2\\-2\end{pmatrix}\begin{pmatrix}0 & 0 & -2 & -4\end{pmatrix} = \begin{pmatrix}1 & 0\\3 & -2\\1 & -2\end{pmatrix}\begin{pmatrix}1 & 2 & 3 & 4\\0 & 0 & 1 & 2\end{pmatrix}.$$

3.3. **Rank inequalities.** As a first application of minimal rank-one decomposition we prove some inequalities about the rank.

Proposition 13. $\operatorname{rk}(A+B) \le \operatorname{rk} A + \operatorname{rk} B$

Proof. Let $A = \sum_{k=1}^{\operatorname{rk} A} u_k v_k^T$ and $B = \sum_{\ell=1}^{\operatorname{rk} B} w_\ell z_\ell^T$ be minimal rank-one decompositions for A and B, respectively. Then

$$A + B = \sum_{k=1}^{\operatorname{rk} A} u_k v_k^T + \sum_{\ell=1}^{\operatorname{rk} B} w_\ell z_\ell^T$$

is a (not necessarily minimal) rank-one decomposition for $A+B$ with $\operatorname{rk} A + \operatorname{rk} B$ rank-one matrices. Hence, $\operatorname{rk}(A+B) \le \operatorname{rk} A + \operatorname{rk} B$, as in a minimal rank-one decomposition of $A+B$ the number of rank-one matrices is exactly $\operatorname{rk}(A+B)$. □

Proposition 14. $\operatorname{rk}(AB) \le \min(\operatorname{rk} A, \operatorname{rk} B)$

Proof. Let $A = \sum_{k=1}^{\operatorname{rk} A} u_k v_k^T$ be a minimal rank-one decomposition for A. Then

$$AB = \sum_{k=1}^{\operatorname{rk} A} u_k (v_k^T B)$$

is a (not necessarily minimal) rank-one decomposition for AB, which implies that $\operatorname{rk}(AB) \le \operatorname{rk} A$.

Similarly, by taking a minimal rank-one decomposition $B = \sum_{\ell=1}^{\operatorname{rk} B} w_\ell z_\ell^T$ for B we get the (not necessarily minimal) rank-one decomposition

$$AB = \sum_{\ell=1}^{\operatorname{rk} B} (A w_\ell) z_\ell^T$$

for AB with $\operatorname{rk} B$ terms, implying that $\operatorname{rk}(AB) \le \operatorname{rk} B$ also holds. □

4. Inverse

Let A be an $n \times n$ matrix. If $AX = I(= I_n)$, then we say that X is the right inverse of A. Analogously, if $YA = I$, then Y is the left inverse of A. Note that if both the right and left inverse exist, then they are the same, since by the associativity of matrix multiplication we get

$$X = IX = (YA)X = Y(AX) = YI = Y.$$

In fact we will prove soon that $AX = I \iff XA = I$, so the left inverse exists if and only if the right inverse exists, they are unique (and equal to each other). Because of this, in case of matrices we won't use the expressions left inverse and right inverse (just: inverse), however, a priori it was not clear that these two notions coincide, and in case of arbitrary rings, it may happen that only the left (or right) inverse exists, and unicity could also fail to hold.

Definition 6. *Let A be an $n \times n$ matrix. We say that X is the inverse of A, if $AX = XA = I$. The inverse matrix (if it exists) is denoted by A^{-1}.*

Example 7. *Let $D = \begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & d_n \end{pmatrix}$, where the diagonal entries are all different from 0. Then*

$$D^{-1} = \begin{pmatrix} 1/d_1 & 0 & \dots & 0 \\ 0 & 1/d_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1/d_n \end{pmatrix}.$$

Now, we give a necessary and sufficient condition for a square matrix to be invertible.

Proposition 15. *The square matrix A is invertible if and only if* $\det A \neq 0$.

Proof. First we prove that the condition $\det A \neq 0$ is necessary. Indeed, if $\det A = 0$, then for any square matrix X (of the same size as A)

$$\det(AX) = (\underbrace{\det A}_{0})(\det X) = 0 \neq 1 = \det I,$$

thus $AX \neq I$.

Now, we prove that the condition $\det A \neq 0$ is sufficient. By Corollaries 8 and 9 we get that $A \cdot \operatorname{adj} A = \operatorname{adj} A \cdot A = (\det A)I$, thus for $X = \frac{\operatorname{adj} A}{\det A}$ we have $AX = XA = I$, that is, X is the inverse of A. $\square$

Remark 2. For an invertible square matrix we may alternatively use the expressions regular or nonsingular. Otherwise, when the matrix is not invertible, we say that it is singular.

Now we prove some elementary properties of the inverse.

Proposition 16. *The following statements hold:*

- $(AB)^{-1} = B^{-1}A^{-1}$
- $(A^T)^{-1} = (A^{-1})^T$

Proof. To prove the first statement let us write

$$(AB)(B^{-1}A^{-1}) = A(\underbrace{BB^{-1}}_{I})A^{-1} = AA^{-1} = I.$$

The second statement holds, since

$$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I.$$

□

Proposition 17. *The matrix $I - AB$ is invertible if and only $I - BA$ is invertible.*

Proof. Let us assume that $I - BA$ is invertible. We claim that $I + A(I - BA)^{-1}B$ is the inverse of $I - AB$. Indeed:

$$\begin{aligned}(I - AB)(I + A(I - BA)^{-1}B) &= I - AB + A(I - BA)^{-1}B - ABA(I - BA)^{-1}B = \\ &= I + A[-I + \underbrace{(I - BA)^{-1} - BA(I - BA)^{-1}}_{(I-BA)(I-BA)^{-1}=I}]B = I.\end{aligned}$$

Hence, $I - AB$ is invertible. The other direction can be proved in a similar way. □

Remark 3. The formula $I+A(I-BA)^{-1}B$ can be conjectured for instance in the following way. We would like to find $(I - AB)^{-1}$. If x is a real (or complex) number with absolute value smaller than 1, then $(1 - x)^{-1}$ is the sum of the series $1 + x + x^2 + x^3 + \dots$. Let us write the analogue for this in case of $(I - AB)^{-1}$:

$$\begin{aligned}(I-AB)^{-1} \text{“}=\text{”}\, I+AB+(AB)^2+(AB)^3+\dots \text{“}=\text{”}\, I+AB+ABAB+ABABAB+\dots \text{“}=\text{”} \\ \text{“}=\text{”}\, I + A[I + BA + BABA + \dots]B \text{“}=\text{”}\, I + A(I - BA)^{-1}B.\end{aligned}$$

This is not yet a rigorous proof in this form, as we did not define what is meant by an infinite sum of matrices, furthermore, convergence is not justified either. (Even for 1×1 matrices, that is, for numbers $A = a, B = b$ convergence can fail to hold.) Although this argument could be made precise (by considering *formal* power series), for us it suffices to arrive at a plausible conjecture, as the conjectured inverse could be checked easily (as we did above).

Proposition 18. *If B is invertible, then* $\operatorname{rk}(AB) = \operatorname{rk} A$.

Proof. According to Proposition 14 we know that $\operatorname{rk}(AB) \le \operatorname{rk} A$ holds. However, by writing $A = (AB)(B^{-1})$ we get by Proposition 14 that $\operatorname{rk} A \le \operatorname{rk}(AB)$ also holds. □

Now, we give a formula for the inverse of a matrix changed by a rank-one matrix.

Proposition 19 (Sherman-Morrison formula). *Let A be an invertible matrix and u, v column vectors. If $1 + v^T A^{-1} u \neq 0$, then*

$$(A + uv^T)^{-1} = A^{-1} - \frac{(A^{-1}u)(v^T A^{-1})}{1 + v^T A^{-1} u}.$$

Remark 4. Note that by changing the matrix A by a rank-one matrix resulted in a change in its inverse by a rank-one matrix, too.

Proof.

$$\left(A^{-1} - \frac{(A^{-1}u)(v^T A^{-1})}{1 + v^T A^{-1} u}\right)(A + uv^T) = I + A^{-1}uv^T - \frac{A^{-1}uv^T}{1 + v^T A^{-1}u} - \frac{A^{-1}u(v^T A^{-1}u)v^T}{1 + v^T A^{-1}u} =$$

$$= I + A^{-1}uv^T - A^{-1}uv^T \left(\underbrace{\frac{1}{1 + v^T A^{-1}u} + \frac{v^T A^{-1}u}{1 + v^T A^{-1}u}}_{1} \right) = I.$$

□

Remark 5. The Sherman-Morrison formula is not applicable when $1 + v^T A^{-1} u = 0$. In fact, in this case $A + uv^T$ is not invertible. To prove this it suffices to show that the columns of $A + uv^T$ are linearly dependent. (If the column rank is smaller than n, then so does the determinant rank, which implies that the determinant of the matrix is 0.) Observe now that

$$(A + uv^T)(A^{-1}u) = \underbrace{AA^{-1}}_{I} u + u \underbrace{v^T A^{-1} u}_{-1} = u - u = 0.$$

This linear combination is nontrivial, as $u \neq 0$ implies that $A^{-1}u \neq 0$. (Note that $u \neq 0$, since otherwise $1 + v^T A^{-1} u = 0$ would fail to hold.)
Hence, $A + uv^T$ is singular.

As a special case of the Sherman-Morrison formula we can calculate the inverse if one entry of the matrix is modified.

Corollary 20. *Let A be an invertible matrix. Let $1 \leq i \leq n$ and $1 \leq j \leq n$ be two (fixed) indices. Let us change a_{ij} to $a_{ij} + c$, obtaining the matrix A'. If $1 + ce_j^T A^{-1} e_i \neq 0$, then*

$$(A')^{-1} = (A + ce_i e_j^T)^{-1} = A^{-1} - \frac{A^{-1} ce_i e_j^T A^{-1}}{1 + e_j^T A^{-1} ce_i}.$$

Proof. The statement follows from the Sherman-Morrison formula applied for $u = ce_i$ and $v = e_j$. □

Remark 6. For brevity, let $B := A^{-1}$. Let us denote by $b_i = Be_i$ the ith column and by $b^j = e_j^T B$ the jth row of B. Then we may reformulate the result as

$$(A + ce_i e_j^T)^{-1} = B - \frac{b_i b^j}{\frac{1}{c} + b_{ji}},$$

assuming that $c \neq 0$. (Note that in case of $c = 0$ the matrix is not modified at all.)

Example 8. *The task is to calculate the inverse of the following matrix:*

$$\begin{pmatrix} 1+x & 1 & \dots & 1 \\ 1 & 1+x & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \dots & 1 & 1+x \end{pmatrix}.$$

Let j denote the all-one column vector. Then the matrix in question can be expressed az $xI + jj^T$. Let us apply the Sherman-Morrison formula with the choice $A = xI$, $u = v = j$. For the matrix A to be invertible, we need to assume that $x \neq 0$. The condition on the rank-one matrix uv^T reads as

$$0 \neq 1 + v^T A^{-1} u = 1 + j^T \frac{1}{x} I j = 1 + \frac{n}{x},$$

thus $x \neq -n$ should also be assumed. If $x \neq 0, -n$, then we obtain that

$$(xI + jj^T)^{-1} = (xI)^{-1} - \frac{(xI)^{-1} jj^T (xI)^{-1}}{1 + j^T (xI)^{-1} j} = \frac{1}{x} I - \frac{\frac{1}{x} jj^T \frac{1}{x}}{1 + \frac{1}{x} j^T j} = \frac{1}{x} \left(I - \frac{jj^T}{x + \underbrace{j^T j}_{n}} \right) =$$

$$= \frac{1}{x(x+n)} \left((x+n) I - jj^T \right).$$

Note that

$$(x+n)I - jj^T = \begin{pmatrix} x+n-1 & -1 & \dots & -1 \\ -1 & x+n-1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 \\ -1 & \dots & -1 & x+n-1 \end{pmatrix}.$$

Remark 7. If $x = 0$, then the matrix is the all-1 matrix which has rank one, thus it is not invertible (except the trivial case $n = 1$). If $x = -n$, then the sum of the rows is 0, thus the matrix is singular. Note that this latter implication also follows from Remark 5.

4.1. **Equivalent transformations.** Let us say that an equivalent transformation of A is multiplying it from both sides by nonsingular square matrices: $\tilde{A} = QAR$. Our aim is to transform A to the "simplest" form that is possible. Note that

$$\operatorname{rk}(\tilde{A}) = \operatorname{rk}(QAR) = \operatorname{rk} A =: r.$$

To achieve this goal we shall use the following three types of elementary transformations.

- The first type of elementary transformations are going to be used to permute the columns or the rows of a matrices. For this we shall use permutation matrices.

Definition 7. *An $n \times n$ matrix P is a permutation matrix if its columns form a permutation of the standard basis vectors, $e_1, \dots, e_n$. That is, if $P = \begin{pmatrix} e_{i_1} & \dots & e_{i_n} \end{pmatrix}$ for some $\{i_1, \dots, i_n\} = \{1, \dots, n\}$. Note that in a permutation matrix each row and each column contains exactly one nonzero entry which is a 1.*

Note that

$$PP^T = \begin{pmatrix} e_{i_1} & \dots & e_{i_n} \end{pmatrix} \begin{pmatrix} e_{i_1}^T \\ \vdots \\ e_{i_n}^T \end{pmatrix} = \sum_{j=1}^{n} e_{i_j} e_{i_j}^T = I,$$

thus $P^T = P^{-1}$. This implies that P is nonsingular, and $\det P = \pm 1$.

Note that

$$AP = \begin{pmatrix} a_1 & \dots & a_n \end{pmatrix} \begin{pmatrix} e_{i_1} & \dots & e_{i_n} \end{pmatrix} = \begin{pmatrix} a_{i_1} & \dots & a_{i_n} \end{pmatrix}$$

and

$$P^T A = \begin{pmatrix} e_{i_1}^T \\ \vdots \\ e_{i_n}^T \end{pmatrix} \begin{pmatrix} a^1 \\ \vdots \\ a^n \end{pmatrix} = \begin{pmatrix} a^{i_1} \\ \vdots \\ a^{i_n} \end{pmatrix}.$$

- The second type of elementary transformations are going to be used to multiply a column or a row by a nonzero scalar λ.
  To multiply the ith column of a matrix A by λ, we shall multiply it from the right hand side by the matrix

$$\begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & \lambda & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix},$$

  which is obtained from the identity by changing the ith entry on the main diagonal to λ.

  Multiplying by such a matrix from the left hand side multiplies the ith row by λ (and keep the other rows unchanged).
  Note that the determinant of this matrix is λ.
- The third type of elementary transformations are going to be used to add λ times the ith column (row) to the jth column (row).
  This effect can be achieved by multiplying by the matrix $I + \lambda E_{ij}$ from the right hand side. (In case of rows: from the left hand side by $I + \lambda E_{ji}$.)
  Note that $\det(I + \lambda E_{ij}) = 1$.

Let us consider now a minimal rank-one decomposition of a $k \times n$ matrix A. Let us permute first the columns in such a way that the first $r = \operatorname{rk} A$ columns are linearly independent:

$$AP,$$

where P is a permutation matrix. Then

$$AP = UV^T,$$

where V^T is of the form

$$V^T = \begin{pmatrix} 1 & * & \dots & * & * & \dots & * \\ 0 & 1 & \ddots & \vdots & * & \dots & * \\ \vdots & \ddots & \ddots & * & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & * & \dots & * \end{pmatrix}.$$

With the help of elementary transformations we may transform V^T to the following form:

$$V^T\tilde{R} = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 1 & \ddots & \vdots & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & 0 & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & 0 & \dots & 0 \end{pmatrix}.$$

Hence:

$$AP\tilde{R} = UV^T\tilde{R},$$

where A is $k \times n$, P is $n \times n$, $\tilde{R}$ is $n \times n$, U is $k \times r$, V^T is $r \times n$.

Let us extend U to a $k \times k$ nonsingular matrix $\tilde{U}$ by adding $k - r$ columns in a suitable way. (Note that the columns of U are linearly independent, thus they can be extended to a basis.) Let us extend $V^T\tilde{R}$ to a $k \times n$ matrix W by adding $k - r$ all-zero rows. Note that

$$AP\tilde{R} = UV^T\tilde{R} = \tilde{U}W.$$

After multiplying by the inverse of $\tilde{U}$ we obtain that

$$\tilde{U}^{-1}AP\tilde{R} = W,$$

where

$$W = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 1 & \ddots & \vdots & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & 0 & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Here W is a $k \times n$ matrix, where the top left $r \times r$ is I_r, and all the remaining entries are zeros.

By setting

$$Q := \tilde{U}^{-1}, \quad R := P\tilde{R}$$

we obtain

$$QAR = W = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}.$$

Example 9. *The task is transform the following matrix into this form:*

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 6 & 7 & 8 \\ 1 & 2 & 1 & 0 \end{pmatrix}.$$

First of all, the rank of the matrix is $r = \operatorname{rk} A = 2$*, since the first two rows are linearly independent, but* $row_3 = row_2 - 2row_1$*. The first two columns are linearly dependent, but the 1st and 3rd columns are linearly independent, so let us choose these two columns. Consequently, let us swap the 2nd and 3rd columns with the help of the permutation matrix*

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now, let us determine a rank-one decomposition for AP *(see Example 6)*

$$AP = \begin{pmatrix} 1 & 3 & 2 & 4 \\ 3 & 7 & 6 & 8 \\ 1 & 1 & 2 & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 3 & -2 \\ 1 & -2 \end{pmatrix}}_{:=U} \underbrace{\begin{pmatrix} 1 & 3 & 2 & 4 \\ 0 & 1 & 0 & 2 \end{pmatrix}}_{:=V^T} = UV^T.$$

Now, we continue by determining the matrix $\tilde{R}$ *that satisfies*

$$\begin{pmatrix} 1 & 3 & 2 & 4 \\ 0 & 1 & 0 & 2 \end{pmatrix} \tilde{R} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

To each of the four columns we will add a suitable chosen linear combination of the first two columns to get the required column. This means that in $\tilde{R}$ *the entries on the main diagonal will be 1s, and with the exception of the entries in the intersection of the first two rows and the upper triangular part of* $\tilde{R}$ *must be 0s. We obtain the following:*

$$\tilde{R} = \begin{pmatrix} 1 & -3 & -2 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let us extend U to a nonsingular 3×3 matrix and $V^T \tilde{R}$ to a 3×4 matrix:

$$\tilde{U} := \begin{pmatrix} 1 & 0 & 0 \\ 3 & -2 & 0 \\ 1 & -2 & 1 \end{pmatrix}, \quad W := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is obtained that
$$\tilde{U}^{-1}AP\tilde{R} = W,$$
where
$$\tilde{U}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 3/2 & -1/2 & 0 \\ 2 & -1 & 1 \end{pmatrix}, \quad P\tilde{R} = \begin{pmatrix} 1 & -3 & -2 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$
That is:
$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 3/2 & -1/2 & 0 \\ 2 & -1 & 1 \end{pmatrix}}_{\tilde{U}^{-1}} \underbrace{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 6 & 7 & 8 \\ 1 & 2 & 1 & 0 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} 1 & -3 & -2 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{P\tilde{R}} = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{W}.$$

4.2. **Sylvester's Law of Nullity.** As an application we shall prove Sylvester's Law of Nullity.

Definition 8. *The nullity of an $n \times n$ square matrix A is defined to be $d(A) = n - \operatorname{rk} A$.*

Theorem 21 (Sylvester's Law of Nullity)**.** *Let A and B be $n \times n$ matrices. Then*
$$\min(\operatorname{rk} A, \operatorname{rk} B) - \min(d(A), d(B)) \le \operatorname{rk}(AB) \le \min(\operatorname{rk} A, \operatorname{rk} B).$$

1st Proof. We have already proved the upper bound $\operatorname{rk}(AB) \le \min(\operatorname{rk} A, \operatorname{rk} B)$, thus it suffices to prove the lower bound.

Without the loss of generality we may assume that $\operatorname{rk} A \ge \operatorname{rk} B$ (and consequently $\mathrm{d}(A) \le \mathrm{d}(B)$). (Otherwise we can consider $B^T A^T$ in place of AB and use that $\operatorname{rk}(B^T A^T) = \operatorname{rk}((AB)^T) = \operatorname{rk}(AB)$ and $\operatorname{rk} A^T = \operatorname{rk} A, \operatorname{rk} B^T = \operatorname{rk} B$.)

Let us take invertible matrices Q, R such that
$$QAR = \begin{pmatrix} I_{\operatorname{rk} A} & 0 \\ 0 & 0 \end{pmatrix}.$$
Observe that
$$\operatorname{rk}(AB) = \operatorname{rk}(QAB) = \operatorname{rk}((QAR) \cdot (R^{-1}B)).$$
Note that $\operatorname{rk}(R^{-1}B) = \operatorname{rk} B$. The effect of multiplying this matrix by $QAR = \begin{pmatrix} I_{\operatorname{rk} A} & 0 \\ 0 & 0 \end{pmatrix}$ is that the first $\operatorname{rk} A$ rows are unchanged, and the remaining rows are changed to all-zero rows. We are interested in the rank of the resulting matrix. Let us choose $\operatorname{rk} B$ rows in $R^{-1}B$ in such a way that they form a linearly independent system. Out of these $\operatorname{rk} B$ rows at most $\mathrm{d}(A)$ can have indices larger than $\operatorname{rk} A$, thus at least $\operatorname{rk} B - \mathrm{d}(A)$ of them are *not* changed to all-zero rows in $(QAR)(R^{-1}B)$. Hence,
$$\operatorname{rk}(AB) \ge \operatorname{rk} B - \mathrm{d}(A) = \min(\operatorname{rk} A, \operatorname{rk} B) - \min(\mathrm{d}(A), \mathrm{d}(B)),$$
as we claimed. ∎

Remark 8. As
$$\min(\operatorname{rk} A, \operatorname{rk} B) - \min(\mathrm{d}(A), \mathrm{d}(B)) = \operatorname{rk} A + \operatorname{rk} B - n = n - \mathrm{d}(A) - \mathrm{d}(B),$$
we may formulate Sylvester's Law of Nullity in the following, equivalent forms:
$$\operatorname{rk} A + \operatorname{rk} B - n \le \operatorname{rk}(AB)$$
$$\mathrm{d}(AB) \le \mathrm{d}(A) + \mathrm{d}(B)$$

2nd Proof. Let us choose a matrix A' in such a way that $A + A'$ is nonsingular and $\operatorname{rk}(A') = n - \operatorname{rk} A = \mathrm{d}(A)$. (Such a matrix can be found as follows. Let us choose $\operatorname{rk} A$ linearly independent columns of A and write in the corresponding columns of A' zeros. Let us modify the remaining columns of A with the help of A' in such a way that this partial independent system is extended to a basis.)

Now, we have
$$\operatorname{rk} B = \operatorname{rk}((A + A')B) \le \operatorname{rk}(AB) + \operatorname{rk}(A'B) \le \operatorname{rk}(AB) + \operatorname{rk}(A') \le \operatorname{rk}(AB) + n - \operatorname{rk} A,$$
after rearranging
$$\operatorname{rk} A + \operatorname{rk} B - n \le \operatorname{rk}(AB),$$
as needed. □

Corollary 22. *If A and B are $n \times n$ matrices satisfying $AB = 0$, then $\operatorname{rk} A + \operatorname{rk} B \le n$.*

Proof. Directly follows from (any of) the previous bounds. □

5. Projections

Definition 9. *A square matrix P is a projection, if $P^2 = P$.*

Remark 9. That is, in the ring of square matrices the projections are the idempotent elements.

Example 10. *If $n = 1$, then $P = (a)$ is a projection if and only if $a^2 = a$, that is, if $a = 0$ or $a = 1$. Therefore, among the 1×1 matrices there are two projections, namely, (0) and (1).*

Example 11. *Built on the previous example we can easily construct the following four 2×2 projections:*

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

However, there are much (in fact infinitely many) more projections, for instance, $\begin{pmatrix} 5 & 2 \\ -10 & -4 \end{pmatrix}$ is also a projection.

With the help of the following observation we can obtain rank-one projections.

Proposition 23. *If $v^T u = 1$, then uv^T is a projection.*

Proof. Indeed,

$$(uv^T)^2 = u(\underbrace{v^T u}_{1})v^T = uv^T.$$

□

This already implies that there are a lot of rank-one projections. On the other hand, there is only one projection with full rank:

Proposition 24. *If the projection P is nonsingular, then $P = I$.*

Proof.

$$P = IP = (P^{-1}P)P = P^{-1}P^2 = P^{-1}P = I.$$

□

Proposition 25. *If P is an $n \times n$ projection, then $I - P$ is also a projection, furthermore, $\operatorname{rk}(I - P) = n - \operatorname{rk} P$.*

Proof. The claim that $I - P$ is also a projection follows from the following simple calculation:

$$(I - P)^2 = I - 2P + P^2 = I - 2P + P = I - P.$$

For determining the rank of $I - P$, let us observe first that

$$n = \operatorname{rk} I = \operatorname{rk}(P + I - P) \le \operatorname{rk} P + \operatorname{rk}(I - P).$$

Thus $n - \operatorname{rk} P \le \operatorname{rk}(I - P)$.

Note that $P(I-P) = P - P^2 = 0$. Now, Corollary 22 yields that $\operatorname{rk} P + \operatorname{rk}(I-P) \le n$, that is, $\operatorname{rk}(I-P) \le n - \operatorname{rk} P$ also holds.

Hence, $\operatorname{rk}(I-P) = n - \operatorname{rk} P$, as we claimed.

□

Definition 10. *We say that the vectors $u_1, \dots, u_r, v_1, \dots, v_r \in \mathbb{R}^n$ form a biorthogonal system, if $v_i^T u_j = \delta_{ij}$ for every $i, j \in \{1, 2, \dots, r\}$.*

If $r = n$, then the biortogonal system is complete.

Remark 10. The system $u_1, \dots, u_r, v_1, \dots, v_r \in \mathbb{R}^n$ is biorthogonal, if

$$\underbrace{\begin{pmatrix} v_1^T \\ \vdots \\ v_r^T \end{pmatrix}}_{=:V^T} \underbrace{\begin{pmatrix} u_1 & \dots & u_r \end{pmatrix}}_{=:U} = I_r.$$

If $r = n$, then U and V^T are square matrices that are the inverses of each other: $V^T U = I (= I_n)$, that is, $V^T = U^{-1}$.

The following simple lemma will be applied several times.

Lemma 26. *The following statements hold.*

(i) *If the columns of a matrix A are linearly independent and $AB = 0$, then $B = 0$.*
(ii) *If the rows of a matrix B are linearly independent and $AB = 0$, then $A = 0$.*

Proof. Let us prove statement (i), statement (ii) can be proved similarly. Let b be a column of B. The condition $AB = 0$ implies that $Ab = 0$. We can think of Ab as a linear combination of the columns of A (where the coefficients are the entries of b). However, since the columns of A are linearly independent, only the trivial linear combination can be equal to 0, therefore $b = 0$. This holds for all of the columns of B, hence, $B = 0$. □

Theorem 27 (Egerváry)**.** *In a minimal rank-one decomposition of a projection the column and row vectors form a biorthogonal system.*

Proof. Let $P = \sum_{k=1}^{r} u_k v_k^T = UV^T$ be a minimal rank-one decomposition of the projection P.

Since P is a projection we have

$$P^2 - P = 0,$$

that is,

$$UV^TUV^T - UV^T = 0,$$
$$U(V^TU - I_r)V^T = 0.$$

Note that the columns of U are linearly independent and the rows of V^T are also linearly independent. Thus by Lemma 26 we get that

$$U(V^TU - I_r)V^T = 0 \implies (V^TU - I_r)V^T = 0 \implies V^TU - I_r = 0,$$

that is $V^TU = I_r$, which means that the system $u_1, \dots, u_r, v_1, \dots, v_r$ is biorthogonal. □

Now, let us combine this with Proposition 25. If P is a projection, then so is the matrix $I-P$, furthermore, its rank is $\operatorname{rk}(I-P)=n-\operatorname{rk}P$. Let us take a rank-one decomposition of $I-P$:

$$I-P=\sum_{\ell=1}^{n-r} w_\ell z_\ell^T = WZ^T.$$

Since $P(I-P)=0$, we have

$$0=P(I-P)=(UV^T)(WZ^T)=U(V^TW)Z^T,$$

yielding $V^TW=0$ by Lemma 26.

Analogously,

$$0=(I-P)P=W(Z^TU)V^T$$

implies that $Z^TU=0$.

We may summarize these in the matrix equation

$$\begin{pmatrix} V^T \\ Z^T \end{pmatrix}\begin{pmatrix} U & W \end{pmatrix}=\begin{pmatrix} I_r & 0 \\ 0 & I_{n-r} \end{pmatrix}=I_n.$$

This way the biorthogonal system $u_1,\dots,u_r,v_1,\dots,v_r$ is extended to the complete biorthogonal system $u_1,\dots,u_r,w_1,\dots,w_{n-r},v_1,\dots,v_r,z_1,\dots,z_{n-r}$.

Note that this method works for *every* biorthogonal system, since, if $u_1,\dots,u_r,v_1,\dots,v_r$ is a biorthogonal system (that is, $V^TU=I$), then $P=UV^T$ is a projection:

$$P^2=(UV^T)(UV^T)=U(V^TU)V^T=UV^T=P.$$

Example 12. *The task is to extend the following biorthogonal system to a complete biorthogonal system:*

$$u_1=\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix},\quad u_2=\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix},\quad v_1=\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},\quad v_2=\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}.$$

Let us consider the projection

$$P=\begin{pmatrix} 2 & 1 \\ 1 & 0 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 0 \end{pmatrix}.$$

Since $\operatorname{rk}P=2$*, the projection* $I-P$ *has rank 1, and by taking its rank-one decomposition we achieve the needed extension:*

$$I-P=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & 1 & 1 \end{pmatrix}=\underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{=:u_3}\underbrace{\begin{pmatrix} -2 & 1 & 1 \end{pmatrix}}_{=:v_3^T}.$$

Now, we continue with the complex case.

Definition 11. *We say that the vectors $u_1, \dots, u_r \in \mathbb{C}^n$ form a unitary system, if $u_i^* u_j = \delta_{ij}$ for every $i, j \in \{1, 2, \dots, r\}$.*

If $r = n$, then the unitary system is complete.

The real analogue of the above definition is the following:

Definition 12. *We say that the vectors $u_1, \dots, u_r \in \mathbb{R}^n$ form an orthogonal system, if $u_i^T u_j = \delta_{ij}$ for every $i, j \in \{1, 2, \dots, r\}$.*

If $r = n$, then the orthogonal system is complete.

Definition 13. *We say that P is an Hermitian projection if $P^2 = P$ and $P^* = P$.*

Definition 14. *An Hermitian rank-one matrix is a matrix of the form uu^*, where $u \neq 0$.*

Definition 15. *The trace of a square matrix is the sum of the entries on the main diagonal:*

$$\operatorname{Tr} A = \sum_{k=1}^{n} a_{kk}.$$

Remark 11. $\operatorname{Tr}(A + B) = \operatorname{Tr} A + \operatorname{Tr} B$.

Proposition 28. $\operatorname{Tr}(AB) = \operatorname{Tr}(BA)$

Proof.

$$\operatorname{Tr}(AB) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b_{ji} = \sum_{j=1}^{n} \sum_{i=1}^{n} b_{ji} a_{ij} = \operatorname{Tr}(BA)$$

□

Corollary 29. *If A and B are $n \times n$ matrices, then $AB - BA \neq I$.*

Corollary 30. $\operatorname{Tr}(ABC) = \operatorname{Tr}(BCA) = \operatorname{Tr}(CAB)$

Proposition 31.

$$\operatorname{Tr}(AA^*) = \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2$$

Corollary 32. *If A is self-adjoint, then $\operatorname{Tr}(A^2) \geq 0$. Furthermore, $\operatorname{Tr}(A^2) = 0$ if and only if $A = 0$.*

Proposition 33. *If $P \neq 0$ is an Hermitian projection, then on the main diagonal of P there is at least one positive element.*

Proof. Since, P is self-adjoint, all the diagonal entries are real. As

$$\operatorname{Tr} P = \operatorname{Tr} P^2 = \operatorname{Tr}(PP^*) > 0,$$

at least one of them must be positive. □

Now, we are ready to prove the following theorem about the rank-one decomposition of an Hermitian projection.

Theorem 34. *Every Hermitian projection P can be expressed as a sum of Hermitian rank-one matrices in such a way that the number of summands is the rank of P.*

Proof. According to Proposition 33 there is at least one positive element on the main diagonal of P. For simplicity, let us assume that $a_{11} > 0$. Let the first Hermitian rank-one matrix in the decomposition be

$$\frac{1}{a_{11}} a_1 a_1^* = \left(\frac{1}{\sqrt{a_{11}}} a_1\right)\left(\frac{1}{\sqrt{a_{11}}} a_1\right)^*,$$

where a_1 is the first column of P (and consequently, a_1^* is the first row of P). After subtracting this rank-one matrix, we are left with the following matrix:

$$P - \frac{1}{a_{11}} a_1 a_1^*.$$

Note that this matrix is still Hermitian, as it is obtained as the difference of two Hermitian matrices:

$$\left(P - \frac{1}{a_{11}} a_1 a_1^*\right)^* = P^* - \frac{1}{a_{11}} a_1^{**} a_1^* = P - \frac{1}{a_{11}} a_1 a_1^*.$$

Furthermore, the matrix is still a projection! To be able to check this, we will use that $P^2 = P$ implies that

$$Pa_1 = a_1, \quad a_1^* P = a_1^*, \quad a_1^* a_1 = a_{11}.$$

Now, the calculation is as follows:

$$\left(P - \frac{1}{a_{11}} a_1 a_1^*\right)^2 = P^2 - \frac{1}{a_{11}}(Pa_1)a_1^* - \frac{1}{a_{11}} a_1(a_1^* P) + \frac{1}{a_{11}^2} a_1(a_1^* a_1)a_1^* =$$

$$= P - \frac{1}{a_{11}} a_1 a_1^* - \frac{1}{a_{11}} a_1 a_1^* + \frac{1}{a_{11}} a_1 a_1^* = P - \frac{1}{a_{11}} a_1 a_1^*.$$

For finding the Hermitian rank-one decomposition we shall continue iteratively in a similar way, till the remaining matrix is the zero matrix. Finally, we obtain a decomposition $P = \sum_{k=1}^{r} u_k u_k^* = UU^*$. Note that the columns of U form a unitary system: $U^*U = I_r$.

The proof of the claim that the number of summands r in the obtained decomposition is $\operatorname{rk} P$ is similar to the proof of Corollary 12.

□

5.1. **Extending a unitary system to a complete unitary system.** Now, if $u_1, \ldots, u_r$ is a unitary system, then we can extend it to a complete unitary system as follows. Let $P := \sum_{k=1}^{r} u_k u_k^* = UU^*$. Note that P is an Hermitian projection, since $U^*U = I_r$ implies that $P^2 = U(U^*U)U^* = P$.

Now, we may take an Hermitian rank-one decomposition of the Hermitian projection $I - P$:

$$I - P = \sum_{\ell=1}^{n-r} w_\ell w_\ell^* = WW^*.$$

This way a complete unitary system $u_1, \dots, u_r, w_1, \dots, w_{n-r}$ is obtained. Note that this also means that $\begin{pmatrix} U & W \end{pmatrix}$ is a unitary matrix.

Remark 12. In the real case, for a symmetric matrix we obtain a rank-one decomposition to symmetric rank-one matrices. Similarly to the above, this may be used to extend orthogonal systems to complete orthogonal systems.

Example 13. *Let P be the following Hermitian projection:*

$$P := \begin{pmatrix} \frac{1}{2} & \frac{1+i}{12} & \frac{5+3i}{12} \\ \frac{1-i}{12} & \frac{35}{36} & \frac{-4+i}{36} \\ \frac{5-3i}{12} & \frac{-4-i}{36} & \frac{19}{36} \end{pmatrix}.$$

The task is to find an Hermitian rank-one decomposition for P.

After subtracting the first Hermitian rank-one matrix we are left with the following Hermitian projection:

$$P - 2\begin{pmatrix} \frac{1}{2} \\ \frac{1-i}{12} \\ \frac{5-3i}{12} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1+i}{12} & \frac{5+3i}{12} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{17}{18} & \frac{-4+i}{18} \\ 0 & \frac{-4-i}{18} & \frac{1}{18} \end{pmatrix}.$$

After subtracting the second Hermitian rank-one matrix the resulting Hermitian projection is the zero matrix:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{17}{18} & \frac{-4+i}{18} \\ 0 & \frac{-4-i}{18} & \frac{1}{18} \end{pmatrix} - \frac{18}{17} \begin{pmatrix} 0 \\ \frac{17}{18} \\ \frac{-4-i}{18} \end{pmatrix} \begin{pmatrix} 0 & \frac{17}{18} & \frac{-4+i}{18} \end{pmatrix} = 0.$$

Therefore, the following Hermitian rank-one decomposition is obtained:

$$P = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}-\sqrt{2}i}{12} & \sqrt{\frac{17}{18}} \\ \frac{5\sqrt{2}-3\sqrt{2}i}{12} & \frac{1}{\sqrt{17\cdot 18}}(-4-i) \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}+\sqrt{2}i}{12} & \frac{5\sqrt{2}+3\sqrt{2}i}{12} \\ 0 & \sqrt{\frac{17}{18}} & \frac{1}{\sqrt{17\cdot 18}}(-4+i) \end{pmatrix}.$$

6. Moore-Penrose pseudoinverse

In this section we will look at the Moore-Penrose pseudoinverse and further generalized inverses. If the $n \times n$ matrices A and X are the inverses of each other, then $AX = XA = I$ is a projection of rank n (which is self-adjoint). Now, these requirements will be relaxed. Namely, we will consider the following requirements:

(1) $AXA = A$
(2) $XAX = X$
(3) $(AX)^* = AX$
(4) $(XA)^* = XA$

Definition 16. *If (1) is satisfied, then we say that X is a generalized inverse of A, and denote it by A^g.*

Remark 13. If X is a generalized inverse of A, then AX is a projection. Indeed, $(AX)^2 = \underbrace{AXA}_{A} X = AX$.

Example 14. *If A is a nonsingular square matrix, then $A^g = A^{-1}$. Since, $AXA = A$ implies that $X = A^{-1}AA^{-1} = A^{-1}$.*

Example 15. *Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$. The task is to determine A^g.*

Let $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. As $AXA = (a + 2c + 3b + 6d) \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$, condition (1) is satisfied if and only if $a + 2c + 3b + 6d$=1. This example shows that the generalized inverse might not be unique, in this case there are infinitely many generalized inverses for A.

Definition 17. *If (1) and (2) are satisfied, then we say that X is a reflexive generalized inverse of A, and denote it by $X = A^r$.*

Remark 14. If (1) and (2) are satisfied, then A is also a reflexive generalized inverse for X.

Example 16. *Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$. The task is to determine A^r.*

Let $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we already know that (1) is satisfied if and only if $a + 2c + 3b + 6d = 1$. Now, we shall continue with condition (2). After rearranging the equation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we get

$$\begin{pmatrix} a + 3b - 1 & 2a + 6b \\ c + 3d & 2c + 6d - 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 0,$$

which, by using that $a + 2c + 3b + 6d = 1$, may be written as

$$\begin{pmatrix} -2c-6d & 2a+6b \\ c+3d & -a-3b \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 0.$$

After calculating the product on the left hand side, we obtain that the equation holds if and only if $ad = bc$. Therefore, X is a generalized reflexive inverse if and only if $a + 2x + 3b + 6d = 1$ and $ad = bc$.

Definition 18. *If (1), (2) and (3) are satisfied, then we say that X is a normalized generalized inverse of A, and denote it by $X = A^n$.*

Definition 19. *If (1)-(4) are all satisfied, then X is the Moore-Penrose pseudoinverse of A, which is denoted by $X = A^+$.*

Remark 15. We will see that the Moore-Penrose pseudoinverse always exists and uniquely determined.

Remark 16. If A is an $n \times k$ matrix, then A^+ is a $k \times n$ matrix.

Example 17. *Let $A = 0_{n\times k}$ be the $n\times k$ zero matrix. Condition (2) implies that $A^+ = 0_{k\times n}$, which satisfies conditions (1), (3), (4), as well.*

Example 18. *Let $A = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$. The task is to determine $X = A^+$.*

Let $X = A^+ = \begin{pmatrix} x & y \end{pmatrix}$. Note that $XA = 3x + 4y$.
Condition (1) may be written as $A(XA) = A$, which holds if and only if $3x + 4y = 1$.
Condition (2) may be written as $(XA)X = X$, which also holds if $3x + 4y = 1$.
Condition (4) also holds, since $XA = (3x + 4y) = (1)$, which is a 1×1 matrix whose unique entry is a real number (namely, 1), thus XA is self-adjoint.

Condition (3) requires that $AX = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \begin{pmatrix} x & y \end{pmatrix} = \begin{pmatrix} 3x & 3y \\ 4x & 4y \end{pmatrix}$ is self-adjoint. That is, $x, y \in \mathbb{R}$ and $3y = 4x$. From the equations

$$3x + 4y = 1$$

$$3y = 4x$$

we get that $x = \frac{3}{25}$, $y = \frac{4}{25}$, thus $A^+ = \begin{pmatrix} \frac{3}{25} & \frac{4}{25} \end{pmatrix}$.

Now, our aim is to show the existence of these generalized inverses.

Proposition 35. *Every matrix has a generalized inverse.*

Proof. Let us denote the matrix by A and its rank by $r := \operatorname{rk} A$. Let us take nonsingular matrices Q and R such that

$$QAR = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} =: T_r.$$

Then $A = Q^{-1}T_r R^{-1}$. We claim that $X = R\begin{pmatrix} I_r & U \\ V & W \end{pmatrix} Q$ is an appropriate choice for the generalized inverse for every U, V, W. Indeed:

$$AXA = Q^{-1}T_r \underbrace{R^{-1}R}_{I} \begin{pmatrix} I_r & U \\ V & W \end{pmatrix} \underbrace{QQ^{-1}}_{I} T_r R^{-1} = Q^{-1}T_r R^{-1} = A.$$

□

Remark 17. The proof also shows that there are infinitely many generalized inverses for A with the exception of the case when A is a nonsingular square matrix.

Remark 18. $\mathrm{rk}(A) \le \mathrm{rk}(A^g)$ and $\mathrm{rk}(A) = \mathrm{rk}(A^r)$.

Proposition 36. *Every matrix has a reflexive generalized inverse.*

Proof. The choice $U = 0, V = 0, W = 0$ is fine in the construction from the proof of Proposition 35:

$$A^r = R\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} Q.$$

□

Theorem 37. *Every matrix has a Moore-Penrose pseudoinverse.*

Proof. Let us denote the matrix by A and its rank by $r := \mathrm{rk}\, A$. Furthermore, let us take a minimal rank-one decomposition of A:

$$A = UV^*.$$

We claim that $A^+ = X = V(V^*V)^{-1}(U^*U)^{-1}U^*$ satisfies all the requirements (1) – (4). Note that the matrices arising here are of the following types:

$$A:\ k \times n, \quad U:\ k \times r, \quad V^*:\ r \times n, \quad V^*V:\ r \times r, \quad U^*U:\ r \times r, \quad A^+:\ n \times k.$$

Let us check first that condition (1) holds:

$$AXA = U \underbrace{V^*V(V^*V)^{-1}}_{I} \underbrace{(U^*U)^{-1}U^*U}_{I} V^* = UV^* = A.$$

Condition (2) can be checked similarly:

$$XAX = V(V^*V)^{-1} \underbrace{(U^*U)^{-1}U^*U}_{I} \underbrace{V^*V(V^*V)^{-1}}_{I} (U^*U)^{-1}U^* = V(V^*V)^{-1}(U^*U)^{-1}U^* = X.$$

For checking (3) let us calculate AX first:

$$AX = U \underbrace{V^*V(V^*V)^{-1}}_{I} (U^*U)^{-1}U^* = U(U^*U)^{-1}U^*.$$

Therefore, AX is indeed self-adjoint:

$$(AX)^* = (U(U^*U)^{-1}U^*)^* = U^{**}((U^*U)^{-1})^*U^* = U(U^*U)^{-1}U^* = AX.$$

It can be checked in a similar way that XA is also self-adjoint. Hence, $A^+ = X = V(V^*V)^{-1}(U^*U)^{-1}U^*$ satisfies all the requirements (1) – (4). $\square$

Now, we also prove the uniqueness.

Theorem 38. *The Moore-Penrose pseudoinverse is unique.*

Proof. Let us assume that X_1 and X_2 are both Moore-Penrose pseudoinverses for the matrix A. We are aiming to show that $X_1 = X_2$.

We know that X_1 and X_2 satisfy all the following conditions.

(1) $AX_iA = A$
(2) $X_iAX_i = X_i$
(3) $(AX_i)^* = AX_i$
(4) $(X_iA)^* = X_iA$

(Here $i \in \{1,2\}$.)

Observe that

$$A^* \underbrace{=}_{(1)_i} (AX_iA)^* = (X_iA)^*A^* \underbrace{=}_{(4)_i} X_iAA^*$$

holds for $i \in \{1,2\}$. By taking the difference, we obtain that

$$0 = A^* - A^* = (X_1 - X_2)AA^*.$$

That is, we already know that $(X_1 - X_2)AA^* = 0$, it remains to show that the first matrix, $X_1 - X_2$, is itself 0. Let $M := (X_1 - X_2)A$. Then

$$MM^* = ((X_1 - X_2)A)((X_1 - X_2)A)^* = \underbrace{(X_1 - X_2)AA^*}_{0}(X_1 - X_2)^* = 0.$$

According to Proposition 31 we have $\mathrm{Tr}(MM^*) = \sum |m_{ij}|^2$, thus $MM^* = 0$ implies that $M = 0$. Hence, $M = (X_1 - X_2)A = 0$.

Now, observe that

$$A^* \underbrace{=}_{(1)_i} (AX_iA)^* = A^*(AX_i)^* \underbrace{=}_{(3)_i} A^*AX_i$$

holds for $i \in \{1,2\}$. By taking the difference, we obtain that

$$0 = A^* - A^* = A^*A(X_1 - X_2).$$

Now, for the matrix $N := A(X_1 - X_2)$ we have

$$N^*N = (X_1 - X_2)^* \underbrace{A^*A(X_1 - X_2)}_{0} = 0$$

yielding that $N = A(X_1 - X_2) = 0$.

We are ready to show that $X_1 = X_2$:

$$X_1 \underbrace{=}_{(2)} X_1AX_1 = X_1AX_1 - X_1AX_2 + X_1AX_2 - X_2AX_2 + X_2AX_2 =$$

$$= X_1 \underbrace{A(X_1 - X_2)}_{=N=0} + \underbrace{(X_1 - X_2)A}_{=M=0} X_2 + X_2AX_2 = X_2AX_2 \underbrace{=}_{(2)} X_2.$$

□

2nd Proof.

$$X_1 = X_1AX_1 = X_1(AX_1)^* = X_1X_1^*A^* = X_1(AX_1)^*(AX_2)^* = X_1AX_2 =$$
$$= (X_1A)^*(X_2A)^*X_2 = A^*X_2^*X_2 = (X_2A)^*X_2 = X_2$$

□

Now, we shall discuss an extremal property of the Moore-Penrose pseudoinverse. If A is a nonsingular square matrix, then the unique solution of the equation $Ax = b$ is $x = A^{-1}b$. If A is a singular square matrix, or not even a square matrix, then it's natural to consider $x = A^+b$. The following propositions will show some extremal property of this vector.

Definition 20. *Let $\|x\|_2$ denote the euclidean norm, that is, $\|x\|_2 := \sqrt{x^*x}$.*

Proposition 39. $\|Ax - b\|_2 \geq \|AA^+b - b\|_2$

Remark 19. This means that for $x = A^+b$ the vector Ax is as close to b as it can be (according to the euclidean distance). Specially, if $Ax = b$ is solvable, then $x = A^+b$ is one of the solutions.

Proof. If $AA^+b = b$, then the statement trivially holds.

To prove the general case it suffices to show that

$$b - AA^+b \perp \operatorname{Im} A,$$

where $\operatorname{Im} A$ is the column space of A. That is, we would like to prove that $b - AA^+b \perp Aw$ for every w:

$$(Aw)^*(b - AA^+b) = 0.$$

This holds, if $w^*A^*(I - AA^+)b = 0$. Thus it is enough to show that $A^*(I - AA^+) = 0$ which may also be written as

$$A^* = A^*AA^+.$$

This holds, since $A^*AA^+ = A^*(AA^+)^* = (AA^+A)^* = A^*$. □

Remark 20. Equation holds in Proposition 39 if and only if $x \in A^+b + \operatorname{Ker} A$, where $\operatorname{Ker} A$ is the kernel of A.

Proposition 40. $\operatorname{Ker} A = \operatorname{Im}(I - A^+A)$

Proof. Since $A(I - A^+A) = 0$, we clearly have $\mathrm{Im}(I - A^+A) \subseteq \mathrm{Ker}\, A$. Note that

$$\dim \mathrm{Ker}\, A = n - \mathrm{rk}\, A$$

and

$$\dim \mathrm{Im}(I - A^+A) = n - \mathrm{rk}(A^+A) = n - \mathrm{rk}\, A.$$

Therefore, $\mathrm{Ker}\, A = \mathrm{Im}(I - A^+A)$. □

Corollary 41. $\|Ax - b\|_2 = \|AA^+b - b\|_2$ *if and only if* $x = A^+b + (I - A^+A)w$ *for some* w.

Proposition 42. *If* $\|Ax - b\|_2 = \|AA^+b - b\|_2$, *then* $\|x\|_2 \geq \|A^+b\|_2$ *and equation holds if and only if* $x = A^+b$.

Proof. According to Remark 20 it suffices to show that $A^+b \perp \mathrm{Ker}\, A$. That is, we would like to show that $A^+b \perp (I - A^+A)w$ for every w:

$$b^*(A^+)^*(I - A^+A)w = 0.$$

This holds, if $(A^+)^* = (A^+)^*A^+A$. However, $(A^+)^*A^+A = (A^+AA^+)^* = (A^+)^*$ indeed holds. □

Remark 21. According to the above statement $x_0 = A^+b$ is the vector which minimizes $\|Ax - b\|_2$, furthermore, among the vectors minimizing $\|Ax - b\|_2$ it minimizes $\|x\|_2$.

7. Special matrices

In this section we define and investigate some special types of matrices.

A square matrix is *diagonal*, if only entries lying on the main diagonal are allowed to be different from 0. If we allow two more diagonals, then the matrix is called tridiagonal:

Definition 21. *A square matrix A is tridiagonal if all the nonzero entries lie on the main diagonal or the superdiagonal or the subdiagonal:*

$$A = \begin{pmatrix} a_1 & b_1 & 0 & \dots & 0 \\ c_1 & a_2 & b_2 & \ddots & \vdots \\ 0 & c_2 & a_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & b_{n-1} \\ 0 & \dots & 0 & c_{n-1} & a_n \end{pmatrix}.$$

Specially, if $a_1 = \dots = a_n = a$, $b_1 = \dots = b_{n-1} = b$, $c_1 = \dots = c_{n-1} = c$, then the matrix is a uniform tridiagonal matrix:

$$A = \begin{pmatrix} a & b & 0 & \dots & 0 \\ c & a & b & \ddots & \vdots \\ 0 & c & a & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & b \\ 0 & \dots & 0 & c & a \end{pmatrix}.$$

If $b_i = c_i$ for every i, then A is a symmetric tridiagonal matrix.

Combining these, if $a_1 = \dots = a_n = a$, $b_1 = \dots = b_{n-1} = b = c_1 = \dots = c_{n-1} = c$, then the matrix is a symmetric uniform tridiagonal matrix.

Analogously to the case of tridiagonal matrices we may allow that nonzero entries can occur on the main diagonal and two-two diagonals right above and below the main diagonal (that is, on 5 diagonals in total), then we say that the matrix is pentadiagonal. In general:

Definition 22. *If nonzero entries all lie on the main diagonal, and on a specified number, say p, diagonals above, and on q diagonals below the main diagonal, then we say that the matrix is a band matrix.*

If $a_{ij} = a_{j-i}$, that is, the values on the same diagonal are always equal to each other, then we say that the matrix is a Toeplitz matrix.

If $a_{ij} = a_{|j-i|}$, then the matrix is a symmetric Toeplitz matrix.

Example 19. *The following matrix is a Toeplitz matrix:*

$$H = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & \ddots & \\ & & & \ddots & 1 \\ & & & & 0 \end{pmatrix}.$$

Remark 22. The square of the previous matrix H is

$$H^2 = \begin{pmatrix} 0 & 0 & 1 & & \\ & 0 & 0 & \ddots & \\ & & 0 & \ddots & 1 \\ & & & \ddots & 0 \\ & & & & 0 \end{pmatrix},$$

that is, the diagonal containing the 1's is shifted. In H^3, $H^4, \dots$ the diagonal is shifted again and again, finally, in H^{n-1} (if H is an $n \times n$ matrix) contains one single nonzero entry, namely, a 1 in the upper right corner and $H^n = 0$.

Definition 23. *A square matrix A is nilpotent if there exists some positive integer k such that $A^k = 0$. The smallest such k is called the degree of nilpotency of A.*

Example 20. *The matrix H from the previous example is nilpotent, its degree of nilpotency is n.*

A natural question at this point as follows:

Question 1. *What can be the degree of nilpotency of an $n \times n$ nilpotent matrix A?*

Remark 23. If H is nilpotent and $H^k = 0$, then for every $\ell > k$ we have $H^\ell = 0$.

Proposition 43. *If A is nilpotent, then* $\det A = 0$.

Proof. If $A^k = 0$, then $(\det A)^k = \det 0 = 0$ which implies that $\det A = 0$. □

Remark 24. The reverse implication would be false, for instance, the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

has determinant 0, but it is not nilpotent.

Proposition 44. *If A is nilpotent, then $I - A$ is invertible.*

Proof. Since A is nilpotent, there exists some k such that $A^k = 0$. Then

$$(I - A)(I + A + A^2 + \dots + A^{k-1}) = I - A^k = I,$$

thus $I - A$ is invertible. □

Definition 24. *Let us denote by $N(c_0, c_1, \dots, c_{n-1})$ the $n \times n$ upper triangular Toeplitz matrix $c_0 I + c_1 H + \dots + c_{n-1} H^{n-1}$:*

$$N(c_0, c_1, \dots, c_{n-1}) := c_0 I + c_1 H + \dots + c_{n-1} H^{n-1} = \begin{pmatrix} c_0 & c_1 & \dots & c_{n-1} \\ 0 & c_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & c_1 \\ 0 & \dots & 0 & c_0 \end{pmatrix}.$$

Remark 25. Any two matrices of the above type are interchangeable with each other, as they are expressed as a polynomial of the same matrix H.

Let Ω be the following $n \times n$ permutation matrix:

$$\Omega = \begin{pmatrix} & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ 1 & & & & \end{pmatrix} = \begin{pmatrix} e_n & e_1 & e_2 & \dots & e_{n-1} \end{pmatrix}.$$

Note that

$$\Omega^2 = \begin{pmatrix} & & 1 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \\ 1 & & & & & \\ & 1 & & & & \end{pmatrix},$$

so the 1's are cyclically shifted by one. In $\Omega^3,\ \Omega^4, \dots$ the 1's are shifted again and again, finally $\Omega^n = I$.

Definition 25. *Let us denote by $C(c_0, c_1, \dots, c_{n-1})$ the $n \times n$ matrix $c_0 I + c_1 \Omega + \dots + c_{n-1}\Omega^{n-1}$:*

$$C(c_0, c_1, \dots, c_{n-1}) := c_0 I + c_1 \Omega + \dots + c_{n-1}\Omega^{n-1} = \begin{pmatrix} c_0 & c_1 & \dots & c_{n-1} \\ c_{n-1} & c_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & c_1 \\ c_1 & \dots & c_{n-1} & c_0 \end{pmatrix}.$$

These matrices are called circulant matrices.

Remark 26. Any two matrices of the above type are interchangeable with each other, as they are expressed as a polynomial of the same matrix Ω.

Example 21. *Let us consider the circulant matrix*

$$I + x\Omega + x^2\Omega^2 + \dots + x^{n-1}\Omega^{n-1} = \begin{pmatrix} 1 & x & \dots & x^{n-1} \\ x^{n-1} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & x \\ x & \dots & x^{n-1} & 1 \end{pmatrix}.$$

Observe that

$$(I + x\Omega + x^2\Omega^2 + \dots + x^{n-1}\Omega^{n-1})(I - x\Omega) = I - x^n\Omega^n = (1 - x^n)I.$$

That is, if $1 - x^n \neq 0$, then $I + x\Omega + x^2\Omega^2 + \dots + x^{n-1}\Omega^{n-1}$ is invertible:

$$(I + x\Omega + x^2\Omega^2 + \dots + x^{n-1}\Omega^{n-1})^{-1} = \frac{1}{1 - x^n}(I - x\Omega) = \frac{1}{1 - x^n}\begin{pmatrix} 1 & -x & & \\ & 1 & \ddots & \\ & & \ddots & -x \\ -x & & & 1 \end{pmatrix}.$$

If $1 - x^n = 0$, then $x^n = 1$, that is, x is an nth root of unity:

$$x = x_k = \cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n}$$

for some $k \in \{0, 1, \ldots, n-1\}$. In this case the matrix is not invertible, since the first row, $(1, x, \ldots, x^{n-1})$, is x times the second row, $(x^{n-1}, 1, \ldots, x^{n-2})$.

7.1. **Chebyshev polynomials.** Our aim is to determine the determinant and the inverse of the symmetric tridiagonal matrix

$$\begin{pmatrix} a & b & 0 & \ldots & 0 \\ b & a & b & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & b & a & b \\ 0 & \ldots & 0 & b & a \end{pmatrix}.$$

For this, we will need to use the so-called Chebyshev polynomials that we briefly review in this subsection.

Definition 26. *The Chebyshev polynomials $T_n(x)$ of the first kind are defined by the following recurrence:*

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x).$$

The Chebyshev polynomials $U_n(x)$ of the second kind are defined by the following recurrence:

$$U_0(x) = 1, \quad U_1(x) = 2x, \quad U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x).$$

If $|x| \le 1$, then we may express x as $x = \cos\theta$. Then $T_1(x) = \cos\theta$ and $T_2(x) = 2\cos\theta\cos\theta - 1 = \cos(2\theta)$, by using the addition rule of cosine. Motivated by this we may conjecture that $T_n(x) = \cos(n\theta)$, which indeed turns out to be true:

Proposition 45. $T_n(\cos\theta) = \cos(n\theta)$

Proof. We have already seen that the statement holds for $n = 0, 1, 2$. We may proceed by induction on n:

$$T_n(\cos\theta) = 2(\cos\theta)T_{n-1}(\cos\theta) - T_{n-2}(\cos\theta) = 2\cos\theta\cos((n-1)\theta) - \cos((n-2)\theta).$$

We need that $2\cos\theta\cos((n-1)\theta) - \cos((n-2)\theta) = \cos(n\theta)$. This follows from the identity

$$\cos(\alpha + \beta) + \cos(\alpha - \beta) = 2\cos\alpha\cos\beta$$

applied for $\alpha = (n-1)\theta$ and $\beta = \theta$.

□

Corollary 46. $T_n(T_m(x)) = T_{nm}(x)$

For Chebyshev polynomials of the second kind there exists a similar trigonometric formula:

Proposition 47. $U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}$

Proof. We prove the statement by induction on n. For $n = 0, 1$ the statement holds. The induction step is as follows:

$$U_n(\cos\theta) = 2\cos\theta\frac{\sin(n\theta)}{\sin\theta} - \frac{\sin((n-1)\theta)}{\sin\theta}.$$

We would like to prove that this is equal to $\frac{\sin((n+1)\theta)}{\sin\theta}$, which follows from the identity

$$\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin\alpha\cos\beta$$

applied for $\alpha = n\theta$ and $\beta = \theta$. □

Corollary 48. *Let $z = a + bi$, where $a, b \in \mathbb{R}$. Then*

$$z^n = |z|^n T_n(a/|z|) + ib|z|^{n-1}U_{n-1}(a/|z|).$$

Proof. Let us choose θ in such a way that $\cos\theta = a/|z|$ and $\sin\theta = b/|z|$. Then

$$z = |z|\left(\frac{a}{|z|} + \frac{b}{|z|}i\right) = |z|(\cos\theta + i\sin\theta).$$

As $z^n = |z|^n(\cos(n\theta) + i\sin(n\theta))$, the statement follows. □

Remark 27. Here we list some connections between Chebyshev polynomials of the first and second kind.

- $T_n'(x) = nU_{n-1}(x)$
- $T_n(x) = \frac{1}{2}(U_n(x) - U_{n-2}(x))$
- $T_{n+1}(x) = xT_n(x) - (1-x^2)U_{n-1}(x)$
- $T_n(x) = U_n(x) - xU_{n-1}(x)$
- $U_n(x) = 2\sum_{\substack{2\nmid j\\ j\le n}} T_j(x)$ if $2 \nmid n$
- $U_n(x) = 2\sum_{\substack{2\mid j\\ j\le n}} T_j(x) - 1$ if $2 \mid n$
- $T_n(x)^2 - T_{n-1}(x)T_{n+1}(x) = 1 - x^2$
- $U_n(x)^2 - U_{n-1}(x)U_{n+1}(x)$

7.1.1. *Calculating $T_n(x)$.* For $|x| \le 1$ we may write $x = \cos\theta$ and we have seen that $T_n(x) = T_n(\cos\theta) = \cos(n\theta)$.

Assume now that $x > 1$. Then we write $x = \cosh\theta$, where cosh the hyperbolic cosine function defined as $\cosh\theta = \frac{e^\theta + e^{-\theta}}{2}$. (Note that the hyperbolic sine function is defined as $\sinh\theta = \frac{e^\theta - e^{-\theta}}{2}$.)

Proposition 49. $T_n(\cosh\theta) = \cosh(n\theta)$

Proof. The statement holds for $n = 0, 1$. We may proceed by induction on n:

$$T_n(\cosh\theta) = 2(\cosh\theta)T_{n-1}(\cosh\theta) - T_{n-2}(\theta) = 2\cosh\theta\cosh((n-1)\theta) - \cosh((n-2)\theta).$$

We need that $2\cosh\theta\cosh((n-1)\theta) - \cosh((n-2)\theta) = \cosh(n\theta)$. This follows from the identity

$$\cosh(\alpha+\beta) + \cosh(\alpha-\beta) = 2\cosh\alpha\cosh\beta$$

applied for $\alpha = (n-1)\theta$ and $\beta = \theta$.

(The above identity follows from the addition rule of cosh:

$$\cosh(\alpha \pm \beta) = \cosh\alpha\cosh\beta \pm \sinh\alpha\sinh\beta.)$$

□

Similarly, for $x < -1$ we may write $x = -\cosh\theta$, and by covering all possible cases the following is obtained:

Corollary 50.

$$T_n(x) = \begin{cases} \cos(n\arccos x) & \text{if } |x| \le 1 \\ \cosh(n\operatorname{arccosh} x) & \text{if } x > 1 \\ (-1)^n\cosh(n\operatorname{arccosh}(-x)) & \text{if } x < -1 \end{cases}$$

For every fixed x the sequence $T_n(x)$ satisfies the linear recurrence

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x).$$

We may express $T_n(x)$ as $T_n(x) = \alpha z_1^n + \beta z_2^n$, where $z_{1,2}$ are the roots of $z^2 - 2xz + 1 = 0$ and α, β are chosen in a suitable way. Note that $z_{1,2} = x \pm \sqrt{x^2-1}$, so α and β have to satisfy

$$\alpha + \beta = 1,$$

$$\alpha(x + \sqrt{x^2-1}) + \beta(x - \sqrt{x^2-1}) = x$$

to get the right values in the two initial cases: $n = 0$ and $n = 1$.

The appropriate choice is $\alpha = \beta = \frac{1}{2}$ yielding that

$$T_n(x) = \frac{(x+\sqrt{x^2-1})^n + (x-\sqrt{x^2-1})^n}{2}.$$

Similarly, for $U_n(x)$ we get that

$$U_n(x) = \frac{(x+\sqrt{x^2-1})^{n+1} - (x-\sqrt{x^2-1})^{n+1}}{2\sqrt{x^2-1}}.$$

7.2. **Calculating the inverse of a tridiagonal symmetric Toeplitz matrix.** Let us consider first the following matrix:

$$\begin{pmatrix} x & -1 & & \\ -1 & x & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & x \end{pmatrix} = xI - K,$$

where

$$K = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 0 \end{pmatrix}.$$

We start with determining the determinant of this matrix,

$$D_n(x) := \det(xI_n - K).$$

Note that $D_1(x) = x$ and $D_2(x) = x^2 - 1$. By expanding the determinant along the last row we obtain that the recurrence

$$D_n(x) = xD_{n-1}(x) - D_{n-2}(x)$$

holds for $n \geq 3$. Let us define $D_0(x) := 1$, then the recurrence also applies for $n = 2$.

Comparing these with the recurrence for the Chebyshev polynomials we may observe that

$$D_n(x) = U_n(x/2).$$

Let us assume that $|x| < 2$. Let us write $x = 2\cos\theta$. Then $D_n(x) = \frac{\sin[(n+1)\theta]}{\sin\theta}$.

Now, we continue with determining the inverse. Observe that the adjugate of $xI - K$ is:

$$(\mathrm{adj}(xI - K))_{ij} = \begin{cases} (-1)^{i+j}(-1)^{j-i}D_{i-1}D_{n-j} = D_{i-1}D_{n-j} & \text{if } i \leq j \\ D_{j-1}D_{n-i} & \text{if } i \geq j \end{cases}.$$

That is,

$$(\mathrm{adj}(xI - K))_{ij} = \begin{cases} \frac{\sin(i\theta)}{\sin\theta} \cdot \frac{\sin((n+1-j)\theta)}{\sin\theta} & \text{if } i \leq j \\ \frac{\sin(j\theta)}{\sin\theta} \cdot \frac{\sin((n+1-i)\theta)}{\sin\theta} & \text{if } i \geq j \end{cases}.$$

Therefore, by using that $\det(xI - K) = \frac{\sin(n+1)\theta}{\sin\theta}$ we get that

$$(xI - K)^{-1}_{ij} = \begin{cases} \frac{\sin(i\theta)}{\sin\theta} \cdot \frac{\sin((n+1-j)\theta)}{\sin(n+1)\theta} & \text{if } i \leq j \\ \frac{\sin(j\theta)}{\sin\theta} \cdot \frac{\sin((n+1-i)\theta)}{\sin(n+1)\theta} & \text{if } i \geq j \end{cases}.$$

Remark 28. This is a one-pair matrix, that is, the upper triangular part of the matrix looks like as the upper triangular part of a rank-one matrix, and the lower triangular part is obtained by reflecting the upper part to the main diagonal (see the definition below).

Definition 27. *An $n \times n$ matrix A is a one-pair matrix if there exist vectors u, v such that*

$$(A)_{ij} = \begin{cases} u_i v_j & \text{if } i \leq j \\ u_j v_i & \text{if } i > j \end{cases}.$$

Till now we considered the case $|x| < 2$. If $x > 2$, then we shall use the substitution $x = 2\cosh\theta$, where $\cosh\theta = \frac{e^\theta + e^{-\theta}}{2}$ is the hyperbolic cosine function. A very similar calculation yields the following results:

$$(xI-K)^{-1}_{ij} = \begin{cases} \frac{\sinh(i\theta)}{\sinh\theta}\cdot\frac{\sinh((n+1-j)\theta)}{\sinh(n+1)\theta} & \text{if } i \le j \\ \frac{\sinh(j\theta)}{\sinh\theta}\cdot\frac{\sinh((n+1-i)\theta)}{\sinh(n+1)\theta} & \text{if } i \ge j \end{cases}.$$

Finally, if $x < -2$, then we shall express x as $x = -2\cosh\theta$. Then

$$\begin{pmatrix} -2\cosh\theta & -1 & 0 & \dots & 0 \\ -1 & -2\cosh\theta & -1 & \ddots & \vdots \\ 0 & -1 & -2\cosh\theta & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \dots & 0 & -1 & -2\cosh\theta \end{pmatrix} =$$

$$= -\begin{pmatrix} 2\cosh\theta & 1 & 0 & \dots & 0 \\ 1 & 2\cosh\theta & 1 & \ddots & \vdots \\ 0 & 1 & 2\cosh\theta & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & 1 & 2\cosh\theta \end{pmatrix},$$

so the 1's along the subdiagonal and superdiagonal are changed to -1's. By doing a similar calculation as above it turns out that there is no change in the determinant: $D_n = \frac{\sinh(n+1)\theta}{\sinh\theta}$, but there is a (sign) change in the adjugate. The inverse is the following:

$$(xI-K))^{-1}_{ij} = \begin{cases} (-1)^{i+j-1}\frac{\sinh(i\theta)}{\sinh\theta}\cdot\frac{\sinh((n+1-j)\theta)}{\sinh(n+1)\theta} & \text{if } i \le j \\ (-1)^{i+j-1}\frac{\sinh(j\theta)}{\sinh\theta}\cdot\frac{\sinh((n+1-i)\theta)}{\sinh(n+1)\theta} & \text{if } i \ge j \end{cases}.$$

Let us take a look at the special case $x = 2$. This can be considered as a limit case of $|x| < 2$. Note that $\lim_{\theta\to 0}\frac{\sin(i\theta)}{\sin\theta}\cdot\frac{\sin((n+1-j)\theta)}{\sin(n+1)\theta} = \frac{i(n+1-j)}{n+1}$. Therefore:

$$\begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -1 & 2 & -1 \\ 0 & \dots & 0 & -1 & 2 \end{pmatrix}^{-1}_{ij} = \begin{cases} \frac{i(n+1-j)}{n+1} & \text{if } i \le j \\ \frac{j(n+1-i)}{n+1} & \text{if } i \ge j \end{cases}.$$

Similarly, the case of $x = -2$ can be also handled as a limit case:

$$\begin{pmatrix} -2 & -1 & 0 & \dots & 0 \\ -1 & -2 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -1 & -2 & -1 \\ 0 & \dots & 0 & -1 & -2 \end{pmatrix}^{-1}_{ij} = \begin{cases} (-1)^{i+j-1}\,\frac{i(n+1-j)}{n+1} & \text{if } i \le j \\ (-1)^{i+j-1}\,\frac{j(n+1-i)}{n+1} & \text{if } i \ge j \end{cases}.$$

To summarize:

$$(xI-K))^{-1}_{ij} = \begin{cases} \frac{D_{i-1}(x)D_{n-j}(x)}{D_n(x)} & \text{if } i \le j \\ \frac{D_{j-1}(x)D_{n-i}(x)}{D_n(x)} & \text{if } i \ge j \end{cases}.$$

So far we always had -1's along the sub- and superdiagonal, however the general case can be easily handled now. Let us take a symmetric tridiagonal Toeplitz matrix:

$$A := \begin{pmatrix} a & b & 0 & \dots & 0 \\ b & a & b & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & b & a & b \\ 0 & \dots & 0 & b & a \end{pmatrix}.$$

The case $b = 0$ is trivial, otherwise let $x = -\frac{a}{b}$, then $-\frac{1}{b}A$ has -1's along the sub- and superdiagonal, and the inverse of A can be expressed as:

$$A^{-1} = -\frac{1}{b}\begin{pmatrix} x & -1 & 0 & \dots & 0 \\ -1 & x & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -1 & x & -1 \\ 0 & \dots & 0 & -1 & x \end{pmatrix}^{-1}.$$

As a final remark:

Proposition 51.

$$\det\begin{pmatrix} \cos\varphi & -1 & 0 & \dots & 0 \\ -1 & 2\cos\varphi & -1 & \ddots & \vdots \\ 0 & -1 & 2\cos\varphi & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \dots & 0 & -1 & 2\cos\varphi \end{pmatrix} = \cos(n\varphi)$$

8. Block matrices

A block matrix (or a partitioned matrix) is a matrix that is interpreted as having been broken into sections called blocks. Notation:

$$\begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ \vdots & \vdots & & \vdots \\ A_{m1} & A_{m2} & \dots & A_{mn} \end{pmatrix} = (A_{ij}).$$

Here the matrices $A_{11}, A_{12}, \dots, A_{1n}$ must have the same number of rows and they form a *block row*. Similary, e.g. the matrices $A_{11}, \dots, A_{m1}$ must have the same number of columns and they form a *block column*.

If each A_{ii} is quadratic, then we say that the block matrix is *symmetrically partitioned*.

If the block matrices (A_{ij}) and (B_{ij}) are partitioned in the same way, then we can perform addition blockwise:

$$(A_{ij}) + (B_{ij}) = (A_{ij} + B_{ij}).$$

If the partitioning of the columns in (A_{ik}) and the partitioning of the rows in (B_{kj}) is the same, then the product can be calculated blockwise:

$$(A_{ik})(B_{kj}) = (\sum_{k=1}^{n} A_{ik}B_{kj}).$$

The product of a block column and block row (the analogue of a rank-one matrix) looks like this:

$$\begin{pmatrix} A_1 \\ \vdots \\ A_m \end{pmatrix} \begin{pmatrix} B_1 & B_2 & \dots & B_n \end{pmatrix} = \begin{pmatrix} A_1B_1 & A_1B_2 & \dots & A_1B_n \\ \vdots & \vdots & & \vdots \\ A_mB_1 & A_mB_2 & \dots & A_mB_n \end{pmatrix}.$$

(Note that this multiplication is defined only when the number of columns in the matrices A_i is the same as the number of rows in the matrices B_j.)

Let us consider a symmetrically partitioned 2×2 block matrix:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

where A is a $k \times k$ matrix and D is an $(n-k) \times (n-k)$ matrix.

Theorem 52. *If* $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *is a symmetrically partitioned* 2×2 *block matrix and* A *is nonsingular, then*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{pmatrix} \begin{pmatrix} I & A^{-1}B \\ 0 & I \end{pmatrix}.$$

Proof. The claimed equality can be easily checked by computing the product on the right hand side. However, we shall illustrate how one can arrive at this formula. Let us perform the blockwise analogue of the algorithm for finding the rank-one decomposition. First we substract the product of the first (block) column and first (block) row:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} - \begin{pmatrix} A \\ C \end{pmatrix} A^{-1} \begin{pmatrix} A & B \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & D - CA^{-1}B \end{pmatrix}.$$

Then, by continuing in a similar fashion:

$$\begin{pmatrix} 0 & 0 \\ 0 & D - CA^{-1}B \end{pmatrix} = \begin{pmatrix} 0 \\ I \end{pmatrix} (D - CA^{-1}B) \begin{pmatrix} 0 & I \end{pmatrix}.$$

Hence, we obtained that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A \\ C \end{pmatrix} A^{-1} \begin{pmatrix} A & B \end{pmatrix} + \begin{pmatrix} 0 \\ I \end{pmatrix} (D - CA^{-1}B) \begin{pmatrix} 0 & I \end{pmatrix}.$$

Or equivalently,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{pmatrix} \begin{pmatrix} I & A^{-1}B \\ 0 & I \end{pmatrix},$$

as we claimed. □

The identity that we used at the end of the previous proof can be formulated in general:

Proposition 53.

$$\begin{pmatrix} A_{11} & \dots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \dots & A_{nn} \end{pmatrix} \begin{pmatrix} B_1 & & \\ & \ddots & \\ & & B_n \end{pmatrix} \begin{pmatrix} C_{11} & \dots & C_{1n} \\ \vdots & \ddots & \vdots \\ C_{n1} & \dots & C_{nn} \end{pmatrix} = \begin{pmatrix} A_{11} \\ \vdots \\ A_{n1} \end{pmatrix} B_1 \begin{pmatrix} C_{11} & \dots & C_{1n} \end{pmatrix} + \dots$$

We can formulate an analogue of Theorem 52 for the case when D happens to be nonsingular:

Theorem 54. *If* $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *is a symmetrically partitioned* 2×2 *block matrix and* D *is nonsingular, then*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & BD^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I & 0 \\ D^{-1}C & I \end{pmatrix}.$$

Corollary 55. *If* $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *is a symmetrically partitioned* 2×2 *block matrix and* A *is nonsingular, then*

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A) \det(D - CA^{-1}B).$$

In particular, $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *is nonsingular iff* $\det(D - CA^{-1}B) \neq 0$.

Proof. This follows immediately from Theorem 52. □

The following variant when D is nonsingular also holds:

Corollary 56. *If* $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *is a symmetrically partitioned* 2×2 *block matrix and* D *is nonsingular, then*

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A - BD^{-1}C)\det(D).$$

In particular, $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *is nonsingular iff* $\det(A - BD^{-1}C) \neq 0$.

Let us consider the special case when $B = b$ is a column vector, $C = c^T$ is a row vector and $D = d$ is a scalar. According to Corollary 55 we obtain that

$$\det \begin{pmatrix} A & b \\ c^T & d \end{pmatrix} = \det(A)(d - c^T A^{-1} b) = d\det(A) - c^T \operatorname{adj}(A) b.$$

Therefore, this matrix is singular if and only if $d - c^T A^{-1} b = 0$. If A is singular, then by using continuity we obtain that

$$\det \begin{pmatrix} A & b \\ c^T & d \end{pmatrix} = -c^T \operatorname{adj}(A) b.$$

8.1. **Inverse of a block matrix.** The inverse of a block diagonal matrix can be obtained by taking the inverses of the matrices on the main diagonal, for instance:

Proposition 57. *If* A *and* B *are invertible square matrices, then*

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & 0 \\ 0 & B^{-1} \end{pmatrix}.$$

If the block matrix is lower triangular, and the diagonal blocks are identity matrices, then the inverse can be obtained as follows:

Proposition 58.

$$\begin{pmatrix} I & 0 \\ X & I \end{pmatrix}^{-1} = \begin{pmatrix} I & 0 \\ -X & I \end{pmatrix}.$$

Proof. Observe that

$$\begin{pmatrix} I & 0 \\ X & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ X & 0 \end{pmatrix},$$

where

$$N := \begin{pmatrix} 0 & 0 \\ X & 0 \end{pmatrix}$$

is nilpotent, moreover $N^2 = 0$. Thus $(I + N)^{-1} = I - N$, as we claimed.

□

Now, using Theorem 52 we can calculate the inverse of a 2×2 block matrix, as follows:

Theorem 59. *If* $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *is a symmetrically partitioned* 2×2 *block matrix and* A *is nonsingular, then*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} + A^{-1}B(D-CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D-CA^{-1}B)^{-1}) \\ -(D-CA^{-1}B)^{-1}CA^{-1} & (D-CA^{-1}B)^{-1} \end{pmatrix}.$$

Proof. According to Theorem 52 we have

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & D-CA^{-1}B \end{pmatrix}\begin{pmatrix} I & A^{-1}B \\ 0 & I \end{pmatrix}.$$

Now, using Propositions 57 and 58 we obtain that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} I & -A^{-1}B \\ 0 & I \end{pmatrix}\begin{pmatrix} A^{-1} & 0 \\ 0 & (D-CA^{-1}B)^{-1} \end{pmatrix}\begin{pmatrix} I & 0 \\ -CA^{-1} & I \end{pmatrix} =$$

$$= \begin{pmatrix} A^{-1} + A^{-1}B(D-CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D-CA^{-1}B)^{-1} \\ -(D-CA^{-1}B)^{-1}CA^{-1} & (D-CA^{-1}B)^{-1} \end{pmatrix}.$$

□

Similarly, when the other diagonal block is nonsingular, we obtain the following theorem:

Theorem 60. *If* $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *is a symmetrically partitioned* 2×2 *block matrix and* D *is nonsingular, then*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} (A-BD^{-1}C)^{-1} & -(A-BD^{-1}C)^{-1}BD^{-1} \\ -D^{-1}C(A-BD^{-1}C)^{-1} & D^{-1} + D^{-1}C(A-BD^{-1}C)^{-1}BD^{-1} \end{pmatrix}.$$

Now, we present an application of Theorem 59.

Example 22. *Let* M *be the following* $n\times n$ *matrix:*

$$M := \begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -1 & 2 & -1 \\ 0 & \dots & 0 & -1 & 1 \end{pmatrix}.$$

(That is, by changing M_{nn} *to 2 we would obtain a symmetric tridiagonal Toeplitz matrix with 2's along the main diagonal and* -1*'s along the sub- and superdiagonal.) Our goal is to determine* M^{-1}*.*

Let us consider M as a block matrix in the following way:

$$M = \begin{pmatrix} A & b \\ c^T & d \end{pmatrix},$$

where $b = c = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix}$ *and* $d = 1$. *We have determined earlier (see Subsection 7.2) that the inverse of the* $(n-1)\times(n-1)$ *matrix A is as follows:*

$$(A^{-1})_{ij} = \begin{cases} \frac{i(n-j)}{n} & \text{if } i \le j \\ \frac{j(n-i)}{n} & \text{if } i \ge j \end{cases}.$$

Note that

$$d - c^T A^{-1} b = 1 - \frac{n-1}{n} = \frac{1}{n},$$

$$A^{-1}b = \begin{pmatrix} -1/n \\ \vdots \\ -(n-1)/n \end{pmatrix},$$

$$c^T A^{-1} = \begin{pmatrix} -1/n & \dots & -(n-1)/n \end{pmatrix}.$$

According to Theorem 59 we have

$$\begin{pmatrix} A & b \\ c^T & d \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} + A^{-1}b(d - c^T A^{-1}b)^{-1}c^T A^{-1} & -A^{-1}b(d - c^T A^{-1}b)^{-1} \\ -(d - c^T A^{-1}b)^{-1}c^T A^{-1} & (d - c^T A^{-1}b)^{-1} \end{pmatrix}.$$

Since

$$\left(\frac{(A^{-1}b)(c^T A^{-1})}{d - c^T A^{-1}b} \right)_{ij} = \frac{ij}{n},$$

we get that

$$(M^{-1})_{ij} = \begin{cases} i & \text{if } i \le j \\ j & \text{if } i \ge j \end{cases}.$$

Hence:

$$M^{-1} = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 2 & \dots & 2 \\ 1 & 2 & 3 & \dots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & \dots & n \end{pmatrix}$$

As another application we prove Woodbury's theorem.

Theorem 61. *(Woodbury) Let A be a nonsingular $n \times n$ matrix and let $M = BC$ be a minimal rank-one decomposition of M. If* $\det(I - CA^{-1}B) \neq 0$, *then $A - M$ is nonsingular and* $(A-M)^{-1} = A^{-1} + A^{-1}B(I - CA^{-1}B)^{-1}CA^{-1}$.

Proof. Let us consider the following 2×2 block matrix:

$$X := \begin{pmatrix} A & B \\ C & I \end{pmatrix}.$$

According to Corollaries 55 and 56 we know that

$$\det X = \det(I - CA^{-1}B) \det A = \det(A - BC),$$

thus $A - M = A - BC$ is indeed nonsingular.

Furthermore, by Theorems 59 and 60 we get that

$$X^{-1} = \begin{pmatrix} A & B \\ C & I \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} + A^{-1}B(I - CA^{-1}B)^{-1}CA^{-1} & * \\ * & * \end{pmatrix} = \begin{pmatrix} (A - BC)^{-1} & * \\ * & * \end{pmatrix},$$

where only the relevant (top-left) blocks are indicated.

Since X^{-1} is unique, $(A - M)^{-1} = (A - BC)^{-1} = A^{-1} + A^{-1}B(I - CA^{-1}B)^{-1}CA^{-1}$, as we claimed. □

As a special case this yields a new proof for the Sherman-Morrison formula:

Corollary 62. *(Sherman-Morrison formula) Let A be a nonsingular $n \times n$ matrix and let $M = uv^T$ be an $n \times n$ rank-one matrix. If $1 - v^T A^{-1} u \neq 0$, then*

$$(A - uv^T)^{-1} = A^{-1} + \frac{(A^{-1}u)(v^T A^{-1})}{1 - v^T A^{-1} u}.$$

Proof. This is a direct consequence of Theorem 61. Note that the assumption $\det(I - v^T A^{-1} u) \neq 0$ holds, since $1 - v^T A^{-1} u \neq 0$. □

Remark 29. The Sherman-Morrison formula tells us how the inverse changes, if we add a rank-one matrix to an invertible matrix. Now, we will investigate how the adjugate changes, if we add a rank-one matrix to an $n \times n$ matrix of rank $n - 1$.

Let $\operatorname{rk}(A) = n - 1$, where A is an $n \times n$ matrix. We would like to study $\operatorname{adj}(A - uv^T)$. We shall distinguish four cases:

(a) u is contained in the column space of A, but v^T is not in the row space of A.

According to the condition there exists some x such that $Ax = u$. Then

$$\operatorname{adj}(A - uv^T) = \operatorname{adj}(A - Axv^T) = \operatorname{adj}(A(I - xv^T)).$$

By the Sherman-Morrison formula we get that $(I - xv^T)^{-1} = I + \frac{xv^T}{1 - v^T x}$. Also, $\det(I - xv^T) = 1 - v^T x$, thus $\operatorname{adj}(I - xv^T) = (1 - v^T x)I + xv^T$. Therefore, $\operatorname{adj}(A - uv^T) = \operatorname{adj}(I - xv^T)\operatorname{adj}(A) = ((1 - v^T x)I + xv^T)\operatorname{adj} A$. Since $\operatorname{rk}(A) = n - 1$, we have $A \cdot \operatorname{adj} A = 0$, and so $\operatorname{rk}(\operatorname{adj} A) = 1$. Let us write $\operatorname{adj} A$ as $\operatorname{adj} A = rs^T$. Then

$$\operatorname{adj}(A - uv^T) = ((1 - v^T x)I + xv^T)rs^T = ((1 - v^T x)r + (v^T r)x)s^T.$$

We can interpret this as follows. The matrix A is a singular matrix, so 0 is an eigenvalue of A, r is a right eigenvector (since $A \cdot \operatorname{adj} A = 0$) and s^T is a left eigenvector (since $(\operatorname{adj} A) \cdot A = 0$). According to our calculation 0 is also an eigenvalue of $A - uv^T$, the left eigenvector s^T remains an eigenvector, but the right eigenvector r changes to $(1 - v^T x)r + (v^T r)x$.

(b) v^T is contained in the row space of A, but u is not in the column space of A.

Let $y^T A = v^T$. The calculation is very similar to case (a), and the following is obtained:

$$\operatorname{adj}(A - uv^T) = (\operatorname{adj} A)(\operatorname{adj} I - uv^T) = r((1 - y^T u)s^T + (s^T u)y^T).$$

So 0 remains an eigenvalue, r remains a right eigenvector, but the left eigenvector s changes to $((1 - y^T u)s^T + (s^T u)y^T)$.

(c) u is in the column space of A and v^T is in the row space of A.

Let $Ax = u$ and $y^T A = v^T$. Then:

$$\operatorname{adj}(A - uv^T) = \operatorname{adj}(A - Axy^T A) = \operatorname{adj} A \operatorname{adj}(I - Axy^T) = rs^T((1 - y^T Ax)I + Axy^T).$$

Note that $s^T A = 0$.
If $1 - y^T Ax \neq 0$, then $\operatorname{adj}(A - uv^T) = (1 - y^T Ax)rs^T$, so 0 remains an eigenvalue, moreover r remains a right eigenvector and s^T remains a left eigenvector.
If $1 - y^T Ax = 0$, then $\operatorname{adj}(A - uv^T) = 0$, that is, the rank decreases: $\operatorname{rk}(A - uv^T) = n - 2$.

(d) u is not in the column space of A and v^T is not in the row space of A.

We will show that $\operatorname{rk}(A - uv^T) = n$, so the rank increases in this case. Let $\operatorname{adj} A = rs^T$. For the sake of contradiction let us assume that $(A - uv^T)w = 0$ for some $w \neq 0$. Then $Aw = (v^T w)u$. If $v^T w \neq 0$, then this would imply that $u = A\left(\frac{1}{v^T w} w\right)$ is in the column space of A, thus we must have $v^T w = 0$. Consequently, $Aw = 0$ also holds. Note that $\{x : x^T w = 0\}$ is an $(n-1)$-dimensional subspace. However, all the rows of A are contained in this subspace, and $\operatorname{rk} A = n - 1$, thus $\{x : x^T w = 0\}$ is the row space of A. As $v^T w = 0$, we get that v^T is in the row space of A, which contradicts our assumption.
Hence, in case (d), $\operatorname{rk}(A - uv^T) = n$, the rank increases by 1.

Theorem 63. *Let* $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *be a symmetrically partitioned nonsingular* 2×2 *block matrix, let* $\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} P & Q \\ S & T \end{pmatrix}$*, and assume that* A *is nonsingular. Then* $A^{-1} = P - QT^{-1}S$*.*

Proof. According to Corollary 55 the matrix $D - CA^{-1}B$ is nonsingular, let $X := (D - CA^{-1}B)^{-1}$.

According to Theorem 59 we obtain the following:

$$\begin{aligned} P &= A^{-1} + A^{-1}BXCA^{-1}, \\ Q &= -A^{-1}BX, \\ S &= -XCA^{-1}, \\ T &= X. \end{aligned}$$

Hence,

$$P - QT^{-1}S = A^{-1} + A^{-1}BXCA^{-1} - (-A^{-1}BX)X^{-1}(-XCA^{-1}) = A^{-1},$$

as claimed. □

The following analogous theorem can be proven similarly:

Theorem 64. *Let* $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *be a symmetrically partitioned nonsingular* 2×2 *block matrix, let* $\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} P & Q \\ S & T \end{pmatrix}$, *and assume that* D *is nonsingular. Then* $D^{-1} = T - SP^{-1}Q$.

Corollary 65. *Let* $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *be a symmetrically partitioned nonsingular* 2×2 *block matrix, let* $\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} P & Q \\ S & T \end{pmatrix}$, *and assume that* A *is nonsingular.*
Then $\det A = \det \begin{pmatrix} A & B \\ C & D \end{pmatrix} \det(T)$.

Proof. By applying Corollary 56 to the matrix $\begin{pmatrix} P & Q \\ S & T \end{pmatrix}$ we obtain that $\det \begin{pmatrix} P & Q \\ S & T \end{pmatrix} = \det(T)\det(P - QT^{-1}S)$. By Theorem 63 we know that $\det(P - QT^{-1}S) = \det(A)^{-1}$, so after multiplying both sides by $\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} \det(A)$ the required equality is obtained. □

Analogously:

Corollary 66. *Let* $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *be a symmetrically partitioned nonsingular* 2×2 *block matrix, let* $\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} P & Q \\ S & T \end{pmatrix}$, *and assume that* D *is nonsingular. Then* $\det D = \det \begin{pmatrix} A & B \\ C & D \end{pmatrix} \det(P)$.

Definition 28. *The partitioning of a block matrix* $\begin{pmatrix} A_{11} & \dots & A_{1n} \\ \vdots & & \vdots \\ A_{n1} & \dots & A_{nn} \end{pmatrix} = (A_{ij})$ *is persymmetric if* $A_{i,n+1-i}$ *is quadratic for every* $1 \le i \le n$.

Let $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a persymmetric partitioning, where the four blocks are of the following types:

$$A : p \times q,\ \ B : p \times p,\ \ C : q \times q,\ \ D : q \times p.$$

Assume that the block matrix is invertible and let

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} P & Q \\ S & T \end{pmatrix}.$$

Here the partitioning of $\begin{pmatrix} P & Q \\ S & T \end{pmatrix}$ is also persymmetric:

$$P : q \times p,\ \ Q : q \times q,\ \ S : p \times p,\ \ T : p \times q.$$

By permuting the order of rows and columns in $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ we can get a symmetrically partitioned block matrix, then we can apply Theorem 63. Namely:

$$\begin{pmatrix} P & Q \\ S & T \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \left\{ \begin{pmatrix} B & A \\ D & C \end{pmatrix} \begin{pmatrix} 0 & I_p \\ I_q & 0 \end{pmatrix} \right\}^{-1} = \begin{pmatrix} 0 & I_q \\ I_p & 0 \end{pmatrix} \begin{pmatrix} B & A \\ D & C \end{pmatrix}^{-1}.$$

Now, by applying Theorem 63 to the symmetrically partitioned block matrix $\begin{pmatrix} B & A \\ D & C \end{pmatrix}$ we get that:

$$P = -(C - DB^{-1}A)^{-1}DB^{-1},$$
$$Q = (C - DB^{-1}A)^{-1},$$
$$S = B^{-1} + B^{-1}A(C - DB^{-1}A)^{-1}DB^{-1},$$
$$T = -B^{-1}A(C - DB^{-1}A)^{-1}.$$

From these we conclude the following:

Theorem 67. $B^{-1} = S - TQ^{-1}P$

In a similar fashion, we can prove that

Theorem 68. $C^{-1} = Q - PS^{-1}T$

Note that $\det \begin{pmatrix} 0 & I_q \\ I_p & 0 \end{pmatrix} = (-1)^{pq}$. Therefore, the following analogue of Corollaries 65 and 66 can be deduced:

Corollary 69.

$$\det(C) = (-1)^{pq} \det \begin{pmatrix} A & B \\ C & D \end{pmatrix} \det(S)$$

$$\det(B) = (-1)^{pq} \det \begin{pmatrix} A & B \\ C & D \end{pmatrix} \det(Q)$$

Now, let us turn our attention at tridiagonal matrices.

Theorem 70. *Let A be a symmetric nonsingular matrix. Then the following are equivalent:*

(i) *A is tridiagonal and all the elements on the subdiagonal (and superdiagonal) are nonzero,*

(ii) *A^{-1} is a one-pair matrix.*

Remark 30. See Definition 27 for the definition of a one-pair matrix.

Proof. We start by proving the direction $(i) \implies (ii)$.

Let

$$A := \begin{pmatrix} a_1 & b_1 & 0 & \dots & 0 \\ b_1 & a_2 & b_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & b_{n-2} & a_{n-1} & b_{n-1} \\ 0 & \dots & 0 & b_{n-1} & a_n \end{pmatrix}.$$

According to our assumption $b_i \neq 0$ (for every $1 \leq i \leq n-1$). Let us extend A by adding an additional row and column as below:

$$\begin{pmatrix} e_1^T & 0 \\ A & e_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ a_1 & b_1 & 0 & \dots & 0 & 0 \\ b_1 & a_2 & b_2 & \ddots & \vdots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 & 0 \\ \vdots & \ddots & b_{n-2} & a_{n-1} & b_{n-1} & 0 \\ 0 & \dots & 0 & b_{n-1} & a_n & 1 \end{pmatrix}.$$

Note that this is a lower triangular matrix, and this 2×2 block partitioning is persymmetric. Let us write

$$\begin{pmatrix} e_1^T & 0 \\ A & e_n \end{pmatrix}^{-1} = \begin{pmatrix} u & X \\ -u_0 & u_0 v^T \end{pmatrix},$$

where $u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$, $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ and u_0 is scalar. Now, according to Theorem 68 we have

$$A^{-1} = X - u(-u_0)^{-1}(u_0 v^T) = X + uv^T.$$

Note that, since the inverse of a lower triangular matrix is lower triangular, X is a lower triangular matrix with 0's on its main diagonal:

$$X = \begin{pmatrix} 0 & 0 & \dots & 0 \\ * & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ * & \dots & * & 0 \end{pmatrix}.$$

Therefore $(A^{-1})_{ij} = u_i v_j$, if $i \le j$. However, A is symmetric, hence, so is its inverse A^{-1}. That is,

$$(A^{-1})_{ij} = \begin{cases} u_i v_j & \text{if } i \le j \\ \\ u_j v_i & \text{if } i > j \end{cases}.$$

So the matrix A^{-1} is a one-pair matrix, as we claimed.

Now, we continue with the direction $(ii) \implies (i)$.
Let B be a nonsingular one-pair matrix:

$$(B)_{ij} = \begin{cases} u_i v_j & \text{if } i \le j \\ \\ u_j v_i & \text{if } i > j \end{cases}.$$

Let us define

$$(X)_{ij} = \begin{cases} 0 & \text{if } i \le j \\ \\ u_j v_i - u_i v_j & \text{if } i > j \end{cases}.$$

Then $B = X + uv^T$, since $u_j v_i = u_i v_j + (u_j v_i - u_i v_j)$.
Note that

$$\begin{pmatrix} u & X \\ -1 & v^T \end{pmatrix} = \begin{pmatrix} u_1 & 0 & 0 & \dots & 0 & 0 \\ u_2 & x_{21} & 0 & \dots & 0 & 0 \\ u_3 & x_{31} & x_{32} & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & 0 \\ u_n & x_{n1} & \dots & x_{n,n-2} & x_{n,n-1} & 0 \\ -1 & v_1 & \dots & v_{n-2} & v_{n-1} & v_n \end{pmatrix}$$

is a lower triangular matrix. It is nonsingular if and only if the main diagonal contains only nonzero numbers:

- $u_1 \ne 0$,
- $v_n \ne 0$,
- $x_{i+1,i} \ne 0$ for every $1 \le i \le n-1$.

Since B is nonsingular, $u_1 \ne 0$, otherwise, the first row of B would be all-0. Similarly, $v_n \ne 0$, since the last column of B is not all-0. Finally, $x_{i+1,i} = u_i v_{i+1} - u_{i+1} v_i \ne 0$, otherwise the ith and $(i+1)$st rows of B would be linearly dependent. Since, these two rows are

$$(u_1 v_i, u_2 v_i, \dots, u_i v_i, u_i v_{i+1}, \dots, u_i v_n)$$

$$(u_1 v_{i+1}, u_2 v_{i+1}, \dots, u_i v_{i+1}, u_{i+1} v_{i+1}, \dots, u_{i+1} v_n)$$

and $x_{i+1,i} = 0$ would imply that $v_{i+1} \cdot (\text{row } i) - v_i \cdot (\text{row } i+1) = 0$.
Hence, the above lower triangular matrix is nonsingular.
Let us write

$$\begin{pmatrix} u & X \\ -1 & v^T \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{u_1} e_1^T & 0 \\ H & \frac{1}{v_n} e_n \end{pmatrix}.$$

Now, by Theorem 68 we have

$$H^{-1} = X - u(-1)v^T,$$

thus

$$H = (X + uv^T)^{-1} = B^{-1}.$$

As B is symmetric, H is also symmetric. Also, $H_{ij} = 0$ for $j - i > 1$, therefore H is indeed tridiagonal, as claimed.

The entries on the superdiagonal of H lie on the main diagonal of $\begin{pmatrix} \frac{1}{u_1}e_1^T & 0 \\ H & \frac{1}{v_n}e_n \end{pmatrix}$ which is a nonsingular lower triangular matrix. Consequently, the entries on the superdiagonal of H are all different from 0 (and, since H is symmetric the entries on the subdiagonal of H are also nonzero). This completes the proof of the direction $(ii) \implies (i)$.

□

8.2. **Recursive formula for calculating the inverse of a symmetric tridiagonal matrix.** We will look at the equation

$$\begin{pmatrix} e_1^T & 0 \\ A & e_n \end{pmatrix} \begin{pmatrix} u & X \\ -u_0 & u_0 v^T \end{pmatrix} = I,$$

that is,

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ a_1 & b_1 & 0 & \dots & 0 \\ b_1 & a_2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & b_{n-1} & 0 \\ 0 & \dots & b_{n-1} & a_n & 1 \end{pmatrix} \begin{pmatrix} u_1 & 0 & 0 & \dots & 0 \\ u_2 & x_{21} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ u_n & x_{n,1} & \dots & x_{n,n-1} & 0 \\ -u_0 & u_0 v_1 & \dots & u_0 v_{n-1} & u_0 v_n \end{pmatrix} = I.$$

From the first column of the product we learn the following:

$$1 \cdot u_1 = 1 \implies u_1 = 1,$$

$$a_1 u_1 + b_1 u_2 = 0 \implies u_2 = -\frac{1}{b_1} a_1 u_1,$$

$$b_1 u_1 + a_2 u_2 + b_2 u_3 = 0 \implies u_3 = -\frac{1}{b_2}(a_2 u_2 + b_1 u_1),$$

in general, for $i \le n$:

$$b_{i-2} u_{i-2} + a_{i-1} u_{i-1} + b_{i-1} u_i = 0 \implies u_i = -\frac{1}{b_{i-1}}(a_{i-1} u_{i-1} + b_{i-2} u_{i-2}),$$

and finally,

$$b_{n-1} u_{n-1} + a_n u_n - u_0 = 0 \implies u_0 = a_n u_n + b_{n-1} u_{n-1}.$$

That is, $u_1, u_2, \dots, u_n, u_0$ can be easily calculated recursively (in this order).

Now, we shall concentrate on the last row of the product

$$\begin{pmatrix} u & X \\ -u_0 & u_0 v^T \end{pmatrix} \begin{pmatrix} e_1^T & 0 \\ A & e_n \end{pmatrix} = I.$$

From the last entry we obtain that

$$-u_0 \cdot 0 + u_0 v^T e_n = 1 \iff u_0 v_n = 1 \iff v_n = \frac{1}{u_0}.$$

Then, by continuing (from right to left in the last row)

$$b_{n-1} u_0 v_{n-1} + a_n u_0 v_n = 0 \iff v_{n-1} = -\frac{1}{b_{n-1}} a_n v_n,$$

$$b_{n-2} v_{n-2} + a_{n-1} v_{n-1} + b_{n-1} v_n = 0 \iff v_{n-2} = -\frac{1}{b_{n-2}} (a_{n-1} v_{n-1} + b_{n-1} v_n),$$

and in general:

$$v_{n-i} = -\frac{1}{b_{n-i}} \left(a_{n-i+1} v_{n-i+1} + b_{n-i+1} v_{n-i+2} \right),$$

finally,

$$v_1 = -\frac{1}{b_1} \left(a_2 v_2 + b_2 v_3 \right).$$

That is, $v_n, v_{n-1}, \ldots, v_1$ can also be easily calculated recursively (in this order), thus A^{-1} is determined.

Let us consider the following example.

Example 23. *Let* $A = \begin{pmatrix} 2 & 2 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 2 & 1 \end{pmatrix}$*, we will calculate* A^{-1}*.*

According to the general method, we calculate $u_1, u_2, u_3, u_4, u_0, v_4, v_3, v_2, v_1$ *in this order.*

$$u_1 = 1$$
$$u_2 = -\frac{1}{b_1} a_1 = -\frac{2}{2} = -1$$
$$u_3 = -\frac{1}{b_2} (b_1 + a_2 u_2) = -(2 + 3(-1)) = 1$$
$$u_4 = -\frac{1}{b_3} (a_3 u_3 + b_2 u_2) = -\frac{1}{2} (3 \cdot 1 + 1(-1)) = -1$$
$$u_0 = a_4 u_4 + b_3 u_3 = 1(-1) + 2 \cdot 1 = 1$$
$$v_4 = \frac{1}{u_0} = 1$$
$$v_3 = -\frac{1}{b_3} a_4 u_4 = -\frac{1}{2} \cdot 1 \cdot 1 = -\frac{1}{2}$$

$$v_2 = -\frac{1}{b_2}(a_3v_3 + b_3v_4) = -\left(3 \cdot \frac{-1}{2} + 2 \cdot 1\right) = -\frac{1}{2}$$

$$v_1 = -\frac{1}{b_1}(a_2v_2 + b_2v_3) = -\frac{1}{2}\left(3 \cdot \frac{-1}{2} + 1 \cdot \frac{-1}{2}\right) = 1$$

Therefore,

$$u = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \quad v = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix},$$

thus

$$A^{-1} = \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} & 1 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -1 \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}.$$

Now, we slightly modify the matrix A from the previous example.

Example 24. *What happens, if we change a_4 to 2?*

$$A' = \begin{pmatrix} 2 & 2 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 2 & 2 \end{pmatrix}$$

We get the same u_1, u_2, u_3, u_4, but then for u_0 we obtain

$$u_0 = a_4u_4 + b_3u_3 = 2 \cdot (-1) + 2 \cdot 1 = 0.$$

This implies that A' is singular. Since, $\begin{pmatrix} e_1^T & 0 \\ A' & e_n \end{pmatrix}\begin{pmatrix} u & X \\ 0 & v^T \end{pmatrix} = I$ implies that $A'u = 0$, thus A' is singular (as $u \neq 0$).

8.3. **Formula for calculating the inverse of a one-pair matrix.** Let B be a non-singular one-pair matrix:

$$(B)_{ij} = \begin{cases} u_iv_j & \text{if } i \leq j \\ u_jv_i & \text{if } i > j \end{cases}.$$

We know that its inverse is tridiagonal:

$$B^{-1} := \begin{pmatrix} a_1 & b_1 & 0 & \dots & 0 \\ b_1 & a_2 & b_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & b_{n-2} & a_{n-1} & b_{n-1} \\ 0 & \dots & 0 & b_{n-1} & a_n \end{pmatrix}.$$

Let us look at the equation

$$\begin{pmatrix} \frac{1}{u_1} e_1^T & 0 \\ B^{-1} & \frac{1}{v_n} e_n \end{pmatrix} \begin{pmatrix} u & X \\ -1 & v^T \end{pmatrix} = I.$$

Note that both matrices are lower triangular:

$$\begin{pmatrix} \frac{1}{u_1} e_1^T & 0 \\ B^{-1} & \frac{1}{v_n} e_n \end{pmatrix} = \begin{pmatrix} \frac{1}{u_1} & 0 & 0 & \dots & 0 & 0 \\ a_1 & b_1 & 0 & \dots & 0 & 0 \\ b_1 & a_2 & b_2 & \ddots & \vdots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 & 0 \\ \vdots & \ddots & b_{n-2} & a_{n-1} & b_{n-1} & 0 \\ 0 & \dots & 0 & b_{n-1} & a_n & \frac{1}{v_n} \end{pmatrix},$$

$$\begin{pmatrix} u & X \\ -1 & v^T \end{pmatrix} = \begin{pmatrix} u_1 & 0 & 0 & \dots & 0 \\ u_2 & x_{21} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ u_n & x_{n,1} & \dots & x_{n,n-1} & 0 \\ -u_0 & v_1 & \dots & v_{n-1} & v_n \end{pmatrix}.$$

By looking at the diagonal entries we obtain that

$$b_i = \frac{1}{u_i v_{i+1} - u_{i+1} v_i} = \frac{1}{x_{i+1,i}}$$

for every $1 \le i \le n-1$.

Now, for calculating the a_i values let us look at the first column:

$$\begin{pmatrix} \frac{1}{u_1} & 0 & 0 & \dots & 0 & 0 \\ a_1 & b_1 & 0 & \dots & 0 & 0 \\ b_1 & a_2 & b_2 & \ddots & \vdots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 & 0 \\ \vdots & \ddots & b_{n-2} & a_{n-1} & b_{n-1} & 0 \\ 0 & \dots & 0 & b_{n-1} & a_n & \frac{1}{v_n} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}.$$

Then we can calculate the a_i values:

$$a_1 u_1 + b_1 u_2 = 0 \implies a_1 = -\frac{1}{u_1} b_1 u_2,$$

$$b_1 u_1 + a_2 u_2 + b_2 u_3 = 0 \implies a_2 = -\frac{1}{u_2}(b_1 u_1 + b_2 u_3),$$

and in general:

$$b_{i-1}u_{i-1} + a_i u_i + b_i u_{i+1} = 0 \implies a_i = -\frac{1}{u_i}(b_{i-1}u_{i-1} + b_i u_{i+1}).$$

Finally,

$$b_{n-1}u_{n-1} + a_n u_n - \frac{1}{v_n} = 0 \implies a_n = -\frac{1}{u_n}(b_{n-1}u_{n-1} - \frac{1}{v_n}) = -\frac{1}{v_n} b_{n-1}v_{n-1},$$

where the last equality holds, since $b_{n-1} = \frac{1}{u_{n-1}v_n - u_n v_{n-1}}$.

Example 25. *Let us calculate the inverse of the following matrix:*

$$B = \begin{pmatrix} 1 & x & x^2 & \dots & x^{n-1} \\ x & 1 & x & \ddots & \vdots \\ x^2 & x & \ddots & \ddots & x^2 \\ \vdots & \ddots & \ddots & 1 & x \\ x^{n-1} & \dots & x^2 & x & 1 \end{pmatrix}.$$

Note that B is a one-pair matrix. Indeed, for $u = \begin{pmatrix} 1 \\ 1/x \\ \vdots \\ 1/x^{n-1} \end{pmatrix}$ and $v = \begin{pmatrix} 1 \\ x \\ \vdots \\ x^{n-1} \end{pmatrix}$ we have

$$(B)_{ij} = \begin{cases} u_i v_j & \text{if } i \le j \\ u_j v_i & \text{if } i > j \end{cases}.$$

Now, by using the above recursive approach:

$$b_i = \frac{1}{u_i v_{i+1} - u_{i+1} v_i} = \frac{1}{x - 1/x} = \frac{x}{x^2 - 1}.$$

Note that for $x = \pm 1$ we would divide by 0, and according to the above proof $u_i v_{i+1} - u_{i+1} v_i = 0$ implies that B is singular. Hence, for $x = \pm 1$ the matrix B is not invertible, and from now on, we assume that $x \neq \pm 1$. In this case the formula for the b_i values is correct.

Now, let us turn our attention at the a_i values. First of all,

$$a_1 = -\frac{b_1 u_2}{u_1} = -\frac{x}{x^2 - 1} \cdot \frac{1}{x} = -\frac{1}{x^2 - 1}.$$

Then, for $1 < i < n$:

$$a_i = -\frac{1}{u_i}(b_i u_{i+1} + b_{i-1}u_{i-1}) = -x^{i-1}\left(\frac{x}{x^2-1}\frac{1}{x^i} + \frac{x}{x^2-1}\frac{1}{x^{i-2}}\right) = -\frac{x}{x^2-1}\left(\frac{1}{x} + x\right) = -\frac{x^2+1}{x^2-1}.$$

Finally,

$$a_n = -\frac{1}{v_n} b_{n-1} v_{n-1} = -\frac{1}{x^{n-1}} \cdot \frac{x}{x^2 - 1} \cdot x^{n-2} = -\frac{1}{x^2 - 1}.$$

Therefore,

$$B^{-1} = \begin{pmatrix} -\frac{1}{x^2-1} & \frac{x}{x^2-1} & 0 & \cdots & 0 \\ \frac{x}{x^2-1} & -\frac{x^2+1}{x^2-1} & \frac{x}{x^2-1} & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \frac{x}{x^2-1} & -\frac{x^2+1}{x^2-1} & \frac{x}{x^2-1} \\ 0 & \cdots & 0 & \frac{x}{x^2-1} & -\frac{1}{x^2-1} \end{pmatrix}.$$

9. Systems of linear equations

A system of linear equations is a collection of linear equations, as follows.

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n & = & b_2 \\ & \vdots & \\ a_{k1}x_1 + a_{k2}x_2 + \dots + a_{kn}x_n & = & b_k \end{array}$$

By writing $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ for the column vector of the unknowns, $b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \end{pmatrix}$ for the scalars appearing on the right hand sides of the equations and $A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \dots & a_{kn} \end{pmatrix}$ for the coefficient matrix the linear system can be written in the following concise way: $Ax = b$.

A linear system is *homogeneous* if $b = 0$. In this case $x = 0$ is always a (trivial) solution of the system. Let us determine all the solutions with the help of the minimal rank-one decomposition.

Let $A = UV^T$ be a minimal rank-one decomposition of A. Since the columns of U are linearly independent, according to Lemma 26 we have

$$Ax = 0 \iff UV^Tx = 0 \iff V^Tx = 0.$$

Note that we can take a minimal rank-one decomposition in such a way that V^T looks like as below:

$$V^T = \begin{pmatrix} 1 & * & \dots & * & * & * & * & \dots & * \\ 0 & 1 & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * & * & * & * & \dots & * \\ 0 & \dots & 0 & 1 & * & * & * & \dots & * \end{pmatrix}.$$

That is, the first $r = \operatorname{rk} A$ columns of V^T form an upper triangular matrix with 1's along the main diagonal. We call $V^Tx = 0$ a *reduced* system. Here $x_{r+1}, \dots, x_n$ are *free variables*, they can take any value, and then $x_1, \dots, x_r$ are uniquely determined. (For instance, x_i is determined by the equation corresponding to the ith row.) Let us collect $n - r$ linearly independent solutions in the columns of the following matrix:

$$X_{n-r} := \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1,n-r} \\ x_{21} & x_{22} & \dots & x_{2,n-r} \\ \vdots & \vdots & & \vdots \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}.$$

The solutions of $Ax = 0$ are the linear combinations of these column vectors. That is, the general formula for a solution of $Ax = 0$ is

$$x = X_{n-r} \begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_{n-r} \end{pmatrix}.$$

If $b \neq 0$, then the system $Ax = b$ is *inhomogeneous*. However, we can think of an inhomogeneous system as a homogeneous system

$$\begin{pmatrix} A & b \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ x_{n+1} \end{pmatrix},$$

where we look for solutions satisfying $x_{n+1} = -1$. (Since then the above homogeneous system translates to $0 = Ax + bx_{n+1} = Ax - b$ with $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$.) To get a solution (or all the solutions) we shall try to solve it in such a way (if possible) that x_{n+1} becomes a free variable. If this is successful, then let us consider the matrix

$$Y := \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1,n-r} & x_{1,n-r+1} \\ x_{21} & x_{22} & \dots & x_{2,n-r} & x_{2,n-r+1} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & -1 \end{pmatrix}.$$

Let the last column of Y be $\begin{pmatrix} x_0 \\ -1 \end{pmatrix}$, then x_0 is a solution (*particular solution*) of $Ax = b$. Then the general solution of $Ax = b$ can be expressed as

$$x = X_{n-r} \begin{pmatrix} t_1 \\ \vdots \\ t_{n-r} \end{pmatrix} + x_0,$$

where X_{n-r} is the matrix obtained by deleting the last row and last column of Y. (Note that the columns of X_{n-r} form a linearly independent system of solutions of the homogeneous equation $Ax = 0$.)

Theorem 71. *$Ax = b$ is solvable* $\iff \operatorname{rk}(A) = \operatorname{rk}(A \mid b)$

Proof. $\operatorname{rk}(A) = \operatorname{rk}(A \mid b)$ if and only if b is contained in the column space of A, that is, if $Ax = b$ is solvable. $\square$

Remark 31. The variable x_{n+1} can be a free variable if during the rank-one decompisition we do not need to choose any (nonzero) entry from the last column. We can avoid choosing a nonzero entry from the last column, if there is no row (throughout the process) in which only the last entry is nonzero.

Example 26. *Let*

$$A = \begin{pmatrix} 2 & 5 & 1 & 3 \\ 4 & 6 & 3 & 5 \\ 4 & 14 & 1 & 7 \\ 2 & -3 & 3 & \lambda \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ 4 \\ 4 \\ \mu \end{pmatrix},$$

where λ and μ are parameters. Let us determine the solutions of the inhomogeneous system $Ax = b$.

Let us start with finding a minimal rank-one decomposition of $(A \mid b)$ by selecting the 3rd entry from the first row:

$$\begin{pmatrix} 2 & 5 & 1 & 3 & 2 \\ 4 & 6 & 3 & 5 & 4 \\ 4 & 14 & 1 & 7 & 4 \\ 2 & -3 & 3 & \lambda & \mu \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 1 \\ 3 \end{pmatrix} \begin{pmatrix} 2 & 5 & 1 & 3 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -2 & -9 & 0 & -4 & -2 \\ 2 & 9 & 0 & 4 & 2 \\ -4 & -18 & 0 & \lambda - 9 & \mu - 6 \end{pmatrix}.$$

Now, we select the 1st entry of the 2nd row:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -2 & -9 & 0 & -4 & -2 \\ 2 & 9 & 0 & 4 & 2 \\ -4 & -18 & 0 & \lambda - 9 & \mu - 6 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 2 \\ -4 \end{pmatrix} \begin{pmatrix} 1 & 9/2 & 0 & 2 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda - 1 & \mu - 2 \end{pmatrix}.$$

Hence, we shall solve the system

$$\begin{pmatrix} 2 & 5 & \mathbf{1} & 3 & 2 \\ \mathbf{1} & 9/2 & 0 & 2 & 1 \\ 0 & 0 & 0 & \lambda-1 & \mu-2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = 0.$$

After rearranging the columns (and also the entries of x accordingly):

$$\begin{pmatrix} 1 & 2 & 3 & 5 & 2 \\ 0 & 1 & 2 & 9/2 & 1 \\ 0 & 0 & \lambda-1 & 0 & \mu-2 \end{pmatrix} \begin{pmatrix} x_3 \\ x_1 \\ x_4 \\ x_2 \\ x_5 \end{pmatrix} = 0,$$

and we look for solutions where $x_5 = -1$.
Note that x_2 is a free variable.

Case 1. $\lambda = 1, \mu \neq 2$. Then x_5 is not free, so there is no solution.

Case 2. $\lambda = 1, \mu = 2$. In this case x_4, x_2, x_5 are free variables. Let us plug in a linearly independent system:

$$x_4 = 1,\ x_2 = 0,\ x_5 = 0 \implies x_1 = -2,\ x_3 = 1,$$
$$x_4 = 0,\ x_2 = 1,\ x_5 = 0 \implies x_1 = -9/2,\ x_3 = 4,$$
$$x_4 = 0,\ x_2 = 0,\ x_5 = -1 \implies x_1 = 1,\ x_3 = 0.$$

The general solution in Case 2 can be given as:

$$\begin{pmatrix} -2 & -9/2 \\ 0 & 1 \\ 1 & 4 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Case 3: $\lambda \neq 1$. Then x_2 and x_5 are free variables.

$$x_2 = 1, x_5 = 0 \implies x_4 = 0, x_1 = -9/2, x_3 = 4$$

$$x_2 = 0, x_5 = -1 \implies x_4 = \frac{\mu-2}{\lambda-1}, x_1 = \frac{-2(\mu-2)}{\lambda-1} + 1, x_3 = \frac{\mu-2}{\lambda-1}$$

The general solution in Case 3 can be given as:

$$\begin{pmatrix} -9/2 \\ 1 \\ 4 \\ 0 \end{pmatrix} t_1 + \begin{pmatrix} \frac{-2(\mu-2)}{\lambda-1} + 1 \\ 0 \\ \frac{\mu-2}{\lambda-1} \\ \frac{\mu-2}{\lambda-1} \end{pmatrix}.$$

Let us consider some special cases of linear systems.

- If A is a nonsingular square matrix, then the unique solution of $Ax = b$ is $x = A^{-1}b$.

- If A is an $n \times n$ matrix with $\operatorname{rk} A = n - 1$, then the columns of $\operatorname{adj} A$ are solutions of $Ax = 0$. Indeed, $A \cdot \operatorname{adj} A = (\det A)I = 0$.
  Furthermore, $\operatorname{rk}(A) + \operatorname{rk}(\operatorname{adj} A) \le n$, thus $\operatorname{rk}(\operatorname{adj} A) \le 1$. Note that $\operatorname{rk}(\operatorname{adj} A) = 0$ would mean that $\operatorname{adj} A = 0$, which holds only if $\operatorname{rk} A < n - 1$, thus $\operatorname{rk}(\operatorname{adj} A) = 1$. Let $\operatorname{adj} A = uv^T$, where u and v are nonzero vectors. Since $v \ne 0$, from $Auv^T = 0$ we obtain $Au = 0$. Hence, the solutions to $Ax = 0$ are of the form $x = \alpha u$ for some scalar α. (Note that there can be no other solution, since the dimension of the orthogonal complement of the row space A is 1.)
- Let $A = P$ be a projection: $P^2 = P$.
  Let us consider first the homogeneous case: $Px = 0$. Since $P(I - P) = 0$, the columns of $I - P$ are solutions. The rank of $I - P$ is $n - \operatorname{rk} P$, so all solutions are contained in the column space of $I - P$. Hence, a general solution can be given as $x = (I - P)v$. (Note that for different vectors v we don't necessarily get different solution x.)
  Now, we consider the inhomogeneous case: $Px = b$. If x_0 is a solution, then $Pb = P(Px_0) = Px_0 = b$, since P is a projection. Hence, if the system is solvable, then $x = b$ is a solution. Therefore, if $b = Pb$, then $x_0 = b$ is a particular solution, and a general solution can be expressed as $x = b + (I - P)v$. If $b \ne Pb$, then the system is not solvable.

9.1. **Linear mappings.** In this subsection we give a brief introduction to linear mappings.

Definition 29. *A linear mapping is a function $\mathcal{A} : \mathbb{R}^n \to \mathbb{R}^m$ satisfying the following conditions:*

- $\mathcal{A}(x + y) = \mathcal{A}(x) + \mathcal{A}(y)$ *for every* $x, y \in \mathbb{R}^n$,
- $\mathcal{A}(\alpha x) = \alpha \mathcal{A}(x)$ *for every* $x \in \mathbb{R}^n, \alpha \in \mathbb{R}$.

In the special case when $m = n$ we say that the linear mapping is a linear transformation.

A linear transformation is uniquely determined, if we specify the images of the elements of a basis. Let $b_1, \dots, b_n$ be a basis. For every $1 \le i \le n$ let us require

$$\mathcal{A}(b_i) = \sum_{j=1}^{n} b_j a_{ji}.$$

Let us write this equation as a coordinate (column) vector:

$$\mathcal{A}(b_i) = \begin{pmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{ni} \end{pmatrix}.$$

Finally, let us assign the matrix $(\mathcal{A}(b_1),\dots,\mathcal{A}(b_n))$ to the linear mapping $\mathcal{A}$. That is, the columns of the corresponding matrix are the coordinate vectors of the images:

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}.$$

We can summarize the above notions as the *formal* product

$$(\mathcal{A}(b_1),\dots,\mathcal{A}(b_n)) = (b_1,\dots,b_n)\cdot A.$$

Let $x = \sum_{i=1}^{n} b_i x_i$. We can represent x as the formal product $\begin{pmatrix} b_1,\dots,b_n \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$. The image of x is determined as follows:

$$\mathcal{A}(x) = \mathcal{A}(\sum_{i=1}^{n} b_i x_i) = (\mathcal{A}(b_1),\dots,\mathcal{A}(b_n))\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1,\dots,b_n \end{pmatrix} A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1,\dots,b_n \end{pmatrix} Ax.$$

Now, let us take another basis, $c_1,\dots,c_n$. Let $(c_1,\dots,c_n) = (b_1,\dots,b_n)C$. Let the coordinates of x according to the two bases be as follows:

$$x = \begin{pmatrix} b_1,\dots,b_n \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} c_1,\dots,c_n \end{pmatrix}\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}.$$

Now, we determine the connection between the matrices of the mapping $\mathcal{A}$ according to these two bases. We have

$$(\mathcal{A}(b_1),\dots,\mathcal{A}(b_n)) = (b_1,\dots,b_n)\cdot A,$$
$$(\mathcal{A}(c_1),\dots,\mathcal{A}(c_n)) = (c_1,\dots,c_n)\cdot \tilde{A},$$

where $\tilde{A}$ is the matrix according to the basis $c_1,\dots,c_n$. Observe that

$$(\mathcal{A}(c_1),\dots,\mathcal{A}(c_n)) = (c_1,\dots,c_n)\tilde{A} = (b_1,\dots,b_n)C\tilde{A}$$

and by using the linearity

$$(\mathcal{A}(c_1),\dots,\mathcal{A}(c_n)) = (\mathcal{A}(b_1),\dots,\mathcal{A}(b_n))C = (b_1,\dots,b_n)AC.$$

Therefore, $C\tilde{A} = AC$, that is,

$$\tilde{A} = C^{-1}AC.$$

This motivates the following definition:

Definition 30. *The $n \times n$ matrices A and B are similar, if there exists an invertible matrix C such that $B = C^{-1}AC$. Notation: $A \sim B$.*

Example 27. *The 0 matrix is similar only to itself. Indeed, $C^{-1}0C = 0$ for every C. The identity matrix is similar only to itself. Indeed, $C^{-1}IC = I$ for every C.*

Example 28. *Let $\mathcal{A} : \mathbb{R}^3 \to \mathbb{R}^3$ be the (perpendicular) projection to the plane $x+y+\sqrt{2}z = 0$. If we choose a basis c_1, c_2, c_3 in such a way that c_1 is perpendicular to the plane and c_2, c_3 lie in the plane, then the matrix of $\mathcal{A}$ in this basis is $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. For instance, an appropriate choice is $c_1 = \begin{pmatrix} 1 \\ 1 \\ \sqrt{2} \end{pmatrix}$ (a normal vector of the plane), $c_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $c_3 = \begin{pmatrix} 1 \\ 1 \\ -\sqrt{2} \end{pmatrix}$. We can calculate the matrix of $\mathcal{A}$ according to the standard basis:*

$$C = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \end{pmatrix}, \quad C^{-1} = \begin{pmatrix} 1/4 & 1/4 & \sqrt{2}/4 \\ 1/2 & -1/2 & 0 \\ 1/4 & 1/4 & -\sqrt{2}/4 \end{pmatrix}.$$

Therefore,

$$A = C\tilde{A}C^{-1} = \begin{pmatrix} 3/4 & 1/4 & -\sqrt{2}/4 \\ -1/4 & 1/2 & -\sqrt{2}/4 \\ \sqrt{2}/4 & \sqrt{2}/4 & 1/2 \end{pmatrix}.$$

10. Eigenvalues and eigenvectors

Definition 31. *Let A be a square matrix. We say that λ is an eigenvalue of A if there exists some vector $u \neq 0$ such that $Au = \lambda u$. In this case u is an eigenvector corresponding to the eigenvalue λ.*

Remark 32. According to the definition $u = 0$ is not considered to be an eigenvector, but $\lambda = 0$ can be an eigenvalue if $Au = 0$ for some $u \neq 0$. That is, 0 is an eigenvalue if and only if A is singular.

Proposition 72. *λ is an eigenvalue of A if and only if* $\det(\lambda I - A) = 0$

Proof. λ is an eigenvalue of A if and only if the system $Au = \lambda u$ has a nontrivial solution, that is, if $(\lambda I - A)u = 0$ has a nontrivial solution. Such a solution exists if and only if $\det(\lambda I - A) = 0$. □

Definition 32. $D(\lambda) := \det(\lambda I - A)$ *is the characteristic polynomial of the square matrix A. Note that the main coefficient is 1, and if A is an $n \times n$ matrix, then the degree of the characteristic polynomial is n.*

Remark 33. The eigenvalues are exactly the roots of the characteristic polynomial. In particular, the number of eigenvalues of an $n \times n$ matrix is at most n. Over $\mathbb{C}$ every matrix has an eigenvalue, however, over $\mathbb{R}$ this might not be the case. For instance, the rotation around 0 by $\alpha \in (0, \pi)$ (which is a linear transformation) does not have any real eigenvalues. (Since, for any $u \neq 0$ the image of u is not parallel with u.) Note that the rotation by $\alpha = 0$ is the identical transformation, $\lambda = 1$ is the unique eigenvalue and all the nonzero vectors are eigenvectors. Also, for the rotation by $\alpha = \pi$ the unique eigenvalue is -1, and all the nonzero vectors are eigenvectors. (The image of any u is $-u$.)

Proposition 73. *If A and B are similar to each other, then their characteristic polynomials are the same: $D_A(\lambda) = D_B(\lambda)$.*

Proof. Let $B = C^{-1}AC$. Then:

$$D_B(\lambda) = \det(\lambda I - B) = \det(\lambda I - C^{-1}AC) = \det(C^{-1}(\lambda I - A)C) =$$
$$= (\det C^{-1}) \det(\lambda I - A) \det C = \det(\lambda I - A) = D_A(\lambda).$$

□

Corollary 74. *If $A \sim B$, then A and B have the same eigenvalues.*

Proposition 75. *The set of eigenvectors corresponding to the same eigenvalue λ together with 0 form a linear subspace.*

Proof. Let $V_\lambda \coloneqq \{u : Au = \lambda u\}$. Clearly, $0 \in V_\lambda$, we will show that V_λ is closed under addition and multiplication by a scalar.
If $u, v \in V_\lambda$, then $Au = \lambda u, Av = \lambda v$, thus

$$A(u+v) = Au + Av = \lambda u + \lambda v = \lambda(u+v).$$

Also, $A(\alpha u) = \alpha Au = \alpha(\lambda u) = \lambda(\alpha u)$. ☐

Definition 33. *If λ is an eigenvalue of A, then*

$$V_\lambda = \{u : \ Au = \lambda u\}$$

is the eigenspace corresponding to the eigenvalue λ.

Proposition 76. *Eigenvectors corresponding to different eigenvalues always form a linearly independent system.*

Proof. Let $\lambda_1, \ldots, \lambda_k$ be different eigenvalues and $u_1, \ldots, u_k$ be eigenvectors, respectively. That is, $Au_i = \lambda_i u_i$ for every $1 \le i \le k$. We prove that $u_1, \ldots, u_i$ form a linearly independent system by induction on i.

If $i = 1$, then u_1 forms a linearly independent system, since $u_1 \neq 0$.

Let us assume now that $2 \le i \le k$ and $u_1, \ldots, u_{i-1}$ are linearly independent. We shall prove that $u_1, \ldots, u_i$ are also linearly independent.

Assume that $\sum\limits_{j=1}^{i} \alpha_j u_j = 0$. Observe that

$$0 = A \cdot 0 = A(\sum_{j=1}^{i} \alpha_j u_j) = \sum_{j=1}^{i} \alpha_j A(u_j) = \sum_{j=1}^{i} \alpha_j \lambda_j u_j$$

and

$$0 = 0 \cdot \lambda_i = (\sum_{j=1}^{i} \alpha_j u_j)\lambda_i = \sum_{j=1}^{i} \alpha_j \lambda_i u_j.$$

By taking the difference:

$$0 = \sum_{j=1}^{i} \alpha_j(\lambda_j - \lambda_i)u_j = \sum_{j=1}^{i-1} \alpha_j(\lambda_j - \lambda_i)u_j.$$

According to the induction hypothesis, $u_1, \ldots, u_{i-1}$ are linearly independent, thus

$$\alpha_j(\lambda_j - \lambda_i) = 0$$

for every $1 \le j \le i-1$. Note that $\lambda_1, \ldots, \lambda_i$ are distinct, hence, $\alpha_j = 0$ for every $1 \le j \le i-1$. Finally, $\alpha_i u_i = 0$ implies that $\alpha_i = 0$ too, the system $u_1, \ldots, u_i$ is indeed linearly independent. ☐

Corollary 77. *If an $n \times n$ matrix A has n different eigenvalues, then the corresponding eigenvectors form a basis. In this basis the matrix of the corresponding linear transformation is diagonal:*

$$\begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \lambda_n \end{pmatrix}.$$

That is, A is similar to a diagonal matrix.

Example 29. *The characteristic polynomial of $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is $(\lambda - 1)^2$, thus it has one eigenvalue: $\lambda = 1$. Since $Iu = 1 \cdot u$ for every u, all the nonzero vectors are eigenvectors.*

Example 30. *The characteristic polynomial of $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ is also $(\lambda - 1)^2$, thus it has one eigenvalue: $\lambda = 1$. To get the corresponding eigenvectors we shall solve the following linear system.*

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 1 \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x \\ x + y \end{pmatrix} = 1 \begin{pmatrix} x \\ y \end{pmatrix}$$

That is, $x = 0$. So the eigenvectors are $\begin{pmatrix} 0 \\ y \end{pmatrix}$ (where $y \neq 0$).

Example 31. *The characteristic polynomial of $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is*

$$(\lambda - 1)^2 - 1 = \lambda^2 - 2\lambda = \lambda(\lambda - 2),$$

thus it has two eigenvalues: $\lambda_1 = 0$ and $\lambda_2 = 2$. The corresponding eigenvectors are

$$u_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ and } u_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

(Note that any nonzero scalar multiple of u_i is also an eigenvector for λ_i.)

Hence, $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$, since by taking the matrix C formed by the eigenvectors, $C = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, we have

$$C^{-1} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} C = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}.$$

Definition 34. *We say that a square matrix A is diagonalizable if there exists a diagonal matrix D such that $A \sim D$.*

According to Corollary 77 a square matrix having n distinct eigenvalues is diagonalizable. This statement can be strengthened as follows:

Corollary 78. *An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors. Namely, if U contains (as columns) the (linearly independent) eigenvectors $u_1, \ldots, u_n$, then*

$$U^{-1}AU = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & \lambda_n \end{pmatrix},$$

where $Au_i = \lambda_i u_i$.

Proof. Let us assume first that A has n linearly independent eigenvectors, and consider the matrix U containing these eigenvectors as columns. Note that U is invertible, since its columns form a linearly independent system. It suffices to prove that

$$AU = U \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & \lambda_n \end{pmatrix}.$$

The ith column on the left hand side is $Au_i = \lambda_i u_i$, since u_i is an eigenvector for the eigenvalue λ_i. Since the ith column on the right hand side is also $\lambda_i u_i$, the two products are indeed the same.

For the other direction, if

$$U^{-1}AU = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & \lambda_n \end{pmatrix},$$

then

$$AU = U \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & \lambda_n \end{pmatrix},$$

which shows that the columns of U are eigenvectors of A. They must form a linearly independent system, since U is invertible. $\square$

Let us define two types of multiplicities for eigenvalues.

Definition 35. *If λ_0 is an eigenvalue of a matrix A, then the algebraic multiplicity of λ_0 is the multiplicity of the root λ_0 in the characteristic polynomial $D_A(\lambda)$. That is, if $(\lambda - \lambda_0)^k \mid D_A(\lambda)$ but $(\lambda - \lambda_0)^{k+1} \nmid D_A(\lambda)$, then the algebraic multiplicity of λ_0 is k.*

Definition 36. *The geometric multiplicity of the eigenvalue λ_0 is the dimension of the eigenspace V_λ.*

We will show that the geometric multiplicity is always smaller or equal than the algebraic multiplicity. We start with some simple observations.

Proposition 79. *Let λ_i be an eigenvalue of an $n \times n$ matrix A. Then the geometric multiplicity of λ_i is $n - \mathrm{rk}(\lambda_i I - A)$.*

Proof. The eigenvectors corresponding to λ_i (and the vector 0) are the solutions of the linear system

$$Au = \lambda_i u$$

which is equivalent with

$$0 = (\lambda_i I - A)u.$$

The dimension of the subspace containing the solutions (which is the eigenspace V_λ) is the number of free variables, which is $n - \mathrm{rk}(\lambda_i I - A)$. $\square$

Proposition 80. *The algebraic multiplicity of λ_i is α_i if and only if*

$$D(\lambda_i) = 0,\ D'(\lambda_i) = 0,\ \ldots, D^{(\alpha_i - 1)}(\lambda_i) = 0,\ D^{(\alpha_i)}(\lambda_i) \neq 0.$$

Proof. This directly follows from the definition of algebraic multiplicity. $\square$

In order to be able to calculate the derivatives of the characteristic polynomial we formulate a more general lemma.

Lemma 81. *Let*

$$B(\lambda) = \begin{pmatrix} b_{11}(\lambda) & b_{12}(\lambda) & \ldots & b_{1n}(\lambda) \\ b_{21}(\lambda) & b_{22}(\lambda) & \ldots & b_{2n}(\lambda) \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1}(\lambda) & b_{n2}(\lambda) & \ldots & b_{nn}(\lambda) \end{pmatrix},$$

where each b_{ij} is a differentiable function. Then

$$(\det B(\lambda))' = \det \begin{pmatrix} b'_{11}(\lambda) & b'_{12}(\lambda) & \ldots & b'_{1n}(\lambda) \\ b_{21}(\lambda) & b_{22}(\lambda) & \ldots & b_{2n}(\lambda) \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1}(\lambda) & b_{n2}(\lambda) & \ldots & b_{nn}(\lambda) \end{pmatrix} +$$

$$+ \det \begin{pmatrix} b_{11}(\lambda) & b_{12}(\lambda) & \ldots & b_{1n}(\lambda) \\ b'_{21}(\lambda) & b'_{22}(\lambda) & \ldots & b'_{2n}(\lambda) \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1}(\lambda) & b_{n2}(\lambda) & \ldots & b_{nn}(\lambda) \end{pmatrix} + \cdots + \det \begin{pmatrix} b_{11}(\lambda) & b_{12}(\lambda) & \ldots & b_{1n}(\lambda) \\ b_{21}(\lambda) & b_{22}(\lambda) & \ldots & b_{2n}(\lambda) \\ \vdots & \vdots & \ddots & \vdots \\ b'_{n1}(\lambda) & b'_{n2}(\lambda) & \ldots & b'_{nn}(\lambda) \end{pmatrix}.$$

Proof. According to the definition of the determinant,

$$\det B(\lambda) = \sum_{\pi:\ \text{permutation}} (-1)^{I(\pi)} b_{1\pi(1)}(\lambda) b_{2\pi(2)}(\lambda) \dots b_{n\pi(n)}(\lambda),$$

where $I(\pi)$ denotes the number of inversions in the permutation π.

Now, by taking the derivative:

$$\begin{aligned}
(\det B(\lambda))' &= \sum_{\pi} (-1)^{I(\pi)} (b_{1\pi(1)}(\lambda) b_{2\pi(2)}(\lambda) \dots b_{n\pi(n)}(\lambda))' = \\
&= \sum_{\pi} (-1)^{I(\pi)} [(b'_{1\pi(1)}(\lambda) b_{2\pi(2)}(\lambda) \dots b_{n\pi(n)}(\lambda)) + (b_{1\pi(1)}(\lambda) b'_{2\pi(2)}(\lambda) \dots b_{n\pi(n)}(\lambda)) + \dots + \\
&\quad + (b_{1\pi(1)}(\lambda) b_{2\pi(2)}(\lambda) \dots b'_{n\pi(n)}(\lambda))] = \sum_{\pi} (-1)^{I(\pi)} (b'_{1\pi(1)}(\lambda) b_{2\pi(2)}(\lambda) \dots b_{n\pi(n)}(\lambda)) + \\
&\quad + \sum_{\pi} (-1)^{I(\pi)} (b_{1\pi(1)}(\lambda) b'_{2\pi(2)}(\lambda) \dots b_{n\pi(n)}(\lambda)) + \dots + \sum_{\pi} (-1)^{I(\pi)} (b_{1\pi(1)}(\lambda) b_{2\pi(2)}(\lambda) \dots b'_{n\pi(n)}(\lambda)),
\end{aligned}$$

which is exactly the sum of the n determinants, as claimed. $\square$

Now, let us turn our attention on the characteristic polynomial and use Lemma 81 to calculate its derivative:

$$D(\lambda) = \det \begin{pmatrix} \lambda - a_{11} & -a_{12} & \dots & -a_{1n} \\ -a_{21} & \lambda - a_{22} & \dots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \dots & \lambda - a_{nn} \end{pmatrix},$$

thus

$$\begin{aligned}
D'(\lambda) = \det \begin{pmatrix} 1 & 0 & \dots & 0 \\ -a_{21} & \lambda - a_{22} & \dots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \dots & \lambda - a_{nn} \end{pmatrix} + \det \begin{pmatrix} \lambda - a_{11} & -a_{12} & \dots & -a_{1n} \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \dots & \lambda - a_{nn} \end{pmatrix} + \\
+ \dots + \det \begin{pmatrix} \lambda - a_{11} & -a_{12} & \dots & -a_{1n} \\ -a_{21} & \lambda - a_{22} & \dots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}.
\end{aligned}$$

Here, after expanding the ith determinant along its ith row we can see that it is equal to the determinant of the matrix we get after deleting the ith row and the ith column of $\lambda I - A$. So, let us denote by $D_i(\lambda)$ the determinant of the matrix obtained by deleting the ith row and ith column of $\lambda I - A$. Then $D'(\lambda) = \sum_{i=1}^{n} D_i(\lambda)$.

We shall continue in a similar fashion. In general, for $1 \le i_1 < i_2 < \dots < i_t \le n$ let $D_{i_1,\dots,i_t}(\lambda)$ denote the determinant of the matrix obtained by deleting the rows $i_1, \dots, i_t$

and the columns $i_1, \dots, i_t$ from $\lambda I - A$. Then

$$D''(\lambda) = \sum_{i_1=1}^{n} D'_{i_1}(\lambda) = \sum_{i_1=1}^{n} \sum_{\substack{1 \le i_2 \le n, \\ i_1 \ne i_2}} D_{i_1,i_2}(\lambda) = 2 \sum_{1 \le i_1 < i_2 \le n} D_{i_1,i_2}(\lambda).$$

In general,

$$D^{(t)}(\lambda) = t! \sum_{1 \le i_1 < \dots < i_t \le n} D_{i_1,\dots,i_t}(\lambda).$$

If the algebraic multiplicity of λ_i is α_i, then by Proposition 80 we have $D^{(\alpha_i)}(\lambda_i) \ne 0$, therefore, there exist $1 \le i_1 < i_2 < \dots < i_{\alpha_i} \le n$ such that $D_{i_1,i_2,\dots,i_{\alpha_i}}(\lambda_i) \ne 0$. This means that $\lambda_i I - A$ has an $(n - \alpha_i) \times (n - \alpha_i)$ submatrix with nonzero determinant, hence $\mathrm{rk}(\lambda_i I - A) \ge n - \alpha_i$. Let us formulate the obtained result in the following statement.

Proposition 82. *If the algebraic multiplicity of λ_i is α_i, then $\mathrm{rk}(\lambda_i I - A) \ge n - \alpha_i$.*

Corollary 83. *The geometric multiplicity of an eigenvalue is always at most its algebraic multiplicity.*

Proof. This follows from Propositions 79 and 82. Indeed, the conclusion of Proposition 82 can be rearranged as $\alpha_i \ge n - \mathrm{rk}(\lambda_i I - A)$, where the right hand side is the geometric multiplicity according to Proposition 79. □

Corollary 84. *Let A be an $n \times n$ square matrix. Over $\mathbb{C}$ its characteristic polynomial factorizes as*

$$D(\lambda) = \prod_{i=1}^{k} (\lambda - \lambda_i)^{\alpha_i},$$

where $\lambda_1, \dots, \lambda_k$ are distinct. The matrix A is diagonalizable if and only if every eigenvalue λ_i has geometric multiplicity α_i.

Proof. According to Corollary 78 the matrix A is diagonalizable if and only if we can choose n linearly independent eigenvectors. The eigenvectors corresponding to λ_i are contained in the eigenspace V_{λ_i}, so from the eigenvectors corresponding to λ_i at most $\dim V_{\lambda_i}$ can be chosen. Note that $\dim V_{\lambda_i}$ is the geometric multiplicity of λ_i and according to Corollary 83 we have $\dim V_{\lambda_i} \le \alpha_i$. Hence,

$$\sum_{i=1}^{k} \dim V_{\lambda_i} \le \sum_{i=1}^{k} \alpha_i = n,$$

as the degree of the characteristic polynomial $D(\lambda) = \prod_{i=1}^{k} (\lambda - \lambda_i)^{\alpha_i}$ is n. Therefore, to be able to choose n linearly independent eigenvectors we must have $\dim V_{\lambda_i} = \alpha_i$ for every $1 \le i \le k$, as claimed.

Finally, assume that this holds, that is, the geometric multiplicity of each eigenvalue is equal to its algebraic multiplicity. Let $u_{i,1}, u_{i,2}, \dots, u_{i,\alpha_i}$ be linearly independent eigenvectors for λ_i (for every $1 \le i \le k$). As $\sum_{i=1}^{k} \alpha_i = n$ it suffices to show that these together,

$$u_{1,1}, \dots, u_{1,\alpha_1}, \dots, u_{k,1}, \dots, u_{k,\alpha_k}$$

still form a linearly independent system. Let us take a linear combination:

$$\sum_{\substack{1 \le i \le k,\\ 1 \le j \le \alpha_i}} \mu_{i,j} u_{i,j} = 0.$$

Then

$$0 = \sum_{i=1}^{k} \left(\sum_{j=1}^{\alpha_i} \mu_{i,j} u_{i,j} \right),$$

where

$$w_i := \sum_{j=1}^{\alpha_i} \mu_{i,j} u_{i,j}$$

lies in the eigenspace V_{λ_i}, since w_i is a linear combination of eigenvectors corresponding to λ_i. We claim that $w_i = 0$ for every $1 \le i \le k$. Let I_0 be the set of those indices for which $w_i \ne 0$. Then $\sum_{i \in I_0} w_i = 0$. Note that each w_i with $i \in I_0$ is an eigenvector for λ_i, since in this case w_i is a nonzero element of the eigenspace V_{λ_i}. However, Proposition 76 yields that eigenvectors corresponding to different eigenvalues always form a linearly independent system, thus I_0 must be empty: $I_0 = \varnothing$. In other words, for every $1 \le i \le k$ we have $w_i = 0$. Therefore, for every i we have

$$w_i = \sum_{j=1}^{\alpha_i} \mu_{i,j} u_{i,j} = 0,$$

however, $u_{i,1}, \dots, u_{i,\alpha_i}$ were chosen to be linearly independent, implying that $\mu_{i,j} = 0$ for every $1 \le j \le \alpha_i$. That is, a linear combination

$$\sum_{i=1}^{k} \left(\sum_{j=1}^{\alpha_i} \mu_{i,j} u_{i,j} \right)$$

can be 0 only if all the $\mu_{i,j}$ coefficients are equal to 0, so the chosen eigenvectors indeed form a linearly independent system. □

We continue with further properties of the eigenvalues and eigenvectors.

Proposition 85. *If λ is an eigenvalue of A, then λ is an eigenvalue of A^T too.*

Proof. According to Proposition 72 a number λ is an eigenvalue of A if and only if $\det(\lambda I - A) = 0$. However, $(\lambda I - A^T) = (\lambda I - A)^T$, thus

$$\det(\lambda I - A^T) = 0 \iff \det(\lambda I - A) = 0,$$

and the claim follows. □

Assume now that a matrix A is diagonalizable:

$$U^{-1}AU = D = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \lambda_n \end{pmatrix}.$$

Let us write $V^T = U^{-1}$, then the equation can be rearranged as

$$V^T A = DV^T.$$

Hence, for the ith row of V^T we have $v_i^T A = \lambda_i v_i^T$, which means that v_i^T is a left eigenvector for the eigenvalue λ_i. Note that this is also equivalent to saying that v_i is a right eigenvector (for λ_i) of the matrix A^T.

To sum up, $V^T AU = D$ and $V^T U = I$. Then we can express A as

$$A = UDV^T = \sum_{i=1}^{n} \lambda_i u_i v_i^T,$$

which is called the *spectral decomposition* of A. Here u_i is a right, v_i^T is a left eigenvector for the eigenvalue λ_i and the $u_1, \dots, u_n, v_1, \dots, v_n$ vectors form a biorthogonal system.

Proposition 86. *If A is self-adjoint, then the eigenvalues of A are real.*

Proof. If λ is an eigenvalue, then for some vector $u \neq 0$ we have $Au = \lambda u$. Then

$$u^* Au = u^*(\lambda u) = \lambda(u^* u),$$

where $u^* u = \sum |u_i|^2$ is a positive real number, since $u \neq 0$. Moreover, $u^* Au$ is a real number, since

$$(u^* Au)^* = u^* A^* u^{**} = u^* Au.$$

As $\lambda(u^* u) \in \mathbb{R}$ and $u^* u \in \mathbb{R}_{>0}$, we conclude that $\lambda \in \mathbb{R}$, as we claimed. □

Corollary 87. *If A is a symmetric real matrix, then the eigenvalues of A are real.*

Proposition 88. *If A is skew-Hermitian, then the eigenvalues of A are purely imaginary.*

Proof. We use again that $u^* Au = \lambda(u^* u)$, where $u^* u$ is a positive real number. However, now

$$(u^* Au)^* = u^* A^* u^{**} = -u^* Au,$$

which means that $u^* Au$ is purely imaginary. (Indeed, $\overline{a + bi} = a - bi$, which is equal to $-(a + bi) = -a - bi$ if and only if the real part is 0, that is, $a = 0$.) From these the claim follows. □

Proposition 89. *If A is unitary, then the eigenvalues of A have absolute value 1.*

Proof. If λ is an eigenvalue, then $Au = \lambda u$ for some $u \neq 0$. Let us consider

$$(Au)^* Au = u^*(A^*A)u = u^* I u = u^* u.$$

Also,

$$(Au)^* Au = (\lambda u)^*(\lambda u) = \overline{\lambda}\lambda u^* u = |\lambda|^2(u^* u).$$

Since $u \neq 0$, we can conclude that $|\lambda| = 1$. □

Corollary 90. *If A is an orthogonal real matrix, then the real eigenvalues of A can only be ± 1.*

Proposition 91. *If P is a projection, then all the eigenvalues of P are equal to 0 or 1.*

Proof. Let us assume that λ is an eigenvalue for P:

$$Pv = \lambda v$$

for some $v \neq 0$. Then

$$P^2 v = P(Pv) = P(\lambda v) = \lambda P v = \lambda^2 v,$$

however, $P = P^2$ implies that $\lambda = \lambda^2$. Hence, $\lambda \in \{0, 1\}$. □

Let us consider a few examples.

Example 32. *Let $A = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$. The characteristic polynomial of A is*

$$D_A(\lambda) = (\lambda - 3)^2,$$

thus A has one eigenvalue: $\lambda_1 = 3$ with algebraic multiplicity $\alpha_1 = 2$. Since

$$\mathrm{rk}(3I - A) = 0 = 2 - \alpha_1,$$

the geometric multiplicity of $\lambda_1 = 3$ is 2. Hence, the matrix is diagonalizable, which is trivial of course, since A itself is a diagonal matrix.

Example 33. *Let $B = \begin{pmatrix} 3 & 0 \\ 1 & 3 \end{pmatrix}$. The characteristic polynomial of B is*

$$D_B(\lambda) = (\lambda - 3)^2,$$

thus B also has one eigenvalue: $\lambda_1 = 3$ with algebraic multiplicity $\alpha_1 = 2$. Since

$$\mathrm{rk}(3I - B) = \mathrm{rk}\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} = 1 > 2 - \alpha_1 = 0,$$

the geometric multiplicity of $\lambda_1 = 3$ is 1 and A is not diagonalizable. By solving the linear system $(3I - B)v = 0$ we get that the eigenvectors are of the form $\begin{pmatrix} 0 \\ y \end{pmatrix}$, where $y \neq 0$.

Example 34. *Let*

$$C = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 1 & t & 2 \end{pmatrix}.$$

The characteristic polynomial of C is

$$D_C(\lambda) = (\lambda - 2)^2(\lambda - 3)$$

(for every choice of the parameter t), thus C has two different eigenvalues:

$$\lambda_1 = 2 \textit{ with algebraic multiplicity } \alpha_1 = 2$$

and

$$\lambda_2 = 3 \textit{ with algebraic multiplicity } \alpha_2 = 1.$$

Note that

$$\mathrm{rk}(3I - C) = \mathrm{rk}\begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & -t & 1 \end{pmatrix} = 2 = 3 - 1 = 3 - \alpha_2.$$

The rank of $2I - C$ depends on the parameter t, as follows.

$$\mathrm{rk}(2I - C) = \mathrm{rk}\begin{pmatrix} 0 & 0 & 0 \\ -1 & -1 & 0 \\ -1 & -t & 0 \end{pmatrix} = \begin{cases} 1 = 3 - \alpha_1, \textit{ if } t = 1 \\ \\ 2, \textit{ if } t \neq 1 \end{cases}$$

Therefore, the matrix is diagonalizable if $t = 1$, and not diagonalizable otherwise.

The eigenvectors for $\lambda_2 = 3$ are of the form $\begin{pmatrix} 0 \\ \alpha \\ \alpha t \end{pmatrix}$ (where $\alpha \neq 0$).

If $t = 1$, then the eigenvectors for $\lambda_1 = 2$ are of the form $\begin{pmatrix} x \\ -x \\ y \end{pmatrix}$ (where $x = y = 0$ is excluded). For instance, $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ are two linearly independent eigenvectors.

(If $t \neq 1$, then the eigenvectors for $\lambda_1 = 2$ are of the form $\begin{pmatrix} 0 \\ 0 \\ x \end{pmatrix}$.)

When $t = 1$, let us consider

$$U = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix},$$

where the 1st and 3rd columns are two linearly independent eigenvectors for $\lambda_1 = 2$ and the 2nd column is an eigenvector for $\lambda_2 = 3$. Then

$$U^{-1}CU = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Now, we consider the case when $A = P$ is a projection: $P^2 = P$. Let a minimal rank-one decomposition of P be $P = UV^T$, where U is an $n \times r$ matrix and V^T is an $r \times n$ matrix (where $r = \operatorname{rk} P$). Note that the columns of U form a linearly independent system and the rows of V^T also form a linearly independent system.

We claim that the columns of U are eigenvectors for the eigenvalue 1 for the matrix P. Indeed,

$$PUV^T = P^2 = P = UV^T$$

implies that

$$(PU - U)V^T = 0,$$

and by using that the rows of V^T are linearly independent Lemma 26 yields that $PU = U$. This means that all the columns of U are eigenvectors of P (for the eigenvalue 1).

Similarly, the rows of V^T are left eigenvectors for the eigenvalue 1 for P, since

$$UV^T P = P^2 = P = UV^T$$

implies that

$$V^T P = V^T.$$

Note that $I - P$ is also a projection, let $I - P = WZ^T$ be a minimal rank-one decomposition of it. The columns of W are eigenvectors of $I - P$ for the eigenvalue 1, thus they are also eigenvalues of P corresponding to the eigenvalue 0. (If $(I - P)u = u$, then $Pu = 0$.)

Analogously, the rows of Z^T are left eigenvectors of P corresponding to the eigenvalue 0.

Since $P = UV^T$, we have

$$P = \begin{pmatrix} U & W \end{pmatrix} \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V^T \\ Z^T \end{pmatrix},$$

where

$$\begin{pmatrix} U & W \end{pmatrix} \begin{pmatrix} V^T \\ Z^T \end{pmatrix} = UV^T + WZ^T = P + (I - P) = I.$$

That is, P is diagonalizable.

The case when P is an Hermitian projection ($P^2 = P$ and $P^* = P$) can be studied similarly. In this case we start with an Hermitian minimal rank-one decomposition $P =$

UU^*, where $U^*U = I_r$. Then, by taking an Hermitian minimal rank-one decomposition of $I - P = WW^*$ we obtain

$$P = \begin{pmatrix} U & W \end{pmatrix} \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} U^* \\ W^* \end{pmatrix},$$

where

$$\begin{pmatrix} U & W \end{pmatrix} \begin{pmatrix} U^* \\ W^* \end{pmatrix} = UU^* + WW^* = P + (I - P) = I.$$

That is, an Hermitian projection can be diagonalized by a unitary matrix.

Definition 37. *The matrix A can be diagonalized by a unitary matrix if there exists some unitary matrix U such that*

$$U^*AU = U^{-1}AU$$

is diagonal. In this case we also say that A is unitary diagonalizable.

We will classify which matrices A are unitary diagonalizable. However, before doing so we show that it is *always* possible to find a unitary matrix U such that U^*AU is a *triangular* matrix.

Theorem 92 (Schur's Theorem)**.** *For every square matrix A there exists a unitary matrix U such that U^*AU is an upper triangular matrix.*

Proof. Let us take an eigenvalue λ_1 of A, then $Av_1 = \lambda_1 v_1$ for some eigenvector v_1 of unit length: $v_1^* v_1 = 1$. (Note that nonzero scalar multiples of an eigenvector are also eigenvectors, thus we may take an eigenvector of unit length.)

Let us extend the one-element unitary system $\{v_1\}$ to a complete unitary system $\{v_1, y_2, \ldots, y_n\}$. Let

$$X_1 = (v_1, y_2, \ldots, y_n),$$

that is, X_1 is a square matrix in which the columns are $v_1, y_2, \ldots, y_n$. Note that

$$X_1^* X_1 = I.$$

Therefore,

$$X_1^* A v_1 = X_1^* \lambda_1 v_1 = \lambda_1 X_1^* v_1 = \lambda_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

since $v_1^* v_1 = 1$ and $y_i^* v_1 = 0$ for every $2 \le i \le n$. Hence:

$$X_1^* A X_1 = \begin{pmatrix} \lambda_1 & * \\ 0 & A_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & * & \ldots & * \\ 0 & * & \ldots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \ldots & * \end{pmatrix},$$

where the block A_2 is an $(n-1) \times (n-1)$ matrix.

Now, we repeat the process with A_2 and obtain some $(n-1)\times(n-1)$ unitary matrix Y_2 such that

$$Y_2^* A_2 Y_2 = \begin{pmatrix} \lambda_2 & * \\ 0 & A_3 \end{pmatrix}.$$

By defining

$$X_2 = \begin{pmatrix} 1 & 0 \\ 0 & Y_2 \end{pmatrix}$$

we get an $n \times n$ unitary matrix such that

$$X_2^* X_1^* A X_1 X_2 = \begin{pmatrix} \lambda_1 & * & * & \dots & * \\ 0 & \lambda_2 & * & \dots & * \\ 0 & 0 & * & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & * & \dots & * \end{pmatrix}.$$

By continuing in this fashion, we get a sequence $X_1, X_2, \dots, X_n$ of $n\times n$ unitary matrices such that by setting $U = X_1 X_2 \dots X_n$ the matrix U^*AU is upper triangular and U is unitary, since it is a product of unitary matrices. Hence, the statement is proved. □

According to Schur's theorem for every square matrix A there exists a unitary U such that U^*AU is a triangular matrix. Now, we will characterise those matrices for which U^*AU can be diagonal, that is, the matrices that are unitary diagonalizable.

Definition 38. *A square matrix A is normal if $A^*A = AA^*$.*

Remark 34. Self-adjoint matrices are normal. Indeed, if $A = A^*$, then clearly,

$$AA^* = A^2 = A^*A.$$

Specially, symmetric real matrices are also normal (since they are self-adjoint.)
Also, unitary matrices are normal, since for a unitary matrix U we have

$$UU^* = I = U^*U.$$

Theorem 93. *A square matrix A is unitary diagonalizable if and only if A is normal.*

Proof. Let us assume first that A is unitary diagonalizable: there exists a unitary U such that $U^*AU = D$ is diagonal. Note that we can express A as

$$A = UDU^*.$$

Since $DD^* = D^*D$ we have

$$AA^* = (UDU^*)(UDU^*)^* = UD\underbrace{U^*U}_{I}D^*U^* = UDD^*U^* =$$

$$= UD^*DU^* = UD^*\underbrace{U^*U}_{I}DU^* = A^*A,$$

so A is indeed normal.

Now let us assume that A is normal:

$$AA^* = A^*A.$$

According to Schur's Theorem there exists some unitary U such that $B = U^*AU$ is an upper triangular matrix:

$$\begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ 0 & b_{22} & \dots & b_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & b_{nn} \end{pmatrix}.$$

Note the B is also normal, since:

$$BB^* = U^*AUU^*A^*U = U^*AA^*U,$$

$$B^*B = U^*A^*UU^*AU = U^*A^*AU,$$

and $AA^* = A^*A$.

Observe that for every $1 \le j \le n-1$

$$(BB^*)_{jj} = |b_{jj}|^2 + \cdots + |b_{jn}|^2$$

and

$$(B^*B)_{jj} = |b_{1j}|^2 + \cdots + |b_{jj}|^2.$$

Since B is normal we obtain from the choice $j = 1$ that

$$|b_{11}|^2 + \cdots + |b_{1n}|^2 = |b_{11}|^2,$$

that is, $b_{12} = \cdots = b_{1n} = 0$.

Now, from the choice $j = 2$ we get that

$$|b_{22}|^2 + \cdots + |b_{2n}|^2 = |b_{12}|^2 + |b_{22}|^2,$$

and by using that $b_{12} = 0$ we conclude that

$$b_{23} = \cdots = b_{2n} = 0.$$

Continuing in this fashion, at a general $1 \le j \le n-1$ from the equation

$$|b_{jj}|^2 + \cdots + |b_{jn}|^2 = |b_{1j}|^2 + \cdots + |b_{jj}|^2$$

we get that

$$b_{j,j+1} = \cdots = b_{jn} = 0$$

for every $1 \le j \le n-1$, since

$$b_{1j} = \cdots = b_{j-1,j} = 0$$

is already determined at this point.

Therefore,

$$b_{j,j+1} = \cdots = b_{jn} = 0$$

for every $1 \le j \le n-1$, meaning that B is a diagonal matrix, which completes the proof.

□

11. Singular value decomposition

Definition 39. *Let A be a $k \times n$ matrix. The singular values of A are $\sigma_i = \sqrt{\lambda_i}$ (for $1 \le i \le n$), where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of $A^* A$.*

The above definition is meaningful, since the eigenvalues of $A^* A$ are indeed nonnegative real numbers according to the following proposition:

Proposition 94. *The eigenvalues of $A^* A$ are nonnegative real numbers.*

Proof. Let λ be an eigenvalue of $A^* A$, then for some $x \ne 0$ we have

$$A^* A x = \lambda x.$$

After multiplying by x^* we obtain that

$$x^* A^* A x = \lambda x^* x.$$

Here the left hand side can be written as $x^* A^* A x = (Ax)^*(Ax)$, which is a nonnegative real number. On the right hand side $x^* x$ is a positive real number, since $x \ne 0$. Therefore,

$$\lambda = \frac{(Ax)^*(Ax)}{x^* x}$$

is a nonnegative real number. $\square$

Remark 35. According to the definition the singular values are nonnegative real numbers.

Example 35. *Let*

$$A = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Then

$$A^* A = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 1,$$

hence the unique singular value of A is $\sigma_1 = 1$.

Example 36. *Let $B = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}$. Then*

$$B^* B = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix},$$

hence the singular values of B are $\sigma_1 = 1, \sigma_2 = \cdots = \sigma_n = 0$.

Example 37. *Let*

$$C = \begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & a_n \end{pmatrix}$$

be a diagonal matrix. Then

$$C^*C = \begin{pmatrix} |a_1|^2 & 0 & \dots & 0 \\ 0 & |a_2|^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & |a_n|^2 \end{pmatrix},$$

hence $\sigma_i = |a_i|$ *(for* $1 \le i \le n$*).*

The last example can be generalized as follows.

Proposition 95. *If the square matrix A is unitary diagonalizable, then the singular values are the absolute values of the eigenvalues:* $\sigma_i = |\lambda_i|$.

Proof. Let $A = UDU^{-1}$, where D is diagonal and U is unitary. Then

$$A^*A = UD^*U^{-1}UDU^{-1} = UD^*DU^{-1}.$$

If the eigenvalues of A (and of D) are $\lambda_1, \dots, \lambda_n$, then the eigenvalues of D^*D are $|\lambda_1|^2, \dots, |\lambda_n|^2$, which proves our claim. ◻

Proposition 96. *Let us assume that $\lambda \neq 0$ is an eigenvalue of A^*A and $v_1, v_2, \dots, v_k$ are linearly independent eigenvectors (corresponding to λ).*

Then λ is also an eigenvalue of AA^*, and $Av_1, \dots, Av_k$ are linearly independent eigenvectors (corresponding to λ).

Proof. We know that

$$A^*Av_i = \lambda v_i.$$

Then

$$AA^*(Av_i) = A(A^*Av_i) = A(\lambda v_i) = \lambda(Av_i).$$

Hence, λ is also an eigenvalue of AA^* and $Av_1, \dots, Av_k$ are all eigenvectors, assuming that $Av_i \neq 0$ for every $1 \le i \le k$. We will prove that

$$Av_1, \dots, Av_k$$

form a linearly independent system, which also shows that $Av_i \neq 0$. Let us multiply all these k vectors by A^*: we obtain the system

$$A^*Av_1, \dots, A^*Av_k.$$

However, these vectors are

$$\lambda v_1, \dots, \lambda v_k,$$

thus they form a linearly independent system, as $\lambda \neq 0$. This implies that $Av_1, \dots, Av_k$ must also form a linearly independent system, since from a linearly dependent system we would have obtained another linearly dependent system.

(Indeed, if

$$\alpha_1 A v_1 + \dots + \alpha_k A v_k = 0,$$

then

$$\alpha_1 A^* A v_1 + \dots + \alpha_k A^* A v_k = 0$$

holds.) □

Corollary 97. *The (geometric) multiplicity of an eigenvalue $\lambda \neq 0$ is the same for A^*A and AA^*.*

Remark 36. Note that A^*A is self-adjoint (since $(A^*A)^* = A^*A^{**} = A^*A$), thus it is normal, and consequently it is (unitary) diagonalizable. Therefore, the algebraic multiplicity of each eigenvalue λ coincides with the geometric multiplicity of λ.

Theorem 98 (Singular Value Decomposition)**.** *Let A be a $k \times n$ matrix, where $k \geq n$. Then there exists a $k \times k$ unitary matrix U and an $n \times n$ unitary matrix V such that $A = UDV^*$, where*

$$D = \begin{pmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \sigma_n \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}.$$

Proof. Note that A^*A is normal (in fact it is self-adjoint), thus it is unitary diagonalizable. Let

$$V^* A^* A V = \begin{pmatrix} \sigma_1^2 & 0 & \dots & 0 \\ 0 & \sigma_2^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \sigma_n^2 \end{pmatrix},$$

where V is unitary and

$$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0 = \sigma_{r+1} = \dots = \sigma_n.$$

(Note that $r = \mathrm{rk}(A^*A)$.)

Let

$$F = AV = (f_1, f_2, \dots, f_n),$$

that is, the columns of F are $f_1, \dots, f_n$. Since

$$V^* A^* A V = F^* F$$

we have

$$f_{r+1} = \cdots = f_n = 0$$

and

$$f_i^* f_j = \delta_{ij}\sigma_i^2 \text{ for } i, j \le r,$$

where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ \\ 0 & \text{if } i \ne j \end{cases}$$

is the Kronecker-symbol.

Let us define $u_i = \frac{1}{\sigma_i} f_i$ for $1 \le i \le r$. Then $u_i^* u_j = \delta_{ij}$ for $i, j \le r$, that is, $u_1, \ldots, u_r$ is a unitary system.

Let us extend this unitary system to a complete unitary system: $u_1, \ldots, u_k$ and let $U = (u_1, \ldots, u_k)$ be the matrix where the columns are the u_i vectors. Note that U is unitary and $F = UD$. Indeed, the jth column of UD is $\sigma_j u_j = f_j$ for $j \le r$ and $\sigma_j u_j = 0 u_j = 0 = f_j$ for $r < j \le n$.

Hence, $AV = F = UD$, thus $A = UDV^*$. □

Remark 37. UDV^* is called the Singular Value Decomposition (SVD) of the matrix A.

Remark 38. Note that $A^* = VD^*U^* = VD^TU^*$. This shows that in the case $k \le n$ an analogous statement holds.

Remark 39. From the equation $AV = UD$ we get some connection between the u_i and v_i vectors. Namely,

$$Av_i = \sigma_i u_i \text{ (for } 1 \le i \le n)$$

and

$$A^* u_i = \sigma_i v_i \text{ (for } 1 \le i \le n).$$

Remark 40. With the help of the singular value decomposition we can find the Moore-Penrose pseudoinverse of matrices. Namely, if $A = UDV^*$, then $A^+ = VD^+U^*$, where D^+ is as follows. To obtain D^+ we transpose D and take the reciprocals of the nonzero singular values (the zero singular values remain unchanged).

12. Matrix polynomials

Let us recall that according to Definition 1 for a square matrix A and a polynomial $p(x) = a_n x^n + \dots + a_1 x + a_0$ we define

$$p(A) := a_n A^n + \dots + a_1 A + a_0 I.$$

Remark 41. We will also work with matrices A whose entries are polynomials (of degree, say, at most d) of a variable λ. Such a matrix A can always be expressed as

$$A = C_0 + C_1 \lambda + \dots + C_d \lambda^d,$$

where $C_0, C_1, \dots, C_d$ are coefficient *matrices*. Namely, to get C_i we shall collect all the coefficients of λ^i arising at the entries. For instance:

$$\underbrace{\begin{pmatrix} \lambda^2 + 2\lambda + 3 & \lambda - 1 \\ 2\lambda^2 + 3\lambda & 1 \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} 3 & -1 \\ 0 & 1 \end{pmatrix}}_{C_0} + \underbrace{\begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix}}_{C_1} \lambda + \underbrace{\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}}_{C_2} \lambda^2.$$

Theorem 99 (Cayley-Hamilton theorem)**.** *Let A be a square matrix and $D(x)$ its characteristic polynomial. Then $D(A) = 0$.*

Example 38. *Let*

$$A = \begin{pmatrix} a & 0 & \dots & 0 \\ 0 & a & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & a \end{pmatrix}.$$

Then $D(x) = \det(xI - A) = (x-a)^n$. In this case the statement $(A - aI)^n = 0$ is indeed true, in fact $A - aI$ itself is the 0 matrix.

Example 39. *Let*

$$A = \begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & a_n \end{pmatrix}.$$

Then $D(x) = \det(xI - A) = (x - a_1)(x - a_2)\dots(x - a_n)$. In this case the statement is

$$(A - a_1 I)(A - a_2 I)\dots(A - a_n I) = 0.$$

All the n matrices appearing on the left-hand side of the equation are diagonal, and in the ith one the ith diagonal entry is 0, thus the total product is the 0 matrix, as claimed.

1st proof (of the Cayley-Hamilton theorem). We have seen earlier that

$$(\lambda I - A)\operatorname{adj}(\lambda I - A) = D(\lambda)I = (\operatorname{adj}(\lambda I - A))(\lambda I - A).$$

In $\operatorname{adj}(\lambda I - A)$ each entry is a polynomial of λ with degree at most $n-1$, therefore we can express this matrix as

$$\operatorname{adj}(\lambda I - A) = C_0 + C_1\lambda + \cdots + C_{n-1}\lambda^{n-1}.$$

Hence, the equation

$$(\lambda I - A)(C_0 + C_1\lambda + \cdots + C_{n-1}\lambda^{n-1}) = (C_0 + C_1\lambda + \cdots + C_{n-1}\lambda^{n-1})(\lambda I - A)$$

holds for every λ, thus the corresponding coefficients are equal to each other.

From the constant term we get

$$-AC_0 = -C_0A,$$

so A and C_0 commute with each other.

From the linear term (coefficient of λ):

$$C_0 - AC_1 = C_0 - C_1A,$$

thus A also commutes with C_1.

In general, from the coefficient of λ^i (where $1 \le i \le n-1$) we get

$$C_{i-1} - AC_i = C_{i-1} - C_iA,$$

thus A and C_i commute with each other.

Therefore, A commutes with all of the coefficient matrices $C_0, C_1, \ldots, C_{n-1}$ and so we can plug the matrix A in the polynomial equation

$$(\lambda I - A)(C_0 + C_1\lambda + \cdots + C_{n-1}\lambda^{n-1}) = D(\lambda)I.$$

This yields that

$$(AI - A)(C_0 + C_1A + \cdots + C_{n-1}A^{n-1}) = D(A)I,$$

which shows that

$$D(A) = D(A)I = (A - A)(C_0 + C_1A + \cdots + C_{n-1}A^{n-1}) = 0,$$

as claimed. □

We also present the proof in a slightly differently formulated way.

2nd proof (of the Cayley-Hamilton theorem). We will use again the equation

$$\operatorname{adj}(\lambda I - A) = C_0 + C_1\lambda + \cdots + C_{n-1}\lambda^{n-1},$$

and let

$$D(\lambda) = d_0 + d_1\lambda + \cdots + d_{n-1}\lambda^{n-1} + \lambda^n$$

be the characteristic polynomial of A.

The equation

$$(\lambda I - A)(C_0 + C_1\lambda + \cdots + C_{n-1}\lambda^{n-1}) = (d_0 + d_1\lambda + \cdots + d_{n-1}\lambda^{n-1} + \lambda^n)I$$

holds for every λ, thus by looking at the coefficients we obtain the following:

$$-AC_0 = d_0 I,$$

$$C_0 - AC_1 = d_1 I.$$

In general, for $1 \leq i \leq n-1$:

$$C_{i-1} - AC_i = d_i I.$$

Finally:

$$C_{n-1} = I.$$

Therefore,

$$\begin{aligned} D(A) &= Id_0 + Ad_1 + A^2 d_2 + \cdots + A^{n-1} d_{n-1} + A^n = \\ &= (-AC_0) + A(C_0 - AC_1) + A^2(C_1 - AC_2) + \cdots + A^{n-1}(C_{n-2} - AC_{n-1}) + (A^n C_{n-1}) = 0, \end{aligned}$$

by observing that this is a telescoping sum. □

We obtained that if we plug a square matrix in its characteristic polynomial, then we get the 0 matrix. A natural question that arises is whether there exists a (nonzero) polynomial with smaller degree having this property.

Example 40. *Let*

$$A = \begin{pmatrix} a & 0 & \dots & 0 \\ 0 & a & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & a \end{pmatrix}.$$

For the polynomial $f(x) = x - a$ we have $f(A) = A - aI = 0$, so it can indeed happen that for a polynomial with smaller degree $f(A) = 0$ holds.

Example 41. *Let*

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

Then $D(x) = (x-2)(x-3)$ and it is easy to see that $f(A) = 0$ can hold for some $f \not\equiv 0$ only if f has degree at least 2, since A is not a scalar matrix. Indeed, for a first degree polynomial $f(x) = x - aI$ we have

$$f(A) = \begin{pmatrix} 2-a & 0 \\ 0 & 3-a \end{pmatrix} \neq 0,$$

since a can not be equal to 2 and 3 at the same time.

Example 42. *Let*
$$A = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}.$$
Then $D(x) = (x-2)^2$. Here again, $f(A) = 0$ can not hold for a polynomial with degree 1, since A is not a scalar matrix. Indeed, for a first degree polynomial $f(x) = x - aI$ we have
$$f(A) = \begin{pmatrix} 2-a & 0 \\ 1 & 2-a \end{pmatrix} \neq 0,$$
since the 1st entry of the 2nd row will always remain a 1.

Definition 40. *Let A be an $n \times n$ matrix. The minimal polynomial of A is the polynomial $\Delta(x)$ with the smallest possible degree and main coefficient 1 such that $\Delta(A) = 0$.*

Remark 42. The minimal polynomial is well-defined. First, according to the Cayley-Hamilton theorem there exists a polynomial f (namely, the characteristic polynomial) such that $f(A) = 0$ (and $f \not\equiv 0$). Second, by fixing the main coefficient to be 1 the choice is unique, since $f_1(A) = f_2(A) = 0$ for polynomials $f_1 \neq f_2$ with the same (smallest possible) degree and main coefficient 1 would imply that $(f_1 - f_2)(A) = 0$, however, $f_1 - f_2$ would have a smaller degree. After multiplying $f_1 - f_2$ by a suitable nonzero scalar, the main coefficient could also be changed to 1.

Now, our aim is to understand the minimal polynomial, the characteristic polynomial better and to classify those polynomials f for which $f(A) = 0$. First, we shall review some facts about polynomial division.

Let f be a polynomial of degree n with complex coefficients:
$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0,$$
where $a_n, a_{n-1} \ldots, a_0 \in \mathbb{C}$ and $a_n \neq 0$. Since the field of complex numbers is algebraically closed, f factorizes as
$$f(z) = a_n(z - z_1)\ldots(z - z_n),$$
where $z_1, \ldots, z_n$ are the (not necessarily distinct) roots of the polynomial f.

Let g be another polynomial of degree $k \leq n$. Then there exists a polynomial h of degree $n - k$ and a polynomial r of degree $d < k$ such that
$$f(z) = g(z)h(z) + r(z).$$

Example 43. *Let $f(z) = 3z^5 + 2z^2 - 1$ and $g(z) = 12z + 38$. Then*
$$f(z) = (12z+38)\left(\frac{1}{4}z^4 - \frac{38}{4\cdot 12}z^3 + \frac{38^2}{4\cdot 12^2}z^2 + \left(-\frac{38^3}{4\cdot 12^3} + \frac{2}{12}\right)z + \left(\frac{38^4}{4\cdot 12^4} - \frac{2\cdot 38}{12^2}\right)\right) - 1 - \frac{38^5}{4\cdot 12^4} + \frac{2\cdot 38^2}{12^2},$$

where it is also indicated how we calculate the coefficients one by one.

Remark 43. If g has degree 1, that is, $g(z) = \alpha(z-a)$ for some α and a, then

$$f(a) = \alpha\underbrace{(a-a)}_{0}h(a) + r(a) = r(a).$$

Since r is a constant polynomial, we get $r \equiv f(a)$. As $f(z) - f(a)$ vanishes at $z = a$, we have $z - a \mid f(z) - f(a)$, that is,

$$f(z) - f(a) = (z-a)h(z)$$

for some polynomial h. Hence,

$$f(z) = (z-a)h(z) + f(a).$$

After rearranging this equation and dividing by $f(a) \neq 0$ we obtain that for a polynomial f of degree n (and satisfying $f(a) \neq 0$) there exists a polynomial $\tilde{h}$ of degree $n-1$ and a constant c such that

$$(z-a)\tilde{h}(z) + cf(z) = 1.$$

Namely,

$$c = \frac{1}{f(a)} \text{ and } \tilde{h}(z) = -\frac{h(z)}{f(a)} = -\frac{f(z)-f(a)}{(z-a)f(a)}.$$

Now, we are ready to characterize those polynomials f for which $f(A) = 0$.

Proposition 100. *Let A be a square matrix and $\Delta(x)$ its minimal polynomial. For a polynomial f we have $f(A) = 0$ if and only if $\Delta \mid f$.*

Proof. If $\Delta \mid f$, say $f(z) = \Delta(z)h(z)$, then

$$f(A) = \underbrace{\Delta(A)}_{0}h(A) = 0$$

obviously holds.

Now, let us assume that $f(A) = 0$. Then either $f \equiv 0$ (when $\Delta \mid f$ holds), or the degree of f is at least $\deg\Delta$ according to the definition of the minimal polynomial. Let us divide f by Δ:

$$f(z) = \Delta(z)h(z) + r(z),$$

where $\deg r < \deg\Delta$. After plugging in A we obtain

$$0 = f(A) = \underbrace{\Delta(A)}_{0}h(A) + r(A) = 0h(A) + r(A) = r(A).$$

Therefore, $r(A) = 0$ and $\deg r < \deg\Delta$, thus according to the definition of the minimal polynomial we must have $r \equiv 0$, and the statement follows. □

Note that each entry of the adjugate of the characteristic matrix, that is, of $\operatorname{adj}(\lambda I - A)$ is a polynomial of degree at most $n-1$. Let $\Theta(\lambda)$ be the greatest common divisor of these n^2 polynomials (in such a way that the main coefficient of $\Theta(\lambda)$ is 1). Let us define the reduced adjugate as

$$F(\lambda) = \frac{1}{\Theta(\lambda)} \operatorname{adj}(\lambda I - A).$$

Then

$$(\lambda I - A)F(\lambda) = \frac{D(\lambda)}{\Theta(\lambda)} I,$$

where the rational function $\frac{D(\lambda)}{\Theta(\lambda)}$ is a polynomial. We will show that in fact this polynomial is the minimal polynomial.

Example 44. *Let*

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 1 & 1 & 2 \end{pmatrix}.$$

Then $D(\lambda) = (\lambda - 2)^2(\lambda - 3)$. *The adjugate matrix is*

$$\operatorname{adj}(\lambda I - A) = \begin{pmatrix} (\lambda-2)(\lambda-3) & 0 & 0 \\ \lambda - 2 & (\lambda-2)^2 & 0 \\ \lambda - 2 & \lambda - 2 & (\lambda-2)(\lambda-3) \end{pmatrix},$$

thus $\Theta(\lambda) = \lambda - 2$ *and the reduced adjugate is:*

$$F(\lambda) = \begin{pmatrix} \lambda - 3 & 0 & 0 \\ 1 & \lambda - 2 & 0 \\ 1 & 1 & \lambda - 3 \end{pmatrix}.$$

Finally, the minimal polynomial is

$$\Delta(\lambda) = (\lambda - 2)(\lambda - 3).$$

Theorem 101. $\frac{D(\lambda)}{\Theta(\lambda)} = \Delta(\lambda)$

For the proof we will need the following lemma:

Lemma 102. *Let f be a polynomial and A a square matrix such that $f(A) = 0$. Then there exists a polynomial $B(\lambda)$ with matrix coefficients such that*

$$f(\lambda)I = (\lambda I - A)B(\lambda).$$

Proof. Let $f(\lambda) = f_k\lambda^k + \cdots + f_1\lambda + f_0$. Then

$$f(\lambda)I = (\lambda I - A)[(If_k\lambda^{k-1}) + (Af_k\lambda^{k-2} + If_{k-1}\lambda^{k-2}) + (A^2 f_k\lambda^{k-3} + Af_{k-1}\lambda^{k-3} + If_{k-2}\lambda^{k-3}) + \\ + \cdots + (A^{k-2}f_k + A^{k-3}f_{k-1} + \cdots + If_2)\lambda + (A^{k-1}f_k + A^{k-2}f_{k-1} + \cdots + If_1)\lambda^0] + \underbrace{f(A)}_{0}.$$

□

Proof of Theorem 101. Let us denote the polynomial $\frac{D(\lambda)}{\Theta(\lambda)}$ by $\tilde{\Delta}(\lambda)$.

First, we prove that $\tilde{\Delta}(A) = 0$. Let the reduced adjugate be

$$F(\lambda) = F_0 + \lambda F_1 + \cdots + \lambda^k F_k,$$

where $k \leq n-1$. We know that

$$(\lambda I - A)F(\lambda) = \tilde{\Delta}(\lambda)I = F(\lambda)(\lambda I - A).$$

We would like to plug A in this identity, for being able to do so, we shall prove that $AF_i = F_i A$ for every $0 \leq i \leq k$. Note that

$$(\lambda I - A)F(\lambda)(= \tilde{\Delta}(A)I) = F(\lambda)(\lambda I - A),$$

thus by looking at the constant term we get

$$AF_0 = F_0 A,$$

from the linear term we get

$$F_0 - AF_1 = F_0 - F_1 A,$$

and in general, from the coefficient of λ^s we obtain

$$F_{s-1} - AF_s = F_{s-1} - F_s A$$

for every $1 \leq s \leq k$. Therefore, $AF_i = F_i A$ for every $0 \leq i \leq k$, and we can plug A in this identity:

$$\underbrace{(AI - A)}_{0} F(A) = \tilde{\Delta}(A)I,$$

implying that $\tilde{\Delta}(A) = 0$.

Now we show that $\tilde{\Delta}(\lambda)$ is indeed the minimal polynomial Δ.

Proposition 100 implies that $\Delta \mid \tilde{\Delta}$, let

$$\tilde{\Delta}(\lambda) = \Delta(\lambda)p(\lambda).$$

Since $\Delta(A) = 0$, we obtain by Lemma 102 that

$$\Delta(\lambda)I = (\lambda I - A)B(\lambda).$$

Then

$$(\lambda I - A)F(\lambda) = \tilde{\Delta}(\lambda)I = \Delta(\lambda)p(\lambda)I = (\lambda I - A)B(\lambda)p(\lambda).$$

Hence,

$$(\lambda I - A)[F(\lambda) - B(\lambda)p(\lambda)] = 0$$

holds for every λ. If λ is not an eigenvalue of A, then the matrix $(\lambda I - A)$ is invertible, thus

$$F(\lambda) - B(\lambda)p(\lambda) = 0.$$

Since this equation holds with the exception of finitely many values (namely, the eigenvalues), we conclude that

$$F(\lambda) \equiv B(\lambda)p(\lambda),$$

thus all entries of $F(\lambda)$ (note that these entries are polynomials) are divisible by $p(\lambda)$. However, F was already reduced, so $p(\lambda)$ must be constant. Therefore, Δ and $\tilde{\Delta}$ are constant multiples of each other, and by using that they both have main coefficient 1, we get that $\Delta = \tilde{\Delta}$, as claimed. □

Proposition 103. *Let f be a polynomial, A a square matrix and assume that $f(A) = 0$. If for some invertible C we define $B := C^{-1}AC$, then $f(B) = 0$ also holds.*

Proof. Let
$$f(z) = f_k z^k + \dots + f_1 z + f_0.$$
Then
$$f(B) = f_k B^k + \dots + f_1 B + f_0 I.$$
Note that $B^s = (C^{-1}AC)^s = C^{-1}A^sC$, thus
$$f(B) = f_k C^{-1}A^k C + \dots + f_1 C^{-1}AC + f_0 I = C^{-1}[f_k A^k + \dots + f_1 A + f_0 I]C = C^{-1}f(A)C = 0.$$
□

Corollary 104. *If $A \sim B$, then $\Delta_A = \Delta_B$, that is, similar matrices have the same minimal polynomial.*

Proposition 105. *Let the characteristic polynomial of an $n \times n$ square matrix A be*
$$D(\lambda) = \prod_{k=1}^{s}(\lambda - \lambda_k)^{\alpha_k},$$
where the eigenvalues $\lambda_1, \dots, \lambda_s$ are distinct and the exponents $\alpha_1, \dots, \alpha_s$ are positive integers.

Then the minimal polynomial of A is
$$\Delta(\lambda) = \prod_{k=1}^{s}(\lambda - \lambda_k)^{\beta_k}$$
for some $1 \le \beta_k \le \alpha_k$.

Remark 44. Note that $\alpha_1, \dots, \alpha_s$ are the algebraic multiplicities of the corresponding eigenvalues and $\alpha_1 + \dots + \alpha_s = n$. The proposition states that each eigenvalues is also a root of the minimal polynomial, and the multiplicity of an eigenvalue (as a root of Δ) can not exceed its algebraic multiplicity.

Proof. Since $D(A) = 0$ (by the Cayley-Hamilton theorem), $\Delta \mid D$ according to Proposition 100. Hence, $\beta_k \le \alpha_k$ for every $1 \le k \le s$.

It is left to show that $\beta_k \ge 1$ for each k. Let $\Theta(\lambda) = \prod_{k=1}^{s}(\lambda - \lambda_k)^{\gamma_k}$. According to Theorem 101 we have $\alpha_k = \beta_k + \gamma_k$ for every $1 \le k \le s$. We know that $D'(\lambda) = \sum_{i=1}^{n} D_i(\lambda)$ (see the proof of Proposition 82, which is just before the statement itself) and the multiplicity of λ_k in D' is exactly $\alpha_k - 1$. Since $\Theta(\lambda)$ is the greatest common divisor of $D_1, \ldots, D_n$ and $n^2 - n$ other polynomials, we get that the multiplicity of λ_k in Θ is also at most $\alpha_k - 1$. That is, $\gamma_k \leq \alpha_k - 1$, thus $\beta_k \geq 1$, as claimed.

□

Now we give another necessary and sufficient condition for a matrix to be diagonalizable.

Theorem 106. *A square matrix is diagonalizable if and only if its minimal polynomial has only single roots.*

Let us consider two examples before proving Theorem 106.

Example 45. *Let*

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}.$$

Then the characteristic polynomial is

$$D_A(\lambda) = (\lambda - 2)^3(\lambda - 3)(\lambda - 4),$$

while the minimal polynomial is

$$\Delta_A(\lambda) = (\lambda - 2)(\lambda - 3)(\lambda - 4).$$

The minimal polynomial has only single roots, and A itself is a diagonal matrix.

Example 46. *Let*

$$B = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}.$$

Then $D_B(\lambda) = (\lambda - 2)^2$. According to Proposition 105 the minimal polynomial is either $\lambda - 2$ or $(\lambda - 2)^2$. By plugging B in $\lambda - 2$ we get

$$B - 2I = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \neq 0,$$

thus $\Delta_B(\lambda) = (\lambda - 2)^2$. Then Theorem 106 implies that B is not diagonalizable.

Proof of Theorem 106. First we show that the minimal polynomial of a diagonalizable matrix has only single roots. If the matrix A is diagonalizable, then $A \sim D$ for some diagonal matrix D. If the (distinct) eigenvalues of A are $\lambda_1, \dots, \lambda_s$ with algebraic multiplicities $\alpha_1, \dots, \alpha_s$, then the main diagonal of D contains α_1 copies of λ_1, α_2 copies of λ_2, and so on... It is easy to see that the minimal polynomial of D is $\prod_{k=1}^{s} (\lambda - \lambda_k)$. (See Example 45.) By Corollary 104 we obtain that

$$\Delta_A(\lambda) = \Delta_D(\lambda) = \prod_{k=1}^{s} (\lambda - \lambda_k),$$

since $A \sim D$. Thus the minimal polynomial of a diagonalizable matrix has only single roots.

Now, we continue with the reverse direction. Let the characteristic polynomial of the $n \times n$ matrix A be $D(\lambda) = \prod_{k=1}^{s} (\lambda - \lambda_k)^{\alpha_k}$, where $\lambda_1, \dots, \lambda_s$ are the distinct eigenvalues and $\alpha_1, \dots, \alpha_s$ denote their algebraic multiplicities. By Corollary 84 the matrix A is diagonalizable if the geometric multiplicity of λ_k is also α_k for every $1 \le k \le s$. Alternatively, it suffices to show that the sum of the geometric multiplicities is

$$\sum_{k=1}^{s} \dim V_{\lambda_k} = n.$$

If we show that every vector v can be written as a sum $v = w_1 + \dots + w_s$, where $Aw_k = \lambda_k w_k$ (for every $1 \le k \le s$), then the sum of the eigenspaces V_{λ_k} is the whole space and $\sum_{k=1}^{s} \dim V_{\lambda_k} = n$ follows.

According to Proposition 105 the condition that the minimal polynomial Δ has no multiple roots implies that

$$\Delta(\lambda) = \prod_{k=1}^{s} (\lambda - \lambda_k).$$

Let us write

$$\Delta(\lambda) = (\lambda - \lambda_1) f_1(\lambda).$$

Since Δ has no multiple roots, we have $f_1(\lambda_1) \neq 0$. Therefore, there exists some polynomial $g_1(\lambda)$ and a constant c_1 such that

$$(\lambda - \lambda_1) g_1(\lambda) + c_1 f_1(\lambda) = 1.$$

(See Remark 43.) After plugging in the matrix A we obtain that

$$(A - \lambda_1 I) g_1(A) + c_1 f_1(A) = I.$$

For every vector v we have:

$$v = Iv = \underbrace{(A - \lambda_1 I) g_1(A) v}_{=:v_1} + \underbrace{c_1 f_1(A) v}_{=:w_1}.$$

Note that

$$(A - \lambda_1 I)w_1 = c_1 \underbrace{(A - \lambda_1 I)f_1(A)}_{\Delta(A)} v = 0,$$

hence w_1 belongs to the eigenspace V_{λ_1}:

$$Aw_1 = \lambda_1 w_1.$$

Now, for $f_2(\lambda) = \frac{\Delta(\lambda)}{\lambda - \lambda_2}$ we have $f_2(\lambda_2) \neq 0$, thus for some polynomial $g_2(\lambda)$ and constant c_2 we have

$$(\lambda - \lambda_2)g_2(\lambda) + c_2 f_2(\lambda) = 1.$$

Hence,

$$(A - \lambda_2 I)g_2(A) + c_2 f_2(A) = I,$$

implying that

$$v_1 = Iv_1 = \underbrace{(A - \lambda_2 I)g_2(A)v_1}_{=:v_2} + \underbrace{c_2 f_2(A)v_1}_{=:w_2}.$$

Note that

$$v = w_1 + v_1 = w_1 + w_2 + v_2$$

and

$$(A - \lambda_2 I)w_2 = c_2 \underbrace{(A - \lambda_2 I)f_2(A)}_{\Delta(A)} v_1 = 0,$$

hence w_2 belongs to the eigenspace V_{λ_2}:

$$Aw_2 = \lambda_2 w_2.$$

By continuing in a similar fashion we obtain the decomposition $v_2 = w_3 + v_3$, then $v_3 = w_4 + v_4$, and so on..., finally $v_{s-1} = w_s + v_s$. Hence, we obtained the decomposition

$$v = w_1 + \cdots + w_s + v_s.$$

Now we will concentrate on the vectors $v_1, \ldots, v_s$ to show that $v_s = 0$:

$$v_1 = (A - \lambda_1 I)g_1(A)v,$$

$$\begin{aligned} v_2 = (A - \lambda_2 I)g_2(A)v_1 &= (A - \lambda_2 I)g_2(A)(A - \lambda_1 I)g_1(A)v = \\ &= (A - \lambda_1 I)(A - \lambda_2 I)\underbrace{g_1(A)g_2(A)}_{=:h_2(A)} v = (A - \lambda_1 I)(A - \lambda_2 I)h_2(A)v, \end{aligned}$$

$$v_3 = (A - \lambda_1 I)(A - \lambda_2 I)(A - \lambda_3 I)h_3(A)v,$$

$$\vdots$$

$$v_s = \underbrace{(A - \lambda_1 I)(A - \lambda_2 I)\ldots(A - \lambda_s I)}_{\Delta(A)=0} h_s(A)v = 0,$$

Since $v_s = 0$, we obtain the desired decomposition

$$v = w_1 + \cdots + w_s,$$

where $Aw_k = \lambda_k w_k$ for every $1 \le k \le s$. This implies that the matrix A is diagonalizable.

□

Example 47. *Let*

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 1 & 1 & 2 \end{pmatrix}.$$

Then the characteristic polynomial of A is $D(\lambda) = (\lambda-2)^2(\lambda-3)$. According to Proposition 105 the minimal polynomial is either $(\lambda-2)(\lambda-3)$ or $(\lambda-2)^2(\lambda-3)$. As

$$(A-2I)(A-3I) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & -1 \end{pmatrix} = 0,$$

we have $\Delta(\lambda) = (\lambda-2)(\lambda-3)$.

Let $\lambda_1 = 2$, then

$$f_1(\lambda) = \frac{\Delta(\lambda)}{\lambda - \lambda_1} = \lambda - \lambda_2 = \lambda - 3.$$

To obtain

$$(\lambda-2)g_1(\lambda) + c_1(\lambda-3) = 1$$

we may choose $g_1(\lambda) \equiv 1$ and $c_1 = -1$:

$$(\lambda-2)\cdot 1 + (-1)(\lambda-3) = 1.$$

Then

$$v = \underbrace{(A-2I)v}_{=:w_2} \underbrace{-(A-3I)v}_{=:w_1}.$$

Here $Aw_1 = 2w_1$, $Aw_2 = 3w_2$.

Let $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Then

$$w_1 = -(A-3I)v = -\begin{pmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ -x \\ -x-y+z \end{pmatrix},$$

$$w_2 = (A-2I)v = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ x+y \\ x+y \end{pmatrix}.$$

With the help of the minimal polynomial we can calculate the powers of a square matrix A as follows. Let the minimal polynomial of A be

$$\Delta(\lambda) = \lambda^k + \alpha_{k-1}\lambda^{k-1} + \cdots + \alpha_0.$$

Since $\Delta(A) = 0$, the kth power of A can be expressed as

$$A^k = -\alpha_{k-1}A^{k-1} - \cdots - \alpha_0 I.$$

Then the $(k+1)$st power is

$$\begin{aligned} A^{k+1} = (-\alpha_{k-1}A^{k-1} - \alpha_{k-2}A^{k-2} - \cdots - \alpha_0 I)A = -\alpha_{k-1}A^k - \alpha_{k-2}A^{k-1} \cdots - \alpha_0 A = \\ = -\alpha_{k-1}(-\alpha_{k-1}A^{k-1} - \cdots - \alpha_0 I) - \alpha_{k-2}A^{k-1} \cdots - \alpha_0 A = \\ = (\alpha_{k-1}^2 - \alpha_{k-2})A^{k-1} + \cdots + (\alpha_{k-1}\alpha_1 - \alpha_0)A + \alpha_{k-1}\alpha_0 I. \end{aligned}$$

This recursive calculation can be continued and we obtain that for every t the tth power of A can be expressed as

$$A^t = \beta_{k-1}A^{k-1} + \cdots + \beta_0 I$$

with suitable coefficients. Hence, for any vector v we have $A^t v = \beta_{k-1}A^{k-1}v + \cdots + \beta_0 v$, that is, $A^t v$ lies in the linear subspace spanned by the vectors $v, Av, \ldots, A^{k-1}v$.

13. Matrix functions

In the previous section we have seen that we can plug (square) matrices in polynomials. Now our aim is to plug (square) matrices into more general functions, and define for instance e^A or $\sin A$, and so on...

First we shall review the so-called Lagrange-polynomials that can be used for polynomial interpolation.

Lagrange polynomials.

Let $z_1, z_2, \dots, z_s$ be data points in the complex plane. Let us define the Lagrange basis polynomials as

$$L_k(z) = \frac{(z-z_1)(z-z_2)\dots(z-z_{k-1})(z-z_{k+1})\dots(z-z_s)}{(z_k-z_1)(z_k-z_2)\dots(z_k-z_{k-1})(z_k-z_{k+1})\dots(z_k-z_s)}.$$

Observe that the degree of L_k is exactly $s-1$ and

$$L_k(z_j) = \begin{cases} 1 & \text{if } j = k \\ \\ 0 & \text{if } j \neq k \end{cases}.$$

Any polynomial g with degree at most $s-1$ can be expressed as a linear combination of $L_1, \dots, L_s$. Namely,

$$g(z) = \sum_{k=1}^{s} g(z_k) L_k(z),$$

since on both sides of the equation we have a polynomial of degree at most $s-1$ which attains the values $g(z_1), \dots, g(z_s)$ at the points $z_1, \dots, z_s$. Indeed,

$$\sum_{k=1}^{s} g(z_k) \underbrace{L_k(z_j)}_{\delta_{kj}} = g(z_j) \cdot 1 = g(z_j).$$

A polynomial of degree at most $s-1$ is uniquely determined by s values at given points.

Remark 45. Note that by writing $\Delta(z) = (z-z_1)\dots(z-z_s)$ and $\Delta_k(z) = \frac{\Delta(z)}{z-z_k}$ we have

$$L_k(z) = \frac{\Delta_k(z)}{\Delta_k(z_k)}.$$

Alternatively,

$$L_k(z) = \frac{\Delta(z)}{(z-z_k)\Delta'(z_k)}.$$

Remark 46. We have

$$\sum_{k=1}^{s} L_k(z) \equiv 1$$

since on the two sides of the equation we have polynomials of degree at most $s-1$ that coincide at s distinct points: $z_1,\dots,z_s$. After multiplication by z we get the identity

$$\sum_{k=1}^{s} zL_k(z) \equiv z.$$

Let us consider now a Taylor series $\sum\limits_{k=0}^{\infty} c_k z^k$, where the coefficients are complex numbers. Let $S_N(z) = \sum\limits_{k=0}^{N} c_k z^k$ denote the Nth Taylor polynomial. For every Taylor series there exists a radius of convergence R such that $S_N(z)$ converges if $|z| < R$ and diverges if $|z| > R$. (Note that $R = 0$ means that $S_N(z)$ converges only at $z = 0$ and $R = \infty$ means that $S_N(z)$ converges at every z.)

For $|z| < R$ let

$$f(z) \coloneqq \lim_{N\to\infty} S_N(z).$$

Now, we would like to define

$$f(A) \coloneqq \lim_{N\to\infty} S_N(A).$$

In order to do so we shall study when $S_N(A)$ converges.

Theorem 107. *Let A be an $n\times n$ matrix, whose minimal polynomial Δ has only single roots. Moreover, each eigenvalue λ_j has absolute value smaller than R: $|\lambda_j| < R$, where R is the radius of convergence of the power series $f(z) = \sum c_k z^k$. Then $S_N(A)$ converges, moreover, the limit $f(A)$ can be expressed as a polynomial of A with degree smaller than* $\deg\Delta$.

Proof. Let us write

$$S_N(z) = \Delta(z)q_N(z) + R_N(z),$$

where $\deg\Delta = s$ and $\deg R_N \le s-1$.

For every eigenvalue λ_j we have

$$S_N(\lambda_j) = \underbrace{\Delta(\lambda_j)}_{0} q_N(\lambda_j) + R_N(\lambda_j) = R_N(\lambda_j).$$

Therefore, $S_N(\lambda_j) = R_N(\lambda_j) \to f(\lambda_j)$.

Note that $S_N(A) = R_N(A)$ and the polynomial R_N has degree at most $s-1$, furthermore, R_N is given at s different points, namely, at the eigenvalues of A. This uniquely determines the polynomial R_N. Let $L_1(z),\dots,L_s(z)$ be the Lagrange basis polynomials satisfying that

$$L_k(\lambda_j) = \begin{cases} 1 & \text{if } j = k \\ \\ 0 & \text{if } j \ne k \end{cases}.$$

Let us express the polynomial R_N as

$$R_N(z) = \sum_{j=1}^{s} R_N(\lambda_j) L_j(z).$$

Hence:

$$R_N(A) = \sum_{j=1}^{s} R_N(\lambda_j) L_j(A) = \sum_{j=1}^{s} S_N(\lambda_j) L_j(A),$$

where $L_j(A)$ does *not* depend on N. Therefore,

$$R_N(A) \to f(A) = \sum_{j=1}^{s} f(\lambda_j) L_j(A),$$

since $S_N(\lambda_j) \to f(\lambda_j)$. Since

$$\sum_{j=1}^{s} f(\lambda_j) L_j(z)$$

is a polynomial of degree at most $s - 1$, the theorem is proven. $\square$

Proposition 108. *Let A be a square matrix. If the minimal polynomial of A has only single roots, $\lambda_1, \dots, \lambda_s$, and $L_1, \dots, L_s$ are the Lagrange basis polynomials, respectively, then $L_i(A)$ is a projection for every $1 \le i \le s$.*

Proof. Observe that

$$\underbrace{L_i(\lambda_j)}_{0, \text{ if } i \neq j} \underbrace{(L_i(\lambda_j) - 1)}_{0, \text{ if } i = j} = 0.$$

Consider the polynomial $L_i(z)(L_i(z) - 1)$. The eigenvalues $\lambda_1, \dots, \lambda_s$ are all roots of this polynomial. Therefore,

$$\Delta(z) \mid L_i(z)(L_i(z) - 1)$$

for every $1 \le i \le s$. Hence,

$$L_i(A)(L_i(A) - I) = 0,$$

which implies that $L_i(A)^2 = L_i(A)$, that is, $L_i(A)$ is a projection, as we claimed. $\square$

Proposition 109. *Let A be a square matrix. If the minimal polynomial of A has only single roots, $\lambda_1, \dots, \lambda_s$, and $L_1, \dots, L_s$ are the Lagrange basis polynomials, respectively, then $L_j(A) L_k(A) = 0$ for $j \neq k$.*

Proof. Note that

$$L_j(\lambda) = \frac{\Delta(\lambda)}{(\lambda - \lambda_j)\Delta'(\lambda_j)}$$

and

$$L_k(\lambda) = \frac{\Delta(\lambda)}{(\lambda - \lambda_k)\Delta'(\lambda_k)}.$$

(See Remark 45.)

Therefore,

$$L_j(\lambda)L_k(\lambda) = \frac{\Delta(\lambda)}{(\lambda-\lambda_j)\Delta'(\lambda_j)} \cdot \frac{\Delta(\lambda)}{(\lambda-\lambda_k)\Delta'(\lambda_k)},$$

and this polynomial is divisible by $\Delta(\lambda)$, since the factors $\lambda-\lambda_j$ and $\lambda-\lambda_k$ appear once, all the other $\lambda-\lambda_\ell$ factors appear twice in the product.

Hence, $L_j(A)L_k(A) = 0$ by Proposition 100.

□

Theorem 110. *Let A be a square matrix. If the minimal polynomial of A has only single roots, $\lambda_1,\ldots,\lambda_s$, and $L_1,\ldots,L_s$ are the Lagrange basis polynomials, respectively, then $\operatorname{rk} L_k(A) = \alpha_k$, where α_k is the algebraic multiplicity of λ_k.*

Proof. Note that

$$L_k(\lambda) = \frac{\Delta(\lambda)}{(\lambda-\lambda_k)\Delta'(\lambda_k)},$$

hence

$$L_k(\lambda)\cdot(\lambda-\lambda_k) = \frac{\Delta(\lambda)}{\Delta'(\lambda_k)}.$$

After plugging in A we obtain that

$$L_k(A)\cdot(A-\lambda_k I) = \frac{\Delta(A)}{\Delta'(\lambda_k)} = 0.$$

Thus, by Corollary 22 we get that

$$\operatorname{rk}(L_k(A)) + \operatorname{rk}(A-\lambda_k I) \le n.$$

According to Proposition 82 we have $\operatorname{rk}(A-\lambda_k I) \ge n-\alpha_k$, therefore, $\operatorname{rk}(L_k(A)) \le \alpha_k$.

On the other hand, we know that

$$\sum_{k=1}^{s} L_k(A) = I$$

(see Remark 46), therefore:

$$n = \operatorname{rk} I = \operatorname{rk}\left(\sum_{k=1}^{s} L_k(A)\right) \le \sum_{k=1}^{s} \alpha_k = n.$$

Hence, we must have equality everywhere, which implies that $\operatorname{rk}(L_k(A)) = \alpha_k$. □

13.1. **Spectral decomposition of $f(A)$.** Let us consider the minimal rank-one decomposition of $L_k(A)$:

$$L_k(A) = U_k V_k^T,$$

where the number of columns of U_k (and V_k) is $\alpha_k = \mathrm{rk}(L_k(A))$.

Then $\sum_{k=1}^{s} U_k V_k^T = I$. Let

$$U := (U_1, \dots, U_s)$$

and

$$V := (V_1, \dots, V_s).$$

Note that U and V are $n \times n$ matrices satisfying that $UV^T = I$. Then $V^T U = I$ also holds, that is, the columns of U and the columns of V form a biorthogonal system.

According to Theorem 107 we have

$$f(A) = \sum_{k=1}^{s} f(\lambda_k) L_k(A) = \sum_{k=1}^{s} f(\lambda_k) U_k V_k^T =$$

$$= U \begin{pmatrix} f(\lambda_1) I_{\alpha_1} & 0 & \dots & 0 \\ 0 & f(\lambda_2) I_{\alpha_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & f(\lambda_s) I_{\alpha_s} \end{pmatrix} V^T.$$

This is the *spectral decomposition* of $f(A)$. For instance, by taking $f(z) = z$ we get the spectral decomposition of A:

$$A = U \begin{pmatrix} \lambda_1 & & & & & & \\ & \ddots & & & & & \\ & & \lambda_1 & & & & \\ & & & \ddots & & & \\ & & & & \lambda_s & & \\ & & & & & \ddots & \\ & & & & & & \lambda_s \end{pmatrix} V^T.$$

According to the above calculation the matrices U and V^T appearing in the spectral decomposition of A (or $f(A)$) can be simply obtained by taking a minimal rank-one decomposition for each $L_k(A)$.

Example 48. *Let*

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 1 & 1 & 2 \end{pmatrix}.$$

Then the characteristic polynomial of A is $D(\lambda) = (\lambda - 2)^2(\lambda - 3)$ and the minimal polynomial of A is $\Delta(\lambda) = (\lambda - 2)(\lambda - 3)$. (See Example 47.)

The eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 3$. The Lagrange basis polynomials are

$$L_1(\lambda) = -(\lambda - 3) = -\lambda + 3$$

and

$$L_2(\lambda) = \lambda - 2.$$

Let us determine a minimal rank-one decomposition for $L_1(A)$ and $L_2(A)$:

$$L_1(A) = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \begin{pmatrix} 0 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix},$$

$$L_2(A) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}.$$

Therefore:

$$A = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ -1 & -1 & 1 \end{pmatrix}}_{U} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{pmatrix}}_{V^T = U^{-1}}.$$

For a function f we have

$$f(A) = U \begin{pmatrix} f(2) & 0 & 0 \\ 0 & f(2) & 0 \\ 0 & 0 & f(3) \end{pmatrix} V^T = \begin{pmatrix} f(2) & 0 & 0 \\ -f(2) + f(3) & f(3) & 0 \\ -f(2) + f(3) & -f(2) + f(3) & f(2) \end{pmatrix}.$$

Example 49. *Let P be a projection ($P^2 = P$) such that $P \neq 0$ and $P \neq I$. Since $P(P - I) = P^2 - P = 0$, we get that the minimal polynomial of P is $\Delta(\lambda) = \lambda(\lambda - 1)$. The eigenvalues of P are $\lambda_1 = 1$ and $\lambda_2 = 0$. The Lagrange basis polynomials for P are $L_1(\lambda) = \lambda$ and $L_2(\lambda) = 1 - \lambda$.*

Let us consider a minimal rank-one decomposition of $L_1(P)$ and $L_2(P)$:

$$L_1(P) = P = UV^T,$$

$$L_2(P) = I - P = WZ^T.$$

By taking $f(z) = z$ we obtain the spectral decomposition of P:

$$P = \begin{pmatrix} U & W \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V^T \\ Z^T \end{pmatrix}.$$

More generally, by taking a function $f(z)$, say $f(z) = e^z$, we obtain that:

$$f(P) = e^P = \begin{pmatrix} U & W \end{pmatrix} \begin{pmatrix} f(1)I & 0 \\ 0 & f(0)I \end{pmatrix} \begin{pmatrix} V^T \\ Z^T \end{pmatrix} = \begin{pmatrix} U & W \end{pmatrix} \begin{pmatrix} eI & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} V^T \\ Z^T \end{pmatrix} =$$
$$= eUV^T + WZ^T = eP + I - P = I + (e - 1)P.$$

Note that this can be calculated in the following way, too:

$$e^P = I + P + \frac{1}{2!}P^2 + \frac{1}{3!}P^3 + \dots = I + P\underbrace{\left(1 + \frac{1}{2!} + \frac{1}{3!} + \dots\right)}_{=e-1} = I + (e-1)P.$$

Example 50. *Let us consider now the case of a skew-symmetric 3×3 real matrix:*

$$A = \begin{pmatrix} 0 & a_1 & a_2 \\ -a_1 & 0 & a_3 \\ -a_2 & -a_3 & 0 \end{pmatrix}.$$

We will calculate e^A when $A \neq 0$. (Note that $A = 0$ is a projection, hence $e^0 = I$ according to the previous example.)

We start with calculating the characteristic polynomial of A.

$$D(\lambda) = \det\begin{pmatrix} \lambda & -a_1 & -a_2 \\ a_1 & \lambda & -a_3 \\ a_2 & a_3 & \lambda \end{pmatrix} = \lambda^3 + \lambda\underbrace{(a_1^2 + a_2^2 + a_3^2)}_{=:a^2} = \lambda(\lambda^2 + a^2),$$

where $a = \sqrt{a_1^2 + a_2^2 + a_3^2} \neq 0$.

The roots of the characteristic polynomial are

$$\lambda_1 = 0, \quad \lambda_2 = ia \text{ and } \lambda_3 = -ia.$$

Since the roots are distinct, the minimal polynomial coincides with the characteristic polynomial: $\Delta(\lambda) = D(\lambda)$. The Lagrange basis polynomials are as follows:

$$L_1(\lambda) = \frac{(\lambda + ia)(\lambda - ia)}{-ia(ia)} = \frac{\lambda^2 + a^2}{a^2},$$

$$L_2(\lambda) = \frac{\lambda(\lambda + ia)}{ia(2ia)} = \frac{\lambda^2 + ia\lambda}{-2a^2},$$

$$L_3(\lambda) = \frac{\lambda(\lambda - ia)}{-ia(-2ia)} = \frac{\lambda^2 - ia\lambda}{-2a^2}.$$

Now,

$$e^A = e^{\lambda_1}L_1(A) + e^{\lambda_2}L_2(A) + e^{\lambda_3}L_3(A) = \frac{A^2 + a^2 I}{a^2} + e^{ia}\frac{A^2 + iaA}{-2a^2} + e^{-ia}\frac{A^2 - iaA}{-2a^2} =$$
$$= A^2\left(\frac{1}{a^2} - \frac{e^{ia} + e^{-ia}}{2a^2}\right) + A \cdot \frac{-e^{ia}ia + e^{-ia}ia}{2a^2} + I = \frac{1 - \cos a}{a^2} \cdot A^2 + \frac{\sin a}{a} \cdot A + I.$$

(Note that $e^{ia} = \cos a + i \sin a$, since $a \in \mathbb{R}$.)

Let us consider the case of circulant matrices. We will consider the permutation matrix

$$\Omega = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 1 & 0 \\ 0 & \dots & 0 & 0 & 0 & 1 \\ 1 & \dots & 0 & 0 & 0 & 0 \end{pmatrix}.$$

and more generally, arbitrary circulant matrices:

$$C = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \dots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & a_2 & \dots & a_{n-2} \\ a_{n-2} & a_{n-1} & a_0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & a_1 & a_2 \\ a_2 & \dots & a_{n-2} & a_{n-1} & a_0 & a_1 \\ a_1 & \dots & a_{n-3} & a_{n-2} & a_{n-1} & a_0 \end{pmatrix} = a_0 I + a_1\Omega + a_2\Omega^2 + \dots + a_{n-1}\Omega^{n-1}.$$

The characteristic polynomial of Ω is $D(\lambda) = \lambda^n - 1$. Consequently, the eigenvalues of Ω are the nth roots of unity: $\lambda_k = e^{\frac{2k\pi i}{n}}$ for $k = 0, 1, \dots, n-1$. Specially, $\lambda_0 = 1$.

Since the eigenvalues are distinct, the minimal polynomial coincides with the characteristic polynomial:

$$\Delta(\lambda) = D(\lambda) = \lambda^n - 1.$$

Let us calculate the Lagrange basis polynomials:

$$L_k(\lambda) = \frac{\Delta(\lambda)}{(\lambda - \lambda_k)\Delta'(\lambda_k)} = \frac{\lambda^n - 1}{(\lambda - \lambda_k)n\lambda_k^{n-1}} =$$
$$= \frac{1}{n\lambda_k^{n-1}}(\lambda^{n-1} + \lambda_k\lambda^{n-2} + \lambda_k^2\lambda^{n-3} + \dots + \lambda_k^{n-1}) = \frac{1}{n}(\overline{\lambda}_k^{n-1}\lambda^{n-1} + \overline{\lambda}_k^{n-2}\lambda^{n-2} + \dots + 1),$$

by using that $\lambda_k^{-1} = \overline{\lambda}_k$, since $|\lambda_k| = 1$.

Therefore,

$$L_k(\Omega) = \frac{1}{n}(\overline{\lambda}_k^{n-1}\Omega^{n-1} + \overline{\lambda}_k^{n-2}\Omega^{n-2} + \dots + I) = \frac{1}{n}\begin{pmatrix} 1 & \overline{\lambda}_k & \overline{\lambda}_k^2 & \dots & \overline{\lambda}_k^{n-1} \\ \overline{\lambda}_k^{n-1} & 1 & \overline{\lambda}_k & \ddots & \vdots \\ \overline{\lambda}_k^{n-2} & \overline{\lambda}_k^{n-1} & \ddots & \ddots & \overline{\lambda}_k^2 \\ \vdots & \ddots & \ddots & 1 & \overline{\lambda}_k \\ \overline{\lambda}_k & \dots & \overline{\lambda}_k^{n-2} & \overline{\lambda}_k^{n-1} & 1 \end{pmatrix} =$$
$$= \frac{1}{n}\begin{pmatrix} 1 \\ \overline{\lambda}_k^{n-1} \\ \vdots \\ \overline{\lambda}_k \end{pmatrix}\begin{pmatrix} 1 & \overline{\lambda}_k & \dots & \overline{\lambda}_k^{n-1} \end{pmatrix}.$$

That is,

$$(L_k(\Omega))_{st} = \overline{\lambda}_k^{(n-s)} \cdot \overline{\lambda}_k^{t} = \overline{\lambda}_k^{(n-s+t)},$$

if the rows and columns are indexed by $s, t \in \{0, 1, \ldots, n-1\}$.

For brevity, let ω denote the first nth root of unity: $\omega = e^{\frac{2\pi i}{n}}$. Then $\lambda_k = \omega^k$ for $0 \le k \le n-1$. By using the rank-one decomposition of $L_0(\Omega), \ldots, L_{n-1}(\Omega)$ we obtain the spectral decomposition of Ω:

$$\Omega = \frac{1}{\sqrt{n}} \underbrace{\begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & \omega & \omega^2 & \ldots & \omega^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \ldots & \omega^{(n-1)(n-1)} \end{pmatrix}}_{=:U} \begin{pmatrix} 1 & & & \\ & \omega & & \\ & & \ddots & \\ & & & \omega^{n-1} \end{pmatrix} U^* \frac{1}{\sqrt{n}}.$$

Note that

$$\frac{1}{\sqrt{n}} U U^* \frac{1}{\sqrt{n}} = I.$$

For a polynomial p we have

$$p(\Omega) = \frac{1}{\sqrt{n}} U \begin{pmatrix} p(1) & & & \\ & p(\omega) & & \\ & & \ddots & \\ & & & p(\omega^{n-1}) \end{pmatrix} U^* \frac{1}{\sqrt{n}}.$$

Let us now take a circulant matrix C. Note that C can be expressed as $C = p(\Omega)$ for a polynomial p (with degree at most $n-1$). Now, for a function f we can calculate $f(C)$ as follows:

$$f(C) = f(p(\Omega)) = \frac{1}{\sqrt{n}} U \begin{pmatrix} f(p(1)) & & & \\ & f(p(\omega)) & & \\ & & \ddots & \\ & & & f(p(\omega^{n-1})) \end{pmatrix} U^* \frac{1}{\sqrt{n}}$$

That is,

$$(f(C))_{st} = \frac{1}{n} \sum_{k=0}^{n-1} \omega^{k(s-t)} f(p(\omega^k)),$$

where $0 \le s, t \le n-1$ (the rows and columns are indexed by $0, 1, \ldots, n-1$).

Theorem 111. *Let the minimal polynomial of an $n \times n$ matrix A be $\Delta(\lambda) = \prod_{k=1}^{s} (\lambda - \lambda_k)$, where the eigenvalues $\lambda_1, \ldots, \lambda_s$ are distinct. Then*

$$L_t(A) = \frac{1}{\Delta'(\lambda_t)} \cdot \left(\frac{\mathrm{adj}(\lambda I - A)}{\Theta(\lambda)} \right)_{\lambda = \lambda_t}.$$

Remark 47. Note that $\Theta(\lambda)$ was defined to be the greatest common divisor of those polynomials that are the entries of $\operatorname{adj}(\lambda I - A)$. Thus $\frac{\operatorname{adj}(\lambda I - A)}{\Theta(\lambda)}$ is a matrix whose entries are still polynomials of λ and we may plug in $\lambda = \lambda_t$. Directly writing $\frac{\operatorname{adj}(\lambda_k I - A)}{\Theta(\lambda_t)}$ would be incorrect, as $\Theta(\lambda_t)$ may be 0.

Proof. Let us consider the following 2-variable polynomial:

$$F(x, y) = \frac{\Delta(x) - \Delta(y)}{x - y}.$$

(Note that for any nonnegative integer j we have

$$\frac{x^j - y^j}{x - y} = x^{j-1} + x^{j-2}y + \cdots + y^{j-1},$$

so $F(x, y)$ is indeed a polynomial.) The degree of F is $\deg F = \deg \Delta - 1 = s - 1$.

For a fixed x the function $F(x, y)$ becomes a polynomial with one variable (y). Let us express $F(x, y)$ as

$$F(x, y) = \sum_{k=1}^{s} F(x, \lambda_k) L_k(y),$$

where $L_1, \ldots, L_s$ are the Lagrange basis polynomials for the points $\lambda_1, \ldots, \lambda_s$. The two sides of the above equation are equal to each other, since they coincide at the points $\lambda_1, \ldots, \lambda_s$ and they have degree at most $s - 1$.

That is,

$$\frac{\Delta(x) - \Delta(y)}{x - y} = \sum_{k=1}^{s} \frac{\Delta(x) - \Delta(\lambda_k)}{x - \lambda_k} L_k(y) = \sum_{k=1}^{s} \underbrace{\frac{\Delta(x)}{x - \lambda_k}}_{\Delta'(\lambda_k) L_k(x)} L_k(y) - \sum_{k=1}^{s} \frac{\overbrace{\Delta(\lambda_k)}^{0}}{x - \lambda_k} L_k(y) =$$

$$= \sum_{k=1}^{s} \Delta'(\lambda_k) L_k(x) L_k(y),$$

since

$$\frac{\Delta(x)}{x - \lambda_k} = \Delta'(\lambda_k) L_k(x)$$

according to Remark 45.

After multiplying by $x - y$ we obtain the following polynomial identity:

$$\Delta(x) - \Delta(y) = \sum_{k=1}^{s} \Delta'(\lambda_k) L_k(x) L_k(y)(x - y).$$

The matrices λI and A commute with each other ($\lambda I A = A \lambda I$), thus we may plug $x = \lambda I$, $y = A$ in this identity:

$$\underbrace{\Delta(\lambda I)}_{\Delta(\lambda) I} - \underbrace{\Delta(A)}_{0} = \sum_{k=1}^{s} \Delta'(\lambda_k) \underbrace{L_k(\lambda I)}_{L_k(\lambda) I} L_k(A)(\lambda I - A).$$

If λ is *not* an eigenvalue of A, then $\lambda I - A$ is invertible and we obtain that

$$\Delta(\lambda)(\lambda I - A)^{-1} = \sum_{k=1}^{s} \Delta'(\lambda_k) L_k(\lambda) L_k(A).$$

We know that

$$(\lambda I - A)\operatorname{adj}(\lambda I - A) = D(\lambda) I,$$

thus

$$(\lambda I - A)^{-1} = \frac{\operatorname{adj}(\lambda I - A)}{D(\lambda)}.$$

Hence:

$$\underbrace{\frac{\Delta(\lambda)}{D(\lambda)}}_{\frac{1}{\Theta(\lambda)}} \operatorname{adj}(\lambda I - A) = \sum_{k=1}^{s} \Delta'(\lambda_k) L_k(\lambda) L_k(A).$$

Therefore, we obtained that

$$\frac{\operatorname{adj}(\lambda I - A)}{\Theta(\lambda)} = \sum_{k=1}^{s} \Delta'(\lambda_k) L_k(\lambda) L_k(A)$$

holds for every λ which is *not* an eigenvalue of A. On both sides of the equation we have a matrix whose entries are polynomials, thus, by continuity, we get that the equation also holds when λ is one of the (finitely many) eigenvalues. After plugging in $\lambda = \lambda_t$:

$$\left(\frac{\operatorname{adj}(\lambda I - A)}{\Theta(\lambda)} \right)_{\lambda = \lambda_t} = \sum_{k=1}^{s} \Delta'(\lambda_k) \underbrace{L_k(\lambda_t)}_{\delta_{kt}} L_k(A) = \Delta'(\lambda_t) L_t(A),$$

which is equivalent to the statement of the theorem. □

Example 51. *Let*

$$K = \begin{pmatrix} 0 & 1 & \dots & 0 \\ 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \dots & 1 & 0 \end{pmatrix}.$$

We have seen in subsection 7.2 that

$$D(x) = \det(xI - K) = \frac{\sin(n+1)\theta}{\sin\theta},$$

if $x = 2\cos\theta$. Also, we proved that

$$(\operatorname{adj}(xI - K))_{ij} = \begin{cases} \frac{\sin(i\theta)}{\sin\theta} \cdot \frac{\sin((n+1-j)\theta)}{\sin\theta} & \text{if } i \le j \\[2ex] \frac{\sin(j\theta)}{\sin\theta} \cdot \frac{\sin((n+1-i)\theta)}{\sin\theta} & \text{if } i \ge j \end{cases}$$

and

$$(xI-K)^{-1}_{ij} = \begin{cases} \frac{\sin(i\theta)}{\sin\theta}\cdot\frac{\sin((n+1-j)\theta)}{\sin(n+1)\theta} & \textit{if } i\le j \\ \frac{\sin(j\theta)}{\sin\theta}\cdot\frac{\sin((n+1-i)\theta)}{\sin(n+1)\theta} & \textit{if } i\ge j \end{cases}.$$

To determine the eigenvalues of K we determine the roots of

$$\frac{\sin(n+1)\theta}{\sin\theta} = 0.$$

These are $\theta_k = \frac{k\pi}{n+1}$ for $k = 1,\dots,n$, so the eigenvalues of K are

$$\lambda_k = 2\cos\theta_k = 2\cos\frac{k\pi}{n+1}$$

for $k = 1,\dots,n$. Since the eigenvalues are distinct, we have $\Delta(x) = D(x)$: the minimal polynomial is the same as the characteristic polynomial.

Since the derivative of the characteristic polynomial is

$$D'(x) = \frac{(n+1)\cos(n+1)\theta\sin\theta - \sin(n+1)\theta\cos\theta}{\sin^2\theta(-2\sin\theta)},$$

which is obtained by applying the chain rule to $x\to\theta(x)\to D(x)$, that is:

$$\frac{d(D(x))}{dx} = \frac{d(D(x))}{d\theta}\cdot\frac{d\theta}{dx}.$$

After plugging in $x = \lambda_k$ we get that

$$\Delta'(\lambda_k) = D'(\lambda_k) = \frac{(n+1)\cos(k\pi)\sin\frac{k\pi}{n+1} - \overbrace{\sin(k\pi)}^{0}\cos\frac{k\pi}{n+1}}{-2\sin^3\frac{k\pi}{n+1}} = \frac{(-1)^{k+1}(n+1)}{2\sin^2\frac{k\pi}{n+1}}.$$

Now, by Theorem 111 we obtain that

$$L_k(K) = \frac{2\sin^2\frac{k\pi}{n+1}}{(-1)^{k+1}(n+1)}\operatorname{adj}(\lambda_k I - K).$$

Therefore, for $i\le j$ we have

$$(L_k(K))_{ij} = \frac{2\sin^2\frac{k\pi}{n+1}}{(-1)^{k+1}(n+1)}\cdot\frac{\sin\frac{ik\pi}{n+1}\overbrace{\sin\frac{(n-j+1)k\pi}{n+1}}^{(-1)^{k+1}\sin\frac{jk\pi}{n+1}}}{\sin^2\frac{k\pi}{n+1}}.$$

That is,

$$(L_k(K))_{ij} = \frac{2}{n+1}\sin\frac{ik\pi}{n+1}\sin\frac{jk\pi}{n+1}.$$

Note that this formula also applies for the case $i > j$.

Consequently, the spectral decomposition of K is as follows:

$$K = \underbrace{\sqrt{\frac{2}{n+1}}\begin{pmatrix} \sin\frac{\pi}{n+1} & \sin\frac{2\pi}{n+1} & \dots & \sin\frac{n\pi}{n+1} \\ \sin\frac{2\pi}{n+1} & \sin\frac{4\pi}{n+1} & \dots & \sin\frac{2n\pi}{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \sin\frac{n\pi}{n+1} & \sin\frac{2n\pi}{n+1} & \dots & \sin\frac{n^2\pi}{n+1} \end{pmatrix}}_{=:U} \begin{pmatrix} 2\cos\frac{\pi}{n+1} & & & \\ & 2\cos\frac{2\pi}{n+1} & & \\ & & \ddots & \\ & & & 2\cos\frac{n\pi}{n+1} \end{pmatrix} U^*.$$

14. Hermite-interpolation

Now, we will continue with the general case when the minimal polynomial might have multiple roots. Let the characteristic polynomial of an $n \times n$ matrix be

$$D(\lambda) = \prod_{k=1}^{s} (\lambda - \lambda_k)^{\alpha_k},$$

and the minimal polynomial be

$$\Delta(\lambda) = \prod_{k=1}^{s} (\lambda - \lambda_k)^{\beta_k}.$$

Here the sum of the algebraic multiplicities is $\sum \alpha_k = n$. For the multiplicities in the minimal polynomial we have $1 \le \beta_k \le \alpha_k$, let $m := \sum \beta_k$. Note that $m \le n$.

For a function

$$f(z) = \sum_{\ell=0}^{\infty} c_\ell z^\ell$$

let $S_N(z) = \sum\limits_{\ell=0}^{N} c_\ell z^\ell$ be the Nth Taylor polynomial. Let us perform the polynomial division

$$S_N(z) = \Delta(z) q(z) + R_N(z),$$

where $\deg R_N \le \deg \Delta - 1 = m - 1$.

For every eigenvalue λ_k we have

$$S_N(\lambda_k) = \underbrace{\Delta(\lambda_k)}_{0} q(\lambda_k) + R_N(\lambda_k) = R_N(\lambda_k),$$

that is, S_N and R_N attain the same values at λ_k.

Now, let us take the derivative:

$$S_N'(z) = \Delta'(z) q(z) + \Delta(z) q'(z) + R_N'(z).$$

If an eigenvalue λ_k is a multiple root of Δ, then it is also a root of Δ', thus

$$S_N'(\lambda_k) = \underbrace{\Delta'(\lambda_k)}_{0} q(\lambda_k) + \underbrace{\Delta(\lambda_k)}_{0} q'(\lambda_k) + R_N'(\lambda_k) = R_N'(\lambda_k),$$

that is, at multiple roots the first derivative of S_N and R_N also attain the same value. More generally, with a similar calculation it follows that $S_N^{(t)}(\lambda_k) = R_N^{(t)}(\lambda_k)$ for every $0 \le t \le \beta_k - 1$. This way we obtain $\sum \beta_k = m$ linear equations for the coefficients of the polynomial R_N. The polynomial R_N has degree at most $m - 1$, and we will see that the m coefficients of it can be uniquely determined in such a way that all of these constraints are satisfied.

This type of interpolation is called *Hermite-interpolation.* Let us formulate the main question. The data points $x_1, \ldots, x_s$ are given. The value of a function f is given at these points:

$$f(x_1) = y_1, \ldots, f(x_s) = y_s.$$

At some of these points the first derivative,

$$f'(x_k) = y'_k$$

is also prescribed. More generally, for each point x_k a positive integer β_k is given, and

$$f^{(t)}(x_k) = y_k^{(t)}$$

is prescribed for each $0 \le t \le \beta_k - 1$. The total number of constraints is $\beta_1 + \cdots + \beta_s =: m$, and we will show that there exists a uniquely determined polynomial with degree at most $m-1$, satisfying all of these constraints.

Analogously to Lagrange basis polynomials here we are going to use Hermite basis polynomials that satisfy the following requirements:

$$H_{k,\nu}^{(\mu)}(x_\ell) = \delta_{k\ell}\delta_{\mu\nu},$$

where the polynomial is defined for every $1 \le k \le s$ and $0 \le \nu \le \beta_k - 1$ and the condition needs so be satisfied for every $1 \le \ell \le s$ and $0 \le \mu \le \beta_\ell - 1$.

Lemma 112. *Let $x_1, \ldots, x_s$ be distinct points and $\beta_1, \ldots, \beta_s$ be positive integers. Then there exists a uniquely determined polynomial $H_{k,\nu}$ with degree at most $\beta_1 + \cdots + \beta_s - 1$ satisfying*

$$H_{k,\nu}^{(\mu)}(x_\ell) = \delta_{k\ell}\delta_{\mu\nu}$$

for every $1 \le \ell \le s$ and $0 \le \mu \le \beta_\ell - 1$.

Proof. We omit the proof. □

Remark 48. In other words, the above lemma states that the Hermite basis polynomials always exist and they are uniquely determined.

Corollary 113. *Let $x_1, \ldots, x_s$ be distinct points and $\beta_1, \ldots, \beta_s$ be positive integers. Let $y_k^{(\nu)}$ be given constants for $1 \le k \le s$ and $0 \le \nu \le \beta_k - 1$. Then there exists a uniquely determined polynomial f with degree at most $\beta_1 + \cdots + \beta_s - 1$ satisfying*

$$f^{(\mu)}(x_\ell) = y_\ell^{(\mu)}$$

for every $1 \le \ell \le s$ and $0 \le \mu \le \beta_\ell - 1$.

Namely,

$$f(x) = \sum_{k=1}^{s} \sum_{\nu=0}^{\beta_k - 1} y_k^{(\nu)} H_{k,\nu}(x).$$

Example 52. *Let $s = 1$, that is, only one data point x_1 is given. Let $t := \beta_1 + 1$, and we are looking for a polynomial $p(x)$ with degree at most t satisfying the following constraints:*

$$p(x_1) = y_1,$$
$$p'(x_1) = y'_1,$$
$$\vdots$$

$$p^{(t)}(x_1) = y_1^{(t)},$$

where $y_1, y_1', \dots, y^{(t)}$ are given parameters. Let

$$p(x) = a_t x^t + a_{t-1}x^{t-1} + \dots + a_0.$$

Then we get the following system of linear equations:

$$\begin{aligned} a_t x_1^t + a_{t-1}x_1^{t-1} + \dots + a_0 &= y_1, \\ t a_t x_1^{t-1} + (t-1)a_{t-1}x_1^{t-2} + \dots + a_1 &= y_1', \\ &\vdots \\ t! a_t &= y_1^{(t)}. \end{aligned}$$

The matrix form of this linear system is as follows:

$$\begin{pmatrix} x_1^t & x_1^{t-1} & \dots & x_1^2 & x_1 & 1 \\ t x_1^{t-1} & (t-1)x_1^{t-2} & \dots & 2x_1 & 1 & 0 \\ t(t-1)x_1^{t-2} & (t-1)(t-2)x_1^{t-3} & \dots & 2 & 0 & 0 \\ & & \iddots & & & \\ t! x_1 & (t-1)! & \dots & 0 & 0 & 0 \\ t! & 0 & \dots & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_t \\ a_{t-1} \\ \vdots \\ a_2 \\ a_1 \\ a_0 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_1' \\ y_1'' \\ \vdots \\ y_1^{(t-1)} \\ y_1^{(t)} \end{pmatrix}.$$

Note that the solution is unique: a_t can be determined with the help of the last equation, then a_{t-1} can be determined with the help of the t-th equation, and so on..., finally a_0 is determined by the first equation.

Example 53. *Let*

$$x_1 = -1, \; x_2 = 2 \text{ and } \beta_1 = 1, \; \beta_2 = 2.$$

This choice corresponds to the case when $f(-1)$, $f(2)$, $f'(2)$ are prescribed. The task is to find a polynomial with degree at most 2 such that

$$f(-1) = 6, \; f(2) = 3, \; f'(2) = 2.$$

Let us determine the Hermite basis polynomials H_{10}, H_{20}, H_{21}. Note that these polynomials have degree at most $\beta_1 + \beta_2 - 1 = 2$.

The polynomial H_{10} has to satisfy the following constraints:

$$-1 \xmapsto{H_{10}} 1, \quad 2 \xmapsto{H_{10}} 0, \quad 2 \xmapsto{H_{10}'} 0.$$

Since $H_{10}(2) = H_{10}'(2) = 0$, the value 2 is a multiple root of H_{10}, thus

$$H_{10}(x) = c(x-2)^2$$

for some constant c. As $H_{10}(-1) = 1$ we get that $c = 1/9$. Therefore,

$$H_{10}(x) = \frac{1}{9}(x-2)^2 = \frac{1}{9}(x^2 - 4x + 4).$$

The polynomial H_{20} *has to satisfy the following constraints:*

$$-1 \overset{H_{20}}{\longmapsto} 0, \quad 2 \overset{H_{20}}{\longmapsto} 1, \quad 2 \overset{H_{20}'}{\longmapsto} 0.$$

Since $H_{20}(-1) = 0$*, the value* -1 *is a root of* H_{20}*, thus*

$$H_{20}(x) = (x+1)(ax+b)$$

for some constants a, b*. Note that*

$$H_{20}'(x) = ax + b + (x+1)a.$$

Therefore:

$$H_{20}(2) = 1 \implies 3(2a+b) = 1,$$
$$H_{20}'(2) = 0 \implies 2a + b + 3a = 0.$$

From the first equation $2a + b = \frac{1}{3}$*, so the second equation implies that* $a = -\frac{1}{9}$*. Then we get* $b = \frac{5}{9}$*. Therefore,*

$$H_{20}(x) = \frac{1}{9}(x+1)(-x+5) = \frac{1}{9}(-x^2 + 4x + 5).$$

The polynomial H_{21} *has to satisfy the following constraints:*

$$-1 \overset{H_{21}}{\longmapsto} 0, \quad 2 \overset{H_{21}}{\longmapsto} 0, \quad 2 \overset{H_{21}'}{\longmapsto} 1.$$

Since $H_{21}(-1) = H_{21}(2) = 0$*, the values* -1 *and 2 are roots of* H_{21}*, thus*

$$H_{21}(x) = d(x+1)(x-2)$$

for some constant d*. As* $H_{21}'(x) = d(2x-1)$*, from* $H_{21}'(2) = 1$ *we obtain that* $d = \frac{1}{3}$*. Therefore,*

$$H_{21}(x) = \frac{1}{3}(x+1)(x-2) = \frac{1}{3}(x^2 - x - 2).$$

Therefore, if f *is a polynomial with degree at most 2 satisfying that*

$$f(-1) = 6, \; f(2) = 3, \; f'(2) = 2,$$

then

$$f(x) = 6H_{10}(x) + 3H_{20}(x) + 2H_{21}(x) = \frac{2}{3}\left(x^2 - 4x + 4\right) + \frac{1}{3}\left(-x^2 + 4x + 5\right) + \frac{2}{3}\left(x^2 - x - 2\right) = $$
$$= x^2 - 2x + 3.$$

Now, we start collecting some properties of the Hermite basis polynomials.

Proposition 114. *Let* A *be an* $n \times n$ *matrix with characteristic polynomial* $D(\lambda) = \prod_{k=1}^{s} (\lambda - \lambda_k)^{\alpha_k}$*, where the eigenvalues* $\lambda_1, \ldots, \lambda_s$ *are distinct. Furthermore, let the minimal polynomial of* A *be* $\Delta(\lambda) = \prod_{k=1}^{s} (\lambda - \lambda_k)^{\beta_k}$*. Let* $H_{k\nu}$ *denote the Hermite basis polynomial for* $1 \le k \le s$ *and* $0 \le \nu \le \beta_k - 1$*.*

Then

$$\sum_{k=1}^{s} H_{k0}(A) = I.$$

Proof. Let

$$f(x) := \sum_{k=1}^{s} H_{k0}(x).$$

Then

$$f(\lambda_j) = \sum_{k=1}^{s} \underbrace{H_{k0}(\lambda_j)}_{\delta_{kj}} = 1$$

and for every $1 \leq j \leq \beta_k - 1$ we have $f^{(j)}(\lambda_k) = 0$. Hence, $f(x) \equiv 1$, according to Corollary 113.

By plugging in $x = A$ we obtain that

$$f(A) = \sum_{k=1}^{s} H_{k0}(A) = I.$$

□

Proposition 115. *Let A be an $n \times n$ matrix with characteristic polynomial $D(\lambda) = \prod_{k=1}^{s} (\lambda - \lambda_k)^{\alpha_k}$, where the eigenvalues $\lambda_1, \ldots, \lambda_s$ are distinct. Furthermore, let the minimal polynomial of A be $\Delta(\lambda) = \prod_{k=1}^{s} (\lambda - \lambda_k)^{\beta_k}$. We define the Hermite basis polynomials $H_{k\nu}$ for $1 \leq k \leq s$ and $0 \leq \nu \leq \beta_k - 1$.*

Then

$$\sum_{k=1}^{s} (\lambda_k H_{k0}(A) + \underbrace{H_{k1}(A)}_{\text{only for } k \text{ with } \beta_k \geq 2}) = A.$$

Proof. Let

$$f(x) = \sum_{k=1}^{s} (\lambda_k H_{k0}(x) + H_{k1}(x)).$$

Since $f(\lambda_k) = \lambda_k$ for every $1 \leq k \leq s$ and $f'(\lambda_k) = 1$ for every $1 \leq k \leq s$ with $\beta_k \geq 2$, furthermore, $f^{(j)}(\lambda_k) = 0$ for $2 \leq j \leq \beta_k - 1$, by Corollary 113 we get that $f(x) \equiv x$. From this the statement follows. □

This can be further continued for higher powers too:

Proposition 116.

$$\frac{A^2}{2!} = \sum_{k=1}^{s} \left(\frac{\lambda_k^2}{2} H_{k0}(A) + \lambda_k \underbrace{H_{k1}(A)}_{\text{only for } k \text{ with } \beta_k \geq 2} + \underbrace{H_{k2}(A)}_{\text{only for } k \text{ with } \beta_k \geq 3} \right)$$

Proof. Similarly to the previous proofs we can show that

$$\frac{x^2}{2} = \sum_{k=1}^{s} \left(\frac{\lambda_k^2}{2} H_{k0}(x) + \lambda_k H_{k1}(x) + H_{k2}(x) \right).$$

From this the statement follows. □

More generally, the following theorem holds:

Theorem 117. *Let A be an $n \times n$ matrix. Let us assume that each eigenvalue λ_j has absolute value smaller than R: $|\lambda_j| < R$, where R is the radius of convergence of the power series $f(z) = \sum c_k z^k$. Then $f(A)$ can be expressed as*

$$f(A) = \sum_{k=1}^{s} \sum_{\nu=0}^{\beta_k - 1} f^{(\nu)}(\lambda_k) H_{k\nu}(A).$$

Proof. The proof is similar to the proof of Theorem 107. We omit the details. □

14.1. **Jordan normal form.** Now, our aim is to find the so-called Jordan normal form for an arbitrary matrix A. This is a matrix similar to A, consisting of several diagonal blocks that can decomposed to the sum of a scalar matrix and a nilpotent matrix. We will arrive at this form after a series of propositions that describe how the blocks, and then the scalar matrix and nilpotent "part" can be found.

Proposition 118. *$H_{k0}(A)$ is a projection with rank $\mathrm{rk}(H_{k0}(A)) = \alpha_k$. Furthermore, $H_{k0}(A)H_{\ell 0}(A) = 0$ for $k \neq \ell$.*

Proof. Let us consider the polynomial $H_{k0}(x)H_{\ell 0}(x)$. For every $j \in \{1, \dots, s\} \setminus \{k, \ell\}$ the value λ_j is a root of this polynomial with multiplicity at least $2\beta_j$. Also, λ_k is a root with multiplicity at least β_k and λ_ℓ is a root with multiplicity at least β_ℓ. Thus

$$\Delta(x) \mid H_{k0}(x)H_{\ell 0}(x),$$

so for some polynomial $q(x)$ we have

$$H_{k0}(x)H_{\ell 0}(x) = \Delta(x)q(x).$$

Hence,

$$H_{k0}(A)H_{\ell 0}(A) = \underbrace{\Delta(A)}_{0} q(A) = 0,$$

as we claimed.

Observe that by Proposition 114

$$\sum_{\ell=1}^{s} H_{\ell 0}(A) = I.$$

After multiplying by $H_{k0}(A)$ we get that

$$\sum_{\ell=1}^{s} \underbrace{H_{\ell 0}(A)H_{k0}(A)}_{0, \text{ if } k \neq \ell} = H_{k0}(A),$$

implying that

$$H_{k0}^2(A) = H_{k0}(A).$$

That is, $H_{k0}(A)$ is a projection.

Now, observe that

$$(A - \lambda_k I)^{\beta_k} H_{k0}(A) = 0,$$

since $(x - \lambda_k)^{\beta_k} H_{k0}(x)$ is divisible by $\Delta(x)$.

Therefore,

$$\mathrm{rk}((A - \lambda_k I)^{\beta_k}) + \mathrm{rk}(H_{k0}(A)) \le n. \tag{1}$$

Note that 0 is an eigenvalue of $A - \lambda_k I$ with algebraic multiplicity α_k. According to Theorem 92 the matrix $A - \lambda_k I$ is similar to an upper triangular matrix B. As similar matrices have the same characteristic polynomial, the algebraic multiplicity of 0 as an eigenvalue of B is also α_k, hence on the main diagonal of B we have exactly α_k many 0's. Therefore, the algebraic multiplicity of 0 as an eigenvalue of B^{β_k} (which is also an upper triangular matrix having exactly α_k many 0's on its main diagonal) is also α_k. As $(A - \lambda_k I)^{\beta_k} \sim B^{\beta_k}$, it is obtained that the algebraic multiplicity of 0 as an eigenvalue of $(A - \lambda_k I)^{\beta_k}$ is α_k, thus $\mathrm{rk}((A - \lambda_k I)^{\beta_k}) \ge n - \alpha_k$ according to Proposition 82.

Therefore, we get from (1) that $\mathrm{rk}(H_{k0}(A)) \le \alpha_k$. However, we have already seen that

$$\sum_{k=1}^{s} H_{k0}(A) = I,$$

so

$$\sum_{k=1}^{s} \mathrm{rk}(H_{k0}(A)) \ge n$$

also holds. As $\sum_{k=1}^{s} \alpha_k = n$, this implies that $\mathrm{rk}(H_{k0}(A)) = \alpha_k$ for every k. □

Proposition 119. *If $\beta_k \ge 2$, then*

$$H_{k1}(A) = (A - \lambda_k I) H_{k0}(A),$$

and more generally, for every $\nu \ge 1$ and $\beta_k \ge \nu + 1$

$$H_{k\nu}(A) = \frac{1}{\nu!}(A - \lambda_k I)^{\nu} H_{k0}(A).$$

Proof. Let us start with the case $\nu = 1$. It suffices to show that the polynomial

$$f(x) := H_{k1}(x) - (x - \lambda_k) H_{k0}(x)$$

is divisible by $\Delta(x)$.

First we show that f vanishes at the eigenvalues:

$$f(\lambda_k) = H_{k1}(\lambda_k) - (\lambda_k - \lambda_k) H_{k0}(\lambda_k) = 0 - 0 = 0,$$

and for $j \ne k$:

$$f(\lambda_j) = H_{k1}(\lambda_j) - (\lambda_j - \lambda_k) \underbrace{H_{k0}(\lambda_j)}_{0} = 0 - 0 = 0.$$

Now, we continue with the first derivatives. Note that

$$f'(x) = H'_{k1}(x) - H_{k0}(x) - (x - \lambda_k)H'_{k0}(x).$$

By taking $x = \lambda_k$:

$$f'(\lambda_k) = \underbrace{H'_{k1}(\lambda_k)}_{1} - \underbrace{H_{k0}(\lambda_k)}_{1} - (\lambda_k - \lambda_k)H'_{k0}(\lambda_k) = 1 - 1 - 0 = 0.$$

Now, if $j \neq k$ with $\beta_j \geq 2$, then

$$f'(\lambda_j) = \underbrace{H'_{k1}(\lambda_j)}_{0} - \underbrace{H_{k0}(\lambda_j)}_{0} - (\lambda_j - \lambda_k)\underbrace{H'_{k0}(\lambda_j)}_{0} = 0 - 0 - 0 = 0.$$

Finally, for every j (including the case $j = k$) if $2 \leq \mu \leq \beta_j - 1$, then we have $f^{(\mu)}(\lambda_j) = 0$. Therefore, the multiplicity of $x - \lambda_j$ in $f(x)$ is at least β_j (for every $1 \leq j \leq s$), which implies that $\Delta \mid f$. Hence, $f(A) = 0$.

We omit the proof of the case $\nu > 1$, which is very similar to the case $\nu = 1$. Here, it suffices to show that

$$H_{k\nu}(x) - \frac{1}{\nu!}(x - \lambda_k I)^{\nu} H_{k0}(x)$$

is divisible by $\Delta(x)$.

□

Proposition 120. *$H_{k\nu}(A)$ is nilpotent if $\nu > 0$.*

Proof. According to Proposition 119 we have

$$H_{k\nu}(A) = \frac{1}{\nu!}(A - \lambda_k I)^{\nu} H_{k0}(A).$$

By raising this to the power of β_k we obtain that

$$H_{k\nu}^{\beta_k}(A) = \frac{1}{(\nu!)^{\beta_k}}(A - \lambda_k I)^{\nu\beta_k}(H_{k0}(A))^{\beta_k},$$

since $A - \lambda_k I$ and $H_{k0}(A)$ are interchangeable.

Note that

$$(H_{k0}(A))^{\beta_k} = H_{k0}(A),$$

since $H_{k0}(A)$ is a projection. Let us observe that $(x - \lambda_k)^{\nu\beta_k} H_{k0}(x)$ is divisible by $\Delta(x)$, since the multiplicity of λ_k is at least $\nu\beta_k \geq \beta_k$, and for any other eigenvalue λ_j the factor $H_{k0}(x)$ is divisible by $(x - \lambda_j)^{\beta_j}$. Hence, $(H_{k\nu}(A))^{\beta_k} = 0$, the matrix $H_{k\nu}(A)$ is indeed nilpotent.

□

Remark 49. The proof also shows that $(H_{k\nu}(A))^t = 0$, if $\nu t \geq \beta_k$.

Remark 50. A matrix $N \neq 0$ was defined to be nilpotent, if for some exponent t we have $N^t = 0$. This also implies that the minimal polynomial of N is a divisor of x^t. The eigenvalues of N are the roots of the minimal polynomial, so all the eigenvalues of N are equal to 0. As a consequence, a nilpotent matrix can not be diagonalizable, since the diagonal form could only be the zero matrix 0, but $U0U^{-1} = 0$, so the everywhere zero matrix is not similar to any other matrix.

According to Proposition 115 we have

$$A = \sum_{k=1}^{s} (\lambda_k H_{k0}(A) + H_{k1}(A)),$$

where $H_{k0}(A)$ is a projection and $H_{k1}(A)$ is a nilpotent matrix for every k. Let us consider a minimal rank-one decomposition for $H_{k0}(A)$, namely, let

$$H_{k0}(A) = U_k V_k^T.$$

By using that

$$H_{k1}(A) = (A - \lambda_k I) H_{k0}(A)$$

(according to Proposition 119) we obtain the following:

$$\begin{aligned} A = \sum_{k=1}^{s} (\lambda_k H_{k0}(A) + H_{k1}(A)) &= \sum_{k=1}^{s} (\lambda_k H_{k0}(A) + (A - \lambda_k I) H_{k0}(A)) = \\ &= \sum_{k=1}^{s} (\lambda_k H_{k0}(A) + (A - \lambda_k I) H_{k0}^2(A)) = \\ = \sum_{k=1}^{s} (\lambda_k H_{k0}(A) + H_{k0}(A)(A - \lambda_k I) H_{k0}(A)) &= \sum_{k=1}^{s} U_k (\lambda_k I + V_k^T (A - \lambda_k I) U_k) V_k^T, \end{aligned} \tag{2}$$

since $H_{k0}(A)$ and $A - \lambda_k I$ are interchangeable.

As $H_{k0}(A) = U_k V_k^T$ is a minimal rank-one decomposition we have $V_k^T U_k = I_{\alpha_k}$ (where $\alpha_k = \mathrm{rk}(H_{k0}(A))$ is the algebraic multiplicity of the eigenvalue λ_k).

Let us consider the $n \times n$ matrices

$$U = (U_1, U_2, \ldots, U_s)$$

and

$$V^T = \begin{pmatrix} V_1^T \\ V_2^T \\ \vdots \\ V_s^T \end{pmatrix}.$$

We claim that U and V^T are the inverses of each other. Let us consider the product

$$V^T U = \begin{pmatrix} V_1^T \\ V_2^T \\ \vdots \\ V_s^T \end{pmatrix} \begin{pmatrix} U_1 & U_2 & \ldots & U_s \end{pmatrix} = \begin{pmatrix} V_1^T U_1 & V_1^T U_2 & \ldots & V_1^T U_s \\ V_2^T U_1 & V_2^T U_2 & \ldots & V_2^T U_s \\ \vdots & \vdots & \ddots & \vdots \\ V_s^T U_1 & V_s^T U_2 & \ldots & V_s^T U_s \end{pmatrix}.$$

If $i \neq j$, then $H_{i0}(A)H_{j0}(A) = 0$ according to Proposition 118. Hence,

$$0 = H_{i0}(A)H_{j0}(A) = (U_iV_i^T)(U_jV_j^T) = U_i(V_i^TU_j)V_j^T,$$

which implies by Lemma 26 that $V_i^TU_j = 0$, since the columns of U_i are linearly independent and the rows of V_j^T are also linearly independent.

Note that for every i we have $V_i^TU_i = I_{\alpha_i}$, hence

$$V^TU = \begin{pmatrix} I_{\alpha_1} & 0 & \dots & 0 \\ 0 & I_{\alpha_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & I_{\alpha_s} \end{pmatrix} = I.$$

Now, we will look at V^TAU:

$$V^TAU = \begin{pmatrix} V_1^TAU_1 & V_1^TAU_2 & \dots & V_1^TAU_s \\ V_2^TAU_1 & V_2^TAU_2 & \dots & V_2^TAU_s \\ \vdots & \vdots & \ddots & \vdots \\ V_s^TAU_1 & V_s^TAU_2 & \dots & V_s^TAU_s \end{pmatrix}.$$

By using (2) we obtain the following for $V_i^TAU_j$.

$$V_i^TAU_j = \sum_{k=1}^{s} \underbrace{V_i^TU_k}_{\delta_{ik}I}(\lambda_kI + V_k^T(A-\lambda_kI)U_k)\underbrace{V_k^TU_j}_{\delta_{kj}I} = \begin{cases} 0 & \text{if } i \neq j \\ \lambda_iI + V_i^T(A-\lambda_iI)U_i & \text{if } i = j \end{cases}$$

That is, with the exception of the diagonal blocks all the blocks are zero matrices. Let us investigate further the diagonal blocks:

$$V_i^TAU_i = \lambda_iI + V_i^T(A-\lambda_iI)U_i. \tag{3}$$

Here λ_iI is a scalar matrix, and now we show that the second matrix is nilpotent:

Proposition 121. *$V_i^T(A-\lambda_iI)U_i$ is nilpotent.*

Proof. Observe that

$$(V_i^T(A-\lambda_iI)U_i)^{\beta_i+1} = V_i^T(A-\lambda_iI)\underbrace{U_iV_i^T}_{H_{i0}(A)}(A-\lambda_iI)U_i\dots V_i^T(A-\lambda_iI)U_i =$$

$$= V_i^T\underbrace{(A-\lambda_iI)^{\beta_i+1}(H_{i0}(A))^{\beta_i}}_{0}U_i = 0.$$

□

Remark 51. In fact we can change the exponent β_i+1 to β_i. Indeed,

$$U_i(V_i^T(A-\lambda_iI)U_i)^{\beta_i} = \underbrace{(A-\lambda_iI)^{\beta_i}(H_{i0}(A))^{\beta_i}}_{0}U_i = 0$$

implies by Lemma 26 that

$$(V_i^T(A - \lambda_i I)U_i)^{\beta_i} = 0,$$

since the columns of U_i are linearly independent.

Therefore, if the minimal polynomial has only single roots ($\beta_i = 1$ for every i), then

$$V^T A U = \begin{pmatrix} \lambda_1 I_{\alpha_1} & 0 & \dots & 0 \\ 0 & \lambda_2 I_{\alpha_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \lambda_s I_{\alpha_s} \end{pmatrix},$$

a diagonal matrix.

Now, our aim is to find a normal form for nilpotent matrices.

Definition 41. *A (lower) Jordan block is a square matrix of the following form:*

$$\begin{pmatrix} \lambda & 0 & 0 & \dots & 0 \\ 1 & \lambda & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & 1 & \lambda & 0 \\ 0 & \dots & 0 & 1 & \lambda \end{pmatrix}.$$

That is, all the entries on the main diagonal are equal to a value λ and all the entries on the subdiagonal are equal to 1. Finally, all other entries are zeros.

Definition 42. *The Jordan normal form is a symmetric partitioning of a square matrix, where the diagonal blocks are Jordan blocks and every other block is a zero matrix:*

$$\begin{pmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & J_t \end{pmatrix}.$$

(Note that the diagonal entries of two different Jordan blocks might differ from each other.)

First we show the Jordan normal form of a $k \times k$ nilpotent matrix satisfying $N^{k-1} \neq 0$ consists of one single Jordan block, where the diagonal entries are equal to (the unique eigenvalue) 0.

Proposition 122. *Let N be a $k \times k$ nilpotent matrix such that $N^{k-1} \neq 0$. Then*

$$N = T \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & 1 & 0 & 0 \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix} T^{-1}$$

for an invertible matrix T.

Proof. First of all, note that $N^k = 0$, since the characteristic polynomial of N is λ^k, so the Cayley-Hamilton theorem implies that $N^k = 0$.

We will find vectors x and y such that

$$y^T x = 0, \quad y^T N x = 0, \quad \dots, y^T N^{k-2} x = 0, \quad y^T N^{k-1} x = 1.$$

After finding such an x and y we will be done, because of the following:

$$\begin{pmatrix} y^T N^{k-1} \\ y^T N^{k-2} \\ \vdots \\ y^T \end{pmatrix} N \begin{pmatrix} x & Nx & \dots & N^{k-1}x \end{pmatrix} = \begin{pmatrix} y^T N^k x & y^T N^{k+1} x & \dots & y^T N^{2k-1} x \\ y^T N^{k-1} x & y^T N^k x & \dots & y^T N^{2k-2} x \\ \vdots & \vdots & \ddots & \vdots \\ y^T N x & y^T N^2 x & \dots & y^T N^k x \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & 1 & 0 & 0 \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix},$$

where the upper triangular part (including the main diagonal) consists only of zeros, since $N^k = 0$, while the lower triangular part looks like this because of the choice of x and y.

Moreover, with a similar reasoning we get that

$$\begin{pmatrix} y^T N^{k-1} \\ y^T N^{k-2} \\ \vdots \\ y^T \end{pmatrix} \begin{pmatrix} x & Nx & \dots & N^{k-1}x \end{pmatrix} = I,$$

thus

$$T = \begin{pmatrix} y^T N^{k-1} \\ y^T N^{k-2} \\ \vdots \\ y^T \end{pmatrix}$$

is an appropriate choice.

It remains to find such an x and y. Let us choose first y and z in such a way that

$$y^T N^{k-1} z = 1.$$

Since $N^{k-1} \neq 0$, we may choose indices i and j such that $(N^{k-1})_{ij} = c \neq 0$. Then the choice

$$y = e_i \text{ and } z = \frac{1}{c} e_j$$

is fine, where e_ℓ denotes the ℓth standard basis vector.

We will look for x in the form

$$x = (I + c_1 N + c_2 N^2 + \dots + c_{k-1} N^{k-1}) z.$$

Note that

$$y^T N^{k-1} x = y^T N^{k-1}(I + c_1 N + c_2 N^2 + \cdots + c_{k-1} N^{k-1})z = y^T N^{k-1} z = 1.$$

Now, the condition

$$y^T N^{k-2} x = y^T N^{k-2} z + c_1 \underbrace{y^T N^{k-1} z}_{1} = 0$$

holds if and only if

$$c_1 = -y^T N^{k-2} z.$$

We can continue in a similar fashion, the condition

$$y^T N^{k-3} x = y^T N^{k-3} z + c_1 y^T N^{k-2} z + c_2 \underbrace{y^T N^{k-1} z}_{1}$$

holds if and only if

$$c_2 = -y^T N^{k-3} z - c_1 y^T N^{k-2} z.$$

And similarly to the above, the c_i coefficients can be determined one by one in such a way that all requirements are satisfied. □

Remark 52. Note that the above proof is constructive in the sense that we can actually find the corresponding matrix T by following the steps of the proof.

Now, we continue with the case of a general nilpotent matrix and find its Jordan normal form.

Theorem 123. *Let N be a $k \times k$ nilpotent matrix. Then there exists an invertible matrix T such that*

$$N = TJT^{-1},$$

where J is in Jordan normal form, that is, all the entries of J are zeros with the exception of some elements on the subdiagonal that are 1's.

Remark 53. Since all eigenvalues of a nilpotent matrix are equal to 0, the diagonal elements in the Jordan blocks, and also in the Jordan normal form are 0's.

Proof. Let $N^\beta = 0$, where β is minimal. The case $\beta = k$ was covered in Proposition 122, in this case we have only one Jordan block. Let us consider now the case $\beta < k$.

Again, we start by choosing vectors x and y such that

$$y^T x = 0, \quad y^T N x = 0, \quad \ldots, y^T N^{\beta-2} x = 0, \quad y^T N^{\beta-1} x = 1.$$

Such vectors can be found similarly to the previous proof. First we take y and z such that

$$y^T N^{\beta-1} z = 1,$$

then we look for x in the form

$$x = (I + c_1 N + c_2 N^2 + \cdots + c_{\beta-1} N^{\beta-1})z$$

and determine the coefficients $c_1, c_2, \dots, c_{\beta-1}$.

Let us consider the matrices

$$U_1 := \begin{pmatrix} x & Nx & \dots & N^{\beta-1}x \end{pmatrix}, \quad V_1^T := \begin{pmatrix} y^T N^{\beta-1} \\ y^T N^{\beta-2} \\ \vdots \\ y^T \end{pmatrix}.$$

Similarly to the previous proof we get that $V_1^T U_1 = I_\beta$, that is, the columns of U and the columns of V form a (partial) biorthogonal system. Let us extend it to a complete biorthogonal system:

$$U := \begin{pmatrix} U_1 & U_2 \end{pmatrix}, \quad V^T := \begin{pmatrix} V_1^T \\ V_2^T \end{pmatrix},$$

where $V^T U = I_k$.

Let us consider now the matrix

$$V^T N U = \begin{pmatrix} V_1^T \\ V_2^T \end{pmatrix} N \begin{pmatrix} U_1 & U_2 \end{pmatrix} = \begin{pmatrix} V_1^T N U_1 & V_1^T N U_2 \\ V_2^T N U_1 & V_2^T N U_2 \end{pmatrix}.$$

By the choice of U_1 and V_1^T we obtain that the block

$$V_1^T N U_1 = J_\beta = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & 1 & 0 & 0 \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix}$$

is a Jordan block (with 0's on the diagonal).

Note that

$$V_1^T N = \begin{pmatrix} y^T N^\beta \\ y^T N^{\beta-1} \\ \vdots \\ y^T N \end{pmatrix}.$$

The first row is 0, since $N^\beta = 0$. The remaining rows are also rows of V_1^T. Since $V_1^T U_2 = 0$, all the rows of $V_1^T N$ are perpendicular to all the columns of U_2, which yields that $V_1^T N U_2 = 0$. In a similar way, it can also be shown that $V_2^T N U_1 = 0$. Let the fourth block be $\tilde{N} = V_2^T N U_2$. Then:

$$V^T N U = \begin{pmatrix} J_\beta & 0 \\ 0 & \tilde{N} \end{pmatrix}.$$

That is, we found the first Jordan block J_β, and we can continue in a similar way with the smaller nilpotent matrix $\tilde{N}$. $\square$

Example 54. *Let*

$$N = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}.$$

Note that $N^2 = 0$*, so* N *is nilpotent.*

We may choose

$$y = z = e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

then $y^T N z = 1$*. Next, we find* $x = (I + c_1 N)z$*. The condition* $y^T N x = 1$ *holds, and from*

$$0 = y^T x = y^T (I + c_1 N) z = \underbrace{y^T z}_{1} + c_1 \underbrace{y^T N z}_{1} = 1 + c_1$$

we get $c_1 = -1$*. Thus*

$$x = (I - N)z = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Therefore,

$$\begin{pmatrix} x & Nx \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}, \quad \begin{pmatrix} y^T N \\ y^T \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

and

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} N \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

the Jordan normal form of N *consists of one Jordan block.*

Note that it is easy to see that every 2×2 *nilpotent matrix* $A \neq 0$ *has the above Jordan normal form.*

Example 55. *Let*

$$N = \begin{pmatrix} 2 & -1 & 1 & -1 \\ -3 & 4 & -5 & 4 \\ 8 & -4 & 4 & -4 \\ 15 & -10 & 11 & -10 \end{pmatrix},$$

then

$$N^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ -2 & 1 & -1 & 1 \end{pmatrix}$$

and $N^3 = 0$*.*

It can be immediately seen, that there must be a 3×3 *block, as* $N^2 \neq 0$ *but* $N^3 = 0$*. Then the other block can only be a* 1×1 *block. Hence, the Jordan normal form is*

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad or \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

(depending on the order of the two blocks), since there must be a 3×3 *Jordan block, so the other block can only be a* 1×1 *block. These are both Jordan normal forms for the given matrix, we will find the first one, since by our method the larger block is found first.*

Let us choose first a suitable pair of vectors y *and* z*, let*

$$y = e_2, z = e_3,$$

then $y^T N^2 z = 1$ *holds. Next, we find*

$$x = (I + c_1 N + c_2 N^2) z = (I + c_1 N + c_2 N^2) e_3.$$

The condition $y^T N^2 x = 1$ *holds, and from*

$$0 = y^T N x = y^T N (I + c_1 N + c_2 N^2) z = \underbrace{e_2^T N e_3}_{-5} + c_1 \underbrace{e_2^T N^2 e_3}_{1} = -5 + c_1$$

we get $c_1 = 5$.

Finally, from the condition

$$0 = y^T x = y^T (I + c_1 N + c_2 N^2) z = \underbrace{e_2^T e_3}_{0} + 5 \underbrace{e_2^T N e_3}_{-5} + c_2 \underbrace{e_2^T N^2 e_3}_{1} = -25 + c_2$$

we get that $c_2 = 25$*, thus*

$$x = (I + 5N + 25N^2) e_3 = \begin{pmatrix} 5 \\ 0 \\ 21 \\ 30 \end{pmatrix}.$$

Therefore,

$$U_1 = \begin{pmatrix} x & Nx & N^2 x \end{pmatrix} = \begin{pmatrix} 5 & 1 & 0 \\ 0 & 0 & 1 \\ 21 & 4 & 0 \\ 30 & 6 & -1 \end{pmatrix}, \quad V_1^T = \begin{pmatrix} y^T N^2 \\ y^T N \\ y^T \end{pmatrix} = \begin{pmatrix} 2 & -1 & 1 & -1 \\ -3 & 4 & -5 & 4 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

The columns of U *and the columns of* V *form a partial biorthogonal system that we shall extend to a complete biorthogonal system:*

$$I - U_1 V_1^T = \begin{pmatrix} -6 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ -30 & 5 & 0 & 5 \\ -42 & 7 & 0 & 7 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 5 \\ 7 \end{pmatrix} \begin{pmatrix} -6 & 1 & 0 & 1 \end{pmatrix},$$

yielding the complete biorthogonal system

$$U = \begin{pmatrix} 5 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 21 & 4 & 0 & 5 \\ 30 & 6 & -1 & 7 \end{pmatrix}, \quad V^T = \begin{pmatrix} 2 & -1 & 1 & -1 \\ -3 & 4 & -5 & 4 \\ 0 & 1 & 0 & 0 \\ -6 & 1 & 0 & 1 \end{pmatrix}.$$

Hence,

$$V^T N U = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where $V^T U = I$.

Now, turn our attention again at $V^T A U$. According to (3) and Proposition 121 the ith block is

$$V_i^T A U_i = \lambda_i I + V_i^T (A - \lambda_i I) U_i,$$

where

$$V_i^T (A - \lambda_i I) U_i =: N_i$$

is nilpotent, Theorem 123 implies that there exists an invertible T_i such that $T_i N_i T_i^{-1}$ is in Jordan normal form. Then

$$T_i (\lambda_i I + N_i) T_i^{-1} = \lambda_i I + T_i N_i T_i^{-1}$$

is also in Jordan normal form for each i. Therefore, every matrix can be transformed to a Jordan normal form.

Remark 54. Note that the Jordan normal form is unique up to the order of the blocks. Let $\ell_{k,1} \times \ell_{k,1}, \dots, \ell_{k,m_k} \times \ell_{k,m_k}$ be the sizes of the Jordan blocks, where the diagonal elements are equal to the eigenvalue λ_k. The number of such blocks (that is, m_k) is the geometric multiplicity of λ_k, while $\ell_{k,1} + \dots + \ell_{k,m_k}$ is the algebraic multiplicity of λ_k. The multiplicity of the root λ_k in the minimal polynomial is

$$\max(\ell_{k,1}, \dots, \ell_{k,m_k}).$$

Let us take a function

$$f(z) = \sum_{k=0}^{\infty} c_k z^k.$$

If N is nilpotent matrix, for instance, $N^{\ell+1} = 0$, then

$$f(N) = \sum_{k=0}^{\infty} c_k N^k = \sum_{k=0}^{\ell} c_k N^k.$$

Let us assume now that N is similar to a Jordan block (that is, the Jordan normal form of N consists of only one block). Let $N = T^{-1} J T$, where

$$J = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & 1 & 0 & 0 \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix}.$$

Then

$$f(N) = \sum_{k=0}^{\ell} c_k N^k = \sum_{k=0}^{\ell} c_k (T^{-1}JT)^k = T^{-1} \left(\sum_{k=0}^{\ell} c_k J^k \right) T,$$

where

$$\sum_{k=0}^{\ell} c_k J^k = \begin{pmatrix} c_0 & 0 & 0 & \dots & 0 \\ c_1 & c_0 & 0 & \dots & 0 \\ c_2 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & c_1 & c_0 & 0 \\ c_\ell & \dots & c_2 & c_1 & c_0 \end{pmatrix}.$$

If the Jordan normal form of N consists of several blocks, then each block looks like this.

Now, let A be an arbitrary square matrix. According to Theorem 117 we can express $f(A)$ as

$$f(A) = \sum_{k=1}^{s} (f(\lambda_k)I + f'(\lambda_k)(A - \lambda_k I) + \dots + \frac{f^{(\beta_k - 1)}(\lambda_k)}{(\beta_k - 1)!}(A - \lambda_k I)^{\beta_k - 1}) H_{k0}(A).$$

Now, by taking a rank-one decomposition for each $H_{k0}(A)$, say $H_{k0}(A) = U_k V_k^T$, we obtain the complete biorthogonal system

$$U = \begin{pmatrix} U_1 & U_2 & \dots & U_s \end{pmatrix}, \quad V = \begin{pmatrix} V_1 & V_2 & \dots & V_s \end{pmatrix}.$$

Then

$$f(A) = \sum_{k=1}^{s} U_k Q_k V_k^T,$$

where

$$Q_k = f(\lambda_k)I + f'(\lambda_k)V_k^T(A - \lambda_k I)U_k + \dots + \frac{f^{(\beta_k - 1)}(\lambda_k)}{(\beta_k - 1)!} V_k^T (A - \lambda_k I)^{\beta_k - 1} U_k$$

is a polynomial of the matrix

$$V_k^T (A - \lambda_k I) U_k,$$

which is nilpotent by Proposition 121.

Let the Jordan normal form of this matrix be A_k, where

$$V_k^T (A - \lambda_k I) U_k = T_k A_k T_k^{-1}.$$

Therefore,

$$f(A) = UT \sum_{k=1}^{s} \sum_{i=0}^{\beta_k - 1} \frac{f^{(i)}(\lambda_k)}{i!} A_k^i T^{-1} U^{-1}.$$

15. Applications to systems of linear differential equations

Let

$$x = x(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

be a vector-valued function: $x : \mathbb{R} \to \mathbb{R}^n$. A linear system of differential equations is a system of the following type:

$$\begin{aligned} x_1' &= a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + f_1(t) \\ x_2' &= a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + f_2(t) \\ &\vdots \\ x_n' &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + f_n(t) \end{aligned}$$

where $a_{ij}(t), f_i(t)$ are all functions.
Initial values

$$x_i(t_0) = x_{i0} \text{ (for } 1 \le i \le n)$$

might also be prescribed.

For brevity, we may write $x' = Ax + f$, where

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad x' = \begin{pmatrix} x_1' \\ \vdots \\ x_n' \end{pmatrix}, \quad f = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}, \quad x(t_0) = \begin{pmatrix} x_1(t_0) \\ \vdots \\ x_n(t_0) \end{pmatrix}, \quad A = (a_{ij})_{i,j=1}^n.$$

The system is homogeneous, if $f \equiv 0$ and inhomogeneous otherwise.

Let us study first the homogeneous case without initial conditions. It is known that there are n linearly independent solutions in the common interval of continuity of the a_{ij} functions (assuming that t_0 is contained in this interval).

Let us choose n linearly independent solutions in such a way, that at t_0 the ith one is the ith standard basis vector. Let us form an $n \times n$ matrix from these solutions (as columns), obtaining the so-called resolvent matrix X. To obtain the solution we shall determine the matrix X which, in the homogeneous case, satisfies that

$$X' = A(t)X$$

with the initial condition

$$X(t_0) = I.$$

From this the solution is $x = Xx_0$.

In the inhomogeneous case the solution can be written as $x = x_p + z$, where x_p is a particular solution with $x_p(t_0) = 0$ and z is the solution of the homogeneous version. Let us look for a particular solution in the form

$$x_p = Xh(t).$$

Then

$$x_p' = X'h(t) + Xh'(t)$$

and we need $x_p' = Ax_p + f$, that is,

$$\underbrace{X'}_{AX} h(t) + Xh'(t) = AXh(t) + f(t),$$

which holds if and only if

$$Xh'(t) = f(t),$$

that is, $h'(t) = X^{-1}f(t)$. Therefore,

$$h(t) = \int_{t_0}^{t} X^{-1}(\tau)f(\tau)d\tau.$$

This yields that a general solution can be expressed as

$$x = \underbrace{X(t)x_0}_{z} + \underbrace{X(t)\int_{t_0}^{t} X^{-1}(\tau)f(\tau)d\tau}_{x_p}.$$

15.1. **Constant coefficients.** Let us consider the case when the coefficient matrix $A(t)$ does not depend on t, that is,

$$a_{ij}(t) \equiv a_{ij}, \quad A(t) = A.$$

Then the system is $X' = AX$. Note that for $n = 1$ we get the equation $x' = ax$, where $e^{a(t-t_0)}$ is the solution giving the value 1 at t_0. Analogously, here, the matrix

$$X = e^{A(t-t_0)} = I + (t - t_0)A + \frac{(t - t_0)^2}{2!}A^2 + \dots$$

satisfies the constraints, since

$$X' = 0 + A + (t - t_0)A^2 + \dots = AX, \quad X(t_0) = I$$

Hence, the solution is

$$x = e^{A(t-t_0)}x_0.$$

In the inhomogeneous case:

$$x = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}f(\tau)d\tau,$$

since $X(t) = e^{A(t-t_0)}$ and $X^{-1}(\tau) = e^{-A(\tau-t_0)}$.

Let us consider the special case when the minimal polynomial of A has only single roots. Then

$$x = \sum_{k=1}^{s} e^{\lambda_k(t-t_0)}L_k(A)x_0 + \sum_{k=1}^{s}\int_{t_0}^{t} e^{\lambda_k(t-\tau)}L_k(A)f(\tau)d\tau.$$

Let us consider a rank-one decomposition for each $L_k(A)$:

$$L_k(A) = \sum_{\nu=1}^{\alpha_k} u_{\nu k} v_{\nu k}^T.$$

Instead of the double indices νk (for $u_{\nu k}$ and $v_{\nu k}$) and indices k (for λ_k) let us use the indices $j = 1, 2, \ldots, n$. Then

$$x = \sum_{j=1}^{n} e^{\lambda_j (t-t_0)} u_j v_j^T x_0 + \sum_{j=1}^{n} \int_{t_0}^{t} e^{\lambda_j (t-\tau)} u_j v_j^T f(\tau) d\tau.$$

In the general case, when there can be multiple roots in the minimal polynomial:

$$x = \sum_{k=1}^{s} e^{\lambda_k (t-t_0)} \sum_{\nu=0}^{\beta_k - 1} (t-t_0)^\nu H_{k\nu}(A) x_0 + \sum_{k=1}^{s} \int_{t_0}^{t} e^{\lambda_k (t-\tau)} \sum_{\nu=0}^{\beta_k - 1} (t-\tau)^\nu H_{k\nu}(A) f(\tau) d\tau.$$

Example 56. *Let us consider the homogeneous system*

$$x_1' = -3x_1 + 2x_2 - 5x_3$$

$$x_2' = -6x_1 - 5x_2 - 30x_3$$

$$x_3' = x_1 + 3x_3$$

with the initial condition

$$x(0) = \begin{pmatrix} x_{10} \\ x_{20} \\ x_{30} \end{pmatrix}.$$

Note that the characteristic polynomial of the coefficient matrix is

$$D(\lambda) = \det \begin{pmatrix} \lambda + 3 & -2 & 5 \\ 6 & \lambda + 5 & 30 \\ -1 & 0 & \lambda - 3 \end{pmatrix} = (\lambda + 1)(\lambda + 2)^2$$

and the minimal polynomial coincides with the characteristic polynomial:

$$\Delta(\lambda) = D(\lambda) = (\lambda + 1)(\lambda + 2)^2.$$

The eigenvalues are $\lambda_1 = -2, \lambda_2 = -1$. The Hermite basis polynomials are

$$H_{10}(z) = -(z+3)(z+1), \quad H_{20}(z) = (z+2)^2, \quad H_{11}(z) = -(z+2)(z+1).$$

Then

$$\begin{aligned} x = e^{At} x_0 &= (e^{-2t} H_{10}(A) + e^{-t} H_{20}(A) + t e^{-2t} H_{11}(A)) x_0 = \\ &= (-e^{-2t}(A^2 + 4A + 3I) + e^{-t}(A^2 + 4A + 4I) + t e^{-2t}(-A^2 - 3A - 2I)) x_0 = \\ &= [(-e^{-2t} + e^{-t} - t e^{-2t}) A^2 + (-4e^{-2t} + 4e^{-t} - 3t e^{-2t}) A + (-3e^{-2t} + 4e^{-t} - 2t e^{-2t}) I] x_0. \end{aligned}$$

We have seen that with the help of the resolvent X the solution can be expressed as

$$x = \underbrace{X(t)x_0}_{z} + \underbrace{X(t)\int_{t_0}^{t} X^{-1}(\tau)f(\tau)d\tau}_{x_p}.$$

Now, our aim is to determine the resolvent X.

The resolvent matrix X satisfies that $X' = AX$ and $X(t_0) = I$. Therefore, for every t we have

$$X(t) = I + \int_{t_0}^{t} A(\tau)X(\tau)d\tau.$$

The idea of Picard iteration is to start with a matrix, say, $X_0 \equiv I$ and define recursively X_k by

$$X_{k+1}(t) = I + \int_{t_0}^{t} A(\tau)X_k(\tau)d\tau.$$

Without proof we use that the limit of X_k is the resolvent matrix:

Theorem 124. *$X_k(t) \to X(t)$ uniformly.*

Remark 55. By repeatedly applying the recurrence formula we obtain the following:

$$X(t) = I + \int_{t_0}^{t} A(\tau)d\tau + \int_{t_0}^{t}\int_{t_0}^{\tau_1} A(\tau_1)A(\tau_2)d\tau_2 d\tau_1 + \cdots +$$
$$+ \int_{t_0}^{t}\int_{t_0}^{\tau_1}\int_{t_0}^{\tau_2}\cdots\int_{t_0}^{\tau_k} A(\tau_1)A(\tau_2)\dots A(\tau_k)d\tau_k\dots d\tau_2 d\tau_1 + \dots$$

Let us write $B(t) = \int_{t_0}^{t} A(\tau)d\tau$. Let us further assume that $A(t)$ and $B(t)$ are interchangeable:

$$A(t)B(t) = B(t)A(t). \tag{$*$}$$

Then

$$\int_{t_0}^{t}\int_{t_0}^{\tau_1} A(\tau_1)A(\tau_2)d\tau_2 d\tau_1 = \int_{t_0}^{t} A(\tau_1)B(\tau_1)d\tau_1 = \int_{t_0}^{t} B'(\tau_1)B(\tau_1)d\tau_1 =$$
$$= \frac{1}{2}B^2(t) - \frac{1}{2}B^2(t_0) = \frac{1}{2}B^2(t),$$

since

$$(B^2(\tau))' = B'(\tau)B(\tau) + B(\tau)B'(\tau) = 2B'(\tau)B(\tau)$$

by using that $B(\tau)$ and $A(\tau)$ are interchangeable.

Similarly, we obtain that

$$\int_{t_0}^{t}\int_{t_0}^{\tau_1}\int_{t_0}^{\tau_2} A(\tau_1)A(\tau_2)A(\tau_3)d\tau_3 d\tau_2 d\tau_1 = \frac{1}{3!}B^3(t),$$

and in general,

$$\int_{t_0}^{t}\int_{t_0}^{\tau_1}\cdots\int_{t_0}^{\tau_{k-1}} A(\tau_1)A(\tau_2)\dots A(\tau_k)d\tau_k\dots d\tau_2 d\tau_1 = \frac{1}{k!}B^k(t).$$

Therefore,

$$X(t) = I + B(t) + \frac{1}{2!}B^2(t) + \frac{1}{3!}B^3(t) + \cdots + \frac{1}{k!}B^k(t) + \cdots = e^{B(t)} = e^{\int_{t_0}^{t} A(\tau)d\tau}$$

and

$$x = e^{\int_{t_0}^{t} A(\tau)d\tau} x_0 + e^{\int_{t_0}^{t} A(\tau)d\tau}\int_{t_0}^{t} e^{-\int_{t_0}^{\tau} A(\tau_1)d\tau_1} f(\tau)d\tau = e^{\int_{t_0}^{t} A(\tau)d\tau} x_0 + \int_{t_0}^{t} e^{\int_{\tau}^{t} A(\tau_1)d\tau_1} f(\tau)d\tau.$$

Proposition 125. *If $A(t) = h_0(t)I + h_1(t)C + \cdots + h_m(t)C^m$, where C is a constant matrix, then the interchanging property $(*)$ holds.*

Proof. $\int A(\tau)d\tau$ is a polynomial of C, and so does $A(t)$, thus they are interchangeable. □

Remark 56. The circulant matrices can be expressed this way:

$$\begin{pmatrix} c_0(t) & c_1(t) & c_2(t) & c_3(t) & \dots & c_{n-1}(t) \\ c_{n-1}(t) & c_0(t) & c_1(t) & c_2(t) & \dots & c_{n-2}(t) \\ c_{n-2}(t) & c_{n-1}(t) & c_0(t) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \dots & c_1(t) & c_2(t) \\ c_2(t) & c_3(t) & c_4(t) & c_5(t) & \dots & c_1(t) \\ c_1(t) & c_2(t) & c_3(t) & c_4(t) & \dots & c_0(t) \end{pmatrix} = c_0 I + c_1\Omega + c_2\Omega^2 + \cdots + c_{n-1}\Omega^{n-1}.$$

Example 57. *Let us solve the system*

$$x_1' = x_1 \sinh t + x_2 \cosh t$$

$$x_2' = x_1 \cosh t + x_2 \sinh t$$

with the initial condition

$$x(0) = \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix}.$$

Here $A(t)=\begin{pmatrix}\sinh t & \cosh t\\ \cosh t & \sinh t\end{pmatrix}=(\sinh t)I+(\cosh t)\Omega$. *Note that*

$$\int\limits_0^t A(\tau)d\tau=\begin{pmatrix}\cosh t & \sinh t\\ \sinh t & \cosh t\end{pmatrix}-\begin{pmatrix}\cosh 0 & \sinh 0\\ \sinh 0 & \cosh 0\end{pmatrix}=(\cosh t-1)I+(\sinh t)\Omega.$$

Since

$$\Omega=\frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix},$$

we get that

$$B(t)=\frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}\cosh t-1+\sinh t & 0\\ 0 & \cosh t-1-\sinh t\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}.$$

As

$$\cosh t+\sinh t=\frac{e^t+e^{-t}}{2}+\frac{e^t-e^{-t}}{2}=e^t$$

and

$$\cosh t-\sinh t=e^{-t}$$

we obtain that

$$B(t)=\frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}e^t-1 & 0\\ 0 & e^{-t}-1\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}.$$

Hence,

$$e^{B(t)}=\frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}e^{e^t-1} & 0\\ 0 & e^{e^{-t}-1}\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}=\begin{pmatrix}\frac{e^{e^t-1}+e^{e^{-t}-1}}{2} & \frac{e^{e^t-1}-e^{e^{-t}-1}}{2}\\ \frac{e^{e^t-1}-e^{e^{-t}-1}}{2} & \frac{e^{e^t-1}+e^{e^{-t}-1}}{2}\end{pmatrix},$$

yielding

$$x_1(t)=\frac{1}{2}e^{e^t-1}(x_{10}+x_{20})+\frac{1}{2}e^{e^{-t}-1}(x_{10}-x_{20})$$
$$x_2(t)=\frac{1}{2}e^{e^t-1}(x_{10}+x_{20})+\frac{1}{2}e^{e^{-t}-1}(x_{20}-x_{10})$$

.

15.2. **Periodic solution.** Let us consider the differential equation

$$x'=Ax+f,$$

and look for a solution which is periodic with period p:

$$x(t+p)=x(t).$$

Let us assume that A is a constant coefficient matrix. Note that we have

$$x(t)=e^{A(t-t_0)}x_0+\int_{t_0}^t e^{A(t-\tau)}f(\tau)d\tau$$

and similarly,

$$x(t+p) = e^{A(t+p-t_0)}x_0 + \int_{t_0}^{t+p} e^{A(t+p-\tau)} f(\tau)d\tau.$$

From these we get that

$$(I - e^{Ap})x(t) = \underbrace{x(t+p)}_{=x(t)} - e^{Ap}x(t) =$$

$$= \int\limits_{t_0}^{t+p} e^{A(t+p-\tau)} f(\tau)d\tau - \int\limits_{t_0}^{t} e^{A(t+p-\tau)} f(\tau)d\tau = \int\limits_{t}^{t+p} e^{A(t+p-\tau)} f(\tau)d\tau.$$

Hence,

$$x(t) = (I - e^{Ap})^{-1} \int\limits_{t}^{t+p} e^{A(t+p-\tau)} f(\tau)d\tau,$$

if $I - e^{Ap}$ is invertible.

The initial value has to be

$$x(0) = (I - e^{Ap})^{-1} \int\limits_{0}^{p} e^{A(p-\tau)} f(\tau)d\tau.$$

15.3. **Differential equations of order** n**.** A differential equation of order n is an equation of the form

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \cdots + p_1(t)y' + p_0(t)y = f(t),$$

$$y(t_0) = y_0, \quad y'(t_0) = y_0', \quad y''(t_0) = y_0'', \quad \ldots, \quad y^{(n-1)}(t_0) = y_0^{(n-1)},$$

where the functions $p_{n-1}, \ldots, p_0, f$ and the initial values $y_0, y_0', y_0'', \ldots, y_0^{(n-1)}$ are given and we are looking for a function y satisfying the equation.

We can transform such an equation to a linear system as follows. Let us consider the following n functions:

$$y, \quad y_1 = y', \quad y_2 = y'', \quad \ldots, \quad y_{n-1} = y^{(n-1)}.$$

In terms of these functions the problem is to solve the following system.

$$\begin{aligned} y' &= y_1 \\ y_1' &= y_2 \\ &\vdots \\ y_{n-2}' &= y_{n-1} \\ y_{n-1}' &= -p_{n-1}(t)y_{n-1} - \cdots - p_1(t)y_1 - p_0(t)y + f(t) \end{aligned}$$

That is, the coefficient matrix and f are as follows:

$$A = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ -p_0(t) & -p_1(t) & \dots & -p_{n-1}(t) \end{pmatrix}, \quad f = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ f(t) \end{pmatrix}.$$

The initial condition is

$$\begin{pmatrix} y(t_0) \\ y_1(t_0) \\ \vdots \\ y_{n-1}(t_0) \end{pmatrix} = \begin{pmatrix} y_0 \\ y_0' \\ \vdots \\ y_0^{(n-1)} \end{pmatrix}.$$

The solution can be calculated as

$$Xx_0 + X \int_{t_0}^{t} X^{-1}(\tau) f(\tau) d\tau.$$

Note that we are interested only in the first entry, since this entry contains the function y in question. To calculate the 1st entry it suffices to calculate the first row of X and the last column of X^{-1}.

Remark 57. The characteristic polynomial of the matrix

$$\begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ -p_0 & -p_1 & \dots & -p_{n-1} \end{pmatrix}$$

is

$$\lambda^n + p_{n-1}\lambda^{n-1} + \dots + p_2\lambda^2 + p_1\lambda + p_0.$$