Quantum Computing and its Application (BMEVHIAD00)

Programming Quantum Computers 4 - Simple Quantum algorithms

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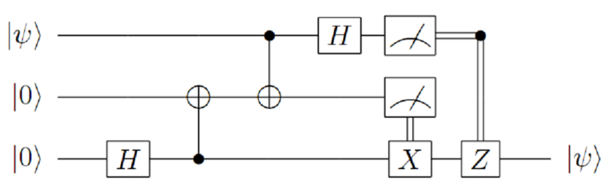
1 Content

Once again we will be solving problems with the `qiskit`. The calculations can be done on paper, but if anyone has a computer, feel free to follow the steps.

2 Teleportation

Let's implement the teleportation protocol in `qiskit`. As a reminder the circuit corresponding to it is:



If we would like to control a quantum gate with a classical bit (a measurement result), we can do this in `qiskit` as:

`qc.x(index_of_qubit).c_if(index_of_classical_bit, value_for_which_it_should_be_used)`

The teleportation protocol in `qiskit`:

[2]:
```
from qiskit import ClassicalRegister, QuantumRegister, QuantumCircuit
from qiskit.quantum_info import random_statevector
from qiskit_aer import Aer

crx = ClassicalRegister(1, 'crx')
crz = ClassicalRegister(1, 'crz')

backend = Aer.get_backend("aer_simulator")

qbits = QuantumRegister(3, 'q')
qc = QuantumCircuit(qbits, crz, crx)
```

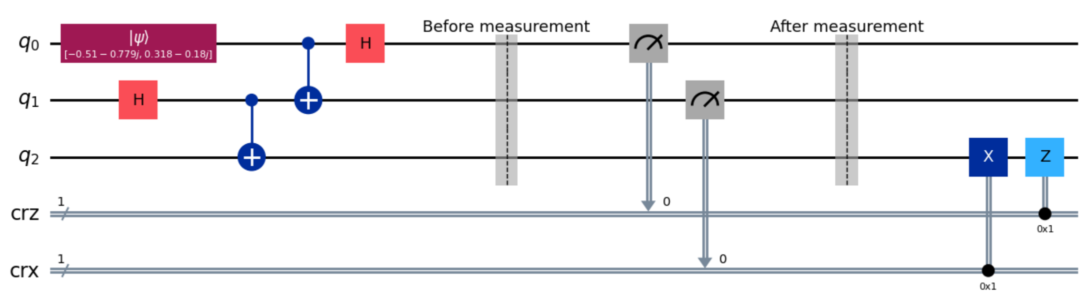
```
svector = random_statevector(2)
qc.initialize(svector, 0)

qc.h(1)
qc.cx(1,2)
qc.cx(0,1)
qc.h(0)
qc.save_statevector(label="Before measurement")
qc.measure(0, crz)
qc.measure(1, crx)
qc.save_statevector(label="After measurement")

qc.x(2).c_if(crx, 1)
qc.z(2).c_if(crz, 1)
```

```
/tmp/ipykernel_3636/3679911133.py:25: DeprecationWarning: The method
``qiskit.circuit.instructionset.InstructionSet.c_if()`` is deprecated as of
qiskit 1.3.0. It will be removed in 2.0.0.
  qc.x(2).c_if(crx, 1)
/tmp/ipykernel_3636/3679911133.py:26: DeprecationWarning: The method
``qiskit.circuit.instructionset.InstructionSet.c_if()`` is deprecated as of
qiskit 1.3.0. It will be removed in 2.0.0.
  qc.z(2).c_if(crz, 1)
```

[2]: `<qiskit.circuit.instructionset.InstructionSet at 0x7ffb8164ef20>`

[3]:
```
qc.draw("mpl")
```

[3]:



[4]:
```
def get_statevector_from_result(result, label: str):
    result_dict = result.to_dict()
    data = result_dict["results"][0]["data"]
    if label in data.keys():
        return data[label]
    return None
```

[5]:
```
sim = Aer.get_backend("statevector_simulator")
result = sim.run(qc).result()
```

[6]:
```
print(get_statevector_from_result(result, "Before measurement"))
print(get_statevector_from_result(result, "After measurement"))
print(get_statevector_from_result(result, "statevector"))
```

```
Statevector([-0.25478892-0.38953266j, -0.25478892-0.38953266j,
              0.15899575-0.08981796j, -0.15899575+0.08981796j,
              0.15899575-0.08981796j, -0.15899575+0.08981796j,
             -0.25478892-0.38953266j, -0.25478892-0.38953266j],
            dims=(2, 2, 2))
Statevector([ 0.        +0.j        ,  0.        -0.j        ,
              0.        +0.j        , -0.3179915 +0.17963591j,
              0.        +0.j        , -0.        +0.j        ,
              0.        +0.j        , -0.50957785-0.77906531j],
            dims=(2, 2, 2))
Statevector([ 0.        +0.j        , -0.        +0.j        ,
              0.        +0.j        , -0.50957785-0.77906531j,
             -0.        +0.j        ,  0.        +0.j        ,
             -0.        +0.j        ,  0.3179915 -0.17963591j],
            dims=(2, 2, 2))
```

Or in an other way:

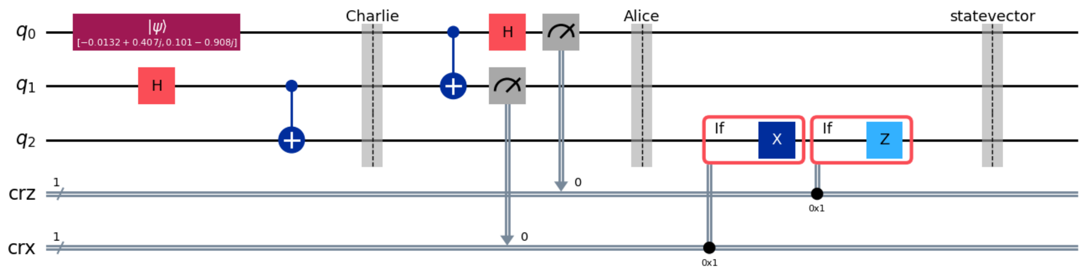
[7]:
```
crx = ClassicalRegister(1, 'crx')
crz = ClassicalRegister(1, 'crz')

backend = Aer.get_backend("aer_simulator")

qbits = QuantumRegister(3, 'q')
qc = QuantumCircuit(qbits, crz, crx)

svector = random_statevector(2)
qc.initialize(svector, 0)

qc.h(1)
qc.cx(1,2)
qc.barrier(label="Charlie")

qc.cx(0,1)
qc.h(0)
qc.measure(0, crz)
qc.measure(1, crx)
qc.barrier(label="Alice")

with qc.if_test((crx, 1)):
    qc.x(2)
```

```
with qc.if_test((crz, 1)):
    qc.z(2)

qc.save_statevector()
```

[7]: <qiskit.circuit.instructionset.InstructionSet at 0x7ffb3c6cc880>

[8]:
```
qc.draw("mpl")
```

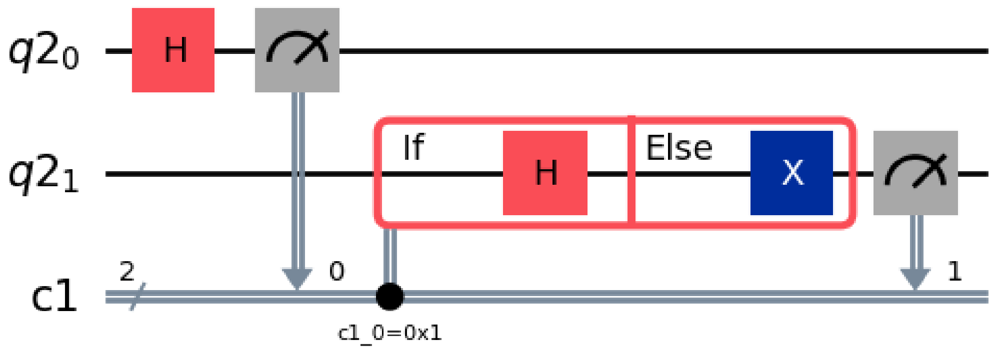
[8]:



Classical control structures:

[10]:
```
# Of course, qiskit has the if-else as well
qubits = QuantumRegister(2)
clbits = ClassicalRegister(2)
circuit = QuantumCircuit(qubits, clbits)
(q0, q1) = qubits
(c0, c1) = clbits

circuit.h(q0)
circuit.measure(q0, c0)
with circuit.if_test((c0, 1)) as else_:
    circuit.h(q1)
with else_:
    circuit.x(q1)
circuit.measure(q1, c1)

circuit.draw("mpl")
```

[10]:



[11]:
```python
# We can also use a for and a while loop:
qubits = QuantumRegister(1)
clbits = ClassicalRegister(1)
circuit = QuantumCircuit(qubits, clbits)
(q0,) = qubits
(c0,) = clbits

with circuit.for_loop(range(5)) as _:
    circuit.x(q0)
circuit.measure(q0, c0)

circuit.draw("mpl")
```

[11]:

3 Measurement in a given basis

During projective measurement, we've learned that we can make a measurement in any orthogonal basis.

For example: We can make projections corresponding to the $|0\rangle$, $|1\rangle$ states: $P_0 = |0\rangle\langle 0|$, $P_1 = |1\rangle\langle 1|$, which statisfy the completeness relation $P_0 + P_1 = I$

3.1 $|0\rangle$ in another basis

What are the measurement probabilites when perform a projective measurement of $|0\rangle$ in the $|+\rangle, |-\rangle$ basis?

Hint: We can of course calculate the necessary projections and get the probabilites using the Born-rule, but try to find a different method.

Solution:

As we know, $|+\rangle = H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|-\rangle = H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.

From this we can conculude that $|0\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle$

Then the probabilities are: * $|+|0|^2 = 1/2$ for measuring $|+\rangle$ * $|-|0|^2 = 1/2$ for measuring $|-\rangle$

3.2 Measurement in a different basis

What are the measurement probabilites for $0.6|0\rangle + 0.8|1\rangle$ in the $|i\rangle, |-i\rangle$ basis?

Where $|i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$ and $|-i\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$.

Hint: Now use the steps learned for the projective measurements!

Solution:

Let's calculate the two projections P_i -t és P_{-i} -t :

$$P_i = ii = \frac{1}{2}\begin{bmatrix}1\\ i\end{bmatrix}\begin{bmatrix}1 & -i\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & -i\\ i & 1\end{bmatrix},$$

$$P_{-i} = -i-i = \frac{1}{2}\begin{bmatrix}1\\ -i\end{bmatrix}\begin{bmatrix}1 & i\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & i\\ -i & 1\end{bmatrix}.$$

After this the probabilites are:

$$\begin{aligned}P(i|\psi\rangle) = \langle\psi|P_{+i}\psi &= \begin{bmatrix}0.6 & 0.8\end{bmatrix} \times \frac{1}{2}\begin{bmatrix}1 & -i\\ i & 1\end{bmatrix} \times \begin{bmatrix}0.6\\ 0.8\end{bmatrix}\\ &= \begin{bmatrix}0.6 & 0.8\end{bmatrix} \times \frac{1}{2}\begin{bmatrix}0.6 - 0.8i\\ 0.8 + 0.6i\end{bmatrix} = \frac{1}{2}\end{aligned}$$

$$\begin{aligned}P(i|\psi\rangle) = \langle\psi|P_{-i}\psi &= \begin{bmatrix}0.6 & 0.8\end{bmatrix} \times \frac{1}{2}\begin{bmatrix}1 & i\\ -i & 1\end{bmatrix} \times \begin{bmatrix}0.6\\ 0.8\end{bmatrix}\\ &= \begin{bmatrix}0.6 & 0.8\end{bmatrix} \times \frac{1}{2}\begin{bmatrix}0.6 + 0.8i\\ 0.8 - 0.6i\end{bmatrix} = \frac{1}{2}\end{aligned}$$

What happens if we can only measure in the $|0\rangle, |1\rangle$ basis, but the state is currently represented in another?

Formally, we have a state $|\psi\rangle = c_0|b_0\rangle + c_1|b_1\rangle$, where $|b_0\rangle$ and $|b_1\rangle$ orthogonal and $|c_0|^2 + |c_1|^2 = 1$. But we can only measure in the $|b_0\rangle, |b_1\rangle$ basis. How can we still get the same measurement probabilities and corresponding states (as per the measurement postulate)?

We will create a transformation U, which takes $|\psi\rangle$ from one basis to the other (that is from b_0, b_1 to $0, 1$). Then we measure it there and transform it back. How does the matrix of U look like?

We know that is should behave in the following way:

$$\begin{aligned}
U|b_0\rangle &= |0\rangle \\
U|b_1\rangle &= |1\rangle \\
U^\dagger|0\rangle &= |b_0\rangle \\
U^\dagger|1\rangle &= |b_1\rangle
\end{aligned}$$

Putting it together:

$$\begin{aligned}
U &= |0\rangle\langle b_0| + |1\rangle\langle b_1| \\
U^\dagger &= |b_0\rangle\langle 0| + |b_1\rangle\langle 1|
\end{aligned}$$

Why does this work?

Example:

$$U|b_0\rangle = |0\rangle\langle b_0|b_0\rangle + |1\rangle\langle b_1|b_0\rangle = |0\rangle \times 1 + |1\rangle \times 0 = |0\rangle$$

$$U^\dagger|0\rangle = |b_0\rangle\langle 0|0\rangle + |b_1\rangle\langle 1|0\rangle = |b_0\rangle \times 1 + |b_1\rangle \times 0 = |b_0\rangle$$

General case:

$$\begin{aligned}
U(c_0|b_0\rangle + c_1|b_1\rangle) &= |0\rangle\langle b_0|(c_0|b_0\rangle + c_1|b_1\rangle) + |1\rangle\langle b_1|(c_0|b_0\rangle + c_1|b_1\rangle) \\
&= c_0|0\rangle\langle b_0|b_0\rangle + c_1|0\rangle\langle b_0|b_1\rangle + c_0|1\rangle\langle b_1|b_0\rangle + c_1|1\rangle\langle b_1|b_1\rangle \\
&= c_0|0\rangle \times 1 + c_1|0\rangle \times 0 + c_0|1\rangle \times 0 + c_1|1\rangle \times 1 \\
&= c_0|0\rangle + c_1|1\rangle
\end{aligned}$$

You can also check that U is unitary.

In summary:

1. With the help of U we transform our state from a given basis to the $0, 1$ basis.
2. We perform the measurements there, for which the probabilities do not change.
3. Using $U^\dagger$ we transform the state back to the original basis and the state will correspond to a measurement in the original basis.

With these steps we can perform a projective measurement in any basis without calculating the necessary projections!

3.3 Distinguishing orthogonal states

We found a black box with the following note: This box outputs i or $-i$ with equal probability.

Create an algorithm that is able to distinguish the outputs of this box with 100% accuracy.

Reminder:

$$U = |0\rangle\langle b_0| + |1\rangle\langle b_1|$$
$$U^\dagger = |b_0\rangle\langle 0| + |b_1\rangle\langle 1|$$

Solution:

Because i and $-i$ are orthogonal, we can use the subsitution $b_0 = i$ and $b_1 = -i$ to write up U as follows:

$$U = |0\rangle\langle i| + |1\rangle\langle -i| = \begin{bmatrix}1\\0\end{bmatrix} \times \frac{1}{\sqrt{2}}\begin{bmatrix}1 & i\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix} \times \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -i\end{bmatrix}$$
$$= \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -i\\0 & 0\end{bmatrix} + \frac{1}{\sqrt{2}}\begin{bmatrix}0 & 0\\1 & i\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -i\\1 & i\end{bmatrix}$$

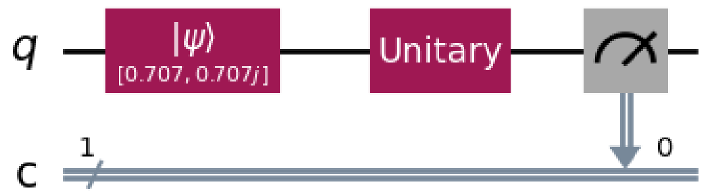
[12]:
```
from qiskit.circuit.library import UnitaryGate
import numpy as np
import random

qc = QuantumCircuit(1, 1)

ket_i = np.array([1, 1j]) * 1/np.sqrt(2)
ket_minus_i = np.array([1, -1j]) * 1/np.sqrt(2)

u_gate_matrix = np.array([[1, -1j], [1, 1j]]) * 1/np.sqrt(2)
u_gate = UnitaryGate(u_gate_matrix)

if random.randint(0,1):
    qc.initialize(ket_i)
else:
    qc.initialize(ket_minus_i)

qc.append(u_gate, [0])
qc.measure(0, 0)
```

[12]:
```
<qiskit.circuit.instructionset.InstructionSet at 0x7ffb3c6cde10>
```

Let's check it:

[13]:
```
qc.draw("mpl")
```

[13]:



[17]:
```
backend = AerSimulator()
result = backend.run(qc, shots=1000).result()
counts = result.get_counts()
print(counts)
```

```
{'0': 1000}
```

Performing the complete measurement Let's add one more step to the previous exercise: After the measurement we would like to have the state i or $-i$, based on the result.

Solution:

We only have to calculate the adjoint of U:

$$U^\dagger = (U^*)^T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$$

Let's extend the circuit:

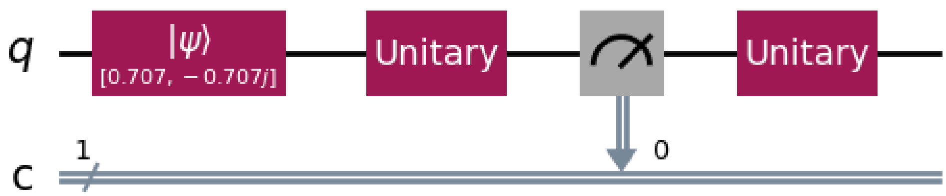
[14]:
```
qc = QuantumCircuit(1,1)

ket_i = np.array([1, 1j]) * 1/np.sqrt(2)
ket_minus_i = np.array([1, -1j]) * 1/np.sqrt(2)

u_gate_matrix = np.array([[1, -1j], [1, 1j]]) * 1/np.sqrt(2)
u_gate = UnitaryGate(u_gate_matrix)

if random.randint(0,1):
    qc.initialize(ket_i)
else:
    qc.initialize(ket_minus_i)

qc.append(u_gate, [0])
qc.measure(0, 0)
qc.append(u_gate.adjoint(), [0])
```

```
[14]: <qiskit.circuit.instructionset.InstructionSet at 0x7ffb3b7e40d0>
```

Check the circuit:

```
[15]: qc.draw("mpl")
```

[15]:




```
[16]: sim = Aer.get_backend("statevector_simulator")
      result = sim.run(qc).result().get_statevector()
      print(result)
```

```
Statevector([0.70710678+0.j        , 0.        -0.70710678j],
            dims=(2,))
```

3.4 Dinstinguishing orthogonal states - Solution 2.

Transform the state 0 to i using the gates from the previous lectures!

Reminder: $i = \frac{1}{\sqrt{2}}(0 + i1)$

The matrix of the S gate is:

$$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$$

The matrix of the T gate is:

$$T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$$

Solution:

First apply the H gate to get $+ = \frac{1}{\sqrt{2}}(0 + 1)$, then use the S gate to get the desired state $i = \frac{1}{\sqrt{2}}(0 + i1)$.
The matrix of the S gate is:

$$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$$

The whole transformation is:

$$SH0 = i$$

We can also check that the adjoint of SH takes 1 -to $-i$.

Furthermore SH is unitary as both S and H are unitary, so it can be reversed and the matrix of the adjoint is:

$$(SH)^\dagger = H^\dagger S^\dagger = HS^\dagger = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}$$

But this is the same matrix we got for U (and also $U^\dagger = SH$).

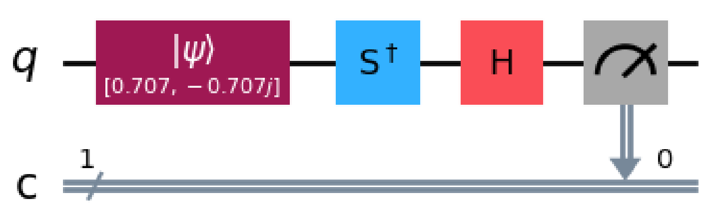
We can simplify the circuit:

[17]:
```
qc = QuantumCircuit(1,1)

ket_i = np.array([1, 1j]) * 1/np.sqrt(2)
ket_minus_i = np.array([1, -1j]) * 1/np.sqrt(2)

if random.randint(0,1):
    qc.initialize(ket_i)
else:
    qc.initialize(ket_minus_i)

qc.sdg(0)
qc.h(0)
qc.measure(0, 0)
```

[17]:
```
<qiskit.circuit.instructionset.InstructionSet at 0x7ffb3b2b0a60>
```

[18]:
```
qc.draw("mpl")
```

[18]:



[19]:
```
sim = Aer.get_backend("aer_simulator")
result = sim.run(qc, shots=1000).result().get_counts()
print(result)
```

```
{'1': 1000}
```

4 Parametrization of circuits in `qiskit`

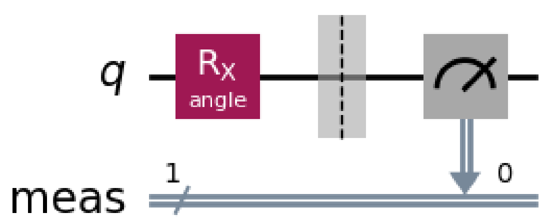
$R_x(\theta)$	$\begin{bmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}$	$R_x(\theta)\lvert\psi\rangle = (\alpha\cos\frac{\theta}{2} - i\beta\sin\frac{\theta}{2})\lvert 0\rangle + (\beta\cos\frac{\theta}{2} - i\alpha\sin\frac{\theta}{2})\lvert 1\rangle$	$R_x(\theta)\lvert 0\rangle = \cos\frac{\theta}{2}\lvert 0\rangle - i\sin\frac{\theta}{2}\lvert 1\rangle$ $R_x(\theta)\lvert 1\rangle = \cos\frac{\theta}{2}\lvert 1\rangle - i\sin\frac{\theta}{2}\lvert 0\rangle$
$R_y(\theta)$	$\begin{bmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}$	$R_y(\theta)\lvert\psi\rangle = (\alpha\cos\frac{\theta}{2} - \beta\sin\frac{\theta}{2})\lvert 0\rangle + (\beta\cos\frac{\theta}{2} + \alpha\sin\frac{\theta}{2})\lvert 1\rangle$	$R_y(\theta)\lvert 0\rangle = \cos\frac{\theta}{2}\lvert 0\rangle + \sin\frac{\theta}{2}\lvert 1\rangle$ $R_y(\theta)\lvert 1\rangle = \cos\frac{\theta}{2}\lvert 1\rangle - \sin\frac{\theta}{2}\lvert 0\rangle$
$R_z(\theta)$	$\begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix}$	$R_z(\theta)\lvert\psi\rangle = \alpha e^{-i\theta/2}\lvert 0\rangle + \beta e^{i\theta/2}\lvert 1\rangle$	$R_z(\theta)\lvert 0\rangle = e^{-i\theta/2}\lvert 0\rangle$ $R_z(\theta)\lvert 1\rangle = e^{i\theta/2}\lvert 1\rangle$

Often, we want to use the same circuit (or algorithm) more than once, but in a slightly different way. For example, the rotation angle used for the R_y gate above changes in different executions of a circuit.

We can solve this problem with `qiskit`'s `Parameter` class.

[20]:
```python
from qiskit.circuit import Parameter

angle = Parameter("angle")  # We do not specify a value upon declaration.

qc = QuantumCircuit(1)
qc.rx(angle, 0)
qc.measure_all()
qc.draw("mpl")
```

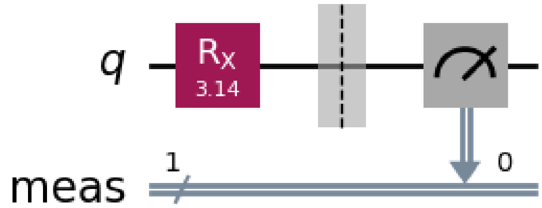
[20]:



A `Parameter` must a value specified during the execution of the circuit. For this we use the `assign_parameters` function.

[21]:
```python
bc = qc.assign_parameters({angle: 3.14})
bc.draw('mpl')
```

[21]:



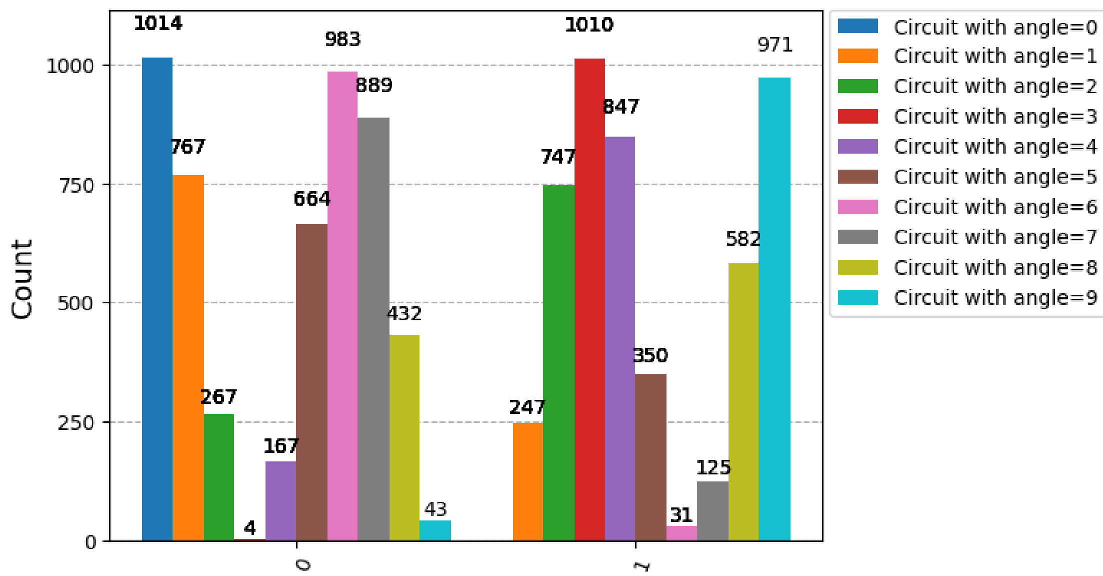
5 Running multiple circuits at the same time

If there are numerous circuits that we would like to simulate together, we can do the following:

[22]:
```
circuits = []
circuit_names = []
for value in range(10):
    circuits.append(qc.assign_parameters({ angle: value }))
    circuit_names.append(f"Circuit with angle={value}")
```

[23]:
```
from qiskit.visualization import plot_histogram
from qiskit import transpile

backend = Aer.get_backend("qasm_simulator")
shots = 1014

job = backend.run([transpile(circ, backend) for circ in circuits], shots=shots)
results = job.result()
plot_histogram(results.get_counts(), legend=circuit_names)
```

[23]:



6 The effect of `transpile`

The `transpile` function was used in the previous lesson. With the help of this function `qiskit` transforms our circuit to the instruction set supported by the backend. What does this circuit look like?

6.1 Small detour: What kind of gates are we really using under the hood?

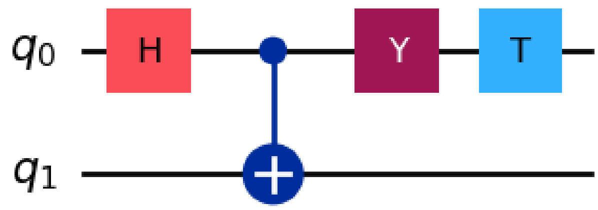
As we saw in the previous lesson, the $SWAP$ gate actually corresponds to three cleverly used $CNOT$s. Perhaps $CNOT$ can be resolved in the same way? What are the elementary gates we can work with (also known as universal gates)?

A universal gate set can be used to build any quantum circuit.

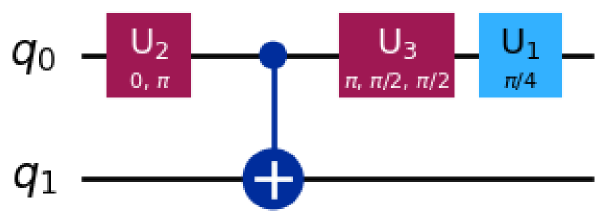
It can be proven that, for example, by adding the gates R_x, R_y, R_z to the gates P and $CNOT$ we obtain an universal gate-set. (By the way, the set $CNOT, H, S, T$ is also universal).

[24]:
```
qc = QuantumCircuit(2)

qc.h(0)
qc.cx(0, 1)
qc.y(0)
qc.t(0)
qc.draw("mpl")
```

[24]:



[25]:
```
qc.decompose().draw("mpl")
```
[25]:



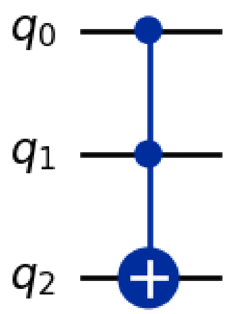
6.2 Let's get back to the original problem

The `transpile` process has 6 steps in `qiskit`:

1. `init` - All gates and instructions are converted to one- or two-bit gates.
2. `layout` - The logic quantum bits of the circuit are mapped to physical quantum bits.
3. `routing` - $SWAP$ gates are inserted based on the physical connections on the backend.
4. `translation` - Translation of the gates to the instruction set used by the backend.
5. `optimization` - Optimization of the circuit.
6. `scheduling` - Here we can specify scheduling related tasks.

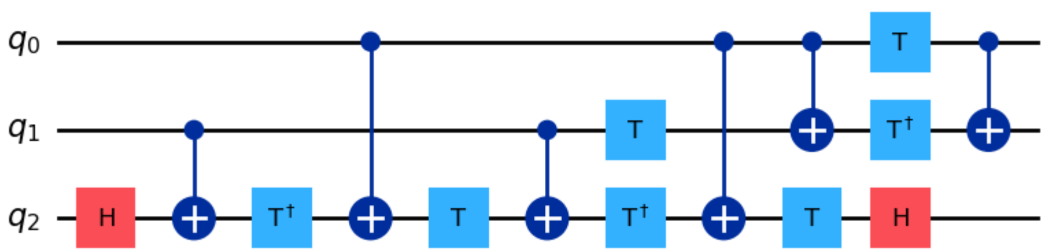
A more thorough description of these steps can be found here.

[26]:
```
from qiskit_ibm_runtime.fake_provider import FakeAuckland

backend = FakeAuckland()

qc = QuantumCircuit(3)
qc.ccx(0, 1, 2)
qc.draw("mpl")
```
[26]:



[27]:
```
qc.decompose().draw("mpl")
```

[27]:



[28]:
```
print("native gates:" + str(backend.operation_names))
```

```
native gates:['for_loop', 'if_else', 'id', 'x', 'rz', 'reset', 'delay', 'sx',
'cx', 'switch_case', 'measure']
```

[29]:
```
transpile(qc, backend).draw("mpl", idle_wires=False)
```

[29]: