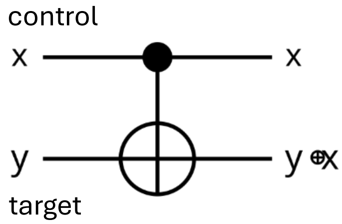
CNOT gate



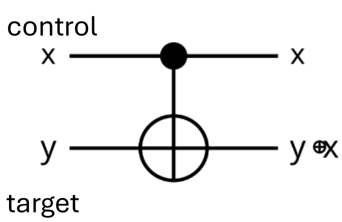
The CNOT gate is a two-qubit gate. It inverts the value of the target bit if the control bit is 1 and does nothing on the target bit if the control bit is 0.

With classical inputs

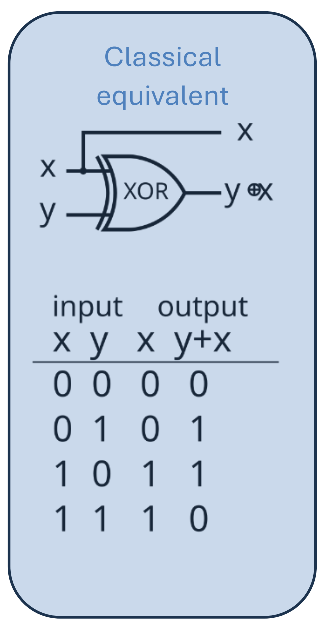
Truth table for classical inputs:

Before CNOT		**After CNOT**	
Control	**Target**	**Control**	**Target**
$\|0\rangle$	$\|0\rangle$	$\|0\rangle$	$\|0\rangle$
$\|0\rangle$	$\|1\rangle$	$\|0\rangle$	$\|1\rangle$
$\|1\rangle$	$\|0\rangle$	$\|1\rangle$	$\|1\rangle$
$\|1\rangle$	$\|1\rangle$	$\|1\rangle$	$\|0\rangle$

Circuit diagram:



input		output	
x	y	x	y+x
$\|0\rangle$	$\|0\rangle$	$\|0\rangle$	$\|0\rangle$
$\|0\rangle$	$\|1\rangle$	$\|0\rangle$	$\|1\rangle$
$\|1\rangle$	$\|0\rangle$	$\|1\rangle$	$\|1\rangle$
$\|1\rangle$	$\|1\rangle$	$\|1\rangle$	$\|0\rangle$



Generally, for a classical input $|xy\rangle$ the output is:

$$CNOT(|xy\rangle) = |x(x \oplus y)\rangle$$

Where $\oplus$ denotes the XOR operation.

With quantum inputs

Bra-ket notation:

A general 2 qubit state with bra-ket notation (before CNOT):

$$|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$$

CNOT is a linear operation (like all unitary operations), therefore:

$$\begin{aligned} &CNOT(a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle) \\ &\qquad = a\,CNOT(|00\rangle) + b\,CNOT(|01\rangle) + c\,CNOT(|10\rangle) + d\,CNOT(|11\rangle) \end{aligned}$$

However, we know the effect for every term in the superposition based on the truth table:

$$a\,CNOT(|00\rangle) + b\,CNOT(|01\rangle) + c\,CNOT(|10\rangle) + d\,CNOT(|11\rangle) = a|00\rangle + b|01\rangle + c|11\rangle + d|10\rangle$$

Vector notation:

A general 2 qubit state with bra-ket notation (before CNOT):

$$|\psi\rangle = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

The matrix of a CNOT operation:

$$CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} I & \hat{0} \\ \hat{0} & X \end{bmatrix}$$

Where $\hat{0}$ is the 2x2 zero submatrix.

Therefore, the result of a CNOT operation on an arbitrary state $|\psi\rangle$ is:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a \\ b \\ d \\ c \end{bmatrix}$$

(Note that the last two components of the state vector were switched.)