Book: G. Fodor:
Signals, Systems and Networks
ISBN 963 05 7924 3

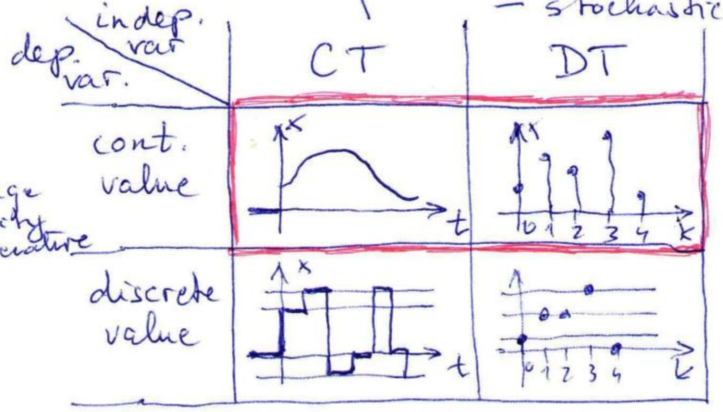
System Theory

HORVÁTH ZOLTÁN
HORVI@EVT.BME.HU

Signal : mathematical description of the for us interesting part of a physical process

deterministic signals ⟷ stochastic signals

- deterministic signals:
  - value at all time value known/can be calculated
  - repeatable
- stochastic signals:
  - not repeatable
  - similar seeming process gives different result
  - stochastic features



ex. voltage velocity temperature

CT signal : $x = x(t)$, where $t \in \mathbb{R}$ (or $-\infty < t < \infty$)

DT signal : $x = x[k]$, where $k \in \mathbb{Z}$

rectangular!

production:

a, from CT singual $x(t) \xrightarrow[\text{(regular)}]{\text{sampling}} x[k] = x(kT)$ ← sampling period

b, independently from CT signal

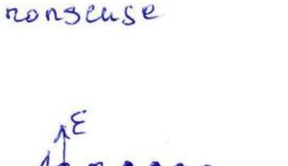
ex throwing with cube : k – number for throwing
$x[k]$ – result of throwing
$k = 1{,}5$ nonsense

ex. $e^{\alpha t} \longleftrightarrow a^k$ exp. signal

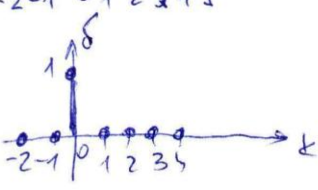
$A \cos \omega t \longleftrightarrow A \cos \vartheta k$

important

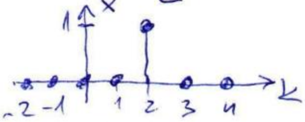
unit step $\varepsilon[k] = \begin{cases} 0, & \text{if } k \in \mathbb{Z}_- \\ 1, & \text{if } k \in \mathbb{N} \end{cases}$



unit impulse $\delta[k] = \begin{cases} 0, & \text{if } k \in \mathbb{Z}_- \\ 1, & \text{if } k = 0 \\ 0, & \text{if } k \in \mathbb{Z}_+ \end{cases}$



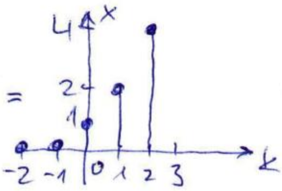
Operations:

– shifting  $x[k] = \delta[k-2]$ ← shifted into + direct.

– windowing: later

– production with convolution: $\boxed{x[k] = \sum_{i=-\infty}^{\infty} x[i] \cdot \delta[k-i]}$ $k \in \mathbb{Z}$

Examples

1, $x[k] = \varepsilon[k] \cdot 2^k =$  $= 1 \cdot \delta[k] + 2 \cdot \delta[k-1] + 4 \cdot \delta[k-2] + \dots$

2, unit step with unit impulse

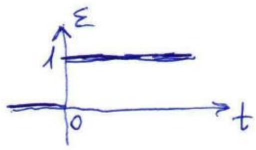
$$\varepsilon[k] = \delta[k] + \delta[k-1] + \delta[k-2] + \dots = \sum_{i=0}^{\infty} \delta[k-i]$$

3, unit impulse with unit step

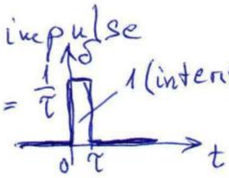
$$\delta[k] = \varepsilon[k] - \varepsilon[k-1]$$

CT signals

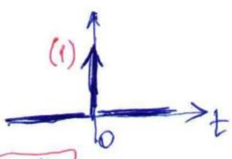
ex $e^{\alpha t}$; $A \cos \omega t$; $A e^{-\alpha t} \cos \omega t$

unit step $\varepsilon(t) = \begin{cases} 0, & \text{if } t \in \mathbb{R}_- \text{ or } t < 0 \\ 1, & \text{if } t \in \mathbb{R}_+ \text{ or } t \geq 0 \end{cases}$ 

Dirac-impulse

$\delta(t, \tau) =$  1 (intensity) $= \dfrac{\varepsilon(t) - \varepsilon(t-\tau)}{\tau} = \begin{cases} \frac{1}{\tau} & \text{if } 0 < t < \tau \\ 0 & \text{otherwise} \end{cases}$

$\Downarrow$ if $\tau \to 0$

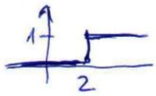
$\delta(t) \approx \dfrac{\varepsilon(t) - \varepsilon(t-\tau)}{\tau} =$  $= \begin{cases} \text{„infinity"}, & \text{if } t \approx 0 \\ 0, & \text{otherwise} \end{cases}$

BUT: $\boxed{\int_{-\infty}^{\infty} \delta(t)\, dt = 1}$!!

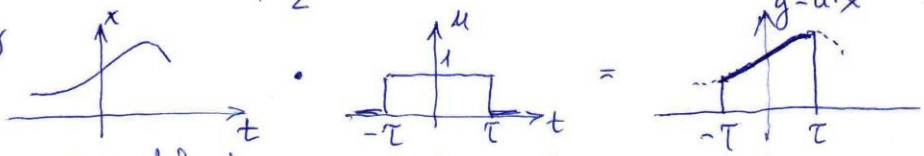
„generalised function"

Operations

- shifting $\varepsilon(t-2) =$

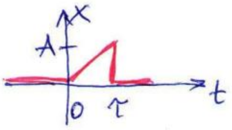


- windowing



original function · windowing f. „window" = cutting out

Ex 1)



$$\boxed{x(t) = \frac{A}{\tau} t \left(\varepsilon(t) - \varepsilon(t-\tau)\right)}$$

2) $x(t) = \begin{cases} 1 & t<0 \\ 2t & 0<t<10 \\ e^{-2t} & t>10 \end{cases}$

$$\boxed{x(t) = 2t \cdot \left(\varepsilon(t) - \varepsilon(t-10)\right) + e^{-2t}\varepsilon(t-10) + \underbrace{\left(1-\varepsilon(t)\right)}_{\varepsilon(-t)}}$$

- generalised derivative:

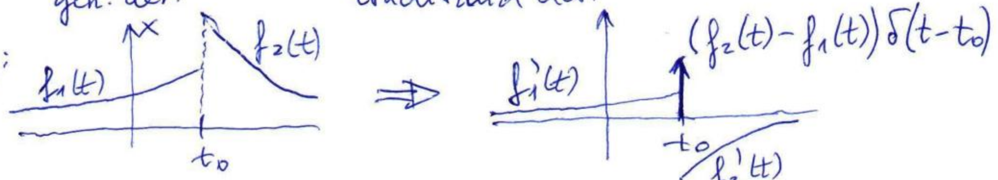
the generalised derivative of an $x(t)$ signal is that $x'(t)$ signal („prime"), from which $x(t)$ can be reconstructed

$$\boxed{x(t) = \int_{t_0}^{t} x'(\tau)\,d\tau + x(t_0)} \qquad \text{typically } t_0 = -\infty$$

$$\varepsilon(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau \Rightarrow \boxed{\varepsilon'(t) = \delta(t)} \Rightarrow$$ function incl. jump can be also differentiated !!

$$\varepsilon(t) \cdot x(t) \Rightarrow \underset{\substack{\uparrow \\ \text{gen. der.}}}{x'(t)} = \left(x(t)\cdot\varepsilon(t)\right)' = \underset{\substack{\uparrow \\ \text{traditional der.}}}{x'(t)} \cdot \varepsilon(t) + x(+0)\cdot\delta(t)$$

generally:



Ex 1) $x(t) = \left(2 + e^{-4t}\right)\varepsilon(t)$

$$\hookrightarrow \boxed{x'(t) = -4e^{-4t}\cdot\varepsilon(t) + 3\cdot\delta(t)}$$

2) $x(t) = \left(\varepsilon(t) - \varepsilon(t-\tau)\right)\cdot \sin\frac{\pi t}{\tau}$

$$\hookrightarrow x'(t) = \cancel{\left(\delta(t) - \delta(t-\tau)\right)\cdot \sin\frac{\pi t}{\tau}} + \left(\varepsilon(t) - \varepsilon(t-\tau)\right)\frac{\pi}{\tau}\cos\frac{\pi}{\tau}t$$

Further classification of signals:

– stepping in (causal) signal: $x(t) \equiv 0$, if $t \in \mathbb{R}_-$
$x[k] \equiv 0$, if $t \in \mathbb{Z}_-$

– even signal: $x(-t) = x(t)$ symm. to the vertical axe
odd signal: $x(t) = -x(-t)$ symm. to the origo

– bounded signal: $\exists\ M < \infty$ so, that $|x(t)| < M$

– abs. integrable, if $\boxed{\int_{-\infty}^{\infty} |x(t)|\,dt < \infty}$
abs. summable, if $\boxed{\sum_{k=-\infty}^{\infty} |x[k]| < \infty}$

– finite energy $\equiv$ quadratially integrable / summable

$$\boxed{E_x = \int_{-\infty}^{\infty} |x(t)|^2\,dt < \infty}$$

– finite time: $x(t) = 0$, if $t < t_1$ and $t > t_2$

Ex 1, stepping in?

a, $\varepsilon[k-2]$ Yes $(k \geq 2)$
b, $\sin 2(t-1)$ No
c, $\varepsilon(t-1)\sin 2(t-1)$ Yes $(t \geq 1)$
d, $\varepsilon[k+1]\sin 3(k+1)$ Yes! $(k \geq 0)$

2, periodic?

a, $\cos(4t+5)$ Yes $\omega = 4 = \frac{2\pi}{T} \to T = \frac{\pi}{2} \in \mathbb{R}$
b, $\cos 0{,}2k$ No $\vartheta = 0{,}2 = \frac{2\pi}{L} \to L = 10\pi \notin \mathbb{Z}$
c, $\cos 0{,}17\pi \cdot k$ Yes $\vartheta = 0{,}17\pi = \frac{2\pi}{L} \to L = \frac{200}{17} \notin \mathbb{Z}$, $\in \mathbb{Q}$

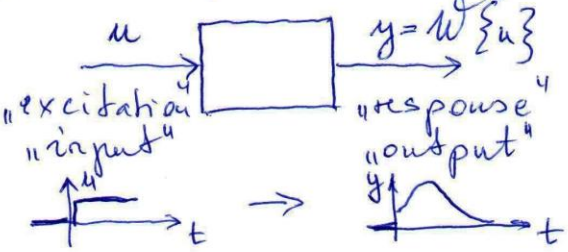
$\downarrow$

$L = 200$

Definition

practical: System = Model of a physical („real") Object

theoretical: System = Transformation: to given excitations orders responses



SISO, MIMO etc.

DT, CT

$\boxed{\text{Linearity}}$ The system is linear,
- if the operator $\mathcal{W}$ is linear: $\mathcal{W}\{c_a u_a + c_b u_b\} = c_a \mathcal{W}\{u_a\} + c_b \mathcal{W}\{u_b\}$
- if for the system the superposition theorem is valid:

u_1, u_2, $a u_1 + b u_2$ → [ ] → y_1, y_2, $a y_1 + b y_2$

The system is $\boxed{\text{invariant}}$, if a shifting in the excitation causes the same shifting in the response

$u_1(t)$, $u_1(t-\tau)$ → [ ] → $y_1(t)$, $y_1(t-\tau)$

The system is $\boxed{\text{causal}}$, if
- the response does not depend on the future values of the exc.
- if $y(t_1)$ depends only on such values of $u(t)$, where $t \le t_1$.
- so to stepping in (causal) excitation belongs „earlier" stepping in response

Ex. noncausal = predictive: („saying the future") $y[k] = u[k+1] \Rightarrow y[0] = u[1]$ reaction is earlier than the excitation

$\boxed{\text{Stability}}$, that means BIBO-stable, if any bounded excitation implies bounded response

Ex. linear? invariant?

a) CT: $y(t) = 5u^2(t)$ — $\cancel{L}\,I$

b) DT: $y[k] = 4^k u[k]$ — $L\,\cancel{I}$ (because of 4^k)

c) CT: $y(t) = 5u(t) + 4$ — $\cancel{L}\,I$ (because of 4)

Impulse response of DT system

for LI systems

$$\boxed{y[k] = h[k] \quad , \text{ if } \; u[k] = \delta[k]}$$

$$\delta[k] \longrightarrow \boxed{} \longrightarrow h[k]$$

– causal system, if imp. response $h[k]$ is stepping in,
that means $h[k] \equiv 0$, if $k \in \mathbb{Z}_-$

– FIR (finite impulse response) system
$h[k] \equiv 0$, if $k \geq L$ (and $k \leq -1$)
L: length of $h[k]$ in time

– IIR : all the others
I: infinite

Ex.
$$y[k] = \frac{1}{4} u[k+1] + \frac{1}{2} u[k] + \frac{1}{4} u[k-1]$$

a, $h[k] = ?$
b, causal?
c, FIR?
d, BIBO-stable?

a, $h[k] = \frac{1}{4} \delta[k+1] + \frac{1}{2} \delta[k] + \frac{1}{4} \delta[k-1]$
b, no (because of $\delta[k+1]$)
c, yes (after $k=2$ $h[k]$ is 0)
d, yes (since it is FIR)

Response with convolution

already used:
$$u[k] = \sum_{i=-\infty}^{\infty} u[i] \cdot \delta[k-i]$$

$\delta[k] \longrightarrow h[k]$

$\delta[k-i] \xrightarrow{\text{invar.}} h[k-i]$

because of linearity

CONVOLUTION SUM

$$\boxed{y[k] = \sum_{i=-\infty}^{\infty} u[i] \cdot h[k-i]} \Rightarrow \; \xrightarrow{u[k]} \boxed{h[k]} \xrightarrow{y[k]}$$

$\Rightarrow$ weighted sum:
$u[i]$ is with weight $h[k-i]$ in $y[k]$
$\Rightarrow$ „weight function"

if $p := k-i \rightarrow \boxed{y[k] = \sum_{p=-\infty}^{\infty} u[k-p]\, h[p]}$

short writing way: $\boxed{y = h * u}$

convolution is:
– commutative $\quad f*g = g*f$
– associative $\quad f*(g+h) = f*g + f*h$
– distributive $\quad f*g*h = (f*g)*h = f*(g*h)$

if causal system and stepping in excitation:

$$y[k] = \sum_{i=0}^{k} u[i] \cdot h[k-i] = \sum_{p=0}^{k} u[k-p]\, h[p]$$

(first sum: upper limit k ← causal system, lower limit $i=0$ ← stepping in exc.; second sum: upper limit k ← stepping in exc., lower limit $p=0$ ← causal syst.)

Unit step response

$$\boxed{u[k] = \varepsilon[k] \longrightarrow y[k] = g[k]}$$

Relation between impulse response and unit step response:

1) $\delta[k] = \varepsilon[k] - \varepsilon[k-1] \xrightarrow{\text{lin, inv}} h[k] = g[k] - g[k-1]$

2) $g[k] = \sum\limits_{p=-\infty}^{\infty} \overset{\varepsilon[k-p]}{u[k-p]} \cdot h[p] = \sum\limits_{p=-\infty}^{k} h[p]$

$0 \leftarrow$ if causal system

Examples

1)
$$h[k] = -1{,}5 \cdot 0{,}8^k \cdot \varepsilon[k] + 2{,}5 \cdot \delta[k]$$
$$u[k] = \varepsilon[k] \cdot 0{,}6^k \longrightarrow y[k] = ?$$

$$y[k] = \sum_{i=-\infty}^{\infty} u[i] \cdot h[k-i] = \sum_{i=0}^{k} 0{,}6^i \left(-1{,}5 \cdot 0{,}8^{k-i} + 2{,}5 \cdot \delta[k-i]\right) =$$

$k \leftarrow$ causal syst.; $i=0 \leftarrow$ stepping in exc.

$$= -1{,}5 \cdot 0{,}8^k \cdot \sum_{i=0}^{k} \left(\tfrac{3}{4}\right)^i + \underbrace{\sum_{i=0}^{k} 2{,}5 \cdot 0{,}6^i \, \delta[k-i]}_{2{,}5 \cdot (1 \cdot \delta[k] + 0{,}6\,\delta[k-1] + 0{,}6^2 \delta[k-2]) = 2{,}5 \cdot} =$$

$$= -1{,}5 \cdot 0{,}8^k \cdot \frac{1-\left(\frac{3}{4}\right)^{k+1}}{1-\frac{3}{4}} + 2{,}5 \cdot 0{,}6^k = \left(\underbrace{-6 \cdot 0{,}8^k}_{\text{free}} + \underbrace{(4{,}5+2{,}5) \cdot 0{,}6^k}_{\text{generated}}\right) \cdot \underbrace{\varepsilon[k]}_{\text{causal!}}$$

solution

2) Numerical evaluation of convolution

$$h[k] = 5\,\varepsilon[k-1]\left(0{,}5^{k-1} - 0{,}1^{k-1}\right)$$
$$u[k] = \varepsilon[k] \longrightarrow y[0] = ?,\ y[1] = ?,\ y[2] = ?,\ y[3] = ?$$

$h[0] = 0 \quad h[1] = 0 \quad h[2] = 2 \quad h[3] = 1{,}2$

$$y[k] = \sum_{i=0}^{k} u[i] \cdot h[k-i]$$

$y[0] = u[0] \cdot h[0] = \underline{\underline{0}}$

$y[1] = u[0] \cdot h[1] + u[1] \cdot h[0] = \underline{\underline{0}}$

$y[2] = u[0] \cdot h[2] + u[1] \cdot h[1] + u[2] \cdot h[0] = 2+0+0 = \underline{\underline{2}}$

$y[3] = \cancel{k}\, u[0] \cdot h[3] + u[1] \cdot h[2] + u[2] \cdot h[1] + u[3] \cdot h[0] =$
$= 1{,}2 + 2 + 0 + 0 = \underline{\underline{3{,}2}}$

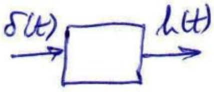
advantage: with digital computer 100000 solutions in 1 sec

disadv.: no closed mathematical form

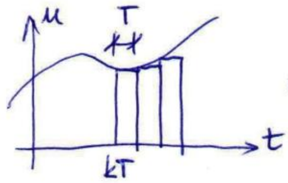
Impulse response of a CT system

for LI system

$$\boxed{y(t) = h(t)\ ,\ \text{if}\quad u(t) = \delta(t)}$$



causal system, if $h(t) \equiv 0$, if $t \in \mathbb{R}_-$



shifted to the given time value

$$\Rightarrow u(t) \cong \sum_{k=-\infty}^{\infty} u(kT) \cdot \underset{\text{between } kT \text{ and } kT+T}{\text{window}} = \sum_{k=-\infty}^{\infty} u(kT) \cdot \delta_T(t-kT) \cdot T$$

$$\delta(t) = \delta_T(t) = \frac{\varepsilon(t) - \varepsilon(t-T)}{T}$$

lin. inv.

$$y(t) \approx \sum_{k=-\infty}^{\infty} u(kT)\, h(t-kT) \cdot T$$

$T \to d\tau$ ← differentially small change

CONVOLUTION

$$\boxed{y(t) = \int_{-\infty}^{\infty} u(\tau) \cdot h(t-\tau)\, d\tau} = \int_{-\infty}^{\infty} u(t-\tau)\, h(\tau)\, d\tau \Leftrightarrow \boxed{y = h * u}$$

commutative, associative, distributive ← causal system

for causal system and stepping in excitation $y(t) = \int_{(-)0}^{t} u(\tau)\, h(t-\tau)\, d\tau$ ← stepping in excitation

can be proofed:

BIBO-stable system $\Longleftrightarrow$ $h(t)$ abs. integrable $\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$

necessary condition: $h(t) \to 0$, if $t \to \infty$ (practically also enough)

Example $\quad h(t) = \varepsilon(t) \cdot \dfrac{1}{t+1}$

similarly for DT system

BIBO-stable system $\Longleftrightarrow$ $h[k]$ abs. summable $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$

necessary condition: $h[k] \to 0$, if $k \to \infty$

unit step response: $\boxed{u(t) = \varepsilon(t) \to y(t) = g(t)}$

gener. derivative

$$g(t) = \int_{-\infty}^{\infty} h(\tau) \cdot \varepsilon(t-\tau)\, d\tau = \int_{-\infty}^{t} h(\tau)\, d\tau \Rightarrow \boxed{h(t) = g'(t)}$$

Examples

1) CT: $h(t) = \varepsilon(t)\left(8e^{-0{,}5t} - 4e^{-0{,}1t}\right)$

a) $u(t) = \varepsilon(t)$ checking with generalised derivative

b) $u(t) = \varepsilon(t) - \varepsilon(t-2)$ $y(t) = ?$

c) $u(t) = 2 + 3\varepsilon(t)$

a) $y(t) = \int_0^t 1\left(8e^{-0{,}5(t-\tau)} - 4e^{-0{,}1(t-\tau)}\right)d\tau = 8e^{-0{,}5t}\left[\frac{e^{0{,}5\tau}}{0{,}5}\right]_0^t - 4e^{-0{,}1t}\left[\frac{e^{0{,}1\tau}}{0{,}1}\right]_0^t =$

$= 16 - 16e^{-0{,}5t} - 40 + 40e^{-0{,}1t} = \boxed{\Big(\underbrace{-24}_{\substack{\text{generated}\\ \text{particular}\\ \text{steady state}}} \underbrace{- 16e^{-0{,}5t} + 40e^{-0{,}1t}}_{\substack{\text{free} \quad \text{free}\\ \text{homogeneous}\\ \text{transient}}}\Big)\varepsilon(t) = g(t)}$ causal system

checking:

$h(t) = g'(t) = \left(8e^{-0{,}5t} - 4e^{-0{,}1t}\right)\varepsilon(t) + \underbrace{0}_{} \cdot \delta(t)$ ✓

b) $y(t) = g(t) \underset{\text{lin.}}{-} g(t \underset{\text{inv.}}{-} 2) = \left(-24 - 16e^{-0{,}5t} + 40e^{-0{,}1t}\right)\varepsilon(t) - \left(-24 - 16e^{-0{,}5(t-2)} + 40e^{-0{,}1(t-2)}\right)\varepsilon(t-2)$

c) $y(t) = -48 + 3\left(-24 - 16^{-0{,}5t} + 40e^{-0{,}1t}\right)\varepsilon(t) = -48 + 3g(t)$

2) $h(t) = 5 \cdot \varepsilon(t) \cdot e^{-2t}$

a) $u(t) = 10\,\varepsilon(t) \cdot \cos 4t$

b) $u(t) = 10\cos 4t$ $\Big\}\ y(t) = ?$

a) $y(t) = \int_{-\infty}^{t} u(\tau)h(t-\tau)d\tau = \int_0^t 10\cos 4\tau \cdot 5 \cdot e^{-2(t-\tau)}d\tau =$

$= 50 \cdot e^{-2t} \cdot \frac{1}{2} \cdot \int_0^t \left(e^{j4\tau} + e^{-j4\tau}\right)e^{2\tau}d\tau = 25 \cdot e^{-2t}\int_0^t \left(e^{(2+j4)\tau} + e^{(2-j4)\tau}\right)d\tau =$

$= 25e^{-2t} \cdot \left[\frac{e^{(2+j4)\tau}}{2+j4} + \frac{e^{(2-j4)\tau}}{2-j4}\right]_0^t =$

$= 25e^{-2t} \cdot \left(\frac{e^{(2+j4)t}}{2+j4} + \frac{e^{(2-j4)t}}{2-j4} - \left(\frac{1}{2+j4} + \frac{1}{2-j4}\right)\right) =$

$= 25 \cdot \left(\frac{e^{j4t}}{\sqrt{20} \cdot e^{j1{,}1\,\text{rad}}} + \frac{e^{-j4t}}{\sqrt{20}\, e^{-j1{,}1}} - \frac{1}{5} \cdot e^{-2t}\right) =$

$= 25 \cdot \left(\frac{1}{\sqrt{5}} \cdot \frac{1}{2}\left(e^{j(4t-1{,}1)} + e^{-j(4t-1{,}1)}\right) - \frac{1}{5}e^{-2t}\right) = \left(5\sqrt{5} \cdot \cos(4t - 1{,}1) - 5e^{-2t}\right)\varepsilon(t)$

b) $y(t) = \underbrace{5 \cdot \sqrt{5}\cos(4t - 1{,}1)}_{\text{steady state!}}$

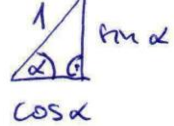
3)

$$h(t) = 10\,\varepsilon(t)\cdot e^{-0,5t}\cos(2t+0,6)$$

$$u(t) = 5 \;\rightarrow\; y(t) = ?$$

$$y(t) = \int_{-\infty}^{\infty} u(\tau)\,h(t-\tau)\,d\tau =$$

$$= \int_{-\infty}^{t} 5\cdot 10\,e^{-0,5(t-\tau)}\underbrace{\cos(2(t-\tau)+0,6)}_{Re\left\{e^{j(2(t-\tau)+0,6)}\right\}}\,d\tau =$$



$$= 50\cdot e^{-0,5t}\,Re\left\{e^{j2t}\cdot e^{j0,6}\int_{-\infty}^{t} e^{(0,5-2j)\tau}\,d\tau\right\} =$$

$$= 50\cdot e^{-0,5t}\,Re\left\{e^{j2t}\cdot e^{j0,6}\cdot\left[\frac{e^{(0,5-2j)\tau}}{0,5-2j}\right]_{-\infty}^{t}\right\} =$$

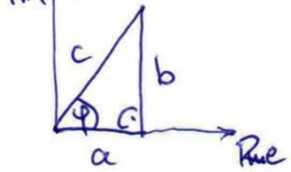
$$= 50\cdot \cancel{e^{-0,5t}}\cdot Re\left\{\frac{1}{0,5-j2}\,e^{\cancel{j2t}}\cdot e^{j0,6}\cdot \cancel{e^{0,5t}}\cdot e^{\cancel{-2jt}}\right\} =$$

$$= 50\,Re\left\{\frac{e^{j0,6}}{2,06\,e^{-j1,33}}\right\} = 50\cdot Re\left\{0,485\cdot e^{j1,93}\right\} =$$

$$= 24,3\cdot\cos 1,93 = \underline{\underline{-\,8,54}}$$

help:

exponential form of complex number



$$a + jb = \sqrt{a^2+b^2}\cdot e^{j\,\arctan\frac{b}{a}}$$ $= \varphi$

STATE VARIABLE DESCRIPTION (State Space Descr.)

- for „dynamic" systems
- N amount of differential equations
- inner variables: in case of network realisation easy
- implicite form with new variables

State variable is $x_i = x_i[k]$ and $x_i = x_i(t)$ $i = 1 \dots N$

for which with $x_i[k_a]$ or $x_i(t_a)$ can be given

a, state variables for $k_b > k_a$ or $t_b > t_a$

b, response in k_a or t_a

⇓

Normal form of state var. descript.

„k+1"!

DT:
$$\boxed{\begin{aligned} \underline{x}[k+1] &= \underline{\underline{A}}\,\underline{x}[k] + \underline{\underline{B}}\,\underline{u}[k] \\ \underline{y}[k] &= \underline{\underline{C}}\,\underline{x}[k] + \underline{\underline{D}}\,\underline{u} \end{aligned}} \iff \begin{aligned} \underline{x}' &= \underline{\underline{A}}\,\underline{x} + \underline{\underline{B}}\,\underline{u} \\ \underline{y} &= \underline{\underline{C}}\,\underline{x} + \underline{\underline{D}}\,\underline{u} \end{aligned}$$

state equation

response normal form

$\underline{x}$ = state vector (state variables in a vector)

$\underline{\underline{A}}$ = system matrix $N \times N$ $\left[\vdots\right]$

if SISO:
$$\underline{x}[k+1] = \underline{\underline{A}} \cdot \underline{x}[k] + \underline{B}\, u[k]$$
$$y[k] = \underline{c}^T \cdot \underline{x}[k] + D \cdot u[k]$$
← first order difference equation system

$[\dots]$ scalar

1st derivative

CT:
$$\underline{x}' = \underline{\underline{A}}\,\underline{x} + \underline{\underline{B}}\,\underline{u}$$
$$\underline{y} = \underline{\underline{C}}\,\underline{x} + \underline{\underline{D}}\,\underline{u}$$

$$x(t_b) = x(t_a + dt) \simeq \underset{\text{known}}{x(t_a)} + \underset{\text{from state eq.}}{x'(t_a)} \cdot dt \quad \text{and} \quad \underline{y}(t_a) = \underline{\underline{C}}\,\underline{x}(t_a) + \underline{\underline{D}}\,\underline{u}(t_a)$$

so the condition for state variables are fulfilled

normal form is important:

a, if A, B, C, D are time independent, then invariant system

b, if it can be given, then lin-system

c, if LI network representation, and cannot be given, then regularity problems

Solutions

- step-by-step solution (DT) or numerical solution (CT)
- putting together with components (free and generated sol.)
- with matrix function

Ex Step-by-step solution

1) DT: $x_1[k+1] = u[k]$

$x_2[k+1] = x_1[k] - \frac{1}{3}x_2[k] + \frac{1}{12}x_3[k] - \frac{1}{2}u[k]$

$x_3[k+1] = x_2[k]$

$y[k] = x_1[k] - \frac{1}{3}x_2[k] + \frac{1}{12}x_3[k] - \frac{1}{2}u[k]$

$u[k] = \varepsilon[k] \downarrow y[k] = ?$

k	$u[k]$	$x_1[k]$	$x_2[k]$	$x_3[k]$	$y[k]$
0	1	0	0	0	$-\frac{1}{2}$
1	1	1	$-\frac{1}{2}$	0	$\frac{2}{3}$
2	1	1	$\frac{2}{3}$	$-\frac{1}{2}$	$\frac{17}{72}$
3	1	1	$\frac{17}{72}$	$\frac{2}{3}$	$\frac{103}{216}$

adv.: easy
exact sol.
automatical (program)

disadv.: no closed math. form

2) With two components

CT: $x_1' = 0{,}1x_2$

$x_2' = -0{,}3x_1 - 0{,}4x_2 + 0{,}7u$

$y = x_2$

$u(t) = \varepsilon(t)\cdot 10 \downarrow y(t) = ?$

HOMOGENOUS sol. (free sol.):

$$\underline{\underline{A}} = \begin{bmatrix} 0 & 0{,}1 \\ -0{,}3 & -0{,}4 \end{bmatrix} \Rightarrow |\lambda \underline{\underline{I}} - \underline{\underline{A}}| = \begin{vmatrix} \lambda & -0{,}1 \\ 0{,}3 & \lambda+0{,}4 \end{vmatrix} = \lambda^2 + 0{,}4\lambda + 0{,}3 = 0 \begin{matrix} \nearrow \lambda_1 = -0{,}1 \\ \searrow \lambda_2 = -0{,}3 \end{matrix}$$

$$\begin{matrix} x_{1h}(t) = M_1 e^{\lambda t} \\ x_{2h}(t) = M_2 e^{\lambda t} \end{matrix} \rightarrow M_1 \lambda \cancel{e^{\lambda t}} = 0{,}1 M_2 \cancel{e^{\lambda t}} \rightarrow M_2 = 10\lambda M_1 \begin{matrix} \nearrow M_{21} = -M_{11} \\ \searrow M_{22} = -3M_{12} \end{matrix}$$

PARTICULAR sol. (generated sol.)

$$\begin{matrix} x_{1p}(t) := X_1 \\ x_{2p}(t) := X_2 \end{matrix} \rightarrow \begin{matrix} 0 = 0{,}1X_2 \rightarrow X_2 = 0 \\ 0 = -0{,}3X_1 - 0{,}4\cdot 0 + 0{,}7\cdot 10 \rightarrow X_1 = \frac{70}{3} \end{matrix}$$

$$x_1(t) = M_{11}e^{-0{,}1t} + M_{12}e^{-0{,}3t} + \frac{70}{3}$$

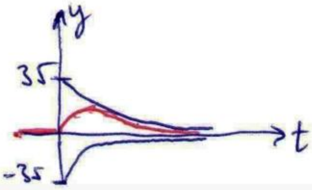
$$x_2(t) = -M_{11}e^{-0{,}1t} - 3M_{12}e^{-0{,}3t}$$

INITIAL CONDITIONS

$$\left.\begin{matrix} x_1(0) = 0 = M_{11} + M_{12} + \frac{70}{3} \\ x_2(0) = 0 = -M_{11} - 3M_{12} \end{matrix}\right\} \quad \begin{matrix} M_{11} = -35 \\ M_{12} = \frac{35}{3} \end{matrix}$$

+1: SUBSTITUTION INTO THE RESPONSE FORMULA

$$\boxed{y(t) = x_2(t) = 35\left(e^{-0{,}1t} - e^{-0{,}3t}\right)\cdot\varepsilon(t)}$$



3) DT: $x_1[k+1] = 0{,}4x_1[k] + 0{,}03x_2[k] + u[k]$
$x_2[k+1] = x_1[k] + 0{,}2x_2[k]$

$y[k] = x_2[k]$

a) $u[k] = \varepsilon[k]$
b) $u[k] = (-0{,}1)^k \cdot \varepsilon[k]$ $\rightarrow y[k] = ?$

---

a) HOM.: $|\lambda\underline{\underline{I}} - \underline{\underline{A}}| = \begin{vmatrix} \lambda - 0{,}4 & -0{,}03 \\ -1 & \lambda - 0{,}2 \end{vmatrix} = \lambda^2 - 0{,}6\lambda + 0{,}05 = 0$ $\nearrow \lambda_1 = 0{,}1$, $\searrow \lambda_2 = 0{,}5$

$y_h[k] = c_1 \cdot 0{,}1^k + c_2 \cdot 0{,}5^k$

PART.: $x_{1p}[k] := A$, $x_{2p}[k] := B$ $\rightarrow$ $\left.\begin{matrix} A = 0{,}4A + 0{,}03B + 1 \\ B = A + 0{,}2B \end{matrix}\right\}$ $B = \frac{20}{9}$

whole sol.: $y[k] = c_1 \cdot 0{,}1^k + c_2 \cdot 0{,}5^k + \frac{20}{9}$

IN. COND.: $x_1[0] = x_2[0] = 0 = y[0]$ $\rightarrow 0 = c_1 + c_2 + \frac{20}{9}$
$y_1[1] = x_1[0] + 0{,}2x_2[0] = 0 \rightarrow 0 = 0{,}1c_1 + 0{,}5c_2 + \frac{20}{9}$
$\Big\}$ $c_1 = \frac{25}{9}$, $c_2 = -5$

$$\boxed{y[k] = \left(\frac{25}{9} \cdot 0{,}1^k - 5 \cdot 0{,}5^k + \frac{20}{9}\right)\varepsilon[k]}$$

b) HOM.: Same

PART.: $x_{1p}(t) := A \cdot (-0{,}1)^k$
$x_{2p}(t) := B\,(-0{,}1)^k$

$\hookrightarrow A \cdot (-0{,}1)^{\cancel{k}+1} = 0{,}4A \cdot \cancel{(-0{,}1)^k} + 0{,}03 \cdot B \cdot \cancel{(-0{,}1)^k} + 1 \cdot \cancel{(-0{,}1)^k}$
$B \cdot (-0{,}1)^{\cancel{k}+1} = A \cdot \cancel{(-0{,}1)^k} + 0{,}2B \cdot \cancel{(-0{,}1)^k}$ $\quad B = \frac{25}{3}$

whole sol.: $y[k] = c_1 \cdot 0{,}1^k + c_2 \cdot 0{,}5^k + \frac{25}{3}(-0{,}1)^k$

IN. COND.: $\left.\begin{matrix} 0 = c_1 + c_2 + \frac{25}{3} \\ 0 = 0{,}1c_1 + 0{,}5c_2 + \frac{25}{3} \cdot (-0{,}1) \end{matrix}\right\}$ $c_1 = -\frac{25}{2}$, $c_2 = \frac{25}{6}$

$$\boxed{y[k] = \left(-\frac{25}{2} \cdot 0{,}1^k + \frac{25}{6} \cdot 0{,}5^k + \frac{25}{3}(-0{,}1)^k\right) \cdot \varepsilon[k]}$$

Solution of state var. description with matrix functions

DT: $\underline{x}[k+1] = \underline{\underline{A}}\,\underline{x}[k] + \underline{\underline{B}}\,\underline{u}[k]$

$\underline{y}[k] = \underline{\underline{C}}\,\underline{x}[k] + \underline{\underline{D}}\,\underline{u}[k]$

Proof: $\underline{x}[1] = \underline{\underline{A}}\,\underline{x}[0] + \underline{\underline{B}}\,\underline{u}[0]$

$\underline{x}[2] = \underline{\underline{A}}\,\underline{x}[1] + \underline{\underline{B}}\,\underline{u}[1] = \underline{\underline{A}}^2\underline{x}[0] + \underline{\underline{A}}\,\underline{\underline{B}}\,\underline{u}[0] + \underline{\underline{B}}\,\underline{u}[1]$

$\underline{x}[3] = \dots = \underline{\underline{A}}^3\underline{x}[0] + \underline{\underline{A}}^2\underline{\underline{B}}\,\underline{u}[0] + \underline{\underline{A}}\cdot\underline{\underline{B}}\,\underline{u}[1] + \underline{\underline{B}}\,\underline{u}[2]$

generalised:

$$\underline{x}[k] = \underline{\underline{A}}^k\underline{x}[0] + \sum_{i=0}^{k-1}\underline{\underline{A}}^{k-1-i}\underline{\underline{B}}\,\underline{u}[i] \qquad k\in\mathbb{Z}_+$$

Response with simple substitution:

$$\boxed{\underline{y}[k] = \begin{cases} \underline{\underline{C}}\,\underline{x}[0] + \underline{\underline{D}}\,\underline{u}[0] & \text{if } k=0 \\ \underline{\underline{C}}\,\underline{\underline{A}}^k\underline{x}[0] + \underline{\underline{C}}\cdot\sum_{i=0}^{k-1}\underline{\underline{A}}^{k-1-i}\underline{\underline{B}}\,\underline{u}[i] + \underline{\underline{D}}\,\underline{u}[k] & \text{if } k\in\mathbb{Z}_+ \end{cases}}$$

solution with Lagrange-matrices without proof:

if single eigenvalues and $\lambda_i \neq 0$

$$\boxed{\underline{\underline{A}}^k = \sum_{i=1}^{N}\lambda_i^k\,\underline{\underline{L}}_i \ , \text{ where } \underline{\underline{L}}_i = \prod_{\substack{p=1\\ p\neq i}}^{N}\frac{\underline{\underline{A}} - \lambda_p\underline{\underline{I}}}{\lambda_i - \lambda_p}}$$

(for multiple eigenvalues with the so called Hermite-matrices)

checking possibility: $\sum_{i=1}^{N}\underline{\underline{L}}_i = \underline{\underline{I}}$

Simplification:

Impulse response

- SISO
- $\underline{x}[0] = \underline{0}$ since $\delta[k]$ is stepping in
- $u[i] = \delta[i]$ is only in $i=0$ nonzero

$$\boxed{h[k] = D\cdot\delta[k] + \varepsilon[k-1]\,\underline{c}^T\cdot\underline{\underline{A}}^{k-1}\underline{b}}$$

because of $k\in\mathbb{Z}_+$

Example

DT:
$$\begin{bmatrix} x_1[k+1] \\ x_2[k+1] \end{bmatrix} = \begin{bmatrix} 0 & -0{,}24 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1[k] \\ x_2[k] \end{bmatrix} + \begin{bmatrix} -0{,}24 \\ 1{,}5 \end{bmatrix} u[k]$$
$$y[k] = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1[k] \\ x_2[k] \end{bmatrix} + u[k]$$
$$u[k] = \varepsilon[k] \cdot 0{,}5^k \longrightarrow y[k] = ?$$

1, eigenvalues:
$$|\lambda \underline{\underline{I}} - \underline{\underline{A}}| = \begin{vmatrix} \lambda & 0{,}24 \\ -1 & \lambda - 1 \end{vmatrix} = \lambda^2 - \lambda + 0{,}24 = 0 \quad \begin{matrix} \nearrow \lambda_1 = 0{,}6 \\ \searrow \lambda_2 = 0{,}4 \end{matrix}$$

2, Lagrange-matrices: $\underline{\underline{L}}_i = \prod\limits_{\substack{p=1 \\ p \neq i}}^{N} \dfrac{\underline{\underline{A}} - \lambda_p \underline{\underline{I}}}{\lambda_i - \lambda_p}$

the other eigenv.

$$\lambda_1 \xrightarrow[p=2]{i=1} \underline{\underline{L}}_1 = \frac{1}{0{,}6-0{,}4} \begin{bmatrix} 0 - 0{,}4 \cdot 1 & -0{,}24 \\ 1 & 1 - 0{,}4 \cdot 1 \end{bmatrix} = \begin{bmatrix} -2 & -1{,}2 \\ 5 & 3 \end{bmatrix}$$

checking: $\underline{\underline{L}}_1 + \underline{\underline{L}}_2 = \underline{\underline{I}}$ ✓

$$\lambda_2 \xrightarrow[p=1]{i=2} \underline{\underline{L}}_2 = \frac{1}{0{,}4-0{,}6} \begin{bmatrix} 0 - 0{,}6 \cdot 1 & -0{,}24 \\ 1 & 1 - 0{,}6 \cdot 1 \end{bmatrix} = \begin{bmatrix} 3 & 1{,}2 \\ -5 & -2 \end{bmatrix}$$

3, response formula: a) $\underline{c}^T \cdot \underline{\underline{A}}^m \cdot \underline{B} = \underline{c}^T \cdot \left(\lambda_1^m \underline{\underline{L}}_1 + \lambda_2^m \cdot \underline{\underline{L}}_2\right) \underline{B} =$
$$= \begin{bmatrix} 0 & 1 \end{bmatrix} \left\{ 0{,}6^m \begin{bmatrix} -2 & -1{,}2 \\ 5 & 3 \end{bmatrix} + 0{,}4^m \begin{bmatrix} 3 & 1{,}2 \\ -5 & -2 \end{bmatrix} \right\} \begin{bmatrix} -0{,}24 \\ 1{,}5 \end{bmatrix} = 3{,}3 \cdot 0{,}6^m - 1{,}8 \cdot 0{,}4^m$$

"number"

b) $k = 0 \longrightarrow y[0] = \underline{c}^T \cdot \underline{x}[0] + D \cdot u[0] = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \text{"}1\text{"} \cdot 1 = 1$

$$k \in \mathbb{Z}_+ \longrightarrow y[k] = \underline{C}\, \underline{\underline{A}}^k \underline{x}[0] + \underline{C} \cdot \sum_{i=0}^{k-1} \underline{\underline{A}}^{k-1-i} \underline{B}\, \underline{u}[i] + \underline{D}\, \underline{u}[k] =$$
$$= 0 + \sum_{i=0}^{k-1} \left\{ \left(3{,}3 \cdot 0{,}6^{k-1-i} - 1{,}8 \cdot 0{,}4^{k-1-i}\right) \cdot 0{,}5^i \right\} + 1 \cdot 0{,}5^k =$$
$$= 3{,}3 \cdot \frac{0{,}6^k}{0{,}6} \sum_{i=0}^{k-1} \left(\frac{0{,}5}{0{,}6}\right)^i - 1{,}8 \cdot \frac{0{,}4^k}{0{,}4} \sum_{i=0}^{k-1} \left(\frac{0{,}5}{0{,}4}\right)^i + 0{,}5^k =$$
$$= 3{,}3 \cdot \frac{0{,}6^k}{0{,}6} \cdot \frac{1 - \left(\frac{0{,}5}{0{,}6}\right)^k}{1 - \frac{0{,}5}{0{,}6}} - 1{,}8 \cdot \frac{0{,}4^k}{0{,}4} \cdot \frac{1 - \left(\frac{0{,}5}{0{,}4}\right)^k}{1 - \frac{0{,}5}{0{,}4}} + 0{,}5^k =$$

$0{,}6 \cdot \left(1 - \frac{0{,}5}{0{,}6}\right) = 0{,}1 \qquad 0{,}4 \cdot \left(1 - \frac{0{,}5}{0{,}4}\right) = -0{,}1$

$$= \boxed{\left(33 \cdot 0{,}6^k + 18 \cdot 0{,}4^k - 50 \cdot 0{,}5^k\right) \varepsilon[k \cancel{-1}]}$$

$-33 - 1{,}8 + 1$!

in $k = 0$ is also good!

Solution with Matrixfunctions for CT system

given: $\underline{u}(t)$ for $t>0$

$\underline{x}(-0)$

formula for $\underline{x}(t) = e^{\underline{\underline{A}}t}\underline{x}(-0) + e^{\underline{\underline{A}}t}\cdot\int_{-0}^{t} e^{-\underline{\underline{A}}\tau}\underline{\underline{B}}\,\underline{u}(\tau)\,d\tau$ for $t>0$

proof:

$$\underline{\dot{x}}(t) = \underline{\underline{A}}\,e^{\underline{\underline{A}}t}\underline{x}(-0) + \underline{\underline{A}}\,e^{\underline{\underline{A}}t}\int_{-0}^{t} e^{-\underline{\underline{A}}\tau}\underline{\underline{B}}\,\underline{u}(\tau)\,d\tau + \cancel{e^{\underline{\underline{A}}t}}\cdot\cancel{e^{-\underline{\underline{A}}t}}\,\underline{\underline{B}}\,\underline{u}(t) =$$

$$= \underline{\underline{A}}\cdot\underline{x}(t) + \underline{\underline{B}}\,\underline{u}(t) \quad \text{qu.e.d.}$$

with simple substitution:

$$\boxed{\underline{y}(t) = \underline{\underline{C}}\,e^{\underline{\underline{A}}t}\underline{x}(-0) + \underline{\underline{C}}\int_{-0}^{t} e^{\underline{\underline{A}}(t-\tau)}\underline{\underline{B}}\,\underline{u}(\tau)\,d\tau + \underline{\underline{D}}\,\underline{u}(t)}$$

where for single eigenvalues using Taylor-series:

$$e^{\underline{\underline{A}}t} = \sum_{i=1}^{N} e^{\lambda_i t}\cdot\underline{\underline{L}}_i$$

where the Lagrange-matrices: $\underline{\underline{L}}_i = \prod_{\substack{p=1\\p\neq i}}^{N} \frac{\underline{\underline{A}} - \lambda_p\underline{\underline{I}}}{\lambda_i - \lambda_p}$

Simplified solution:

1, Impulse response

- SISO
- $\underline{x}(-0) = \underline{0}$
- $\int \text{const}\cdot\delta(\tau)\,d\tau = \text{const}$

$$\boxed{h(t) = D\cdot\delta(t) + \varepsilon(t)\cdot\underline{c}^T e^{\underline{\underline{A}}t}\underline{B}}$$

2, Convolution

Example

CT: $$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad u(t) = \varepsilon(t) \rightarrow y(t) = ?$$

---

eigenvalues: $$\begin{vmatrix} \lambda & -1 \\ 3 & \lambda+4 \end{vmatrix} = \lambda^2 + 4\lambda + 3 = 0 \rightarrow \lambda_1 = -1, \quad \lambda_2 = -3$$

Lagrange: $$\underline{\underline{L}}_i = \prod_{\substack{p=1 \\ p \neq i}}^{N} \frac{\underline{\underline{A}} - \lambda_p \underline{\underline{I}}}{\lambda_i - \lambda_p}$$

$$\lambda_1 \xrightarrow[p=2]{i=1} \underline{\underline{L}}_1 = \frac{1}{-1-(-3)} \begin{bmatrix} 0-(-3) & 1 \\ -3 & -4-(-3) \end{bmatrix} = \begin{bmatrix} 1,5 & 0,5 \\ -1,5 & -0,5 \end{bmatrix}$$

$$\underline{\underline{L}}_2 = \underline{\underline{I}} - \underline{\underline{L}}_1 = \begin{bmatrix} -0,5 & -0,5 \\ 1,5 & 1,5 \end{bmatrix}$$

Impulse response:

$$h(t) = 0 \cdot \delta(t) + \varepsilon(t) \cdot \begin{bmatrix} 1 & 5 \end{bmatrix} \left\{ e^{-t} \begin{bmatrix} 1,5 & 0,5 \\ -1,5 & -0,5 \end{bmatrix} + e^{-3t} \begin{bmatrix} -0,5 & -0,5 \\ 1,5 & 1,5 \end{bmatrix} \right\} \begin{bmatrix} 0 \\ 1 \end{bmatrix} =$$

$$= \left(-2e^{-t} + 7e^{-3t}\right) \varepsilon(t)$$

Convolution: .... $\rightarrow$ $$\boxed{y(t) = \left(\frac{1}{3} + 2e^{-t} - \frac{7}{3} e^{-3t}\right) \varepsilon(t)}$$

Checking of the solution:

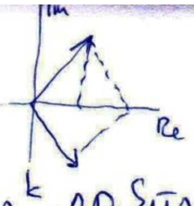
$y_{st} = ?$

$$0 = x_{2st}$$

$$0 = -3x_{1st} - 4 \cdot 0 + 1 \cdot 1 \rightarrow x_{1st} = \frac{1}{3}$$

$$y_{st} = x_{1st} + 5x_{2st} = \frac{1}{3} \checkmark$$

$$y(+0) = x_1(+0) + 5x_2(+0) = x_1(-0) + 5 \cdot x_2(-0) = 0 \checkmark$$



Double (multiple) eigenvalue ...

Complex conjugate eigenvalues

DT: let's assume: $\lambda_2 := \lambda_1^* \Rightarrow y_{tr}[k] = \bar{M}\cdot\lambda_1^k + \bar{M}^*\lambda_2^k = 2\,\mathrm{Re}\{\bar{M}\lambda_1^k\}$

ex: $\lambda_1 := -0{,}5 + j0{,}2 = 0{,}54\, e^{j2{,}76}$ → stable $\forall |\lambda_i| < 1$ $\vartheta = \mathrm{arc}\,\lambda_i$

$M := 5 \cdot e^{j0{,}3}$

DT: $y_{tr}[k] = 2\,\mathrm{Re}\{5\cdot e^{j0{,}3}\cdot(0{,}54\cdot e^{j2{,}76})^k\} = 10\cdot 0{,}54^k\cdot\cos(2{,}76k + 0{,}3)$

CT: $y_{tr}(t) = 2\,\mathrm{Re}\{5\cdot e^{j0{,}3}\cdot e^{(-0{,}5+j0{,}2)t}\} = 10\, e^{-0{,}5t}\cdot\cos(0{,}2t + 0{,}3)$

$2\,\mathrm{Re}\{\bar{M} e^{\lambda_1 t}\}$ $\forall\, \mathrm{Re}\,\lambda_i < 0$ $\omega = \mathrm{Im}\,\lambda_i$

ASYMPTOTIC STABILITY

A LI system is asymptotically stable, if all state variables of the system without excitation ("transient solution") goes to 0 in infinity for any initial values:

DT: $u[k] \equiv 0,\ k \in \mathbb{N} \longrightarrow \lim\limits_{k\to\infty} x[k] = 0$

CT: $u(t) \equiv 0,\ t \in \mathbb{R}_+ \longrightarrow \lim\limits_{t\to\infty} x(t) = 0$

---

asymptotical stability $\Longleftrightarrow$ DT: $\forall |\lambda_i| < 1$ (in the unit circle)
CT: $\forall\, \mathrm{Re}\,\lambda_i < 0$ (left side)

limiting condition of stability: DT: $\exists\, |\lambda_i| = 1$ / CT: $\exists\, \mathrm{Re}\,\lambda_i = 0$ and single (remark: constant in impulse response...)

nonstable: DT: $\exists\, |\lambda_i| > 1$ / CT: $\exists\, \mathrm{Re}\,\lambda_i > 0$ or multiple eigenvalue on unit circle / on imag. axis

---

theorem:

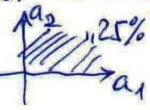
asymptotic stability $\underset{?}{\Longrightarrow}$ BIBO-stability (ex) $0\cdot e^{5t}$ ...

Decidings of stability <u>without</u> calculation of eigenvalues:

CT: $F(\lambda) = \lambda^n + a_1 \lambda^{n-1} + \ldots + a_n = 0$

if $F(\lambda)$ is „Hurwitz-polynomial" then $Re\lambda_i < 0 \rightarrow$ stable

necessary condition: $\forall a_i > 0$ (for $N=2$ also enough)



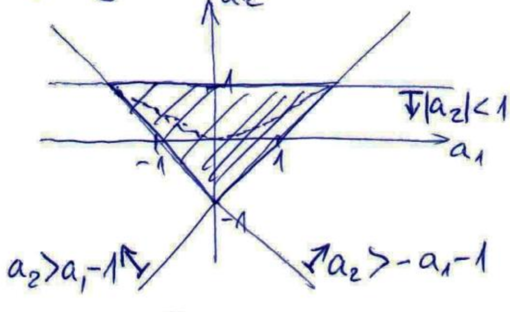
„25%" why...

DT: $N = 2 \rightarrow F(\lambda) = \lambda^2 + a_1\lambda + a_2 = 0$

Jury-criterion: stable, if a, $1 + a_1 + a_2 > 0$

b, $1 - a_1 + a_2 > 0$

c, $|a_2| < 1$



Ex. 1, Given: $\lambda^2 - m\lambda + 0{,}05 = 0$ characteristic equation incl. parameter

CT: Hurwitz: $a_1 > 0 \rightarrow -m > 0 \rightarrow \underline{\underline{m < 0}}$

$a_2 = 0{,}05 > 0$

DT: $\left.\begin{array}{ll} a,\ 1 - m + 0{,}05 > 0 & \rightarrow m < 1{,}05 \\ b,\ 1 + m + 0{,}05 > 0 & \rightarrow m > -1{,}05 \\ c,\ |0{,}05| < 1 & \end{array}\right\} \underline{\underline{-1{,}05 < m < 1{,}05}}$

2, CT: $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} u$

a, asymptotically stable?

b, BIBO-stable?

---

a, $\lambda_1 = -2$

$\lambda_2 = 2 \rightarrow$ ~~non~~ asympt. non stable

b, we do not know, but possible!!

Signal flow networks

realisations, representations of systems

components:
- source: $\triangleright\!\!\rightarrow u[k],\ u(t)$ given; excitation of the system
- sink: $\rightarrow\!\circ\ y[k],\ y(t)$ to be calculated; response of the system

multiplier (amplifier): $p_i \xrightarrow{K_i} q_i$ $\quad q_i[k] = K_i\, p_i[k]$, $\quad q_i(t) = K_i\, p_i(t)$

dynamic comp.:
- DT: delayer (memory): $p_i\ x[k+1] \rightarrow \boxed{>} \rightarrow q_i\ x[k]$ $\quad q_i[k] = p_i[k-1]$ — state var.
- CT: integrator: $x'(t) \rightarrow \boxed{>} \rightarrow x(t)$ $\quad x(t) = \int_{-\infty}^{t} x'(\tau)\,d\tau$ — state var.

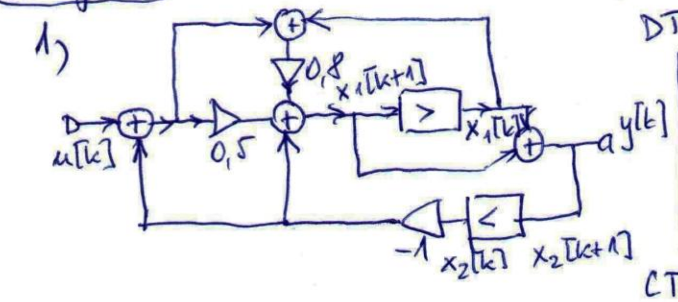
binding rules:

summing (adding) node: $p_1, p_2, p_3 \rightarrow \oplus \rightarrow q$ $\quad q = p_1 + p_2 + p_3$

branching node: $p \rightarrow q_1, q_2, q_3$ $\quad q_1 = p$, $q_2 = p$, $q_3 = p$

Examples

1)

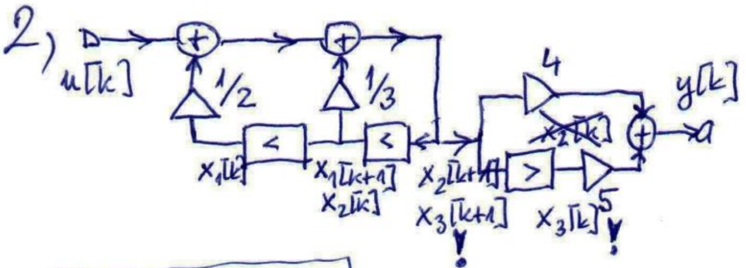


DT: !! !

$$\boxed{\begin{aligned} x_1[k+1] &= 0{,}8\,x_1[k] - 2{,}3\,x_2[k] + 1{,}3\,u[k] \\ x_2[k+1] &= x_1[k+1] + x_1[k] = \\ &= 1{,}8\,x_1[k] - 2{,}3\,x_2[k] + 1{,}3\,u[k] \\ y[k] &= 1{,}8\,x_1[k] - 2{,}3\,x_2[k] + 1{,}3\,u[k] \end{aligned}}$$

CT: $x_1' = \dots$

2)



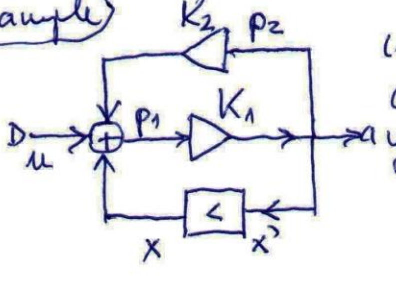
Remark: Order N of the system = = Amount of integrators/delayers of the network!

$$\boxed{\begin{aligned} x_1[k+1] &= x_2[k] \\ x_2[k+1] &= \tfrac{1}{2}\,x_1[k] + \tfrac{1}{3}\,x_2[k] + u[k] \\ x_3[k+1] &= \tfrac{1}{2}\,x_1[k] + \tfrac{1}{3}\,x_2[k] + u[k] \\ y[k] &= \boxed{4\,x_2[k+1] + 5\,x_3[k]} = \boxed{2\,x_1[k] + \tfrac{4}{3}\,x_2[k] + 5\,x_3[k] + 4\,u[k]} \end{aligned}}$$

<u>Regularity of the network</u>

Nonregular, if the normal form of the stat var. descr. to the network cannot be produced

(Example)



helping variables

(1) $p_1 = K_2 p_2 + x + u = K_2 K_1 p_1 + x + u$ $\boxed{\text{if } K_1K_2 \neq 1}$

(2) $p_2 = K_1 p_1$

$$p_1 = \underbrace{\frac{1}{1-K_1K_2}}_{K_0}(x+u)$$

$x' = K_1 p_1 \rightarrow x' = K_1K_0x + K_1K_0u$

$y = K_1 p_1 \rightarrow y = K_1K_0x + K_1K_0u$ ✓ OK.

BUT:

$\boxed{\text{if } K_1K_2 \equiv 1}$ → (1) $p_1 - \frac{1}{K_1}p_2 = x+u$

(2) $p_1 - \frac{1}{K_1}p_2 = 0$

??? $\Rightarrow x = -u$ NO state equation!! → „parametically nonregular"

CT: $y = x' = -u'$ Differtiator

DT: $y[k] = -u[k+1]$ Predictor

No <u>memoryless loop</u> should be ~~into~~ the network!

$\boxed{\text{if } K_1K_2 \approx 1}$ for ex: $K_1 = 1$ and $K_2 = 1+\varepsilon$

$$\lambda = K_1K_0 = K_1 \cdot \frac{1}{1-K_1K_2} = -\frac{1}{\varepsilon}$$

if $\varepsilon = +0{,}001 \Rightarrow \lambda = -1000$

if $\varepsilon = -0{,}001 \Rightarrow \lambda = +1000$

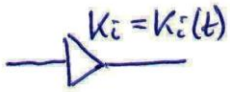
⇓

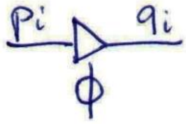
small change in parameter causes <u>drastic</u> changes (f.e. in stability)

Stability of the network

The network is then and only then stable, if the system represented by the network is asymptotically stable

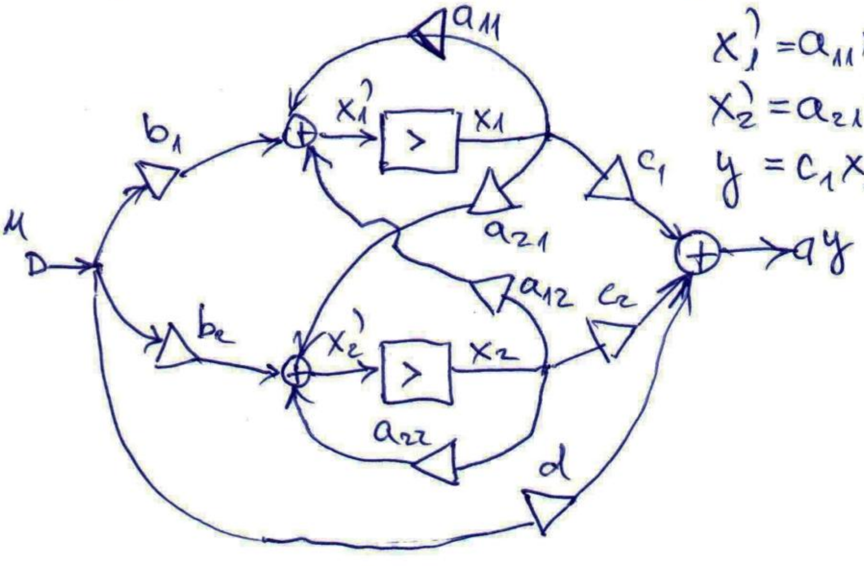
↳ same condition for the eigenvalues

Variant network  ex.: switch $K_i = \varepsilon(t)$

Nonlinear network  $q_i = \phi(p_i)$ ex.: $q_i = p_i^2$

$q_i = 2p_i + 5$

Signal flow network realisation of state var. descr.



$$x_1' = a_{11}x_1 + a_{12}x_2 + b_1u$$
$$x_2' = a_{21}x_1 + a_{22}x_2 + b_2u$$
$$y = c_1x_1 + c_2x_2 + du$$

SINUSOIDAL STEADY STATE SOLUTION

$u(t) = U\cos(\omega t + \varrho)$ $\qquad \omega = \frac{2\pi}{T}$ ← period

$u[k] = U\cos(\vartheta k + \varrho)$ $\qquad \vartheta = \frac{2\pi}{L}$ ← length of period

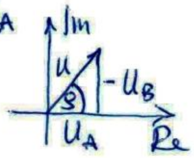
amplitude (U), angular frequency (ω, ϑ), initial phase (ϱ)

other writing way (without initial phase)

$u(t) = U_A\cos\omega t + U_B\sin\omega t = U\cos(\omega t + \varrho) = U\cos\varrho\cos\omega t - U\sin\varrho\sin\omega t$

$U_A = U\cos\varrho$

$U_B = -U\sin\varrho$ $\Rightarrow$ $U = \sqrt{U_A^2 + U_B^2}$; $\operatorname{tg}\varrho = -\frac{U_B}{U_A}$



$U\cos(\omega t + \varrho) = \operatorname{Re}\{\underbrace{U e^{j\varrho}}_{\overline{U}} \cdot e^{j\omega t}\}$

$\overline{U}$ complex amplitude

$e^{j\omega t}$ ← because of linearity in all variables $\Rightarrow$ complex amplitude includes all specialities of the signal

CT: $u(t) = \operatorname{Re}\{\overline{U} e^{j\omega t}\} \rightarrow u'(t) = \operatorname{Re}\{j\omega\overline{U} \cdot e^{j\omega t}\}$

$j\omega\overline{U}$: complex amplitude of derivative → derivative is a multiplication with $j\omega$

DT: $u[k] = \operatorname{Re}\{\overline{U} e^{j\vartheta k}\} \rightarrow u[k-1] = \operatorname{Re}\{\overline{U} \cdot e^{-j\vartheta} \cdot e^{j\vartheta k}\}$

delayer is a multiplication with $e^{-j\vartheta}$

<u>transfer coefficient:</u> $\overline{H} = \frac{\overline{Y}}{\overline{U}}$ ← complex ampl.

<u>transfer characteristic:</u> if ω or ϑ remains as parameter $\rightarrow$ $H(j\omega)$, $H(e^{j\vartheta})$ since ...

(frequency response)

$H(j\omega) = |H(j\omega)|\, e^{j\operatorname{arc} H(j\omega)}$

$|H(j\omega)|$: amplitude charact. (ampl. response) (even function)

$\operatorname{arc} H(j\omega)$: phase characteristic (phase response) (odd function)

calculation of response: $u(t) = U\cos(\omega_0 t + \varrho) \rightarrow \overline{U} = U e^{j\varrho}$

$H(j\omega) \rightarrow H(j\omega_0) = \overline{H} = H e^{j\varphi}$

$\overline{Y} = \overline{H}\,\overline{U} = HU \cdot e^{j(\varphi + \varrho)}$

$\hookrightarrow y(t) = U \cdot H \cos(\omega_0 t + \varrho + \varphi)$

H: amplification; φ: phase shifting

Transfer characteristic from state variable description

$$\underline{x}' = \underline{\underline{A}}\,\underline{x} + \underline{\underline{B}}\,\underline{u}$$

$$\underline{y} = \underline{\underline{C}}\,\underline{x} + \underline{\underline{D}}\,\underline{u}$$

DT: $e^{j\vartheta}$

CT: $(j\omega\underline{\underline{I}} - \underline{\underline{A}})\overline{\underline{X}} = \underline{\underline{B}}\,\overline{\underline{U}} \rightarrow \overline{\underline{X}} = (j\omega\underline{\underline{I}} - \underline{\underline{A}})^{-1}\underline{\underline{B}}\,\overline{\underline{U}}$

$\overline{\underline{Y}} = \underline{\underline{C}}\,\overline{\underline{X}} + \underline{\underline{D}}\,\overline{\underline{U}} \rightarrow \overline{\underline{Y}} = \underbrace{\left(\underline{\underline{C}}(j\omega\underline{\underline{I}} - \underline{\underline{A}})^{-1}\underline{\underline{B}} + \underline{\underline{D}}\right)}_{\underline{\underline{H}}(j\omega)}\overline{\underline{U}}$

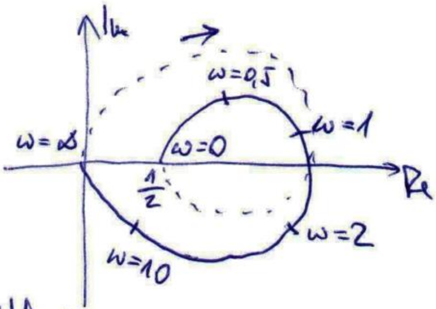
so: CT $\boxed{H(j\omega) = \underline{C}^T(j\omega\underline{\underline{I}} - \underline{\underline{A}})^{-1}\underline{B} + D}$

DT $\boxed{H(e^{j\vartheta}) = \underline{C}^T(e^{j\vartheta}\underline{\underline{I}} - \underline{\underline{A}})^{-1}\cdot\underline{B} + D}$

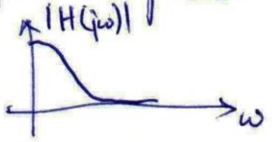
Plot for transfer characteristic

1, Nyquist plot

ex. $H(j\omega) = \dfrac{5(j\omega)+1}{(j\omega)^2 + 4j\omega + 3}$



2, Ampl. and phase char. nonlogarithmic



3, Bode plot: ampl. and phase char logarithmic
later more detailed

Examples

1) CT: $x_1' = -5x_1 + x_2$
$x_2' = -4x_1 + 3u$ $\qquad u(t) = 10\cos(2t+0{,}5) \rightarrow y(t) = ?$
$y = x_1$

$j\omega\overline{X}_1 + 5\overline{X}_1 = \overline{X}_2$

$j\omega\overline{X}_2 + 4\overline{X}_1 = 3\overline{U} \rightarrow (j\omega)^2\cdot\overline{X}_1 + 5j\omega\overline{X}_1 + 4\overline{X}_1 = 3\overline{U}$

tr. char. $H(j\omega) = \dfrac{3}{\underbrace{(j\omega)^2 + 5j\omega + 4}} \rightarrow$ tr. coeff. $H(j2) = \dfrac{3}{-4+10j+4} = 0{,}3e^{-j\frac{\pi}{2}}$

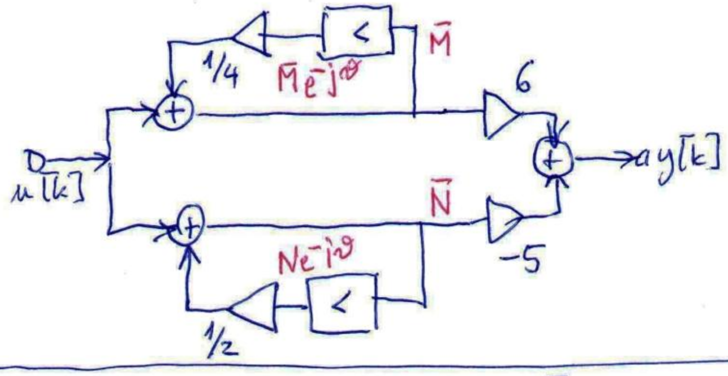
$\overline{U} = 10e^{j0{,}5}$ $\quad \overline{Y} = \overline{U}\cdot H(j2)$

amplific. phase shifting

$\boxed{y(t) = 10\cdot 0{,}3\cdot\cos(2t + 0{,}5 - \frac{\pi}{2})}$

- pol./pol. form
- denom.: highest power is the ~~...~~ order (or less)
- numerator: highest power is the denom. power
- denom must be Hurwitz-pol.
- normal form: denom. 1st coeff is 1

2) DT:



$u[k] = 5\cos\left(\frac{\pi}{6}k + \frac{\pi}{4}\right)$

$\downarrow$

$y[k] = ?$

$$\bar{M} = \frac{1}{4}\bar{M}e^{-j\vartheta} + \bar{U} \rightarrow \bar{M} = \frac{\bar{U}}{1 - \frac{1}{4}e^{-j\vartheta}}$$

$$\bar{N} = \frac{1}{2}\bar{N}e^{-j\vartheta} + \bar{U} \rightarrow \bar{N} = \frac{\bar{U}}{1 - \frac{1}{2}e^{-j\vartheta}}$$

$$\bar{Y} = 6\bar{M} - 5\bar{N} \rightarrow H(e^{j\vartheta}) = \frac{\bar{Y}}{\bar{U}} = \frac{6}{1 - \frac{1}{4}e^{-j\vartheta}} - \frac{5}{1 - \frac{1}{2}e^{-j\vartheta}} = \frac{1 - \frac{7}{4}e^{-j\vartheta}}{1 - \frac{3}{4}e^{-j\vartheta} + \frac{1}{8}e^{-j2\vartheta}}$$

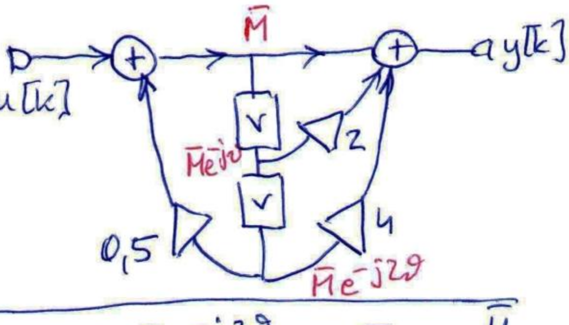
$$H(e^{j\frac{\pi}{6}}) = \frac{1 - \frac{7}{4}e^{-j\frac{\pi}{6}}}{1 - \frac{3}{4}e^{-j\frac{\pi}{6}} + \frac{1}{8}e^{-j\frac{\pi}{3}}} = \frac{1 - \frac{7}{4}\cos\frac{\pi}{6} + j\frac{7}{4}\sin\frac{\pi}{6}}{1 - \frac{3}{4}\cos\frac{\pi}{6} + \frac{1}{8}\cos\frac{\pi}{3} + j\left(\frac{3}{4}\sin\frac{\pi}{6} - \frac{1}{8}\sin\frac{\pi}{3}\right)} =$$

$$= \frac{-0{,}516 + j0{,}875}{0{,}419 + j0{,}267} = \frac{1{,}03e^{j2{,}1}}{0{,}492e^{j0{,}54}} = 2{,}093e^{j1{,}53}$$

$$\bar{U} = 5 \cdot e^{j\frac{\pi}{4}} \Rightarrow \bar{Y} = \bar{U} \cdot H(e^{j\frac{\pi}{6}}) = 5 \cdot 2{,}093\, e^{j\left(\frac{\pi}{4} + 1{,}53\right)}$$

$$\boxed{y[k] = 10{,}465\cos\left(\frac{\pi}{6}k + 2{,}32\right)}$$

3) DT:



$H(e^{j\vartheta}) = ?$

$$\bar{M} = \bar{U} + 0{,}5 \cdot \bar{M}e^{-j2\vartheta} \rightarrow \bar{M} = \frac{\bar{U}}{1 - 0{,}5e^{-j2\vartheta}}$$

$$\bar{Y} = \bar{M}\left(1 + 2e^{-j\vartheta} + 4e^{-j2\vartheta}\right)$$

$$H(e^{j\vartheta}) = \frac{\bar{Y}}{\bar{U}} = \frac{1 + 2e^{-j\vartheta} + 4e^{-j2\vartheta}}{1 - 0{,}5e^{-j2\vartheta}}$$

canonical network representation

CT: $$H(j\omega) = \frac{1 + 2(j\omega)^{-1} + 4(j\omega)^{-2}}{1 - 0{,}5(j\omega)^{-2}} = \boxed{\frac{(j\omega)^2 + 2j\omega + 4}{(j\omega)^2 + 0{,}5}}$$

Bode plot (diagram)

logarithmic drawing for the transfer characteristic

1, Why is it comfortable?

$H(j\omega) = |H(j\omega)| \cdot e^{j \, arc \, H(j\omega)}$ $/ \ln$

$\ln |H(j\omega)| + j \, arc \, H(j\omega)$ „separated"

2, $H(j\omega) = H_1(j\omega) \cdot H_2(j\omega) = |H_1(j\omega)| \cdot |H_2(j\omega)| \, e^{j(arc H_1(j\omega) + arc H_2(j\omega))}$

$\ln |H_1(j\omega)| + \ln |H_2(j\omega)|$

„plot of elementary parts, and after it added"

3, elementary parts

a, real pole

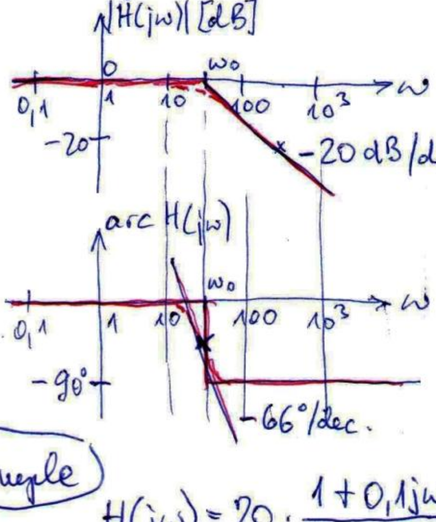
$H(j\omega) = \frac{1}{1 + j\frac{\omega}{\omega_0}}$ where $\omega_0 > 0$; given

asymptotic behaviors

if $\omega \to 0$ or $\omega \ll \omega_0 \Rightarrow H(j\omega) \approx 1 \longrightarrow 20 \lg 1 = 0 dB$; $0°$

if $\omega \to \infty$ or $\omega \gg \omega_0 \Rightarrow |H(j\omega)| \approx \frac{\omega_0}{\omega}$

$20 \lg \omega_0 - 20 \lg \omega$ $-90°$

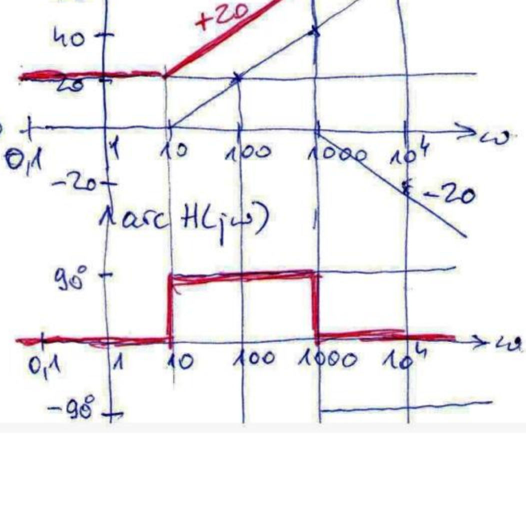


if $\omega = \omega_0 \Rightarrow H(j\omega_0) = \frac{1}{1+j}$

$20 \lg \frac{1}{\sqrt{2}} \approx -3 dB$; $-45°$

Example

$H(j\omega) = 20 \cdot \frac{1 + 0{,}1 j\omega}{2 + 0{,}002 j\omega} \Rightarrow 10 \frac{1 + \frac{j\omega}{10}}{1 + \frac{j\omega}{1000}}$



b, complex conj. poles

$$H(j\omega)=\frac{1}{1+2\xi\frac{j\omega}{\omega_0}+\frac{(j\omega)^2}{\omega_0^2}}$$

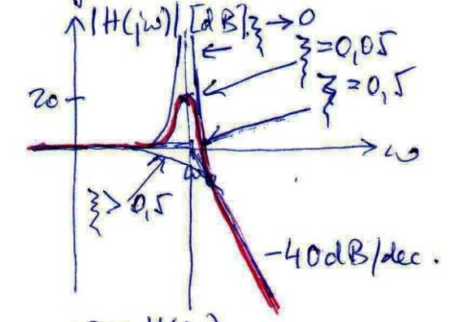
$0<\xi<1$

$\omega_0>0$ braking angular fr.

asymptotic behavior:

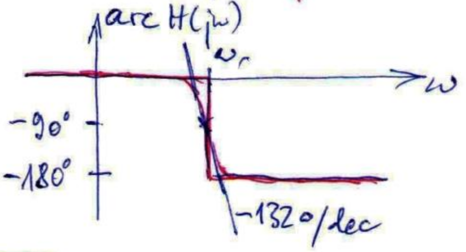
if $\omega\to 0$ or $\omega\ll\omega_0 \Rightarrow H(j\omega)\approx 1 \Rightarrow 0\,dB$; 0°

if $\omega\to\infty$ or $\omega\gg\omega_0 \Rightarrow |H(j\omega)|\approx\frac{\omega_0^2}{\omega^2} \Rightarrow 40\lg\omega_0-40\lg\omega$; -180°



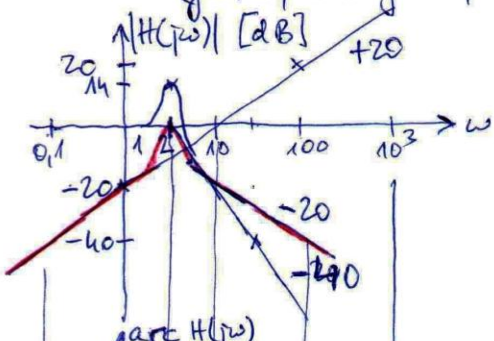
if $\omega=\omega_0 \Rightarrow H(j\omega_0)=\frac{1}{j2\xi}$

$20\lg\frac{1}{2\xi}$; -90°



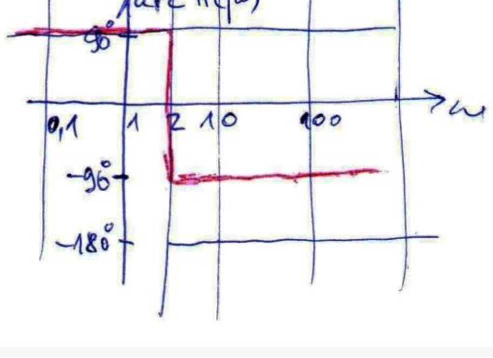
Example

$$H(j\omega)=\frac{j\omega\,0{,}1}{(j\omega)^2 0{,}25+j\omega\cdot 0{,}1+1}=\frac{\frac{j\omega}{10}}{1+2\cdot 0{,}1\frac{j\omega}{2}+\frac{(j\omega)^2}{2^2}}$$



$20\lg\frac{1}{2\cdot 0{,}1}=14\,dB$

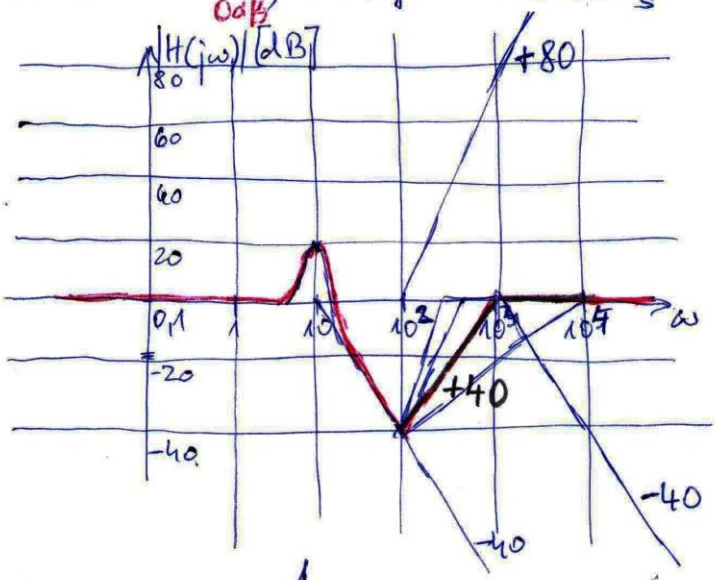
$H(j2)=\frac{j0{,}2}{j0{,}2}=1\to 20\lg 1=0$!!



Example Designing of the transfer of a system

$$u \longrightarrow \boxed{H_1(j\omega)=\frac{1}{1+0{,}01j\omega+0{,}01(j\omega)^2}} \longrightarrow \boxed{H_2(j\omega)} \longrightarrow y$$

Let us design $H(j\omega)$ so, that the transfer for $\omega > 10\,\frac{krad}{s}$ should be 1, 0dB and for $\omega < 100\,\frac{rad}{s}$ $H_1(j\omega)$ unchanged!

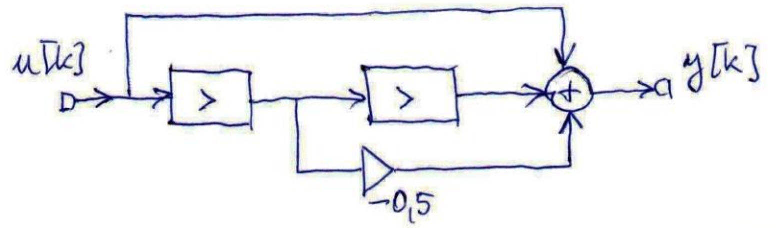


$$H_1(j\omega) = \frac{1}{1+2\cdot 0{,}05\cdot\frac{j\omega}{10}+\frac{(j\omega)^2}{10^2}}$$

$$H_2(j\omega) = \frac{\left(1+\frac{j\omega}{100}\right)^4}{\left(1+\frac{j\omega}{1000}\right)^2}$$

$$\boxed{H(j\omega) = \frac{\left(1+\frac{j\omega}{100}\right)^4}{\left(1+0{,}01j\omega+0{,}01(j\omega)^2\right)\left(1+\frac{j\omega}{1000}\right)^2}}$$

(Example) Plot for amplitude charakteristic of DT system



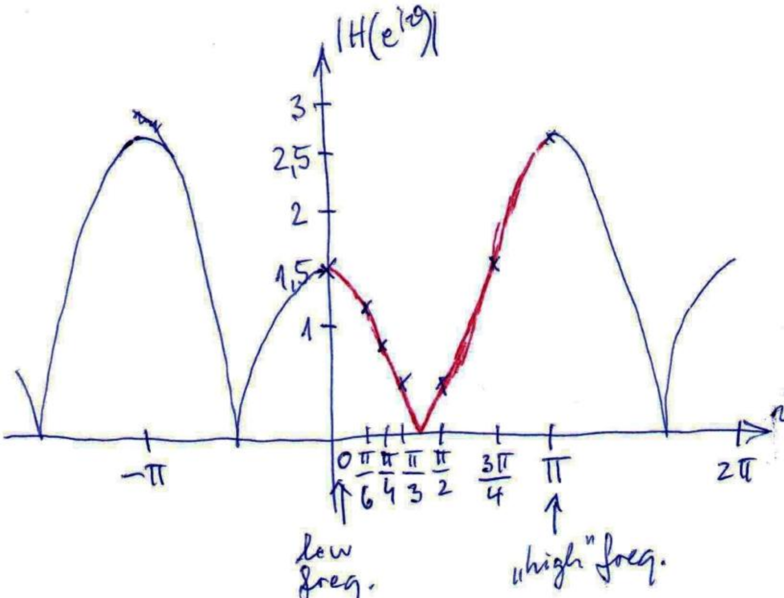
$|H(e^{j\vartheta})| = ?$

$$H(e^{j\vartheta}) = \frac{\bar{Y}}{\bar{U}} = 1 - 0{,}5e^{-j\vartheta} + e^{-j2\vartheta} \Leftarrow \bar{Y} = \bar{U}\left(1 - 0{,}5e^{-j\vartheta} + e^{-j2\vartheta}\right)$$

$$|H(e^{j\vartheta})| = \sqrt{(1 - 0{,}5 \cdot \cos\vartheta + \cos 2\vartheta)^2 + (0{,}5 \cdot \sin\vartheta - \sin 2\vartheta)^2}$$

$$\mathrm{arc}\, H(e^{j\vartheta}) = \mathrm{arctg}\frac{0{,}5\sin\vartheta - \sin 2\vartheta}{1 - 0{,}5\cos\vartheta + \cos 2\vartheta}$$

ϑ	$\lvert H(e^{j\vartheta})\rvert$
0	1,5
$\pi/6$	1,23
$\pi/4$	0,9
$\pi/3$	0,5
$\pi/2$	0,5
$3\pi/4$	1,5
π	2,5
$\arccos\frac{1}{4} = 1{,}33$	0



- even (because of abs. value)
- periodic! (2π)
- continous f. ($\vartheta \in \mathbb{R}$)

RESPONSE FOR GENERAL PERIODIC EXCITATION

DT case: $x[k+L] = x[k]$, where $k \in \mathbb{Z}$, $L \in \mathbb{N}$

periodic signals can be written as sum of sinusoidal signals (Fourier-series)

$$\boxed{x[k] = \sum_{p=\langle L\rangle \;(p=0)}^{L-1} X_p^c e^{jp\vartheta_0 k}} \qquad k \in \mathbb{Z};\ \vartheta_0 = \frac{2\pi}{L}$$

where the so-called Fourier-coeffitients:

$$\boxed{X_p^c = \frac{1}{L} \sum_{k=\langle L\rangle \;(k=0)}^{L-1} x[k] e^{-jp\vartheta_0 k}} \quad p \in \mathbb{Z}$$

1, „going around" $\Rightarrow X_{L+p}^c = X_p^c$

2, X_0^c is average value

proof:

$$\sum_{k=0}^{L-1} x[k] e^{-jrk\vartheta_0} = \sum_{k=0}^{L-1} \left\{ \sum_{p=\langle L\rangle} X_p^c e^{jpk\vartheta_0} \right\} e^{-jrk\vartheta_0} = \sum_{p=\langle L\rangle} X_p^c \left\{ \sum_{k=0}^{L-1} e^{j(p-r)k\vartheta_0} \right\} = L \cdot X_p^c$$

independent
⇓
changable

1, $p=r \Rightarrow \sum_{k=0}^{L-1} 1 = L$

2, $p \neq r \Rightarrow \sum_{k=0}^{L-1} \left(e^{j(p-r)\vartheta_0}\right)^k = \dfrac{1 - e^{j(p-r)\overbrace{\vartheta_0 \cdot L}^{2\pi}}}{1 - e^{j(p-r)\vartheta_0}} = 0$!!

only the $p=r$ element nonzero
↓
no sum any more

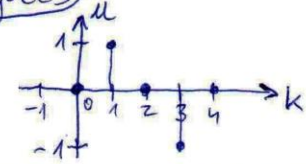
calculation of the response:

1, Fourier-series of the excitation → sum of sinusoidal functions

2, sinusoidal response on all frequencies with the well-know complex amplitude calculation

3, adding of the part responses because of linearity

Examples

1)  $L=4$ Fourier-series?

$$U_p^c = \frac{1}{L}\cdot\sum_{k=0}^{L-1} u[k]\cdot e^{-jp\vartheta_0 k} \quad \text{, where } \vartheta_0 = \frac{2\pi}{L} = \frac{\pi}{2}$$

$$U_4^c = U_0^c = \frac{1}{4}\left(0+1+0-1\right) = 0$$

$$U_5^c = U_1^c = \frac{1}{4}\left(0+1\cdot e^{-j\frac{\pi}{2}} + 0\cdot e^{-j\pi} - 1\, e^{-j\frac{3\pi}{2}}\right) = -\frac{j}{2} = \frac{1}{2}e^{-j\frac{\pi}{2}}$$

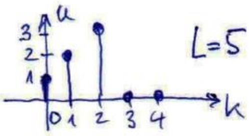
$$U_2^c = \frac{1}{4}\left(0+1\cdot e^{-j\pi} + 0\cdot e^{-j2\pi} - 1\, e^{-j3\pi}\right) = 0$$

$$U_{-1}^c = U_3^c = \frac{1}{4}\left(0+1\cdot e^{-j\frac{3\pi}{2}} + 0\cdot e^{-j3\pi} - 1\cdot e^{-j\frac{9\pi}{2}}\right) = \frac{j}{2} = \frac{1}{2}e^{+j\frac{\pi}{2}} = U_1^{c*}$$

$$u[k] = \sum_{p=0}^{L-1} U_p^c\cdot e^{jp\vartheta_0 k} = 0 + \frac{1}{2}e^{-j\frac{\pi}{2}}\cdot e^{j\frac{\pi}{2}k} + 0\cdot e^{j\pi k} + \frac{1}{2}e^{j\frac{\pi}{2}}\cdot e^{j\overset{-\frac{\pi}{2}}{\frac{3\pi}{2}}k} =$$

$$= 2\,\mathrm{Re}\left\{\frac{1}{2}e^{j\left(\frac{\pi}{2}k-\frac{\pi}{2}\right)}\right\} = \cos\left(\frac{\pi}{2}k-\frac{\pi}{2}\right) = \underline{\underline{\sin\frac{\pi}{2}k}} \quad !$$

1, exact! (no approximation)
2, „approx." $L/2$ coeff. ...

2)  $L=5 \to \vartheta_0 = \frac{2\pi}{5}$ $\quad H(e^{j\vartheta}) = \dfrac{4-2e^{-j\vartheta}}{1-e^{-j\vartheta}+0{,}05e^{-j2\vartheta}} \Rightarrow y[k]=?$

$$U_0^c = \frac{1}{5}\left(1+2+3+0+0\right) = 1{,}2$$

$$U_1^c = \frac{1}{5}\left(1+2e^{-j\frac{2\pi}{5}}+3e^{-j\frac{4\pi}{5}}\right) = -0{,}16-j0{,}73 = 0{,}75\,e^{-j1{,}79} = U_4^{c*}$$

$$U_2^c = \frac{1}{5}\left(1+2e^{-j\frac{4\pi}{5}}+3\cdot e^{-j\frac{8\pi}{5}}\right) = 0{,}062+j0{,}336 = 0{,}34\,e^{j1{,}39} = U_3^{c*}$$

$$u[k] = 1{,}2 + 1{,}5\cos\left(\frac{2\pi}{5}k-1{,}79\right) + 0{,}68\cos\left(\frac{4\pi}{5}k+1{,}39\right)$$

$$H(e^{j0}) = \frac{4-2}{1-1+0{,}05} = 40$$

$$H(e^{j\frac{2\pi}{5}}) = \ldots = 5{,}1\,e^{-j0{,}75} \;;\; H(e^{j\frac{4\pi}{5}}) = \ldots = 2\,e^{-j0{,}85}$$

$$\underline{\underline{y[k]}} = 1{,}2\cdot 40 + 1{,}5\cdot 5{,}1\cdot\cos\left(\frac{2\pi}{5}k-1{,}79-0{,}75\right) + 0{,}68\cdot 2\cdot\cos\left(\frac{4\pi}{5}k+1{,}39-0{,}85\right) =$$

$$= \underline{\underline{48 + 7{,}6\cdot\cos\left(\frac{2\pi}{5}k-2{,}54\right) + 1{,}37\cos\left(\frac{4\pi}{5}k+0{,}54\right)}}$$

CT case: complex form of Fourier-series (from DT with „letter exchange similarity")

$$\boxed{x(t) = \sum_{p=-\infty}^{\infty} X_p^c \cdot e^{jp\omega_0 t}} \qquad \omega_0 = \frac{2\pi}{T}$$

where $$\boxed{X_p^c = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\cdot e^{-jp\omega_0 t}\,dt}$$

real formulas:

$$x(t) \approx X_0 + \sum_{p=1}^{\cancel{\infty} N}\left(X_p^A \cos p\omega_0 t + X_p^B \sin p\omega_0 t\right)$$

where $X_0 = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,dt$ (mean value, av.)

$$X_p^A = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos p\omega_0 t\,dt$$

$$X_p^B = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin p\omega_0 t\,dt$$

Examples

1) $x(t) = \begin{cases} -1 & -\frac{T}{2} < t < 0 \\ 1 & 0 < t < \frac{T}{2} \end{cases}$ $\qquad x(t+T) = x(t) \Rightarrow$ Fourier-series?

$$X_p^c = \frac{1}{T}\left(\int_{-T/2}^{0} -1\cdot e^{-jp\omega_0 t}\,dt + \int_{0}^{T/2} 1\cdot e^{-jp\omega_0 t}\,dt\right) =$$

$$= \frac{1}{T}\left(\left[\frac{e^{-jp\omega_0 t}}{jp\omega_0}\right]_{-T/2}^{0} + \left[\frac{e^{-jp\omega_0 t}}{-jp\omega_0}\right]_{0}^{T/2}\right) =$$

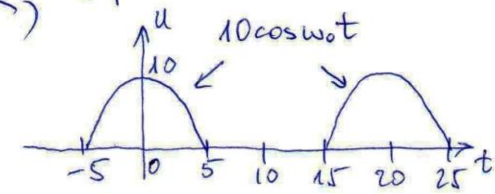
$$= \frac{1}{jp2\pi} - \frac{1}{jp2\pi}e^{+jp\pi} - \frac{1}{jp2\pi}e^{-jp\pi} + \frac{1}{jp2\pi} = \underline{\frac{1-\cos p\pi}{jp\pi}}$$

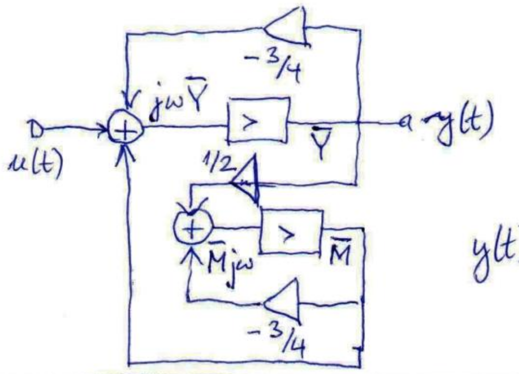
$$x(t) = \underset{p=0}{0} + \underset{p=1}{\frac{2}{j\pi}e^{j\omega_0 t}} + \frac{2}{-j\pi}e^{-j\omega_0 t} + \frac{2}{j3\pi}e^{j3\omega_0 t} + \frac{2}{-j3\pi}e^{-j3\omega_0 t} + \dots \overset{\text{„2Re"}}{=}$$

$$= \frac{4}{\pi}\cos\left(\omega_0 t - \frac{\pi}{2}\right) + \frac{4}{3\pi}\cos\left(3\omega_0 t - \frac{\pi}{2}\right) + \frac{4}{5\pi}\cos\left(5\omega_0 t - \frac{\pi}{2}\right) + \dots =$$

$$= \underline{\underline{\frac{4}{\pi}\sin\omega_0 t + \frac{4}{3\pi}\sin 3\omega_0 t + \frac{4}{5\pi}\sin 5\omega_0 t + \dots}}$$

2) CT:





$y(t) = ?$

Fourier-series:

$$U_p^c = \frac{1}{T}\int_{-T/2}^{T/2} u(t)\cdot e^{-jp\omega_0 t}dt = \frac{10}{T}\cdot\frac{1}{2}\cdot\int_{-T/4}^{T/4}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)e^{-jp\omega_0 t}dt =$$

$$= \frac{5}{T}\int_{-T/4}^{T/4}\left(e^{j(1-p)\omega_0 t} + e^{-j(1+p)\omega_0 t}\right)dt = \frac{5}{T}\cdot\left[\frac{e^{j(1-p)\omega_0 t}}{j(1-p)\omega_0} + \frac{e^{-j(1+p)\omega_0 t}}{-j(1+p)\omega_0}\right]_{-T/4}^{T/4} =$$

$$= \frac{5}{2\pi\cdot j}\left(\frac{e^{j(1-p)\frac{\pi}{2}}}{1-p} - \frac{e^{-j(1-p)\frac{\pi}{2}}}{1-p} + \frac{e^{-j(1+p)\frac{\pi}{2}}}{-(1+p)} - \frac{e^{j(1+p)\frac{\pi}{2}}}{-(1+p)}\right) =$$

$$= \underbrace{\frac{5}{\pi}\left(\frac{\sin(1-p)\frac{\pi}{2}}{1-p} + \frac{\sin(1+p)\frac{\pi}{2}}{1+p}\right)}\Rightarrow$$

$$U_0^c = \frac{5}{\pi}\cdot(1+1) = \frac{10}{\pi} = 3{,}18$$

$$U_1^c = \lim_{p\to1}\frac{5}{\pi}\,\frac{\sin(1-p)\frac{\pi}{2}}{\cancel{\frac{1-p}{2}}\cdot 2} = 2{,}5 = U_{-1}^c$$

$$U_2^c = \frac{5}{\pi}\cdot\left(1-\frac{1}{3}\right) = \frac{10}{3\pi} = U_{-2}^c$$

$$u(t) = \sum_{p=-\infty}^{\infty} U_p^c\cdot e^{jp\omega_0 t} = \frac{10}{3\pi}e^{-j2\omega_0 t} + 2{,}5\cdot e^{-j\omega_0 t} + 3{,}18 + 2{,}5\cdot e^{j\omega_0 t} + \frac{10}{3\pi}\cdot e^{j2\omega_0 t} + \ldots =$$

$$= \underbrace{3{,}18 + 5\cos\omega_0 t + 2{,}12\cos 2\omega_0 t + \ldots}$$

Transfer characteristic:

$$j\omega\bar{Y} = -\frac{3}{4}\bar{Y} + \bar{M} + \bar{U}$$

$$j\omega\bar{M} = \frac{1}{2}\bar{Y} - \frac{3}{4}\bar{M} \rightarrow \bar{M} = \bar{Y}\cdot\frac{\frac{1}{2}}{j\omega+\frac{3}{4}}$$

$$\longrightarrow H(j\omega) = \frac{\bar{Y}}{\bar{U}} = \underbrace{\frac{j\omega + 0{,}75}{(j\omega)^2 + 1{,}5j\omega + 0{,}0625}}$$

Transfer coefficient<u>s</u>:

$$H(j0) = \frac{0{,}75}{0{,}0625} = \underbrace{12}\ ;\quad H(j\,0{,}314) = \frac{0{,}75 + j0{,}314}{-0{,}036 + j0{,}471} = \ldots = \underbrace{1{,}72e^{-j1{,}25}}$$

$$H(j0{,}628) = \frac{0{,}75 + j0{,}628}{-0{,}33 + j0{,}942} = \ldots = 0{,}98\cdot e^{-j1{,}2}$$

Response time function:

$$y(t) = 3{,}18\cdot 12 + 5\cdot 1{,}72\cdot\cos(0{,}314t - 1{,}25) + 2{,}12\cdot 0{,}98\cos(0{,}628t - 1{,}2) =$$

$$= \underline{\underline{38{,}16 + 8{,}6\cdot\cos(0{,}314t - 1{,}25) + 2{,}1\cdot\cos(0{,}628t - 1{,}2)}}$$

FOURIER-TRANSFORMATION = Analysis in frequency domain

Found from the Fourier-series formula, for general <u>non</u>periodic signals:

CT: $X(j\omega) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)\cdot e^{-j\omega t}\,dt$, where $x(t)$ abs. integrable

Fourier-transf. = spectrum ("frequency content" of the signal) = complex spectrum

$$\underset{\text{spectrum}}{X(j\omega)} = \underset{\substack{\text{amplitude}\\ \text{spectrum (even)}}}{|X(j\omega)|}\, e^{j\,\underset{\substack{\text{phase}\\ \text{spectrum (odd)}}}{\underline{\operatorname{arc} X(j\omega)}}}$$

inv. Fourier-transformation: $x(t) = \mathcal{F}^{-1}\{X(j\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\,d\omega$

DT case with letter exchange similarity:

$X(e^{j\vartheta}) = \mathcal{F}\{x[k]\} = \sum_{k=-\infty}^{\infty} x[k]\cdot e^{-j\vartheta k}$, where $x[k]$ abs. summable

$x[k] = \mathcal{F}^{-1}\{X(e^{j\vartheta})\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\vartheta})\, e^{j\vartheta k}\,d\omega$

the spectrum of DT signal is periodic

Theorems:

<u>Linearity</u>: $\mathcal{F}\{c_1x_1 + c_2x_2\} = c_1\mathcal{F}\{x_1\} + c_2\mathcal{F}\{x_2\}$ ← from integration (summing)

<u>Shifting theorem</u>:

$\mathcal{F}\{x(t-\tau)\} = \int_{-\infty}^{\infty} x(t-\tau)\, e^{-j\omega(t-\tau)}\cdot \overbrace{e^{-j\omega\tau}}^{\text{const}}\, d(t-\tau) = e^{-j\omega\tau}\cdot X(j\omega)$

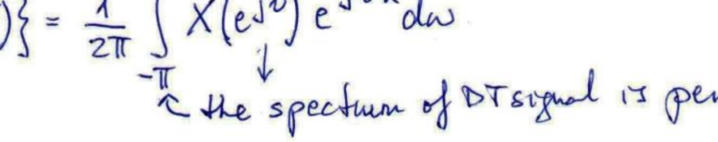
$\mathcal{F}\{x[k-k_0]\} = e^{-j\vartheta k_0}\cdot X(e^{j\vartheta})$ → we knew it already

<u>Modulation theorem</u>

$\mathcal{F}\{x(t)\cdot e^{j\omega_0 t}\} = \int_{-\infty}^{\infty} x(t)\cdot e^{-j\overbrace{(\omega-\omega_0)}^{\text{"const."}}t}\,dt = X(j(\omega-\omega_0))$

$\mathcal{F}\{x[k]\cdot e^{j\vartheta_0 k}\} = X(e^{j(\vartheta-\vartheta_0)})$

typical: $\mathcal{F}\{x(t)\cos\omega_0 t\} = \frac{1}{2}\cdot\left(X(j(\omega-\omega_0)) + X(j(\omega+\omega_0))\right)$



importance ...

<u>Differetiation theorem</u>: $\mathcal{F}\{x'(t)\} = j\omega X(j\omega)$ as earlier...

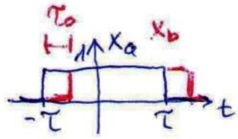
<u>Convolution theorem</u>: $\mathcal{F}\{u(t) * v(t)\} = U(j\omega)\cdot V(j\omega)$; $\mathcal{F}\{u[k] * v[k]\} = U(e^{j\vartheta})\cdot V(e^{j\vartheta})$

proof: $\mathcal{F}^{-1}\{U(j\omega)\cdot V(j\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} u(\tau)\, e^{-j\omega\tau}\,d\tau\right\}\cdot V(j\omega)\, e^{j\omega t}\,d\omega =$ (independent → changable)

$= \int_{-\infty}^{\infty} u(\tau)\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} V(j\omega)\, e^{j\omega(t-\tau)}\,d\omega\right\}d\tau = \int_{-\infty}^{\infty} u(\tau)\cdot v(t-\tau)\,d\tau$ qu.e.d.

Examples

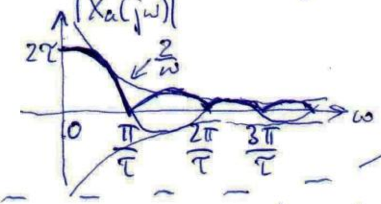
1, a, $x_a(t) = \varepsilon(t+\tau) - \varepsilon(t-\tau)$



symmetrical rectangular signal

$\hookrightarrow |X_a(j\omega)| = ?$

$$X_a(j\omega) = \int_{-\infty}^{\infty} x_a(t)\, e^{-j\omega t} dt = \int_{-\tau}^{\tau} 1 \cdot e^{-j\omega t} dt = \left[\frac{e^{-j\omega t}}{-j\omega}\right]_{-\tau}^{\tau} = \frac{1 \cdot 2}{-j\omega\, 2}\left(e^{-j\omega\tau} - e^{j\omega\tau}\right) = \frac{2}{\omega}\sin\omega\tau$$



b, $x_b(t) = \varepsilon(t+\tau-\tau_0) - \varepsilon(t-\tau-\tau_0)$ shifted rectangular signal

shifting theorem $\rightarrow X_b(j\omega) = X_a(j\omega) \cdot e^{-j\omega\tau_0} \rightarrow |X_b(j\omega)| = |X_a(j\omega)| \cdot 1$

same (phase...)

2, Spectrum of a few basic signals:

$$\underline{\underline{\mathcal{F}\{\delta(t)\}}} = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = \underline{\underline{1}}$$

$\underline{\underline{\mathcal{F}\{1\} = 2\pi\delta(\omega)}}$, since $\mathcal{F}^{-1}\{2\pi\delta(\omega)\} = 2\pi \cdot \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega) \cdot e^{j\omega t} d\omega = 1$ ✓

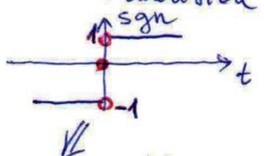
$\mathcal{F}\{\varepsilon(t)\} = ?$ problem: $\varepsilon(t)$ is not abs. integrable!!

based on 1a, $\Rightarrow \mathcal{F}\{\varepsilon(t) - \varepsilon(t-\tau)\} = \frac{1 - e^{-j\omega\tau}}{j\omega}$, if $\tau \to \infty \Rightarrow \frac{1}{j\omega}$

similarly: $\mathcal{F}\{-(\varepsilon(t+\tau) - \varepsilon(t))\} = -\frac{e^{j\omega\tau} - 1}{j\omega}$, if $\tau \to -\infty \Rightarrow \frac{1}{j\omega}$

$\Downarrow$ addition

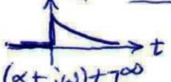
linearity



$$\mathcal{F}\{\mathrm{sgn}(t)\} = \frac{2}{j\omega}$$

$$\varepsilon(t) = \frac{1 + \mathrm{sgn}(t)}{2} \Rightarrow \underline{\underline{\mathcal{F}\{\varepsilon(t)\} = \pi \cdot \delta(\omega) + \frac{1}{j\omega}}}$$

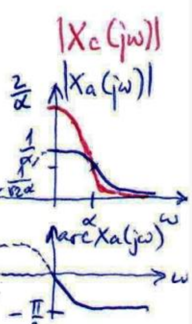
3, a, $x_a(t) = \varepsilon(t) \cdot e^{-\alpha t}$ $\alpha > 0$



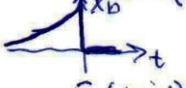
$$X_a(j\omega) = \int_0^{\infty} e^{-\alpha t} \cdot e^{-j\omega t} dt = \left[\frac{e^{-(\alpha+j\omega)t}}{-(\alpha+j\omega)}\right]_0^{\infty} = \frac{1}{\alpha + j\omega}$$

$\nearrow \underline{\underline{|X_a(j\omega)| = \frac{1}{\sqrt{\alpha^2+\omega^2}}}}$

$\searrow \underline{\underline{\mathrm{arc}\, X_a(j\omega) = -\mathrm{arctg}\frac{\omega}{\alpha}}}$



b, $x_b(t) = (1 - \varepsilon(t)) e^{\alpha t}$

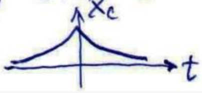


$$X_b(j\omega) = \int_{-\infty}^{0} e^{(\alpha - j\omega)t} dt = \left[\frac{e^{(\alpha-j\omega)t}}{\alpha - j\omega}\right]_{-\infty}^{0} = \frac{1}{\alpha - j\omega}$$

$\nearrow |X_b(j\omega)| = |X_a(j\omega)|$

$\searrow \mathrm{arc}\, X_b(j\omega) = -\mathrm{arc}\, X_a(j\omega)$

c, $x_c(t) = e^{-\alpha|t|} = x_a(t) + x_b(t)$



$\hookrightarrow X_c(j\omega) = \frac{1}{\alpha + j\omega} + \frac{1}{\alpha - j\omega} = \underline{\underline{\frac{2\alpha}{\alpha^2+\omega^2}}} = |X_c(j\omega)|$ $\rightarrow \mathrm{arc}\, X_c(j\omega) \equiv 0$

real spectrum, since $x(t)$ is even

4) a, $\mathcal{F}\{(\varepsilon(t+\tau)-\varepsilon(t-\tau))\cos\omega_0 t\} = ?$

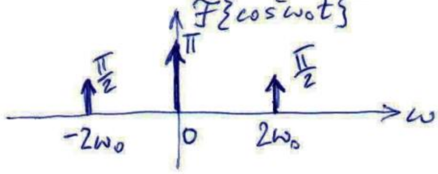
b, $\mathcal{F}\{\cos^2\omega_0 t\} = ?$

c, $\mathcal{F}\{\text{general periodic signal}\}$?

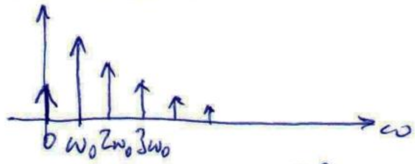
with modulation theorem: $\mathcal{F}\{x(t)\cos\omega_0 t\} = \frac{1}{2}\left(X(j(\omega-\omega_0)) + X(j(\omega+\omega_0))\right)$

a, $\underline{\underline{\frac{1}{2}\left(\frac{2}{\omega-\omega_0}\sin(\omega-\omega_0)\tau + \frac{2}{\omega+\omega_0}\sin(\omega+\omega_0)\tau\right)}}$

b, $\mathcal{F}\{\cos^2\omega_0 t\} = \mathcal{F}\left\{\frac{1+\cos 2\omega_0 t}{2}\right\} = \underline{\underline{\pi\cdot\delta(\omega) + \frac{\pi}{2}\left(\delta(\omega-2\omega_0)+\delta(\omega+2\omega_0)\right)}}$



$\Longrightarrow$ c,



5) DT:

a, $x_a[k] = \delta[k]+\delta[k-1] \Rightarrow X_a(e^{j\vartheta}) = \sum_{k=\cancel{-\infty}\,0}^{\cancel{\infty}\,1} x[k]e^{-j\vartheta k} = \underline{\underline{1+e^{-j\vartheta}}}$

↑ shifting theorem...

b, $x_b[k] = \frac{1}{2}\delta[k+1]+\delta[k]+\frac{1}{2}\delta[k-1]$

$\hookrightarrow X_b(e^{j\vartheta}) = \frac{1}{2}e^{j\vartheta} + 1 + \frac{1}{2}e^{-j\vartheta} = \underline{\underline{\cos\vartheta + 1}}$ periodic!

c, $x_c[k] = 0{,}5^{|k|}$

$\hookrightarrow X_c(e^{j\vartheta}) = \sum_{k=-\infty}^{\infty} 0{,}5^{|k|}e^{-j\vartheta k} = \sum_{k=-\infty}^{\cancel{\infty}\,0} 0{,}5^{-k}\cdot e^{-j\vartheta k} + \sum_{k=0}^{\infty} 0{,}5^k e^{-j\vartheta k} - 1 =$

n, ∞, $n=0$

$= \sum_{n=0}^{\infty}(0{,}5\cdot e^{j\vartheta})^n + \sum_{k=0}^{\infty}(0{,}5e^{-j\vartheta})^k - 1 = \frac{1}{1-0{,}5e^{j\vartheta}} + \frac{1}{1-0{,}5e^{-j\vartheta}} - 1 =$

$= \frac{2-0{,}5(e^{-j\vartheta}+e^{j\vartheta})}{1-0{,}5(e^{j\vartheta}+e^{-j\vartheta})+0{,}25} - 1 = \frac{2-\cos\vartheta}{1{,}25-\cos\vartheta} - 1 = \underline{\underline{\frac{0{,}75}{1{,}25-\cos\vartheta}}}$

summary:

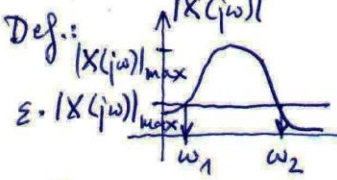
signal		(ampl.) spectrum	
C	NP	C	NP
C	P	D	NP
D	NP	C	P
D	P	D	P

!

Bandwidth of the (CT) signal

that frequency interval, where the dominating amplitudes of the signal are included

Def.:



$\Delta\omega_{signal} = \omega_2 - \omega_1$, where

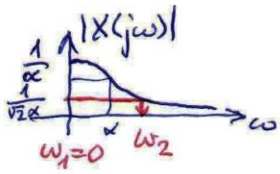
<u>outside of</u> the interval $|X(j\omega)| < \varepsilon\, |X(j\omega)|_{max}$

small, given

(Examples) $\Delta\omega_{signal} = ?$ $\quad \varepsilon = 0{,}1$

1) $x(t) = \varepsilon(t)\, e^{-\alpha t}$

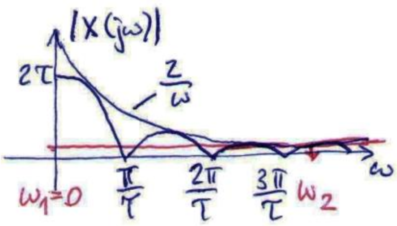
known: $X(j\omega) = \dfrac{1}{\alpha + j\omega} \Rightarrow |X(j\omega)| = \dfrac{1}{\sqrt{\alpha^2+\omega^2}}$



$$\frac{1}{\sqrt{\alpha^2+\omega_2^2}} = \varepsilon\cdot\frac{1}{\alpha} \Rightarrow \Delta\omega_{signal} = \omega_2 = \frac{1}{\varepsilon}\,\alpha\cdot\sqrt{1-\varepsilon^2} \approx \frac{1}{\varepsilon}\cdot\alpha = \underline{\underline{10\alpha}}$$

2) $x(t) = \varepsilon(t+\tau) - \varepsilon(t-\tau)$

known: $|X(j\omega)| = \dfrac{2}{\omega}\,|\sin\omega\tau|$



$$\downarrow$$

$$\frac{2}{\omega_2} = \varepsilon\cdot 2\tau$$

$$\downarrow$$

$$\Delta\omega_{signal} = \omega_2 = \frac{1}{\varepsilon\tau} = \underline{\underline{\frac{10}{\tau}}}$$

Distortion free signal transfer

" excitation and response looks similar

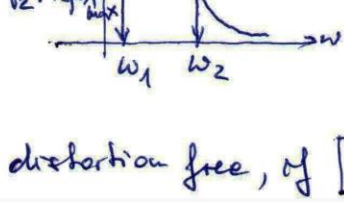
theor.: multiplication with constant and shifting can happen

$$y(t) = C\,u(t-\tau) \Rightarrow h(t) = C\,\delta[t-\tau] \Rightarrow H(j\omega) = C\cdot e^{-j\omega\tau} \Rightarrow |H(j\omega)| = C$$

$$y[k] = C\,u[k-k_0]$$

$$\operatorname{arc} H(j\omega) = -\tau\omega$$

practically:



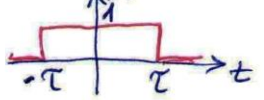
<u>Bandwidth of the system</u>, is that frequency interval, where ampl. char. is practically a constant (close to maximum): $1/\sqrt{2}$ (−3 dB)

$\Delta\omega_{system} = \omega_2 - \omega_1$, where $|H(j\omega)| \geq \sigma\,|H(j\omega)|_{max}$ (30%)

distortion free, if $\boxed{\Delta\omega_{system} \geq \Delta\omega_{signal} \text{ (fully includes)}}$

Example

$$H(j\omega) = \frac{2}{j\omega + \beta}$$



$\varepsilon = 0{,}1$

$\beta = ?$ for distortion free (distortionless) signal transfer

---

Known: $\Delta\omega_{signal} = \frac{10}{\tau}$

$$|H(j\omega)| = \frac{2}{\sqrt{\omega_2^2+\beta^2}} = \frac{1}{\sqrt{2}} \cdot \frac{2}{\beta} \leftarrow |H(j\omega)|_{max} \implies \Delta\omega_{system} = \omega_2 = \beta$$

max. in $\omega = 0 \Rightarrow \omega_1 = 0$

distortion free, if $\boxed{\beta \geq \frac{10}{\tau}}$ ← given

Parseval theorem

Known def: $E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt \leftrightarrow P = R I_{eff}^2 = G U_{eff}^2$

can be proofed, that the energy can be written also with spectrum:

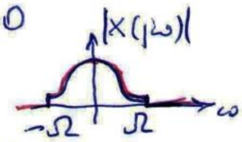
$$E_x = \frac{1}{2\pi} \int |X(j\omega)|^2 d\omega$$

← energy spectrum

Band limited signal

finite energy ⇒ on high frequency spectrum goes to 0

A signal is band limited, if $X(j\omega) \equiv 0$, if $|\omega| > \Omega$ ← band limit



Sampling theorem (Nyquist theorem)

A cont. time, finite energy signal with band limit Ω can be reconstructed from its samples $x(pT)$:

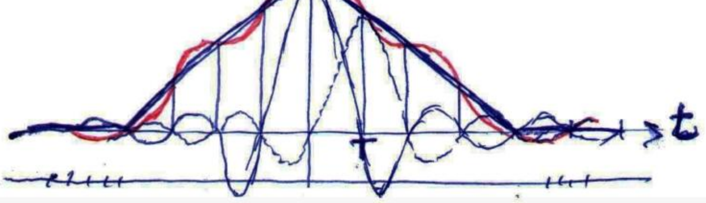
for digital signal processing

$$x(t) = \sum_{p=-\infty}^{\infty} x(pT) \cdot \frac{\sin \pi\left(\frac{t}{T} - p\right)}{\pi\left(\frac{t}{T} - p\right)}, \text{ where } \boxed{T = \frac{\pi}{\Omega}}$$

smaller → better

sampling period

too big...
too small...
optimal...



Spectrum and time function of the response

convolution theorem or Fourierseries generalized:

$$Y(j\omega) = U(j\omega) \cdot H(j\omega) \quad \rightarrow y(t) = \mathcal{F}^{-1}\{Y(j\omega)\}$$

$$Y(e^{j\vartheta}) = U(e^{j\vartheta}) \cdot H(e^{j\vartheta})$$

since $\mathcal{F}\{\delta(t)\} = 1 \rightarrow \boxed{\mathcal{F}\{h(t)\} = H(j\omega)}$

$\mathcal{F}\{\delta[k]\} = 1 \rightarrow \boxed{\mathcal{F}\{h[k]\} = H(e^{j\vartheta})}$

Example $u(t) = U_0 \cdot (1-\varepsilon(t))e^{\alpha t}$ $\quad H(j\omega) = \frac{1}{\beta + j\omega}$

a, $Y(j\omega) = ?$

b, expression for $y(t)$

c, $y(t) = ?$ if $t > 0$ and $\beta = \alpha$

a, we know: $U(j\omega) = \frac{U_0}{\alpha - j\omega} \rightarrow Y(j\omega) = U(j\omega) \cdot H(j\omega) = \frac{U_0}{(\alpha - j\omega)(\beta + j\omega)}$

b, $$y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{U_0}{(\alpha - j\omega)(\beta + j\omega)} \cdot e^{j\omega t} d\omega$$

c, $$y(0) = \frac{U_0}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\alpha^2 + \omega^2} \cdot 1 \, d\omega = \frac{U_0}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\alpha^2} \cdot \frac{1}{1 + \left(\frac{\omega}{\alpha}\right)^2} d\omega = \frac{U_0}{2\pi} \left[\frac{1}{\alpha} \operatorname{arctg} \frac{\omega}{\alpha}\right]_{-\infty}^{\infty} =$$

$$= \frac{U_0}{2\pi\alpha}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = \frac{U_0}{2\alpha}$$

$$\boxed{\text{for } t > 0 \rightarrow y_{tr}(t) = \frac{U_0}{2\alpha} e^{-\beta t}}$$

Remark: big difficulties → Laplace-transformation is the help for stepping in excitation

CT LAPLACE-TRANSFORMATION =

= Analysis in the complex frequency domain

$$X(j\omega) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \xLeftrightarrow{j\omega = s} X(s) = \mathcal{L}\{x(t)\} = \int_{-0}^{\infty} x(t)\, e^{-st}\, dt$$

abs. integrable

if 1, abs. integr. 2, stepping in

for stepping in signal + initial cond + Dirac-imp.

complex angular frequency → complex fr. domain:
- physically not imp.
- very good mathematical calc. ex. diff. equations

Inv. Laplace-transformation: integral we do not use at all!

Theorems:

- Linearity: $\mathcal{L}\{c_1 x_1(t) + c_2 x_2(t)\} = c_1 \mathcal{L}\{x_1(t)\} + c_2 \mathcal{L}\{x_2(t)\}$ $\boxed{\text{like } \mathcal{F}}$
- Damping theorem: $\mathcal{L}\{e^{-\alpha t} x(t)\} = \int_{-0}^{\infty} x(t)\, e^{-(s+\alpha)t}\, dt = X(s+\alpha)$
- Parameter theorem: $\mathcal{L}\left\{\frac{\partial}{\partial p} x(t,p)\right\} = \frac{\partial}{\partial p} X(s,p)$ since p is independent from t and s
- Shifting theorem: $\mathcal{L}\{\varepsilon(t-\tau)\cdot x(t-\tau)\} = e^{-s\tau}\cdot X(s)$ $\boxed{\text{like } \mathcal{F}}$
- Differentiation theorem: $\mathcal{L}\{x'(t)\} = s X(s) - x(-0)$

proof: $\mathcal{L}\{x'(t)\} = \int_{-0}^{\infty} \underbrace{x'(t)\, e^{-st}}_{f'\cdot g} dt = \left[\underbrace{x(t)\cdot e^{-st}}_{f\cdot g}\right]_{-0}^{\infty} - \int_{-0}^{\infty} \underbrace{x(t)}_{f}\cdot \underbrace{(-s)\cdot e^{-st}}_{g'} dt =$ "const."

$= -x(-0) + s X(s)$ ✓

- Initial value theorem: $x(+0) = \lim_{s\to\infty} \{s X(s)\}$
- End value theorem: $x(\infty) = \lim_{s\to 0} \{s X(s)\}$
- Convolution theorem: $\mathcal{L}\{u(t) * v(t)\} = U(s)\cdot V(s)$ $\boxed{\text{like } \mathcal{F}}$

Laplace-transform of a few basic signals:

$$\mathcal{L}\{\delta(t)\} = \underline{\underline{1}} \qquad \boxed{\text{like } \mathcal{F}}$$

$$\mathcal{L}\{\varepsilon(t)\} = \int_{-0}^{\infty} \varepsilon(t)\cdot e^{-st}\,dt = \left[\frac{e^{-st}}{-s}\right]_{0}^{\infty} = \underline{\underline{\frac{1}{s}}}$$

$$\mathcal{L}\{e^{-\alpha t}\cdot\varepsilon(t)\} = \underline{\underline{\frac{1}{s+\alpha}}} \qquad \boxed{\text{like } \mathcal{F}}$$

$$\mathcal{L}\{\varepsilon(t)\cdot t\cdot e^{pt}\} \underset{\text{parameter theorem}}{=} \frac{\partial}{\partial t}\left\{\frac{1}{s-p}\right\} = -\frac{1}{(s-p)^2}\cdot(-1) \Rightarrow \mathcal{L}\{\varepsilon(t)t\} = \underline{\underline{\frac{1}{s^2}}}$$

$$\left(\text{generally: } \mathcal{L}\left\{\varepsilon(t)\,\frac{1}{m!}\,t^m e^{pt}\right\} = \frac{1}{(s-p)^{m+1}}\right)$$

$$\mathcal{L}\{\cos\omega_0 t\cdot\varepsilon(t)\} = \mathcal{L}\left\{\frac{1}{2}\left(e^{-j\omega_0 t}+e^{+j\omega_0 t}\right)\right\} = \frac{1}{2}\left(\frac{1}{s+j\omega_0}+\frac{1}{s-j\omega_0}\right) = \underline{\underline{\frac{s}{s^2+\omega_0^2}}}$$

$x(t)$	$\delta(t)$	$\varepsilon(t)$	t	$e^{-\alpha t}$	$\cos\omega_0 t$	$\sin\omega_0 t$	$x(t-\tau)\varepsilon(t-\tau)$
$X(s)$	1	$\frac{1}{s}$	$\frac{1}{s^2}$	$\frac{1}{s+\alpha}$	$\frac{s}{s^2+\omega_0^2}$	$\frac{\omega_0}{s^2+\omega_0^2}$	$X(s)e^{-s\tau}$

Examples 1) a) $x(t) = A\cdot e^{-\alpha t}\cdot\cos\omega_0 t\cdot\varepsilon(t) \Longrightarrow X(s) = \underline{\underline{A\cdot\dfrac{s+\alpha}{(s+\alpha)^2+\omega_0^2}}}$

b) $x(t) = A\cdot e^{-\alpha t}\cos(\omega_0 t+\varphi)\cdot\varepsilon(t) = A\cdot e^{-\alpha t}(\cos\omega_0 t\cdot\cos\varphi - \sin\omega_0 t\sin\varphi)\varepsilon(t)$

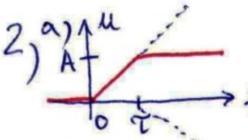
"const." $\qquad X(s) = \ldots$

c) $x(t) = A\cdot t\cdot e^{-\alpha t}\cdot\varepsilon(t) \rightarrow X(s) = \underline{\underline{\dfrac{A}{(s+\alpha)^2}}}$

d) $x(t) = A\cdot t\cdot\cos\omega_0 t\cdot\varepsilon(t) = \dfrac{A}{2}\,t\left(e^{-j\omega_0 t}+e^{j\omega_0 t}\right)\varepsilon(t)$

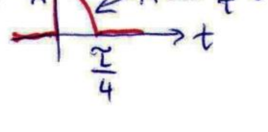
$$X(s) = \frac{A}{2}\cdot\left(\frac{1}{(s+j\omega_0)^2}+\frac{1}{(s-j\omega_0)^2}\right) = \underline{\underline{A\cdot\frac{s^2-\omega_0^2}{(s^2+\omega_0^2)^2}}}$$

2) a)



$$u(t) = \frac{A}{\tau}t\cdot\varepsilon(t) - \frac{A}{\tau}(t-\tau)\varepsilon(t-\tau) \Longrightarrow U(s) = \underline{\underline{\frac{A}{\tau}\cdot\frac{1}{s^2}\left(1-e^{-s\tau}\right)}}$$

b)



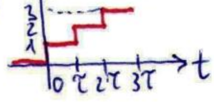
$$u(t) = A\cdot\left(\varepsilon(t)-\varepsilon\left(t-\frac{\tau}{4}\right)\right)\cos\frac{2\pi}{\tau}t =$$

$$- \sin\frac{2\pi}{\tau}\left(t-\frac{\tau}{4}\right)$$

$$= A\cos\frac{2\pi}{\tau}t\cdot\varepsilon(t) - A\cdot\varepsilon\left(t-\frac{\tau}{4}\right)\cdot\cos\frac{2\pi}{\tau}\left[\left(t-\frac{\tau}{4}\right)+\frac{\pi}{2}\right]$$

$$U(s) = \underline{\underline{A\cdot\frac{s}{s^2+\left(\frac{2\pi}{\tau}\right)^2} + A\,\frac{\frac{2\pi}{\tau}}{s^2+\left(\frac{2\pi}{\tau}\right)^2}\cdot e^{-s\frac{\tau}{4}}}}$$

c) "stair-function"



$$u(t) = \varepsilon(t)+\varepsilon(t-\tau)+\varepsilon(t-2\tau)+\ldots$$

$$U(s) = \frac{1}{s}\left(1+e^{-s\tau}+e^{-s2\tau}+\ldots\right) = \underline{\frac{1}{s\left(1-e^{-s\tau}\right)}}$$

Inv. Laplace-transformation with partial fractions

$$Y(s) \Rightarrow y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

for proper rational function:

$$Y(s) = \frac{M(s)}{N(s)} = \frac{M(s)}{(s+\alpha_1)(s+\alpha_2)\ldots(s+\alpha_N)} = \frac{A_1}{s+\alpha_1} + \frac{A_2}{s+\alpha_2} + \ldots + \frac{A_N}{s+\alpha_N} =$$

single poles

$$= \frac{A_1(s+\alpha_2)(s+\alpha_3)\ldots + A_2(s+\alpha_1)(s+\alpha_3)\ldots +}{(s+\alpha_1)(s+\alpha_2)\ldots(s+\alpha_N)} \quad /\cdot \text{denom.} \quad /\cdot \lim_{s\to-\alpha_1}$$

$$\boxed{A_1 = \lim_{s\to-\alpha_1} \frac{M(s)}{\cancel{(s+\alpha_1)}(s+\alpha_2)\ldots(s+\alpha_N)}}$$

„covering method"

Examples

1) $$Y(s) = \frac{3(s+2)}{(s+1)(s+4)} = \frac{1}{s+1} + \frac{2}{s+4} \rightarrow y(t) = \left(e^{-t} + 2e^{-4t}\right)\varepsilon(t)$$

2) nonproper rational function:

$$H(s) = 3\,\frac{(s+2)^2}{(s+1)(s+4)} \rightarrow h(t) = ?$$

$$H(s) = 3\cdot\frac{s^2+4s+4+s-s}{s^2+5s+4} = 3 + \frac{-3s}{(s+1)(s+4)} = 3 + \frac{1}{s+1} + \frac{-4}{s+4}$$

$$h(t) = 3\delta(t) + \varepsilon(t)\left(e^{-t} - 4e^{-4t}\right)$$

3) multiple pole:

$$Y(s) = \frac{3s+2}{(s+1)(s+2)^2} = \frac{-1}{s+1} + \frac{4}{(s+2)^2} + \frac{C}{s+2}$$

$$y(t) = \varepsilon(t)\left(-e^{-t} + 4t\,e^{-2t} + e^{-2t}\right) \qquad s^2:\ 0 = -1 + C \rightarrow C = 1$$

two possibilities: 1) multiple eigenvalue; 2) resonance

4) complex conjugate poles (not really new case..):

$$H(s) = \frac{1}{s^2+8s+25} \rightarrow h(t) = ?$$

$$H(s) = \frac{1}{(s+4+j3)(s+4-j3)} = \frac{A}{s+4+j3} + \frac{B}{s+4-j3}, \quad A = \frac{1}{-j6} = \frac{1}{6}\cdot e^{j\frac{\pi}{2}}, \quad B = A^*$$

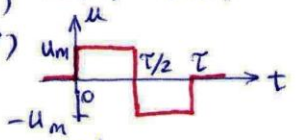
$$h(t) = \varepsilon(t)\cdot 2\mathrm{Re}\left\{\frac{1}{6}\cdot e^{j\frac{\pi}{2}}\cdot e^{-(4+j3)t}\right\} = \varepsilon(t)\cdot\frac{1}{3}e^{-4t}\cos\left(-3t+\frac{\pi}{2}\right) =$$

or: $$H(s) = \frac{1\cdot 3\cdot\frac{1}{3}}{(s+4)^2+3^2} \Rightarrow \quad = \varepsilon(t)\cdot\frac{1}{3}e^{-4t}\sin 3t$$

5) $H(s) = \dfrac{1}{s+\alpha}$

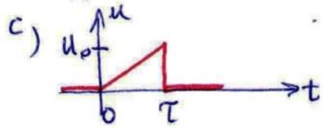
a) $u(t) = \varepsilon(t) \rightarrow g(t) = ?$

b)



$\rightarrow y(t) = ?$

c)



---

a) $U(s) = \dfrac{1}{s} \xRightarrow[\text{th.}]{\text{conv.}} Y(s) = U(s)\cdot H(s) = \dfrac{1}{s(s+\alpha)} = \dfrac{\frac{1}{\alpha}}{s} + \dfrac{-\frac{1}{\alpha}}{s+\alpha}$

$$\boxed{y(t) = \varepsilon(t)\,\frac{1}{\alpha}\,(\underset{\text{st.}}{1} - \underset{\text{tr.}}{e^{-\alpha t}}) = g(t)}$$

b) $u(t) = u_m\cdot\varepsilon(t) - 2u_m\cdot\varepsilon(t-\frac{\tau}{2}) + u_m\cdot\varepsilon(t-\tau)$

$\hookrightarrow U(s) = \dfrac{u_m}{s}\left(1 - 2e^{-s\frac{\tau}{2}} + e^{-s\tau}\right) \Rightarrow Y(s) = \dots \Rightarrow \boxed{y(t) = u_m\cdot g(t) - 2u_m\, g(t-\tfrac{\tau}{2}) + u_m\cdot g(t-\tau) = \dots}$

c) $u(t) = \dfrac{U_0}{\tau}\, t\,(\varepsilon(t) - \varepsilon(t-\tau)) = \dfrac{U_0}{\tau}\, t\cdot\varepsilon(t) - \dfrac{U_0}{\tau}\,\varepsilon(t-\tau)(t-\tau) - U_0\cdot\varepsilon(t-\tau)$

$U(s) = \dfrac{U_0}{\tau}\,\dfrac{1}{s^2}\,(1 - e^{-s\tau}) - \dfrac{U_0}{s}\,e^{-s\tau}$

$Y(s) = \dfrac{U_0}{\tau}\cdot\dfrac{1}{s^2(s+\alpha)}\,(1 - e^{-s\tau}) - \dfrac{U_0}{s(s+\alpha)}\,e^{-s\tau} =$

$= \dfrac{U_0}{\tau}\left(\dfrac{\frac{1}{\alpha}}{s^2} + \dfrac{B}{s} + \dfrac{\frac{1}{\alpha^2}}{s+\alpha}\right)(1 - e^{-s\tau}) - U_0\cdot\left(\dfrac{\frac{1}{\alpha}}{s} + \dfrac{-\frac{1}{\alpha}}{s+\alpha}\right)e^{-s\tau}$ $\quad B = -\dfrac{1}{\alpha^2}$

$s^2:\ 0 = 0 + B + \dfrac{1}{\alpha^2} \rightarrow B = -\dfrac{1}{\alpha^2}$

$$\boxed{y(t) = \frac{U_0}{\tau\alpha}\cdot\left(\varepsilon(t)\left(t - \frac{1}{\alpha} + \frac{1}{\alpha}e^{-\alpha t}\right) - \varepsilon(t-\tau)\left(t - \tau - \frac{1}{\alpha} + \frac{1}{\alpha}e^{-\alpha(t-\tau)}\right)\right) - \frac{U_0}{\alpha}\,\varepsilon(t-\tau)\left(1 - e^{-\alpha(t-\tau)}\right)}$$

6) $\dots H(s) = 0{,}057\,\dfrac{s+0{,}4}{s^2 + 2{,}97s + 0{,}314}$ $\quad u(t) = 170\sin(0{,}314t + 30^\circ)\cdot\varepsilon(t) \rightarrow y(t) = ?$

$u(t) = 170\cdot(\sin 0{,}314t\cdot\cos 30^\circ + \cos 0{,}314t\cdot\sin 30^\circ) \Rightarrow U(s) = 85\,\dfrac{0{,}314\sqrt{3} + s}{s^2 + 0{,}314^2}$

$Y(s) = U(s)\cdot H(s) = \dfrac{(2{,}64 + 4{,}86s)(s+0{,}4)}{(s - j0{,}314)(s + j0{,}314)(s + 0{,}11)(s + 2{,}86)} =$

$= \dfrac{A}{s - j0{,}314} + \dfrac{A^*}{s + j0{,}314} + \dfrac{2}{s + 0{,}11} + \dfrac{-1{,}21}{s + 2{,}86}$ $\quad A = 2{,}6\,e^{-j99^\circ}$

$\boxed{y(t)} = \varepsilon(t)\left(2\,\mathrm{Re}\left\{2{,}6\,e^{-j99^\circ}\cdot e^{j0{,}314t}\right\} + 2\,e^{-0{,}11t} - 1{,}21\,e^{-2{,}86t}\right) =$

$$= \boxed{\varepsilon(t)\left(5{,}2\cdot\cos(0{,}314t - 99^\circ) + 2e^{-0{,}11t} - 1{,}21\,e^{-2{,}86t}\right)}$$

DT LAPLACE-TRANSFORMATION (Z-TRANSFORMATION)

= Analysis in the complex frequency domain

$$x[k] \longrightarrow X(z) = \mathcal{L}\{x[k]\} = \sum_{k=0}^{\infty} x[k] \cdot z^{-k}$$

$\leftarrow$ from CT with letter exchange sim.

z^{-k}: discrete version of exp. function

Theorems:

- Linearity: same
- Damping theorem: $\mathcal{L}\{x[k] \cdot q^k\} = \sum\limits_{k=0}^{\infty} x[k] \cdot (q^{-1} \cdot z)^{-k} = X\left(\frac{z}{q}\right)$
- Parameter theorem: $\mathcal{L}\left\{\frac{\partial}{\partial p} x[k,p]\right\} = \frac{\partial}{\partial p} X(z,p)$ (same)
- Shifting theorem: $\mathcal{L}\{\varepsilon[k-r] \cdot x[k-r]\} = X(z) \cdot z^{-r}$
- (Hurrying signal:
  $\mathcal{L}\{x[k-1]\} = \underbrace{z^{-1}}_{\text{shifting theorem}} X(z) + \underbrace{x(-1)}_{\text{in } x[k-1] \text{ it is } 0 \text{ so remains if it was}} \longrightarrow \mathcal{L}\{x[k+1]\} = \underbrace{z}_{\text{shifting th.}} X(z) - \underbrace{x[0] \cdot z}_{\text{nonzero initial value}}$)
- Initial value theorem: $x[0] = \lim\limits_{z \to \infty} X(z)$
- End value theorem: $x[\infty] = \lim\limits_{z \to 1} \{(z-1)X(z)\}$
- Convolution theorem: $\mathcal{L}\{u[k] * v[k]\} = U(z) \cdot V(z)$ (same)

Z-transform of a few basic signals:

$$\mathcal{L}\{\delta[k]\} = 1$$

$$\boxed{\mathcal{L}\{q^k \cdot \varepsilon[k]\}} = \sum_{k=0}^{\infty} q^k z^{-k} = \sum_{k=0}^{\infty} (q \cdot z^{-1})^k = \frac{1}{1 - q z^{-1}} = \boxed{\frac{z}{z-q}} \quad \Big/ \frac{d}{dq}$$

$$\mathcal{L}\{\varepsilon[k]\} = \frac{z}{z-1}$$

$$\mathcal{L}\{k \cdot q^{k-1} \varepsilon[k]\} = \frac{z}{(z-q)^2} \text{ or } \mathcal{L}\{k q^k \varepsilon[k]\} = \frac{qz}{(z-q)^2}$$

Examples

1. $\mathcal{Z}\{\varepsilon[k-4]\cdot 0{,}5^{k-4+4}\} = \underline{\underline{0{,}5^{4}\cdot z^{-4}\cdot \frac{z}{z-0{,}5}}}$ (→ form of shifting theor.)

2. a, $x_a[k] = (\cos 3k + 5\sin 2k)\cdot \varepsilon[k]$

   b, $x_b[k] = 0{,}7^{k}\cos 5k\, \varepsilon[k]$

we „create":

$$\mathcal{Z}\{\alpha^k(\cos\beta k + j\sin\beta k)\varepsilon[k]\} = \mathcal{Z}\{(\alpha\cdot e^{j\beta})^k\} = \frac{z}{z-\alpha e^{j\beta}} =$$

$$= \frac{z}{(z-\alpha\cos\beta) - j\alpha\sin\beta} = \frac{z(z-\alpha\cos\beta + j\alpha\sin\beta)}{(z-\alpha\cos\beta)^2 + \alpha^2\sin^2\beta} =$$

$$= \underbrace{\frac{z^2 - z\cdot\alpha\cos\beta}{z^2 - 2z\alpha\cos\beta + \alpha^2}}_{\mathcal{Z}\{\alpha^k\cos\beta k\cdot\varepsilon[k]\}} + j\,\underbrace{\frac{z\alpha\sin\beta}{z^2 - 2z\alpha\cos\beta + \alpha^2}}_{\mathcal{Z}\{\alpha^k\cdot\sin\beta k\cdot\varepsilon[k]\}},$$

a, $X_a(z) = \underline{\underline{\frac{z^2 - z\cos 3}{z^2 - 2z\cos 3 + 1} + 5\cdot\frac{z\sin 2}{z^2 - 2z\cos 2 + 1}}}$ $\left(\frac{\text{pol}}{\text{pol}}!\right)$

b, $X_b(z) = \underline{\underline{\frac{z^2 - z\cdot 0{,}7\cdot\cos 5}{z^2 - 2z\cdot 0{,}7\cos 5 + 0{,}49}}}$

3. $Y(z) = \frac{3 - 2{,}6z^{-1}}{1 - 1{,}6z^{-1} + 0{,}6z^{-2}} = \frac{3z^2 - 2{,}6z}{z^2 - 1{,}6z + 0{,}6} = z\cdot\frac{3z - 2{,}6}{(z-1)(z-0{,}6)} =$

$= z\cdot\left(\frac{1}{z-1} + \frac{2}{z-0{,}6}\right) = \frac{z}{z-1} + 2\cdot\frac{z}{z-0{,}6}$

„covering method"

$$\boxed{y[k] = \varepsilon[k]\cdot(1 + 2\cdot 0{,}6^k)}$$

or polynomial division:

$Y(z) = (3 - 2{,}6z^{-1}) : (1 - 1{,}6z^{-1} + 0{,}6z^{-2}) = 3 + 2{,}2z^{-1} + 1{,}72z^{-2} + \ldots$

$\underline{3 - 4{,}8z^{-1} + 1{,}8z^{-2}}$

$2{,}2z^{-1} - 1{,}8z^{-2}$

$\underline{2{,}2z^{-1} - 3{,}52z^{-2} + 1{,}32z^{-3}}$

$1{,}72z^{-2} - 1{,}32z^{-3}$

...

$\downarrow$

$y[k] = \underline{\underline{3\delta[k] + 2{,}2\delta[k-1] + 1{,}72\delta[k-2] + \ldots}}$

4) $H(z) = \dfrac{2z^{-1} - 1{,}2z^{-2} + 1{,}1z^{-3} - 1{,}1z^{-4}}{1 - 0{,}6z^{-1} + 0{,}05z^{-2}}$

a) $h[k] = ?$

b) canonical network realisation

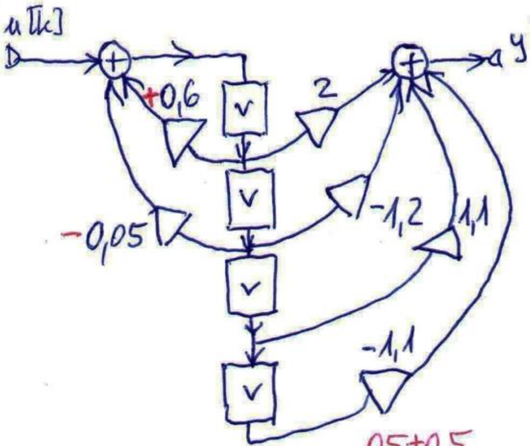
---

a) $H(z) = \dfrac{2z^3 - 1{,}2z^2 + 1{,}1z - 1{,}1}{z^2 - 0{,}6z + 0{,}05} \cdot z^{-2} = z^{-2} \cdot \left(2z + \dfrac{z - 1{,}1}{z^2 - 0{,}6z + 0{,}05}\right) =$

$= 2z^{-1} + z^{-3} \cdot z \cdot \dfrac{z - 1{,}1}{(z-0{,}5)(z-0{,}1)} = 2z^{-1} + z^{-3} \cdot z \left(\dfrac{-1{,}5}{z-0{,}5} + \dfrac{2{,}5}{z-0{,}1}\right)$ *

$$\boxed{h[k] = 2\delta[k-1] + \varepsilon[k-3] \cdot \left(-1{,}5 \cdot 0{,}5^{k-3} + 2{,}5 \cdot 0{,}1^{k-3}\right)}$$

b)



5) $H(z) = \dfrac{z^2 - z + 1}{z^2 - z + 0{,}5} \longrightarrow h[k] = ?$ ($1 = 0{,}5 + 0{,}5$)

$H(z) = 1 + \dfrac{0{,}5}{(z-(0{,}5+0{,}5j))(z-(0{,}5-0{,}5j))} \; z \cdot z^{-1} =$

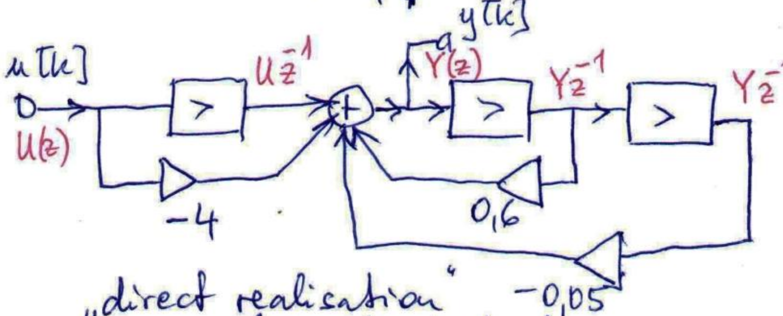
$= 1 + z^{-1} \cdot z \cdot \left(\dfrac{\frac{0{,}5}{j} = 0{,}5e^{-j\frac{\pi}{2}}}{z-(0{,}5+0{,}5j)} + \dfrac{0{,}5e^{j\frac{\pi}{2}}}{z-(0{,}5-0{,}5j)}\right) =$

$h[k] = \delta[k] + \varepsilon[k-1] \cdot 2\,\mathrm{Re}\left\{0{,}5e^{-j\frac{\pi}{2}} \cdot \left(\dfrac{\sqrt{2}}{2} e^{j\frac{\pi}{4}}\right)^{k-1}\right\} =$

$= \delta[k] + \varepsilon[k-1] \cdot \left(\dfrac{\sqrt{2}}{2}\right)^{k-1} \cdot \cos\left(\dfrac{\pi}{4}(k-1) - \dfrac{\pi}{2}\right) =$

$= \delta[k] + \varepsilon[k-1] \cdot \left(\dfrac{\sqrt{2}}{2}\right)^{k-1} \cdot \sin \dfrac{\pi}{4}(k-1)$

6)



„direct realisation“ (not prefered, why...)

a) $h[k] = ?$

b) $u[k] = \varepsilon[k] \cdot 0{,}5^k$ $\Downarrow$ $y[k] = ?$

$$H(z) = \frac{-4 + z^{-1}}{1 - 0{,}6 z^{-1} + 0{,}05 z^{-2}} \leftarrow Y = 0{,}6 z^{-1} Y - 0{,}05 z^{-2} Y - 4U + z^{-1} U$$

a) $$H(z) = z \cdot \frac{-4z + 1}{(z-0{,}5)(z-0{,}1)} = z \cdot \left(\frac{-2{,}5}{z-0{,}5} + \frac{-1{,}5}{z-0{,}1} \right)$$

$$\boxed{h[k] = \varepsilon[k] \cdot \left(-2{,}5 \cdot 0{,}5^k - 1{,}5 \cdot 0{,}1^k \right)}$$

b) $$U(z) = \frac{z}{z-0{,}5} \longrightarrow Y(z) = U(z) \cdot H(z) =$$

$$= z \cdot \frac{-4z^2 + z}{(z-0{,}5)^2 \cdot (z-0{,}1)} = z \cdot \left(\frac{-1{,}25}{(z-0{,}5)^2} + \frac{B}{z-0{,}5} + \frac{0{,}375}{z-0{,}1} \right)$$

$z^2: -4 = 0 + B + 0{,}375 \Rightarrow B = -4{,}375$

$$\boxed{y[k] = \varepsilon[k] \cdot \left(-1{,}25 \cdot k \cdot 0{,}5^{k-1} - 4{,}375 \cdot 0{,}5^k + 0{,}375 \cdot 0{,}1^k \right)}$$

TRANSFER FUNCTION

1, we used already: $H(z) = \dfrac{Y(z)}{U(z)}$, where $Y(z) = \mathcal{L}\{y[k]\}$, $U(z) = \mathcal{L}\{u[k]\}$ H(s) same

2, since $\mathcal{L}\{\delta[k]\} = 1 \rightarrow H(z) = \mathcal{L}\{h[k]\}$ H(s) same

3, from state var. description a, SISO
b, zero initial condition!

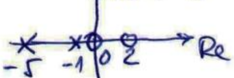
$$\begin{aligned} \dot{\underline{x}} &= \underline{\underline{A}}\,\underline{x} + \underline{B}u \\ y &= \underline{c}^T \underline{x} + Du \end{aligned} \rightarrow \begin{aligned} s\underline{X}(s) &= \underline{\underline{A}}\,\underline{X} + \underline{B}U(s) \\ Y &= \underline{c}^T \cdot \underline{X} + DU \end{aligned} \begin{aligned} &\rightarrow \underline{X}(s) = (s\underline{\underline{I}} - \underline{\underline{A}})^{-1}\underline{\underline{B}}\,U(s) \\ &\rightarrow Y(s) = \underbrace{\left(\underline{c}^T (s\underline{\underline{I}} - \underline{\underline{A}})^{-1}\underline{B} + D\right)}_{H(s)} U(s) \end{aligned}$$

H(z) same

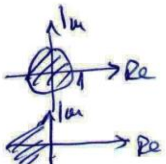
POLE - ZERO PLOT

zero — zero of the numerator polynomial of H
pole — zero of the denominator polynomial of H

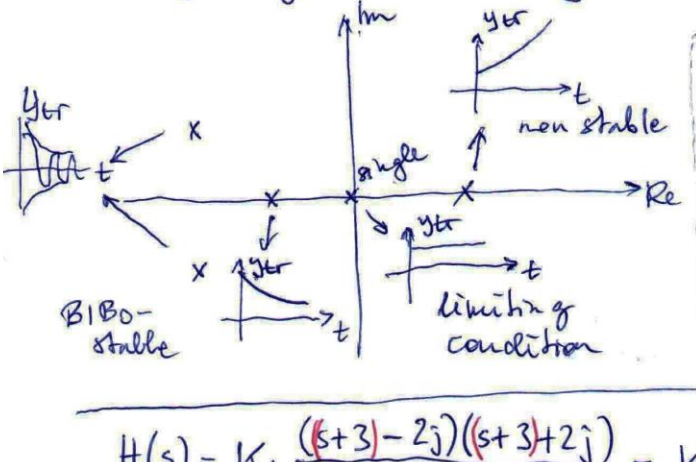
Ex. $H(s) = \dfrac{s(s-2)}{(s+1)(s+5)}$



BIBO - stable system $\Longleftrightarrow$ DT: $\forall |p_i| < 1$
CT: $\forall \operatorname{Re} p_i < 0$



why „only" BIBO-stability? ... $\rightarrow p_i \subseteq \lambda_i$



Example

poles: $p_1 = -4 \quad p_2 = -2$
zeros $z_1 = -3+2j \quad z_2 = -3-2j$
$g(+\infty) = 5$
$H(s) = ?$

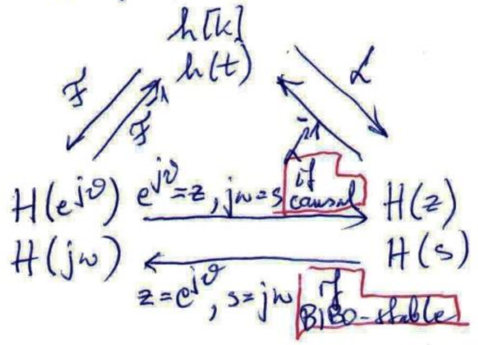
$$H(s) = K \cdot \frac{((s+3)-2j)((s+3)+2j)}{(s+4)(s+2)} = K \cdot \frac{s^2+6s+9+4}{s^2+6s+8}$$

$$H(s)\Big|_{s=0} = K \cdot \frac{13}{8} \equiv 5 \rightarrow K = \frac{40}{13}$$

$$H(s) = \frac{40}{13} \cdot \frac{s^2+6s+13}{s^2+6s+8}$$

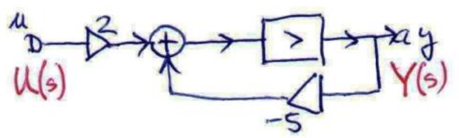
SYSTEM DESCRIBING FUNCTIONS

With the help of these functions can y be calculated to u



Examles

1) CT:

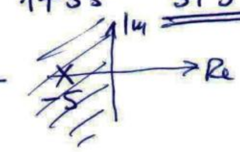


a) H(s) pole-zero plot
b) $H(j\omega)$
c) h(t)

---

a) $Y = U\cdot 2s^{-1} - 5s^{-1}\cdot Y \rightarrow Y(1+5s^{-1}) = U2s^{-1} \rightarrow H(s) = \frac{2s^{-1}}{1+5s^{-1}} = \underline{\underline{\frac{2}{s+5}}}$

b) $\exists\ H(j\omega) = \underline{\underline{\frac{2}{j\omega+5}}}$ ← BIBO-stable ←



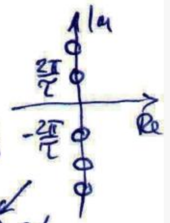
c) $h(t) = \underline{\underline{2\cdot e^{-5t}\cdot \varepsilon(t)}}$

2) $h(t) = \varepsilon(t) - \varepsilon(t-\tau)$
a) H(s) pole-zero plot
b) $H(j\omega)$

---

a) causal $\rightarrow \exists\ H(s) = \underline{\underline{\frac{1-e^{-s\tau}}{s}}}$

poles: —
zeros: $jk\frac{2\pi}{\tau}$, where $k\in\mathbb{Z}\setminus\{0\}$

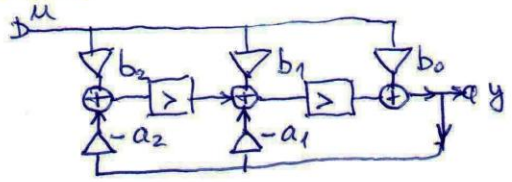


b) $\exists\ H(j\omega) = \underline{\underline{\frac{1-e^{-j\omega\tau}}{j\omega}}}$ ← BIBO-stable

realisation problems...

REALISATIONS OF SYSTEMS

- 2^{nd} canonical realisation was earlier
- 1^{st} canonical realisation

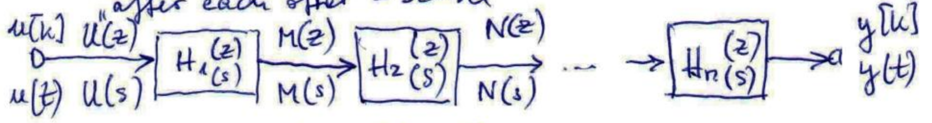


$$Y = -a_1 Y z^{-1} - a_2 Y z^{-2} + U(b_0 + b_1 z^{-1} + b_2 z^{-2})$$

$$\hookrightarrow \quad H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

- cascade realisation

"after each other" = serial



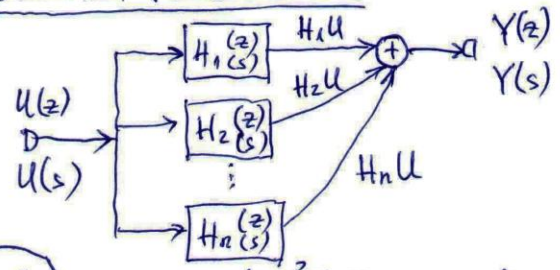
$$M(z) = H_1(z) \cdot U(z)$$

$$N(z) = M(z) \cdot H_2(z) = U(z) \cdot H_1(z) \cdot H_2(z)$$

$$\boxed{H(z) = H_1(z) \cdot H_2(z) \cdot \ldots \cdot H_n(z)}$$

$H(s)$ same

- parallel realisation



$$\boxed{H(z) = H_1(z) + H_2(z) + \ldots + H_n(z)}$$

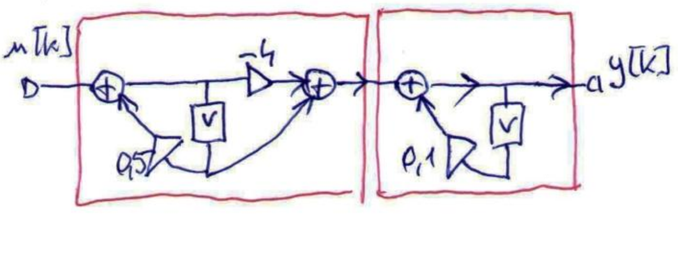
$H(s)$ same

Example $\quad H(z) = \dfrac{-4z^2 + z}{z^2 - 0{,}6z + 0{,}05}$ from earlier example

Let us give the system as a, cascade
b, parallel realisation of 2 first order systems

a, one poss.: $H(z) = \dfrac{-4z+1}{z-0{,}5} \cdot \dfrac{z}{z-0{,}1} =$

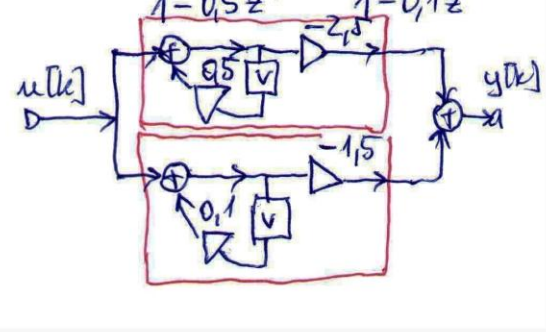
$$= \frac{-4 + z^{-1}}{1 - 0{,}5 z^{-1}} \cdot \frac{1}{1 - 0{,}1 z^{-1}}$$



b, partial fractions earlier

$$H(z) = \frac{-2{,}5z}{z-0{,}5} + \frac{-1{,}5z}{z-0{,}1} =$$

$$= \frac{-2{,}5}{1 - 0{,}5 z^{-1}} + \frac{-1{,}5}{1 - 0{,}1 z^{-1}}$$



SPECIAL SYSTEMS

a) FINITE IMPULSE RESPONSE SYSTEM

$h[k] \equiv 0$, if $k < k_0$ and $k \geq k_0 + L$

$h(t) \equiv 0$, if $k < t_0$ and $t \geq t_0 + T$

- surely BIBO-stable
- no feedback

System describing functions

DT: $h[k] = (\varepsilon[k] - \varepsilon[k - \underset{\substack{\uparrow \\ \text{length}}}{L}]) \cdot f[k] = c_0\delta[k] + c_1\delta[k-1] + \dots + c_{L-1}\delta[k-(L-1)]$

$$\left.\begin{aligned} H(e^{j\vartheta}) &= c_0 + c_1 e^{-j\vartheta} + \dots + c_{L-1} e^{-j(L-1)\vartheta} \\ H(z) &= c_0 + c_1 z^{-1} + \dots + c_{L-1} z^{-(L-1)} \end{aligned}\right\} \text{realisable with the learned elements}$$

CT: $h(t) = (\varepsilon(t) - \varepsilon(t - T)) f(t)$

$\Downarrow$

$e^{-j\omega T}$ or e^{-sT}

$H(j\omega)/H(s)$ is non rational function of $j\omega$ or s

$\Downarrow$

non realisable with the learned elements

(after series evaluation as approximation maybe ...)

only: $h(t) = K \cdot \delta(t) \to H(j\omega) = K \to$ multiplier

$\to H(s) = K$

b) ALL PASS SYSTEM

amplitude charakteristic is constant (independent from frequency)

$$\boxed{|H_{AP}(j\omega)| = K_0 > 0 \qquad |H_{AP}(e^{j\vartheta})| = K_0 > 0}$$ for the phase no prescription

ex: a) if $\text{arc}\, H_{AP} = 0 \to$ amplifier

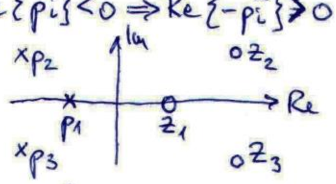
b) if phase linear $\text{arc}\, H_{AP}(j\omega) = -\tau\omega \to K_0 e^{-j\omega\tau} \Rightarrow$ shifting of excitation

valid: c) phase characteristic is always monoton decreasing

$$\frac{d\,\text{arc}\, H_{AP}}{d\vartheta\ (\omega)} \leq 0 \qquad \begin{aligned} 0 &\leq \vartheta < \pi \\ 0 &\leq \omega < \infty \end{aligned}$$

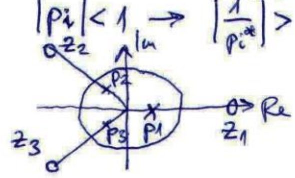
CT: $H_{AP}(s) = K_0 \cdot \dfrac{(s+p_1^*)(s+p_2^*)\dots}{(s-p_1)(s-p_2)\dots}$

$\text{Re}\{p_i\} < 0 \Rightarrow \text{Re}\{-p_i^*\} > 0$



DT: $H_{AP}(z) = K_0 \cdot z^{-r} \dfrac{(z - \frac{1}{p_1^*})(z - \frac{1}{p_2^*})}{(z-p_1)(z-p_2)}$

$|p_i| < 1 \to \left|\frac{1}{p_i^*}\right| > 1$



abs value: $\left|\frac{1}{p_i}\right|$ $\Uparrow$

proof: $\dfrac{j\omega + p_i^*}{j\omega - p_i} = -\dfrac{-j\omega - p_i^*}{j\omega - p_i} = -\dfrac{(j\omega - p_i)^*}{j\omega - p_i}$

abs value: 1

proof: $\dfrac{e^{j\vartheta} - \frac{1}{p_i^*}}{e^{j\vartheta} - p_i} = -\dfrac{1}{p_i^*} e^{j\vartheta} \dfrac{e^{-j\vartheta} - p_i^*}{e^{j\vartheta} - p_i} = -\dfrac{1}{p_i^*} e^{j\vartheta} \dfrac{(e^{j\vartheta} - p_i)^*}{e^{j\vartheta} - p_i}$

C) **MINIMUM PHASE SYSTEM** „smallest phase"

DT: all poles of $H(z)$ inside the unit circle („stable")
no zeros outside the unit circle

CT: all poles of $H(s)$ on the left side
no zeros on the right side

$$|p_i| < 1$$

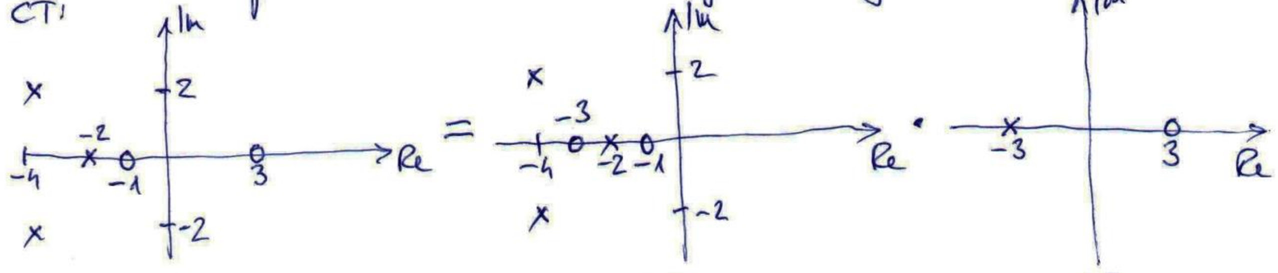
$$H_{MPh}(z) = K \cdot \frac{(z-z_1)(z-z_2)\ldots}{(z-p_1)(z-p_2)\ldots} \qquad \text{minimum phase } 0 < |z_i| \leq 1 \qquad \text{rig. min. phase } 0 < |z_i| < 1$$

$$H_{MPh}(s) = K \cdot \frac{(s-z_1)(s-z_2)\ldots}{(s-p_1)(s-p_2)\ldots} \qquad \text{min phase } Re\{z_i\} \leq 0 \qquad \text{rig. min. phase } Re\{z_i\} < 0$$

$$Re\{p_i\} < 0$$

Example Let us give the system as a cascade realisation of an all pass and a minimum phase system

CT:



MPh AP

$$H_{MPh}(s) = \frac{(s+1)(s+3)}{(s+2)(s+4-2j)(s+4+j2)} \qquad H_{AP}(s) = \frac{s-3}{s+3}$$

$$\underline{\underline{H(s) = H_{MPh}(s) \cdot H_{AP}(s)}} = \frac{(s+1)(s-3)}{(s+2)(s+4-2j)(s+4+2j)}$$