DEPARTMENT OF
NETWORKED SYSTEMS
AND SERVICES

Projective measurement

Quantum Computing and its Applications
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MŰEGYETEM 1782

From the previous lecture

POSTULATES OF QUANTUM MECHANICS FROM ENGINEERING POINT OF VIEW

- 1th postulate: quantum bit
  - Vector in Hilbert space
- 2th postulate : logic gates
  - Unitary transform
  - Elementary logic gates
- 3rd postulate : Q/C conversion
  - Measurement statistics
  - Post measurement state
- 4th postulate : registers
  - Tensor product

$$|\varphi\rangle = \sum_{i=0}^{2^n-1} \varphi_i |i\rangle$$

$$U^\dagger \equiv U^{-1}$$

$$P(m \mid |\varphi\rangle) = \langle\varphi|M_m^\dagger M_m|\varphi\rangle$$

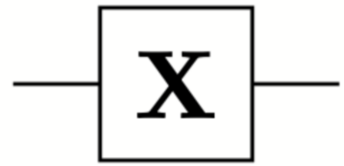
$$|\varphi'\rangle = \frac{M_m|\varphi\rangle}{\sqrt{\langle\varphi|M_m^\dagger M_m|\varphi\rangle}}$$

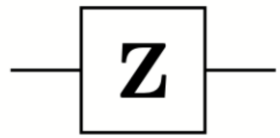
$$|\varphi\rangle = |0\rangle \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

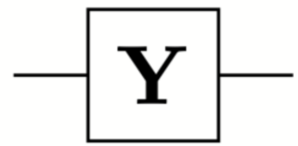
- Dirac's 'ket' and 'bra' notations:
  - Ket: $| \quad \rangle$
  - Bra: $\langle \quad |$
- Connection: $\langle\varphi| = |\varphi\rangle^\dagger$
- What does adjungate (dagger operator) mean?
  - $|\varphi\rangle^\dagger = (|\varphi\rangle^*)^\intercal$

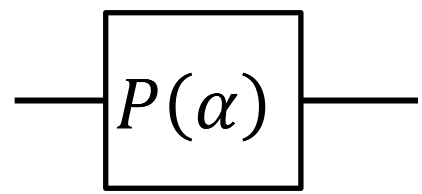
ELEMENTARY GATES

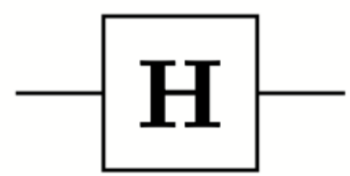
- $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
- $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
- $Y = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$
- $P(\alpha) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{bmatrix}$
- $H = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$











PRODUCT STATES, ENTANGLED STATES

- Product states

$$|\psi\rangle^{\otimes 2} = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|01\rangle$$

- Entangled states

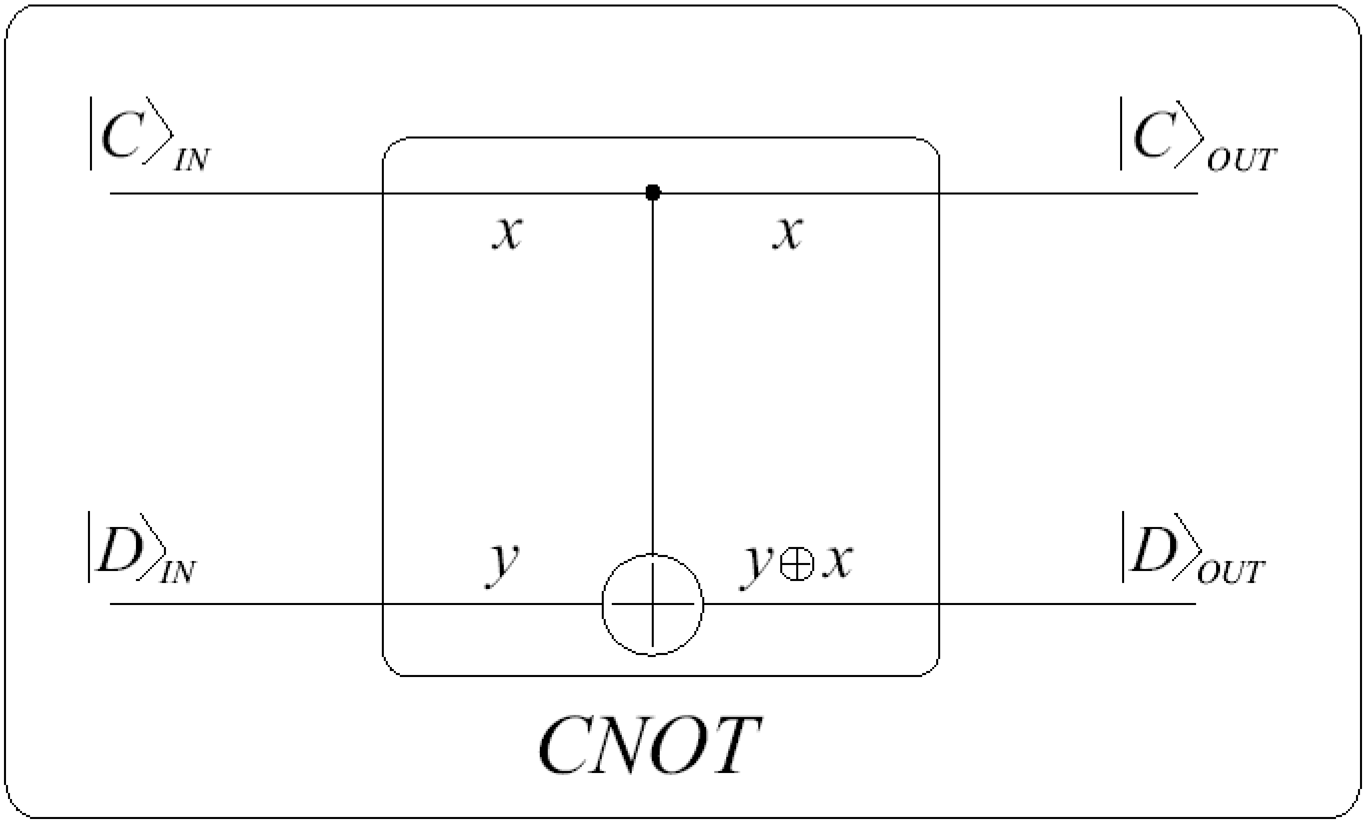
$$|\psi\rangle^{\otimes 2} = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

PRODUCT STATES, ENTANGLED STATES

- Product states:
  - it can be decomposited as a product of two or more qubits

- Entangled states:
  - it cannot be decomposited as product of two or more qubits

CNOT GATE



Upper wire: control

Lower wire: data

BELL STATES

$$|\beta_{00}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}},$$

$$|\beta_{01}\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}},$$

$$|\beta_{10}\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}},$$

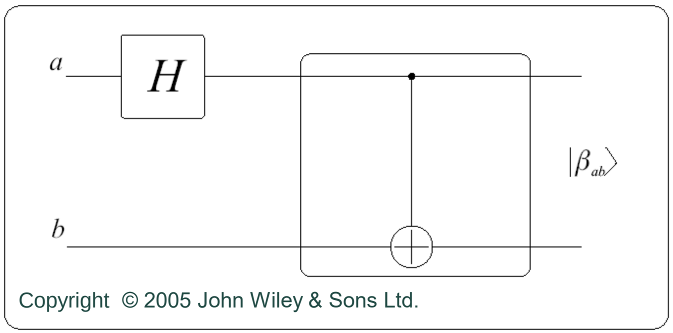
$$|\beta_{11}\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}.$$

The Bell states are orthogonal

BELL STATES

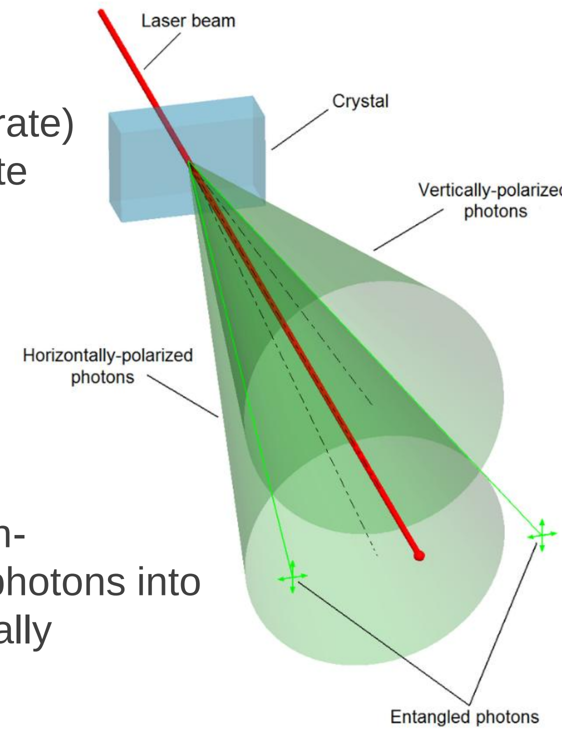
$$|\beta_{ab}\rangle = \frac{|0, b\rangle + (-1)^a |1, NOT(b)\rangle}{\sqrt{2}}$$

$$a, b \in \{0, 1\}$$



IN		OUT	
x	y	x	$y \oplus x$
0	0	0	$0 \oplus 0 = 0$
0	1	0	$1 \oplus 0 = 1$
1	0	1	$0 \oplus 1 = 1$
1	1	1	$1 \oplus 1 = 0$

(beta-barium borate)
or Lithium niobate



Spontaneous parametric down-conversion process can split photons into type II photon pairs with mutually perpendicular polarization.

APPLICATION OF THE HADAMARD GATE

$$|\varphi\rangle = a|0\rangle + b|1\rangle \qquad \begin{bmatrix} a \\ b \end{bmatrix}$$

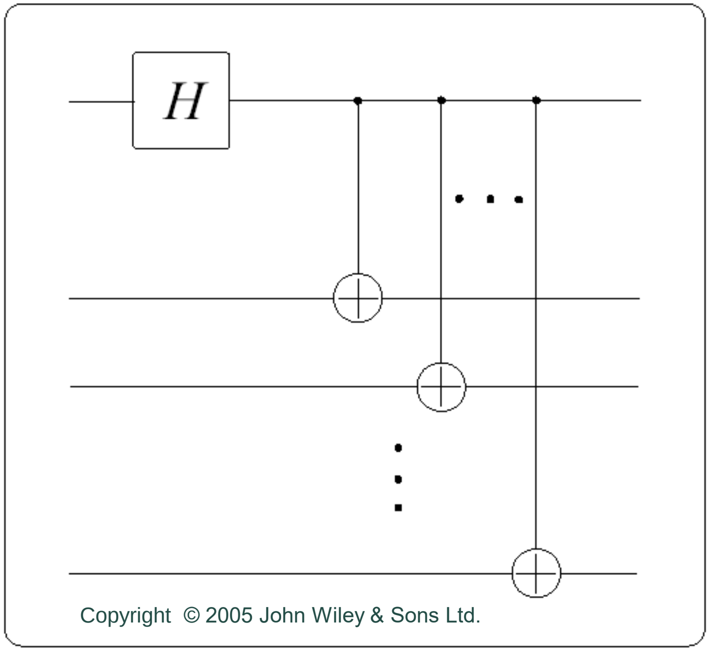
$$|\psi\rangle = H|\varphi\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad = \begin{bmatrix} \frac{a+b}{\sqrt{2}} \\ \frac{a-b}{\sqrt{2}} \end{bmatrix} = \frac{a+b}{\sqrt{2}}|0\rangle + \frac{a-b}{\sqrt{2}}|1\rangle$$

- Hermitian: $H^{\dagger} = H$
- This is why: $HH = I$

- It is worth to know

$$\boxed{\begin{aligned} H|0\rangle &= \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \\ H|1\rangle &= \frac{|0\rangle - |1\rangle}{\sqrt{2}}. \end{aligned}}$$

GENERAL QUANTUM ENTANGLER



- Only one of the entangled qubits is enough to entangle another qubit to the previous set of qubits.
- Entanglement cannot be produced using only classical information/communication

Paradox

TWO MORE BASIS STATES

$$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$$|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

ROTATIONAL INVARIANCE

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle = \frac{1}{\sqrt{2}}|++\rangle + \frac{1}{\sqrt{2}}|--\rangle$$

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle = \frac{1}{\sqrt{2}}|uu\rangle + \frac{1}{\sqrt{2}}\left|u^{orthogonal}u^{orthogonal}\right\rangle$$

Measurement

3RD POSTULATE (MEASUREMENT)

- Any quantum measurement can be described by means of a set of measurement operators $\{M_m\}$, where *m* stands for the possible results of the measurement. The probability of measuring m if the system is in state **v** can be calculated as

$$P(m \mid \mathbf{v}) = \mathbf{v}^\dagger M_m^\dagger M_m \mathbf{v}$$

- and the system after measuring m gets in state

$$\mathbf{v}' = \frac{M_m \mathbf{v}}{\sqrt{\mathbf{v}^\dagger M_m^\dagger M_m \mathbf{v}}}$$

- Because classical probability theory requires that

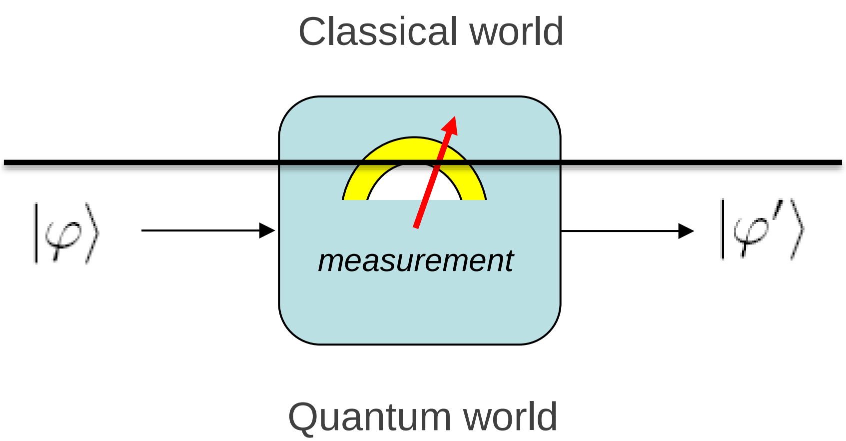
$$\sum_m P(m \mid \mathbf{v}) = \sum_m \mathbf{v}^\dagger M_m^\dagger M_m \mathbf{v} \equiv 1$$

- *Completeness relation:*

$$\boxed{\sum_m M_m^\dagger M_m \equiv I}$$

3RD POSTULATE (MEASUREMENT)

- Projects quantum superpositions to one of its elements with certain probability.
- It gives a classical value back.



$$P(m \mid |\varphi\rangle) = \langle\varphi|M_m^\dagger M_m|\varphi\rangle$$

$$|\varphi'\rangle = \frac{M_m|\varphi\rangle}{\sqrt{\langle\varphi|M_m^\dagger M_m|\varphi\rangle}}$$

MOTTO

"There are two possible outcomes: If the result confirms the hypothesis, then you’ve made a measurement. If the result is contrary to the hypothesis, then you’ve made a discovery.“

Enrico Fermi

General Measurements

"*WHY must I treat the measuring device classically??*
What will happen to me if I don't??"
Eugene Wigner

3RD POSTULATE USING KET NOTATIONS

- Measurement statistic

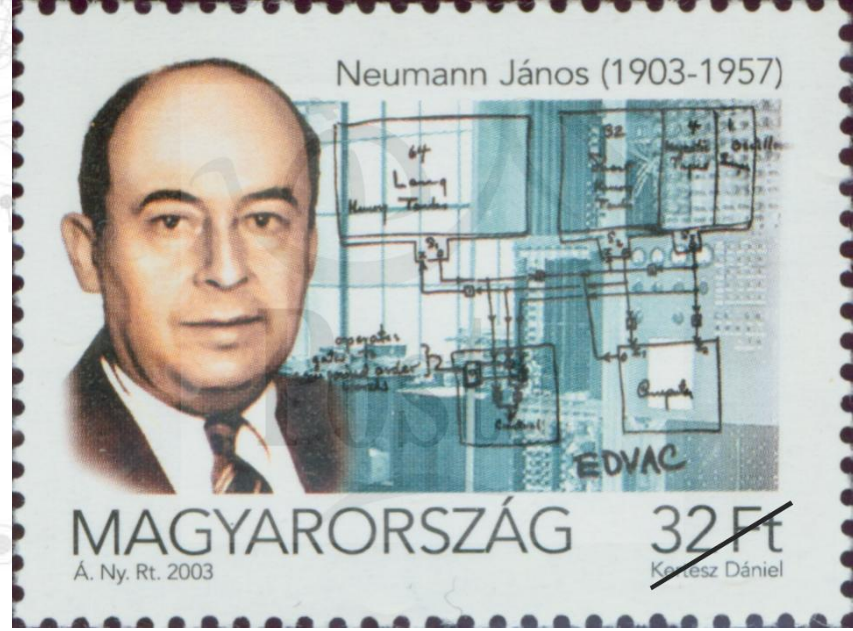
$$P(m \mid |\varphi\rangle) = \langle\varphi|M_m^\dagger M_m|\varphi\rangle$$

- Post measurement state

$$|\varphi'\rangle = \frac{M_m|\varphi\rangle}{\sqrt{\langle\varphi|M_m^\dagger M_m|\varphi\rangle}}$$

- Completeness relation

$$\sum_m M_m^\dagger M_m \equiv I$$



Projective measurement
(Von Neumann measurement)

- Classical computing
- Game theory
- Quantum mechanics

3RD POSTULATE IN CASE OF PROJECTIVE MEASUREMENTS

- Set of two orthogonal states

  $|\varphi_0\rangle = |0\rangle$ or $|\varphi_1\rangle = |1\rangle$

3RD POSTULATE IN CASE OF PROJECTIVE MEASUREMENTS

- Set of two orthogonal states
  $|\varphi_0\rangle = |0\rangle$ or $|\varphi_1\rangle = |1\rangle$
- To find M_0 we need to solve

$$\langle 0|{M_0}^\dagger M_0|0\rangle = 1$$
$$\langle 1|{M_0}^\dagger M_0|1\rangle = 0$$
$$M_0 M_0 = M_0$$

- We are looking for M_0 in the form of

$$\mathbf{M}_0 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

3RD POSTULATE IN CASE OF PROJECTIVE MEASUREMENTS

$$M_0 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

- To find M_0 we need to solve

$$\langle 0|{M_0}^{\dagger} M_0|0\rangle = 1$$

$$\langle 1|{M_0}^{\dagger} M_0|1\rangle = 0$$

$$M_0 M_0 = M_0$$

$$1 = \langle 0|{M_0}^{\dagger} M_0|0\rangle = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{\dagger} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} a^* & c^* \\ b^* & d^* \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Longrightarrow |a|^2 + |c|^2 = 1$$

$$0 = \langle 1|{M_0}^{\dagger} M_0|1\rangle = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{\dagger} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} =$$

$$\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} a^* & c^* \\ b^* & d^* \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Longrightarrow |b|^2 + |d|^2 = 0$$

3RD POSTULATE IN CASE OF PROJECTIVE MEASUREMENTS

$$M_0 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

- To find M_0 we need to solve:

$$\langle 0|{M_0}^\dagger M_0|0\rangle = 1 \rightarrow |a|^2 + |c|^2 = 1$$

$$\langle 1|{M_0}^\dagger M_0|1\rangle = 0 \rightarrow |b|^2 + |d|^2 = 0$$

$$M_0 M_0 = M_0$$

$$\left.\begin{array}{l} |b|^2 + |d|^2 = 0 \\ |b|^2 \geq 0 \\ |d|^2 \geq 0 \end{array}\right\} b = d = 0$$

$$M_0 = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix}$$

$$\begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix}\begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix}$$

$$a^2 = a$$

$$ac = c$$

3RD POSTULATE IN CASE OF PROJECTIVE MEASUREMENTS

$$M_0 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

- To find M_0 we need to solve:

$$\langle 0|{M_0}^{\dagger} M_0|0\rangle = 1 \quad\rightarrow\quad |a|^2 + |c|^2 = 1$$

$$\langle 1|{M_0}^{\dagger} M_0|1\rangle = 0 \quad\rightarrow\quad |b|^2 + |d|^2 = 0$$

$$M_0 M_0 = M_0$$

$$\left.\begin{matrix} |b|^2 + |d|^2 = 0 \\ |b|^2 \geq 0 \\ |d|^2 \geq 0 \end{matrix}\right\} \; b = d = 0$$

$$M_0 = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix}$$

$$\begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix}\begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} \quad\rightarrow\quad a^2 = a, \quad ac = c$$

- The solution:

$$M_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

- Similarly:

$$M_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

3RD POSTULATE IN CASE OF PROJECTIVE MEASUREMENTS

Checking the Completeness relation

$$\sum_m \mathbf{M}_m^\dagger \mathbf{M}_m = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$$

Practical notation

$$M_0 = |0\rangle\langle 0| \text{ and } M_1 = |1\rangle\langle 1|$$

Conclusion

Thus we reached a very simple and practical rule of thumb: *In case we have a set of orthonormal states* $\{|\varphi_m\rangle\}$ *then the corresponding measurement operators which provide exact differentiation among them can be produced by* $M_m = |\varphi_m\rangle\langle\varphi_m|$.

3RD POSTULATE IN CASE OF PROJECTIVE MEASUREMENTS

M_m long to a special set of operators called projectors

$\rightarrow P_m$

Properties

1. Obviously they are self-adjoint operators $P_m^\dagger \equiv P_m$ since $(|\varphi_m\rangle\langle\varphi_m|)^\dagger = \langle\varphi_m|^\dagger|\varphi_m\rangle^\dagger = |\varphi_m\rangle\langle\varphi_m|$.
2. Furthermore $P_m P_m = |\varphi_m\rangle \underbrace{\langle\varphi_m||\varphi_m\rangle}_{\equiv 1} \langle\varphi_m| = P_m$.
3. Finally they are orthogonal which means $P_m P_n = |\varphi_m\rangle \underbrace{\langle\varphi_m||\varphi_n\rangle}_{\equiv 1 \text{or} 0} \langle\varphi_n| = \delta(m-n)P_m$.

3RD POSTULATE IN CASE OF PROJECTIVE MEASUREMENTS

- 3rd Postulate with projectors

$$P(m \mid |\varphi\rangle) = \langle\varphi|P_m|\varphi\rangle$$

$$|\varphi'\rangle = \frac{P_m|\varphi\rangle}{\sqrt{\langle\varphi|P_m|\varphi\rangle}}$$

$$\sum_m P_m \equiv I$$

Exercise 3.1. Construct the measurement operators providing sure success in case of the following set $|\varphi_0\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$ and $|\varphi_1\rangle = \frac{|0\rangle-|1\rangle}{\sqrt{2}}$!

MEASUREMENT USING THE COMPUTATIONAL BASIS STATES (1)

- Let us check what we have learned by means of a simple example

$$|\varphi\rangle = a|0\rangle + b|1\rangle$$

- Basis vectors $|0\rangle$ and $|1\rangle$

$$\begin{bmatrix} a \\ b \end{bmatrix}$$

$$P(0 \mid |\varphi\rangle) = \langle\varphi|P_0|\varphi\rangle = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} a \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a^* & b^* \end{bmatrix} \quad |a|^2$$

- Measurement statistic

$$\begin{bmatrix} a \\ b \end{bmatrix}$$

$$P(1 \,|\, |\varphi\rangle) = \langle\varphi|P_1|\varphi\rangle = \begin{bmatrix} a^* & b^* \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = |b|^2$$

MEASUREMENT USING THE COMPUTATIONAL BASIS STATES (2)

Post measurements states

$$|\varphi_0'\rangle = \frac{P_0|\varphi\rangle}{\sqrt{P(0||\varphi\rangle)}} = \frac{a|0\rangle}{|a|}$$

$$|\varphi_1'\rangle = \frac{P_1|\varphi\rangle}{\sqrt{P(1||\varphi\rangle)}} = \frac{b|1\rangle}{|b|}$$

Remark:

Orthogonal states can always be distinguished via constructing appropriate measurement operators (projectors). This is another explanation why orthogonal (classical) states can be copied because in possession of the exact information about such states we can build a quantum circuit producing them.

REPEATED PROJECTIVE MEASUREMENT

What happens when we repeat a projective measurement on the same qureqister?

- Post measurement state after the first measurement

$$|\varphi_m\rangle = \frac{P_m|\varphi\rangle}{\sqrt{\langle\varphi|P_m|\varphi\rangle}}$$

- Post measurement state after the second measurement

$$P_m = |\varphi_m\rangle\langle\varphi_m|$$

$$\downarrow$$

REPEATED PROJECTIVE MEASUREMENT

What happens when we repeat a projective measurement on the same qureqister?

- Post measurement state after the first measurement

$$|\varphi_m\rangle = \frac{P_m|\varphi\rangle}{\sqrt{\langle\varphi|P_m|\varphi\rangle}}$$

- Post measurement state after the second measurement

$$P_m = |\varphi_m\rangle\langle\varphi_m|$$

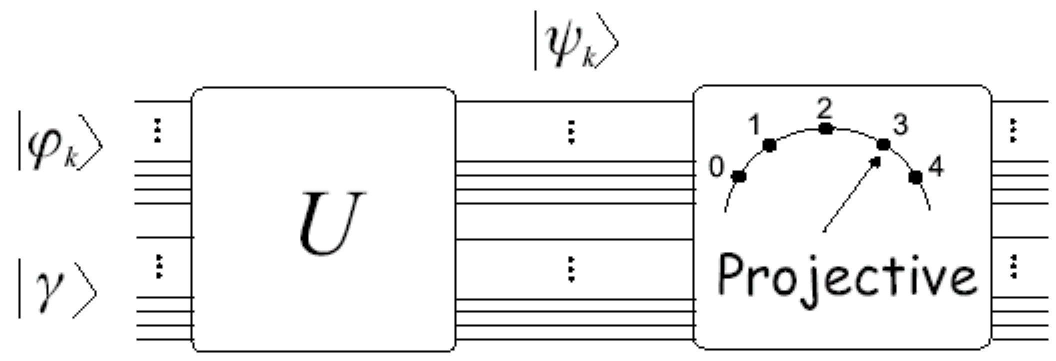
$$|\varphi_m\rangle' = \frac{P_m|\varphi_m\rangle}{\sqrt{\langle\varphi_m|P_m|\varphi_m\rangle}} \Downarrow = \frac{|\varphi_m\rangle\langle\varphi_m||\varphi_m\rangle}{\sqrt{\langle\varphi_m||\varphi_m\rangle\langle\varphi_m||\varphi_m\rangle}} = |\varphi_m\rangle$$

Is projective measurement enough?

NEUMARK'S EXTENSION

Any generalized measurement can be implemented by means of a projective measurement + auxiliary qubits + unitary transform.

NEUMARK'S EXTENSION



DISCLAIMER

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