حل مثال

کانولوشن های زیر را محاسبه کنید:

1) $x(t) = e^{-2t}u(t)$

$h(t) = e^{-3t}u(t)$

$$x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{+\infty} e^{-2\tau}u(\tau)e^{-3(t-\tau)}\boxed{u(t-\tau)}d\tau$$

$$= \boxed{\int_{-\infty}^{0}} x(\tau)h(t-\tau)d\tau + \int_{0}^{t} x(\tau)h(t-\tau)d\tau + \int_{t}^{+\infty} x(\tau)h(t-\tau)d\tau$$

$$\int_{-\infty}^{0} x(\tau)h(t-\tau)d\tau = 0$$

$$\int_{t}^{+\infty} x(\tau)h(t-\tau)d\tau = 0$$

$$\boxed{\int_{0}^{t} x(\tau)h(t-\tau)d\tau} = \boxed{\int_{0}^{t}} e^{-2\tau}e^{-3(t-\tau)}d\tau$$

$$x(t) * h(t) = \int_0^t e^{-2\tau} e^{-3(t-\tau)} d\tau = \int_0^t e^{-2\tau} e^{-3t+3\tau} d\tau$$

$$e^{-3t} \int_0^t e^{-2\tau} e^{+3\tau} d\tau = e^{-3t} \int_0^t e^{\tau} d\tau = e^{-3t} . e^{\tau} \Big|_0^t = e^{-3t} \left(e^{t} - 1 \right)$$

$$= e^{-2t} - e^{-3t}$$

تمرین)

$$x(t) = 2[u(t) - u(t-2)] \qquad h(t) = e^{t}u(-t)$$

$$x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{+\infty} x(\tau)e^{t-\tau}u[-(t-\tau)]d\tau = \int_{-\infty}^{+\infty} x(\tau)e^{t-\tau}u(-t+\tau)d\tau$$

$$= \int_{-\infty}^{0} x(\tau)h(t-\tau)d\tau + \int_{0}^{t} x(\tau)h(t-\tau)d\tau + \int_{t}^{2} x(\tau)h(t-\tau)d\tau + \int_{2}^{+\infty} x(\tau)h(t-\tau)d\tau$$

$$\int_{-\infty}^{0} x(\tau)h(t-\tau)d\tau = 0$$

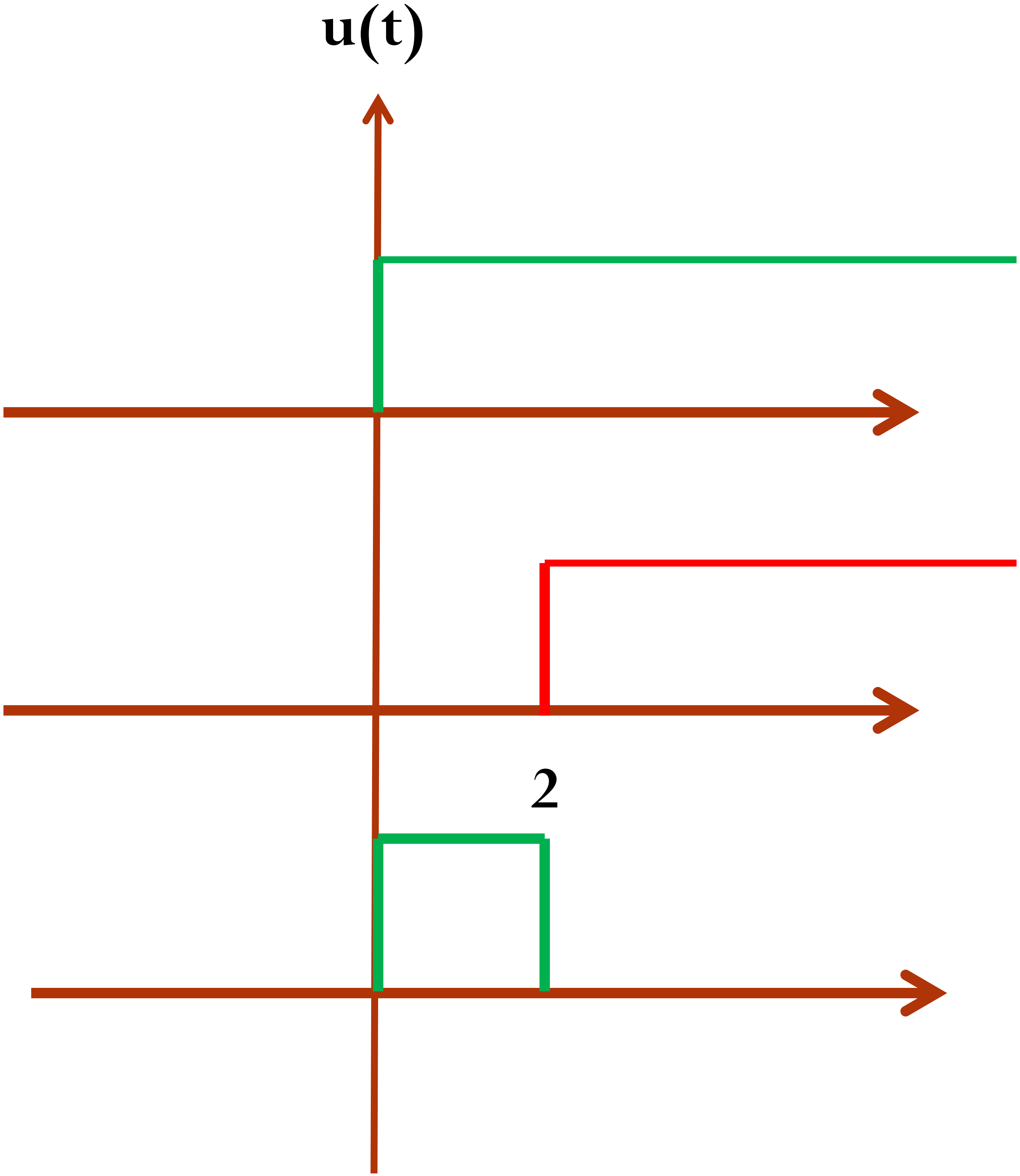
$$\int_{0}^{t} x(\tau)h(t-\tau)d\tau = 0$$

$$\int_{t}^{+\infty} x(\tau)h(t-\tau)d\tau = 0$$

$$\int_{t}^{2} x(\tau)h(t-\tau)d\tau = \int_{t}^{2} 2e^{(t-\tau)}d\tau = 2e^{t}\int_{t}^{2} e^{-\tau}d\tau = -2e^{t}e^{-\tau}\Big|_{t}^{2}$$

$$-2e^{t}\left(e^{-2} - e^{-t}\right) = 2e^{t}\left(e^{-t} - e^{-2}\right) = 2(1 - e^{t-2})$$

u(t)-u(t-2)



1. Express the signals shown in Fig. 1 in terms of unit step functions.

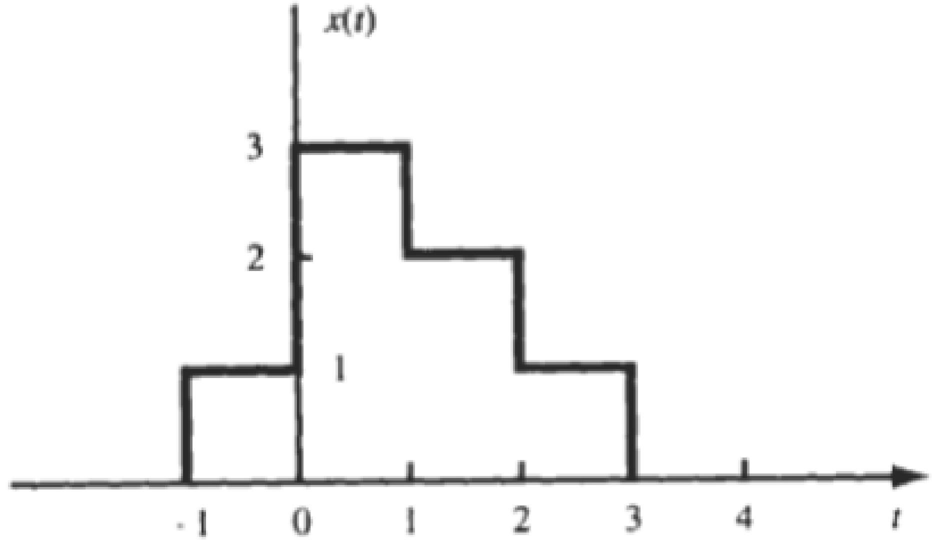
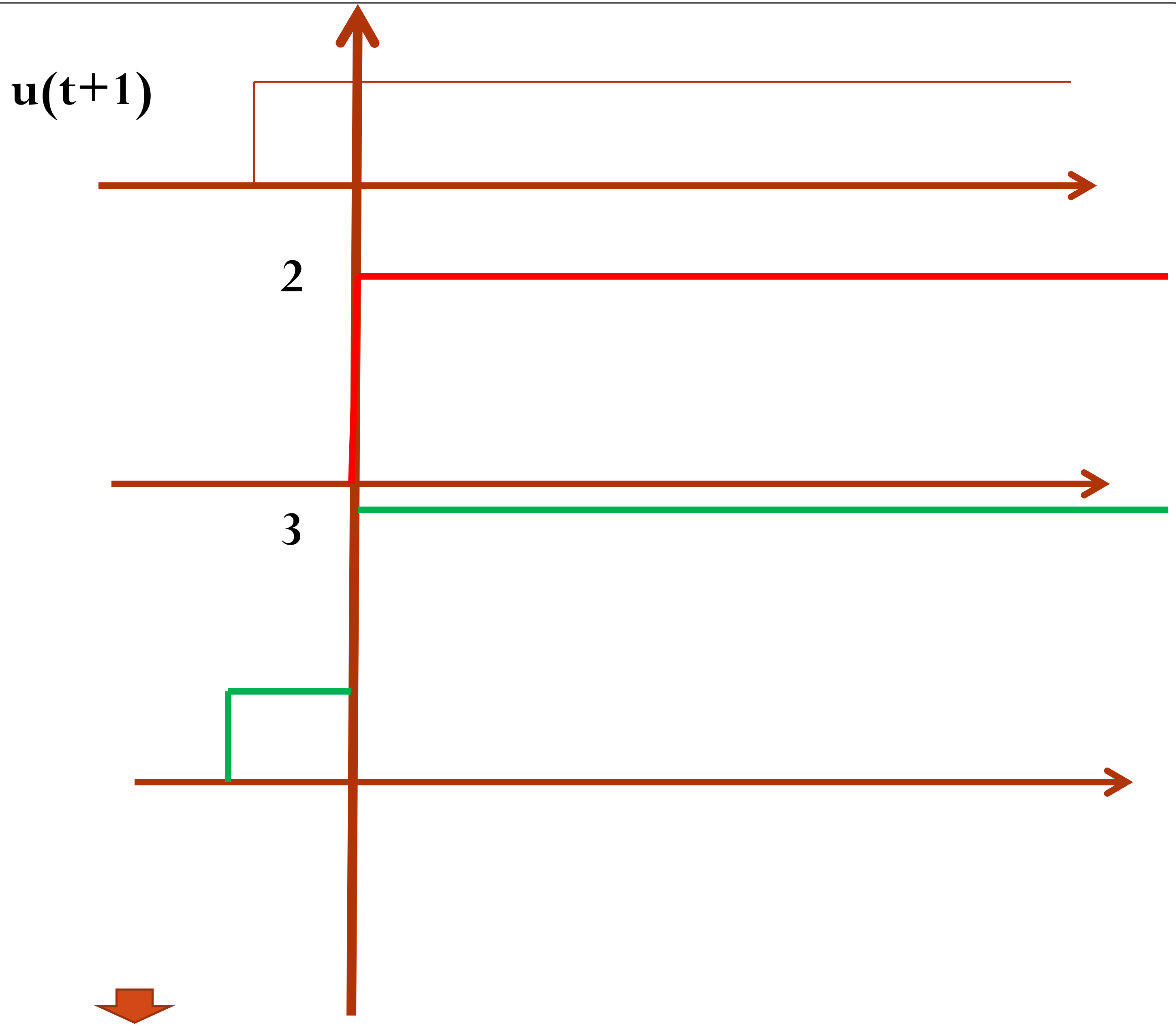


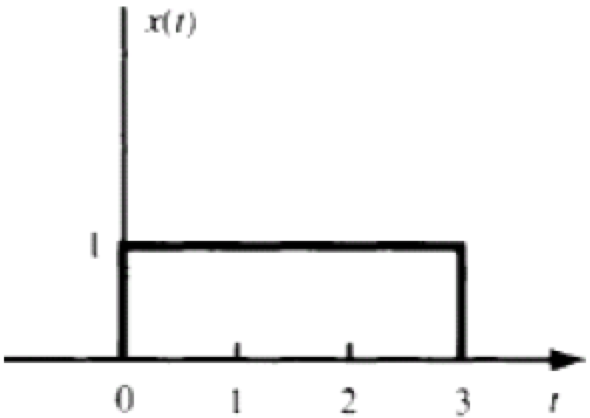
Fig. 1

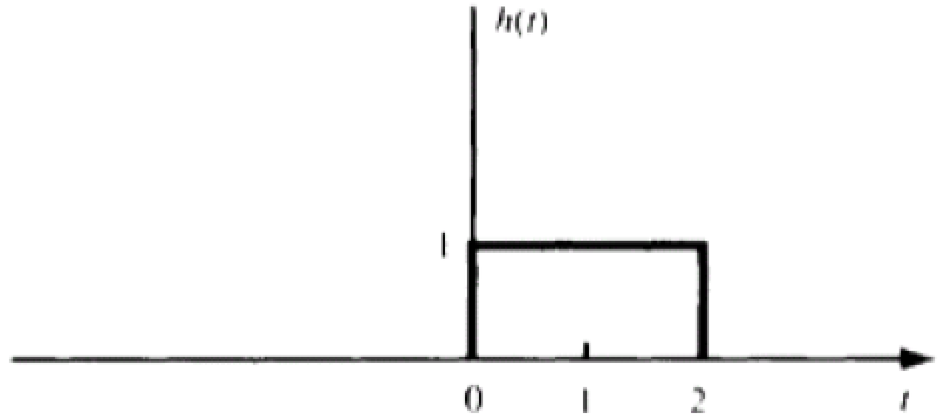
Answer $x(t) = u(t+1) + 2u(t) - u(t-1) - u(t-2) - u(t-3)$



$$x(t) = u(t+1) + 2u(t) - u(t-1) - u(t-2) - u(t-3)$$

3. Evaluate $y(t) = x(t) * h(t)$, where $x(t) = u(t) - u(t-3)$ and $h(t) = u(t) - u(t-2)$





We first express $x(t)$ and $h(t)$ in functional form:

$$x(t) = u(t) - u(t-3) \qquad h(t) = u(t) - u(t-2)$$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} [u(\tau) - u(\tau - 3)][u(t - \tau) - u(t - \tau - 2)]\, d\tau$$

$$= \int_{-\infty}^{\infty} u(\tau)u(t - \tau)\, d\tau - \int_{-\infty}^{\infty} u(\tau)u(t - 2 - \tau)\, d\tau$$

$$- \int_{-\infty}^{\infty} u(\tau - 3)u(t - \tau)\, d\tau + \int_{-\infty}^{\infty} u(\tau - 3)u(t - 2 - \tau)\, d\tau$$

$$u(\tau)u(t - \tau) = \begin{cases} 1 & 0 < \tau < t,\ t > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$u(\tau)u(t - 2 - \tau) = \begin{cases} 1 & 0 < \tau < t - 2,\ t > 2 \\ 0 & \text{otherwise} \end{cases}$$

$$u(\tau - 3)u(t - \tau) = \begin{cases} 1 & 3 < \tau < t,\ t > 3 \\ 0 & \text{otherwise} \end{cases}$$

$$u(\tau - 3)u(t - 2 - \tau) = \begin{cases} 1 & 3 < \tau < t - 2,\ t > 5 \\ 0 & \text{otherwise} \end{cases}$$

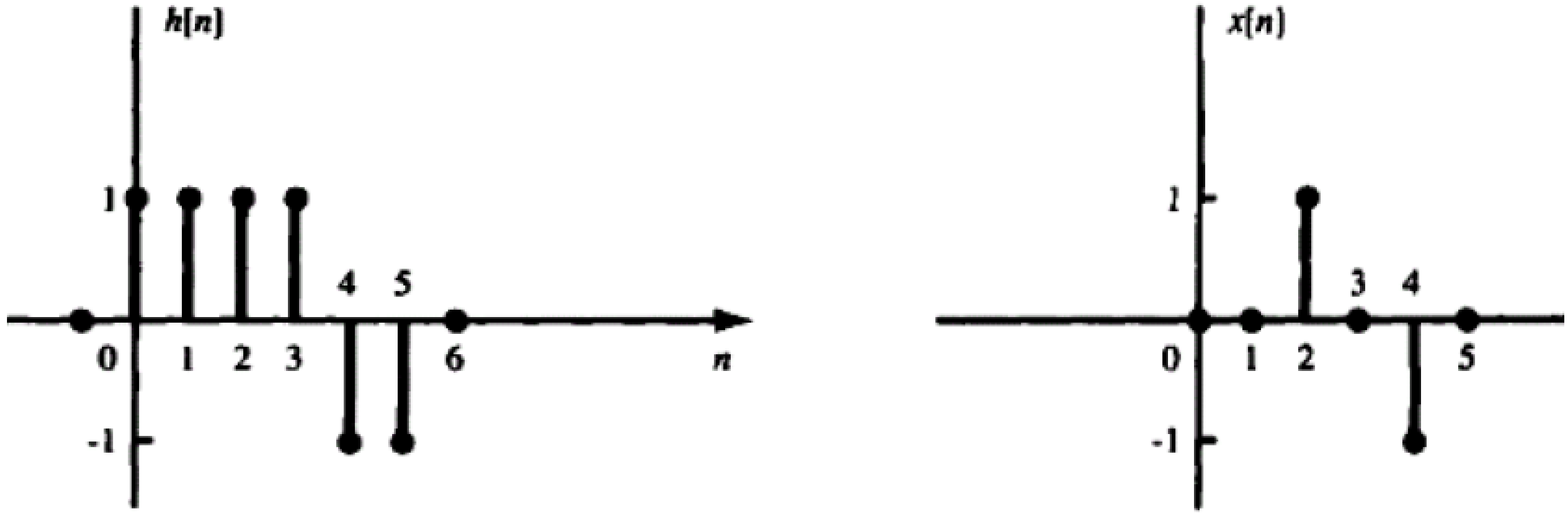
we can express $y(t)$ as

$$y(t) = \left(\int_0^t d\tau\right)u(t) - \left(\int_0^{t-2} d\tau\right)u(t-2)$$

$$- \left(\int_3^t d\tau\right)u(t-3) + \left(\int_3^{t-2} d\tau\right)u(t-5)$$

$$= tu(t) - (t-2)u(t-2) - (t-3)u(t-3) + (t-5)u(t-5)$$

5. The impulse response h[n] of a discrete-time LTI system. (a). Determine and sketch the output y[n] of this system to the input x[n].



$$h[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3] - \delta[n-4] - \delta[n-5],$$

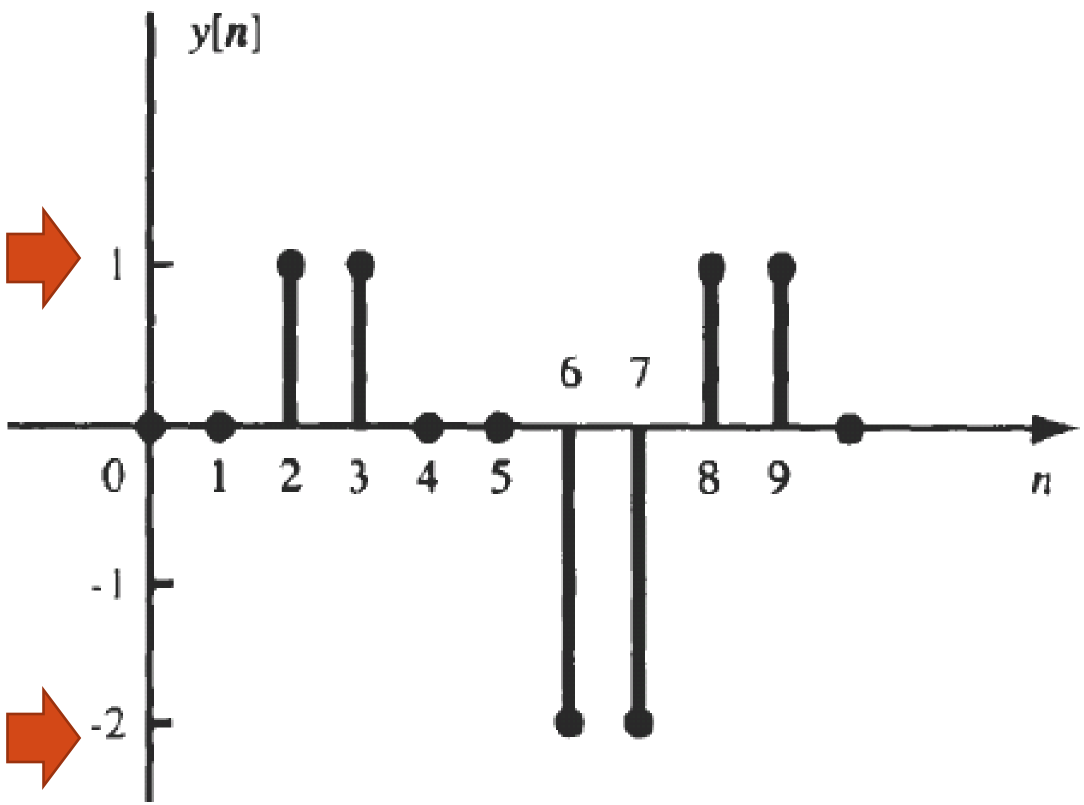
$$x[n] = \delta[n-2] - \delta[n-4]$$

$$x[n] * h[n] = x[n] * \{\delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3] - \delta[n-4] - \delta[n-5]\}$$

$$= x[n] + x[n-1] + x[n-2] + x[n-3] - x[n-4] - x[n-5]$$

$$
\begin{aligned}
y[n] &= \overbrace{\delta[n-2]-\cancel{\delta[n-4]}}^{\mathbf{X[n]}}+\overbrace{\delta[n-3]-\cancel{\delta[n-5]}}^{\mathbf{X[n-1]}}+\cancel{\delta[n-4]}-\delta[n-6]+\cancel{\delta[n-5]}-\delta[n-7] \\
&\quad -\delta[n-6]+\delta[n-8]-\delta[n-7]+\delta[n-9] \\
&= \delta[n-2]+\delta[n-3]-2\delta[n-6]-2\delta[n-7]+\delta[n-8]+\delta[n-9]
\end{aligned}
$$

$$y[n]=\{0,0,1,1,0,0,-2,-2,1,1\}$$



1.13. Consider the continuous-time signal

$$x(t) = \delta(t+2) - \delta(t-2).$$

Calculate the value of E_∞ for the signal

$$y(t) = \int_{-\infty}^{t} \underline{x(\tau)} d\tau.$$

$$y(t) = \int_{-\infty}^{t} x(\tau)dt = \int_{-\infty}^{t} (\delta(\dot{\tau}+2) - \delta(\tau-2))dt = \begin{cases} 0, & t < -2 \\ 1, & -2 \le t \le 2 \\ 0, & t > 2 \end{cases}$$

Therefore,

$$E_\infty = \int_{-2}^{2} dt = 4$$

1.14. Consider a periodic signal

$$x(t) = \begin{cases} 1, & 0 \le t \le 1 \\ -2, & 1 < t < 2 \end{cases}$$

with period $T = 2$. The derivative of this signal is related to the "impulse train"

$$g(t) = \sum_{k=-\infty}^{\infty} \delta(t - 2k)$$

with period $T = 2$. It can be shown that

$$\frac{dx(t)}{dt} = A_1 g(t - t_1) + A_2 g(t - t_2).$$

Determine the values of A_1, t_1, A_2, and t_2.

1.14. The signal $x(t)$ and its derivative $g(t)$ are shown in Figure S1.14.

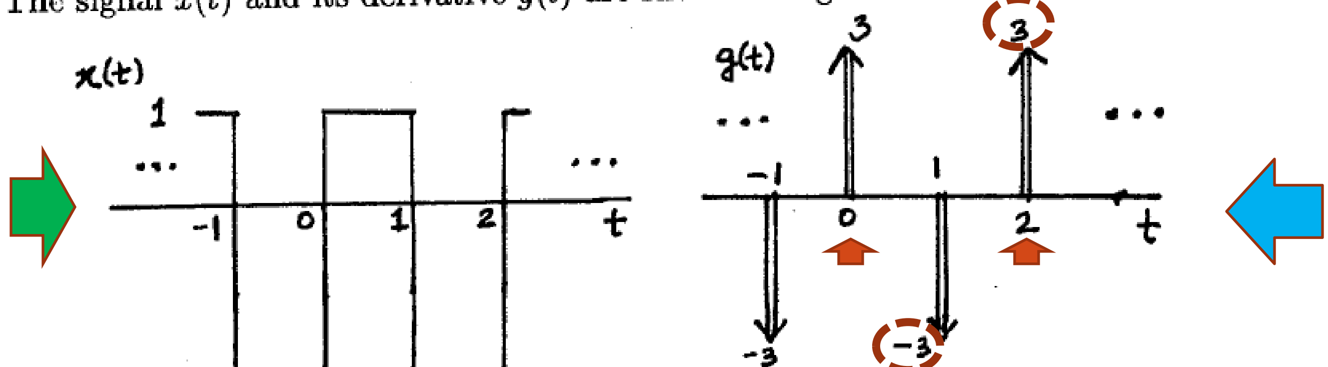


Figure S1.14

Therefore,

$$g(t) = 3 \sum_{k=-\infty}^{\infty} \delta(t - 2k) - 3 \sum_{k=-\infty}^{\infty} \delta(t - 2k - 1)$$

This implies that $A_1 = 3$, $t_1 = 0$, $A_2 = -3$, and $t_2 = 1$.

2.1. Let

$$x[n] = \delta[n] + 2\delta[n-1] - \delta[n-3] \quad \text{and} \quad h[n] = 2\delta[n+1] + 2\delta[n-1].$$

Compute and plot each of the following convolutions:

(a) $y_1[n] = x[n] * h[n]$ **(b)** $y_2[n] = x[n+2] * h[n]$

(c) $y_3[n] = x[n] * h[n+2]$

(a) We know that

$$y_1[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] \tag{S2.1–1}$$

The signals $x[n]$ and $h[n]$ are as shown in Figure S2.1.

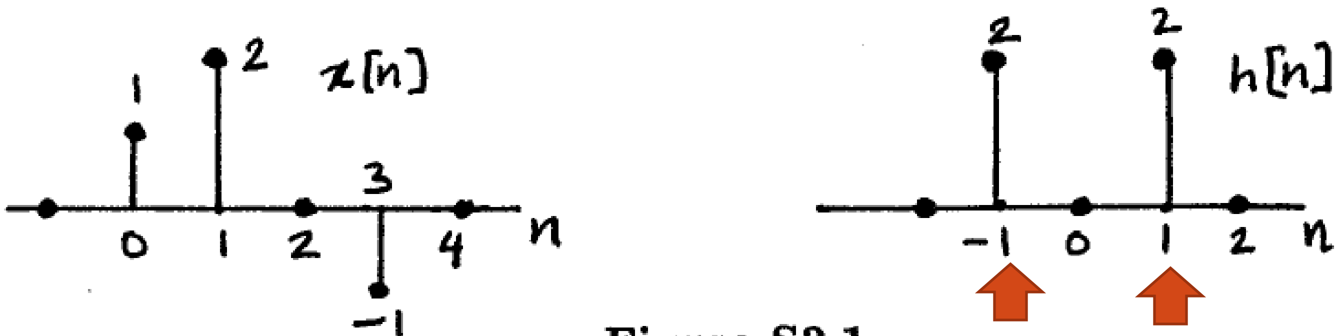


Figure S2.1

From this figure, we can easily see that the above convolution sum reduces to

$$\begin{aligned} y_1[n] &= h[-1]x[n+1] + h[1]x[n-1] \\ &= 2x[n+1] + 2x[n-1] \end{aligned}$$

This gives

$$y_1[n] = 2\delta[n+1] + 4\delta[n] + 2\delta[n-1] + 2\delta[n-2] - 2\delta[n-4]$$

(b) We know that

$$y_2[n] = x[n+2] * h[n] = \sum_{k=-\infty}^{\infty} h[k]x[n+2-k]$$

Comparing with eq. (S2.1-1), we see that

$$y_2[n] = y_1[n+2]$$

(c) We may rewrite eq. (S2.1-1) as

$$y_1[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Similarly, we may write

$$y_3[n] = x[n] * h[n+2] = \sum_{k=-\infty}^{\infty} x[k]h[n+2-k]$$

Comparing this with eq. (S2.1), we see that

$$y_3[n] = y_1[n+2]$$

2.22. For each of the following pairs of waveforms, use the convolution integral to find the response $y(t)$ of the LTI system with impulse response $h(t)$ to the input $x(t)$. Sketch your results.

(a) $\left.\begin{aligned} x(t) &= e^{-\alpha t}u(t) \\ h(t) &= e^{-\beta t}u(t) \end{aligned}\right\}$ (Do this both when $\alpha \neq \beta$ and when $\alpha = \beta$.)

(a) The desired convolution is

$$y(t) = \int_{-\infty}^{\infty} e^{-\alpha\tau}u(\tau)e^{-\beta(t-\tau)}u(t-\tau)d\tau = \int_{0}^{\infty} e^{-\alpha\tau}e^{-\beta(t-\tau)}u(t-\tau)d\tau$$

$$= \int_{0}^{t} e^{-\alpha\tau}e^{-\beta(t-\tau)}d\tau$$

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\ &= \int_{0}^{t} e^{-\alpha t}e^{-\beta(t-\tau)}d\tau, \ t \geq 0 \end{aligned}$$

Then

$$y(t) = \begin{cases} \frac{e^{-\beta t}\{e^{-(\alpha-\beta)t}-1\}}{\beta-\alpha}u(t) & \alpha \neq \beta \\ te^{-\beta t}u(t) & \alpha = \beta \end{cases}.$$

(b) $x(t) = u(t) - 2u(t-2) + u(t-5)$
$h(t) = e^{2t}u(1-t)$

(b) The desired convolution is

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\ &= \int_0^2 h(t-\tau)d\tau - \int_2^5 h(t-\tau)d\tau. \end{aligned}$$

This may be written as

$$y(t) = \begin{cases} \int_0^2 e^{2(t-\tau)}d\tau - \int_2^5 e^{2(t-\tau)}d\tau, & t \le 1 \\ \int_{t-1}^2 e^{2(t-\tau)}d\tau - \int_2^5 e^{2(t-\tau)}d\tau, & 1 \le t \le 3 \\ -\int_{t-1}^5 e^{2(t-\tau)}d\tau, & 3 \le t \le 6 \\ 0, & 6 < t \end{cases}$$

Therefore,

$$y(t) = \begin{cases} (1/2)[e^{2t} - 2e^{2(t-2)} + e^{2(t-5)}], & t \le 1 \\ (1/2)[e^{2} + e^{2(t-5)} - 2e^{2(t-2)}], & 1 \le t \le 3 \\ (1/2)[e^{2(t-5)} - e^{2}], & 3 \le t \le 6 \\ 0, & 6 < t \end{cases}$$

(b) $x(t) = u(t) - 2u(t-2) + u(t-5)$
$h(t) = e^{2t}u(1-t)$

$$y(t) = \int_{-\infty}^{\infty} [u(t) - 2u(t-2) + u(t-5)]\, h(t-\tau)\, d\tau$$

$$= \int_{0}^{2} h(t-\tau)\, d\tau - \int_{2}^{5} h(t-\tau)\, d\tau$$