Quantum Computers and their Applications: Operations with qubits and quantum registers

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Recap

Unitary operators and matrices

An invertible complex square matrix U is unitary if its matrix inverse U^{-1} equals its conjugate transpose $U^\dagger$, that is, if $U^\dagger = U^{-1}$.
Properties:

- $UU^\dagger = U^\dagger U = U^{-1}U = I$, where I is the indentity.
- Given two complex vectors x and y, multiplication by U preserves their inner product; that is, $\langle x|y\rangle = \langle x| U^\dagger U |y\rangle$.
- $|\det U| = 1$.
- Its eigenspaces are orthogonal.

Notations

Important states:

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad |+\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad |-\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$|\psi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \quad \langle\psi| = \begin{bmatrix} \alpha^* & \beta^* \end{bmatrix}, \ |\alpha|^2 + |\beta|^2 = 1$$

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle_A |0\rangle_B + |1\rangle_A |1\rangle_B\right) \quad |\Phi^-\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle_A |0\rangle_B - |1\rangle_A |1\rangle_B\right)$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle_A |1\rangle_B + |1\rangle_A |0\rangle_B\right) \quad |\Psi^-\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle_A |1\rangle_B - |1\rangle_A |0\rangle_B\right)$$

Logic gates:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \quad U = \begin{bmatrix} a & b \\ -\mathrm{e}^{i\phi} b^* & \mathrm{e}^{i\phi} a^* \end{bmatrix}, \ |a|^2 + |b|^2 = 1$$

$$\mathrm{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \mathrm{SWAP} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Comments on tensor product

The tensor product of two qubits is a four dimensional vector. But this vector is not equal to the one that we get if we write the coordinates of each vector after/below each other:

$$|0\rangle \otimes |1\rangle \neq \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

The correct solution is:

$$|0\rangle \otimes |1\rangle = \begin{bmatrix} 1 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ 0 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

The first two coordinates of the product are the first two coordinates of the right vector multiplied by the first coordinate of the left vector (here $|0\rangle$). Similarly the last two coordinates of the product are the last two coordinates of the right vector multiplied by the last coordinate of the left vector). This can be generalized for higher dimensions as well as matrices.

Exercises

Exercise 1: Inner product

Calculate the inner products of following vectors!

$$|a\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ |b\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix},\ |c\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} i \\ 1 \end{bmatrix},\ |d\rangle = \frac{1}{\sqrt{11}} \begin{bmatrix} 3 \\ 1+i \end{bmatrix},\ |e\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

a) $\langle a|a\rangle$ **b**) $\langle b|b\rangle$ **c**) $\langle c|c\rangle$ **d**) $\langle d|d\rangle$
e) $\langle a|b\rangle$ **f**) $\langle b|a\rangle$ **g**) $\langle c|d\rangle$ **h**) $\langle d|c\rangle$
i) $\langle a|c\rangle$ **j**) $\langle d|a\rangle$ **k**) $\langle c|b\rangle$ **l**) $\langle b|d\rangle$

Which vectors are orthogonal to each other?

m) $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$ **n)** $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}$ **o)** $\begin{bmatrix}3\\1\end{bmatrix}, \begin{bmatrix}2\\-5\end{bmatrix}$

p) $\begin{bmatrix}\sqrt{2}\\2\end{bmatrix}, \begin{bmatrix}\sqrt{18}\\-3\end{bmatrix}$ **q)** $\begin{bmatrix}i\\4\end{bmatrix}, \begin{bmatrix}-i\\1/4\end{bmatrix}$ **r)** $\begin{bmatrix}-i\\4\end{bmatrix}, \begin{bmatrix}-i\\1/4\end{bmatrix}$

Exercise 2: Outer product

Calculate the outer product of the following vectors!

The vectors are the same as in Exercise 1.

a) $|a\rangle\langle a|$ **b)** $|b\rangle\langle b|$ **c)** $|c\rangle\langle c|$ **d)** $|d\rangle\langle d|$

e) $|a\rangle\langle e|$ **f)** $|d\rangle\langle b|$ **g)** $|e\rangle\langle c|$ **h)** $|b\rangle\langle a|$

Exercise 3: Vector and matrix products

Calculate the following products!

a) $\begin{pmatrix}3&1\\4&2\end{pmatrix}\begin{bmatrix}5\\0.6\end{bmatrix}$ **b)** $\begin{pmatrix}0&1\\2-i&1\end{pmatrix}\begin{bmatrix}4-i\\i\end{bmatrix}$ **c)** $\begin{pmatrix}1&0\\0&-i\end{pmatrix}\begin{bmatrix}3\\2+i\end{bmatrix}$

Exercise 4: Matrix products

Calculate the following matrix products!

a) $\begin{pmatrix}a&b\\c&d\end{pmatrix}\cdot\begin{pmatrix}e&f\\g&h\end{pmatrix}$ **b)** $\begin{pmatrix}0&1\\1&0\end{pmatrix}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&1\end{pmatrix}$

c) $\begin{pmatrix}1&2+i\\1-i&0\end{pmatrix}\cdot\begin{pmatrix}0&3\\2i&-4i\end{pmatrix}$

Exercise 5: Matrix operations

Calculate the following matrix operations!

$$A = \begin{pmatrix} 1-i & 2 & 3i \\ 0 & 1 & 2+2i \\ -i & 4 & 5i \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}, \quad C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

a) $A^\dagger$ **b**) $B^\dagger$ **c**) $C^\dagger$ **d**) BC **e**) CB

f) $CB^\dagger$ **g**) $(CB)^\dagger$ **h**) $\operatorname{tr} B$ **i**) $\operatorname{tr} BC$ **j**) $\operatorname{tr} CB$

Exercise 6: Tensor product with vectors

Calculate the tensor products for the following vectors!

Normalize the output if needed (an approximation is enough).

$$|\alpha\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |\beta\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad |\gamma\rangle = \frac{1}{2} \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}, \quad |\delta\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

a) $|\alpha\rangle \otimes |\delta\rangle$ **b**) $|\delta\rangle \otimes |\delta\rangle$ **c**) $|\alpha\rangle \otimes |\beta\rangle$ **d**) $|\alpha\rangle \otimes i\,|\alpha\rangle$

e) $|\beta\rangle \otimes |\alpha\rangle$ **f**) $|\alpha\rangle \otimes (2\,|\beta\rangle + |\gamma\rangle)$ **g**) $(3\,|\alpha\rangle + 0.5\,|\beta\rangle) \otimes (|\beta\rangle + 2\,|\gamma\rangle)$

Are the following states product states?

h) $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ **i**) $\frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ **j**) $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ **k**) $\frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$

Exercise 7: Tensor product with matrices

Calculate the tensor product for the following matrices!

The matrices are the same as in Exercise 5.

a) $C \otimes B$ **b**) $B \otimes C$ **c**) $I \otimes I$ **d**) $I \otimes B$ **e**) $C \otimes I$

Exercise 8: Tensor product with matrices and vectors

Calculate the following tensor products!

The vectors and the matrices are same as in Exercise 5. and 6.

a) $(B \otimes C)(|\alpha\rangle \otimes |\delta\rangle)$ **b**) $(B \otimes I)(|\alpha\rangle \otimes |\delta\rangle)$ **c**) $(I \otimes C)(|\alpha\rangle \otimes |\delta\rangle)$

Answers

Exercise 1

a) 1 **b**) 1 **c**) 1 **d**) 1

e) $-\frac{1}{\sqrt{2}}$ **f**) $-\frac{1}{\sqrt{2}}$ **g**) $\frac{1-2i}{\sqrt{22}}$ **h**) $\frac{1+2i}{\sqrt{22}}$

i) $\frac{i}{\sqrt{2}}$ **j**) $\frac{3}{\sqrt{11}}$ **k**) $\frac{1+i}{2}$ **l**) $\frac{-2+i}{\sqrt{22}}$

m) yes **n**) no **o**) no **p**) yes **q**) no **r**) yes

Exercise 2

a) $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ **b**) $\frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ **c**) $\frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$ **d**) $\frac{1}{11}\begin{pmatrix} 9 & 3-3i \\ 3+3i & 2 \end{pmatrix}$

e) $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ **f**) $\frac{1}{\sqrt{22}}\begin{pmatrix} -3 & 3 \\ -1-i & 1+i \end{pmatrix}$ **g**) $\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 \\ -i & 1 \end{pmatrix}$ **h**) $\frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}$

Exercise 3

a) $\begin{bmatrix} 15.6 \\ 21.2 \end{bmatrix}$ **b**) $\begin{bmatrix} i \\ 7-5i \end{bmatrix}$ **c**) $\begin{bmatrix} 3 \\ 1-2i \end{bmatrix}$

Exercise 4

a) $\begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$ **b**) $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ **c**) $\begin{pmatrix} -2+4i & 7-8i \\ 0 & 3-3i \end{pmatrix}$

Exercise 5

a) $\begin{pmatrix} 1+i & 0 & i \\ 2 & 1 & 4 \\ -3i & 2-2i & -5i \end{pmatrix}$ **b**) $\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$ **c**) $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

d) $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$ **e**) $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -1 & -i \end{pmatrix}$

f) $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix}$ **g**) $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ i & i \end{pmatrix}$

h) $1-i$ **i**) $\frac{1}{\sqrt{2}}(1-i)$ **j**) $\frac{1}{\sqrt{2}}(1-i)$

Exercise 6

a) $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ **b)** $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ **c)** $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ **d)** $\begin{bmatrix} i \\ 0 \\ 0 \\ 0 \end{bmatrix}$

e) $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ **f)** $\frac{1}{2.9772} \begin{bmatrix} 2.2802 \\ 1.9142 \\ 0 \\ 0 \end{bmatrix}$ **g)** $\frac{1}{10.0395} \begin{bmatrix} 8.1798 \\ 5.7249 \\ 0.8624 \\ 0.6036 \end{bmatrix}$

h) yes: $|\delta\rangle \otimes |\alpha\rangle$ **i)** yes: $|\beta\rangle \otimes |\beta\rangle$ **j)** no **k)** no

Exercise 7

a) $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & -i & 0 & -i \\ -1 & 0 & 1 & 0 \\ 0 & i & 0 & -i \end{pmatrix}$ **b)** $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -i & -i \\ 0 & 0 & i & -i \end{pmatrix}$

c) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ **d)** $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}$

e) $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$

Exercise 8

a) $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ **b)** $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ **c)** $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

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