DEPARTMENT OF
NETWORKED SYSTEMS
AND SERVICES

Arbitrary state, Superdense coding, Quantum teleportation

March 31, 2025

Quantum Computing and its Applications
BMEVIHIAD00, Spring 2025

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MŰEGYETEM 1782

Revision

POSTULATES

- 1. Postulate: qubit
  - Hilbert-space
- 2. Postulate: logic gates
  - Unitary transform
  - Elementary gates
- 3. Postulate Q/C conversion
  - Measurement statistics
  - Post measurement state
- 4. Postulate: registers
  - Tensor product

$$|\varphi\rangle = \sum_{i=0}^{2^n-1} \varphi_i |i\rangle$$

$$U^\dagger = U^{-1}$$

$$P(m \mid |\varphi\rangle) = \langle\varphi|M_m^\dagger M_m|\varphi\rangle$$

$$|\varphi'\rangle = \frac{M_m|\varphi\rangle}{\sqrt{\langle\varphi|M_m^\dagger M_m|\varphi\rangle}}$$

$$|\varphi\rangle = |0\rangle \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

PAULI GATES

$$|\varphi\rangle = a|0\rangle + b|1\rangle$$

- Pauli X (bit-flip) gate:

$$\begin{bmatrix} a \\ b \end{bmatrix}$$

$$|\psi\rangle = X|\varphi\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} b \\ a \end{bmatrix} = b|0\rangle + a|1\rangle$$

- Pauli Z (phase-flip) gate:

$$\begin{bmatrix} a \\ b \end{bmatrix}$$

$$|\psi\rangle = Z|\varphi\rangle = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ -b \end{bmatrix} = a|0\rangle - b|1\rangle$$

PAULI GATES

$$|\varphi\rangle = a|0\rangle + b|1\rangle \qquad \begin{bmatrix} a \\ b \end{bmatrix}$$

- Pauli Y (double-flip) gate:

$$|\psi\rangle = Y|\varphi\rangle = \begin{bmatrix} 0 & -j \\ j & 0 \end{bmatrix} \begin{bmatrix} -jb \\ ja \end{bmatrix} = -jb|0\rangle + ja|1\rangle$$

- Phase gate

$$\begin{bmatrix} a \\ b \end{bmatrix}$$

$$|\psi\rangle = P(\alpha)|\varphi\rangle = \begin{bmatrix} 1 & 0 \\ 0 & e^{j\alpha} \end{bmatrix} \begin{bmatrix} a \\ e^{j\alpha} b \end{bmatrix} = a|0\rangle + e^{j\alpha} b|1\rangle$$

HADAMARD GATE

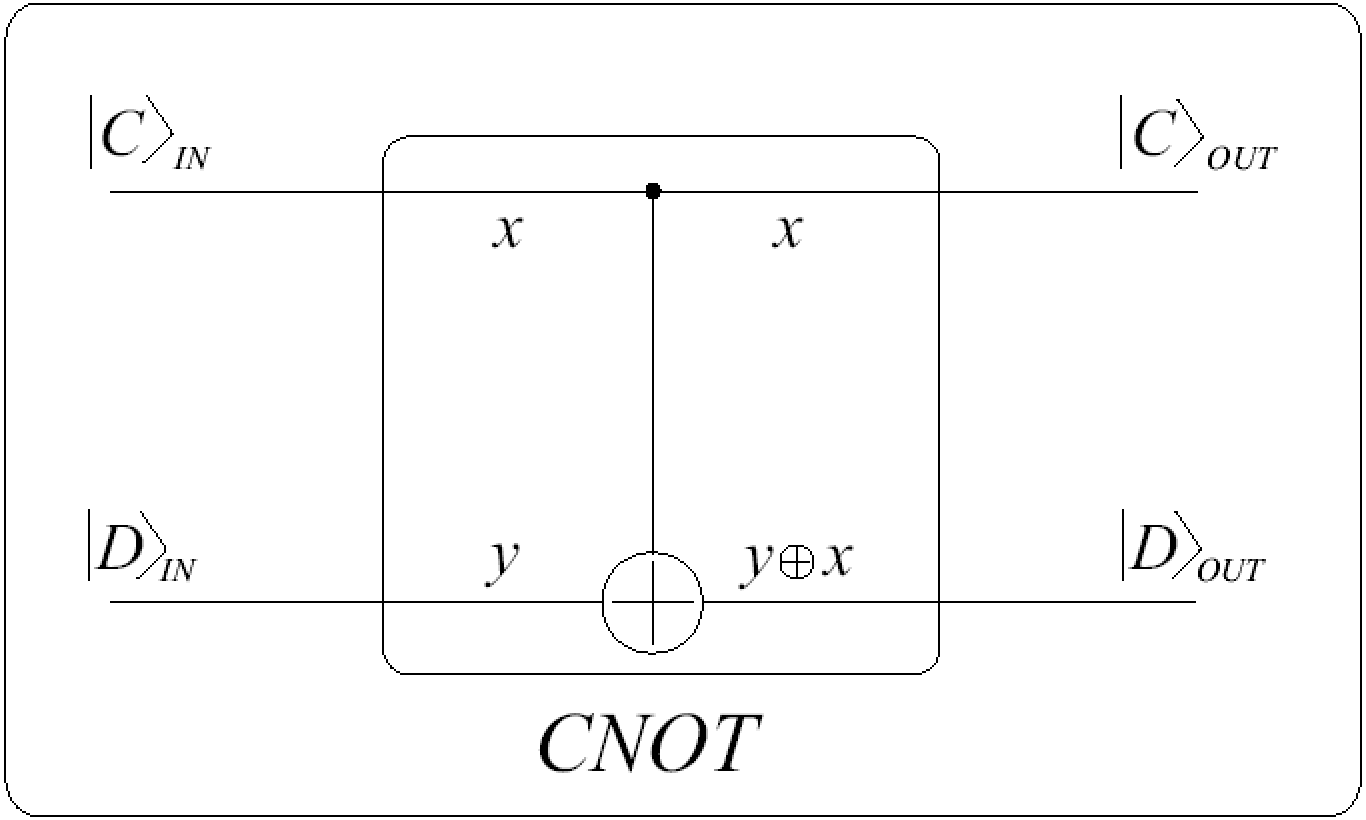
$$|\varphi\rangle = a|0\rangle + b|1\rangle \qquad \begin{bmatrix} a \\ b \end{bmatrix}$$

$$|\psi\rangle = H|\varphi\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \frac{a+b}{\sqrt{2}} \\ \frac{a-b}{\sqrt{2}} \end{bmatrix} = \frac{a+b}{\sqrt{2}}|0\rangle + \frac{a-b}{\sqrt{2}}|1\rangle$$

- Hadamard gate is Hermitian i.e. $H^\dagger = H$
- furthermore: $HH = I$
- *H* gate prepares uniform superposition:

$$\boxed{\begin{aligned} H|0\rangle &= \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \\ H|1\rangle &= \frac{|0\rangle - |1\rangle}{\sqrt{2}}. \end{aligned}}$$

CONTROLLED NOT GATE (CNOT GATE)



Upper wire: control

Lower wire: data

CNOT-GATE

- Truth table

IN		OUT	
x	y	x	$y \oplus x$
0	0	0	$0 \oplus 0 = 0$
0	1	0	$1 \oplus 0 = 1$
1	0	1	$0 \oplus 1 = 1$
1	1	1	$1 \oplus 1 = 0$

- Master equation

$$CNOT : |x\rangle|y\rangle \rightarrow |x\rangle|y \oplus x\rangle$$

Matrix

$$|00\rangle \rightarrow |00\rangle \quad |01\rangle \rightarrow |01\rangle$$
$$|10\rangle \rightarrow |11\rangle \quad |11\rangle \rightarrow |10\rangle$$

$$\mathbf{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

BELL STATES

$$|\beta_{00}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}},$$

$$|\beta_{01}\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}},$$

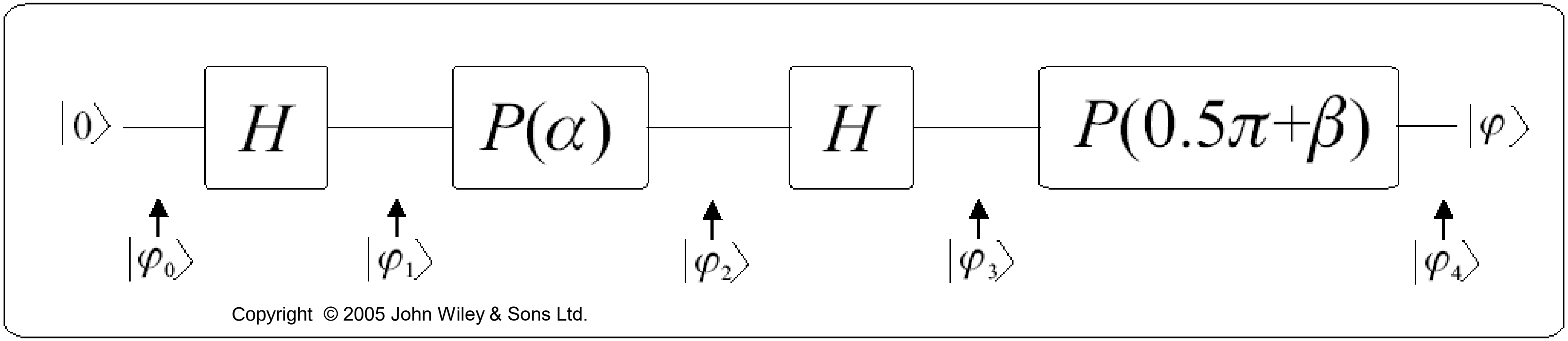
$$|\beta_{10}\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}},$$

$$|\beta_{11}\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}.$$

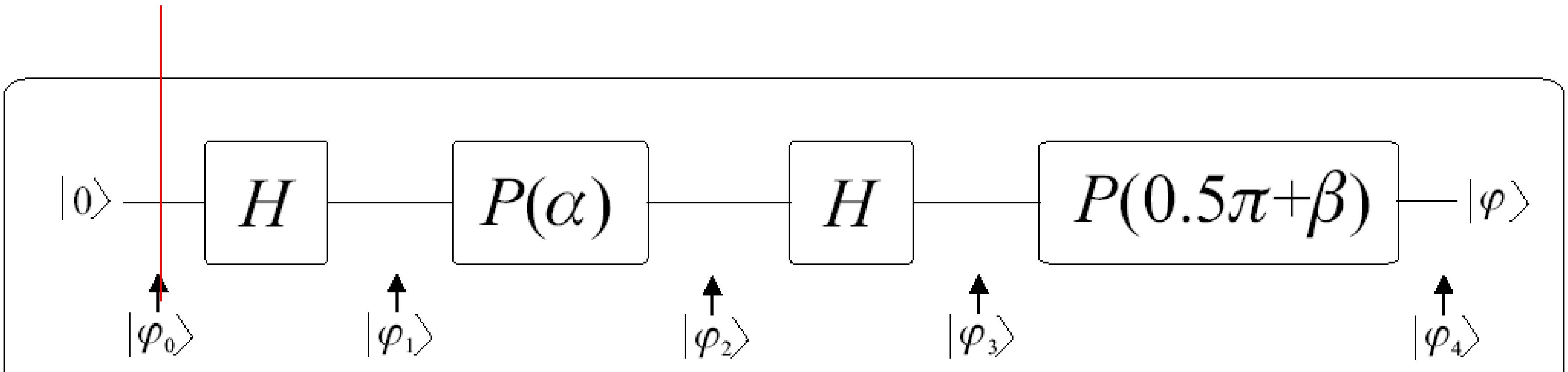
Bell states are orthogonal!

Preparing arbitrary quantum states

ARBITRARY STATE

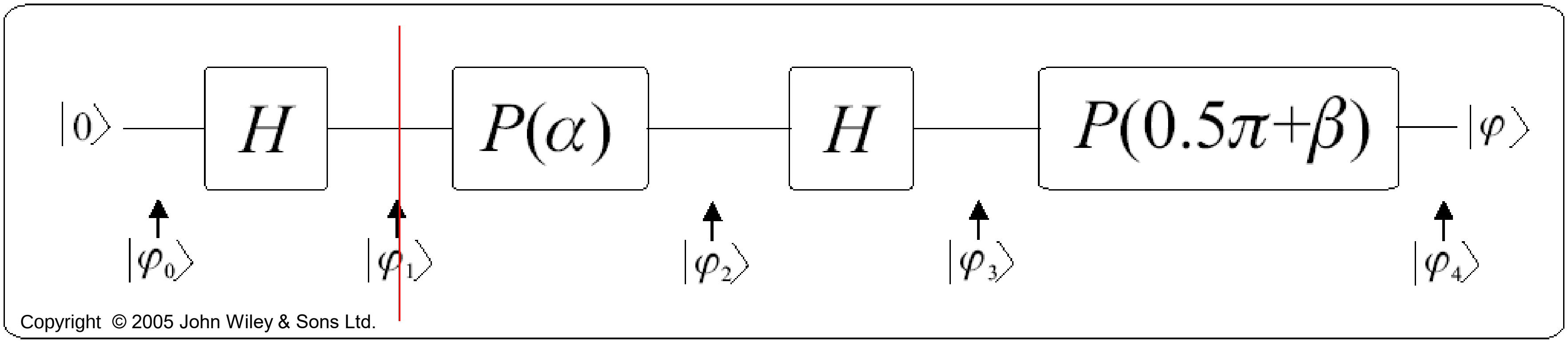


ARBITRARY STATE



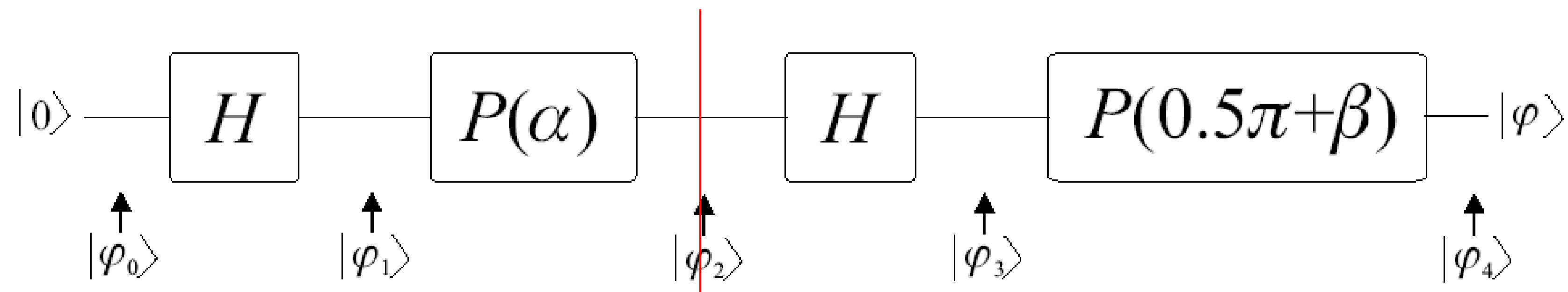
$$|\varphi_0\rangle = |0\rangle$$

ARBITRARY STATE



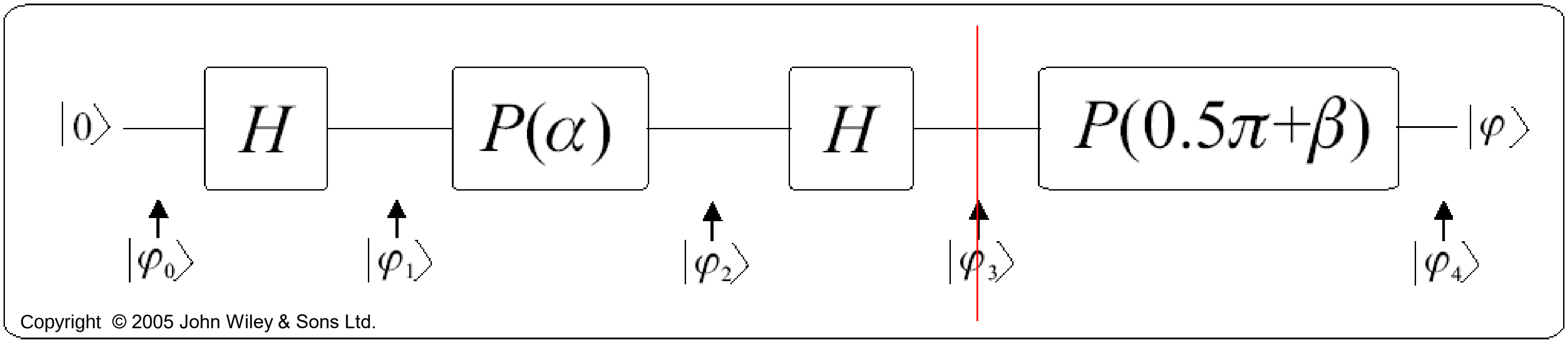
$$|\varphi_1\rangle = H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

ARBITRARY STATE



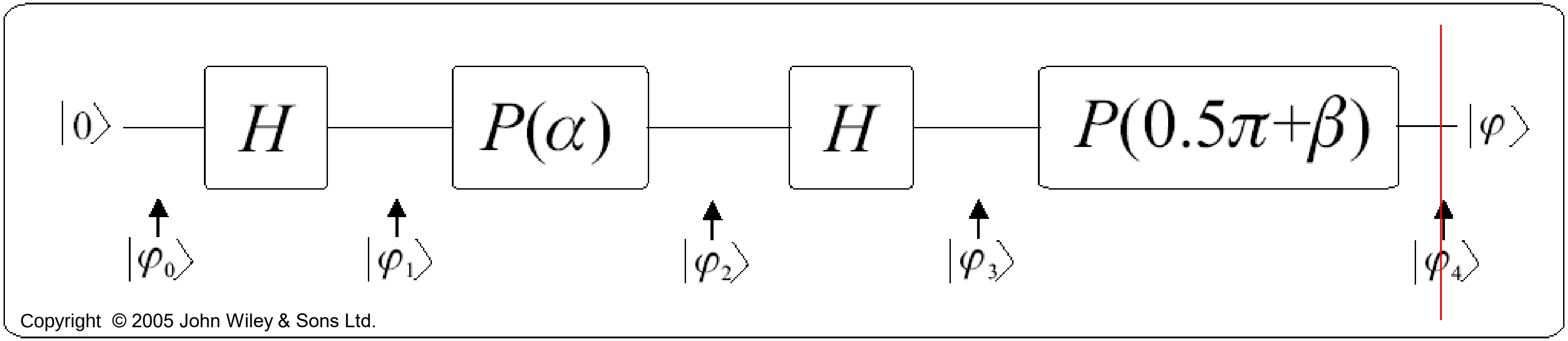
$$|\varphi_2\rangle = P(\alpha)|\varphi_1\rangle = \frac{|0\rangle + e^{j\alpha}|1\rangle}{\sqrt{2}}$$

ARBITRARY STATE



$$|\varphi_3\rangle = H|\varphi_2\rangle = \frac{\frac{|0\rangle+|1\rangle}{\sqrt{2}} + e^{j\alpha}\frac{|0\rangle-|1\rangle}{\sqrt{2}}}{\sqrt{2}} = \frac{1+e^{j\alpha}}{2}|0\rangle + \frac{1-e^{j\alpha}}{2}|1\rangle$$

ARBITRARY STATE



$$|\varphi_4\rangle = e^{j0.5\alpha}\left[\cos\left(\frac{\alpha}{2}\right)|0\rangle + e^{j\beta}\sin\left(\frac{\alpha}{2}\right)|1\rangle\right]$$

Almost general 1-qubit state except the global phase, but this does not influence the measurement statistics!

Exercise 2.6. Show that $\frac{1+e^{j\alpha}}{2} = e^{j0.5\alpha}\cos(0.5\alpha)$ and $e^{j0.5\pi}\frac{1-e^{j\alpha}}{2} = e^{j0.5\alpha}\sin(0.5\alpha)$!

Superdense Coding

SUPERDENSE CODING

Bennett, C. H. & Wiesner, S. J.

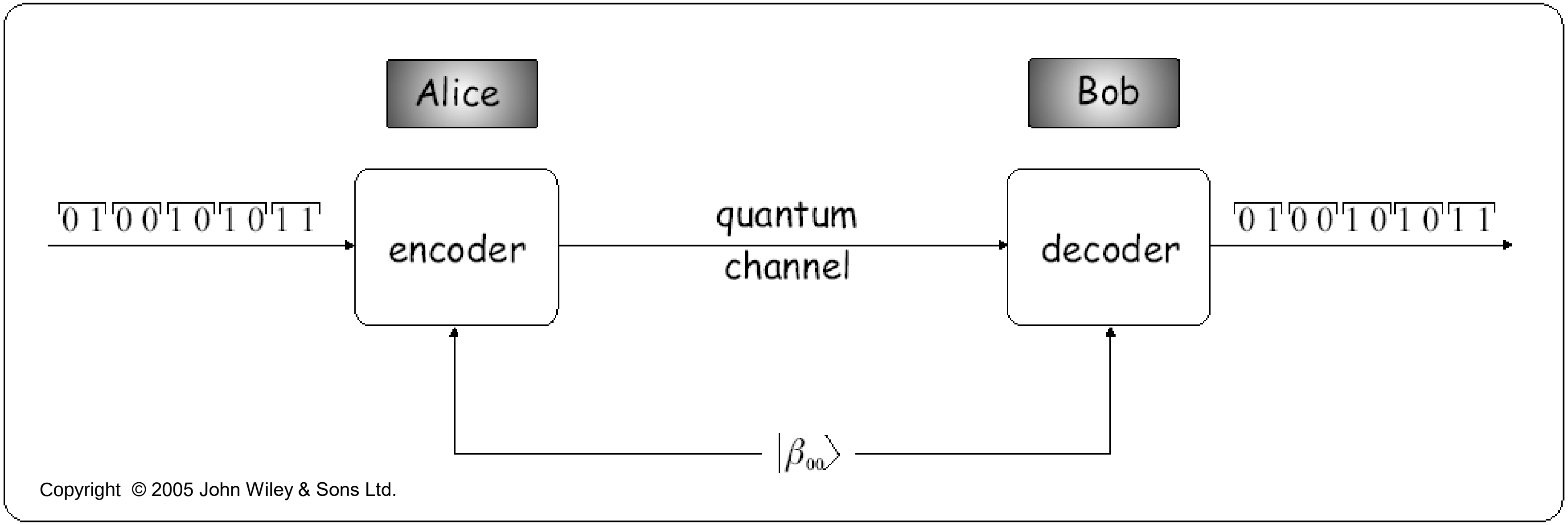
Communication via one- and two-particle operators on Einstein–Podolsky–Rosen states

Phys. Rev. Lett. 69, 2881–2884 (1992).

THE GOAL

- From classical information theory point of view the information transmission rate is limited.
- Alice and Bob would like to increase the rate of information transfer by means of quantum communications exploiting such special properties as entanglement.

THE ARCHITECTURE



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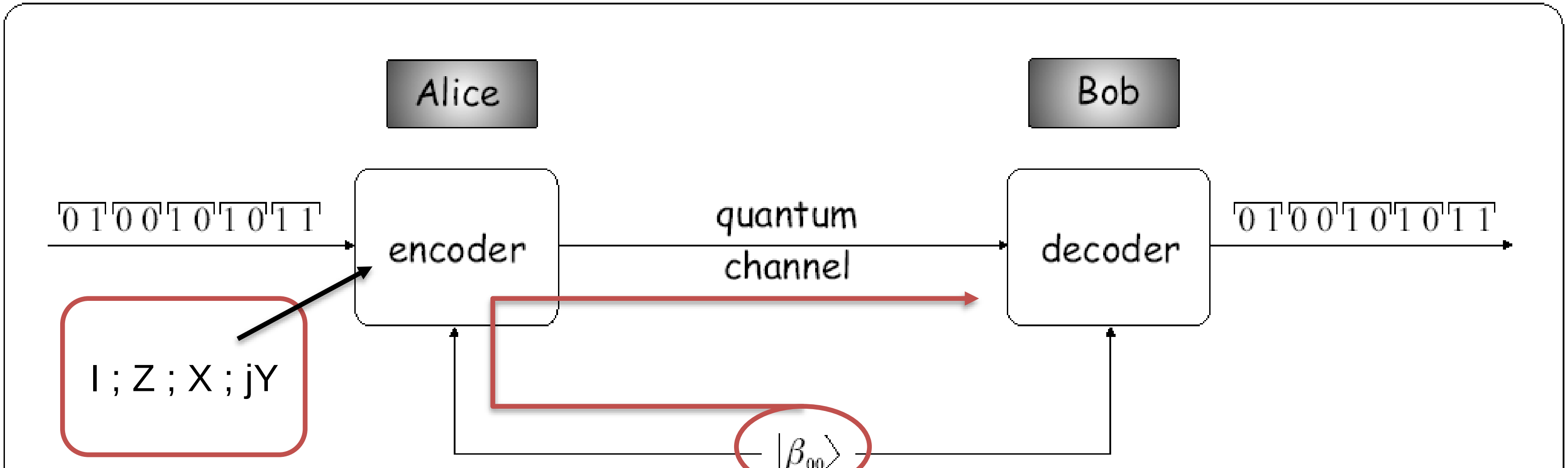
THE ARCHITECTURE



PROTOCOL STEPS (1)

- First they share a $|\beta_{00}\rangle$ entangled pair.
- Next Alice applies the following a special coding scheme on her half pair and sends the her coded qubit to Bob.

dibit	transform	joint state
00	I	$\frac{\|00\rangle+\|11\rangle}{\sqrt{2}}$
01	Z	$\frac{\|00\rangle-\|11\rangle}{\sqrt{2}}$
10	X	$\frac{\|10\rangle+\|01\rangle}{\sqrt{2}}$
11	jY	$\frac{\|01\rangle-\|10\rangle}{\sqrt{2}}$

$$|\beta_{00}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}},$$

$$|\beta_{01}\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}},$$

$$|\beta_{10}\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}},$$

$$|\beta_{11}\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}.$$

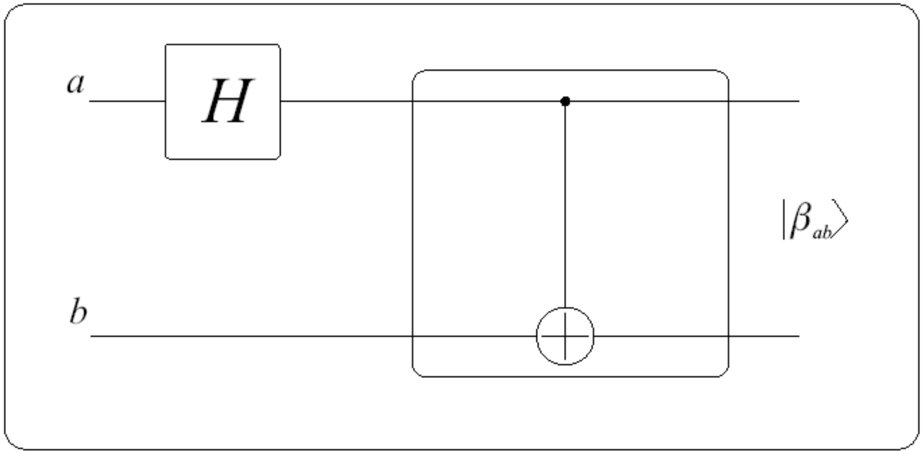
PROTOCOL STEPS (2)

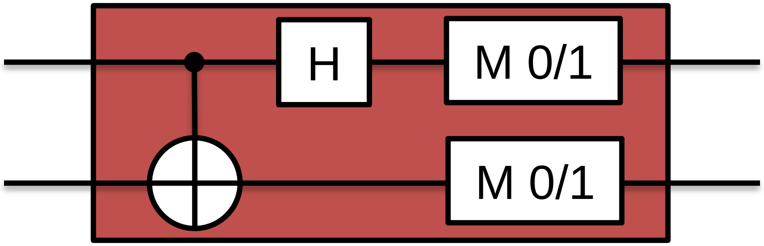
- Bob being an expert of quantum computing realises that the modified pairs represent the four different Bell pairs therefore they for an othonormal set of quantum states.

- The can be unambiguously distinguished by means of a projective measurement.
- However there is another way to solve the detection problem if we exploit the unitary nature of quantum transformations i.e. we are able to compute the inverse

PROTOCOL STEPS (3)

- We know that Bell states can be produced by means of the following circuit





PROTOCOL STEPS (4)

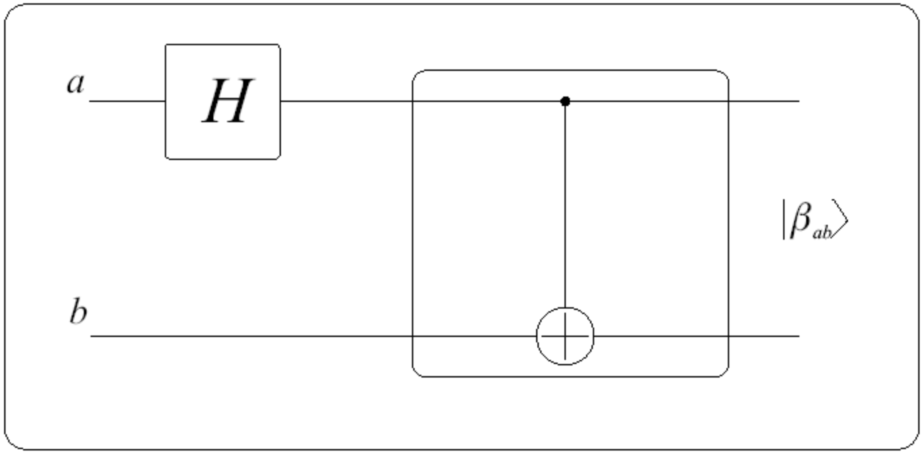
- Since both $(H \otimes I)$ and $CNOT$ gates are Hermitian operators Bob has to implement these gates in the reverse order to build the decoder.

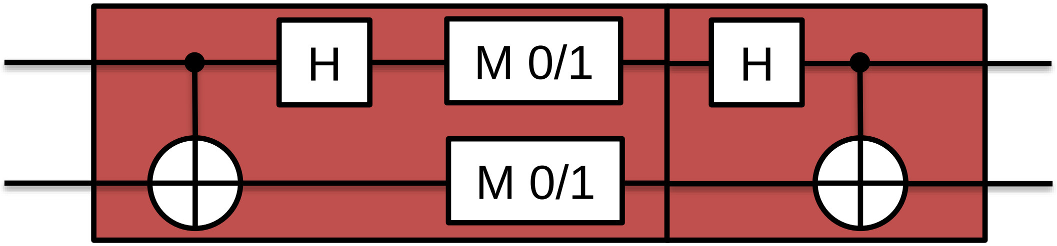
$$((H \otimes I)CNOT)^{-1} = ((H \otimes I)CNOT)^{\dagger} = CNOT^{\dagger}(H \otimes I)^{\dagger}$$

Exercise 4.1. Check whether the $CNOT(H \otimes I)$ gate really returns the wanted classical states!

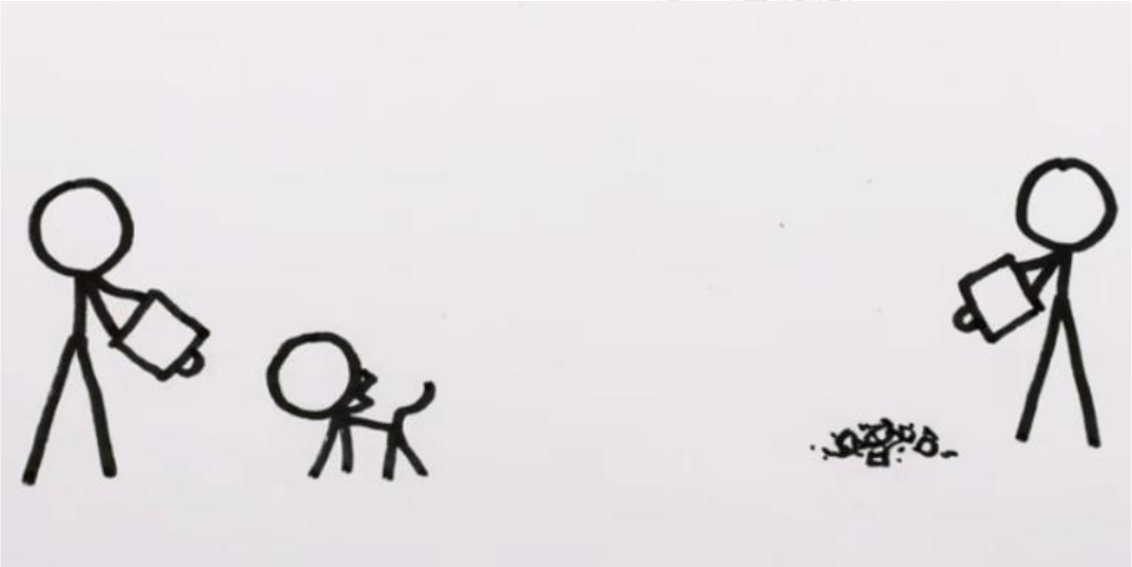
PROTOCOL STEPS (5)

$$|\varphi'\rangle = \frac{M_m|\varphi\rangle}{\sqrt{\langle\varphi|M_m^\dagger M_m|\varphi\rangle}}$$





Quantum teleportation



Avery Thompson: How Quantum Teleportation Actually Works, 2017.

TELEPORTATION

- There is an often repeated scene in most popular science fiction novels and movies.
- The space traveller enters into a cabin on the board of a space ship than he/she suddenly disappears accompanied with colourful lighting effects.
- A few moments later our astronaut appears in another cabin located on a planet hundreds of light-years away from the starting point.
- Let us analyze this futuristic scene scientifically.

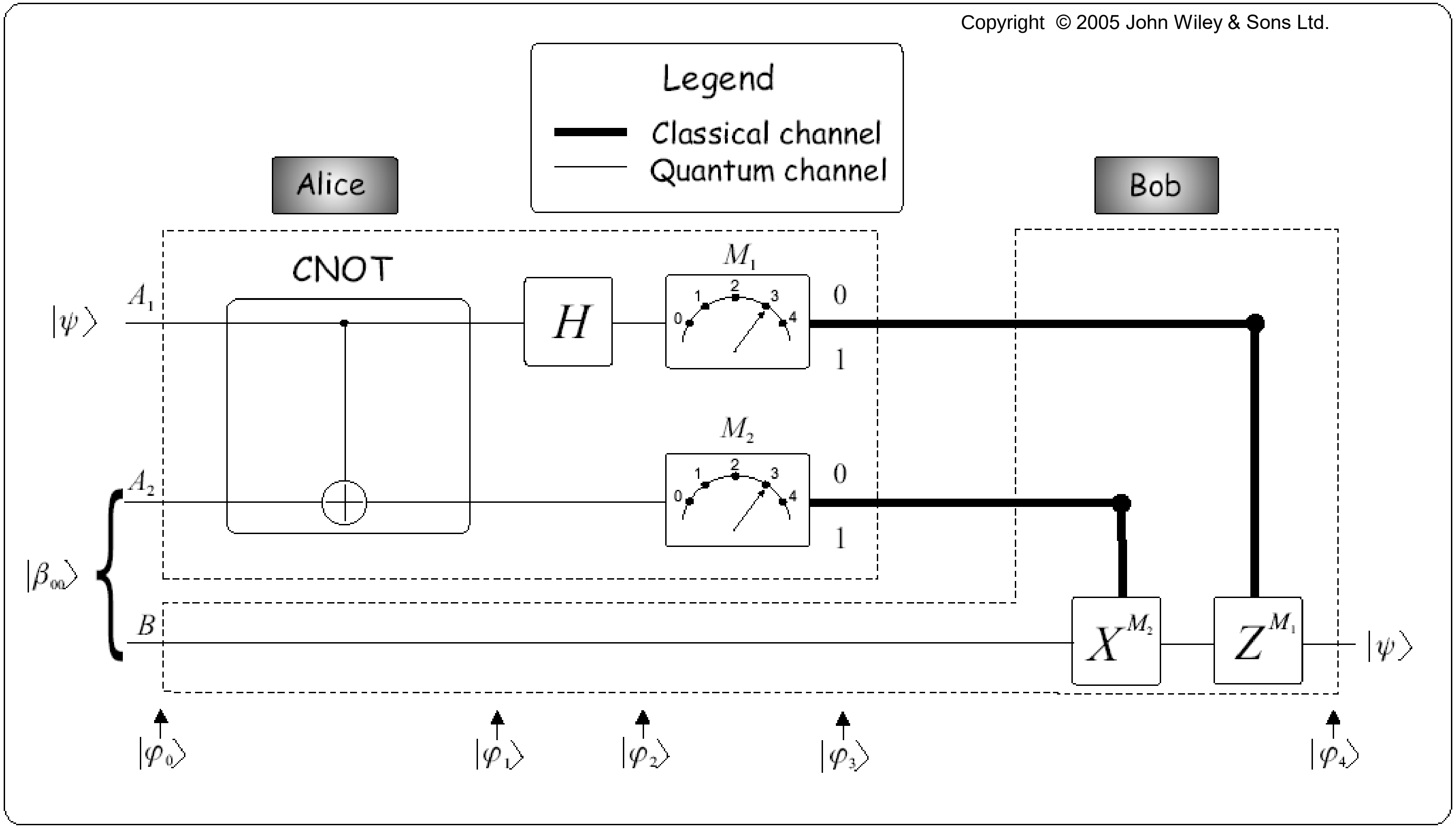
QUANTUM TELEPORTATION

C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, W. K. Wootters

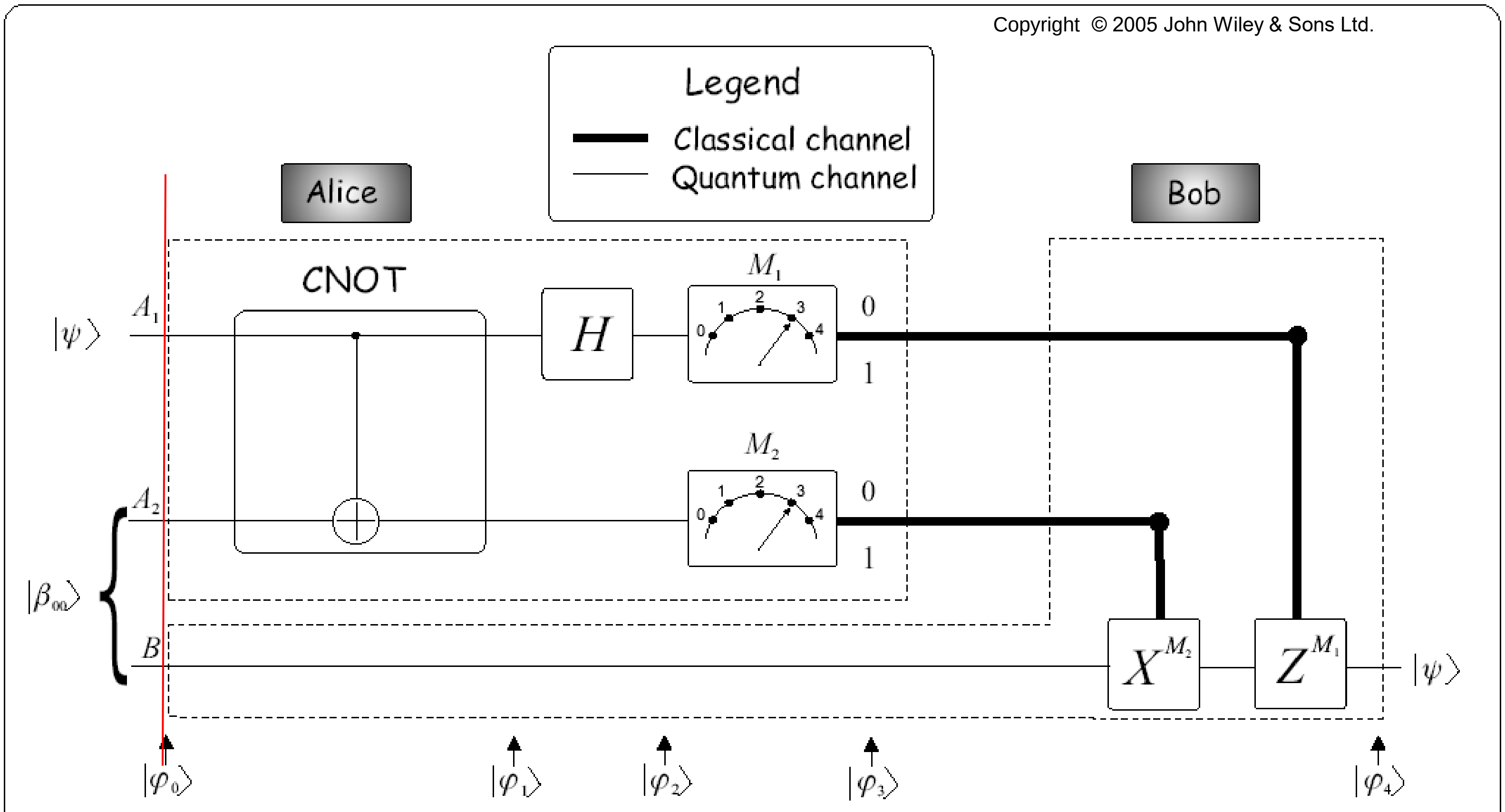
Teleporting an Unknown Quantum State via Dual Classical and Einstein-Podolsky-Rosen Channels,

Phys. Rev. Lett. 70, 1895-1899 (1993)

CIRCUIT OF TELEPORTATION

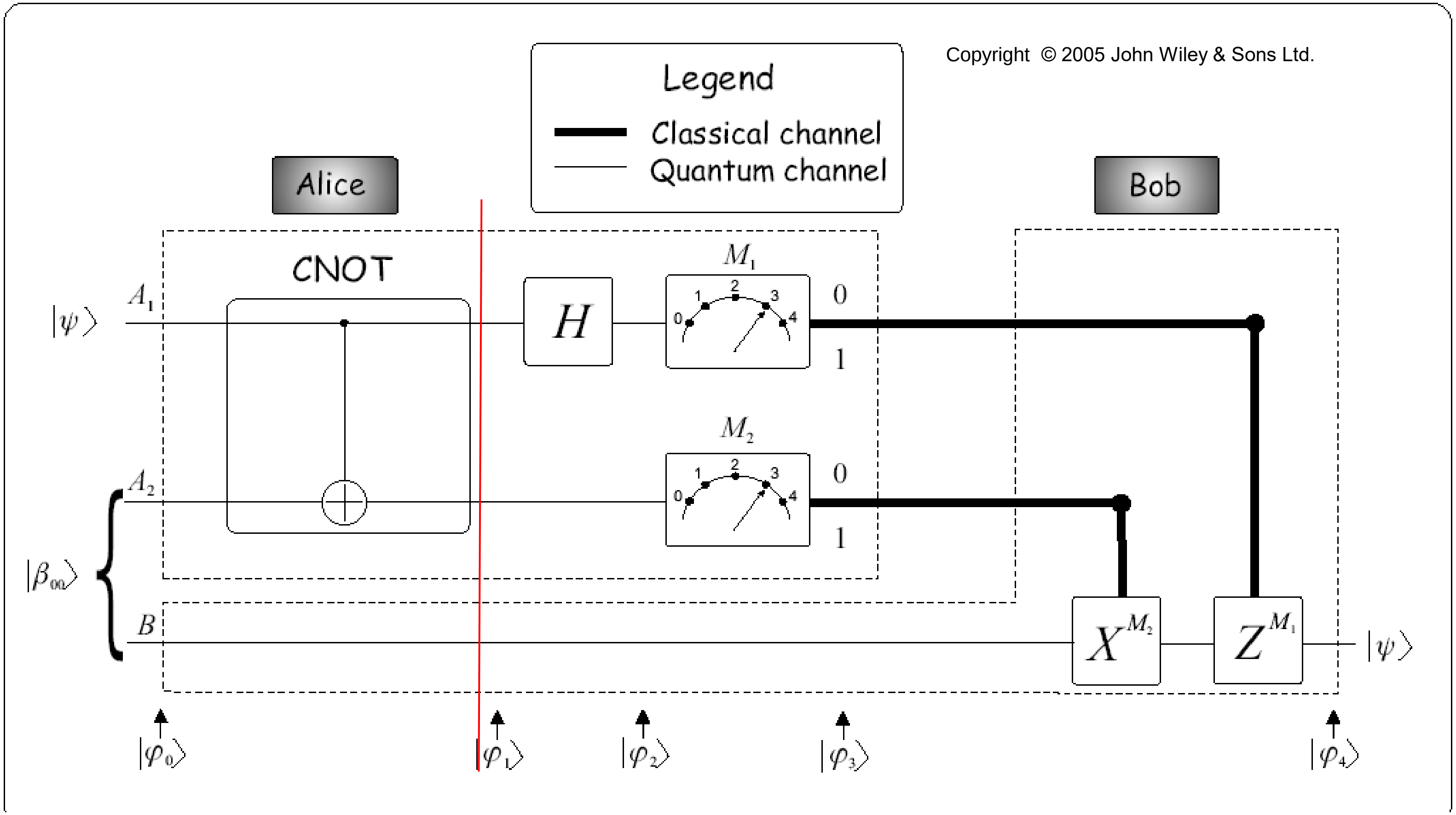


INITIALIZATION



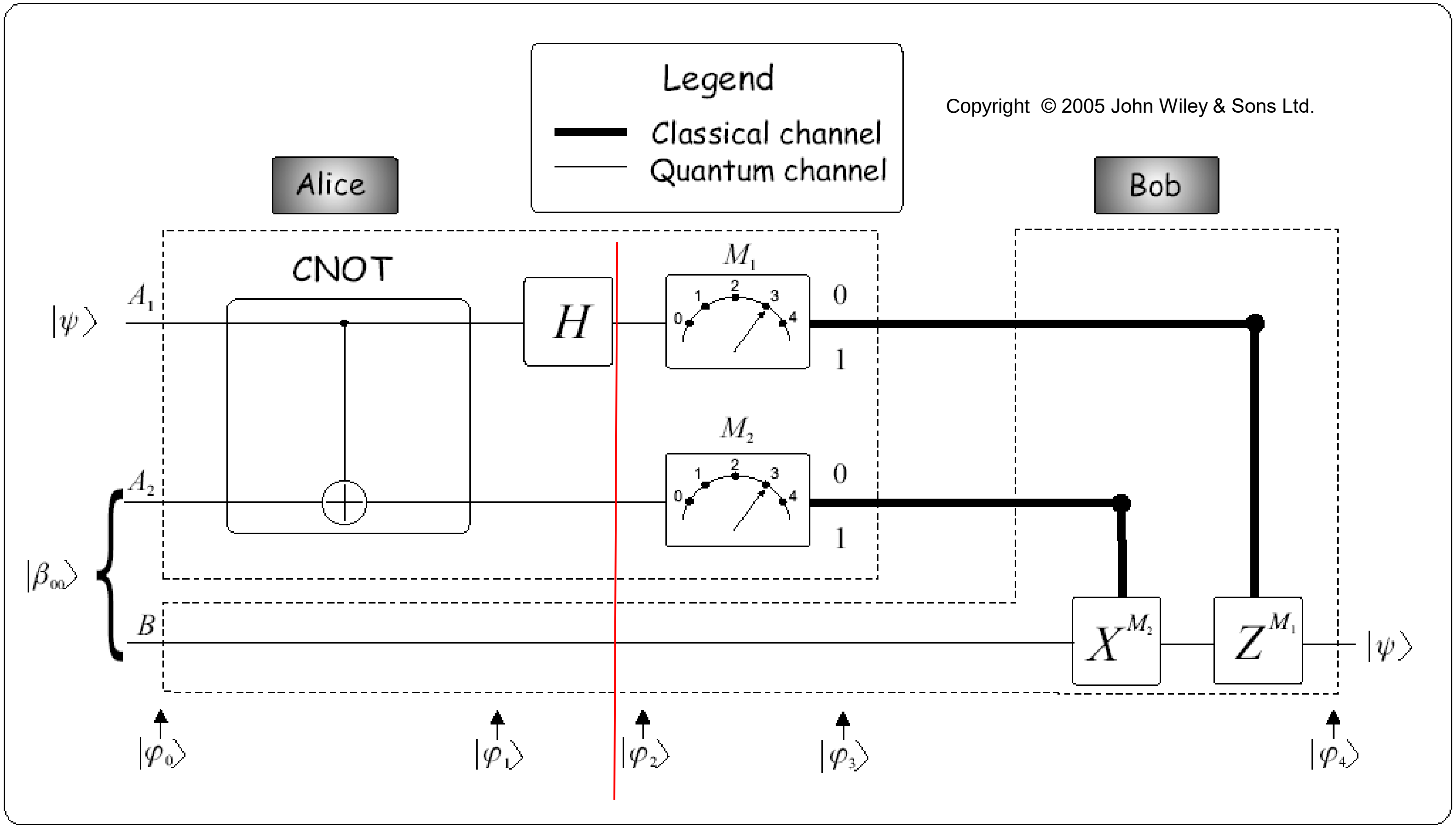
$$|\varphi_0\rangle = |\psi\rangle|\beta_{00}\rangle = \frac{1}{\sqrt{2}}\left[a|\overset{A_1}{0}\rangle\left(|\overset{A_2B}{0\,0}\rangle + |\overset{A_2B}{1\,1}\rangle\right) + b|\overset{A_1}{1}\rangle\left(|\overset{A_2B}{0\,0}\rangle + |\overset{A_2B}{1\,1}\rangle\right)\right]$$

ENTANGLEMENT



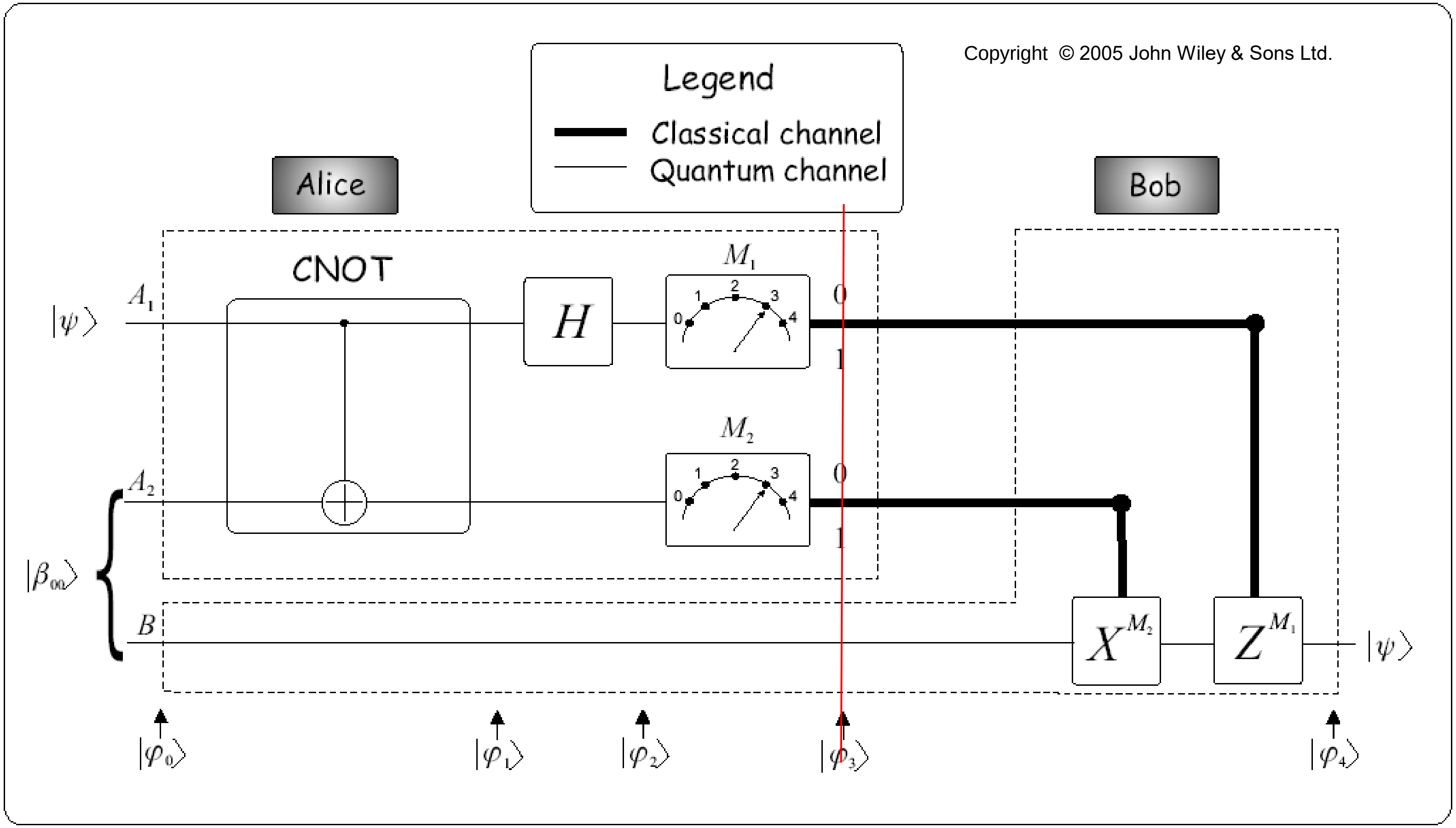
$$|\varphi_1\rangle = \frac{1}{\sqrt{2}}\left[a|\overset{A_1}{0}\rangle\left(|\overset{A_2}{0}\overset{B}{0}\rangle + |\overset{A_2}{1}\overset{B}{1}\rangle\right) + b|\overset{A_1}{1}\rangle\left(|\overset{\mathbf{A_2}}{1}\overset{B}{0}\rangle + |\overset{\mathbf{A_2}}{0}\overset{B}{1}\rangle\right)\right]$$

ENTANGLEMENT



$$|\varphi_2\rangle = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\left[a\left(|\overset{\mathbf{A_1}}{0}\rangle + |\overset{\mathbf{A_1}}{1}\rangle\right)\left(|\overset{A_2 B}{0\,0}\rangle + |\overset{A_2 B}{1\,1}\rangle\right) + b\left(|\overset{\mathbf{A_1}}{0}\rangle - |\overset{\mathbf{A_1}}{1}\rangle\right)\left(|\overset{A_2 B}{1\,0}\rangle + |\overset{A_2 B}{0\,1}\rangle\right)\right]$$

MEASUREMENT



HOW TO REASSEMBLY?

$$
\begin{aligned}
|\varphi_2\rangle &= \frac{1}{2}\Big[|\overset{A_1}{0}\overset{A_2}{0}\rangle\left(a|\overset{B}{0}\rangle + b|\overset{B}{1}\rangle\right) + |\overset{A_1}{0}\overset{A_2}{1}\rangle\left(a|\overset{B}{1}\rangle + b|\overset{B}{0}\rangle\right) \\
&+ |\overset{A_1}{1}\overset{A_2}{0}\rangle\left(a|\overset{B}{0}\rangle - b|\overset{B}{1}\rangle\right) + |\overset{A_1}{1}\overset{A_2}{1}\rangle\left(a|\overset{B}{1}\rangle - b|\overset{B}{0}\rangle\right)\Big].
\end{aligned}
$$

$$
\begin{array}{ccccc}
A_1A_2 & \rightarrow & B & = & U|\psi\rangle \\
00 & \rightarrow & \frac{a|0\rangle+b|1\rangle}{2} & = & I|\psi\rangle \\
01 & \rightarrow & \frac{a|1\rangle+b|0\rangle}{2} & = & X|\psi\rangle \\
10 & \rightarrow & \frac{a|0\rangle-b|1\rangle}{2} & = & Z|\psi\rangle \\
11 & \rightarrow & \frac{a|1\rangle-b|0\rangle}{2} & = & ZX|\psi\rangle
\end{array}
$$

REMARKS

- Alice needs no information about $|\psi\rangle$ to teleport it.
- Without Alice's classically transferred bit pair Bob is not able to produce $|\psi\rangle$ thus no 'faster than light' communication is possible in this way, which is in full harmony with the relativity theory.
- In order to encode and transfer a and b i.e. $|\psi\rangle$ classically Alice may require very large amount of classical bits let alone the measurement problem how to gain them. Contrary teleportation needs only two classical and two quantum bits altogether.

Exercise 4.2. Using teleportation Bob obtains a replica of an arbitrary one-qbit state having in Alice's hand. Explain why quantum teleportation can not be used in this way as a cloning machine!

ACKNOWLEDGEMENTS

The development of this course material has received funding from the European Union under grant agreement No 101081247 (QCIHungary project) and has been implemented with the support provided by the Ministry of Culture and Innovation of Hungary from the National Research, Development and Innovation Fund.

Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or the European Commission. Neither the European Union nor the granting authority can be held responsible for them.

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