DEPARTMENT OF NETWORKED SYSTEMS AND SERVICES

Operations on the Bloch-sphere

2025. 02. 19.

2025 spring

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POSTULATES OF QUANTUM MECHANICS

- 1. postulate: qubit
  - Hilbert-space
- 2. postulate: logical gates
  - Unitary transformation
  - Basic quantum gates
- 3. postulate: Q/C transformation
  - Measurement statistics
  - Post-measurement state
- 4. postulate: registers
  - Tensor product

$$|\varphi\rangle = \sum_{i=0}^{2^n-1} \varphi_i |i\rangle$$

$$U^\dagger \equiv U^{-1}$$

$$P(m \mid |\varphi\rangle) = \langle\varphi|M_m^\dagger M_m|\varphi\rangle$$

$$|\varphi'\rangle = \frac{M_m|\varphi\rangle}{\sqrt{\langle\varphi|M_m^\dagger M_m|\varphi\rangle}}$$

$$|\varphi\rangle = |0\rangle \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

1ST POSTULATE (STATE SPACE)

The actual state of any closed physical system can be described by means of a so called state vector **v** having complex coefficients and unit length in a Hilbert space *V* i.e. a complex linear vector space (state space) equipped with inner product.

QUBIT

- Dirac ‘ket’ and ‘bra’ notation $|\varphi\rangle = (\langle\varphi|)^\dagger$
- Qubit: Contains both classical states (base states): superposition

$$|\varphi\rangle = a|0\rangle + b|1\rangle = a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

- where *a* and *b* are complex probability amplitudes. Their squared absolute values give the probability of a measurement result: $|a|^2 + |b|^2 = 1$

- Operations: inner and outer product

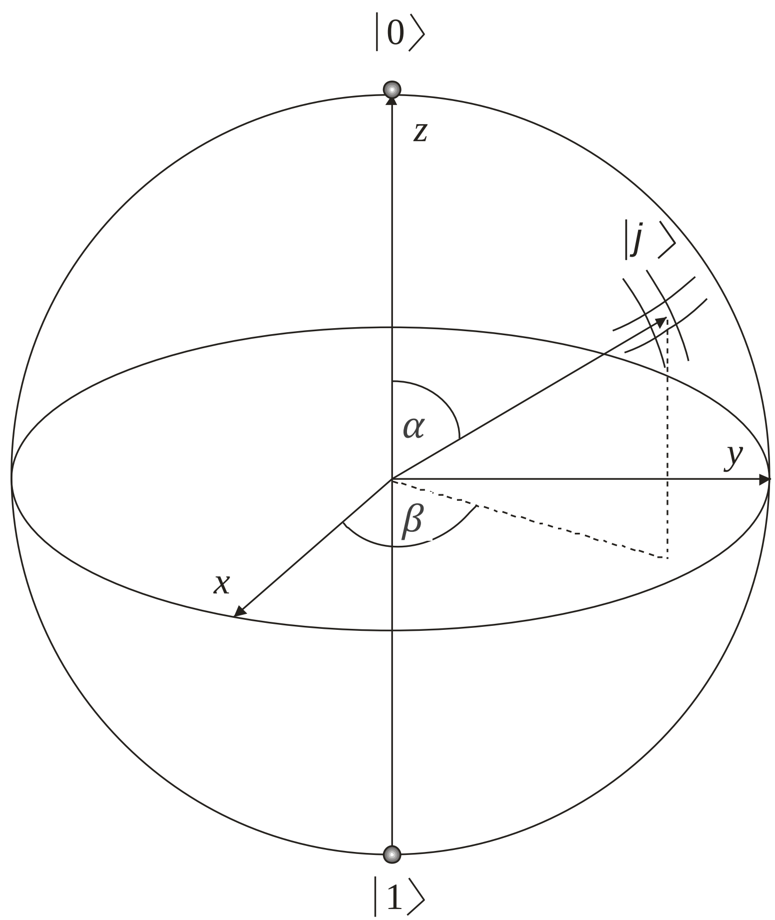
BLOCH-SPHERE (1)

Describes a qubit:

$$|\varphi\rangle = e^{j\gamma}\left[\cos\left(\frac{\alpha}{2}\right)|0\rangle + e^{j\beta}\sin\left(\frac{\alpha}{2}\right)|1\rangle\right] \quad \alpha,\beta,\gamma \in \mathbb{R}$$

$$|\varphi\rangle = [x, y, z]^T = [\cos(\beta)\sin(\alpha), \sin(\beta)\sin(\alpha), \cos(\alpha)]^T$$

BLOCH-SPHERE (2)



$$|\varphi\rangle = a|0\rangle + b|1\rangle$$

$$|\varphi\rangle = e^{j\gamma}\left[\cos\left(\frac{\alpha}{2}\right)|0\rangle + e^{j\beta}\sin\left(\frac{\alpha}{2}\right)|1\rangle\right]$$

$$\alpha, \beta, \gamma \in \mathbb{R}$$

$$|\varphi\rangle = [x, y, z]^T = [\cos(\beta)\sin(\alpha), \sin(\beta)\sin(\alpha), \cos(\alpha)]^T$$

PAULI-GATES

$$|\varphi\rangle = a|0\rangle + b|1\rangle$$

Pauli-X (bit-flip) gate:

$$\begin{bmatrix} a \\ b \end{bmatrix}$$

$$|\psi\rangle = X|\varphi\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} b \\ a \end{bmatrix} = b|0\rangle + a|1\rangle$$

PAULI-GATES

$$|\varphi\rangle = a|0\rangle + b|1\rangle$$

- Pauli X (bit-flip) gate:

$$\begin{bmatrix} a \\ b \end{bmatrix}$$

$$|\psi\rangle = X|\varphi\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} b \\ a \end{bmatrix} = b|0\rangle + a|1\rangle$$

- Pauli *Z* (phase-flip) gate:

$$\begin{bmatrix} a \\ b \end{bmatrix}$$

$$|\psi\rangle = Z|\varphi\rangle = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ -b \end{bmatrix} = a|0\rangle - b|1\rangle$$

PAULI-GATES

$$|\varphi\rangle = a|0\rangle + b|1\rangle$$

- Pauli *Y* (???-flip) gate:

$$\begin{bmatrix} a \\ b \end{bmatrix}$$

$$|\psi\rangle = Y|\varphi\rangle = \begin{bmatrix} 0 & -j \\ j & 0 \end{bmatrix} \begin{bmatrix} -jb \\ ja \end{bmatrix} = -jb|0\rangle + ja|1\rangle$$

- Geometrical interpretation of Pauli *X* gate: rotation around axis *x* in the Bloch sphere

$$e^{-j\frac{\alpha}{2}X} = \cos\left(\frac{\alpha}{2}\right) I - j\sin\left(\frac{\alpha}{2}\right) X$$

PHASE-GATE

$$|\varphi\rangle = a|0\rangle + b|1\rangle$$

$$\begin{bmatrix} a \\ b \end{bmatrix}$$

$$|\psi\rangle = P(\alpha)|\varphi\rangle = \begin{bmatrix} 1 & 0 \\ 0 & e^{j\alpha} \end{bmatrix} \begin{bmatrix} a \\ e^{j\alpha}b \end{bmatrix} = a|0\rangle + e^{j\alpha}b|1\rangle$$

HADAMARD-GATE

$$|\varphi\rangle = a|0\rangle + b|1\rangle$$

$$\begin{bmatrix} a \\ b \end{bmatrix}$$

$$|\psi\rangle = H|\varphi\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \frac{a+b}{\sqrt{2}} \\ \frac{a-b}{\sqrt{2}} \end{bmatrix} = \frac{a+b}{\sqrt{2}}|0\rangle + \frac{a-b}{\sqrt{2}}|1\rangle$$

- Hadamard-gate is hermitian: $H^\dagger = H$
- továbbá: $HH = I$

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HADAMARD-GATE

$$|\varphi\rangle = a|0\rangle + b|1\rangle$$

$$\begin{bmatrix} a \\ b \end{bmatrix}$$

$$|\psi\rangle = H|\varphi\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \frac{a+b}{\sqrt{2}} \\ \frac{a-b}{\sqrt{2}} \end{bmatrix} = \frac{a+b}{\sqrt{2}}|0\rangle + \frac{a-b}{\sqrt{2}}|1\rangle$$

- Hadamard-kapu is hermitian: $H^\dagger = H$
- furthermore: $HH = I$

- Worth noting:

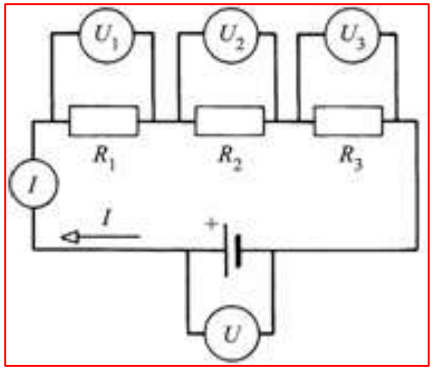
$$\boxed{\begin{aligned} H|0\rangle &= \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \\ H|1\rangle &= \frac{|0\rangle - |1\rangle}{\sqrt{2}}. \end{aligned}}$$

HADAMARD GATE AND SUPERPOSITION

$$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}},$$

$$H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}.$$

$$|\varphi\rangle = a|0\rangle + b|1\rangle$$



$$|\psi\rangle = H|\varphi\rangle = a\frac{|0\rangle + |1\rangle}{\sqrt{2}} + b\frac{|0\rangle - |1\rangle}{\sqrt{2}} = \frac{a+b}{\sqrt{2}}|0\rangle + \frac{a-b}{\sqrt{2}}|1\rangle$$

Exercise 2.1. Prove in several different ways that $HH = I$!

Exercise 2.2. Prove that $HXH = Z$, $HYH = -Y$ and $HZH = X$!