Preparing arbitrary quantum state

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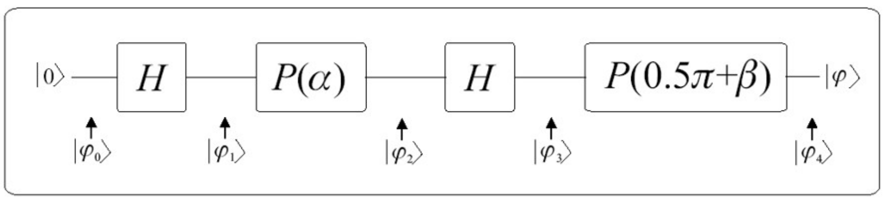
April 2025



When we want to prepare an arbitrary state, we choose an initial state (in this case: $|\varphi_0\rangle = |0\rangle$). The next step is to use a Hadamard gate to superpose this state:

$$|\varphi_1\rangle = H\,|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}.$$

The general formula of the phase shift gate:

$$P(\alpha) = \begin{bmatrix} 1 & 0 \\ 0 & e^{j\alpha} \end{bmatrix},$$

after we use $P(\alpha)$ gate to modify the phase of the quantum state. It means a rotation about the z-axis on the Bloch sphere. We will get the following state:

$$|\varphi_2\rangle = P(\alpha)\,|\varphi_1\rangle = \frac{|0\rangle + e^{j\alpha}\,|1\rangle}{\sqrt{2}}.$$

We use a Hadamard gate to superpose this modified state:

$$|\varphi_3\rangle = H\,|\varphi_2\rangle = \frac{\frac{|0\rangle+|1\rangle}{\sqrt{2}} + e^{j\alpha}\frac{|0\rangle-|1\rangle}{\sqrt{2}}}{\sqrt{2}} = \frac{1+e^{j\alpha}}{2}\,|0\rangle + \frac{1-e^{j\alpha}}{2}\,|1\rangle$$

After using another phase change gate $P(0.5\pi + \beta)$, we will get the following state:

$$
\begin{aligned}
|\varphi_4\rangle &= P(0.5\pi+\beta)|\varphi_3\rangle \\
&= P(0.5\pi+\beta)\left[\frac{1+e^{j\alpha}}{2}|0\rangle + \frac{1-e^{j\alpha}}{2}|1\rangle\right] \\
&= \frac{1+e^{j\alpha}}{2}|0\rangle + e^{j(0.5\pi+\beta)}\left(\frac{1-e^{j\alpha}}{2}\right)|1\rangle \\
&= e^{j0.5\alpha}\left[cos\left(\frac{\alpha}{2}\right)|0\rangle + e^{j\beta}e^{j0.5\alpha}sin\left(\frac{\alpha}{2}\right)|1\rangle\right] \\
&\boxed{= e^{j0.5\alpha}\left[cos\left(\frac{\alpha}{2}\right)|0\rangle + e^{j\beta}sin\left(\frac{\alpha}{2}\right)|1\rangle\right]}
\end{aligned}
$$

To understand the result on which the previous equation of state was transformed, we need to introduce Euler's formula. First, we need to prove that the following two equations 1, 2, are correct. (Exercise 2.6. in the lecture slides)

$$\frac{1+e^{j\alpha}}{2} = e^{j0.5\alpha}cos(0.5\alpha) \tag{1}$$

$$e^{j0.5\pi}\frac{1-e^{j\alpha}}{2} = e^{j0.5\alpha}sin(0.5\alpha) \tag{2}$$

Proof of equation 1:

$$e^{j0.5\alpha}\cos(0.5\alpha) = e^{j0.5\alpha}\left(\frac{e^{j0.5\alpha}+e^{-j0.5\alpha}}{2}\right) = \frac{e^{j\alpha}+e^{j0}}{2} = \frac{1+e^{j\alpha}}{2}$$

Proof of equation 2:

$$
\begin{aligned}
e^{j0.5\pi}\left(\frac{1-e^{j\alpha}}{2}\right) &= (\cos(\pi/2) + i\sin(\pi/2))\left(\frac{1-e^{j\alpha}}{2}\right) \\
&= i\left(\frac{1-e^{j\alpha}}{2}\right) = \frac{i}{i}\left[i\left(\frac{1-e^{j\alpha}}{2}\right)\right] \\
&= \boxed{\frac{e^{j\alpha}-1}{2i}}
\end{aligned}
$$

$$e^{j0.5\alpha}\sin(0.5\alpha) = e^{j0.5\alpha}\left(\frac{e^{j0.5\alpha}-e^{-j0.5\alpha}}{2i}\right) = \boxed{\frac{e^{j\alpha}-1}{2i}}$$