P-s - this is so important

System theory (Budapesti Muszaki és Gazdaságtudományi Egyetem)

SIGNALS AND SYSTEMS
Problems and solutions

1. The DT system is given with the following impulse response:
   $h[k] = 4\delta[k] + \varepsilon[k-1]\left\{5(-0,6)^k \cos(2k-0,8) - 3\cdot 0,9^k\right\}$.

   a) Decide the causality and the BIBO stability of the system!

   b) Find the response of the system for $k=0$, $k=1$, $k=2$ and for $k=3$ strokes if the input signal of the system is: $u[k] = \varepsilon[k]\left[3 - 8(-0,5)^k\right]$!

   c) Find the response to the $u[k] = 10 - 5\cdot 2^k$ input signal!

   Solution

   a) The system is causal as the impulse response is a causal signal.
   The system is BIBO stable if the impulse response is absolute summ able signal.
   $\sum_{k=-\infty}^{\infty} |h[k]| = \sum_{k=-\infty}^{\infty} \left|4\delta[k] + \varepsilon[k-1]\left\{5(-0,6)^k\cos(2k-0,8) - 3\cdot 0,9^k\right\}\right| \leq$
   $\leq 4 + \sum_{k=1}^{\infty}\left\{\left|5(-0,6)^k\right| \; |\cos(2k-0,8)| + 3\cdot 0,9^k\left(\right\} \leq$
   $\leq 4 + \sum_{k=1}^{\infty}\left(5\cdot 0,6^k + 3\cdot 0,9^k\right)$, as $|\cos(2k-0,8)| \leq 1$.
   So $\sum_{k=-\infty}^{\infty}|h[k]| \leq 4 + 3\frac{1}{1-0,6} + 2.7\frac{1}{1-0,9} = 38,5 < \infty$.
   The absolute sum is finite, the system is BIBO stable.

   b) The system is causal, the input signal is causal signal so $y[k] = \varepsilon[k]\sum_{p=0}^{k} u[k-p]h[p]$.

k	$h[k]$	$u[k]$	
0	4	-5	$y[0] = u[0]h[0] = -20$
1	$-3,7871$	7	$y[1] = u[1]h[0] + u[0]h[1] = 45,9354$
2	$-4,2269$	1	$y[2] = u[2]h[0] + u[1]h[1] + u[0]h[2] = -1,3749$
3	$-2,6930$	4	$y[3] = u[3]h[0] + u[2]h[1] + u[1]h[2] + u[0]h[3] = -3,9106$

   c) The system is causal, the input signal is acausal, so
   $y[k] = \sum_{p=-\infty}^{k} u[p]h[k-p] = \sum_{p=0}^{\infty} u[k-p]h[p]$.
   The second formula is advisable to apply as the expression of $h[k]$ is more complicated than that of $u[k]$.
   $y[k] = \sum_{p=0}^{\infty}\left(10 - 5\cdot 2^{k-p}\right)\left\{4\delta[p] + \varepsilon[p-1]\left[5(-0,6)^p\cos(2p-0,8) - 3\cdot 0,9^p\right]\right\} =$
   $= 40 - 20\cdot 2^k + 50\sum_{p=1}^{\infty}(-0,6)^p\cos(2p-0,8) - 30\sum_{p=1}^{\infty} 0,9^p -$
   $-25\cdot 2^k\sum_{p=1}^{\infty} 2^{-p}\cdot(-0,6)^p\cos(2p-0,8) + 15\cdot 2^k\sum_{p=1}^{\infty} 2^{-p}\cdot 0,9^p$
   As $\cos(2p-0,8) = Re\left(e^{-j0,8}e^{j2p}\right)$ and $2^{-p} = \left(\frac{1}{2}\right)^p$,
   $y[k] = 40 - 20\cdot 2^k + Re\left\{50e^{-j0,8}\sum_{p=1}^{\infty}\left(-0,6e^{j2}\right)^p\right\} - 30\sum_{p=1}^{\infty} 0,9^p -$
   $-\; 2^k Re\left\{25e^{-j0,8}\sum_{p=1}^{\infty}\left(-0,3e^{j2}\right)^p\right\} + 15\cdot 2^k\sum_{p=1}^{\infty} 0,45^p = 40 - 20\cdot 2^k +$
   $+\; Re\left\{50e^{-j0,8}\frac{-0,6e^{j2}}{1+0,6e^{j2}}\right\} - 30\frac{0,9}{1-0,9} - 2^k Re\left\{25e^{-j0,8}\frac{-0,3e^{j2}}{1+0,3e^{j2}}\right\} + 15\cdot 2^k\frac{0,45}{1-0,45}$
   $Re\dfrac{-30e^{j1.2}}{1+0,6e^{j2}} = -27,2029 \qquad Re\dfrac{-7,5e^{j1,2}}{1+0,3e^{j2}} = -5,0996$
   $y[k] = 40 - 20\cdot 2^k - 27,2029 - 270 + 5,0996\cdot 2^k + 12,2727\cdot 2^k = -257,2029 - 2,6277\cdot 2^k$.

2. The CT system is given with the following impulse response:
   $h(t) = 2\delta(t) + \varepsilon(t)\left(-5e^{-2t} + 4e^{-4t}\cos(3t-0,5)\right)$.

   a) Decide the causality and BIBO stability of the system!

b) Find the response to the $u(t)=10+2e^{t}$ input signal!

Solution

a) The system is causal as the impulse response is a causal signal.
The system is BIBO stable system if the impulse response is absolute integrable signal.
$\int_{-\infty}^{\infty}|h(t)|\mathrm{d}t=\int_{-\infty}^{\infty}|2\delta(t)+\varepsilon(t)\left(-5e^{-2t}+4e^{-4t}\cos(3t-0,5)\right)|\,\mathrm{d}t\leq$
$\leq\int_{-\infty}^{\infty}2\delta(t)\mathrm{d}t+\int_{0}^{\infty}|-5e^{-2t}|\,\mathrm{d}t+\int_{0}^{\infty}|4e^{-4t}\cos(3t-0,5)|\,\mathrm{d}t\leq 2+5\int_{0}^{\infty}e^{-2t}\mathrm{d}t+4\int_{0}^{\infty}e^{-4t}\mathrm{d}t,$
as $|\cos(3t-0,5)|\leq 1$.
$\int_{-\infty}^{\infty}|h(t)|\mathrm{d}t\leq 2+5\left[\frac{e^{-2t}}{-2}\right]_{0}^{\infty}+4\left[\frac{e^{-4t}}{-4}\right]_{0}^{\infty}=2+5\left(0-\frac{1}{-2}\right)+4\left(0-\frac{1}{-4}\right)=5,5<\infty.$
The absolute integral is finite, the system is BIBO stable.

b) The system is causal, the input signal is acausal, so $y(t)=\int_{-\infty}^{+t}u(\tau)h(t-\tau)\mathrm{d}\tau=$
$=\int_{-0}^{\infty}u(t-\tau)h(\tau)\mathrm{d}\tau$. The expression of the impulse response is more complicated than that of the input signal, so the calculation according to the latter formula looks simpler.
$y(t)=\int_{-0}^{\infty}u(t-\tau)h(\tau)\mathrm{d}\tau=$
$=\int_{-0}^{\infty}\left(10+2e^{t-\tau}\right)\left\{2\delta(\tau)+\varepsilon(t-\tau)\left[-5e^{-2\tau}+4e^{-4\tau}\cos(3\tau-0,5)\right]\right\}\mathrm{d}\tau$
$y(t)=20+4e^{t}-50\int_{0}^{\infty}e^{-2\tau}\mathrm{d}\tau-10e^{t}\int_{0}^{\infty}e^{-\tau}e^{-2\tau}\mathrm{d}\tau+40\int_{0}^{\infty}e^{-4\tau}\cos(3\tau-0,5)\mathrm{d}\tau+$
$+8e^{t}\int_{0}^{\infty}e^{-\tau}e^{-4\tau}\cos(3\tau-0,5)\mathrm{d}\tau.$
In the two latter integrals $\cos(3\tau-0,5)=Re\left\{e^{j3\tau}e^{-j0,5}\right\}$, so
$y(t)=20+4e^{t}-50\left[\frac{e^{-2t}}{-2}\right]_{0}^{\infty}-10e^{t}\left[\frac{e^{-3t}}{-3}\right]_{0}^{\infty}+Re\left\{40e^{-j0,5}\int_{0}^{\infty}e^{(-4+j3)\tau}\mathrm{d}\tau\right\}+$
$+e^{t}Re\left\{8e^{-j0,5}\int_{0}^{\infty}e^{(-5+j3)}\tau\mathrm{d}\tau\right\}$
$y(t)=20+4e^{t}-50\left(0-\frac{1}{-2}\right)-10e^{t}\left(0-\frac{1}{-3}\right)+Re\left\{40e^{-j0,5}\left[\frac{e^{(-4+j3)\tau}}{-4+j3}\right]_{0}^{\infty}\right\}+$
$+e^{t}Re\left\{8e^{-j0,5}\left[\frac{e^{(-5+j3)\tau}}{-5+j3}\right]_{0}^{\infty}\right\}$
$y(t)=20+4e^{t}-25-\frac{10}{3}e^{t}+Re\left\{40e^{-j0,5}\left(0-\frac{1}{-4+j3}\right)\right\}+e^{t}Re\left\{8e^{-j0,5}\left(0-\frac{1}{-5+j3}\right)\right\}=$
$=-5+\frac{2}{3}e^{t}+Re\left\{\frac{40e^{-j0,5}}{4-j3}\right\}+e^{t}Re\left\{\frac{8e^{-j0,5}}{5-j3}\right\}$
$Re\left\{\frac{40e^{-j0,5}}{4-j3}\right\}=7,9178,\qquad Re\left\{\frac{8e^{-j0,5}}{5-j3}\right\}=1,3709,$
$y(t)=2,9178+2,0375e^{t}.$

3. A DT system is given with the following state variable description:

$$\begin{bmatrix}x_1[k+1]\\x_2[k+1]\end{bmatrix}=\begin{bmatrix}-0,85&0,22\\-0,5&0,35\end{bmatrix}\begin{bmatrix}x_1[k]\\x_2[k]\end{bmatrix}+\begin{bmatrix}0,5\\-2\end{bmatrix}u[k],$$

$$y[k]=\begin{bmatrix}1,2&0,8\end{bmatrix}\begin{bmatrix}x_1[k]\\x_2[k]]\end{bmatrix}-0,5u[k].$$

a) Find the numerical values of the impulse response for $k=0$, $k=1$, $k=2$ and for $k=3$ strokes!

b) Find the formula of the system impulse response!

c) Find the response of the system in case of $u[k]=\varepsilon[k]\left(5-3(-0,5)^{k}\right)$ input signal! Plot the response for $k=0$, 1, ... 10 strokes!

d) Find the response of the system in case of $u[k]=7\left(\varepsilon[k]-\varepsilon[k-5]\right)$ input signal! Plot the response for $k=0$, 1, ... 10 strokes!

Solution

a) The step-by-step solution may be followed in the next table. (The initial values of the state variables are zeros as the input signal is causal.)

k	$x_1[k]$	$x_2[k]$	$u[k]=\delta[k]$	$y[k]=h[k]$
0	0	0	1	$-0,5$
1	0.5	-2	0	-1
2	$0,865$	$-0,95$	0	$-1,798$
3	$0,5262$	$0,1$	0	$0,7115$

b) Eugenvalue calculation: $det(\mathbf{A}-\lambda\mathbf{I})=0$

$$\begin{vmatrix}-0,85-\lambda & 0,22\\ -0,5 & 0,35-\lambda\end{vmatrix}=\lambda^2+0,5\lambda-0,1875=0,\qquad \lambda_1=-0,75,\qquad \lambda_2=0,25.$$

First solution. $h[k]=D\delta[k]+\varepsilon[k-1]\mathbf{C}^T\mathbf{A}^{k-1}\mathbf{B}=D\delta[k]+\varepsilon[k-1]\left(\mathbf{C}^T\mathbf{L}_1\mathbf{B}\lambda_1^{k-1}+\mathbf{C}^T\mathbf{L}_2\mathbf{B}\lambda_2^{k-1}\right)$

$$\mathbf{A}=\begin{bmatrix}-0,85 & 0,22\\ -0,5 & 0,35\end{bmatrix},\qquad \mathbf{B}=\begin{bmatrix}0,5\\ -2\end{bmatrix},\qquad \mathbf{C}^T=\begin{bmatrix}1,2 & 0,8\end{bmatrix},\qquad D=-0,5.$$

$$\mathbf{L}_1=\tfrac{1}{\lambda_1-\lambda_2}(\mathbf{A}-\lambda_2\mathbf{I})=\tfrac{1}{-0,75-0,25}\begin{bmatrix}-1,1 & 0,22\\ -0,5 & 0,1\end{bmatrix}=\begin{bmatrix}1,1 & -0,22\\ 0,5 & -0,1\end{bmatrix},$$

$$\mathbf{L}_2=\tfrac{1}{\lambda_2-\lambda_1}(\mathbf{A}-\lambda_1\mathbf{I})=\tfrac{1}{0,25-(-0,75)}\begin{bmatrix}-0,1 & 0,22\\ -0,5 & 1,1\end{bmatrix}=\begin{bmatrix}-0,1 & 0,22\\ -0,5 & 1,1\end{bmatrix}.$$

(Checking: $\mathbf{L}_1+\mathbf{L}_2=\mathbf{I}$.)

$$\mathbf{C}^T\mathbf{L}_1\mathbf{B}=\begin{bmatrix}1,2 & 0,8\end{bmatrix}\begin{bmatrix}1,1 & -0,22\\ 0,5 & -0,1\end{bmatrix}\begin{bmatrix}0,5\\ -2\end{bmatrix}=\begin{bmatrix}1,72 & -0,344\end{bmatrix}\begin{bmatrix}0,5\\ -2\end{bmatrix}=1,548$$

$$\mathbf{C}^T\mathbf{L}_2\mathbf{B}=\begin{bmatrix}1,2 & 0,8\end{bmatrix}\begin{bmatrix}-0,1 & 0,22\\ -0,5 & 1,1\end{bmatrix}\begin{bmatrix}0,5\\ -2\end{bmatrix}=\begin{bmatrix}-0,52 & 1,144\end{bmatrix}\begin{bmatrix}0,5\\ -2\end{bmatrix}=-2,548$$

$h[k]=-0,5\delta[k]+\varepsilon[k-1]\left(1,548(-0,75)^{k-1}-2,548(0,25)^{k-1}\right)$.

Second solution. Response calculation to $u[k]=\delta[k]$ input signal with decomposition to free and generated components.

First step. $y_f[k]=M_1'\lambda_1^k+M_2'\lambda_2^k=M_1'(-0,75)^k+M_2'0,25^k$.

Second step. As $u[k]=0$, if $k\geq 1$, so $y_g[k]=0$, if $k\geq 1$.
$y[k]=y_f[k]+y_g[k]=M_1'(-0,75)^k+M_2'0,25^k=M_1(-0,75)^{k-1}+M_2(0,25)^{k-1}$, if $k\geq 1$, (where $M_1=-0,75M_1'$, $M_2=0,25M_2'$)

Third step. Matching for $k=1$ and for $k=2$. From tthe a) point solution $h[0]=-0,5$, $h[1]=-1$, $h[2]=-1,798$.
$k=1 \quad h[1]=M_1+M_2=-1$
$k=2 \quad h[2]=-0,75M_1+0,25M_2=-1,798$
From the solution of the equation system: $M_1=1,548$, $M_2=-2,548$,
$h[k]=M_1(-0,75)^{k-1}+M_2\cdot 0,25^{k-1}$, if $k\geq 1$, so finally
$h[k]=-0,5\delta[k]+\varepsilon[k-1]\left(1,548(-0,75)^{k-1}-2,548(0,25)^{k-1}\right)$.

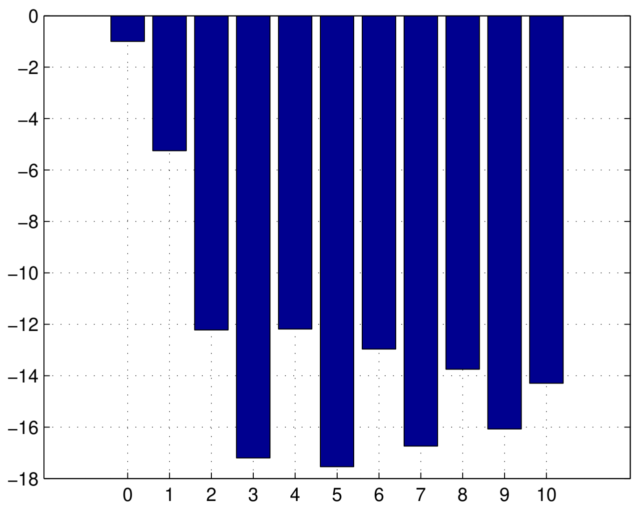
Numerical values with substitution to this formula:

k	0	1	2	3
$h[k]$	$-0,5$	-1	$-1,798$	$0,7115$

The values equal to that in the point a).

c) The system is causal, and the input signal is causal signal, so
$y[k]=\varepsilon[k]\sum_{p=0}^{k}u[p]h[k-p]=\varepsilon[k]\sum_{p=0}^{k}u[k-p]h[p]$.
The latter formula results in simpler calculation. For $k\geq 0$:
$y[k]=\sum_{p=0}^{k}\varepsilon[k-p]\left(5-3(-0,5)^{k-p}\right)\left[-0,5\delta[p]+\varepsilon[p-1]\left(1,548(-0,75)^{p-1}-2,548\cdot 0,25^{p-1}\right)\right]$
$0\leq p\leq k$, so $\varepsilon[k-p]=1$, $\delta[p]=1$ only for $p=0$ and 0 elsewhere, and $\varepsilon[p-1]=1$, only if $p\geq 1$. We have to separately calculate the response for $k=0$ and for $k\geq 1$.

If $k=0$: ($p=0$ only) $y[0]=(5-3)\cdot(-0,5)=-1$.
If $k\geq 1$: $\sum_{p=0}^{k}\cdots=\sum_{p=0}^{0}\cdots+\sum_{p=1}^{k}\cdots$,
$y[k]=-2,5+1,5(-0,5)^k+\sum_{p=1}^{k}\left(5-3(-0,5)^{k-p}\right)\left(1,548(-0,75)^{p-1}-2,548\cdot 0,5^{p-1}\right)=$
$=-2,5+1,5(-0,5)^k+7,74\sum_{p=1}^{k}(-0,75)^{p-1}-12,74\sum_{p=1}^{k}0,25^{p-1}-$
$-4,644(-0,5)^k\sum_{p=1}^{k}\left(\frac{1}{-0,5}\right)^p(-0,75)^{p-1}+7,644(-0,5)^k\sum_{p=1}^{k}\left(\frac{1}{-0,5}\right)^p\cdot 0,25^{p-1}$.
$\left(\frac{1}{-0,5}\right)^p=\frac{1}{-0,5}\left(\frac{1}{-0,5}\right)^{p-1}=-2(-2)^{p-1}$
$y[k]=-2,5+1,5(-0,5)^k+7,74\sum_{p=1}^{k}(-0,75)^{p-1}-12,74\sum_{p=1}^{k}0,25^{p-1}+$
$+9,288(-0,5)^k\sum_{p=1}^{k}1,5^{p-1}-15,288(-0,5)^k\sum_{p=1}^{k}(-0,5)^{p-1}$
$y[k]=-2,5+1,5(-0,5)^k+7,74\frac{(-0,75)^k-1}{-0,75-1}-12,74\frac{0,25^k-1}{0,25-1}+9,288(-0,5)^k\frac{1,5^k-1}{1,5-1}-$
$-15,288(-0,5)^k\frac{(-0,5)^k-1}{-0,5-1}=-2,5+1,5(-0,5)^k-4,4229\left((-0,75)^k-1\right)+16,9867\left(0,25^k-1\right)+$
$+18,576(-0,5)^k\left(1,5^k-1\right)+10,192(-0,5)^k\left(-0,5)^k-1\right)=-15,0638-27,2680(-0,5)^k+$
$+14,1531(-0,75)^k+27,1787\cdot 0,25^k$.
This formula is valid for $k\geq 1$, but putting $k=0$ into this formula the value is -1, that is the formula results in the perfect value for $k=0$ as well. So finally $y[k]=\varepsilon[k]\left(-15,0638-27,2680(-0,5)^k+14,1531(-0,75)^k+27,1787\cdot 0,25^k\right)$.
One possibility to draw the response signal with the help of MATLAB is as follows:
$\ll k=0:10$
$\ll y=-15.0638-27.268*(-.5).\hat{}k+14.1531*(-.75).\hat{}k+27.1787*.25.\hat{}k$
$\ll bar(k,y);grid$



d) First solution.
Let be $u_1[k]=7\varepsilon[k]$ and $y_1[k]$ the response signal in case of $u_1[k]$ input signal! As the system is linear and invariant the response to the $u[k]=u_1[k]-u_1[k-5]$ input signal is $y[k]=y_1[k]-y_1[k-5]$. It is enough to calculate the $y_1[k]$ response signal to the $u_1[k]=7\varepsilon[k]$ input signal.
$y_1[k]=\varepsilon[k]\sum_{p=0}^{k}u_1[k-p]h[p]$
$y_1[k]=\varepsilon[k]\sum_{p=0}^{k}7\left[-0,5\delta[p]+\varepsilon[p-1]\left(1,548(-0,75)^{p-1}-2,548\cdot 0,25^{p-1}\right)\right]$
If $k=0$: ($p=0$ only) $y_1[0]=-3,5$.
If $k\geq 1$: $y_1[k]=\sum_{p=0}^{k}\cdots=\sum_{p=0}^{0}\cdots+\sum_{p=1}^{k}\cdots=-3,5+10,836\sum_{p=1}^{k}(-0,75)^{p-1}-$
$-17,836\sum_{p=1}^{k}0,25^{p-1}=-3,5+10,836\frac{(-0,75)^k-1}{-0,75-1}-17,836\frac{0,25^k-1}{0,25-1}=-3,5-$

$-6,192\left((-0,75)^k-1\right)+23,7813\left(0,25^k-1\right)=-21,0893-6,192(-0,75)^k+23,7813\cdot 0,25^k$. Putting $k=0$ into this formula the value is $-3,5=y_1[0]$, so this formula is valid for $k\geq 0$, not only for $k\geq 1$. So

$y_1[k]=\varepsilon[k]\left(-21,0893-6,192(-0,75)^k+23,7813\cdot 0,25^k\right)$, and

$$y[k]=\varepsilon[k]\left(-21,0893-6,192(-0,75)^k+23,7813\cdot 0,25^k\right)-$$
$$-\varepsilon[k-5]\left(-21,0893-6,192(-0,75)^{k-5}+23,7813\cdot 0,25^{k-5}\right).$$

Second solution

$y[k]=\varepsilon[k]\sum_{p=0}^{k}u[p]h[k-p]$. For $k\geq 0$:

(*) $\qquad y[k]=\sum_{p=0}^{k}7\left(\varepsilon[p]-\varepsilon[p-5]\right)\cdot$
$\cdot\left\{-0,5\delta[k-p]+\varepsilon[k-p-1]\left(1,548(-0,75)^{k-p-1}-2,548\cdot 0,25^{k-p-1}\right)\right\}$

We will separately calculate for $k=0$, for $1\leq k\leq 4$ and for $k\geq 5$.

If $k=0$ ($p=0$ only), $y[0]=-3,5$.

If $1\leq k\leq 4$ for any p $(0\leq p\leq k\leq 4)$ $\quad 7\left(\varepsilon[p]-\varepsilon[p-5]\right)=7$, when $p=k$ $\delta[k-p]=1$ and $\varepsilon[k-p-1]=0$, elsewhere $\delta[k-p]=0$ and $\varepsilon[k-p-1]=1$.

$\sum_{p=0}^{k}\cdots=\sum_{p=0}^{k-1}\cdots+\sum_{p=k}^{k}\ldots$, so $y[k]=\sum_{p=0}^{k-1}7\left(1,548(-0,75)^{k-p-1}-2,548\cdot 0,25^{k-p-1}\right)-$

$-3,5=10,836(-0,75)^k\sum_{p=0}^{k-1}\left(\frac{1}{-0,75}\right)^{p+1}-17,836\cdot 0,25^k\sum_{p=0}^{k-1}\left(\frac{1}{0,25}\right)^{p+1}-3,5$

$$y[k]=10,836(-0,75)^k\frac{1}{-0,75}\frac{\left(\frac{1}{-0,75}\right)^k-1}{\frac{1}{-0,75}-1}-17,836\cdot 0,25^k\frac{1}{0,25}\frac{\left(\frac{1}{0,25}\right)^k-1}{\frac{1}{0,25}-1}-3,5$$

$$y[k]=6,1920(-0,75)^k\left(\left(\frac{1}{-0,75}\right)^k-1\right)-23,7813\cdot 0,25^k\left(\left(\frac{1}{0,25}\right)^k-1\right)-3,5$$

$y[k]=-21,0893-6,1920(-0,75)^k+23,7813\cdot 0,25^k$.

This formula is valid not only for $1\leq k\leq 4$ but for $k=0$ as well (it may be checked with putting $k=0$ into the formula) so finally for $k\leq 4$

$y[k]=\left(\varepsilon[k]-\varepsilon[k-5]\right)\left(-21,0893-6,1920(-0,75)^k+23,7813\cdot 0,25^k\right)$.

If $k\geq 5$, in the formula noted by (*) $7\left(\varepsilon[p]-\varepsilon[p-5]\right)=7$ if $0\leq p\leq 4$, and 0 elsewhere, so $y[k]=\sum_{p=0}^{k}\cdots=\sum_{p=0}^{4}\ldots$. If $k\geq 5$ and $0\leq p\leq 4$ $\delta[k-p]=0$ and $\varepsilon[k-p-1]=1$.

$y[k]=\sum_{p=0}^{4}7\left(1,548(-0,75)^{k-p-1}-2,548\cdot 0,25^{k-p-1}\right)=10,836(-0,75)^k\sum_{p=0}^{4}\left(\frac{1}{-0,75}\right)^{p+1}-$

$-17,836\cdot 0,25^k\sum_{p=0}^{4}\left(\frac{1}{0,25}\right)^{p+1}=10,836(-0,75)^k\frac{1}{-0,75}\frac{\left(\frac{1}{-0,75}\right)^5-1}{\frac{1}{-0,75}-1}-17,836\cdot 0,25^k\frac{1}{0,25}\frac{\left(\frac{1}{0,25}\right)^5-1}{\frac{1}{0,25}-1}=$

$=-32,2850(-0,75)^k-24328\cdot 0,25^k$, if $k\geq 5$. Finally for any k:

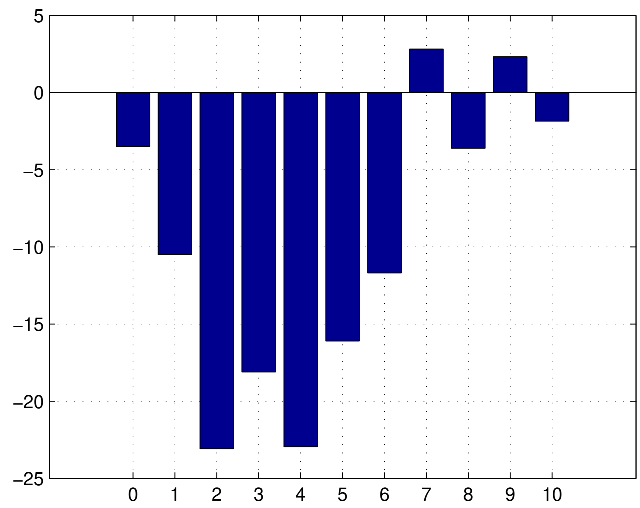
$$y[k]=\left(\varepsilon[k]-\varepsilon[k-5]\right)\left(-21,0893-6,1920(-0,75)^k+23,7813\cdot 0,25^k\right)+$$
$$+\varepsilon[k-5]\left(-32,2850(-0,75)^k-24328\cdot 0,25^k\right).$$

Comparing this formula with the result of the first solution it is trivial, that the two formulae are the same for $k<5$. Let us transform the final formula of the first solution for $k\geq 5$:

$-21,0893-6,1920(-0,75)^k+23,7813\cdot 0,25^k+21,0893+6,192(-0,75)^{k-5}-$
$-23,7813\cdot 0,25^{k-5}=-6,192(-0,75)^k\left(1-(-0,75)^{-5}\right)+23,7813\cdot 0,25^k\left(1-0,25^{-5}\right)=$
$=-32,2850(-0,75)^k-24328\cdot 0,25^k$. The two formulae are common for $k\geq 5$ as well.

The values of the response signal may be plotted with the help of MATLAB on the basis of the form of the first solution final result in the following way:

$\ll k=0:10$
$\ll y1=-21.0893-6.192*(-.75).\hat{}k+23.7813*(.25).\hat{}k$
$\ll y=y1-[zeros(1,5),y1(1:6)]$
$\ll bar(k,y);grid$



4. A CT system is given with the following state variable description:
$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -1 & 1,5 \\ -0,5 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 4 \\ -2 \end{bmatrix} u(t),$$
$$y(t) = \begin{bmatrix} -0,5 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} - 2u(t).$$

a) Find the formula of the system impulse response!

b) Find the response of the system in case of $u(t) = \varepsilon(t)\left(4 - 5e^{-2t}\right)$ input signal! Plot the response for $0 \le t \le 4$!

c) Find the response of the system in case of $u(t) = 2\left(\varepsilon(t) - \varepsilon(t-0,5)\right)$ input signal! Plot the response for $0 \le t \le 2$!

Solution

a) Eugenvalue calculation: $det(\mathbf{A} - \lambda\mathbf{I}) = 0$
$$\begin{vmatrix} -1-\lambda & 1,5 \\ -0,5 & -3-\lambda \end{vmatrix} = \lambda^2 + 4\lambda + 3,75 = 0, \qquad \lambda_1 = -2,5, \qquad \lambda_2 = -1,5.$$
First solution. $h(t) =, D\delta(t) + \varepsilon(t)\mathbf{C}^T e^{\mathbf{A}t}\mathbf{B} = D\delta(t) + \varepsilon(t)\left(\mathbf{C}^T\mathbf{L}_1\mathbf{B}e^{\lambda_1 t} + \mathbf{C}^T\mathbf{L}_2\mathbf{B}e^{\lambda_2 t}\right)$
$$\mathbf{A} = \begin{bmatrix} -1 & 1,5 \\ -0,5 & -3 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}, \qquad \mathbf{C}^T = \begin{bmatrix} -0,5 & 2 \end{bmatrix}, \qquad D = -2.$$
$$\mathbf{L}_1 = \tfrac{1}{\lambda_1 - \lambda_2}(\mathbf{A} - \lambda_2\mathbf{I}) = \tfrac{1}{-2,5+1,5}\begin{bmatrix} 0,5 & 1,5 \\ -0,5 & -1,5 \end{bmatrix} = \begin{bmatrix} -0,5 & -1,5 \\ 0,5 & 1,5 \end{bmatrix},$$
$$\mathbf{L}_2 = \tfrac{1}{\lambda_2 - \lambda_1}(\mathbf{A} - \lambda_1\mathbf{I}) = \tfrac{1}{-1,5+2,5}\begin{bmatrix} 1,5 & 1,5 \\ -0,5 & -0,5 \end{bmatrix} = \begin{bmatrix} 1,5 & 1,5 \\ -0,5 & -0,5 \end{bmatrix}.$$
(Checking: $\mathbf{L}_1 + \mathbf{L}_2 = \mathbf{I}$.)
$$\mathbf{C}^T\mathbf{L}_1\mathbf{B} = \begin{bmatrix} -0,5 & 2 \end{bmatrix}\begin{bmatrix} -0,5 & -1,5 \\ 0,5 & 1,5 \end{bmatrix}\begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} -0,5 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = -2,5$$
$$\mathbf{C}^T\mathbf{L}_2\mathbf{B} = \begin{bmatrix} -0,5 & 2 \end{bmatrix}\begin{bmatrix} 1,5 & 1,5 \\ -0,5 & -0,5 \end{bmatrix}\begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} -0,5 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ -1 \end{bmatrix} = -3,5$$
$h(t) = -2\delta(t) + \varepsilon(t)\left(-2,5e^{-2,5t} - 3,5e^{-1,5t}\right)$.

Second solution. Response calculation to $u(t) = \delta(t)$ input signal with decomposition to free and generated components.

First step. $h(t) = y(t) = y_f(t) + y_g(t), \qquad u(t) = \delta(t).$

$\mathbf{x}(t) = \mathbf{x}_f(t) + \mathbf{x}_g(t) = C_1\mathbf{m}_1 e^{\lambda_1 t} + C_2\mathbf{m}_2 e^{\lambda_2 t} + \mathbf{x}_g(t),\ t \geq 0.$

Eigenvector calculation. $\mathbf{m} = \begin{bmatrix} m_a \\ m_b \end{bmatrix}$

$\mathbf{A}\begin{bmatrix} m_a \\ m_b \end{bmatrix} = \lambda \begin{bmatrix} m_a \\ m_b \end{bmatrix} \quad \begin{matrix} (-1-\lambda)m_a + 1,5m_b = 0 \\ -0,5m_a + (-3-\lambda)m_b = 0 \end{matrix}$ let be $m_a = 1,5$ then $m_b = 1 + \lambda$

$\lambda = \lambda_1 = -2,5, \quad m_b = -1,5 \quad \mathbf{m}_1 = \begin{bmatrix} 1,5 \\ -1,5 \end{bmatrix}$

$\lambda = \lambda_2 = -1,5, \quad m_b = -0,5 \quad \mathbf{m}_2 = \begin{bmatrix} 1,5 \\ -0,5 \end{bmatrix}$

The free component of the state variable vector is then:

$\mathbf{x}_f(t) = C_1 \begin{bmatrix} 1,5 \\ -1,5 \end{bmatrix} e^{-2,5t} + C_2 \begin{bmatrix} 1,5 \\ -0,5 \end{bmatrix} e^{-1,5t}.$

Second step. The generated component of the state variable vector is $\mathbf{x}_g(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, as $u(t) = \delta(t) = 0$, if $t > 0$. So $\mathbf{x}(t) = \mathbf{x}_f(t) = C_1 \begin{bmatrix} 1,5 \\ -1,5 \end{bmatrix} e^{-2,5t} + C_2 \begin{bmatrix} 1,5 \\ -0,5 \end{bmatrix} e^{-1,5t}.$

Third step. The starting value of the state variable vector is $\mathbf{x}(+0) = \mathbf{B} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$ as $u(t) = \delta(t)$.

$\begin{matrix} x_1(+0) = 1,5C_1 + 1,5C_2 = 4 \\ x_2(+0) = -1,5C_1 - 0,5C_2 = -2 \end{matrix}$ From here $\begin{matrix} C_1 = \frac{2}{3} \\ C_2 = 2 \end{matrix}$

The expression of the state variable vector in case of $u(t) = \delta(t)$ is:

$\mathbf{x}(t) = \frac{2}{3} \begin{bmatrix} 1,5 \\ -1,5 \end{bmatrix} e^{-2,5t} + 2 \begin{bmatrix} 1,5 \\ -0,5 \end{bmatrix} e^{-1,5t} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2,5t} + \begin{bmatrix} 3 \\ -1 \end{bmatrix} e^{-1,5t}.$

Fourth step. Substitution into the $y(t) = \mathbf{C}^T\mathbf{x}(t) + Du(t)$ response expression:

$y(t) = \begin{bmatrix} -0,5 & 2 \end{bmatrix} \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2,5t} + \begin{bmatrix} 3 \\ -1 \end{bmatrix} e^{-1,5t} \right) - 2\delta(t),\ (t \geq 0),$

$h(t) = y(t) = -2\delta(t) + \varepsilon(t)\left(-2,5e^{-2,5t} - 3,5e^{-1,5t}\right).$

b) The system is causal, and the input signal is causal signal, so
$y(t) = \varepsilon(t) \int_0^t u(\tau)h(t-\tau)\mathrm{d}\tau = \varepsilon(t) \int_0^t u(t-\tau)h(\tau)\mathrm{d}\tau.$
The latter formula results in simpler calculation. For $t \geq 0$:

$y(t) = \int_0^t \left(4 - 5e^{-2(t-\tau)}\right)\left(-2\delta(\tau) - 2,5e^{-2,5\tau} - 3,5e^{-1,5\tau}\right)\mathrm{d}\tau =$

$= -8 + 10e^{-2t} - 10\int_0^t e^{-2,5\tau}\mathrm{d}\tau - 14\int_0^t e^{-1,5\tau}\mathrm{d}\tau + 12,5e^{-2t}\int_0^t e^{2\tau}e^{-2,5\tau}\mathrm{d}\tau +$

$+17,5e^{-2t}\int_0^t e^{2\tau}e^{-1,5\tau}\mathrm{d}\tau =$

$= -8 + 10e^{-2t} - 10\left[\frac{e^{-2,5\tau}}{-2,5}\right]_0^t - 14\left[\frac{e^{-1,5\tau}}{-1,5}\right]_0^t + 12,5e^{-2t}\left[\frac{e^{-0,5\tau}}{-0,5}\right]_0^t + 17,5e^{-2t}\left[\frac{e^{0,5\tau}}{0,5}\right]_0^t =$

$= -8 + 10e^{-2t} + 4\left(e^{-2,5t} - 1\right) + \frac{28}{3}\left(e^{-1,5t} - 1\right) - 25e^{-2t}\left(e^{-0,5t} - 1\right) + 35e^{-2t}\left(e^{0,5t} - 1\right) =$

$= -\frac{64}{3} - 21e^{-2,5t} + \frac{133}{3}e^{-1,5t}.$

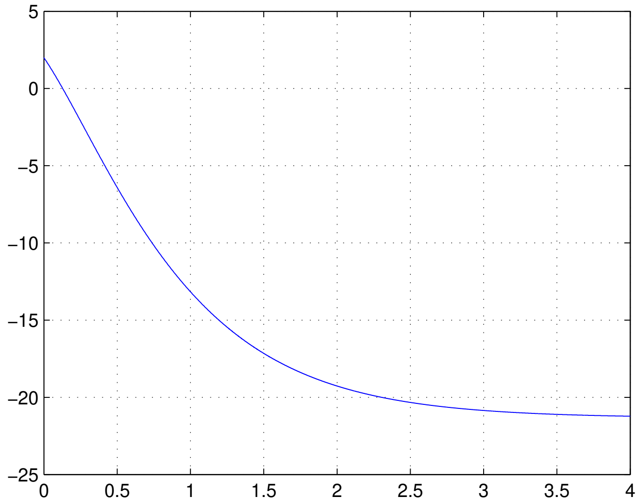
$y(t) = \varepsilon(t)\left(-21,3333 - 21e^{-2,5t} + 44,3333e^{-1,5t}\right).$

Plotting the response signal may be done with the help of MATLAB:

$\ll t = 0 : .001 : 4;$

$\ll y = -64/3 - 21 * exp(-2.5 * t) + 133/3 * exp(-1.5 * t);$

$\ll plot(t, y); grid$



c) First solution.

Let be $u_1(t) = 2\varepsilon(t)$ and $y_1(t)$ the response signal in case of $u_1(t)$ input signal! As the system is linear and invariant the response to the $u(t) = u_1(t) - u_1(t-0,5)$ input signal is $y(t) = y_1(t) - y_1(t-0,5)$. It is enough to calculate the $y_1(t)$ response signal to the $u_1(t) = 2\varepsilon(t)$ input signal.

$y_1(t)] = \varepsilon(t)\int_0^t u_1(t-\tau)h(\tau)\mathrm{d}\tau$. For $t \geq 0$:

$y_1(t) = \int_0^t 2\left(-2\delta(\tau) - 2,5e^{-2,5\tau} - 3,5e^{-1,5\tau}\right)\mathrm{d}\tau = -4 - 5\left[\frac{e^{-2,5\tau}}{-2,5}\right]_0^t - 7\left[\frac{e^{-1,5\tau}}{-1,5}\right]_0^t =$

$= -4 + 2\left(e^{-2,5t} - 1\right) + \frac{14}{3}\left(e^{-1,5t} - 1\right) = -\frac{32}{3} + 2e^{-2,5t} + \frac{14}{3}e^{-1,5t}.$

$y_1(t) = \varepsilon(t)\left(-10,6667 + 2e^{-2,5t} + 4,6667e^{-1,5t}\right)$, so

$y(t) = y_1(t) - y_1(t-0,5) = \varepsilon(t)\left(-10,6667 + 2e^{-2,5t} + 4,6667e^{-1,5t}\right) -$

$-\varepsilon(t-0,5)\left(-10,6667 + 2e^{-2,5(t-0,5)} + 4,6667e^{-1,5(t-0,5)}\right).$

Second solution. With the $y(t) = \varepsilon(t)\int_0^t u(\tau)h(t-\tau)\mathrm{d}\tau$ formula for $t \geq 0$:

(*) $y(t) =$

$= \int_0^t 2\left[\varepsilon(\tau) - \varepsilon(\tau-0,5)\right]\left[-2\delta(t-\tau) + \varepsilon(t-\tau)\left(-2,5e^{-2,5(t-\tau)} - 3,5e^{-1,5(t-\tau)}\right)\right]\mathrm{d}\tau$

We will separately calculate for $0 \leq t < 0,5$, and for $t \geq 0,5$.

If $0 \leq t < 0,5$, $0 \leq \tau < t$: $2\left[\varepsilon(\tau) - \varepsilon(t-\tau)\right] = 2$,

$y(t) = \int_0^t 2\left[-2\delta(t-\tau) + \varepsilon(t-\tau)\left(-2,5e^{-2,5(t-\tau)} - 3,5e^{-1,5(t-\tau)}\right)\right]\mathrm{d}\tau =$

$= -4 - 5e^{-2,5t}\int_0^t e^{2,5\tau}\mathrm{d}\tau - -7e^{-1,5t}\int_0^t e^{1,5\tau}\mathrm{d}\tau = -4 - 5e^{-2,5t}\left[\frac{e^{2,5\tau}}{2,5}\right]_0^t - 7e^{-1,5t}\left[\frac{e^{1,5\tau}}{1,5}\right]_0^t =$

$= -4 - 2e^{-2,5t}\left(e^{2,5t} - 1\right) - -\frac{14}{3}e^{-1,5t}\left(e^{1,5t} - 1\right) = -\frac{32}{3} + 2e^{-2,5t} + \frac{14}{3}e^{-1,5t}.$

That is for $t \leq 0,5$: $y(t) = \left[\varepsilon(t) - \varepsilon(T-0,5)\right]\left(-\frac{32}{3} + 2e^{-2,5t} + \frac{14}{3}e^{-1,5t}\right)$.

If $t \geq 0,5$ in the formula noted by (*): $y(t) = \int_0^t \cdots = \int_0^{0,5} \cdots + \int_{0,5}^t \cdots$.

In the first integral $\tau < t$, so $\delta(t-\tau) = 0$, and $2\left[\varepsilon(\tau) - \varepsilon(t-\tau)\right] = 2$, in the second integral $2\left[\varepsilon(\tau) - \varepsilon(t-\tau)\right] = 0$, so the second integral is 0.

$y(t) = \int_0^{0,5} 2\left(-2,5e^{-2,5(t-\tau)} - 3,5e^{-1,5(t-\tau)}\right)\mathrm{d}\tau = -5e^{-2,5t}\int_0^{0,5} e^{2,5\tau}\mathrm{d}\tau - 7e^{-1,5t}\int_0^{0,5} e^{1,5\tau}\mathrm{d}\tau =$

$= -5e^{-2,5t}\left[\frac{e^{2,5\tau}}{2,5}\right]_0^{0,5} - 7e^{-1,5t}\left[\frac{e^{1,5\tau}}{1,5}\right]_0^{0,5} = -2e^{-2,5t}\left(e^{1,25} - 1\right) - \frac{14}{3}e^{-1,5t}\left(e^{0,75} - 1\right) =$

$= -4,9807e^{-2,5t} - 5,2127e^{-1,5t}.$

Finally

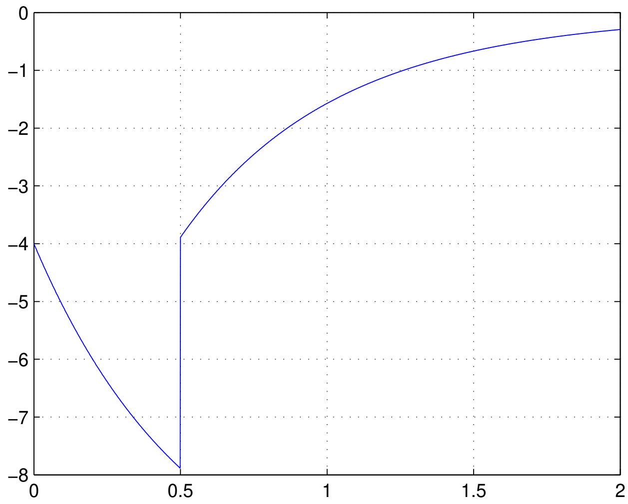
$y(t) = \left[\varepsilon(t) - \varepsilon(t-0,5)\right]\left(-\frac{32}{3} + 2e^{-2,5t} + \frac{14}{3}e^{-1,5t}\right) +$

$+\varepsilon(t-0,5)\left(-4,9807e^{-2,5t} - 5,2127e^{-1,5t}\right).$

Comparing this formula with the result of the first solution it is trivial that the two formulae are common for $t < 0,5$. The final formula of the first solution for $t \geq 0,5$ may be transformed as: $y(t) = -\frac{32}{3} + 2e^{-2,5t} + \frac{14}{3}e^{-1,5t} - \left(-\frac{32}{3} + 2e^{-2,5(t-0,5)} + \frac{14}{3}e^{-1,5(t-0,5)}\right) =$
$= 2e^{-2,5t}\left(1 - e^{1,25}\right) + \frac{14}{3}e^{-1,5t}\left(1 - e^{0,75}\right) = -4,9807e^{-2,5t} - 5,2127e^{-1,5t}$, that is the two results are the same for $t \geq 0,5$ as well.

The values of the response signal may be plotted with the help of MATLAB on the basis of the form of the first solution final result in the following way:

$\ll t = 0 : .001 : 2;$
$\ll y1 = -32/3 + 2 * exp(-2.5 * t) + 14/3 * exp(-1.5 * t);$
$\ll y2[zeros(1, 500), y1(1 : 1501)];$
$\ll y = y1 - y2;$
$\ll plot(t, y); grid$



5. A DT system is given with the following state variable description:

$$\begin{bmatrix} x_1[k+1] \\ x_2[k+1] \end{bmatrix} = \begin{bmatrix} 0,1 & -0,8 \\ 0,4 & -0,7 \end{bmatrix} \begin{bmatrix} x_1[k] \\ x_2[k] \end{bmatrix} + \begin{bmatrix} -0,5 \\ 2 \end{bmatrix} u[k],$$

$$y[k] = \begin{bmatrix} 1,2 & 0,8 \end{bmatrix} \begin{bmatrix} x_1[k] \\ x_2[k]] \end{bmatrix} + 0,5u[k].$$

a) Examine the stability of the system!
b) Find the formula of the system impulse response!
c) The input signal of the system is: $u[k] = \varepsilon[k]\left(5 - 3 \cdot 0,5^k\right)$. Find the numerical value of the response for $k = 0$, $k = 1$, $k = 2$ and for $k = 3$ strokes! Find the formula of the response and plot the response for $k = 0, 1, \ldots 10$ strokes!
d) Find the response of the system in case of $u[k] = 6\left(\varepsilon[k] - \varepsilon[k-4]\right)$ input signal! Plot the response for $k = 0, 1, \ldots 10$ strokes!

Solution

a) Let us find the eigenvalues of the system matrix: $det(\mathbf{A} - \lambda\mathbf{I}) = 0$

$$\begin{vmatrix} 0,1-\lambda & -0,8 \\ 0,4 & -0,7-\lambda \end{vmatrix} = (0,1-\lambda)(-0,7-\lambda) + 0,32 = \lambda^2 + 0,6\lambda + 0,25 = 0,$$

$\lambda_1 = -0,3 + j0,4, \quad \lambda_2 = -0,3 - j0,4.$

$|\lambda_{1,2}| = \sqrt{0,3^2 + 0,4^2} = 0,5 < 1$, so the system is asymptotically stable, consequently it is BIBO stable as well.

b) First solution. $h[k]=D\delta[k]+\varepsilon[k-1]\mathbf{C}^T\mathbf{A}^{k-1}\mathbf{B}=D\delta[k]+\varepsilon[k-1]\left(\mathbf{C}^T\mathbf{L}_1\mathbf{B}\lambda_1^{k-1}+\mathbf{C}^T\mathbf{L}_2\mathbf{B}\lambda_2^{k-1}\right)$

$\mathbf{A}=\begin{bmatrix}0,1 & -0,8\\ 0,4 & -0,7\end{bmatrix}$, $\quad\mathbf{B}=\begin{bmatrix}-0,5\\ 2\end{bmatrix}$, $\quad\mathbf{C}^T=\begin{bmatrix}1,2 & 0,8\end{bmatrix}$, $\quad D=0,5$.

$$\mathbf{L}_1=\frac{1}{\lambda_1-\lambda_2}(\mathbf{A}-\lambda_2\mathbf{I})=\frac{1}{-0,3+j0,4-(-0,3-j0,4)}\begin{bmatrix}0,4+j0,4 & -0,8\\ 0,4 & -0,4+j0,4\end{bmatrix}=$$
$$=\begin{bmatrix}0,5-j0,5 & j\\ -j0,5 & 0,5+j0,5\end{bmatrix},$$

$\mathbf{L}_2=\frac{1}{\lambda_2-\lambda_1}(\mathbf{A}-\lambda_1\mathbf{I})$, as $\lambda_2=\lambda_1^*$ (conjugate): $\mathbf{L}_2=\mathbf{L}_1^*=\begin{bmatrix}0,5+j0,5 & -j\\ j0,5 & 0,5-j0,5\end{bmatrix}$.

(Checking: $\mathbf{L}_1+\mathbf{L}_2=\mathbf{I}$.)

$$\mathbf{C}^T\mathbf{L}_1\mathbf{B}=\begin{bmatrix}1,2 & 0,8\end{bmatrix}\begin{bmatrix}0,5-j0,5 & j\\ -j0,5 & 0,5+j0,5\end{bmatrix}\begin{bmatrix}-0,5\\ 2\end{bmatrix}=\begin{bmatrix}1,2 & 0,8\end{bmatrix}\begin{bmatrix}-0,25+j2,25\\ 1+j1,25\end{bmatrix}=$$
$$=0,5+j3,7,$$

$\mathbf{C}^T\mathbf{L}_2\mathbf{B}=\left(\mathbf{C}^T\mathbf{L}_1\mathbf{B}\right)^*=0,5-j3,7$, and $\mathbf{C}^T\mathbf{L}_2\mathbf{B}\lambda_2^{k-1}=\left(\mathbf{C}^T\mathbf{L}_1\mathbf{B}\lambda_1^{k-1}\right)^*$, so

$$\mathbf{C}^T\mathbf{L}_1\mathbf{B}\lambda_1^{k-1}+\mathbf{C}^T\mathbf{L}_2\mathbf{B}\lambda_2^{k-1}=2Re\left\{\mathbf{C}^T\mathbf{L}_1\mathbf{B}\lambda_1^{k-1}\right\}=Re\left\{(1+j7,4)(-0,3+j0,4)^{k-1}\right\}=$$
$$=Re\left\{7,4673e^{j1,4365}\left(0,5e^{j2,2143}\right)^{k-1}\right\}=Re\left\{7,4673\cdot 0,5^{k-1}e^{j[2,2143(k-1)+1,4365]}\right\}=$$
$$=7,4673\cdot 0,5^{k-1}\cos[2,2143(k-1)+1,4365].$$

Finally $h[k]=0,5\delta[k]+\varepsilon[k-1]\left(7,4673\cdot 0,5^{k-1}\cos[2,2143(k-1)+1,4365]\right)$.

Second solution. Response calculation to $u[k]=\delta[k]$ input signal with decomposition to free and generated components.

First step. $\quad y_f[k]=M_1'\lambda_1^k+M_2'\lambda_2^k=M_1'(-0,3+j0,4)^k+M_2'(-0,3-j0,4)^k$.

Second step. $\quad$ As $u[k]=0$, if $k\geq 1$, so $y_g[k]=0$, if $k\geq 1$.
$y[k]=y_f[k]+y_g[k]=M_1'(-0,3+j0,4)^k+M_2'(-0,3-j0,4)^k=M_1(-0,3+j0,4)^{k-1}+M_2(-0,3-j0,4)^{k-1}$, if $k\geq 1$, (where $M_1=(-0,3+j0,4)M_1'$, $M_2=M_2'(-0,3-j0,4)$).
Becouse of the conjugate relation $h[k]=2Re\left\{M_1(-0,3+j0,4)^{k-1}\right\}$, if $k\geq 1$.

Third step. $\quad$ Matching for $k=1$ and for $k=2$. The calculation of the impulse response numerical values for $k=0$, 1 and 2 may be followed in the next table.

k	$x_1[k]$	$x_2[k]$	$u[k]=\delta[k]$	$y[k]=h[k]$
0	0	0	1	$0,5$
1	$-0,5$	2	0	1
2	$-1,65$	$-1,6$	0	$-3,26$

With the $M_1=m_r+jm_i$ notation
$k=1 \quad h[1]=2m_r=1$,
$k=2 \quad h[2]=-0,6m_r-0,8m_i=-3,26$
From the solution of the equation system: $m_r=0,5$, $m_i=3,7$, $h[0]=0,5$, $h[k]=2Re\left\{(0,5+j3,7)(-0,3+j0,4)^{k-1}\right\}$, if $k\geq 1$, so finally
$h[k]=0,5\delta[k]+\varepsilon[k-1]Re\left\{(1+j7,4)(-0,3+j0,4)^{k-1}\right\}$.
With transformation which was made in the first solution:
$h[k]=0,5\delta[k]+\varepsilon[k-1]\left(7,4673\cdot 0,5^{k-1}\cos[2,2143(k-1)+1,4365]\right)$.

c) The system is causal, the input signal is causal signal, according to the $y[k]=$ $=\varepsilon[k]\sum_{p=0}^{k}u[k-p]h[p]$ formula the numerical values of the response are calculated in the next table:

k	$h[k]$	$u[k]$	
0	$0,5$	2	$y[0]=u[0]h[0]=1$
1	1	$3,5$	$y[1]=u[1]h[0]+u[0]h[1]=3,75$
2	$-3,26$	$4,25$	$y[2]=u[2]h[0]+u[1]h[1]+u[0]h[2]=-0,895$
3	$1,706$	$4,625$	$y[3]=u[3]h[0]+u[2]h[1]+u[1]h[2]+u[0]h[3]=-1,4355$

The response formula for $k \geq 0$: $y[k] =$
$= \sum_{p=0}^{k} \varepsilon[k-p] \left(5 - 3 \cdot 0,5^{k-p}\right) \left(0,5\delta[p] + \varepsilon[p-1] 7,4673 \cdot 0,5^{p-1} \cos[2,2143(p-1) + 1,4365]\right)$.
We will separately calculate for $k = 0$ and for $k \geq 1$.

If $k = 0$ ($p = 0$ only) $y[0] = 2,5 - 1,5 = 1$.

Later on $k \geq 1$. $y[k] = \sum_{p=0}^{k} \cdots = \sum_{p=0}^{0} \cdots + \sum_{p=1}^{k} \ldots$. In the first sum $\delta[p] = 1$, $\varepsilon[p-1] = 0$, in the second one $\delta[p] = 0$, $\varepsilon[p-1] = 1$ and $\varepsilon[k-p] = 1$ in both. So
$y[k] = 2,5 - 1,5 \cdot 0,5^k + \sum_{p=1}^{k} \left(5 - 3 \cdot 0,5^{k-p}\right) 7,4673 \cdot 0,5^{p-1} \cos[2,2143(p-1) + 1,4365]$.
With the $0,5^{-p} = 2^p = 2 \cdot 2^{p-1}$ and $\cos[2,2143(p-1) + 1,4365) = Re\left\{e^{j2,2143(p-1)} e^{j1,4365}\right\}$ substitutions

$$y[k] = 2,5 - 1,5 \cdot 0,5^k + Re\left\{5 \cdot 7,4673 e^{j1,4365} \sum_{p=1}^{k} \left(0,5e^{j2,2143}\right)^{p-1}\right\} -$$
$$-Re\left\{3 \cdot 7,4673 e^{j1,4365} \cdot 0,5^k \sum_{p=1}^{k} 2 \cdot 2^{p-1} \left(0,5e^{j2,2143}\right)^{p-1}\right\}$$
$$y[k] = 2,5 - 1,5 \cdot 0,5^k + Re\left\{(5 + j37) \frac{\left(0,5e^{j2,2143}\right)^k - 1}{0,5e^{j2,2143} - 1}\right\} - 0,5^k Re\left\{(6 + j44,4) \frac{\left(e^{j2,2143}\right)^k - 1}{e^{j2,2143} - 1}\right\} =$$
$$= 2,5 - 1,5 \cdot 0,5^k + Re\left\{(4,4865 - j27,0811)\left(\left(0,5e^{j2,2143}\right)^k - 1\right)\right\} -$$
$$-Re\left\{(8,1 - j23,7)\left(\left(0,5e^{j2,2143}\right)^k - 0,5^k\right)\right\} = 2,5 - 1,5 \cdot 0,5^k +$$
$$+Re\left\{(-3,6135 - j3,3811)\left(0,5e^{j2,2143}\right)^k - (4,4865 - j27,0811) + 0,5^k(8,1 - j23,7)\right\},$$

$-3,6135 - j3,3811 = 4,9487e^{-j2,3894}$, so

$$y[k] = 2,5 - 1,5 \cdot 0,5^k + Re\left\{4,9487 \cdot 0,5^k e^{j[2,2143k - 2,3894]}\right\} - 4,4865 + 8,1 \cdot 0,5^k =$$
$$= -1,9865 + 6,6 \cdot 0,5^k + 4,9487 \cdot 0,5^k \cos(2,2143k - 2,3894).$$

From the above calculation it is clear that this formula is valid for $k \geq 1$. But putting $k = 0$ into the formula the substitution value is 1, that is this formula is valid not only for $k \geq 1$ but for $k = 0$ as well, that is for $k \geq 0$. Finally

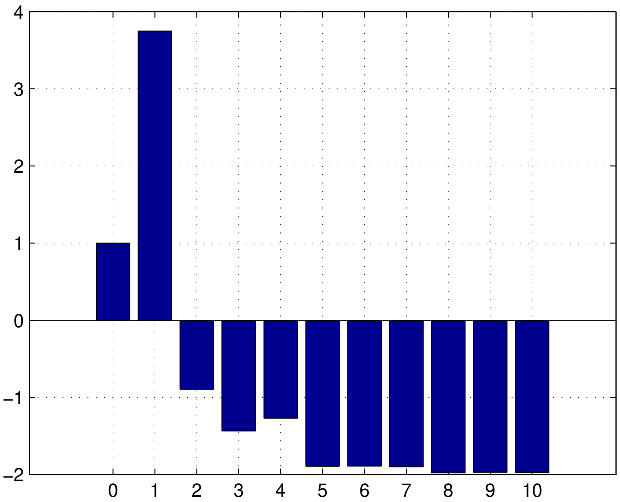
$y[k] = \varepsilon[k]\left(-1,9865 + 6,6 \cdot 0,5^k + 4,9487 \cdot 0,5^k \cos(2,2143k - 2,3894)\right)$.

Plotting the response signal may be done with the help of MATLAB:

$\ll k = 0:10$

$\ll y = -1.9865 + 6.6 * .5.\hat{}k + 4.9487 * .5.\hat{}k. * \cos(2.2143 * k - 2.3894)$

$\ll bar(k, y); grid$



d) Let be $u_1[k] = 6\varepsilon[k]$ and $y_1[k]$ the response signal in case of $u_1[k]$ input signal! As the system is linear and invariant the response to the $u[k] = u_1[k] - u_1[k-4]$ input signal is

$y[k] = y_1[k] - y_1[k-4]$. It is enough to calculate the $y_1[k]$ response signal to the $u_[k] = 6\varepsilon[k]$ input signal.

$y_1[k] = \varepsilon[k] \sum_{p=0}^{k} u_1[k-p]h[p]$. For $k \geq 0$

$y_1[k] = \sum_{p=0}^{k} 6\left\{0,5\delta[p] + \varepsilon[p-1]7,4673 \cdot 0,5^{p-1}\cos[2,2143(p-1)+1,4365]\right\}$

If $k = 0$: ($p = 0$ only) $y_1[0] = 3$.

If $k \geq 0$: $y_1[k] = \sum_{p=0}^{k} \cdots = \sum_{p=0}^{0} \cdots + \sum_{p=1}^{k} \cdots$

In the first sum $\delta[p] = 1$, $\varepsilon[p-1] = 0$, in the second one $\delta[p] = 0$, $\varepsilon[p-1] = 1$, so

$y_1[k] = 3 + \sum_{p=1}^{k} 6 \cdot 7,4673 \cdot 0,5^{p-1}\cos[2,2143(p-1)+1,4365]$.

With the $\cos[2,2143(p-1)+1,4365] = Re\left\{e^{j1,4365}\left(e^{j2,2143}\right)^{p-1}\right\}$ substitution

$y_1[k] = 3 + Re\left\{\sum_{p=1}^{k} 6 \cdot 7,4673 e^{j1,4365}\left(0,5e^{j2,2143}\right)^{p-1}\right\}$.

$y_1[k] = 3 + Re\left\{(6+j44,4)\frac{\left(0,5e^{j2,2143}\right)^{k}-1}{0,5e^{j2,2143}-1}\right\} = 3 + Re\left\{(5,3838 - j32,4973)\left[\left(0,5e^{j2,2143}\right)^{k}-1\right]\right\}$

$5,3839 - j32,4973 = 32,9402e^{-j1,4066}$

$y_1[k] = 3 + Re\left\{32,9402 \cdot 0,5^k e^{j(2,2143k-1,4066)}\right\} - 5,3838 =$

$= -2,3838 + 32,9402 \cdot 0,5^k\cos(2,2143k - 1,4066)$.

Putting $k = 0$ into this formula the value equals to $y_1[0] = 3$, so the formula is valid not only for $k \geq 0$ but for $k = 0$ as well.

Finally $y_1[k] = \varepsilon[k]\left(-2,3838 + 32,9402 \cdot 0,5^k\cos(2,2143k - 1,4066)\right)$, and

$y[k] = y_1[k] - y_1[k-4] = \varepsilon[k]\left(-2,3838 + 32,9402 \cdot 0,5^k\cos(2,2143k - 1,4066)\right) -$
$\quad -\varepsilon[k-4]\left(-2,3838 + 32,9402 \cdot 0,5^{k-4}\cos[2,2143(k-4) - 1,4066]\right)$.

Plotting with the help of MATLAB:

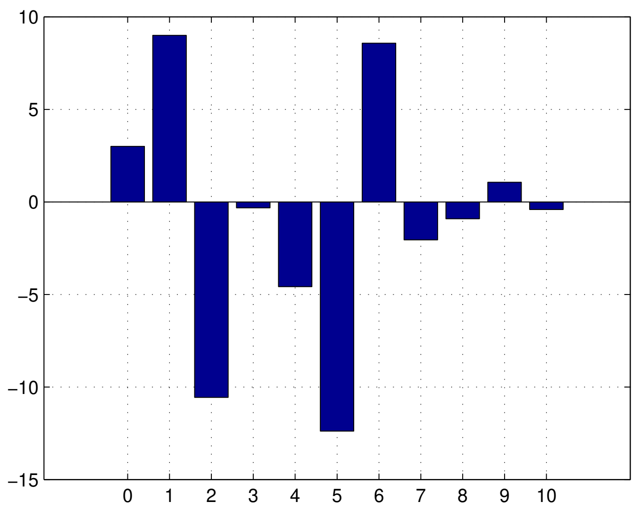
$\ll k = 0:10$

$\ll y1 = -2,3838 + 32,9402 * .5.\hat{}k. * \cos(2.2143 * k - 1.4066)$

$\ll y2 = [zeros(1,4), y1(1:7)]$

$\ll y = y1 - y2$

$\ll bar(k,y); grid$



Other solution method may be applied (see e.g. 3.d) second solution).

6. A CT system is given with the following state variable description:

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -1 & -5 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 4 \\ -4 \end{bmatrix} u(t),$$
$$y(t) = \begin{bmatrix} 0,5 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + 2u(t).$$

a) Find the formula of the system impulse response!

b) Find the response of the system in case of $u(t) = \varepsilon(t)\left(-3 + 4e^{-0,5t}\right)$ input signal! Plot the response for $0 \le t \le 3$!

c) Find the response of the system in case of $u(t) = 5\left(\varepsilon(t) - \varepsilon(t-1)\right)$ input signal! Plot the response for $0 \le t \le 3$!

Solution

a) Eugenvalue calculation: $det(\mathbf{A} - \lambda\mathbf{I}) = 0$

$\begin{vmatrix} -1-\lambda & -5 \\ 5 & -1-\lambda \end{vmatrix} = (-1-\lambda)(-1-\lambda) + 25 = \lambda^2 + 2\lambda + 26 = 0,$

$\lambda_1 = -1 + j5, \qquad \lambda_2 = -1 - j5.$

First solution. $h(t) =, D\delta(t) + \varepsilon(t)\mathbf{C}^T e^{\mathbf{A}t}\mathbf{B} = D\delta(t) + \varepsilon(t)\left(\mathbf{C}^T\mathbf{L}_1\mathbf{B}e^{\lambda_1 t} + \mathbf{C}^T\mathbf{L}_2\mathbf{B}e^{\lambda_2 t}\right)$

$\mathbf{A} = \begin{bmatrix} -1 & -5 \\ 5 & -1 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 4 \\ -4 \end{bmatrix}, \qquad \mathbf{C}^T = \begin{bmatrix} 0,5 & -2 \end{bmatrix}, \qquad D = 2.$

$\mathbf{L}_1 = \frac{1}{\lambda_1 - \lambda_2}(\mathbf{A} - \lambda_2\mathbf{I}) = \frac{1}{-1+j5-(-1-j5)}\begin{bmatrix} j5 & -5 \\ 5 & j5 \end{bmatrix} = \begin{bmatrix} 0,5 & j0,5 \\ -j0,5 & 0,5 \end{bmatrix},$

$\mathbf{L}_2 = \frac{1}{\lambda_2 - \lambda_1}(\mathbf{A} - \lambda_1\mathbf{I})$, as $\lambda_2 = \lambda_1^*$ (conjugate), $\mathbf{L}_2 = \mathbf{L}_1^*$.

(Checking: $\mathbf{L}_1 + \mathbf{L}_2 = \mathbf{I}$.)

$\mathbf{C}^T\mathbf{L}_1\mathbf{B} = \begin{bmatrix} 0,5 & -2 \end{bmatrix}\begin{bmatrix} 0,5 & j0,5 \\ -j0,5 & 0,5 \end{bmatrix}\begin{bmatrix} 4 \\ -4 \end{bmatrix} = \begin{bmatrix} 0,5 & -2 \end{bmatrix}\begin{bmatrix} 2-j2 \\ -2-j2 \end{bmatrix} = 5 + j3$

$\mathbf{C}^T\mathbf{L}_2\mathbf{B} = \left(\mathbf{C}^T\mathbf{L}_1\mathbf{B}\right)^* = 5 - j3$

$\mathbf{C}^T\mathbf{L}_1\mathbf{B}e^{\lambda_1 t} + \mathbf{C}^T\mathbf{L}_2\mathbf{B}e^{\lambda_2 t} = 2Re\left\{\mathbf{C}^T\mathbf{L}_1\mathbf{B}e^{\lambda_1 t}\right\} = 2Re\left\{(5+j3)e^{(-1+j5)t}\right\},$

$Re\left\{(10+j6)e^{(-1+j5)t}\right\} = Re\left\{11,6619e^{j0,5404}e^{-t}e^{j5t}\right\} = 11,6619e^{-t}\cos(5t + 0,5404)$, so

$h(t) = 2\delta(t) + \varepsilon(t)11,6619e^{-t}\cos(5t + 0,5404).$

Second solution. Response calculation to $u(t) = \delta(t)$ input signal with decomposition to free and generated components.

$h(t) = y(t) = y_f(t) + y_g(t), \qquad u(t) = \delta(t).$

First step. $\mathbf{x}(t) = \mathbf{x}_f(t) + \mathbf{x}_g(t) = C_1\mathbf{m}_1e^{\lambda_1 t} + C_2\mathbf{m}_2e^{\lambda_2 t} + \mathbf{x}_g(t), \quad t \ge 0.$

Eigenvector calculation. $\mathbf{m} = \begin{bmatrix} m_a \\ m_b \end{bmatrix}$

$\mathbf{A}\begin{bmatrix} m_a \\ m_b \end{bmatrix} = \lambda\begin{bmatrix} m_a \\ m_b \end{bmatrix} \quad \begin{matrix} (-1-\lambda)m_a - 5m_b = 0 \\ 5m_a + (-1-\lambda)m_b = 0 \end{matrix}$ let be $m_a = 5$ then $m_b = -1 - \lambda$

$\lambda = \lambda_1 = -1 + j5, \quad m_b = -j5 \quad \mathbf{m}_1 = \begin{bmatrix} 5 \\ -j5 \end{bmatrix}$

$\lambda = \lambda_2 = -1 - j5, \quad m_b = j5 \quad \mathbf{m}_2 = \begin{bmatrix} 5 \\ j5 \end{bmatrix}$

The free component of the state variable vector is then:

$\mathbf{x}_f(t) = C_1\mathbf{m}_1e^{\lambda_1 t} + C_2\mathbf{m}_2e^{\lambda_2 t} = 2Re\left\{C_1\mathbf{m}_1e^{\lambda_1 t}\right\}$, with $C_1 = c_r + jc_i$ notation

$\mathbf{x}_f(t) = 2Re\left\{(c_r + jc_i)\begin{bmatrix}5\\-j5\end{bmatrix}e^{(-1+j5)t}\right\}$

Second step. The generated component of the state variable vector is $\mathbf{x}_g(t) = \begin{bmatrix}0\\0\end{bmatrix}$, as $u(t) = \delta(t) = 0$, if $t > 0$. So $\mathbf{x}(t) = \mathbf{x}_f(t) = 2Re\left\{(c_r + jc_i)\begin{bmatrix}5\\-j5\end{bmatrix}e^{(-1+j5)t}\right\}$.

Third step. The starting value of the state variable vector is $\mathbf{x}(+0) = \mathbf{B} = \begin{bmatrix}4\\-4\end{bmatrix}$ as $u(t) = \delta(t)$.

$$\begin{aligned} x_1(+0) &= 10c_r = 4 \\ x_2(+0) &= 10c_i = -4 \end{aligned} \quad \text{From here} \quad \begin{aligned} c_r &= 0,4 \\ c_i &= -0,4 \end{aligned}$$

The expression of the state variable vector in case of $u(t) = \delta(t)$ is:

$\mathbf{x}(t) = 2Re\left\{(0,4 - j0,4)\begin{bmatrix}5\\-j5\end{bmatrix}e^{(-1+j5)t}\right\} = Re\left\{\begin{bmatrix}4-j4\\-4-j4\end{bmatrix}e^{(-1+j5)t}\right\}$

Fourth step. Substitution into the $y(t) = \mathbf{C}^T\mathbf{x}(t) + Du(t)$ response expression:

$y(t) = Re\left\{\begin{bmatrix}0,5 & -2\end{bmatrix}\begin{bmatrix}4-j4\\-4-j4\end{bmatrix}e^{(-1+j5)t}\right\} + 2\delta(t)$, $(t \geq 0)$,

$y(t) = Re\left\{(10 + j6)e^{-t}e^{j5t}\right\} + 2\delta(t) = Re\left\{11,6619e^{-t}e^{j(5t+0,5404)}\right\} + 2\delta(t)$.

$h(t) = y(t) = 2\delta(t) + \varepsilon(t)11,6619e^{-t}\cos(5t + 0,5404)$.

b) The system is causal, and the input signal is causal signal, so

$y(t) = \varepsilon(t)\int_0^t u(\tau)h(t-\tau)\mathrm{d}\tau = \varepsilon(t)\int_0^t u(t-\tau)h(\tau)\mathrm{d}\tau$.

The latter formula results in simpler calculation. For $t \geq 0$:

$$\begin{aligned} y(t) &= \int_0^t \left(-3 + 4e^{-0,5(t-\tau)}\right)\left(2\delta(\tau) + 11,6619e^{-\tau}\cos(5\tau + 0,5404)\right)\mathrm{d}\tau = \\ &= -6 + 8e^{-0,5t} - 3\cdot 11,6619\int_0^t e^{-\tau}\cos(5\tau + 0,5404)\mathrm{d}\tau + \\ &\quad +4\cdot 11,6619e^{-0,5t}\int_0^t e^{0,5\tau}e^{-\tau}\cos(5\tau + 0,5404)\mathrm{d}\tau. \end{aligned}$$

As $\cos(5\tau + 0,5404) = Re\left\{e^{j0,5404}e^{j5\tau}\right\}$

$$\begin{aligned} y(t) &= -6 + 8e^{-0,5t} - Re\left\{34,9857e^{j0,5404}\int_0^t e^{(-1+j5)\tau}\mathrm{d}\tau\right\} + \\ &\quad +Re\left\{46,6476e^{j0,5404}e^{-0,5t}\int_0^t e^{(-0,5+j5)\tau}\mathrm{d}\tau\right\} = \\ &= -6 + 8e^{-0,5t} - Re\left\{\frac{34,9857e^{j0,5404}}{-1+j5}\left[e^{(-1+j5)\tau}\right]_0^t\right\} + Re\left\{\frac{46,6476e^{j0,5404}}{-0,5+j5}e^{-0,5t}\left[e^{(-0,5+j5)\tau}\right]_0^t\right\} = \\ &= -6 + 8e^{-0,5t} - Re\left\{(2,3077 - j6,4615)\left(e^{(-1+j5)t} - 1\right)\right\} + \\ &\quad +Re\left\{(3,9604 - j8,3960)e^{-0,5t}\left(e^{(-0,5+j5)t} - 1\right)\right\} = \\ &= -6 + 8e^{-0,5t} + Re\left\{(1,6527 - j1,9345)e^{-t}e^{j5t}\right\} + Re\left\{2,3077 - j6,4615\right\} - \\ &\quad -Re\left\{(3,9604 - j8,3960)e^{-0,5t}\right\} \end{aligned}$$

$(1,6527 - j1,9345)e^{j5t} = 2,5444e^{j(5t-0,8638)}$

$y(t) = -6 + 8e^{-0,5t} + 2,5444e^{-t}\cos(5t - 0,8638) + 2,3077 - 3,9604e^{-0,5t}$, if $t \geq 0$. Finally

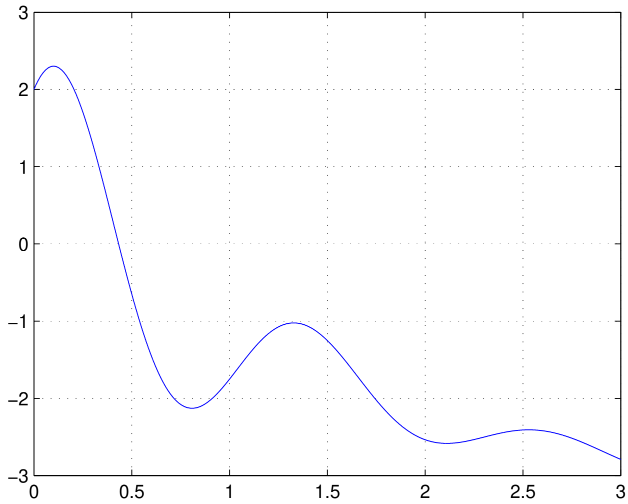
$y(t) = \varepsilon(t)\left(-3,6923 + 4,0396e^{-0,5t} + 2,5444e^{-t}\cos(5t - 0,8638)\right)$.

Plotting the response signal may be done with the help of MATLAB:

$\ll t = 0 : .001 : 3;$

$\ll y = -3.6923 + 4.0396 * exp(-.5 * t) + 2.5444 * exp(-t). * \cos(5 * t - .8638);$

$\ll plot(t, y); grid$



c) Let be $u_1(t) = 5\varepsilon(t)$ and $y_1(t)$ the response signal in case of $u_1(t)$ input signal! As the system is linear and invariant the response to the $u(t) = u_1(t) - u_1(t-1)$ input signal is $y(t) = y_1(t) - y_1(t-1)$. It is enough to calculate the $y_1(t)$ response signal to the $u_1(t) = 5\varepsilon(t)$ input signal.

$y_1(t)] = \varepsilon(t) \int_0^t u_1(t-\tau)h(\tau)\mathrm{d}\tau$. For $t \geq 0$:

$y_1(t) = \int_0^t 5\left[2\delta(\tau) + 11,6619e^{-\tau}\cos(5\tau + 0,5404)\right]\mathrm{d}\tau =$

$\quad = 10 + \int_0^t 58,3095e^{-\tau}\cos(5\tau + 0,5404)\mathrm{d}\tau,$

$\cos(5\tau + 0,5404) = Re\left\{e^{j0,5404}e^{j5\tau}\right\},$

$y_1(t) = 10 + Re\left\{58,3095e^{j0,5404}\int_0^t e^{(-1+j5)\tau}\mathrm{d}\tau\right\} = 10 + Re\left\{\frac{58,3095e^{j0,5404}}{-1+j5}\left[e^{(-1+j5)\tau}\right]_0^t\right\},$

$\frac{58,3095e^{j0,5404}}{-1+j5} = 3,8459 - j10,7693 = 11,4354e^{-j1,2278},$

$y_1(t) = 10 + Re\left\{(3,8459 - j10,7693)\left(e^{(-1+j5)t} - 1\right)\right\} =$

$\quad = 10 + Re\left\{11,4354e^{-t}e^{j(5t-1,2278)} - 3,8459 + j10,7693\right\} =$

$\quad 6,1541 + 11,4354e^{-t}\cos(5t - 1,2278)$, if $t \geq 0$. Finally:

$y_1(t) = \varepsilon(t)\left(6,1541 + 11,4354e^{-t}\cos(5t - 1,2278)\right)$, so

$y(t) = y_1(t) - y_1(t-1) = \varepsilon(t)\left(6,1541 + 11,4354e^{-t}\cos(5t - 1,2278)\right) -$

$\quad -\varepsilon(t-1)\left(6,1541 + 11,4354e^{-(t-1)}\cos[5(t-1) - 1,2278]\right).$

Other solution method may be applied (see e.g. 4.c) second solution).

The values of the response signal may be plotted with the help of MATLAB in the following way:

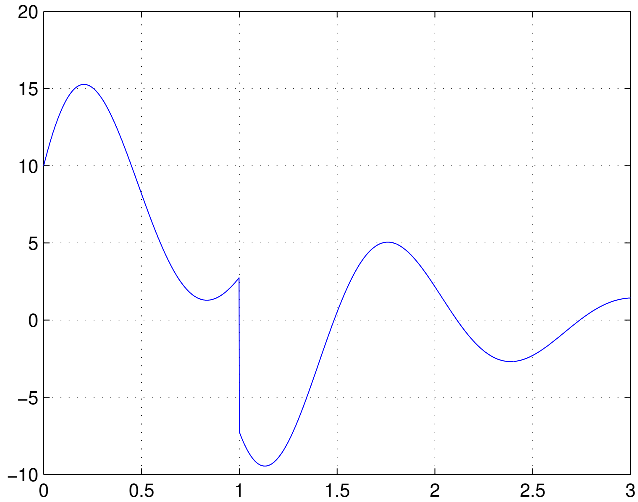
$\ll t = 0 : .001 : 3;$

$\ll y1 = 6.1541 + 11.4354 * exp(-t). * \cos(5 * t - 1.2278);$

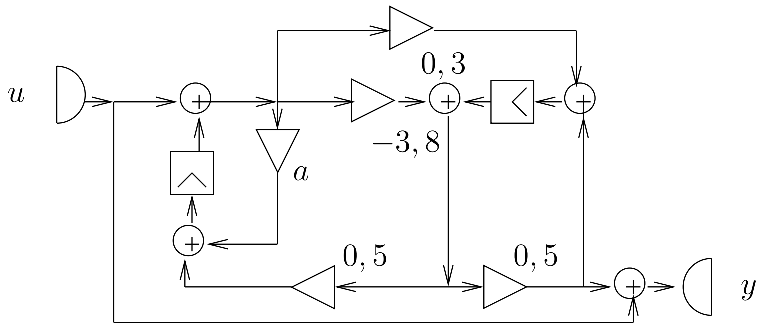
$\ll y2 = [zeros(1, 1000), y1(1 : 2001)];$

$\ll y = y1 - y2;$

$\ll plot(t, y); grid$



7. The CT and the DT systems are given with the following signal flow network:



a) Give the state variable description of both systems in normal form!

b) Examine the stability of the CT and of the DT network as well if $a = 0,8$!

c) Decide the network stability condition of the CT and of the DT network for the a multiplying factor!

Solution

a) The x' notation for the x signal means $\frac{\mathrm{d}x(t)}{\mathrm{d}t}$, if the signal is $x(t)$ CT one and $x[k+1]$ in case of $x[k]$ DT signal.

The x_1 and x_2 state variables are the output signals of the left and of the right dynamic components, respectively. The input signals x_1' and x_2' of that dynamic components and the response signal may be expressed as:

$x_1' = a(x_1 + u) + 0,5[x_2 - 3,8(x_1 + u)]$,

$x_2' = 0,3(x_1 + u) + 0,5[x_2 - 3,8(x_1 + u)]$,

$y = 0,5[x_2 - 3,8(x_1 + u)] + u$.

The normal form of the state variable description of both systems is:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} a - 1,9 & 0,5 \\ -1,6 & 0,5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} a - 1,9 \\ -1,6 \end{bmatrix} u$$

$$y = \begin{bmatrix} -1,9 & 0,5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - 0,9u$$

b) The system matrix with $a = 0,8$ value is: $\mathbf{A} = \begin{bmatrix} -1,1 & 0,5 \\ -1,6 & 0,5 \end{bmatrix}$

Eigenvalue calculation: $det(\mathbf{A} - \lambda\mathbf{I}) = 0$:

$$\begin{vmatrix} -1,1-\lambda & 0,5 \\ -1,6 & 0,5-\lambda \end{vmatrix} = (-1,1-\lambda)(0,5-\lambda)+0,8 = \lambda^2+0,6\lambda+0,25=0,$$
$\lambda_1 = -0,3+j0,4,\ \lambda_2 = -0,3-j0,4.$

The CT system is asymptotically stable, so the CT network is stable as $Re(\lambda_1) < 0$ and $Re(\lambda_2) < 0$.

The DT system is asymptotically stable, so the DT network is stable as $|\lambda_1| < 1$ and $|\lambda_2| < 1$.

c) The characteristic polynomial when a is a parameter:

$$P(\lambda) = \begin{vmatrix} a-1,9-\lambda & 0,5 \\ -1,6 & 0,5-\lambda \end{vmatrix} = (a-1,9-\lambda)(0,5-\lambda)+0,8 =$$
$$= \lambda^2 + (1,4-a)\lambda + 0,5a - 0,15.$$

CT system. According to the Hurwitz criterion:

$a_1 = 1,4-a > 0 \qquad a < 1,4$

$a_2 = 0,5a - 0,15 > 0 \qquad a > 0,3$

So the CT network stability condition for the parameter a is: $0,3 < a < 1,4$.

DT system. According to the Jury criterion:

$P(\lambda=1) = 2,25 - 0,5a > 0 \qquad a < 4,5$

$P(\lambda=-1) = -0,55 + 1,5a > 0 \qquad a > \frac{11}{30}$

$|a_2| = |0,5a - 0,15| < 1 \qquad -1 < 0,5a-0,15 < 1 \qquad -1,7 < a < 2,3$

So the DT network stability condition for the parameter a is: $\frac{11}{30} < a < 2,3$.

8. Find the frequency response of the CT system given with the following state variable description if it exists or give explanation of your "the frequency response does not exist" answer!

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3 & -2,5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 2 \\ -2 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} -0,5 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x2(t) \end{bmatrix} + 0,5u(t)$$

First solution.

$H(j\omega) = \frac{\mathbf{C}^T adj(j\omega\mathbf{I}-\mathbf{A})\mathbf{B}}{det(j\omega\mathbf{I}-\mathbf{A})} + D$, if it exists where $\mathbf{A} = \begin{bmatrix} -2 & 1 \\ 3 & -2,5 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$,

$\mathbf{C}^T = \begin{bmatrix} -0,5 & 2 \end{bmatrix}$, $D = 0,5$.

$j\omega\mathbf{I}-\mathbf{A} = \begin{bmatrix} j\omega+2 & -1 \\ -3 & j\omega+2,5 \end{bmatrix}$, $\quad det(j\omega\mathbf{I}-\mathbf{A}) = (j\omega+2)(j\omega+2,5)-3 = (j\omega)^2+4,5j\omega+2.$

This polynomial is Hurwitz polynomial, the system is asymptotically stable (and BIBO stable as well), so the frequency response exists.

$(j\omega\mathbf{I}-\mathbf{A})^T = \begin{bmatrix} j\omega+2 & -3 \\ -1 & j\omega+2,5 \end{bmatrix}$, $\qquad adj(j\omega\mathbf{I}-\mathbf{A}) = \begin{bmatrix} j\omega+2,5 & 1 \\ 3 & j\omega+2 \end{bmatrix}$,

$$H(j\omega) = \frac{\begin{bmatrix} -0,5 & 2 \end{bmatrix} \begin{bmatrix} j\omega+2,5 & 1 \\ 3 & j\omega+2 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \end{bmatrix}}{(j\omega)^2+4,5j\omega+2} + 0,5 = \frac{\begin{bmatrix} -0,5 & 2 \end{bmatrix} \begin{bmatrix} 2j\omega+3 \\ -2j\omega+2 \end{bmatrix}}{(j\omega)^2+4,5j\omega+2} + 0,5 =$$
$$= \frac{-5j\omega+2,5}{(j\omega)^2+4,5j\omega+2} + 0,5 = \frac{-5j\omega+2,5+0,5(j\omega)^2+2,25j\omega+1}{(j\omega)^2+4,5j\omega+2}$$

In normal form $H(j\omega) = \dfrac{0,5(j\omega)^2 - 2,75j\omega + 3,5}{(j\omega)^2+4,5j\omega+2}$.

Second solution. Assuming the existence of the frequency response the relations of the signals complex amplitudes are:

$$\begin{aligned} j\omega\bar{X}_1 &= -2\bar{X}_1 + \bar{X}_2 + 2\bar{U} \\ j\omega\bar{X}_2 &= 3\bar{X}_1 - 2,5\bar{X}_2 - 2\bar{U} \\ \bar{Y} &= -0,5\bar{X}_1 + 2\bar{X}_2 + 0,5\bar{U} \end{aligned}$$

From the first two equations:

$$\begin{aligned} (j\omega+2)\bar{X}_1 - \bar{X}_2 &= 2\bar{U} \qquad / \cdot (j\omega + 2,5) \qquad / \cdot 3 \\ -3\bar{X}_1 + (j\omega+2,5)\bar{X}_2 &= -2\bar{U} \qquad\qquad\qquad\qquad / \cdot (j\omega+2) \end{aligned}$$

$$\bar{X}_1\left((j\omega)^2 + 4,5j\omega + 5 - 3\right) = \bar{U}(2j\omega + 5 - 2)$$

$$\bar{X}_2\left((j\omega)^2 + 4,5j\omega + 5 - 3\right) = \bar{U}(-2j\omega - 4 + 6)$$

$$\bar{X}_1 = \bar{U}\frac{2j\omega+3}{(j\omega)^2+4,5j\omega+2}, \qquad \bar{X}_2 = \bar{U}\frac{-2j\omega+2}{(j\omega)^2+4,5j\omega+2},$$

$\bar{Y} = -0,5\bar{X}_1 + 2\bar{X}_2 + 0,5\bar{U} = \bar{U}\frac{-0,5(2j\omega+3)+2(-2j\omega+2)+0,5\left((j\omega)^2 4,5j\omega+2\right)}{(j\omega)^2+4,5j\omega+2} = \bar{U}\frac{0,5(j\omega)^2-2,75j\omega+3,5}{(j\omega)^2+4,5j\omega+2}$, so the "formal" frequency response is: $H(j\omega) = \frac{0,5(j\omega)^2-2,75j\omega+3,5}{(j\omega)^2+4,5j\omega+2}$. As the denominator is Hurwitz polynomial of the $j\omega$ variable, the frequency response exists and equals to the formal one.

9. A periodic CT signal is given as $x(t) = -9t - 6 + \varepsilon(t)10,8t$, if $-2 \le t \le 10$, and for any t $x(t+12) = x(t)$.

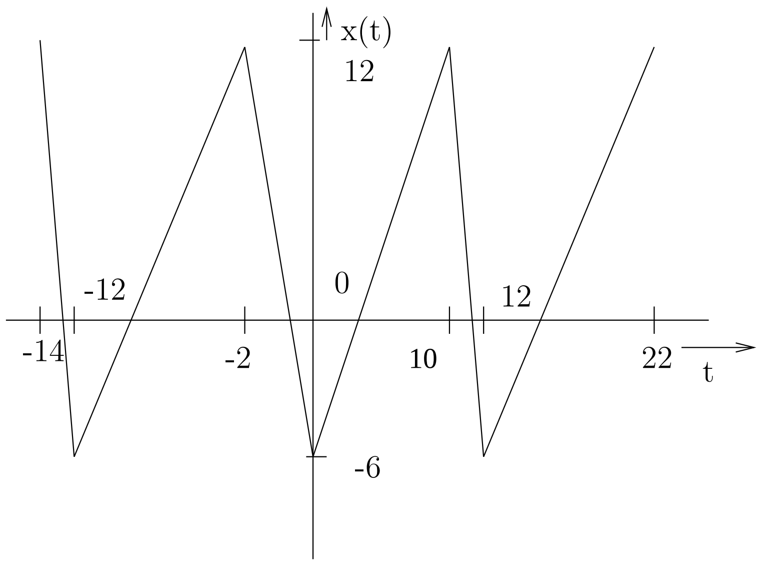
   a) Plot the signal in the $-14 \le t \le 22$ interval!

   b) Find the real form of the Fourier polynomial of the signal which contains - in addition of the constant component - at least 4 non zero harmonics!

   c) Plot the Fourier polynomial values in the $-14 \le t \le 22$ interval!

   Solution

   a) $x(-2) = x(10) = 12$ and $x(0) = -6$ and the signal is linear (the plot is straight line) for $-2 \le t \le 0$ and for $0 \le t \le 10$, and the period of the signal is $T = 12$:



   b) The period is: $T = 12$, the basic angular frequency is: $\omega_0 = \frac{2\pi}{T} = \frac{\pi}{6}$. The formula for the complex Fourier coefficients calculation is: $X_p^C = \frac{1}{T}\int_{t=<T>} x(t)e^{-jp\omega_0 t}\mathrm{d}t$, and the amplitudes and phase angles for one real form of the Fourier series are:
   $U_0 = U_0^C$, and for $p > 0$:
   $U_p = 2\left|U_p^C\right|, \qquad \rho_p = angle\left(U_p^C\right).$

   $p = 0$: $U_0^C = \frac{1}{12}\int_{-2}^{10} x(t)\mathrm{d}t$, but the integral may be calculated with the areas of tree triangles ($x(t) \le 0$, if $-\frac{2}{3} \le t \le \frac{10}{3}$): $\int_{-2}^{10} x(t)\mathrm{d}t = \frac{\frac{4}{3}\cdot 12}{2} - \frac{4\cdot 6}{2} + \frac{\frac{20}{3}\cdot 12}{2} = 36,\ U_0^C = \frac{1}{12}\cdot 36 = 3,$
   $U_0 = U_0^C = 3.$

$p \geq 1$: $U_p^C = \frac{1}{12}\left[\int_{-2}^{0}(-9t-6)e^{-jp\frac{\pi}{6}t}\mathrm{d}t + \int_0^{10}(1,8t-6)e^{-jp\frac{\pi}{6}t}\mathrm{d}t\right] =$

$= \int_{-2}^{0}(-0,75t-0,5)e^{-jp\frac{\pi}{6}t}\mathrm{d}t + \int_0^{10}(0,15t-0,5)e^{-jp\frac{\pi}{6}t}\mathrm{d}t.$

Partial integration may be applied: $\int_a^b u \cdot v' \mathrm{d}x = [u \cdot v]_a^b - \int_a^b u' \cdot v \mathrm{d}x$.

In the first component $u = -0,75t - 0,5$, $u' = -0,75$, in the second one $u = 0,15t - 0,5$, $u' = 0,15$, in both components: $v' = e^{-jp\frac{\pi}{6}t}$, $v = \frac{j6}{\pi p}e^{-jp\frac{\pi}{6}t}$, so

$$U_p^C = \left[(-0,75t-0,5)\tfrac{j6}{\pi p}e^{-jp\frac{\pi}{6}t}\right]_{-2}^{0} + \tfrac{j4,5}{\pi p}\int_{-2}^{0} e^{-jp\frac{\pi}{6}t}\mathrm{d}t + \left[(0,15t-0,5)\tfrac{j6}{\pi p}e^{-jp\frac{\pi}{6}t}\right]_0^{10} -$$

$$-\tfrac{j0,9}{\pi p}\int_0^{10} e^{-jp\frac{\pi}{6}t}\mathrm{d}t = \tfrac{-j3}{\pi p} - \tfrac{j6}{\pi p}e^{jp\frac{\pi}{3}} - \tfrac{27}{\pi^2p^2}\left[e^{-jp\frac{\pi}{6}t}\right]_{-2}^{0} + \tfrac{j6}{\pi p}e^{-jp\frac{\pi}{6}\cdot 10} + \tfrac{j3}{\pi p} + \tfrac{5,4}{\pi^2p^2}\left[e^{-jp\frac{\pi}{6}t}\right]_0^{10} =$$

$$= \tfrac{j6}{\pi p}\left(e^{-jp\frac{5\pi}{3}} - e^{jp\frac{\pi}{3}}\right) + \tfrac{27}{\pi^2p^2}\left(e^{jp\frac{\pi}{3}} - 1\right) + \tfrac{5,4}{\pi^2p^2}\left(e^{-jp\frac{5\pi}{3}} - 1\right).$$

$e^{-jp\frac{5\pi}{3}} = e^{-jp\frac{5\pi}{3}} \cdot e^{jp2\pi} = e^{jp\frac{-5\pi+6\pi}{3}} = e^{jp\frac{\pi}{3}}$, as for any integer p $e^{jp2\pi} = 1$. Finally

$U_p^C = \frac{32,4}{\pi^2p^2}\left(e^{jp\frac{\pi}{3}} - 1\right)$.

The $\left(e^{jp\frac{\pi}{3}} - 1\right)$ term in the expression of U_p^C, is periodic in p, the period is 6. $(e^{j(p+6)\frac{\pi}{3}} = = e^{jp\frac{\pi}{3}} \cdot e^{jp2\pi} = e^{jp\frac{\pi}{3}})$.

If $p = 1, 7, 13, \ldots$ $\quad U_p^C = \frac{32,4}{\pi^2p^2}\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)$; $\quad U_p = \frac{64,8}{\pi^2p^2}$, $\quad \rho_p = \frac{2\pi}{3}$;

if $p = 2, 8, 14, \ldots$ $\quad U_p^C = \frac{32,4}{\pi^2p^2}\left(-\frac{3}{2} + j\frac{\sqrt{3}}{2}\right)$; $\quad U_p = \frac{64,8\sqrt{3}}{\pi^2p^2}$, $\quad \rho_p = \frac{5\pi}{6}$;

if $p = 3, 9, 15, \ldots$ $\quad U_p^C = \frac{32,4}{\pi^2p^2}(-2)$; $\quad U_p = \frac{129,6}{\pi^2p^2}$, $\quad \rho_p = \pi$;

if $p = 4, 10, 16, \ldots$ $\quad U_p^C = \frac{32,4}{\pi^2p^2}\left(-\frac{3}{2} - j\frac{\sqrt{3}}{2}\right)$; $\quad U_p = \frac{64,8\sqrt{3}}{\pi^2p^2}$, $\quad \rho_p = -\frac{5\pi}{6}$;

if $p = 5, 11, 17, \ldots$ $\quad U_p^C = \frac{32,4}{\pi^2p^2}\left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right)$; $\quad U_p = \frac{64,8}{\pi^2p^2}$, $\quad \rho_p = -\frac{2\pi}{3}$;

If $p = 6, 12, 18, \ldots$ $\quad U_p^C = 0$; $\quad U_p = 0$.

The real form of the Fourier polynomial which contains four hamonics is:

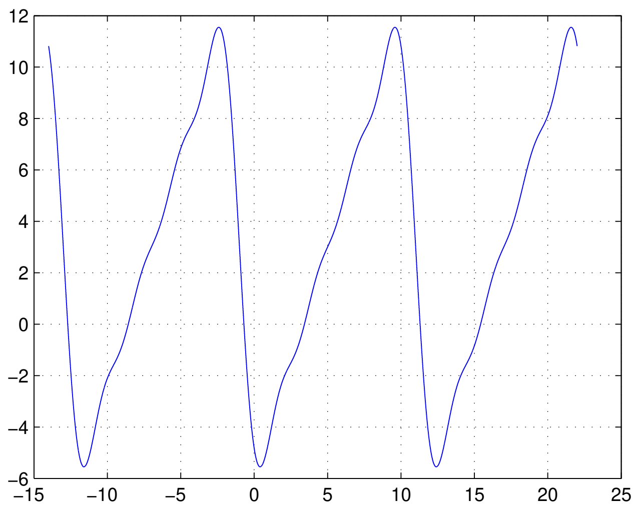
$$x(t) = 3 + 6,5656\cos\left(\tfrac{\pi}{6}t + 2,0944\right) + 2,8430\cos\left(\tfrac{2\pi}{6}t + 2,6180\right) +$$
$$+1,4590\cos\left(\tfrac{3\pi}{6}t + 3,1416\right) + 0,7107\cos\left(\tfrac{4\pi}{6}t - 2,6180\right).$$

c) The time function may be plotted with the following MATLAB statements:

$\ll t = -14 : .001 : 22;$

$\ll x = 3 + 6.5656 * \cos(pi/6 * t + 2 * pi/3) + 2.843 * \cos(pi/3 * t + 5 * pi/6) + 1.459 * \cos(pi/2 * t + pi) + .7107 * \cos(2 * pi/3 * t - 5 * pi/6);$

$\ll plot(t, x); grid$



10. The input signal of the system given with state variable description in the problem 8. is the periodic signal examined in the problem 9. Find the response of the system in four order Fourier polynomial approximation!

Solution

The solution may be followed on the next table:

p	0	1	2	3	4
$p\omega_0$	0	$\frac{\pi}{6}$	$\frac{2\pi}{6}$	$\frac{3\pi}{6}$	$\frac{4\pi}{6}$
U_p	3	$6,5656$	$2,8430$	$1,4590$	$0,7107$
ρ_{up}		$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$-\frac{5\pi}{6}$
$H(p\omega_0)$	$1,75$	$1,2525$	$0,8594$	$0,6886$	$0,6075$
$\varphi(p\omega_0)$	0	$-1,3432$	$-2,1545$	$-2,7245$	$3,1167$
Y_p	$5,25$	$8,2237$	$2,4434$	$1,0047$	$0,4318$
ρ_{yp}		$0,7512$	$0,4635$	$0,4171$	$0,4987$

In the table the second row contains the $p\omega_0$ angular frequencies of the input and the output signal harmonics. The U_p input signal harmonics amplitudes and ρ_p phase angles are copied from the result of the 9th point into the third and into the fourth row of the table. The system frequency response was found in the problem 8. The $H(p\omega_0)$ amplitude characteristic values (the absolute values of the transfer coefficients at the harmonic frequencies) and the $\varphi(p\omega_0)$ phase characteristic values (the angles of the transfer coefficients) are written in the fifth and in the sixth row of the table.

The Y_0 constant coefficient of the response is the product of the U_0 input constant component with the $H(j\omega)|_{\omega=0}$ transfer coefficient at zero frequency. The amplitude and the phase angle of the p-th response harmonics (if $p>0$) are calculated as: $Y_p = U_p \cdot H(p\omega_0)$ and $\rho_{yp} = \rho_{up} + \varphi(p\omega_0)$. The response harmonics amplitudes and phase angles are in the seventh and in the eighth row of the table. So the response signal in four-order Fourier polynomial approximation is:

$$y(t) \simeq 5,25 + 8,2237\cos\left(\tfrac{\pi}{6}t + 0,7512\right) + 2,4434\cos\left(\tfrac{\pi}{3}t + 0,4635\right) + 1,0047\cos\left(\tfrac{\pi}{2}t + 0,4071\right) + \\ + 0,4318\cos\left(\tfrac{2\pi}{3}t + 0,4987\right).$$

11. The input signal of the system given in the problem 8. with state variable description is an impulse which in the $0 \le t \le 12$ interval equals to the periodic signal examined in the problem 9. and that input signal is zero if $t<0$ and if $t>12$.

a) Find the Fourier transform of the response signal and plot the amplitude spectrum in the $0 \le \omega \le 4$ angular frequency interval!

b) Find the bandwidth of the response signal with the condition that the spectrum is negligible if the amplitude spectrum is less than 5% of its maximum!

c) Find the formula of the response time function and plot it in the $0 \le t \le 20$ time interval!

Solution

a) The expression of the input signal is: $u(t) = [\varepsilon(t) - \varepsilon(t-10)](-6+1,8t) + [\varepsilon(t-10) - \varepsilon(t-12)](102-9t)$. For determining the response spectrum we have to find the Fourier transform of the input signal first.

First solution

$$\mathbf{F}\{u(t)\} = U(j\omega) = \int_{-\infty}^{\infty} u(t)e^{-j\omega t}\mathrm{d}t = \int_0^{10}(-6+1,8t)e^{-j\omega t}\mathrm{d}t + \int_{10}^{12}(102-9t)e^{-j\omega t}\mathrm{d}t = \\ = -6\int_0^{10} e^{-j\omega t}\mathrm{d}t + 1,8\int_0^{10} te^{-j\omega t}\mathrm{d}t + 102\int_{10}^{12} e^{-j\omega t}\mathrm{d}t - 9\int_{10}^{12} te^{-j\omega t}\mathrm{d}t.$$

In the second and in the fourth components we apply partial integration: $\int_a^b u \cdot v'\mathrm{d}x = [u \cdot v]_a^b - \int_a^b u' \cdot v\mathrm{d}x$. Here $u=t$, $u'=1$, $v' = e^{-j\omega t}$, $v = \frac{1}{-j\omega}e^{-j\omega t}$,

$U(j\omega) = \frac{6}{j\omega}\left[e^{-j\omega t}\right]_0^{10} + 1,8\left[-\frac{t}{j\omega}e^{-j\omega t}\right]_0^{10} + \frac{1,8}{j\omega}\int_0^{10} e^{-j\omega t}\mathrm{d}t - \frac{102}{j\omega}\left[e^{-j\omega t}\right]_{10}^{12} - 9\left[-\frac{t}{j\omega}e^{-j\omega t}\right]_{10}^{12} -$
$- \frac{9}{j\omega}\int_{10}^{12} e^{-j\omega t}\mathrm{d}t = \frac{6}{j\omega}\left(e^{-j10\omega} - 1\right) - \frac{18}{j\omega}e^{-j10\omega} - \frac{1,8}{(j\omega)^2}\left[e^{-j\omega t}\right]_0^{10} - \frac{102}{j\omega}\left(e^{-j12\omega} - e^{-j10\omega}\right) +$
$+ \frac{9}{j\omega}\left(12e^{-j12\omega} - 10e^{-j10\omega}\right) + \frac{9}{(j\omega)^2}\left[e^{-j\omega t}\right]_{10}^{12} =$
$= -\frac{6}{j\omega} + \frac{1,8}{(j\omega)^2} + e^{-j10\omega}\left(\frac{6}{j\omega} - \frac{18}{j\omega} - \frac{1,8}{(j\omega)^2} + \frac{102}{j\omega} - \frac{90}{j\omega} - \frac{9}{(j\omega)^2}\right) +$
$+ e^{-j12\omega}\left(-\frac{102}{j\omega} + \frac{108}{j\omega} + \frac{9}{(j\omega)^2}\right)$. Finally the formula of the input signal Fourier transform is
$U(j\omega) = \frac{-6j\omega + 1,8 - 10,8e^{-j10\omega} + (6j\omega + 9)e^{-j12\omega}}{(j\omega)^2}$.

Second solution

As the $u(t)$ signal is causal and absolute integrable so the Fourier transform may be got from the Laplace transform: $\mathbf{F}\{u(t)\} = \mathbf{L}\{u(t)\}|_{s=j\omega}$. Transform the signal to such form which is suitable for transforming:

$u(t) = [\varepsilon(t) - \varepsilon(t-10)](-6 + 1,8t) + [\varepsilon(t-10) - \varepsilon(t-12)](102 - 9t) = \varepsilon(t)(-6 + 1,8t) +$
$+ \varepsilon(t-10)(6 - 1,8t + 102 - 9t) + \varepsilon(t-12)(-102 + 9t) = \varepsilon(t)(-6 + 1,8t) +$
$+ \varepsilon(t-10)[108 - 10,8(t-10) - 108] + \varepsilon(t-12)[-102 + 9(t-12) + 108]$,
$u(t) = \varepsilon(t)(-6 + 1,8t) - \varepsilon(t-10)10,8(t-10) + \varepsilon(t-12)[6 + 9(t-12)]$,
$\mathbf{L}\{u(t)\} = -\frac{6}{s} + \frac{1,8}{s^2} - e^{-10s}\frac{10,8}{s^2} + e^{-12s}\left(\frac{6}{s} + \frac{9}{s^2}\right)$.

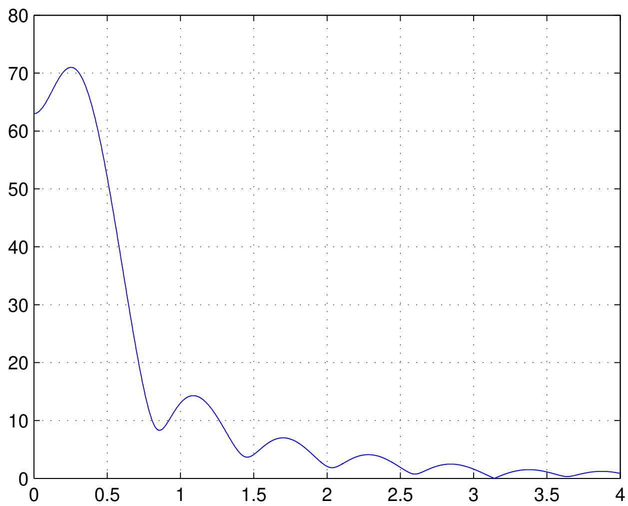
With $s = j\omega$ substitution the result is the same as that of the first solution.

The Fourier transform of the response signal is the product of the input signal Fourier transform and the system frequency response. The latter is the result of the solution of the eighth problem.

$Y(j\omega) = U(j\omega)H(j\omega) = \frac{-6j\omega + 1,8 - 10,8e^{-j10\omega} + (6j\omega + 9)e^{-j12\omega}}{(j\omega)^2} \cdot \frac{0,5(j\omega)^2 - 2,75j\omega + 3,5}{(j\omega)^2 + 4,5j\omega + 2}$.

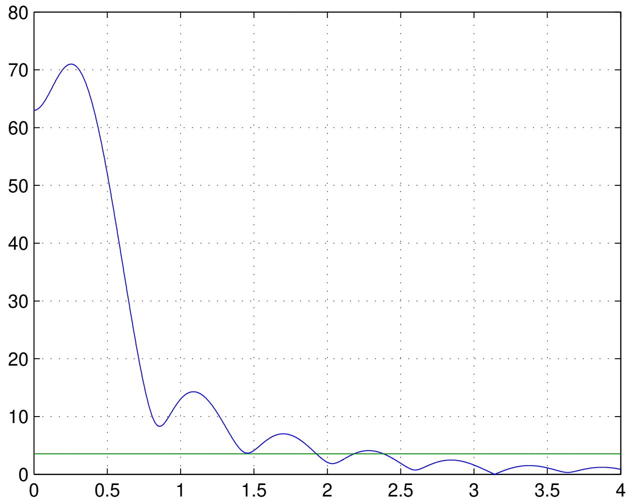
The amplitude spectrum is the absolute value of the Fourier transform. The amplitude spectrum may be plotted with the following MATLAB statements.

$\ll om = .001 : .001 : 4;$
$ujo = (-6j * om + 1.8 - 10.8 * exp(-10j * om) + (6j * om + 9). * exp(-12j * om))./(-om.\hat{}2);$
$hjo = (-.5 * om.\hat{}2 - 2.75j * om + 3.5)./(-om.\hat{}2 + 4.5j * om + 2);$
$\ll yjo = ujo. * hjo;$
$\ll plot(om, abs(yjo)); grid$



b) We will plot in a common figure the amplitude spectrum and a horizontal straight line which corresponds to the value of 5%of the maximal amplitude spectrum value.

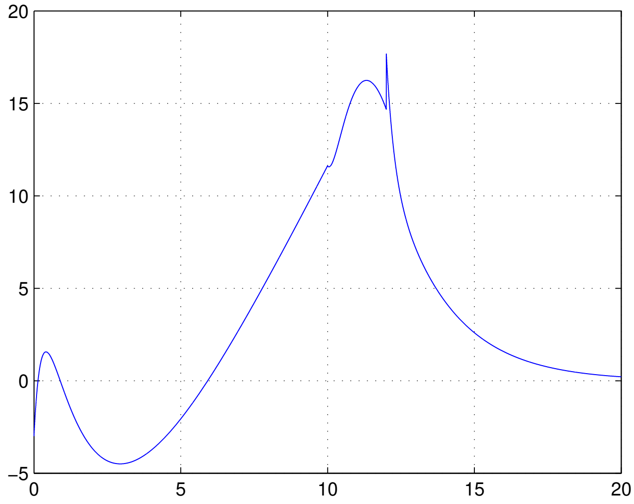
$\ll y5pc = .05 * max(abs(yjo)) * ones(1, 4000);$
$\ll plot(om, abs(yjo), om, y5pc); grid$



From the figure is seen that $\omega_2 \approx 2,3$ is the angular frequency value above which everywhere the line of the amplitude spectrum is under the straight line that is the spectrum is negligible. The lowest frequency of the signal frequency band is $\omega_1 = 0$, so the bandwidth of the signal is: $\Delta\omega = \omega_2 - \omega_1 = 2,3$.

c) The response signal time function may be got from its spectrum with inverse Fourier transformation but the formula of the response signal is easier to find with the help of Laplace transformation. The method is applicable as the system is causal and the input signal is causal signal. The Laplace transform of the response is found as: $Y(s) = U(s) \cdot H(s)$, where the $U(s)$ Laplace transform of the input signal was found in the second solution of the point a), the $H(s)$ transfer function follows from the result of the eighth problem. $U(s) = -\frac{6}{s} + \frac{1,8}{s^2} - e^{-10s}\frac{10,8}{s^2} + e^{-12s}\left(\frac{6}{s} + \frac{9}{s^2}\right)$, and as the system is causal and BIBO stable, $H(s) = H(j\omega)|_{j\omega=s} == \frac{0,5s^2-2,75s+3,5}{s^2+4,5s+2}$. The Laplace transform of the response signal is then: $Y(s) = \frac{-6s+1,8}{s^2}H(s) + e^{-10s}\left(-\frac{10,8}{s^2}H(s)\right) + e^{-12s}\left(\frac{6s+9}{s^2}H(s)\right)$.
In all the three components the rational function term is proper fraction, they can be transformed directly into sum of partial fractions. The $s^2 + 4,5s + 2$ transfer function denominator may be replaced by the $(s+4)(s+0,5)$ product, so the transformation of the three rational functions into sum of partial fractions are as follow:
$\frac{-6s+1,8}{s^2}H(s) = \frac{(-6s+1,8)(0,5s^2-2,75s+3,5)}{s^2(s+4)(s+0,5)} = \frac{-20,0625}{s} + \frac{3,15}{s^2} + \frac{-10,3661}{s+4} + \frac{27,4286}{s+0,5}$
$\frac{-10,8}{s^2}H(s) = \frac{-10,8(0,5s^2-2,75s+3,5)}{s^2(s+4)(s+0,5)} = \frac{57,375}{s} + \frac{-18,9}{s^2} + \frac{4,3393}{s+4} + \frac{-61,7143}{s+0,5}$
$\frac{6s+9}{s^2}H(s) = \frac{(6s+9)(0,5s^2-2,75s+3,5)}{s^2(s+4)(s+0,5)} = \frac{-37,3125}{s} + \frac{15,75}{s^2} + \frac{6,0268}{s+4} + \frac{34,2857}{s+0,5}$
With inverse Laplace transformation the formula of the response signsl is:
$$y(t) = \varepsilon(t)\left[-20,0625 + 3,15t - 10,3661e^{-4t} + 27,4286e^{-0,5t}\right] +$$
$$+\varepsilon(t-10)\left[57,375 - 18,9(t-10) + 4,3393e^{-4(t-10)} - 61,7143e^{-0,5(t-10)}\right] +$$
$$+\varepsilon(t-12)\left[-37,3125 + 15,75(t-12) + 6,0268e^{-4(t-12)} + 34,2857e^{-0,5(t-12)}\right].$$
The time function may be plotted with the help of the following MATLAB statements:
$\ll t = 0 : .001 : 20;$
$\ll y1 = -20.0625 + 3.15 * t - 10.3661 * exp(-4 * t) + 27.4286 * exp(-.5 * t);$
$\ll y2 = 57.375 - 18.9 * t + 4.3393 * exp(-4 * t) - 61.7143 * exp(-.5 * t);$
$\ll y3 = -37.3125 + 15.75 * t + 6.0268 * exp(-4 * t) + 34.2857 * exp(-.5 * t);$
$\ll y = y1 + [zeros(1, 10000), y2(1 : 10001)] + [zeros(1, 12000), y3(1 : 8001)];$

$\ll plot(t, y); grid$



12. The DT system is given with the following state variable description. Find the frequency response of the system or give explanation of your "the frequency response does not exist" answer!

$$\begin{bmatrix} x_1[k+1] \\ x_2[k+1] \end{bmatrix} = \begin{bmatrix} -0,2 & 0,8 \\ 0,3 & -0,4 \end{bmatrix} \begin{bmatrix} x_1[k] \\ x_2[k] \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \end{bmatrix} u[k],$$

$$y[k] = \begin{bmatrix} 0,5 & 2 \end{bmatrix} \begin{bmatrix} x_1[k] \\ x_2[k] \end{bmatrix} - 0,5u[k].$$

First solution.

$H\left(e^{j\vartheta}\right) = \frac{\mathbf{C}^T adj\left(e^{j\vartheta}\mathbf{I}-\mathbf{A}\right)\mathbf{B}}{det\left(e^{j\vartheta}\mathbf{I}-\mathbf{A}\right)} + D$, if it exists where $\mathbf{A} = \begin{bmatrix} -0,2 & 0,8 \\ 0,3 & -0,4 \end{bmatrix}$, $\quad \mathbf{B} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$,

$\mathbf{C}^T = \begin{bmatrix} 0,5 & 2 \end{bmatrix}$, $\quad D = -0,5. \qquad e^{j\vartheta}\mathbf{I} - \mathbf{A} = \begin{bmatrix} e^{j\vartheta}+0,2 & -0,8 \\ -0,3 & e^{j\vartheta}+0,4 \end{bmatrix}$,

$det\left(e^{j\vartheta}\mathbf{I} - \mathbf{A}\right) = \left(e^{j\vartheta}+0,2\right)\left(e^{j\vartheta}+0,4\right) - 0,24 = e^{j2\vartheta} + 0,6e^{j\vartheta} - 0,16 = P\left(e^{j\vartheta}\right)$.

Stability checking:
$$P\left(e^{j\vartheta}=1\right) = 1+0,6-0,16 > 0$$
$$P\left(e^{j\vartheta}=-1\right) = 1-0,6-0,16 > 0$$
$$|a_{22}| = 0,16 < 1$$

According to the Jury criterion the system is asymptotically stable (so BIBO stable as well), the frequency response exists.

$$\left(e^{j\vartheta}\mathbf{I} - \mathbf{A}\right)^T = \begin{bmatrix} e^{j\vartheta}+0,2 & -0,3 \\ -0,8 & e^{j\vartheta}+0,4 \end{bmatrix} \qquad adj\left(e^{j\vartheta}\mathbf{I} - \mathbf{A}\right) = \begin{bmatrix} e^{j\vartheta}+0,4 & 0,8 \\ 0,3 & e^{j\vartheta}+0,2 \end{bmatrix}$$

$$H\left(e^{j\vartheta}\right) = \frac{\begin{bmatrix} 0,5 & 2 \end{bmatrix} \begin{bmatrix} e^{j\vartheta}+0,4 & 0,8 \\ 0,3 & e^{j\vartheta}+0,2 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix}}{e^{j2\vartheta}+0,6e^{j\vartheta}-0,16} - 0,5 = \frac{\begin{bmatrix} 0,5 & 2 \end{bmatrix} \begin{bmatrix} -2e^{j\vartheta} \\ e^{j\vartheta}-0,4 \end{bmatrix}}{e^{j2\vartheta}+0,6e^{j\vartheta}-0,16} - 0,5 =$$

$= \frac{e^{j\vartheta}-0,8}{e^{j2\vartheta}+0,6e^{j\vartheta}-0,16} - 0,5 = \frac{-0,5e^{j2\vartheta}+0,7e^{j\vartheta}-0,72}{e^{j2\vartheta}+0,6e^{j\vartheta}-0,16}$. In normal form:

$H\left(e^{j\vartheta}\right) = \frac{-0,5+0,7e^{-j\vartheta}-0,72e^{-j2\vartheta}}{1+0,6e^{-j\vartheta}-0,16e^{-j2\vartheta}}$.

Second solution.

Assuming the BIBO stability of the system on the basis of the state variable description the relations among the complex amplitudes are as follow:

$$e^{j\vartheta}\bar{X}_1 = -0,2\bar{X}_1 + 0,8\bar{X}_2 - 2\bar{U}$$
$$e^{j\vartheta}\bar{X}_2 = 0,3\bar{X}_1 - 0,4\bar{X}_2 + \bar{U}$$
$$\bar{Y} = 0,5\bar{X}_1 + 2\bar{X}_2 - 0,5\bar{U}$$

From the first two equations:

$$\left(e^{j\vartheta} + 0,2\right)\bar{X}_1 - 0,8\bar{X}_2 = -2\bar{U} \quad / \cdot \left(e^{j\vartheta} + 0,4\right) \qquad / \cdot 0,3$$
$$-0,3\bar{X}_1 + \left(e^{j\vartheta} + 0,4\right)\bar{X}_2 = \bar{U} \quad / \cdot 0,8 \qquad / \cdot \left(e^{j\vartheta} + 0,2\right)$$

$$\bar{X}_1\left(e^{j2\vartheta} + 0,6e^{j\vartheta} + 0,08 - 0,24\right) = \bar{U}\left(-2e^{j\vartheta} - 0,8 + 0,8\right)$$
$$\bar{X}_2\left(e^{j2\vartheta} + 0,6e^{j\vartheta} + 0,08 - 0,24\right) = \bar{U}\left(e^{j\vartheta} + 0,2 - 0,6\right)$$

$\bar{X}_1 = \bar{U}\frac{-2e^{j\vartheta}}{e^{j2\vartheta}+0,6e^{j\vartheta}-0,16}$, $\qquad \bar{X}_2 = \bar{U}\frac{e^{j\vartheta}-0,4}{e^{j2\vartheta}+0,6e^{j\vartheta}-0,16}$.

Substituting into the expression of the response signal complex amplitude:

$\bar{Y} = 0,5\bar{X}_1 + 2\bar{X}_2 - 0,5\bar{U} = \bar{U}\frac{-e^{j\vartheta}+2e^{j\vartheta}-0,8+0,5\left(e^{j2\vartheta}+0,6e^{j\vartheta}-0,16\right)}{e^{j2\vartheta}+0,6e^{j\vartheta}-0,16} = \bar{U}\frac{-0,5e^{j2\vartheta}+0,7e^{j\vartheta}-0,72}{e^{j2\vartheta}+0,6e^{j\vartheta}-0,16}$

The "formal" frequency response expression is then: $H\left(e^{j\vartheta}\right) = \frac{-0,5e^{j2\vartheta}+0,7e^{j\vartheta}-0,72}{e^{j2\vartheta}+0,6e^{j\vartheta}-0,16}$

Examining the denominator polynomial of the variable $e^{j\vartheta}$ according to the Jury criterion the system is BIBO stable, the above expression is not only "formal" frequency response but an existing one, in real form: $H\left(e^{j\vartheta}\right) = \frac{-0,5+0,7e^{-j\vartheta}-0,72e^{-j2\vartheta}}{1+0,6e^{-j\vartheta}-0,16e^{-j2\vartheta}}$.

13. The input signal of the DT system given with state variable description in the twelvth problem is the following periodic signal: $u[k] = -k + \varepsilon[k]4k$ if $-1 \le k \le 4$ and for any k $u[k+6] = u[k]$.

    a) Write the two real forms and the complex form of the Fourier series of that input signal!

    b) Find the response of the system and plot it in the $-6 \le k \le 11$ interval!

    c) The input signal of the examined DT system is an impulse which, in the $0 \le k \le 5$ interval equals to the previous periodic input signal and it equals to zero if $k < 0$ and if $k > 5$. Find the Fourier transform of the response signal and plot the amplitude spectrum in the $0 \le \vartheta \le \pi$ interval!

    d) Find the time function formula of the response and plot the response signal in the $0 \le k \le 15$ interval if the input signal is the same impulse as that in the c) point!

Solution

a) The period is: $L = 6$, the discrete angular frequency is: $\vartheta_0 = \frac{2\pi}{L} = \frac{\pi}{3}$. The formula of the complex Fourier coefficients of an $x[k]$ periodic input signal is: $X_p^C = \frac{1}{L}\sum_{k=0}^{L-1} x[k]e^{-jp\vartheta_0 k}$, and the parameters of the harmonics in the two real forms are:
$X_0 = X_0^C$,
$X_p^A = 2Re\left\{X_p^C\right\}$, $X_p^B = -2Im\left\{X_p^C\right\}$, $X_p = 2\left|X_p^C\right|$, if $p = 1,\ 2$,
$\rho_p = angle\left\{X_p^C\right\}$, if $p = 1,\ 2,\ 3$,
$X_3^A = X_3^C$, $X_3^B = 0$, $X_3 = \left|X_3^C\right|$.
The input signal values for $0 \le k \le 5$ are: $u[0] = 0$, $u[1] = 3$, $u[2] = 6$, $u[3] = 9$, $u[4] = 12$, $u[5] = u[-1] = 1$.
$p = 0$: $U_0^C = \frac{1}{6}(0 + 3 + 6 + 9 + 12 + 1) = \frac{31}{6} \simeq 5,1667$, $\quad U_0 = U_0^C = 5,1667$.
$p = 1$: $U_1^C = \frac{1}{6}\left(0 + 3e^{-j1\cdot\frac{\pi}{3}\cdot 1} + 6e^{-j1\cdot\frac{\pi}{3}\cdot 2} + 9e^{-j1\cdot\frac{\pi}{3}\cdot 3} + 12e^{-j1\cdot\frac{\pi}{3}\cdot 4} + 1e^{-j1\cdot\frac{\pi}{3}\cdot 5}\right) \simeq$
$\simeq -2,6667 + j0,5774 = 2,7285e^{j2,9284}$.
$U_1 = 2\left|U_1^C\right| = 5,4569$, $\qquad \rho_1 = angle\left\{U_1^C\right\} = 2,9284$,
$U_1^A = 2Re\left\{U_1^C\right\} = -5,3333$, $\qquad U_1^B = -2Im\left\{U_1^C\right\} = -1,1547$,
$p = 2$: $U_2^C = \frac{1}{6}\left(0 + 3e^{-j2\cdot\frac{\pi}{3}\cdot 1} + 6e^{-j2\cdot\frac{\pi}{3}\cdot 2} + 9e^{-j2\cdot\frac{\pi}{3}\cdot 3} + 12e^{-j2\cdot\frac{\pi}{3}\cdot 4} + 1e^{-j2\cdot\frac{\pi}{3}\cdot 5}\right) \simeq$
$\simeq -0,3333 - j1,1547 = 1,2019e^{-j1,8518}$.

$U_2 = 2\left|U_2^C\right| = 2,4037,$ $\qquad \rho_2 = angle\left\{U_2^C\right\} = -1,8518,$

$U_2^A = 2Re\left\{U_2^C\right\} = -0,6667,$ $\qquad U_2^B = -2Im\left\{U_2^C\right\} = 2,3094,$

$p = 3$: $U_3^C = \frac{1}{6}\left(0 + 3e^{-j3\cdot\frac{\pi}{3}\cdot 1} + 6e^{-j3\cdot\frac{\pi}{3}\cdot 2} + 9e^{-j3\cdot\frac{\pi}{3}\cdot 3} + 12e^{-j3\cdot\frac{\pi}{3}\cdot 4} + 1e^{-j3\cdot\frac{\pi}{3}\cdot 5}\right) =$

$= \frac{1}{6}(0 - 3 + 6 - 9 + 12 - 1) = \frac{5}{6} \simeq 0,8333.$

$U_3 = \left|U_3^C\right| = 0,8333,$ $\qquad \rho_3 = angle\left\{U_3^C\right\} = 0,$

$U_3^A = U_3^C = 0,8333,$ $\qquad U_3^B = 0.$

For the complex form we need U_4^C and U_5^C Fourier coefficients as well:

$U_4^C = U_{-2}^C = \left(U_2^C\right)^* = -0,3333 + j1,1547,$ $U_5^C = U_{-1}^C = \left(U_1^C\right)^* = -2,6667 - j0,5774.$

The two real forms of the Fourier series are:

$u[k] = 5,1667 + 5,4569\cos\left(\frac{\pi}{3}k + 2,9284\right) + 2,4037\cos\left(\frac{2\pi}{3}k - 1,8518\right) + 0,8333\cos(\pi k),$

$u[k] = 5,1667 - 5,3333\cos\frac{\pi}{3}k - 1,1547\sin\frac{\pi}{3}k - 0,6667\cos\frac{2\pi}{3}k + 2,3094\sin\frac{2\pi}{3}k + 0,8333\cos\pi k.$

The complex form is:

$$u[k] = 5,1667 + (-2,6667 + j0,5774)e^{j\frac{\pi}{3}k} + (-0,3333 - j1,1547)e^{j\frac{2\pi}{3}k} + 0,8333e^{j\pi k} + \\ +(-0,3333 + j1,1547)e^{j\frac{4\pi}{3}k} + (-2,6667 - j0,5774)e^{j\frac{5\pi}{3}k}.$$

b) The solution may be followed on the next table:

p	0	1	2	3
$p\vartheta_0$	0	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π
U_p	$5,1667$	$5,4569$	$2,4037$	$0,8333$
ρ_{up}		$2,9284$	$-1,8518$	0
$H(p\vartheta_0)$	$0,3611$	$0,1472$	$1,2971$	8
$\varphi(p\vartheta_0)$	π	$0,3517$	$-1,2491$	π
Y_p	$-1,8657$	$0,8032$	$3,1178$	$6,6667$
ρ_{yp}	$-$	$3,2801$	$-3,1009$	π

In the table the second row contains the $p\vartheta_0$ discrete angular frequencies of the input and of the output signal harmonics. The U_p input signal harmonics amplitudes and ρ_{up} phase angles are copied from the result of the a)point into the third and into the fourth row of the table. The system frequency response was found in the problem 12. The $H(p\vartheta_0)$ amplitude characteristic values (the absolute values of the transfer coefficients at the harmonic frequencies) and the $\varphi(p\vartheta_0)$ phase characteristic values (the angles of the transfer coefficients) are written in the fifth and in the sixth row of the table.

The Y_0 constant coefficient of the response is the product of the U_0 input constant component with the $H\left(e^{j\vartheta}\right)\big|_{\vartheta=0}$ transfer coefficient at zero frequency. The amplitude and the phase angle of the p-th response harmonics (if $p > 0$) are calculated as: $Y_p = U_p \cdot H(p\vartheta_0)$ and $\rho_{yp} = \rho_{up} + \varphi(p\vartheta_0)$. The response harmonics amplitudes and phase angles are in the seventh and in the eighth row of the table.So the response signal is:

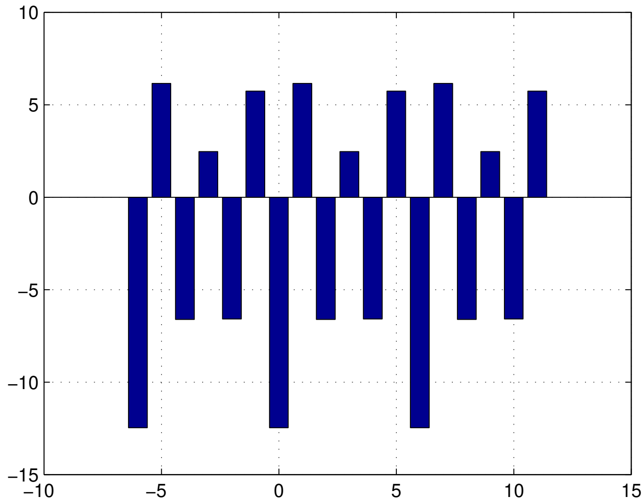
$$y[k] = -1,8657 + 0,8032\cos\left(\frac{\pi}{3}k + 3,2801\right) + 3,1178\cos\left(\frac{2\pi}{3}k - 3,1009\right) + \\ +6,6667\cos(\pi k + \pi).$$

The values of the response signal may be plotted with the help of MATLAB in the following way:

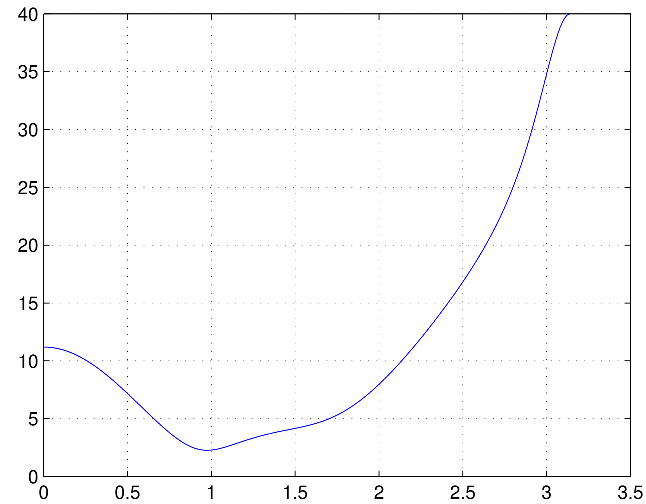
$\ll k = -6 : 11$

$\ll y = -1.8657 + .8032 * \cos(pi/3 * k + 3.2801) + 3.1178 * \cos(2 * pi/3 * k - 3.1009) + 6.6667 * \cos(pi * k + pi)$

$\ll bar(k, y); grid$



c) The input signal is absolute summable so its Fourier transform may be found as $F\left\{u[k]\right\} = U\left(e^{j\vartheta}\right) = \sum_{k=-\infty}^{\infty} u[k]e^{-j\vartheta k} = 0 + 3e^{-j\vartheta} + 6e^{-j2\vartheta} + 9e^{-j3\vartheta} + 12e^{-j4\vartheta} + 1e^{-j5\vartheta}$.
The Fourier transform of the response signal is:
$Y\left(e^{j\vartheta}\right) = U\left(e^{j\vartheta}\right) H\left(e^{j\vartheta}\right)$,
$Y\left(e^{j\vartheta}\right) = \left(3e^{-j\vartheta} + 6e^{-j2\vartheta} + 9e^{-j3\vartheta} + 12e^{-j4\vartheta} + 1e^{-j5\vartheta}\right) \frac{-0{,}5+0{,}7e^{-j\vartheta}-0{,}72e^{-j2\vartheta}}{1+0{,}6e^{-j\vartheta}-0{,}16e^{-j2\vartheta}}$.
The amplitude spectrum is the absolute value of the Fourier transform:
$Y(\vartheta) = \left|U\left(e^{j\vartheta}\right) H\left(e^{j\vartheta}\right)\right|$.
It can be plotted with the following MATLAB statements:
$\ll th = 0 : .001 : pi;$
$\ll ujt = 3{*}exp(-j{*}th){+}6{*}exp(-2j{*}th){+}9{*}exp(-3j{*}th){+}12{*}exp(-4j{*}th){+}exp(-5j{*}th);$
$\ll hjt = (-.5{+}.7{*}exp(-j{*}th){-}.72{*}exp(-2j{*}th))./(1{+}.6{*}exp(-j{*}th){-}.16{*}exp(-2j{*}th));$
$\ll yjt = ujt.*hjt;$
$\ll plot(th, abs(yjt)); grid$



d) The time function of the response equals to the inverse Fourier transform of the response spectrum, but the formula is easier to find with the help of z-transformation. As the system is causal and the input signal is causal signal, the z-transform of the response is: $Y(z) = U(z)H(z)$, where $U(z)$ is the z-transform of the input signal and $H(z)$ is the transfer function of the system. The response signal is the inverse z-transform of $Y(z)$.

The $H(z)$ transfer function is: $H(z) = \frac{-0,5+0,7z^{-1}-0,72z^{-2}}{1+0,6z^{-1}-0,16z^{-2}}$, as the system is BIBO stable and causal, the transfer function - frequency response relation is: $H(z) = H\left(e^{j\vartheta}\right)\Big|_{e^{j\vartheta}=z}$.

First solution.
The input signal may be transformed:
$u[k] = (\varepsilon[k] - \varepsilon[k-5])\,3k + \delta[k-5] = \varepsilon[k]3k - \varepsilon[k-5]\,[3(k-5)+15] + \delta[k-5]$.
The z-transform of the input signal is:
$U(z) = 3\frac{z}{(z-1)^2} - z^{-5}\left(3\frac{z}{(z-1)^2} + 15\frac{z}{z-1}\right) + z^{-5} = 3\frac{z}{(z-1)^2} + z^{-5}\frac{-3z-15z^2+15z+z^2-2z+1}{(z-1)^2}$
$U(z) = 3\frac{z^{-1}}{(1-z^{-1})^2} + z^{-5}\frac{-14+10z^{-1}+z^{-2}}{(1-z^{-1})^2}$.
$Y(z) = U(z)H(z) = 3\frac{z^{-1}}{(1-z^{-1})^2}H(z) + z^{-5}\frac{-14+10z^{-1}+z^{-2}}{(1-z^{-1})^2}H(z)$.
Transformong it to that form which is suitable for direct inverse z-transformation:
$$3\frac{z^{-1}}{(1-z^{-1})^2}H(z) = \frac{3z^{-1}\left(-0,5+0,7z^{-1}-0,72z^{-2}\right)}{(1-z^{-1})^2(1+0,6z^{-1}-0,16z^{-2})}\cdot\frac{z^3}{z^4}z = z\cdot\frac{-1,5z^2+2,1z-2,16}{(z-1)^2(z+0,8)(z-0,2)} =$$
$$= z\left(\frac{1,3310}{z-1} + \frac{-1,0833}{(z-1)^2} + \frac{1,4815}{z+0,8} + \frac{-2,8125}{z-0,2}\right),$$
$$\frac{-14+10z^{-1}+z^{-2}}{(1-z^{-1})^2}H(z) = \frac{\left(-14+10z^{-1}+z^{-2}\right)\left(-0,5+0,7z^{-1}-0,72z^{-2}\right)}{(1-z^{-1})^2(1+0,6z^{-1}-0,16z^{-2})}\frac{z^4}{z^4}zz^{-1} =$$
$$= z^{-1}z\frac{7z^4-14,8z^3+16,58z^2-6,5z-0,72}{z^4-1,4z^3-0,36z^2+0,92z-0,16} = z^{-1}z\left(7 + \frac{-5z^3+19,1z^2-12,94z+0,4}{(z-1)^2(z+0,8)(z-0,2)}\right) =$$
$$= z^{-1}z\left(7 + \frac{5,1690}{z-1} + \frac{1,0833}{(z-1)^2} + \frac{-7,8815}{z+0,8} + \frac{-2,2875}{z-0,2}\right).$$
$$Y(z) = 1,3310\frac{z}{z-1} - 1,0833\frac{z}{(z-1)^2} + 1,4815\frac{z}{z+0,8} - 2,8125\frac{z}{z-0,2} + 7z^{-5}+$$
$$+z^{-6}\left(5,1690\frac{z}{z-1} + 1,0833\frac{z}{(z-1)^2} - 7,8815\frac{z}{z+0,8} - 2,2875\frac{z}{z-0,2}\right).$$
With inverse z-transformation:
$$y[k] = \varepsilon[k]\left(1,3310 - 1,0833k + 1,4815(-0,8)^k - 2,8125\cdot 0,2^k\right) + 7\delta[k-5]+$$
$$+\varepsilon[k-6]\left(5,1690 + 1,0833(k-6) - 7,8815(-0,8)^{k-6} - 2,2875\cdot 0,2^{k-6}\right).$$

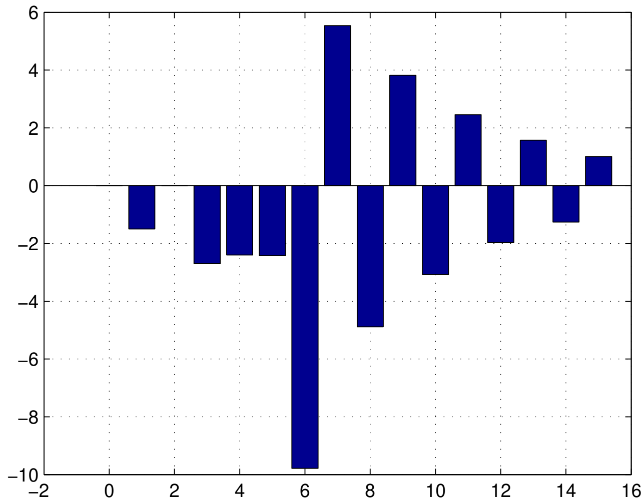
Second solution.
The input signal and its z-transform may be written as:
$u[k] = 3\delta[k-1] + 6\delta[k-2] + 9\delta[k-3] + 12\delta[k-4] + \delta[k-5]$,
$U(z) = 3z^{-1} + 6z^{-2} + 9z^{-3} + 12z^{-4} + z^{-5}$.
The z-transform of the response is:
$$Y(z) = U(z)H(z) = \frac{\left(3z^{-1}+6z^{-2}+9z^{-3}+12z^{-4}+z^{-5}\right)\left(-0,5+0,7z^{-1}-0,72z^{-2}\right)}{1+0,6z^{-1}-0,16z^{-2}}\frac{z^7}{z^2}zz^{-6} =$$
$$= z^{-6}z\frac{-1,5z^6-0,9z^5-2,46z^4-4,02z^3+1,42z^2-7,94z-0,72}{z^2+0,6z-0,16}.$$
After polynomial division:
$$Y(z) = z^{-6}z\left(-1,5z^4 - 2,7z^2 - 2,4z + 2,428 + \frac{-9,7808z-0,3315}{(z+0,8)(z-0,2)}\right) =$$
$$= -1,5z^{-1} - 2,7z^{-3} - 2,4z^{-4} + 2,428z^{-5} + z^{-6}z\left(\frac{-7,4931}{z+0,8} + \frac{-2,2877}{z-0,2}\right) =$$
$$= -1,5z^{-1} - 2,7z^{-3} - 2,4z^{-4} + 2,428z^{-5} + z^{-6}\left(-7,4931\frac{z}{z+0,8} - 2,2877\frac{z}{z-0,2}\right).$$
With inverse z-transformation:
$$y[k] = -1,5\delta[k-1] - 2,7\delta[k-3] - 2,4\delta[k-4] + 2,428\delta[k-5]+$$
$$+\varepsilon[k-6]\left(-7,4931(-0,8)^{k-6} - 2,2877\cdot 0,2^{k-6}\right).$$
This result is the same as that of the first solution. Putting $k = 0, 1, 2, 3, 4$ and 5 values into the two formulae the equivalence is clear. For $k \geq 6$ the final formula of the first solution may be transformed as:
$$y[k] = 1,3310 - 1,083k + 1,4815(-0,8)^k - 2,8125\cdot 0,2^k + 5,1690 + 1,083k - 6,4998-$$
$$-7,8815(-0,8)^{k-6} - 2,2875\cdot 0,2^{k-6} = 0,0002 + 1,4815(-0,8)^{k-6}\cdot(-0,8)^6-$$

$-2,8125 \cdot 0,2^{k-6} \cdot 0,2^6 - 7,8815(-0,8)^{k-6} - 2,2875 \cdot 0,2^{k-6} =$
$= 0,0002 - 7,4931(-0,8)^{k-6} - 2,2879 \cdot 0,2^{k-6}$.

Apart from rounding mistakes the two formulae are common for $k \geq 6$ as well.

To plot the response time function is easier on the basis of the second solution final formula with the following MATLAB statements.

$\ll k = 0:15$
$\ll k2 = 0:9$
$\ll y2 = -7.4931 * (-.8).\hat{}k2 - 2.2877 * (.2).\hat{}k2$
$\ll y = [0, -1.5, 0, -2.7, -2.4, -2.428, y2]$
$\ll bar(k, y); grid$

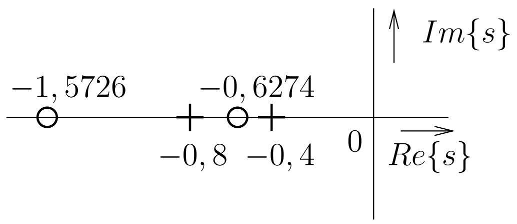


14. The CT system is given with the $h(t) = 6\delta(t) + \varepsilon(t)\left(4e^{-0,4t} + 2e^{-0,8t}\right)$ impulse response.

   a) Find the transfer function of the system and plot the pole-zero map and decide the BIBO stability of the system!
   b) Find the response of the system in case of $u(t) = \varepsilon(t)\left(-5 + 9e^{-0,5t}\right)$ input signal!
   c) Find the response of the system in case of $u(t) = [\varepsilon(t) - \varepsilon(t-2)]\,10$ input signal!

Solution

a) The transfer function is the Laplace-transform of the impulse response:
$H(s) = L\{h(t)\} = 6 + 4\frac{1}{s+0,4} + 2\frac{1}{s+0,8} = \frac{6(s^2+1,2s+0,32)+4(s+0,8)+2(s+0,4)}{s^2+1,2s+0,32} = \frac{6s^2+13,2s+5,92}{s^2+1,2s+0,32}$.
The poles of the system are the zeros of the denominator polynomial:
$s^2 + 1,2s + 0,32 = 0 \qquad s_1 = -0,8; \qquad s_2 = -0,4$.
The zeros of the system are the zeros of the numerator polynomial:
$6s^2 + 13,2s + 5,92 = 0 \qquad s_1 = -1,5726; \qquad s_2 = -0,6274$.
Poles: $\quad p_1 = -0,8; \quad p_2 = -0,4; \qquad$ zeros:$z_1 = -1,5726; \quad z_2 = -0,7264$.
The pole-zero map:



As the poles are situated on the left side of the complex plane the system is BIBO stable.

b) The input signal is causal, so the Laplace transform of the response is: $Y(s) = U(s)H(s)$, where $U(s)$ is the Laplace transform of the $u(t)$ input signal.

$u(t) = \varepsilon(t)(-5) + \varepsilon(t)9e^{-0,5t}$, $\quad U(s) = -5\frac{1}{s} + 9\frac{1}{s+0,5} = \frac{4s-2,5}{s(s+0,5)}$,

$Y(s) = \frac{4s-2,5}{s(s+0,5)} \frac{6s^2+13,2s+5,92}{s^2+1,2s+0,32}$. The fraction is a proper one so it may be transformed directly into sum of partial fractions:

$Y(s) = \frac{(4s-2,5)(6s^2+13,2s+5,92)}{s(s+0,5)(s+0,8)(s+0,4)} = \frac{-47,5}{s+0,8} + \frac{-246}{s+0,5} + \frac{410}{s+0,4} + \frac{-92,5}{s}$. With inverse Laplace transformation:

$y(t) = \varepsilon(t)\left(-92,5 - 246e^{-0,5t} - 47,5e^{-0,8t} + 410e^{-0,4t}\right)$.

c) The Laplace transform of the $u(t) = \varepsilon(t)10 - \varepsilon(t-2)10$ input signal is: $U(s) = \frac{10}{s} - e^{-2s}\frac{10}{s}$, and the Laplace transform of the response is:

$Y(s) = \frac{10(6s^2+13,2s+5,92)}{s(s+0,8)(s+0,4)} - e^{-2s}\frac{10(6s^2+13,2s+5,92)}{s(s+0,8)(s+0,4)}$.

In both components the fraction is a proper one, which may be directly transformed into sum of partial fractions, so

$Y(s) = \frac{185}{s} + \frac{-25}{s+0,8} + \frac{-100}{s+0,4} - e^{-2s}\left(\frac{185}{s} + \frac{-25}{s+0,8} + \frac{-100}{s+0,4}\right)$.

With inverse Laplace transformation:

$y(t) = \varepsilon(t)\left(185 - 25e^{-0,8t} - 100e^{-0,4t}\right) - \varepsilon(t-2)\left(185 - 25e^{-0,8(t-2)} - 100e^{-0,4(t-2)}\right)$.

15. Solve the fourteenth problem if the impulse response of the system is:
$h(t) = \varepsilon(t)5e^{-0,4t}\cos(2,5t - 0,2)$!

Solution

a) With the $\cos\alpha = \frac{1}{2}e^{j\alpha} + \frac{1}{2}e^{-j\alpha}$ identity:

$h(t) = \varepsilon(t)\left(2,5e^{-0,4t}e^{j(2,5t-0,2)} + 2,5e^{-0,4t}e^{-j(2,5t-0,2)}\right) =$

$= \varepsilon(t)2,5e^{-j0,2}e^{(-0,4+j2,5)t} + \varepsilon(t)2,5e^{j0,2}e^{(-0,4-j2,5)t}$.

$H(s) = L\{h(t)\} = \frac{2,5e^{-j0,2}}{s+0,4-j2,5} + \frac{2,5e^{j0,2}}{s+0,4+j2,5} = \frac{2,5e^{-j0,2}(s+0,4+j2,5)+2,5e^{j0,2}(s+0,4-j2,5)}{s^2+s(0,4-j2,5)+s(0,4+j2,5)+0,16+6,25} =$

$= \frac{2,5s\left(e^{j0,2}+e^{-j0,2}\right)+\left(e^{j0,2}+e^{-j0,2}\right)+j6,25\left(e^{-j0,2}-e^{j0,2}\right)}{s^2+0,8s+6,41} = \frac{s5\cos 0,2+2\cos 0,2+12,5\sin 0,2}{s^2+0,8s+6,41}$.

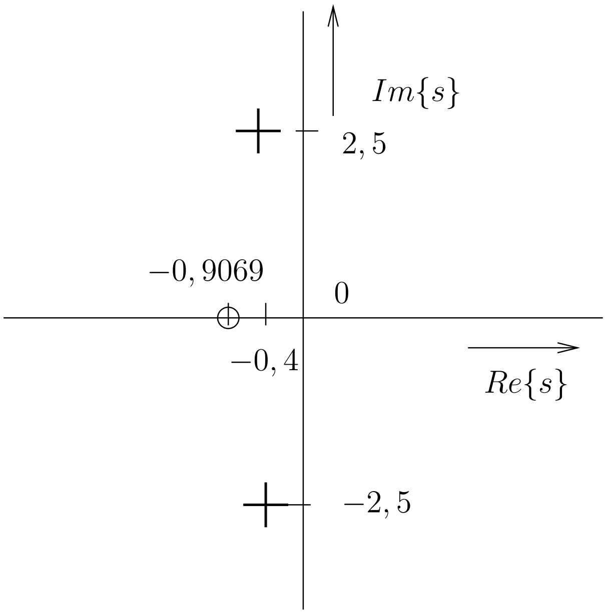
In normal form $H(s) = \frac{4,9003s+4,4435}{s^2+0,8s+6,41}$.

The zeros of the denominator: $\quad s^2+0,8s+6,41 = 0 \quad s_1 = -0,4+j2,5; \quad s_2 = -0,4-j2,5$.

The zero of the numerator: $\quad 4,9003s + 4,4435 = 0 \quad s = -0,9068$.

The poles and the zero of the system are: $\quad p_1 = -0,4 + j2,5; \quad p_2 = -0,4 - j2,5; \quad$ and $z = -0,9068$.

The pole -zero map:



As the poles are situated on the left side of the complex plane the system is BIBO stable.

b) The Laplace transform of the input signal was found in the previous example, the Laplace transform of the response is:
$Y(s) = U(s)H(s) = \frac{4s-2{,}5}{s(s+0{,}5)} \frac{4{,}9003s+4{,}4435}{s^2+0{,}8s+6{,}41}$.
In knowledge of the poles: $s^2 + 0,8s + 6,41 = (s + 0,4 - j2,5)(s + 0,4 + j2,5)$, so
$Y(s) = \frac{(4s-2{,}5)(4{,}9003s+4{,}4435)}{s(s+0{,}5)(s+0{,}4-j2{,}5)(s+0{,}4+j2{,}5)} = \frac{-3{,}4661}{s} + \frac{2{,}8658}{s+0{,}5} + \frac{0{,}3001-j4{,}2549}{s+0{,}4-j2{,}5} + (*)$.
Here the $(*)$ notation represents a component in which the difference from the previous component is only that the j imaginary unit is changed by $-j$. (It does not mean that the two components are conjugate to each other because s is the same complex variable in both components.) With inverse Laplace transformation:
$y(t) = \varepsilon(t)\left(-3,4661 + 2,8658e^{-0{,}5t} + (0,3001 - j4,2549)e^{(-0{,}4+j2{,}5)t} + (*)\right)$.
In the time domain (t is real variable) the component represented by $(*)$ and the previous component are conjugate to each other so their sum equals to the double of the real part of the previous component.
$0,3001 - j4,2549 = 4,2654e^{-j1{,}5004}$,
$(0,3001 - j4,2549)e^{(-0{,}4+j2{,}5)t} + (*) = 2 \cdot Re\left\{(0,3001 - j4,2549)e^{(-0{,}4+j2{,}5)t}\right\} =$
$= 2 \cdot Re\left\{4,2654e^{-0{,}4t}e^{j(2{,}5t-1.5004)}\right\} = 8,5309e^{-0{,}4t}\cos(2,5t - 1,5004)$.
The response signal is finally:
$y(t) = \varepsilon(t)\left[-3,4661 + 2,8658e^{-0{,}5t} + 8,5308e^{-0{,}4t}\cos(2,5t - 1,5004)\right]$.

c) The Laplace transform of the response is the product of the input signal Laplace transform (see the previous problem) and the transfer function:
$Y(s) = \frac{10}{s}H(s) - e^{-2s}\frac{10}{s}H(s)$.
$\frac{10}{s}H(s) = \frac{49{,}0033s+44{,}4350}{s(s+0{,}4-j2{,}5)(s+0{,}4+j2{,}5)} = \frac{6{,}9321}{s} + \frac{-3{,}4661-j9{,}2460}{s+0{,}4-j2{,}5)}3(*)$.
(The explanation in connection with $(*)$ see in the b) point)
$-3,4661 - j9,2460 = 9,8743e^{-j1{,}9295}$,
$L^{-1}\left\{\frac{10}{s}H(s)\right\} = \varepsilon(t)\left(6,9321 + (-3,4661 - j9,2460)e^{(-0{,}4+j2{,}5)t} + (*)\right) =$
$= \varepsilon(t)\left(6,9321 + 9,8743e^{-j1{,}9295}e^{-0{,}4t}e^{j2{,}5t} + (*)\right) =$
$= \varepsilon(t)\left(6,9321 + 2Re\left\{9,8743e^{-0{,}4t}e^{j2{,}5t-1{,}9295}\right\}\right) =$
$= \varepsilon(t)\left(6,9321 + 19,7487e^{-0{,}4t}\cos(2,5t - 1,9295)\right)$.
So finally the response signal is:
$y(t) = \varepsilon(t)\left(6,9321 + 19,7487e^{-0{,}4t}\cos(2,5t - 1,9295)\right) -$
$-\varepsilon(t-2)\left(6,9321 + 19,7487e^{-0{,}4(t-2)}\cos[2,5(t-2) - 1,9295]\right)$.

16. The DT system is given with the $h[k] = 3\delta[k] + \varepsilon[k-1]\left(2 \cdot 0,6^k - 5(-0,4)^k\right)$ impulse response.

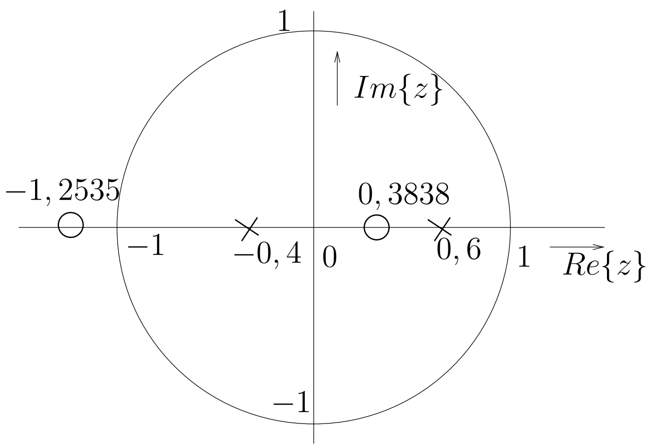
a) Find the transfer function of the system and draw the pole-zero map, and decide the BIBO stability of the system!

b) Find the response of the system in case of $u[k] = \varepsilon[k]\left(10 - 8 \cdot 0,5^k\right)$ input signal!

c) Find the response of the system in case of $u[k] = (\varepsilon[k] - \varepsilon[k-4])\,10$ input signal!

Solution

a) The transfer function is the z-transform of the impulse response: $H(z) = Z\{h[k]\} = $
$= Z\left\{3\delta[k] + 2 \cdot 0,6 \cdot \varepsilon[k-1]0,6^{k-1} - 5(-0,4) \cdot \varepsilon[k-1](-0,4)^{k-1}\right\} = 3 + 1,2z^{-1}\frac{1}{1-0,6z^{-1}} +$
$+2z^{-1}\frac{1}{1+0,4z^{-1}} = \frac{3\left(1-0,2z^{-1}-0,24z^{-2}\right)+1,2z^{-1}\left(1+0,4z^{-1}\right)+2z^{-1}\left(1-0,6z^{-1}\right)}{1-0,2z^{-1}-0,24z^{-2}} = \frac{3+2,6z^{-1}-1,44z^{-2}}{1-0,2z^{-1}-0,24z^{-2}}$.
This is the normal form of the transfer function but for finding the poles and zeros we need the form of transfer function as the ratio of polynomials containing positive powers of the z variable: $H(z) = \frac{3z^2+2,6z-1,44}{z^2-0,2z-0,24}$.
The poles of the system are the zeros of the denominator polynomial:
$z^2 - 0,2z - 0,24 = 0, \qquad z_1 = 0,6, \quad z_2 = -0,4.$
The zeros of the system are the zeros of the numerator polynomial:
$3z^2 + 2,6z - 1,44 = 0, \qquad z_1 = -1,2505, \quad z_2 = 0,3838.$
Poles: $\;p_1 = 0,6,\ p_2 = -0,4; \qquad z_1 = -1,2505,\ z_2 = 0,3838.$
The pole-zero map (containing the unit radius circle):



As the poles are located inside of the unit radius circle the system is BIBO stable.

b) The input signal is causal, the z-transform of the response is:$Y(z) = U(z) \cdot H(z)$, where $U(z)$ is the z-transform of the input signal:
$u[k] = \varepsilon[k]10 - \varepsilon[k]8 \cdot 0,5^k, \quad U(z) = 10\frac{1}{1-z^{-1}} - 8\frac{1}{1-0,5z^{-1}} = \frac{2+3z^{-1}}{(1-z^{-1})(1-0,5z^{-1})},$
$Y(z) = \frac{\left(2+3z^{-1}\right)\left(3+2,6z^{-1}-1,44z^{-2}\right)}{(1-z^{-1})(1-0,5z^{-1})(1-0,2z^{-1}-0,24z^{-2})}\frac{z^3}{z^4}z = z\frac{(2z+3)(3z^2+2,6z-1,44)}{(z-1)(z-0,5)(z^2-0,2z-0,24)}.$
The fraction is a proper one, direct transformation to sum of partial fractions may be done.
$z^2 - 0,2z - 0,24 = (z-0,6)(z+0,4)$, so
$Y(z) = z\frac{(2z+3)(3z^2+2,6z-1,44)}{(z-1)(z-0,5)(z-0,6)(z+0,4)} = z\left(\frac{74,2857}{z-1} + \frac{-126}{z-0,6} + \frac{54,2222}{z-0,5} + \frac{3,4921}{z+0,4}\right) =$
$= 74,2857\frac{z}{z-1} + 54,2222\frac{z}{z-0,5} - 126\frac{z}{z-0,6} + 3,4921\frac{z}{z+0,4}$. With inverse z-transformation:
$y[k] = \varepsilon[k]\left(74,2857 + 54,2222 \cdot 0,5^k - 126 \cdot 0,6^k + 3,4921(-0,4)^k\right).$

c) First solution
The z-transform of the $u[k] = \varepsilon[k]10 - \varepsilon[k-4]10$ input signal is: $U(z) = \frac{10}{1-z^{-1}} - z^{-4}\frac{10}{1-z^{-1}}$, and the z-transform of the response is: $Y(z) = \frac{10}{1-z^{-1}}H(z) - z^{-4}\frac{10}{1-z^{-1}}H(z)$.

$\frac{10}{1-z^{-1}}H(z)=\frac{30+26z^{-1}-14,4z^{-2}}{(1-z^{-1})(1-0,2z^{-1}-0,24z^{-2})}\frac{z^2}{z^3}z=z\frac{30z^2+26z-14,4}{(z-1)(z^2-0,2z-0,24)}$.

With $z^2-0,2z-0,24=(z-0,6)(z+0,4)$ the proper fraction directly may be transformed into sum of partial fractions:

$\frac{10}{1-z^{-1}}H(z)=z\frac{30z^2+26z-14,4}{(z-1)(z-0,6)(z+0,4)}=z\left(\frac{74,2857}{z-1}+\frac{-30}{z-0,6}+\frac{-14,2857}{z+0,4}\right)=$

$=74,2857\frac{z}{z-1}-30\frac{z}{z-0,6}-14,2857\frac{z}{z+0,4}$.

With inverse z-transformation the response signal is:

$y[k]=\varepsilon[k]\left(74,2857-30\cdot 0,6^k-14,2857(-0,4)^k\right)-$

$-\varepsilon[k-4]\left(74,2857-30\cdot 0,6^{k-4}-14,2857(-0,4)^{k-4}\right)$.

Second solution

The input signal and its z-transform may be written as $u[k]=10\delta[k]+10\delta[k-1]+$ $+10\delta[k-2]+10\delta[k-3]$, $\quad U(z)=10+10z^{-1}+10z^{-2}+10z^{-3}$. The z-transform of the response is:

$Y(z)=U(z)H(z)=\frac{\left(10+10z^{-1}+10z^{-2}+10z^{-3}\right)\left(3+2,6z^{-1}-1,44z^{-2}\right)}{1-0,2z^{-1}-0,24z^{-2}}\frac{z^5}{z^2}zz^{-4}=$

$=z^{-4}z\frac{(10z^3+10z^2+10z+10)(3z^2+2,6z-1,44)}{z^2-0,2z-0,24}=z^{-4}z\frac{30z^5+56z^4+41,6z^3+41,6z^2+11,6z-14,4}{z^2-0,2z-0,24}$.

With polynomial division the improper fraction must be transformed into the sum of a polynomial and a proper fraction:

$Y(z)=z^{-4}z\left(30z^3+62z^2+61,2z+68,72+\frac{40,0320z+2,0928}{z^2-0,2z-0,24}\right)$,

with $z^2-0,2z-0,24=(z-0,6)(z+0,4)$

$Y(z)=30+62z^{-1}+61,2z^{-2}+68,72z^{-3}+z^{-4}z\frac{40,0320z+2,0928}{(z-0,6)(z+0,4)}=$

$=30+62z^{-1}+61,2z^{-2}+68,72z^{-3}+z^{-4}z\left(\frac{26,1120}{z-0,6}+\frac{13,9200}{z+0,4}\right)=$

$=30+62z^{-1}+61,2z^{-2}+68,72z^{-3}+z^{-4}\left(26,1120\frac{z}{z-0,6}+13,9200\frac{z}{z+0,4}\right)$.

With inverse z-transformation:

$y[k]=30\delta[k]+62\delta[k-1]+61,2\delta[k-2]+68,72\delta[k-3]+$

$+\varepsilon[k-4]\left(26,1120\cdot 0,6^{k-4}+13,92(-0,4)^{k-4}\right)$.

Putting $k=0$, 0, 2 and 3 into the final formula of the first solution we get 30, 62, $61,2$ and $68,72$, the same values as that of the second solution formula. Transforming the first solution formula for $k\geq 4$:

$y[[k]=74,2857-30\cdot 0,6^{k-4}\cdot 0,6^4-14,2857(-0,4)^{k-4}\cdot(-0,4)^4-74,2857+30\cdot 0,6^{k-4}+$

$+14,2857(-0,4)^{k-4}=$

$=(-30\cdot 0,6^4+30)\cdot 0,6^{k-4}+(-14,2857\cdot(-0,4)^4+14,2857)\cdot(-0,4)^{k-4}=$

$=26,1120\cdot 0,6^{k-4}+13,9200(-0,4)^{k-4}$.

The two formulae are common for $k\geq 4$ as well.

17. Solve the sixteenth problem if the DT system impulse response is:

 $h[k]=\varepsilon[k]20\cdot 0,8^k\cos(0,5k+0,4)$!

 Solution

 a) With the $\cos\alpha=\frac{1}{2}e^{j\alpha}+\frac{1}{2}e^{-j\alpha}$ identity

 $h[k]=\varepsilon[k]\left(10\cdot 0,8^k e^{j(0,5k+0,4)}+10\cdot 0,8^k e^{-j(0,5k+0,4)}\right)=$

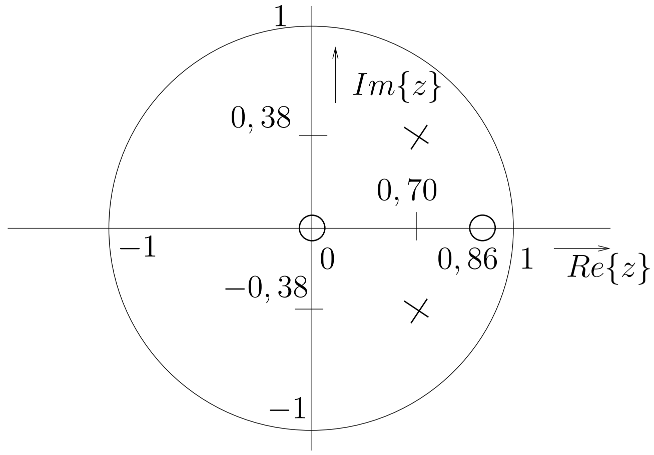
 $=\varepsilon[k]10e^{j0,4}(0,8e^{j0,5})^k+\varepsilon[k]10e^{-j0,4}(0,8e^{-j0,5})^k$.

 The transfer function is the z-transform of the impulse response:

 $H(z)=\frac{10e^{j0,4}}{1-0,8e^{j0,5}z^{-1}}+\frac{10e^{-j0,4}}{1-0,8e^{-j0,5}z^{-1}}=\frac{10e^{j0,4}\left(1-0,8e^{-j0,5}z^{-1}\right)+10e^{-j0,4}\left(1-0,8e^{j0,5}z^{-1}\right)}{1-0,8z^{-1}(e^{j0,5}+e^{-j0,5})+0,64z^{-2}}=$

 $=\frac{10\left(e^{j0,4}+e^{-j0,4}\right)-8z^{-1}\left(e^{j0,1}+e^{-j0,1}\right)}{1-0,8z^{-1}(e^{j0,5}+e^{-j0,5})+0,64z^{-2}}$.

With the $e^{-j\alpha}+e^{-j\alpha}=2\cos\alpha$ identity:
$H(z)=\frac{20\cos 0{,}4-(16\cos 0{,}1)z^{-1}}{1-(1{,}6\cos 0{,}5)z^{-1}+0{,}64z^{-2}}=\frac{18{,}4212-15{,}9201z^{-1}}{1-1{,}4041z^{-1}+0{,}64z^{-2}}$.
This is the normal form of the transfer function but finding the poles and zeros we need the form of ratio of two polynomials with positive powers of z;
$H(z)=\frac{18{,}4212z^2-15{,}9201z}{z^2-1{,}4041z+0{,}64}$.
The poles of the system are the zeros of the denominator polynomials:
$z^2-1,4041z+0,64=0, \quad z_1=0,7021-j0,3835, \quad z_2=0,7021+j0,3835.$
The zeros of the system are the zeros of the numerator polynomial:
$18,4212z^2-15,9201z=0, \quad z_1=0, \quad z_2=0,8642.$
Poles: $\quad p_1=0,7021-j0,3835, \quad p_2=0,7021+j0,3835,$
zeros: $\quad z_1=0, \quad z_2=0,8642.$
The pole-zero map (containing the unit radius circle):



The system is BIBO stable as the poles are inside of the unit radius circle.

b) The z-transform of the input signal was found in the previous example, the z-transform of the response is:
$Y(z)=U(z)H(z)=\frac{\left(2+3z^{-1}\right)\left(18{,}4212-15{,}9201z^{-1}\right)}{(1-z^{-1})(1-0{,}5z^{-1})(1-1{,}4041z^{-1}+0{,}64z^{-2})}\frac{z^2z}{z^4}z=z\frac{(2z+3)(18{,}4212z^2-15{,}9201z)}{(z-1)(z-0{,}5)(z^2-1{,}4041z+0{,}64)}$,
$z^2-1,4041z+0,64=(z-0,7021-j0,3835)(z-0,7021+j0,3835),$
$Y(z)=z\frac{(2z+3)(18{,}4212z^2-15{,}9201z)}{(z-1)(z-0{,}5)(z-0{,}7021-j0{,}3835)(z-0{,}7021+j0{,}3835)}=$
$\quad =z\left(\frac{106{,}0404}{z-1}+\frac{285{,}6091}{z-0{,}5}+\frac{-195{,}8248-j82{,}0791}{z-0{,}7021-j0{,}3835}+(*)\right).$
(The explanation in connection with the $(*)$ notation see the solution of the fifteenth problem.)
$-195,8248-j82,0791=212,3307e^{-j2{,}7447}, \quad 0,7021+j0,3835=0,8e^{j0{,}5},$
$Y(z)=106,0404\frac{z}{z-1}+285,6091\frac{z}{z-0{,}5}+212,3307e^{-j2{,}7447}\frac{z}{z-0{,}8e^{j0{,}5}}+(*)$
With inverse z-transformation:
$y[k]=\varepsilon[k]\left(106,0404+285,6091\cdot 0,5^k+212,3307e^{-j2{,}7447}\left(0,8e^{j0{,}5}\right)^k+(*)\right)$
The last two components are conjugate to each other, so
$y[k]=\varepsilon[k]\left(106,0404+285,6091\cdot 0,5^k+2Re\left\{212,3307\cdot 0,8^k e^{j(0{,}5k-2{,}7447)}\right\}\right)=$
$\quad =\varepsilon[k]\left(106,0404+285,6091\cdot 0,5^k+424,6614\cdot 0,8^k\cos(0,5k-2,7447)\right).$

c) First solution
The z-transform of the input signal (see 16. c solution) and the response are:
$U(z)=\frac{10}{1-z^{-1}}-z^{-4}\frac{10}{1-z^{-1}}, \quad Y(z)=\frac{10}{1-z^{-1}}H(z)-z^{-4}\frac{10}{1-z^{-1}}H(z).$
$\frac{10}{1-z^{-1}}H(z)=\frac{184{,}2122-159{,}2007z^{-1}}{\cdots}\frac{zz}{\cdots}z=z\frac{184{,}2122z^2-159{,}2007z}{\cdots}=$

$= z\frac{184{,}2122z^2-159{,}2007z}{(z-1)(z-0{,}7021-j0{,}3835)(z-0{,}7021+j0{,}3835)} = z\left(\frac{106{,}0404}{z-1}+\frac{39{,}0859-j160{,}0191}{z-0{,}7021-j0{,}3835}+(*)\right),$

$39,0859 - j160,0191 = 164,7235e^{-j1{,}3312}, \quad 0,7021 + j0,3835 = 0,8e^{j0{,}5},$

$\frac{10}{1-z^{-1}}H(z) = 106,0404\frac{z}{z-1} + 164,7235e^{-j1{,}3312}\frac{z}{z-0{,}8e^{j0{,}5}} + (*).$

With inverse z-transformation

$y[k] = \varepsilon[k]\left(106,0404 + 164,7235e^{-j1{,}3312}\left(0,8e^{j0{,}5}\right)^k + (*)\right) -$

$-\varepsilon[k-4]\left(106,0404 + 164,7235e^{-j1{,}3312}\left(0,8e^{j0{,}5}\right)^{k-4} + (*)\right) =$

$= \varepsilon[k]\left(106,0404 + 2Re\left\{164,7235 \cdot 0,8^k e^{j(0{,}5k-1{,}3312)}\right\}\right) -$

$-\varepsilon[k-4]\left(106,0404 + 2Re\left\{164,7235 \cdot 0,8^{k-4} e^{j[0{,}5(k-4)-1{,}3312]}\right\}\right) =$

$= \varepsilon[k]\left(106,0404 + 329,4470 \cdot 0,8^k \cos(0,5k - 1,3312)\right) -$

$-\varepsilon[k-4]\left(106,0404 + 329,4470 \cdot 0,8^{k-4}\cos[0,5(k-4) - 1,3312]\right).$

Second solution

The other form of the input signal and its z-transform are: $u[k] = 10\delta[k] + 10\delta[k-1]+$ $+10\delta[k-2] + 10\delta[k-3]$, $U(z) = 10 + 10z^{-1} + 10z^{-2} + 10z^{-3}$. The z-transform of the response is:

$Y(z) = U(z)H(z) = \frac{\left(10+10z^{-1}+10z^{-2}+10z^{-3}\right)\left(18{,}4212-15{,}9201z^{-1}\right)}{1-1{,}4041z^{-1}+0{,}64z^{-2}}\frac{z^4}{z^2}zz^{-3} =$

$= z^{-3}z\frac{184{,}2122z^4+25{,}0115z^3+25{,}0115z^2+25{,}0115z-159{,}2007}{z^2-1{,}4041z+0{,}64}.$

The improper fraction may be transformed to the sum of a polynomial and a proper fraction:

$Y(z) = z^{-3}z\left(184,2122z^2 + 283,6698z + 305,4256 + \frac{272{,}3207z-354{,}6730}{z^2-1{,}4041z+0{,}64}\right) =$

$= z^{-3}z\left(184,2122z^2 + 283,6698z + 305,4256 + \frac{272{,}3207z-354{,}6730}{(z-0{,}7021-j0{,}3835)(z-0{,}7021+j0{,}3835)}\right) =$

$= 184,2122 + 283,6698z^{-1} + 305,4256z^{-2} + z^{-3}z\left(\frac{136{,}1604+j213{,}1273}{z-0{,}7021-j0{,}3835} + (*)\right) =$

$= 184,2122{+}283,6698z^{-1}{+}305,4256z^{-2}{+}z^{-3}z\left((136,1604 + j213,1273)\frac{z}{z-0{,}7021-j0{,}3835} + (*)\right).$

With inverse z-transformation:

$y[k] = 184,2122\delta[k] + 283,6698\delta[k-1] + 305,4256\delta[k-2]+$

$+\varepsilon[k-3]\left((136,1604 + j213,1273)(0,7021 + j0,3835)^{k-3} + (*)\right).$

The sum of two complex conjugate expression is the double of the real part of oneof them.

$136,1604 + j213,1273 = 252,9089e^{j1{,}0023}, \qquad 0,7021 + j0,3835 = 0,8e^{j0{,}5},$

$y[k] = 184,2122\delta[k] + 283,6698\delta[k-1] + 305,4256\delta[k-2]+$

$+\varepsilon[k-3]2Re\left\{252,9089e^{j1{,}0023}\left(0,8e^{j0{,}5}\right)^{k-3}\right\} =$

$y[k] = 184,2122\delta[k] + 283,6698\delta[k-1] + 305,4256\delta[k-2]+$

$+\varepsilon[k-3]505,8178 \cdot 0,8^{k-3}\cos[0,5(k-3) + 1,0023].$

It may be pointed out that the values of the first and the second solution results are equal for any k.

18. The CT system is given with the following state variable description where a is a parameter:

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} a & 3 \\ -2 & -4 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 2 \\ 6 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} 0,5 & -1 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + 0,8u(t).$$

a) Find the transfer function of the system!

b) Give the condition for the parameter a so the system be a minimum phase (MP) one!

c) Let be $a = 0,5$! Find the poles and zeros of the transfer function! If the system is BIBO stable but not a minimum phase (MP) one, give the transfer function of the system as a product of an MP an all-pass (AP) system transfer functions! Give the poles and the zeros of the two latter systems as well!

Solution

a) First solution. $H(s) = \frac{\mathbf{C}^T adj(s\mathbf{I}-\mathbf{A})\mathbf{B}}{det(s\mathbf{I}-\mathbf{A})} + D$, where

$$\mathbf{A} = \begin{bmatrix} a & 3 \\ -2 & -4 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}, \qquad \mathbf{c}^T = \begin{bmatrix} 0,5 & -1 \end{bmatrix}, \qquad D = 0,8.$$

$$s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s-a & -3 \\ 2 & s+4 \end{bmatrix}, \quad det(s\mathbf{I} - \mathbf{A}) = (s-a)(s+4) + 6 = s^2 + (4-a)s + 6 - 4a,$$

$$(s\mathbf{I} - \mathbf{A})^T = \begin{bmatrix} s-a & 2 \\ -3 & s+4 \end{bmatrix}, \quad adj(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s+4 & 3 \\ -2 & s-a \end{bmatrix},$$

$$H(s) = \frac{\begin{bmatrix} 0,5 & -1 \end{bmatrix} \begin{bmatrix} s+4 & 3 \\ -2 & s-a \end{bmatrix} \begin{bmatrix} 2 \\ 6 \end{bmatrix}}{s^2 + (4-a)s + 6 - 4a} + 0,8 = \frac{\begin{bmatrix} 0,5 & -1 \end{bmatrix} \begin{bmatrix} 2s+26 \\ 6s-6a-4 \end{bmatrix}}{s^2 + (4-a)s + 6 - 4a} + 0,8 =$$

$$= \frac{-5s+17+6a}{s^2+(4-a)s+6-4a} + 0,8 = \frac{0,8s^2+(-1,8-0,8a)s+21,8+2,8a}{s^2+(4-a)s+6-4a}.$$

Second solution. On the basis of the state variable description the relations among the Laplace transforms of the signals are:

$sX_1(s) = aX_1(s) + 3X_2(s) + 2U(s)$
$sX_2(s) = -2X_1(s) - 4X_2(s) + 6U(s)$ From the first two equations:
$Y(s) = 0,5X_1(s) - X_2(s) + 0,8U(s)$

$(s-a)X_1 - 3X_2 = 2U \quad / \cdot (s+4) \quad / \cdot (-2)$
$2X_1 + (s+4)X_2 = 6U \quad / \cdot 3 \quad / \cdot (s-a)$

$X_1(s^2 - as + 4s - 4a + 6) = U(2s + 26)$
$X_2(s^2 + 4s - as - 4a + 6) = U(6s - 6a - 4)$

$X_1 = U\frac{2s+26}{s^2+(4-a)s+6-4a}, \qquad X_2 = U\frac{6s-6a-4}{s^2+(4-a)s+6-4a},$

$Y = 0,5X_1 - X_2 + 0,8U = U\frac{0,5(2s+26)-(6s-6a-4)+0,8\left(s^2+(4-a)s+6-4a\right)}{s^2+(4-a)s+6-4a} = U\frac{0,8s^2+(-1,8-0,8a)s+21,8+2,8a}{s^2+(4-a)s+6-4a}$

$H(s) = \frac{Y(s)}{U(s)} = \frac{0,8s^2+(-1,8-0,8a)s+21,8+2,8a}{s^2+(4-a)s+6-4a}.$

b) The poles and zeros of an MP CT system are situated on the left half of the complex plain, that is their real part is negative, so both the denominator and the numerator polynomials must be Hurwitz polynomials. The denominator is Hurwitz polynomial (the system is BIBO stable) if $4 - a > 0$ and $6 - 4a > 0$. Both are fulfilled if $a < 1,5$. This is the condition of the BIBO stability of the system. The system is an MP one, if additionally the numerator is Hurwitz polynomial, that is when $-1,8 - 0,8a > 0$ and $21,8 + 2,8a > 0$.

$$\left.\begin{array}{r} -1,8 - 0,8a > 0 \\ 21,8 + 2,8a > 0 \end{array}\right\} \qquad \left.\begin{array}{r} -1,8 > 0,8a \\ 2,8a > -21,28 \end{array}\right\} \qquad \left.\begin{array}{r} a < -2,25 \\ a > -\frac{21,28}{2,8} = -7,6 \end{array}\right\},$$

if $-7,6 < a < -2,25$, ($a < 1,5$ is fulfilled as well,) the system is an MP system.

c) If $a = 0,5$, $H(s) = \frac{0,8s^2-2,2s+23,2}{s^2+3,5s+4}$.

The poles of the system are the zeros of the denominator polynomial:

$s^2 + 3,5s + 4 = 0, \qquad s_1 = -1,75 + j0,9682, \quad s_2 = -1,75 - j0,9682.$

The zeros of the system are the zeros of the numerator polynomial:

$0,8s^2 - 2,2s + 23,2 = 0, \qquad s_1 = 1,375 + j5,2067, \quad s_2 = 1,375 - j5,2067.$

Poles: $p_1 = -1,75 + j0,9682,\ p_2 = -1,75 - j0,9682,$
zeros: $z_1 = 1,375 + j5,2067, \quad z_2 = 1,375 - j5,2067.$
The system is BIBO stable (as $Re\{p_1\} < 0$ and $Re\{p_2\} < 0$), but not an MP one, because $Re\{z_1\} < 0$ and $Re\{z_2\} < 0$ are not fulfilled. In the factorization of the transfer function to product of an MP and of an AP system transfer functions the zeros on the right side of the complex plain will be the the zeros of the AP system. For an AP CT system the pole-zero relation is: $z_i = -p_i^*$, that is $p_i = -z_i^*$, the poles of the AP system will be: $p_{1AP} = -(1,375 + j5,2067)^* = -1,375 + j5,2067$, $p_{2AP} = -(1,375 - j5,2067)^* =$
$= -1,375 - j5,2067$. The system transfer function will be multiplied by
$1 = \frac{(s-p_{1AP})(s-p_{2AP})}{(s-p_{1AP})(s-p_{2AP})}$ fraction:
$H(s) = \frac{0{,}8s^2-2{,}2s+23{,}2}{s^2+3{,}5s+4} = \frac{0{,}8(s-1{,}375-j5{,}2067)(s-1{,}375+j5{,}2067)}{(s+1{,}75-j0{,}9682)(s+1{,}75+j0{,}9682)} \cdot \frac{(s+1{,}375-j5{,}2067)(s+1{,}375+j5{,}2067)}{(s+1{,}375-j5{,}2067)(s+1{,}375+j5{,}2067)} =$
$= 0,8\frac{(s+1{,}375-j5{,}2067)(s+1{,}375+j5{,}2067)}{(s+1{,}75-j0{,}9682)(s+1{,}75+j0{,}9682)} \cdot \frac{(s-1{,}375-j5{,}2067)(s-1{,}375+j5{,}2067)}{(s+1{,}375-j5{,}2067)(s+1{,}375+j5{,}2067)}.$
The constant $0,8$ factor may belong to the one or to the other transfer function optionally.
The MP system transfer function is:
$H_{MP}(s) = 0,8\frac{(s+1{,}375-j5{,}2067)(s+1{,}375+j5{,}2067)}{(s+1{,}75-j0{,}9682)(s+1{,}75+j0{,}9682)} = \frac{0{,}8s^2+2{,}2s+23{,}82}{s^2+3{,}5s+4}.$
The poles and zeros of the MP system are:
$p_{12} = -1,75 \pm j0,9682,\ z_{12} = -1,375 \pm j5,2067.$
The AP system transfer function is:
$H_{AP}(s) = \frac{(s-1{,}375-j5{,}2067)(s-1{,}375+j5{,}2067)}{(s+1{,}375-j5{,}2067)(s+1{,}375+j5{,}2067)} = \frac{s^2-2{,}75s+29}{s^2+2{,}75s+29}.$
The poles and zeros of the AP system are:
$p_{12} = -1,375 \pm j5,2067,\ z_{12} = 1,375 \pm j5,2067.$

19. The DT system is given with the following state variable description, where a is a parameter.
$$\begin{bmatrix} x_1[k+1] \\ x_2[k+1] \end{bmatrix} = \begin{bmatrix} -0,5 & 1 \\ -0,9 & a \end{bmatrix} \begin{bmatrix} x_1[k] \\ x_2[k] \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u[k],$$
$$y[k] = \begin{bmatrix} 2 & 0,5 \end{bmatrix} \begin{bmatrix} x_1[k] \\ x_2[k]] \end{bmatrix} + 0,5u[k].$$

a) Find the transfer function of the system!
b) Give the condition for the parameter a so the system be a Minimum phase (MP) one!
c) Let be $a = 1,3$! Find the poles and zeros of the system! If the system is BIBO stable but not an MP one, give the transfer function of the system as a product of an MP and an all-pass (AP) system transfer functions! Give the poles and zeros of the latter systems as well!

Solution

a) First solution. $\quad H(z) = \frac{\mathbf{C}^T adj(z\mathbf{I}-\mathbf{A})\mathbf{B}}{det(z\mathbf{I}-\mathbf{A})} + D$, where
$$\mathbf{A} = \begin{bmatrix} -0,5 & 1 \\ -0,9 & a \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \qquad \mathbf{c}^T = \begin{bmatrix} 2 & 0,5 \end{bmatrix}, \qquad D = 0,5.$$
$$z\mathbf{I} - \mathbf{A} = \begin{bmatrix} z+0,5 & -1 \\ 0,9 & z-a \end{bmatrix},$$
$det(z\mathbf{I} - \mathbf{A}) = (z+0,5)(z-a) + 0,9 = z^2 + (0.5-a)z + 0,9 - 0,5a,$
$$(z\mathbf{I} - \mathbf{A})^T = \begin{bmatrix} z+0,5 & 0,9 \\ -1 & z-a \end{bmatrix} \qquad adj(z\mathbf{I} - \mathbf{A}) = \begin{bmatrix} z-a & 1 \\ -0,9 & z+0,5 \end{bmatrix};$$
$$H(z) = \frac{\begin{bmatrix} 2 & 0,5 \end{bmatrix} \begin{bmatrix} z-a & 1 \\ -0,9 & z+0,5 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix}}{z^2 + (0,5-a)z + 0,9 - 0,5a} + 0,5 = \frac{\begin{bmatrix} 2 & 0,5 \end{bmatrix} \begin{bmatrix} 2 \\ 2z+1 \end{bmatrix}}{z^2 + (0,5-a)z + 0,9 - 0,5a} + 0,5 =$$

$= \frac{z+4{,}5+0{,}5z^2+(0{,}25-0{,}5a)z+0{,}45-0{,}25a}{z^2+(0{,}5-a)z+0{,}9-0{,}5a} = \frac{0{,}5z^2+(1{,}25-0{,}5a)z+4{,}95-0{,}25a}{z^2+(0{,}5-a)z+0{,}9-0{,}5a}$.

In normal form: $H(z) = \frac{0{,}5+(1{,}25-0{,}5a)z^{-1}+(4{,}95-0{,}25a)z^{-2}}{1+(0{,}5-a)z^{-1}+(0{,}9-0{,}5a)z^{-2}}$.

Second solution

Getting the z-transforms of both sides of the equations of the SVD:

$zX_1 = -0,5X_1 + X_2$

$zX_2 = -0,9X_1 + aX_2 + 2U$ From the first two equations:

$Y = 2X_1 + 0,5X_2 + 0,5U$

$$\left.\begin{array}{r} (z+0,5)X_1 - X_2 = 0 \\ 0,9X_1 + (z-a)X_2 = 2U \end{array}\right\} \quad \begin{array}{c} /\cdot(z-a) \\ /\cdot 1 \end{array} \quad \begin{array}{c} /\cdot(-0,9) \\ /\cdot(z+0,5) \end{array}$$

$\left(z^2 + (0,5-a)z + 0,9 - 0,5a\right) X_1 = 2U$

$\left(z^2 + (0,5-a)z + 0,9 - 0,5a\right) X_2 = U(2z+1)$

$X_1 = U\frac{2}{z^2+(0{,}5-a)z+0{,}9-0{,}5a}, \quad X_2 = U\frac{2z+1}{z^2+(0{,}5-a)z+0{,}9-0{,}5a}$.

Substituting X_1 and X_2 expressions to the expression of Y:

$Y = U\frac{4}{z^2+(0{,}5-a)z+0{,}9-0{,}5a} + U\frac{z+0{,}5}{z^2+(0{,}5-a)z+0{,}9-0{,}5a} + U\frac{0{,}5z^2+(0{,}25-0{,}5a)z+0{,}45-0{,}25a}{z^2+(0{,}5-a)z+0{,}9-0{,}5a} =$

$= U\frac{0{,}5z^2+(1{,}25-0{,}5a)z+4{,}95-0{,}25a}{z^2+(0{,}5-a)z+0{,}9-0{,}5a}$

So the transfer function in normal form is: $H(z) = \frac{0{,}5+(1{,}25-0{,}5a)z^{-1}+(4{,}95-0{,}25a)z^{-2}}{1+(0{,}5-a)z^{-1}+(0{,}9-0{,}5a)z^{-2}}$.

b) A DT system is an MP one, if all of the poles and zeros are inside of the unit radius circle (out of the origin) in the complex plain. For that examination we start out from the form of the transfer function containing positive powers of the variable z:

$H(z) = 0,5\frac{z^2+(2{,}5-a)z+9{,}9-0{,}5a}{z^2+(0{,}5-a)z+0{,}9-0{,}5a}$.

The system can not be an MP one, if or the numerator or the denominator polynomials have zero at the origin. So $a \neq 19,8$ and $a \neq 1,8$. The system may be an MP one, if the absolute values of the zeros of the denominator polynomial and that of the numerator polynomial are less than 1. According to the Jury criterions the absolute values of zeros of the $P(z) = z^2 + a_1z + a_2$ polynomial are less than 1, if $P(z=1) > 0$, $P(z=-1) > 0$ and $|a_2| < 1$.

The denominator polynomial is: $P(z) = z^2 + (0,5-a)z + 0,9 - 0,5a$.

$$\left.\begin{array}{r} P(z=1) = 2,4 - 1,5a > 0 \\ P(z=-1) = 1,4 + 0,5a > 0 \\ -1 < 0,9 - 0,5a < 1 \end{array}\right\} \quad \left.\begin{array}{c} a < \frac{2{,}4}{1{,}5} = 1,6 \\ a > -\frac{1{,}4}{0{,}5} = -2,8 \\ -0,2 < a < 3,8 \end{array}\right\} \quad -0,2 < a < 1,6.$$

The numerator polynomial is: $P(z) = z^2 + (2,5-a)z + 9,9 - 0,5a$.

$$\left.\begin{array}{r} P(z=1) = 13,4 - 1,5a > 0 \\ P(z=-1) = 8,4 + 0,5a > 0 \\ -1 < 9,9 - 0,5a < 1 \end{array}\right\} \quad \left.\begin{array}{c} a < \frac{13{,}4}{1{,}5} \approx 8,9333 \\ a > -16,8 \\ 17,8 < a < 21,8 \end{array}\right\}$$

There is no such an a value which is less than $8,9333$ and at the same time is greater than $17,8$, so the system can not be an MP one at all. (The system is BIBO stable, if $-0,2 < a < 1,6$.)

c) If $a = 1,3$, $H(s) = \frac{0{,}5z^2+0{,}6z+4{,}625}{z^2-0{,}8z+0{,}25} = 0,5\frac{z^2+1{,}2z+9{,}25}{z^2-0{,}8z+0{,}25}$.

The poles of the system are the zeros of the denominator polynomial:

$z^2 - 0,8z + 0,25 = 0, \qquad z_1 = 0,4 + j0,3, \quad z_2 = 0,4 - j0,3.$

The zeros of the system are the zeros of the numerator polynomial:

$z^2 + 1,2z + 9,25 = 0, \qquad z_1 = -0,6 + j2,9816, \quad z_2 = -0,6 - j2,9816.$

Poles: $p_{12} = 0,4 \pm j0,3$, $|p_{12}| = 0,5$, zeros: $z_{12} = -0,6 \pm j2,9816$, $|z_{12}| = 3,0414$.

The system is BIBO stable as $|p_1| = |p_2| = 0,5 < 1$, but not an MP one, as $|z_1| = |z_2| =$

$= 3,0414 > 1$. In the factorization of the transfer function to product of an MP and an AP system transfer functions, the zeros will be the zeros of the AP system. For an AP DT system the pole-zero relation (for the poles and zeros out of the origin) is $z_i = \frac{1}{p_i^*}$, that is $p_i = \frac{1}{z_i^*}$, the poles of the AP system are:

$p_{1AP} = \frac{1}{(-0,6+j2,9816)^*} = -0,0649 + j0,3223,$

$p_{2AP} = \frac{1}{(-0,6-j2,9816)^*} = -0,0649 - j0,3223.$

The system transfer function will be multiplied by $1 = \frac{(z-p_{1AP})(z-p_{2AP})}{(z-p_{1AP})(z-p_{2AP})}$ fraction:

$$H(z) = 0,5\frac{z^2+1,2z+9,25}{z^2-0,8z+0,25}\frac{(z-p_{1AP})(z-p_{2AP})}{(z-p_{1AP})(z-p_{2AP})} =$$

$$= 0,5\frac{(z+0,6-j2,9816)(z+0,6+j2,9816)}{(z-0,4-j0,3)(z-0,4+j0,3)}\frac{(z+0,0649-j0,3223)(z+0,0649+j0,3223)}{(z+0,0649-j0,3223)(z+0,0649+j0,3223} =$$

$$= 0,5\frac{(z+0,0649-j0,3223)(z+0,0649+j0,3223)}{(z-0,4-j0,3)(z-0,4+j0,3)}\frac{(z+0,6-j2,9816)(z+0,6+j2,9816)}{(z+0,0649-j0,3223)(z+0,0649+j0,3223)}$$

The $0,5$ constant optionally may belong to one or to the other transfer function, we will connect it to the AP system transfer function.

The MP system transfer function and its poles and zeros are:

$$H_{MP}(z) = \frac{(z+0,0649-j0,3223)(z+0,0649+j0,3223)}{(z-0,4-j0,3)(z-0,4+j0,3)} = \frac{z^2+0,1297z+0,1081}{z^2-0,8z+0,25} = \frac{1+0,1297z^{-1}+0,1081z^{-2}}{1-0,8z^{-1}+0,25z^{-2}},$$

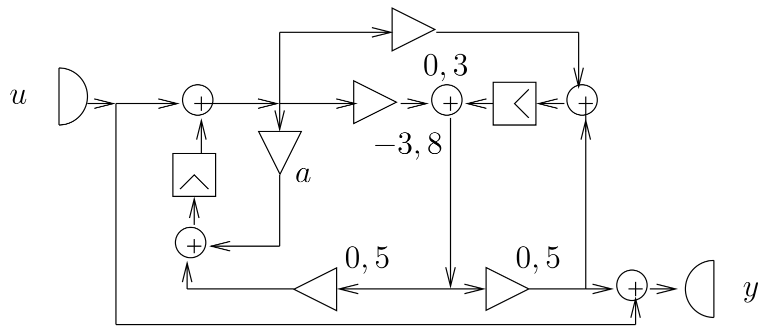
$p_{12MP} = 0,4 \pm j0,3, \qquad z_{12MP} = -0,0649 \pm j0,3223.$

The AP system transfer function and its poles and zeros are:

$$H_{AP}(z) = 0,5\frac{(z+0,6-j2,9816)(z+0,6+j2,9816)}{(z+0,0649-j0,3223)(z+0,0649+j0,3223} = \frac{0,5z^2+0,6z+4,625}{z^2+0,1297z+0,1081} = \frac{0,5+0,6z^{-1}+4,625z^{-2}}{1+0,1297z^{-1}+0,1081z^{-2}},$$

$p_{12AP} = -0,0649 \pm j0,3223, \qquad z_{12AP} = -0,6 \pm j2,9816.$

20. The CT and the DT systems are given with the common signal flow network, where a is a parameter.



a) Find the transfer function of both systems on the basis of the network equations written in the complex frequency domain!

b) Find the poles and zeros of both systems and decide the BIBO stability and the minimum phase (MP) property of the systems if

(1) $a = 0,35$,

(2) $a = 0,8$,

(3) $a = 2$,

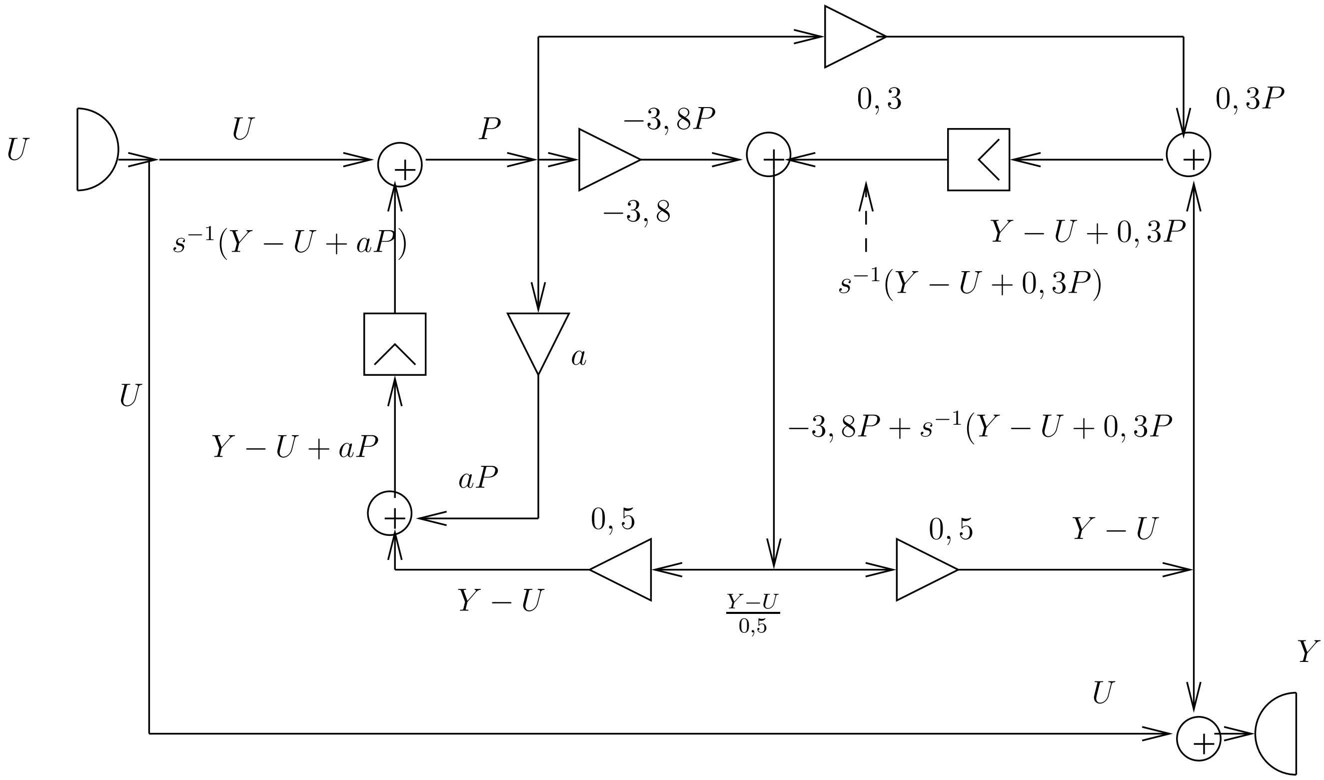
(4) $a = 3$.

c) Let us regard the DT system when $a = 2$! Give a canonical network representation of such a FIR type DT system, which, in cascade connection with the given DT system results in an other FIR type system!

Solution

a) We will find the transfer function of the CT system, and the transfer function of the DT system is found it with $s \to z$ substitution.

The Laplace transforms of signals are noted on the next figure taking into consideration some connection rules and component characteristics. (In addition of the input signal and the output signal we introduced a further signal: $p(t)$.)



The connection rules for the two summing nodes which are not taken into consideration in the figure:

$P = U + s^{-1}(Y - U + aP)$

$\frac{Y-U}{0,5} = -3,8P + s^{-1}(Y - U + 0,3P)$

$$\left.\begin{aligned} P\left(1 - as^{-1}\right) - Ys^{-1} &= U\left(1 - s^{-1}\right) \\ P\left(3,8 - 0,3s^{-1}\right) + Y\left(2 - s^{-1}\right) &= U\left(2 - s^{-1}\right) \end{aligned}\right\} \qquad \begin{aligned} /\cdot\left(-3,8 + 0,3s^{-1}\right) \\ /\cdot\left(1 - as^{-1}\right) \end{aligned}$$

$Y\left(3,8s^{-1} - 0,3s^{-2} + 2 - s^{-1} - 2as^{-1} + as^{-2}\right) =$

$= U\left(-3,8 + 3,8s^{-1} + 0,3s^{-1} - 0,3s^{-2} + 2 - s^{-1} - 2as^{-1} + as^{-2}\right)$

$Y\left(2 + (2,8 - 2a)s^{-1} + (a - 0,3)s^{-2}\right) = U\left(-1,8 + (3,1 - 2a)s^{-1} + (a - 0,3)s^{-2}\right)$

$H(s) = \frac{-1,8+(3,1-2a)s^{-1}+(a-0,3)s^{-2}}{2+(2,8-2a)s^{-1}+(a-0,3)s^{-2}}$

In normal form (simplifying with $2s^{-2}$:

$H(s) = \frac{-0,9s^2+(1,55-a)s+0,5a-0,15}{s^2+(1,4-a)s+0,5a-0,15}$.

The transfer function of the DT system in normal form is:

$H(z) = \frac{-0,9+(1,55-a)z^{-1}+(0,5a-0,15)z^{-2}}{1+(1,4-a)z^{-1}+(0,5a-0,15)z^{-2}}$.

b) (1) CT system: $H(s) = \frac{-0,9s^2+1,2s+0,025}{s^2+1,05s+0,025}$, DT system: $H(z) = H(s)|_{s=z}$

The poles are the zeros of the denominator polynomial: $s^2 + 1,05s + 0,025 = 0$,

Poles: $p_1 = -1,0256$ $p_2 = -0,0244$ (for CT and for DT system as well)

The zeros are the zeros of the numerator polynomial: $-0,9s^2 + 1,2s + 0,025 = 0$,

Zeros: $z_1 = 1,3539$, $z_2 = -0,0205$ (for CT and for DT system as well)

The CT system is BIBO stable as $Re\{p_1\} < 0$ and $Re\{p_2\} < 0$, but not an MP one as $Re\{z_1\} > 0$.

The DT system is BIBO unstable as $|p_1| \not< 1$, so it can not be an MP one.

(2) CT system: $H(s) = \frac{-0,9s^2+0,75s+0,25}{s^2+0,6s+0,25}$, DT system: $H(z) = H(s)|_{s=z}$
The poles are the zeros of the denominator polynomial: $s^2 + 0,6s + 0,25 = 0$,
Poles: $p_1 = -0,3 + j0,4$ $p_2 = -0,3 - j0,4$ (for CT and for DT system as well)
The zeros are the zeros of the numerator polynomial: $-0,9s^2 + 0,75s + 0,25 = 0$,
Zeros: $z_1 = 1,0885$, $z_2 = -0,2552$ (for CT and for DT system as well)

The CT system is BIBO stable as $Re\{p_1\} < 0$ and $Re\{p_2\} < 0$, but not an MP one as $Re\{z_1\} > 0$.

The DT system is BIBO stable as $|p_1| = |p_1| = 0,5 < 1$, but not an MP one as $|z_1| \not< 1$.

(3) CT system: $H(s) = \frac{-0,9s^2+-0,45s+0,85}{s^2-0,6s+0,85}$, DT system: $H(z) = H(s)|_{s=z}$
The poles are the zeros of the denominator polynomial: $s^2 - 0,6s + 0,85 = 0$,
Poles: $p_1 = 0,3 + j0,8718$ $p_2 = 0,3 - j0,8718$ (for CT and for DT system as well)
The zeros are the zeros of the numerator polynomial: $-0,9s^2 - 0,45s + 0,25 = 0$,
Zeros: $z_1 = -1,2535$, $z_2 = 0,7535$ (for CT and for DT system as well)

The CT system is BIBO unstable as for example $Re\{p_1\} \not< 0$. so the system can not be an MP one.

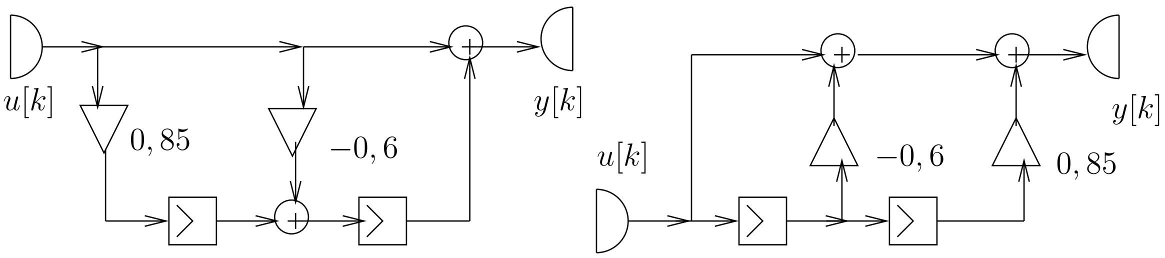
The DT system is BIBO stable as $|p_1| = |p_2| = 0,9220 < 1$, but not be an MP one because $|z_1| \not< 1$.

(4) CT system: $H(s) = \frac{-0,9s^2+-1,45s+1,35}{s^2-1,6s+1,35}$, DT system: $H(z) = H(s)|_{s=z}$
The poles are the zeros of the denominator polynomial: $s^2 - 1,6s + 1,35 = 0$,
Poles: $p_1 = 0,8 + j0,8426$ $p_2 = 0,8 - j0,8426$ (for CT and for DT system as well)
The zeros are the zeros of the numerator polynomial: $-0,9s^2 - 1,45s + 1,35 = 0$,
Zeros: $z_1 = -2,2715$, $z_2 = 0,6604$ (for CT and for DT system as well)

The CT system is BIBO unstable as for example $Re\{p_1\} \not< 0$. so the system can not be an MP one.

The DT system is BIBO unstable as $|p_1| = |p_2| = 1,1619 \not< 1$, so the system can not be an MP one as well.

c) If $a = 2$ the DT system transfer function in normal form is: $H(Z) = \frac{-0,9-0,45z^{-1}+0,85z^{-2}}{1-0,6z^{-1}+0,85z^{-2}}$.
The resultant transfer function of the cascade connection of two systems is the product of the transfer functions of the two systems. The given system in cascade connection with the FIR system of transfer function $H_1(z) = 1 - 0,6z^{-1} + 0,85z^{-2}$ results in $H_{resultant} = H(z) \cdot H_1(z) = -0,9 - 0,45z^{-1} + 0,85z^{-2}$, which is the transfer function of an other FIR system. Two canonical networks of the system of transfer function $H_1(z)$ are seen on the figure.



Checking: For the left network:
$Y(z) = U(z) + z^{-1}[-0,6U(z) + z^{-1}0,85U(z)] = U(z)\,(1 - 0,6z^{-1} + 0,85z^{-2})$.
For the right network:
$Y(z) = 0,85z^{-1}z^{-1}U(z) + (-0,6z^{-1}U(z) + U(z)) = U(z)\,(1 - 0,6z^{-1} + 0,85z^{-2})$.