HÁLÓZATI RENDSZEREK
ÉS SZOLGÁLTATÁSOK
TANSZÉK

Entanglement with the Environment and the No Cloning Theorem

Quantum Computing and its Applications
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MŰEGYETEM 1782

POSTULATES OF QUANTUM MECHANICS

- 1. Postulate: qubit
  - Hilbert-space
- 2. Postulate: logic gates
  - Unitary transform
  - Elementary gates
- 3. Postulate Q/C conversion
  - Measurement statistics
  - Post measurement state
- 4. Postulate: registers
  - Tensor product

$$|\varphi\rangle = \sum_{i=0}^{2^n-1} \varphi_i |i\rangle$$

$$U^\dagger \equiv U^{-1}$$

$$P(m \mid |\varphi\rangle) = \langle\varphi|M_m^\dagger M_m|\varphi\rangle$$

$$|\varphi'\rangle = \frac{M_m|\varphi\rangle}{\sqrt{\langle\varphi|M_m^\dagger M_m|\varphi\rangle}}$$

$$|\varphi\rangle = |0\rangle \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

ENTANGLEMENT - A SURPRISING STATE

- Based on the 4th Postulate, let us split state into two parts!

$$|\varphi\rangle = a|00\rangle + b|11\rangle$$

$|\varphi\rangle = |\varphi_1\rangle \otimes |\varphi_2\rangle \Rightarrow |\varphi_1\rangle = ? \quad |\varphi_2\rangle = ?$

- No such decomposition exists!

- Two different types of quantum states
  - product
  - entangled

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PROJECTIVE MEASUREMENT CONSTRUCTION

- 3rd Postulate with projectors

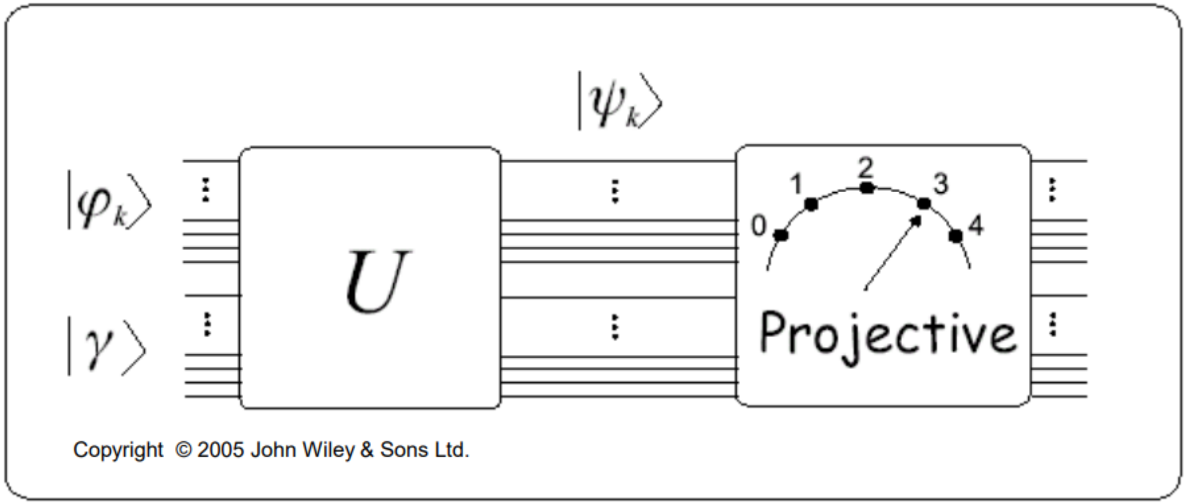
$$P(m \mid |\varphi\rangle) = \langle\varphi|P_m|\varphi\rangle$$

$$|\varphi'\rangle = \frac{P_m|\varphi\rangle}{\sqrt{\langle\varphi|P_m|\varphi\rangle}}$$

$$\sum_m P_m \equiv I$$

NEUMARK'S EXTENSION

- Any general measurement can be traced back to an appropriate projective measurement and a unitary transform by increasing the dimensions of the Hilbert space, i.e., adding extra quantum wires.

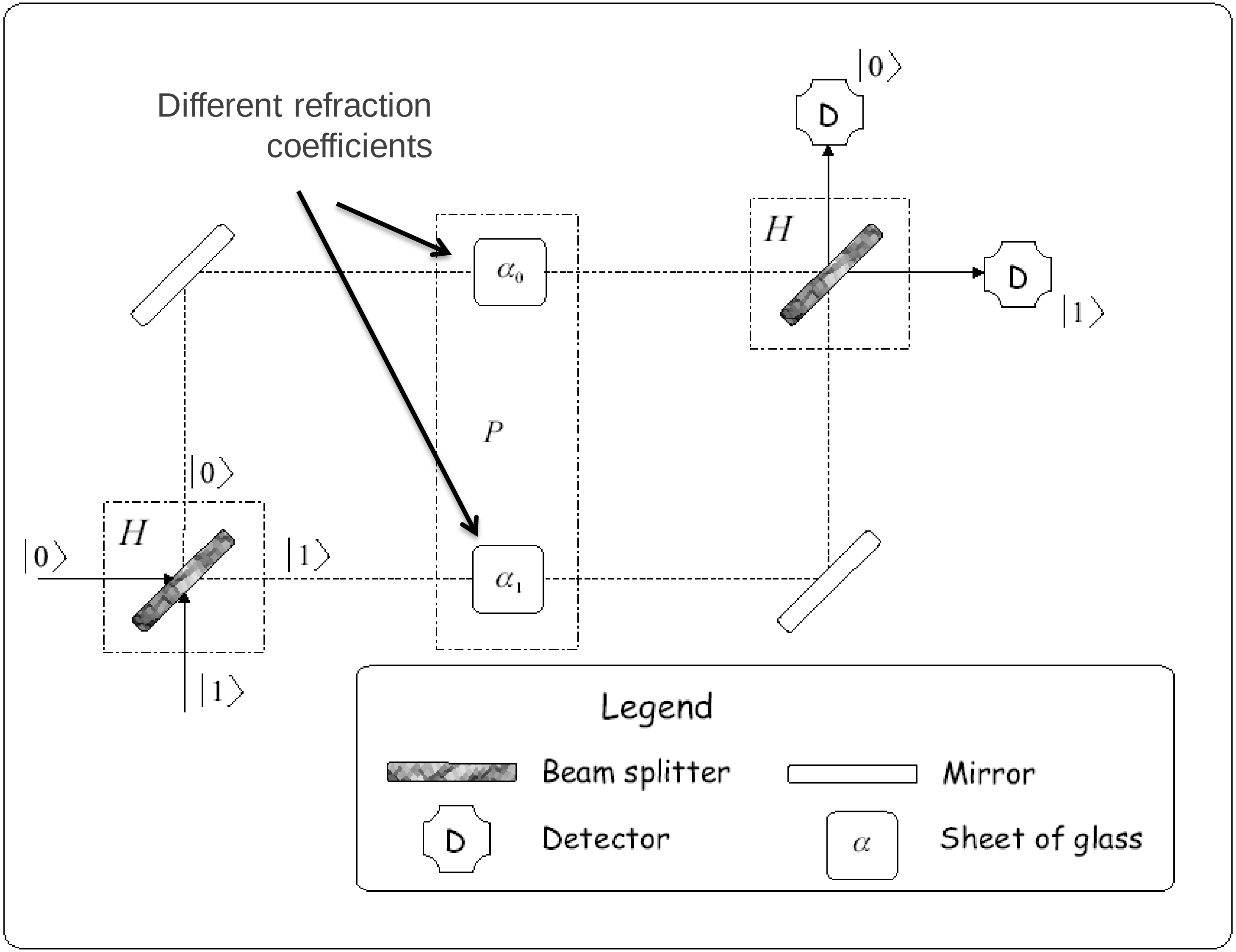


EPR PARADOX

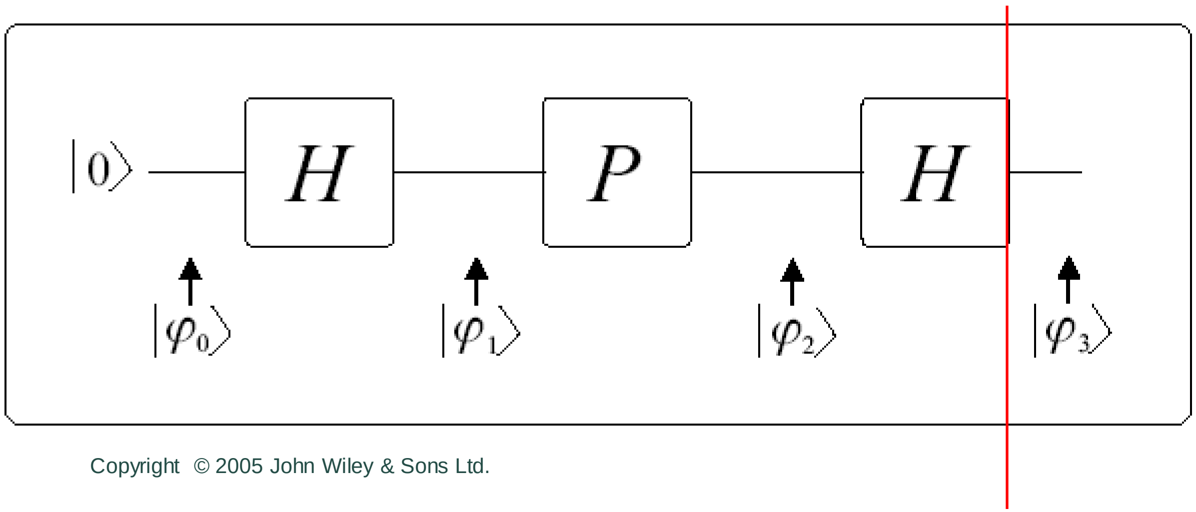
- Einstein: “spooky action at a distance“
- Einstein: “God does not play dice with the universe.”

- Bohr’s answer: "Quit telling God what to do!"
- Born’s answer: "If God has made the world a perfect mechanism, He has at least conceded so much to our imperfect intellect that in order to predict little parts of it, we need not solve innumerable differential equations, but can use dice with fair success".

GENERALIZED INTERFEROMETER



ANALYSIS(4)



$$\begin{bmatrix} \frac{e^{j\alpha_0}}{\sqrt{2}} \\ \frac{e^{j\alpha_1}}{\sqrt{2}} \end{bmatrix}$$

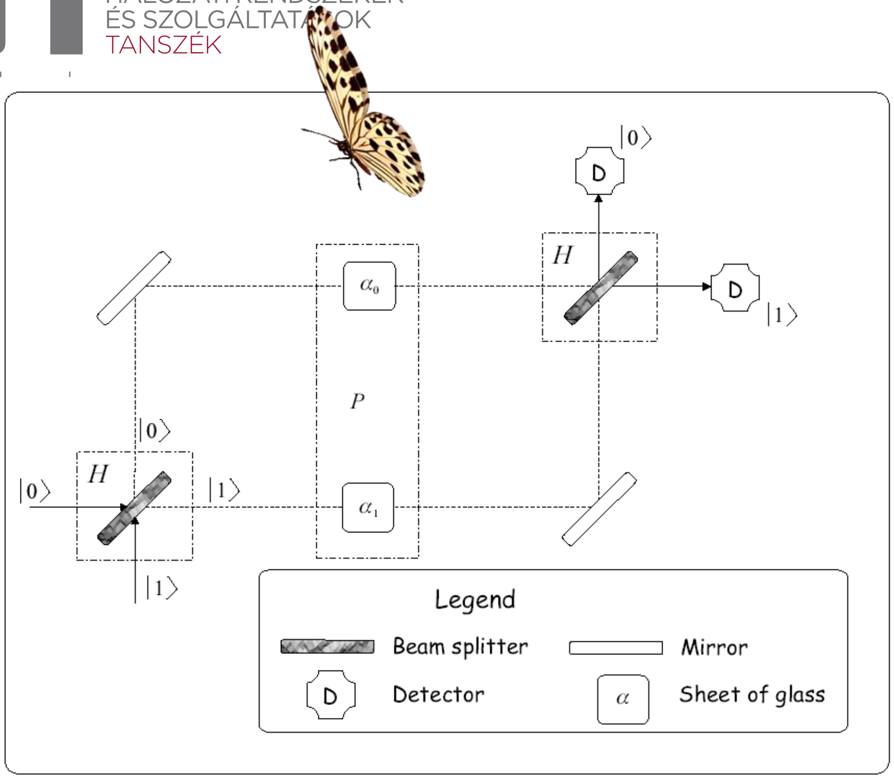
$$|\varphi_3\rangle = H|\varphi_2\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{e^{j\alpha_0} + e^{j\alpha_1}}{2} \\ \frac{e^{j\alpha_0} - e^{j\alpha_1}}{2} \end{bmatrix}$$

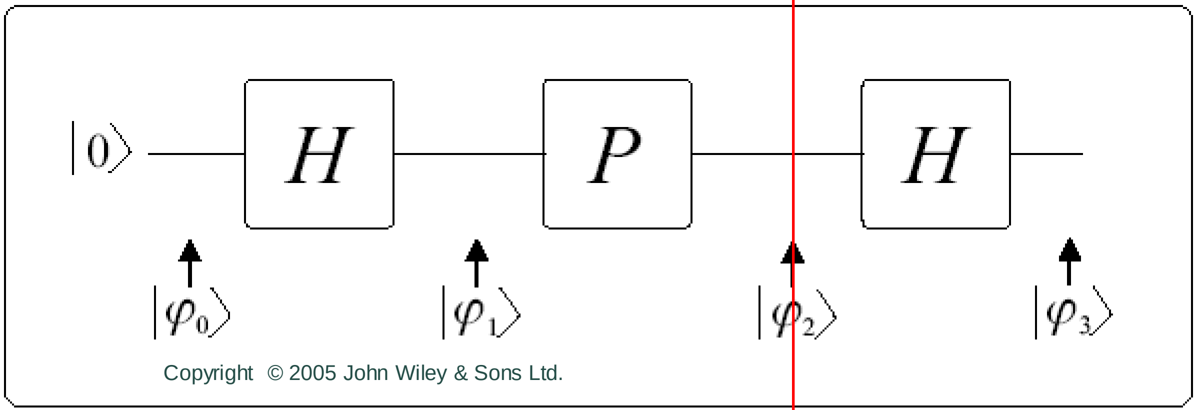
DECOHERENCE – ENTANGLEMENT WITH THE ENVIRONMENT

- The first two postulates are valid only for **closed** systems therefore entanglement with the environment can be very dangerous!
- In order to demonstrate it we use the well-known quantum interferometer and assume that the photon traveling trough the interferometer is entangled with a butterfly outside the laboratory (i.e. the environment).
- The environment is represented by $|\Omega\rangle$
- The flying rule of butterfly

$$\boxed{|0\rangle|\Omega\rangle \to |0\rangle|\Omega_0\rangle, |1\rangle|\Omega\rangle \to |1\rangle|\Omega_1\rangle}$$

DECOHERENCE AND THE INTERFEROMETER





DECOHERENCE AND THE INTERFEROMETER

- Before butterfly's action

$$|\varphi_2\rangle = \frac{e^{j\alpha_0}|0\rangle + e^{j\alpha_1}|1\rangle}{\sqrt{2}}|\Omega\rangle = |\varphi_2\rangle = \frac{e^{j\alpha_0}|0\rangle|\Omega\rangle + e^{j\alpha_1}|1\rangle|\Omega\rangle}{\sqrt{2}}$$

- After butterfly's action

$$|\varphi_2'\rangle = |\varphi_2\rangle = \frac{e^{j\alpha_0}|0\rangle|\Omega_0\rangle + e^{j\alpha_1}|1\rangle|\Omega_1\rangle}{\sqrt{2}}$$

- Now, the Hadamard gate comes

$$\begin{aligned}
|\varphi_3\rangle &= \frac{e^{j\alpha_0}\frac{|0\rangle+|1\rangle}{\sqrt{2}}|\Omega_0\rangle + e^{j\alpha_1}\frac{|0\rangle-|1\rangle}{\sqrt{2}}|\Omega_1\rangle}{\sqrt{2}} \\
&= |0\rangle\frac{e^{j\alpha_0}|\Omega_0\rangle + e^{j\alpha_1}|\Omega_1\rangle}{2} + |1\rangle\frac{e^{j\alpha_0}|\Omega_0\rangle - e^{j\alpha_1}|\Omega_1\rangle}{2} \\
&= e^{j\frac{\alpha_0+\alpha_1}{2}}\left(|0\rangle\frac{e^{j\frac{\alpha_0-\alpha_1}{2}}|\Omega_0\rangle + e^{-j\frac{\alpha_0-\alpha_1}{2}}|\Omega_1\rangle}{2} + |1\rangle\frac{e^{j\frac{\alpha_0-\alpha_1}{2}}|\Omega_0\rangle - e^{-j\frac{\alpha_0-\alpha_1}{2}}|\Omega_1\rangle}{2}\right)
\end{aligned}$$

DECOHERENCE AND THE INTERFEROMETER

$$\Delta\alpha \triangleq \alpha_0 - \alpha_1$$

$$|\varphi_3\rangle = |0\rangle \frac{e^{j\frac{\Delta\alpha}{2}}|\Omega_0\rangle + e^{-j\frac{\Delta\alpha}{2}}|\Omega_1\rangle}{2} + |1\rangle \frac{e^{j\frac{\Delta\alpha}{2}}|\Omega_0\rangle - e^{-j\frac{\Delta\alpha}{2}}|\Omega_1\rangle}{2}$$

- Attention! $|\Omega_0\rangle$ and $|\Omega_1\rangle$ are not necessarily orthogonal $\langle\Omega_0|\Omega_1\rangle \neq 0$

$$|\Omega_1\rangle = \langle\Omega_0|\Omega_1\rangle|\Omega_0\rangle + \sqrt{1 - |\langle\Omega_0|\Omega_1\rangle|^2}|\Omega_0^\perp\rangle$$

- Assuming $\langle\Omega_0|\Omega_1\rangle$ is real

$$|\varphi_3\rangle = \frac{e^{j\frac{\Delta\alpha}{2}} + \langle\Omega_0|\Omega_1\rangle e^{-j\frac{\Delta\alpha}{2}}}{2}|0\rangle|\Omega_0\rangle + \frac{e^{-j\frac{\Delta\alpha}{2}}}{2}\sqrt{1 - |\langle\Omega_0|\Omega_1\rangle|^2}|0\rangle|\Omega_0^\perp\rangle +$$
$$\frac{e^{j\frac{\Delta\alpha}{2}} - \langle\Omega_0|\Omega_1\rangle e^{-j\frac{\Delta\alpha}{2}}}{2}|1\rangle|\Omega_0\rangle - \frac{e^{-j\frac{\Delta\alpha}{2}}}{2}\sqrt{1 - |\langle\Omega_0|\Omega_1\rangle|^2}|1\rangle|\Omega_0^\perp\rangle$$

DECOHERENCE AND THE INTERFEROMETER

$$P_0 = \left| \frac{e^{j\frac{\Delta\alpha}{2}} + \langle \Omega_0 | \Omega_1 \rangle e^{-j\frac{\Delta\alpha}{2}}}{2} \right|^2 + \left| \frac{e^{-j\frac{\Delta\alpha}{2}}}{2} \sqrt{1 - |\langle \Omega_0 | \Omega_1 \rangle|^2} \right|^2$$

$z \in \mathbb{C} : |z|^2 = zz^*$

$\frac{e^{j\Delta\alpha} + e^{-j\Delta\alpha}}{2} = \cos(\Delta\alpha)$

$$P_0 = (1 + \langle \Omega_0 | \Omega_1 \rangle \cos(\Delta\alpha))\frac{1}{2}$$

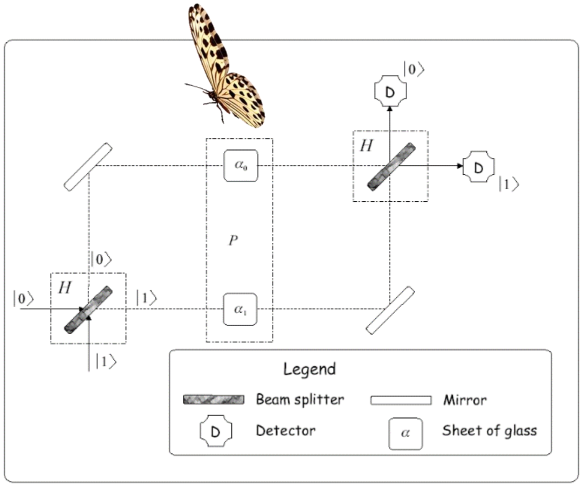
$$P_1 = (1 - \langle \Omega_0 | \Omega_1 \rangle \cos(\Delta\alpha))\frac{1}{2}$$

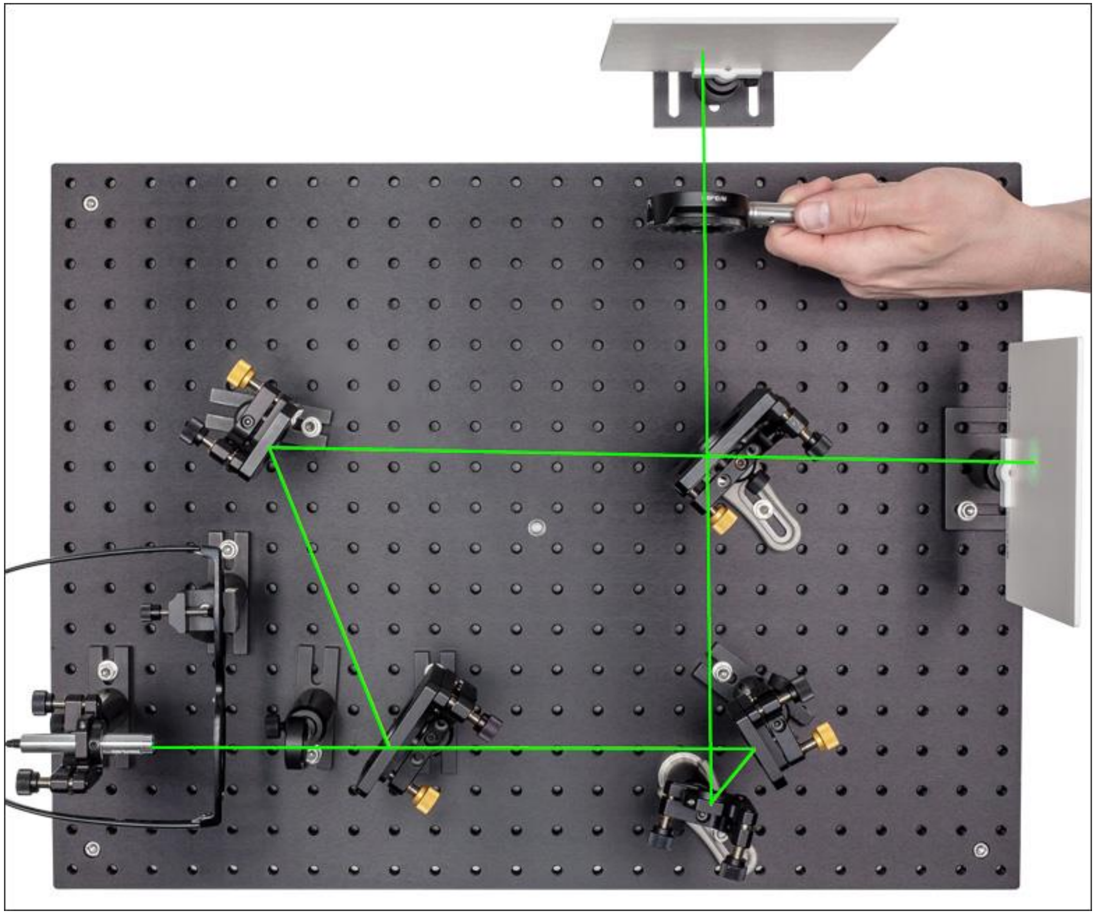
$$P_0 = \cos^2\left(\frac{\Delta\alpha}{2}\right) = (1 + \cos(\Delta\alpha))\frac{1}{2},$$

$$P_1 = \sin^2\left(\frac{\Delta\alpha}{2}\right) = (1 - \cos(\Delta\alpha))\frac{1}{2}.$$

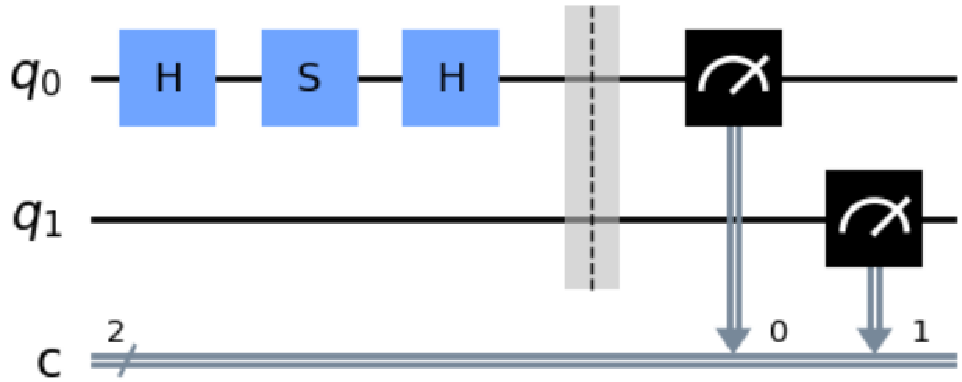
- If $\langle \Omega_0 | \Omega_1 \rangle = 1$ then the entanglement disappears.
- However, if $\langle \Omega_0 | \Omega_1 \rangle = 0$ then the operation becomes fully random instead of being deterministic while the observer becomes miss leaded.

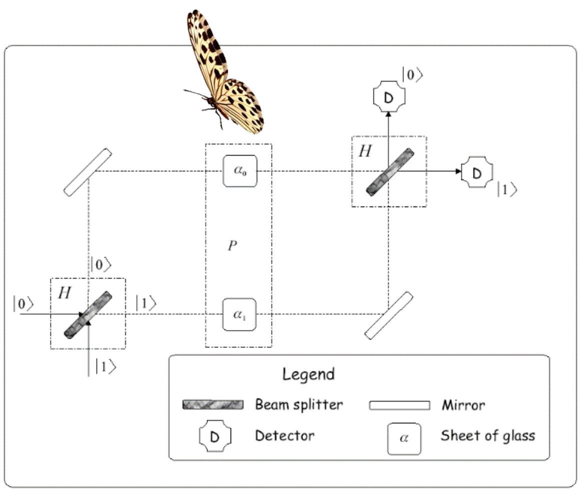
INTERFEROMETER FROM PHYSICIST POINT OF VIEW

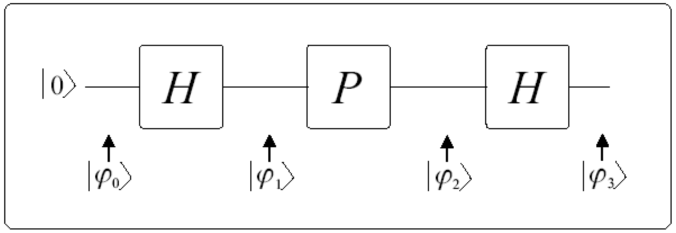




INTERFEROMETER FROM HW ENGINEER POINT OF VIEW





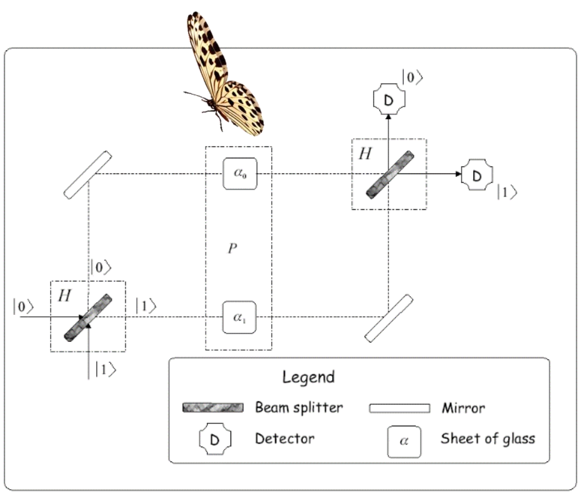


INTERFEROMETER FROM SW ENGINEER POINT OF VIEW

Classical input

Initialization → Quantum parallelism → Amplitude amplification → Measurement → Post-processing

Classical output



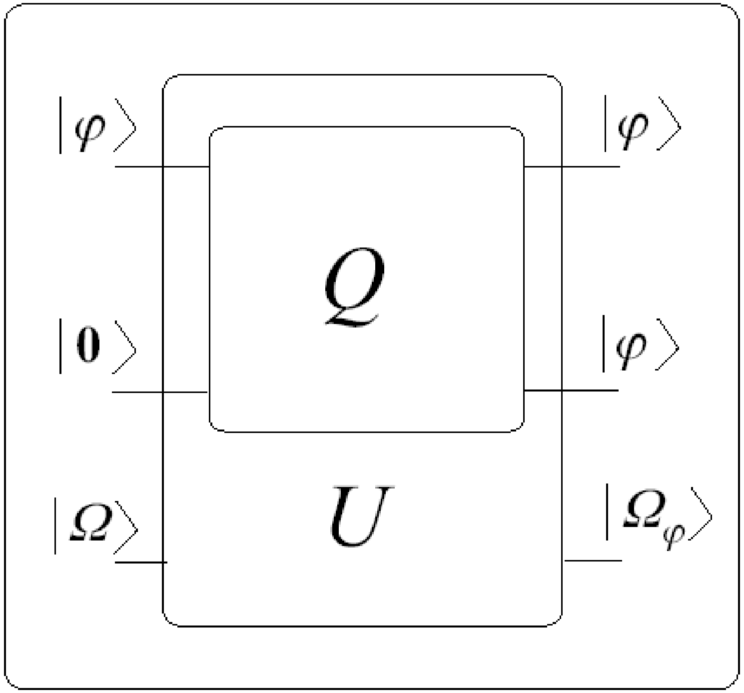

```
qc = QuantumCircuit(1,2)
qc.h(0)
qc.p(Dalpha)
qc.h(0)
qc.measure(0,1)
qc.draw("mpl")
backend = Aer.get_backend("aer_simulator")
result = backend.run(qc, shots=10000).result()
counts = result.get_counts()
print(counts)
```

No Cloning (Copy) Theorem

"Knowledge is what we get when an observer, preferably a scientifically trained observer, provides us with a copy of reality that we can all recognize. "

Christopher Lasch

EXISTENCE OF UNIVERSAL QUANTUM COPY MACHINE (1)



- Upper wire: quantum state to be cloned.
- Middle wire: auxiliary qubits.
- Lower wire: environment.
- We are interested whether such an *U* exists at all? If the answer is YES then we are going to search for *Q*.

EXISTENCE OF UNIVERSAL QUANTUM COPY MACHINE (2)

Making a copy means:

$$U : |\varphi\rangle|\mathbf{0}\rangle|\Omega\rangle \rightarrow |\varphi\rangle|\varphi\rangle|\Omega_\varphi\rangle$$
$$U : |\psi\rangle|\mathbf{0}\rangle|\Omega\rangle \rightarrow |\psi\rangle|\psi\rangle|\Omega_\psi\rangle$$

U must be unitary i.e. it preserves the inner product.

- For input

$$\langle\Omega,\mathbf{0},\psi|\varphi,\mathbf{0},\Omega\rangle = \langle\psi|\varphi\rangle\langle\mathbf{0}|\mathbf{0}\rangle\langle\Omega|\Omega\rangle = \langle\psi|\varphi\rangle$$

- For output

$$\langle\Omega_\psi,\psi,\psi|\varphi,\varphi,\Omega_\varphi\rangle = \langle\psi|\varphi\rangle\langle\psi|\varphi\rangle\langle\Omega_\psi|\Omega_\varphi\rangle = \langle\psi|\varphi\rangle^2\langle\Omega_\psi|\Omega_\varphi\rangle$$

The two inner products are the same if

- $\langle\psi|\varphi\rangle = \pm 1$ which is equivalent to $|\varphi\rangle = |\psi\rangle$ or
- $\langle\psi|\varphi\rangle = 0$ which represents the orthogonality between $|\varphi\rangle$ and $|\psi\rangle$

NO CLONING THEOREM (NCT)

No quantum gate exists which is able to copy quantum states from an unknown and arbitrary set of states!

But

- Orthogonal (e.g., classical states, or Bell pairs) can be copied in compliance with our everyday computer practice.
- Arbitrary but known states can also be copied

DISCLAIMER

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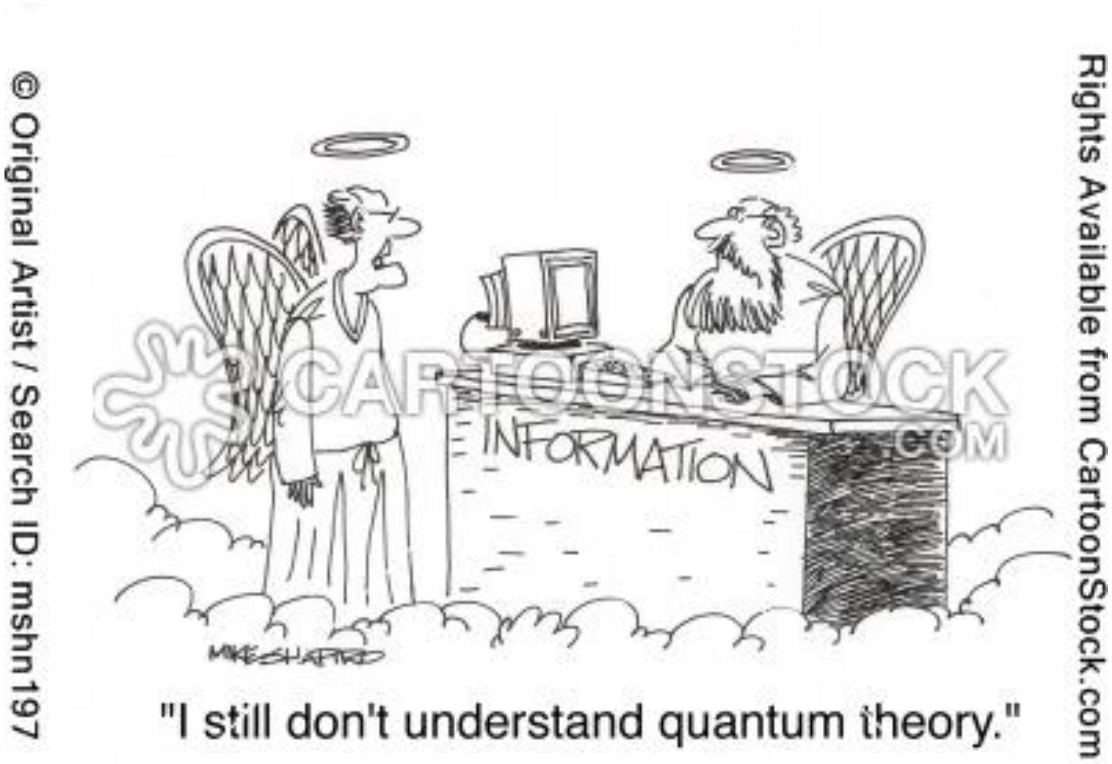
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