Quantum circuit design and analysis

1

Our boss looked at the specifications of two boxes that produce quantum bits and noticed that one produces quantum bits with a different global phase than the other. He could no longer remember which description belonged to which box, so he gave us the task of measuring the difference between the two.

The global phase difference between a $|\psi\rangle$ and a $|\psi'\rangle$ quantum bit is:

$$|\psi\rangle = x|0\rangle + y|1\rangle, \qquad |\psi'\rangle = e^{i\theta}x|0\rangle + e^{i\theta}y|1\rangle$$

Where

$$|x|^2 + |y|^2 = 1, \qquad x, y \in C$$

We can distinguish between the two if the probability of finding the state $|0\rangle$ and $|1\rangle$ is different. For $|\psi\rangle$, the probability of finding the state $|0\rangle$ is $P(0) = |x|^2$, the probability of finding the state $|1\rangle$ is $P(1) = |y|^2$.

What will these probabilities be for $|\psi'\rangle$?

2

Our boss gave us the task to find a gate U that doubles the length of the state vector. Is this possible?

In general, the elements of a 2x2 U unitary matrix can be written as follows:

$$U = \begin{bmatrix} a & b \\ -e^{i\varphi} b^* & e^{i\varphi} a^* \end{bmatrix}, \qquad |a|^2 + |b|^2 = 1 \, ,$$

Consider the following state vector:

$$|\psi\rangle = x|0\rangle + y|1\rangle, \qquad |x|^2 + |y|^2 = 1$$

After performing operation U, what will be the length of the state vector:

$$U|\psi\rangle = x'|0\rangle + y'|1\rangle, \qquad |x'|^2 + |y'|^2 = ?$$

Prove your answer.

3

Our boss is still concerned that the global phase might alter the outcome of some operation. He has given us the task of finding if there is a gate U after which the result is different as a function of the global phase (and which would allow us to detect a global phase). Is this possible?

The global phase can be detected if we can distinguish between a $|\psi\rangle$ and a $|\psi'\rangle$ state vector by a measurement, where

$$|\psi\rangle = x|0\rangle + y|1\rangle, \qquad |\psi'\rangle = e^{i\theta}x|0\rangle + e^{i\theta}y|1\rangle$$

We can distinguish between them if $U|\psi\rangle = x'|0\rangle + y'|1\rangle$ and $U|\psi'\rangle = x''|0\rangle + y''|1\rangle$ gives us $|x'|^2 \neq |x''|^2$ (or $|y'|^2 \neq |y''|^2$). Is this possible? Why or why not? What will $|x'|^2$ and $|x''|^2$ be?

Note that U is still unitary:

$$U = \begin{bmatrix} a & b \\ -e^{i\varphi}b^* & e^{i\varphi}a^* \end{bmatrix}, \qquad |a|^2 + |b|^2 = 1\ ,$$

4

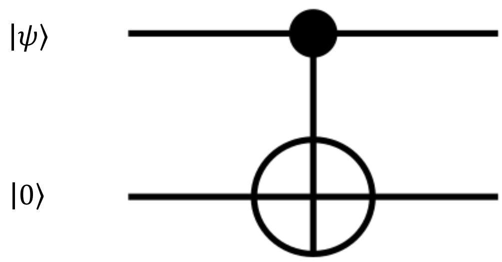
Our boss tried a CNOT gate with classic inputs. He found that when the target bit is $|\mathbf{0}\rangle$, the gate copies the value of the control bit to the target bit. He then asks us to use the CNOT gate to make an independent copy of an arbitrary $|\boldsymbol{\psi}\rangle$ quantum bit.

$$|\psi\rangle = x|0\rangle + y|1\rangle, \qquad |x|^2 + |y|^2 = 1$$

What does two independent copies of $|\psi\rangle$ look like?

$$|\psi\rangle \otimes |\psi\rangle = ?$$

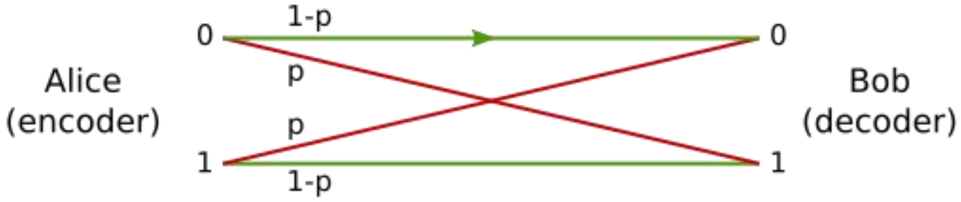
What does the CNOT gate give for arbitrary $|\psi\rangle$ control input?

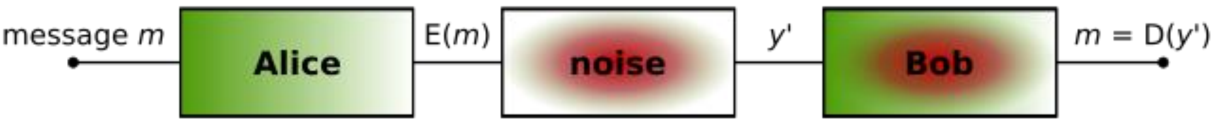


When do the two results match each other?

5

Our boss has heard that in a binary symmetric channel, the classical bits encoded with a Hadamard gate H are insensitive to negation in the channel. Design an encoding and decoding box that can transmit classical data on a quantum channel without error. Justify your claim.





$$H_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$