DEPARTMENT OF
NETWORKED SYSTEMS
AND SERVICES

Superposition and its special case: Entanglement

Quantum Computing and its Applications
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MŰEGYETEM 1782

POSTULATES OF QUANTUM MECHANICS

- 1. Postulate: qubit
  - Hilbert-space
- 2. Postulate: logic gates
  - Unitary transform
  - Elementary gates
- 3. Postulate Q/C conversion
  - Measurement statistics
  - Post measurement state
- 4. Postulate: registers
  - Tensor product

$$|\varphi\rangle = \sum_{i=0}^{2^n-1} \varphi_i |i\rangle$$

$$U^\dagger \equiv U^{-1}$$

$$P(m \mid |\varphi\rangle) = \langle\varphi|M_m^\dagger M_m|\varphi\rangle$$

$$|\varphi'\rangle = \frac{M_m|\varphi\rangle}{\sqrt{\langle\varphi|M_m^\dagger M_m|\varphi\rangle}}$$

$$|\varphi\rangle = |0\rangle \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

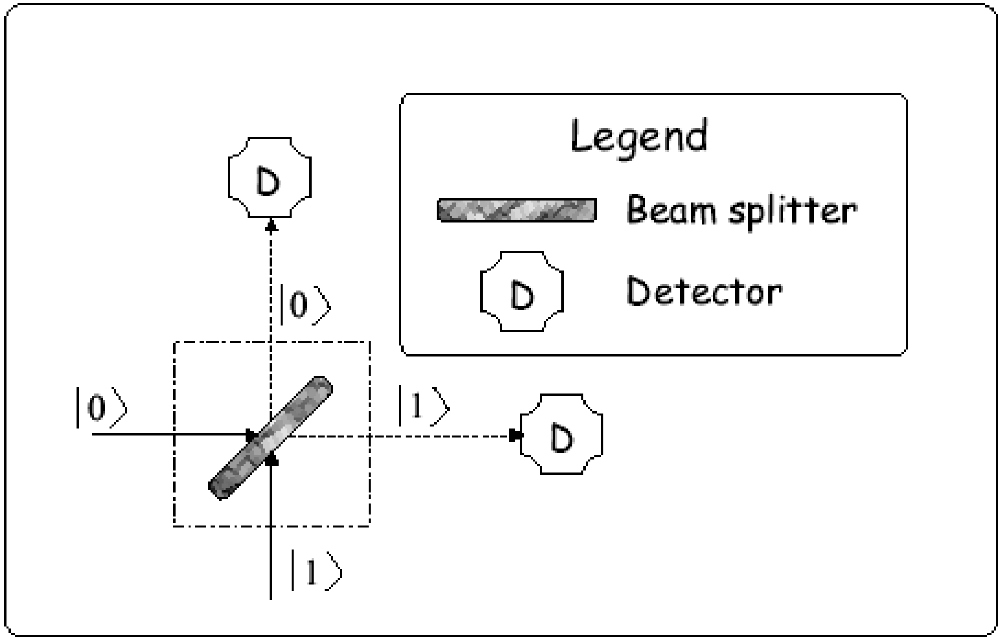
General description of the interferometer

"An idea is always a generalization, and generalization is a property of thinking.

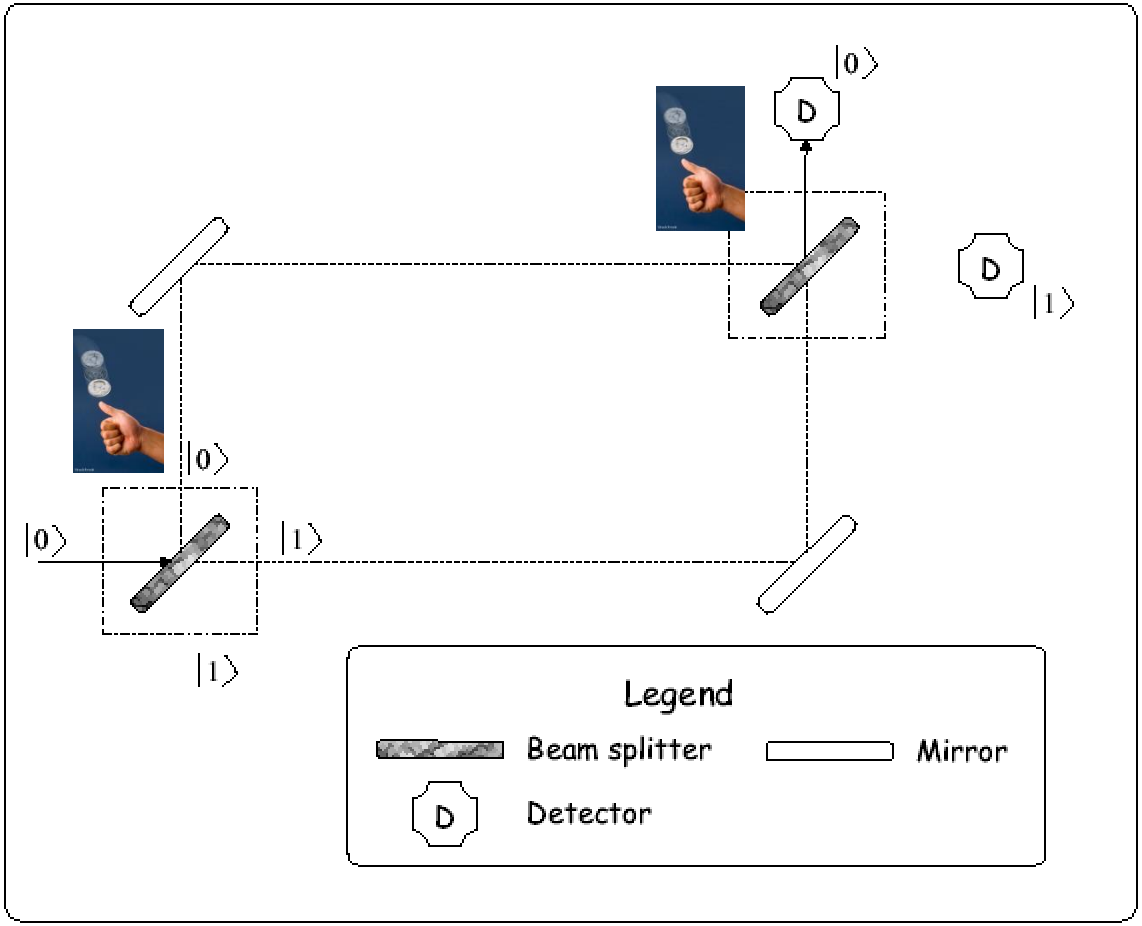
To generalize means to think."

Georg Hegel

HALFSILVERED MIRROR, BEAMSPLITTER



2 BEAMSPLITTERS AS IDENTITY TRANSFORMATION



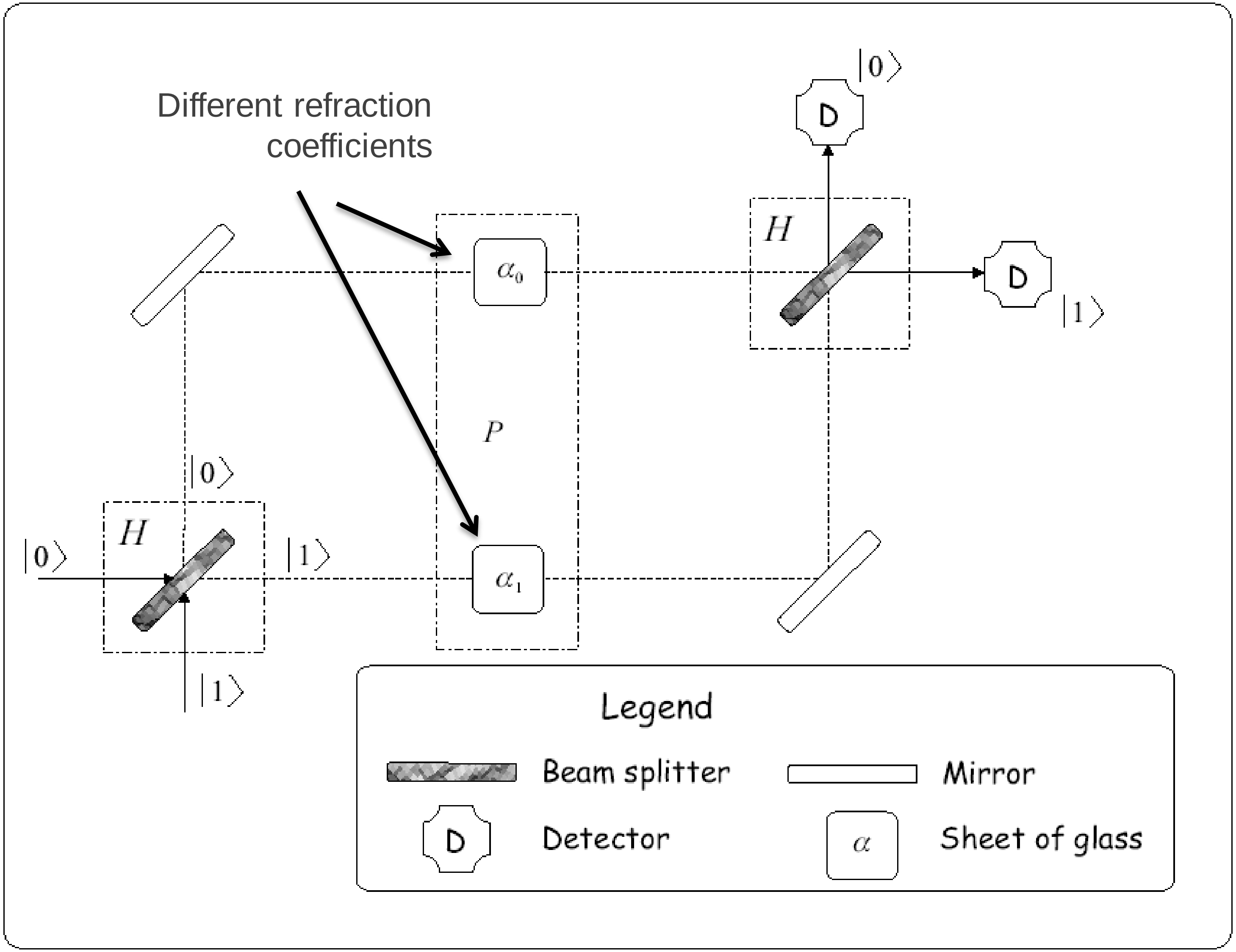
2 BEAMSPLITTERS AS IDENTITY TRANSFORMATION

- Extra detectors placed between the two mirrors make the result again random.

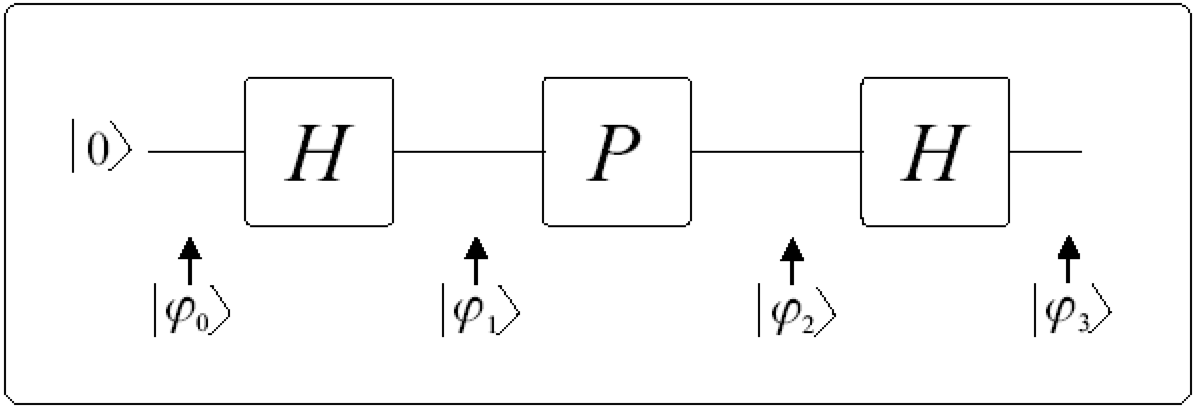
- Remark: *Measurements typically influence the observed system and thus the measurement results themselves*.

- Paradox of the "Observation (measurement) by means of electromagnetic waves": the small is the wavelength the small details can be observed. The small is the wavelength the high is the used frequency and the energy of the photon, thus the big is the impact of the photon onto the observed material.

GENERALIZED INTERFEROMETER

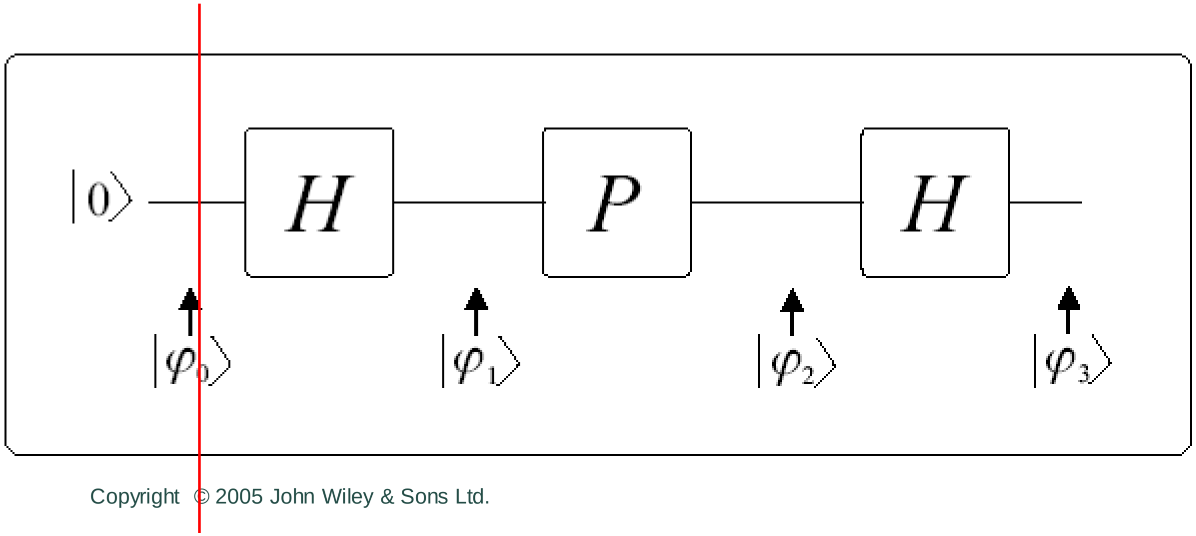


CIRCUIT DESCRIPTION



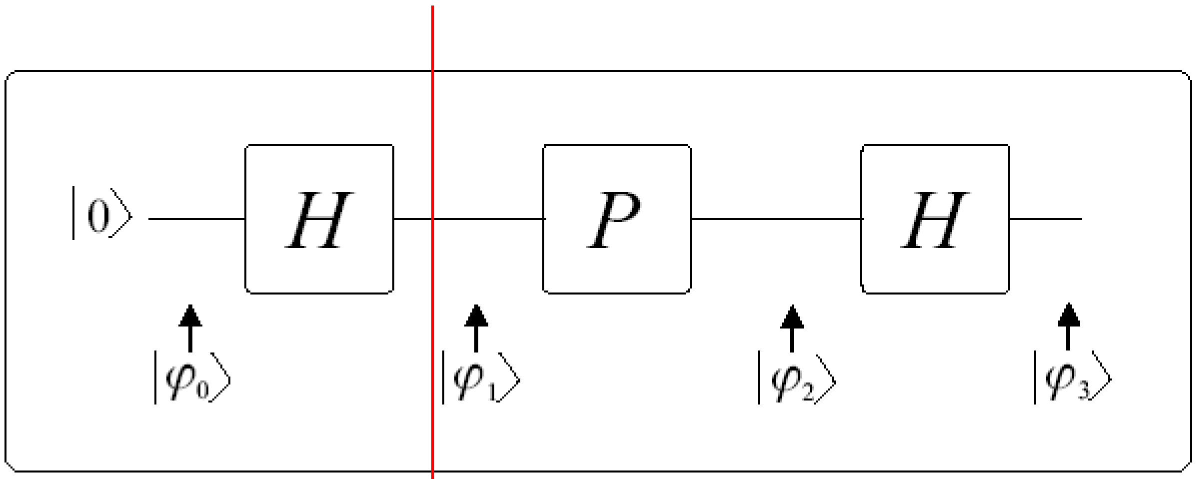
$$\mathbf{H} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad \mathbf{P} = \begin{bmatrix} e^{j\alpha_0} & 0 \\ 0 & e^{j\alpha_1} \end{bmatrix}$$

ANALYSIS(1)



$$|\varphi_0\rangle = |0\rangle$$

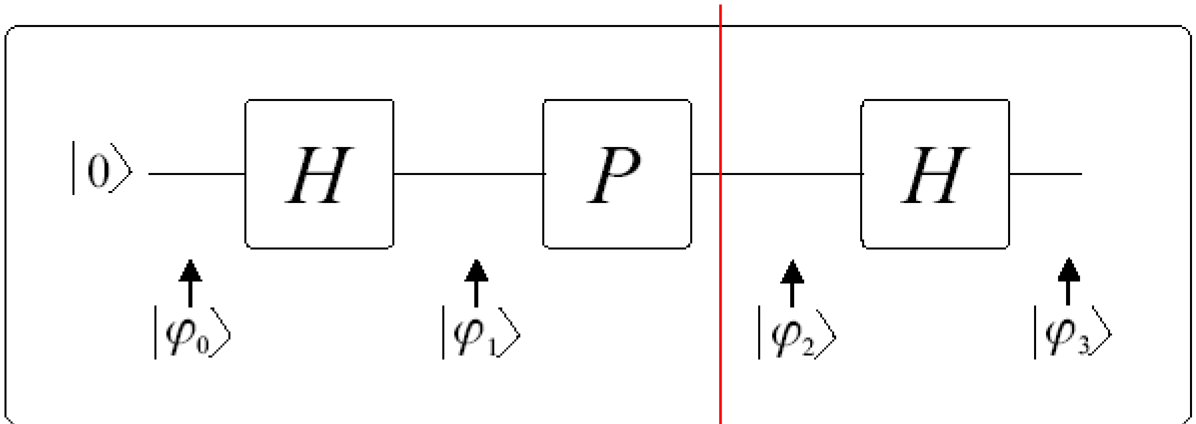
ANALYSIS(2)



$$\begin{bmatrix}1\\0\end{bmatrix}$$

$$|\varphi_1\rangle = H|\varphi_0\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix} \begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}$$

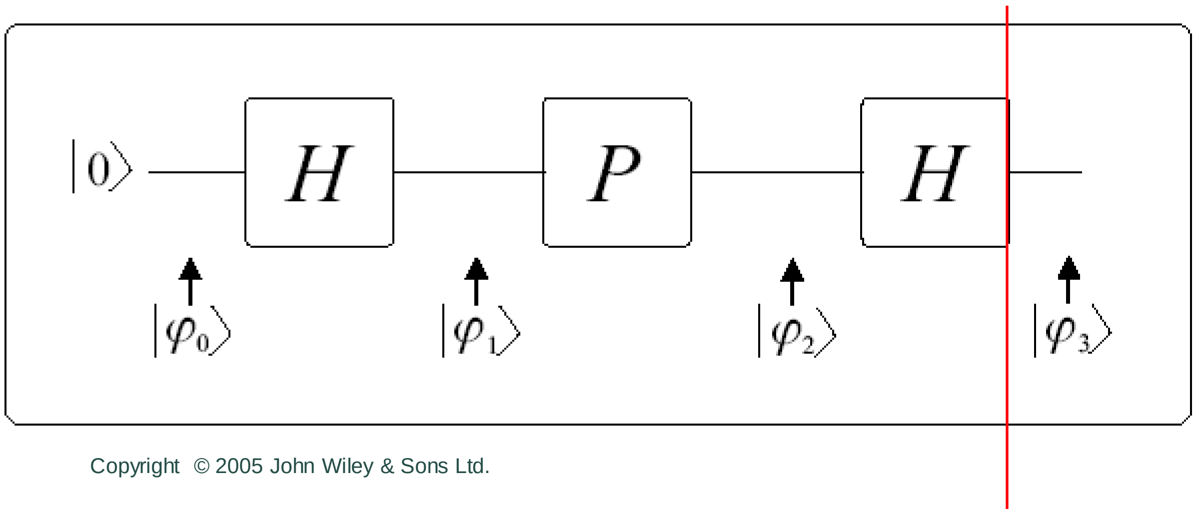
ANALYSIS(3)



$$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$|\varphi_2\rangle = P|\varphi_1\rangle = \begin{bmatrix} e^{j\alpha_0} & 0 \\ 0 & e^{j\alpha_1} \end{bmatrix} \begin{bmatrix} \frac{e^{j\alpha_0}}{\sqrt{2}} \\ \frac{e^{j\alpha_1}}{\sqrt{2}} \end{bmatrix}$$

ANALYSIS(4)



$$\begin{bmatrix} \frac{e^{j\alpha_0}}{\sqrt{2}} \\ \frac{e^{j\alpha_1}}{\sqrt{2}} \end{bmatrix}$$

$$|\varphi_3\rangle = H|\varphi_2\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{e^{j\alpha_0}+e^{j\alpha_1}}{2} \\ \frac{e^{j\alpha_0}-e^{j\alpha_1}}{2} \end{bmatrix}$$

$$|\varphi\rangle = a|0\rangle + b|1\rangle$$

ANALYSIS(5)

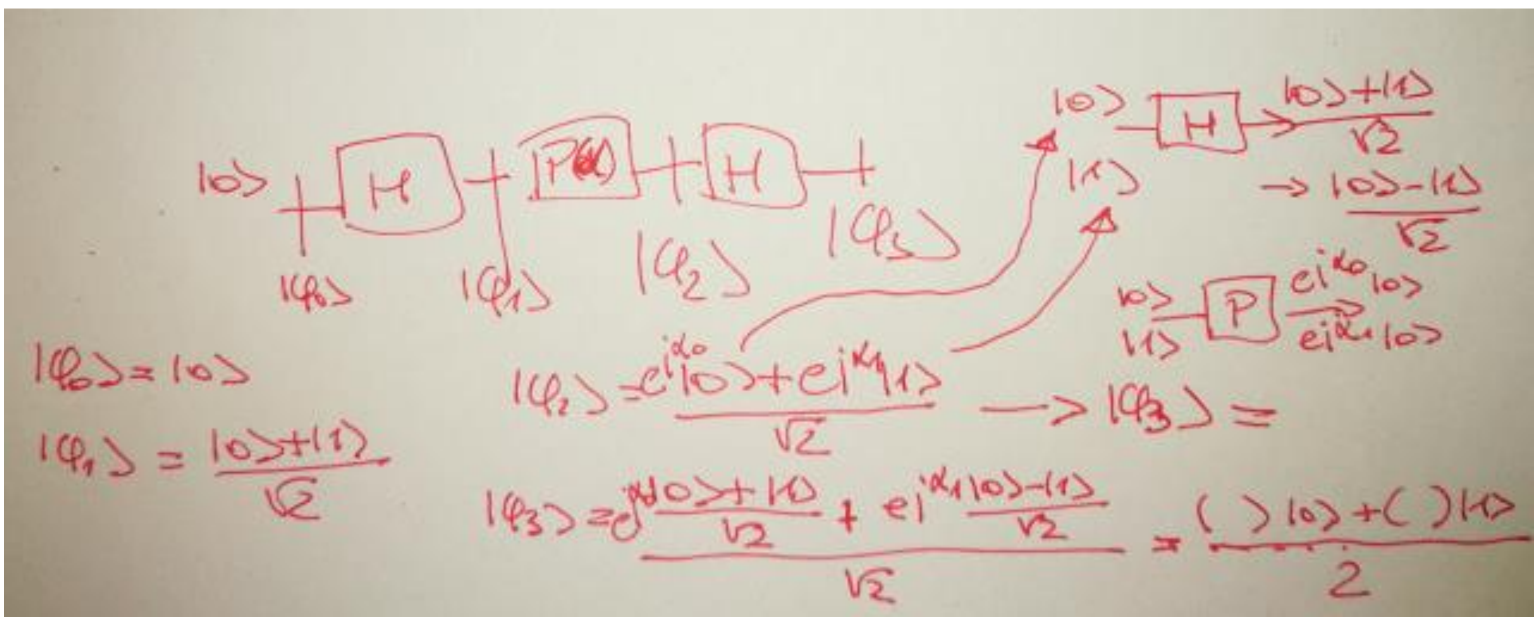
$$\begin{aligned} |\varphi_3\rangle &= \frac{e^{j\alpha_0}+e^{j\alpha_1}}{2}|0\rangle + \frac{e^{j\alpha_0}-e^{j\alpha_1}}{2}|1\rangle \\ &= e^{j\frac{\alpha_0+\alpha_1}{2}}\left(\frac{e^{j\frac{\alpha_0-\alpha_1}{2}}+e^{-j\frac{\alpha_0-\alpha_1}{2}}}{2}|0\rangle + \frac{e^{j\frac{\alpha_0-\alpha_1}{2}}-e^{-j\frac{\alpha_0-\alpha_1}{2}}}{2}|1\rangle\right) \end{aligned}$$

$$\Delta\alpha \triangleq \alpha_0 - \alpha_1$$

$$\frac{e^{j\Delta\alpha}+e^{-j\Delta\alpha}}{2} = \cos(\Delta\alpha) \qquad \frac{e^{j\Delta\alpha}-e^{-j\Delta\alpha}}{2j} = \sin(\Delta\alpha)$$

$$\boxed{\begin{aligned} P_0 &= \cos^2\left(\frac{\Delta\alpha}{2}\right) = (1+\cos(\Delta\alpha))\frac{1}{2}, \\ P_1 &= \sin^2\left(\frac{\Delta\alpha}{2}\right) = (1-\cos(\Delta\alpha))\frac{1}{2}. \end{aligned}}$$

$$
\begin{aligned}
|\varphi_3\rangle &= \frac{e^{j\alpha_0}+e^{j\alpha_1}}{2}|0\rangle + \frac{e^{j\alpha_0}-e^{j\alpha_1}}{2}|1\rangle \\
&= e^{j\frac{\alpha_0+\alpha_1}{2}}\left(\frac{e^{j\frac{\alpha_0-\alpha_1}{2}}+e^{-j\frac{\alpha_0-\alpha_1}{2}}}{2}|0\rangle + \frac{e^{j\frac{\alpha_0-\alpha_1}{2}}-e^{-j\frac{\alpha_0-\alpha_1}{2}}}{2}|1\rangle\right)
\end{aligned}
$$



ANALYSIS(6)

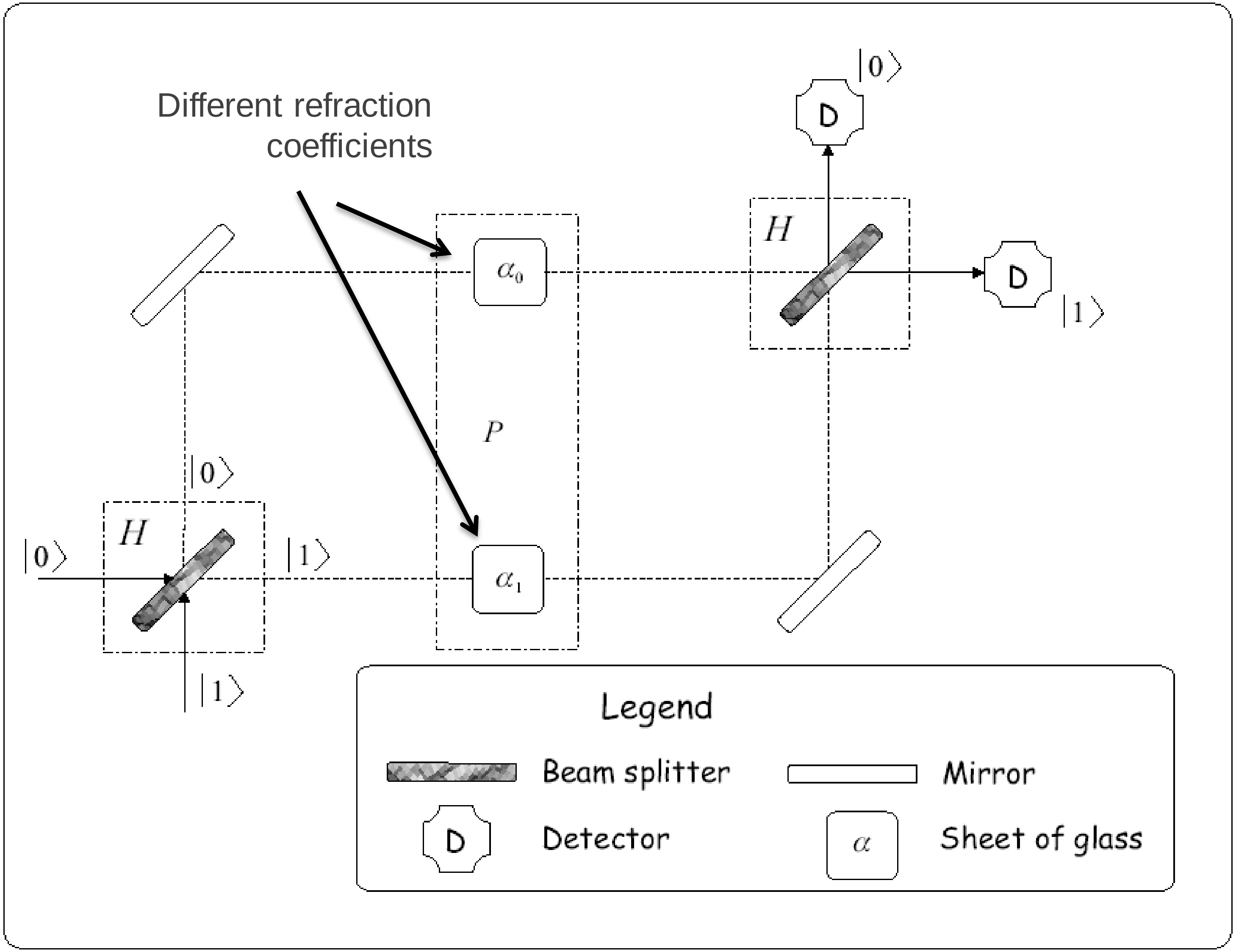
$$\boxed{\begin{aligned} P_0 &= \cos^2\left(\frac{\Delta\alpha}{2}\right) = (1+\cos(\Delta\alpha))\frac{1}{2}, \\ P_1 &= \sin^2\left(\frac{\Delta\alpha}{2}\right) = (1-\cos(\Delta\alpha))\frac{1}{2}. \end{aligned}}$$

$\Delta\alpha = 0$: idealistic scenario

$\Delta\alpha = \frac{\pi}{2}$: fully random operation

Exercise 2.3. Perform the analysis of the generalized interferometer using the superposition principle!

GENERALIZED INTERFEROMETER



Entanglement

"Wonder is from surprise, and surprise stops with experience."

Bishop Robert South

A SURPRISING STATE

- Based on the 4th Postulate, let us split state into two parts!

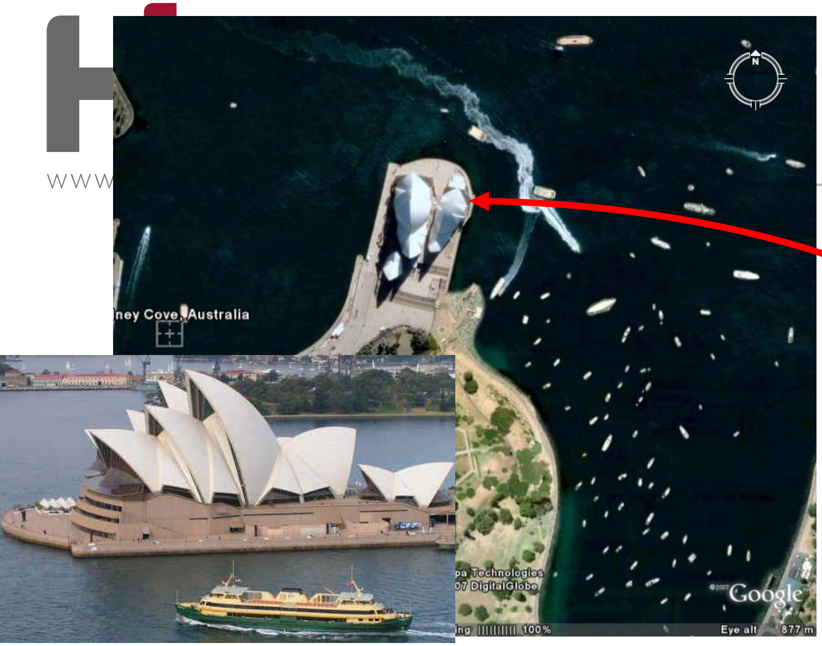
$$|\varphi\rangle = a|00\rangle + b|11\rangle$$

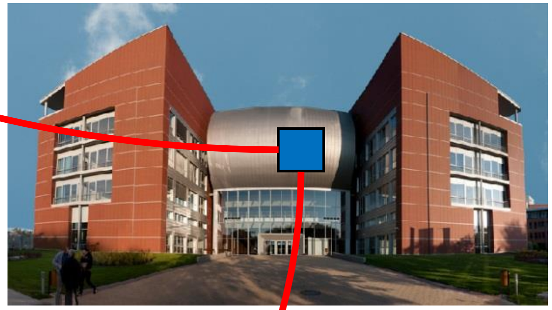
$$|\varphi\rangle = |\varphi_1\rangle \otimes |\varphi_2\rangle \quad \Rightarrow \quad |\varphi_1\rangle = \ ? \quad |\varphi_2\rangle = \ ?$$

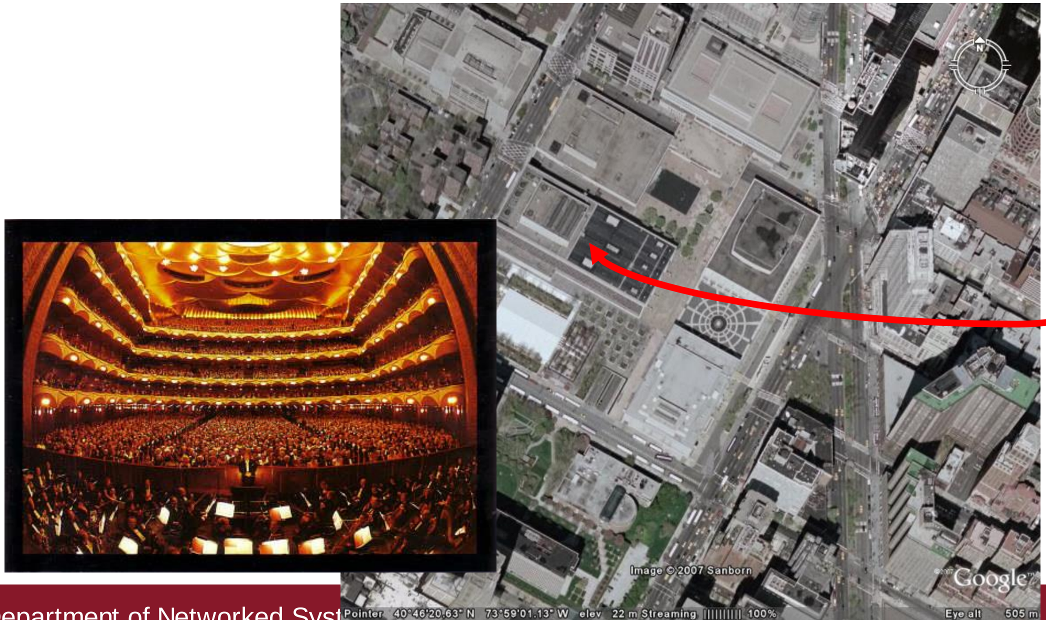
- No such decomposition exists!

- Two different types of quantum states
  - product
  - entangled

A STRANGE EXPERIMENT

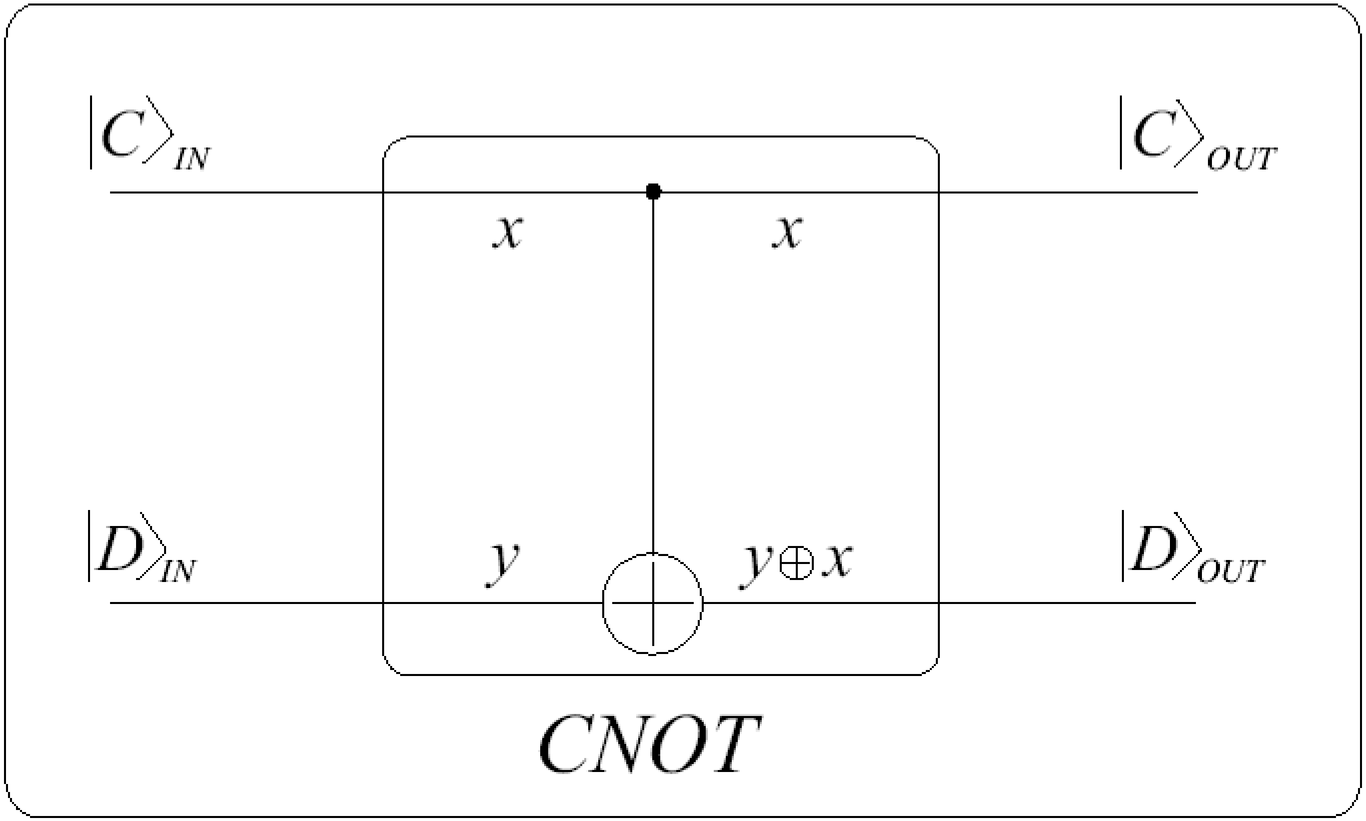






$$\left|\varphi\right\rangle = \varphi_0\left|00\right\rangle + \varphi_3\left|11\right\rangle$$

CONTROLLED NOT GATE (CNOT GATE)



Upper wire: control

Lower wire: data

CNOT-GATE

- Truth table

IN		OUT	
x	y	x	$y \oplus x$
0	0	0	$0 \oplus 0 = 0$
0	1	0	$1 \oplus 0 = 1$
1	0	1	$0 \oplus 1 = 1$
1	1	1	$1 \oplus 1 = 0$

Matrix

$$|00\rangle \rightarrow |00\rangle \quad |01\rangle \rightarrow |01\rangle$$

$$|10\rangle \rightarrow |11\rangle \quad |11\rangle \rightarrow |10\rangle$$

$$\mathbf{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

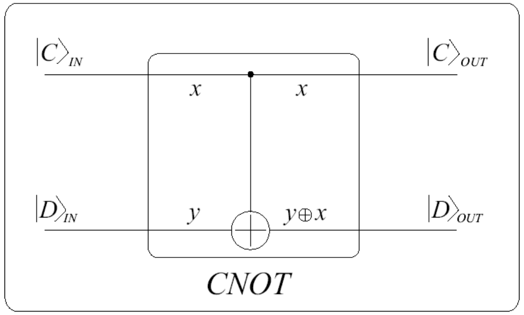
- Master equation

$$CNOT : |x\rangle|y\rangle \rightarrow |x\rangle|y \oplus x\rangle$$

THE CNOT GATE AS CLASSICAL COPY MACHINE

- Provided the data input is initialized permanently with $|0\rangle$ then the CNOT gate emits a copy of the control input on each output!
- Let's try to copy the following state!

$$|C\rangle_{IN} = a|0\rangle + b|1\rangle$$



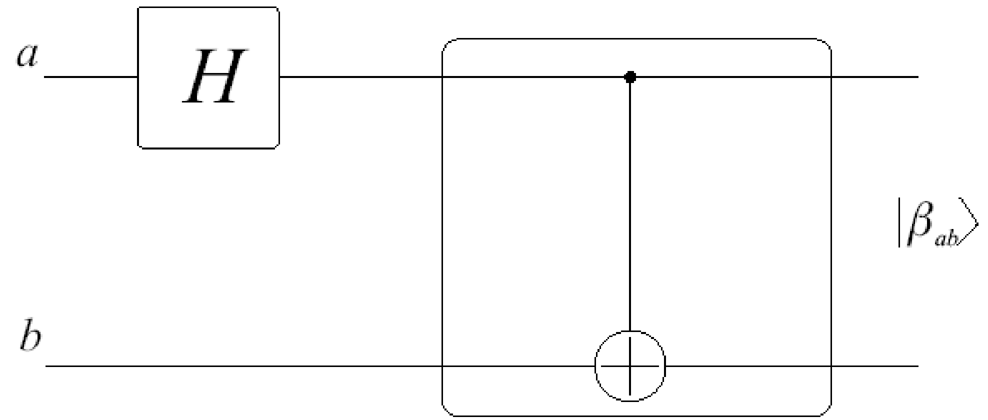
- The input joint state is $|C\rangle_{IN} \otimes |D\rangle_{IN} = a|00\rangle + b|10\rangle$
- Using the superposition principle the output becomes

$$a|0, 0 \oplus 0\rangle + b|1, 1 \oplus 0\rangle = a|00\rangle + b|11\rangle$$

which is nothing else then an entangled pair!

BELL STATES

- Let us investigate the CNOT as an entanglement generator!



$$|\beta_{ab}\rangle = \frac{|0,b\rangle + (-1)^a |1, NOT(b)\rangle}{\sqrt{2}} \qquad a, b \in \{0, 1\}$$

BELL STATES

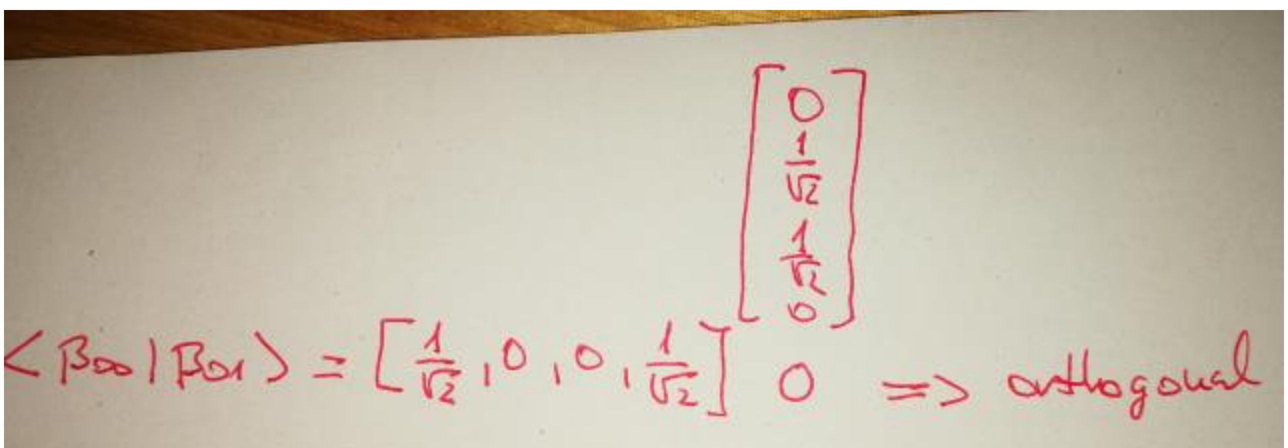
$$|\beta_{00}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}},$$

$$|\beta_{01}\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}},$$

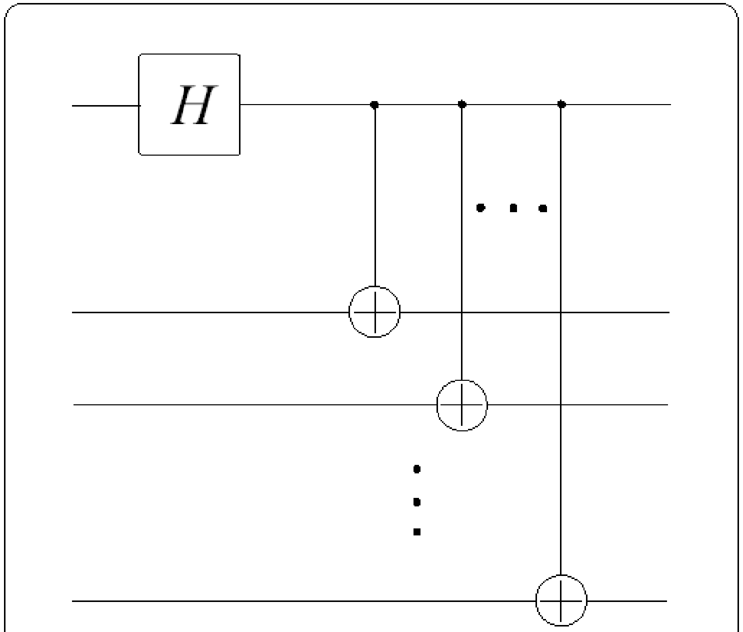
$$|\beta_{10}\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}},$$

$$|\beta_{11}\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}.$$

A Bell states are orthogonal!

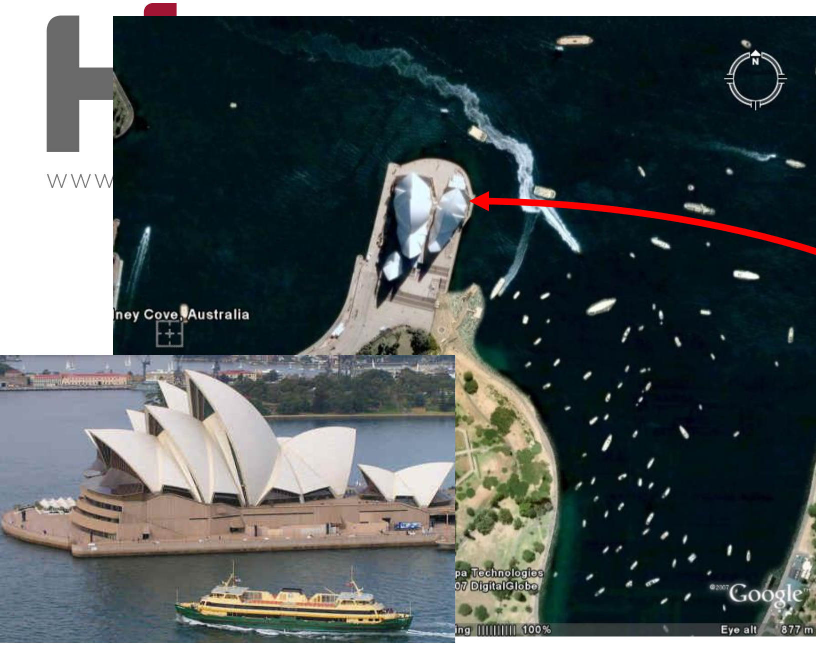


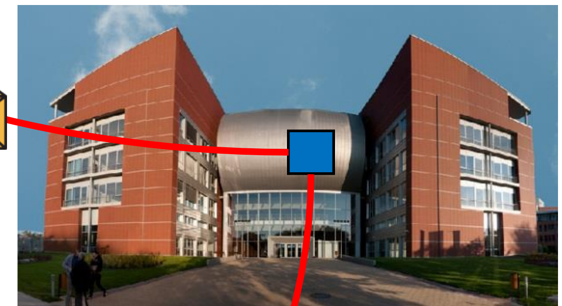
GENERALIZED QUANTUM ENTANGLER

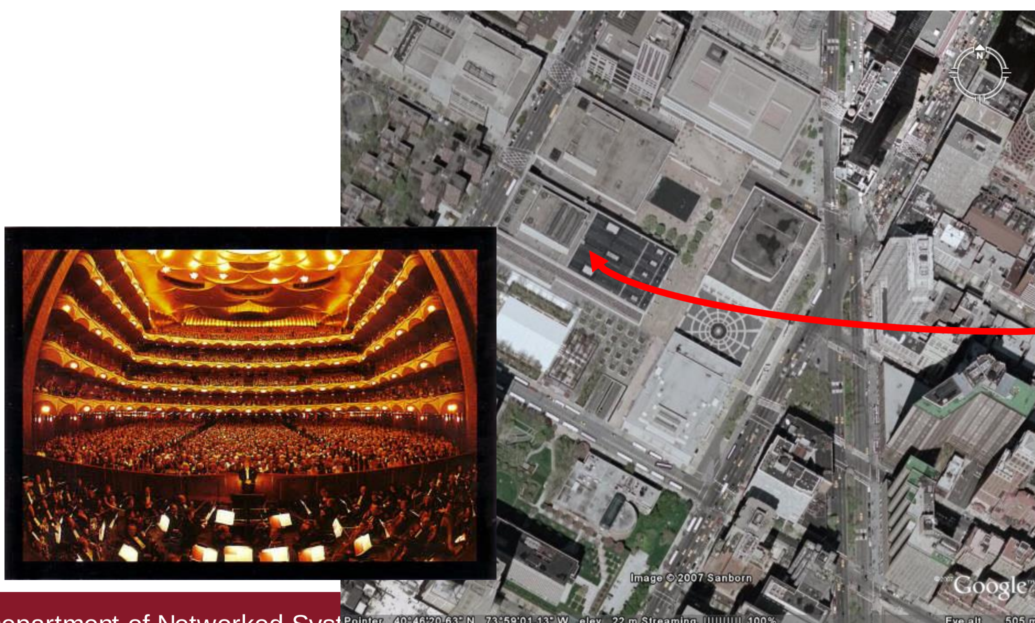


- Only one of the entangled qbits is enough to entangle another qbit to the previous set of qbits.
- Entanglement cannot be produced using only classical communication!

EPR-paradox







$$\left|\varphi\right\rangle = \varphi_0\left|00\right\rangle + \varphi_3\left|11\right\rangle$$

THE EPR PARADOX AND THE BELL INEQUALITY

"'Obvious' is the most dangerous word in mathematics."
Eric Temple Bell

- EPR paradox: Entanglement seems to contradict to limited speed of information transfer (light).
- Einstein: Therefore quantum mechanics provides an incomplete description of the Nature. ⇒ Hidden variables!!!!

- Bell inequality: gives unambiguously different results in case hidden variables exist or not.
- Problem: hard to test it in practice.
- Solution: Clauser-Horne-Schimony-Holt (CHSH) inequality

- NO HIDDEN VARIABLES EXIST!

HOW TO EARN 250 000 USD BY ENTANGLEMENT?

- https://www.youtube.com/watch?v=4q-nBVzFDII&feature=youtu.be

DISCLAIMER

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