Fourth Edition

LINEAR ALGEBRA AND ITS APPLICATIONS

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62. Construct a matrix whose column space contains $(1,1,5)$ and $(0,3.1)$ and whose nullspace contains $(1,1,2)$.

63. Construct a matrix whose column space contains $(1,1,0)$ and $(0,1,1)$ and whose nullspace contains $(1,0,1)$ and $(0,0,1)$.

64. Construct a matrix whose column space contains $(1,1,1)$ and whose nullspace is the line of multiples of $(1,1,1,1)$.

65. Construct a 2 by 2 matrix whose nullspace equals its column space.

66. Why does no 3 by 3 matrix have a nullspace that equals its column space?

67. The reduced form R of a 3 by 3 matrix with randomly chosen entries is almost sure to be ____. What R is virtually certain if the random A is 4 by 3?

68. Show by example that these three statements are generally false:

(a) A and A^{T} have the same nullspace.

(b) A and A^{T} have the same free variables.

(c) If R is the reduced form $\mathsf{rref}(A)$ then R^{T} is $\mathsf{rref}(A^{\mathrm{T}})$.

69. If the special solutions to $Rx = 0$ are in the columns of these N, go backward to find the nonzero rows of the reduced matrices R:

$$N = \begin{bmatrix} 2 & 3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} \\ \\ \end{bmatrix} \quad \text{(empty 3 by 1)}.$$

70. Explain why A and $-A$ always have the same reduced echelon form R.

2.3 Linear Independence, Basis, and Dimension

By themselves, the numbers m and n give an incomplete picture of the true size of a linear system. The matrix in our example had three rows and four columns, but the third row was only a combination of the first two. After elimination it became a zero row, It had no effect on the homogeneous problem $Ax = 0$. The four columns also failed to be independent, and the column space degenerated into a two-dimensional plane.

The important number that is beginning to emerge (the true size) is the ***rank*** r. The rank was introduced as the *number of pivots* in the elimination process. Equivalently, the final matrix U has r nonzero rows. This definition could be given to a computer. But it would be wrong to leave it there because the rank has a simple and intuitive meaning: *The rank counts the number of genuinely independent rows in the matrix A*. We want definitions that are mathematical rather than computational.

The goal of this section is to explain and use four ideas:

1. Linear independence or dependence.

2. Spanning a subspace.

3. Basis for a subspace (a set of vectors).

4. Dimension of a subspace (a number).

The first step is to define ***linear independence***. Given a set of vectors $v_1, \dots, v_k$, we look at their combinations $c_1v_1 + c_2v_2 + \cdots + c_kv_k$. The trivial combination, with all weights $c_i = 0$, obviously produces the zero vector: $0v_1 + \cdots + 0v_k = 0$. The question is whether this is the *only way* to produce zero. If so, the vectors are independent.

If any other combination of the vectors gives zero, they are *dependent*.

> **2E** Suppose $c_1v_1 + \cdots + c_kv_k = 0$ only happens when $c_1 = \cdots = c_k = 0$. Then the vectors $v_1, \dots, v_k$ are ***linearly independent***. If any c's are nonzero, the v's are ***linearly dependent***. One vector is a combination of the others.

Linear dependence is easy to visualize in three-dimensional space, when all vectors go out from the origin. Two vectors are dependent if they lie on the same line. *Three vectors are dependent if they lie in the same plane*. A random choice of three vectors, without any special accident, should produce linear independence (not in a plane). Four vectors are always linearly dependent in $\mathbf{R}^3$.

Example 1. If $v_1 =$ zero vector, then the set is linearly dependent. We may choose $c_1 = 3$ and all other $c_i = 0$; this is a nontrivial combination that produces zero.

Example 2. The columns of the matrix

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 0 \end{bmatrix}$$

are linearly dependent, since the second column is three times the first. The combination of columns with weights -3, 1, 0, 0 gives a column of zeros.

The rows are also linearly dependent; row 3 is two times row 2 minus five times row 1. (This is the same as the combination of b_1, b_2, b_3, that had to vanish on the right-hand side in order for $Ax = b$ to be consistent. Unless $b_3 - 2b_2 + 5b_1 = 0$, the third equation would not become $0 = 0$.)

Example 3. The columns of this triangular matrix are linearly *independent*:

$$\textbf{No zeros on the diagonal} \qquad A = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix}.$$

Look for a combination of the columns that makes zero:

$$\text{Solve } Ac = 0 \qquad c_1 \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

We have to show that c_1, c_2, c_3 ***are all forced to be zero***. The last equation gives $c_3 = 0$. Then the next equation gives $c_2 = 0$, and substituting into the first equation forces $c_1 = 0$. The only combination to produce the zero vector is the trivial combination. ***The nullspace of*** A ***contains only the zero vector*** $c_1 = c_2 = c_3 = 0$.

> ***The columns of*** A ***are independent exactly when*** $N(A) = \{\textbf{zero vector}\}$.

A similar reasoning applies to the rows of A, which are also independent. Suppose

$$c_1(3,4,2) + c_2(0,1,5) + c_3(0,0,2) = (0,0,0).$$

From the first components we find $3c_1 = 0$ or $c_1 = 0$. Then the second components give $c_2 = 0$, and finally $c_3 = 0$.

The nonzero rows of any echelon matrix U must be independent. Furthermore, if we pick out *the columns that contain the pivots*, they also are linearly independent. In our earlier example, with

$$\begin{array}{l}\textbf{Two independent rows} \\ \textbf{Two independent columns}\end{array} \qquad U = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

the pivot columns 1 and 3 are independent. No set of three columns is independent, and certainly not all four. It is true that columns 1 and 4 are also independent, but if that last 1 were changed to 0 they would be dependent. *It is the columns with pivots that are guaranteed to be independent*. The general rule is this:

2F The r nonzero rows of an echelon matrix U and a reduced matrix R are linearly independent. So are the r columns that contain pivots.

Example 4. The columns of the n by n identity matrix are independent:

$$I = \begin{bmatrix} 1 & 0 & \cdot & 0 \\ 0 & 1 & \cdot & 0 \\ \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

These columns $e_1, \dots, e_n$ represent unit vectors in the coordinate directions; in $\mathbf{R}^4$,

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \qquad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \qquad e_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Most sets of four vectors in $\mathbf{R}^4$ are independent. Those e's might be the safest.

To check any set of vectors $v_1,\dots,v_n$ for independence, put them in the columns of A. Then solve the system $Ac=0$; the vectors are dependent if there is a solution other than $c=0$. With no free variables (*rank* n), there is no nullspace except $c=0$; the vectors are independent. If the rank is less than n, at least one free variable can be nonzero and the columns are dependent.

One case has special importance. Let the n vectors have m components, so that A is an m by n matrix. Suppose now that $n>m$. There are too many columns to be independents There cannot be n pivots, since there are not enough rows to hold them. The rank will be less than n. Every system $Ac=0$ with more unknowns than equations has solutions $c \neq 0$.

2G A set of n vectors in $\mathbf{R}^m$ must be linearly dependent if $n>m$.

The reader will recognize this as a disguised form of 2C: Every m by n system $Ax=0$ has nonzero solutions if $n>m$.

Example 5. These three columns in $\mathbf{R}^2$ cannot be independent:

$$A=\begin{bmatrix}1&2&1\\1&3&2\end{bmatrix}.$$

To find the combination of the columns producing zero we solve $Ac=0$:

$$A\to U=\begin{bmatrix}1&2&1\\0&1&1\end{bmatrix}.$$

If we give the value 1 to the free variable c_3, then back-substitution in $Uc=0$ gives $c_2=-1$, $c_1=1$. With these three weights, the first column minus the second plus the third equals zero: Dependence.

Spanning a Subspace

Now we define what it means for a set of vectors to *span a space*. The column space of A is *spanned* by the columns. **Their combinations produce the whole space**:

2H If a vector space $\mathbf{V}$ consists of all linear combinations of $w_1,\dots,w_\ell$, then these vectors ***span*** the space. Every vector v in $\mathbf{V}$ is some combination of the w's:

Every v comes from w's $\qquad v=c_1w_1+\cdots+c_\ell w_\ell$ for some coefficients c_i.

It is permitted that a different combination of w's could give the same vector v. The c's need not be unique, because the spanning set might be excessively large—it could include the zero vector, or even all vectors.

Example 6. The vectors $w_1 = (1,0,0)$, $w_2 = (0,1,0)$, and $w_3 = (-2,0,0)$ span a plane (the x-y plane) in $\mathbf{R}^3$. The first two vectors also span this plane, whereas w_1 and w_3 span only a line.

Example 7. The column space of A is exactly ***the space that is spanned by its columns***. The row space is spanned by the rows. The definition is made to order. Multiplying A by any x gives a combination of the columns; it is a vector Ax in the column space.

The coordinate vectors $e_1,\dots,e_n$ coming from the identity matrix span $\mathbf{R}^n$. Every vector $b = (b_1,\dots,b_n)$ is a combination of those columns. In this example the weights are the components b_i themselves: $b = b_1e_1 + \cdots + b_ne_n$. But the columns of other matrices also span $\mathbf{R}^n$!

Basis for a Vector Space

To decide if b is a combination of the columns, we try to solve $Ax = b$. To decide if the columns are independent, we solve $Ax = 0$. ***Spanning involves the column space, and independence involves the nullspace***. The coordinate vectors $e_1,\dots,e_n$ span $\mathbf{R}^n$ and they are linearly independent. Roughly speaking, ***no vectors in that set are wasted***. This leads to the crucial idea of a ***basis***.

2I A ***basis*** for $\mathbf{V}$ is a sequence of vectors having two properties at once:

1. The vectors are linearly independent (not too many vectors).
2. They span the space $\mathbf{V}$ (not too few vectors).

This combination of properties is absolutely fundamental to linear algebra. It means that every vector in the space is a combination of the basis vectors, because they span. It also means that the combination is unique: If $v = a_1v_1 + \cdots + a_kv_k$ and also $v = b_1v_1 + \cdots + b_kv_k$, then subtraction gives $0 = \sum(a_i - b_i)v_i$. Now independence plays its part; every coefficient $a_i - b_i$ must be zero. Therefore $a_i = b_i$. ***There is one and only one way to write v as a combination of the basis vectors***.

We had better say at once that the coordinate vectors $e_1,\dots,e_n$ are not the only basis for $\mathbf{R}^n$. Some things in linear algebra are unique, but not this. A vector space has ***infinitely many different bases***. Whenever a square matrix is invertible, its columns are independent—and they are a basis for $\mathbf{R}^n$. The two columns of this nonsingular matrix are a basis for $\mathbf{R}^2$:

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$$

Every two-dimensional vector is a combination of those (independent!) columns.

Example 8. The x-y plane in Figure 2.4 is just $\mathbf{R}^2$. The vector v_1 by itself is linearly independent, but it fails to span $\mathbf{R}^2$. The three vectors v_1, v_2, v_3 certainly span $\mathbf{R}^2$, but are not independent. *Any two* of these vectors, say v_1 and v_2, have both properties—they span, and they are independent. So they form a basis. Notice again that *a vector space does not have a unique basis*.

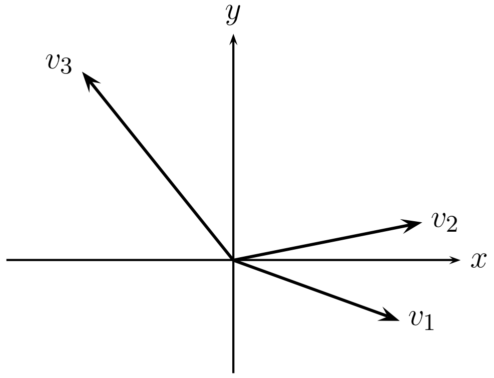

Figure 2.4: A spanning set v_1, v_2, v_3. Bases v_1, v_2 and v_1, v_3 and v_2, v_3.

Example 9. These four columns span the column space of U, but they are not independent:

$$\textbf{Echelon matrix} \qquad U = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

There are many possibilities for a basis, but we propose a specific choice: ***The columns that contain pivots*** (in this case the first and third, which correspond to the basic variables) ***are a basis for the column space***. These columns are independent, and it is easy to see that they span the space. In fact, the column space of U is just the x-y plane within $\mathbf{R}^3$. $C(U)$ is *not the same* as the column space $C(A)$ before elimination—but the *number* of independent columns didn't change.

To summarize: *The columns of any matrix span its column space*. If they are independent, they are a basis for the column space—whether the matrix is square or rectangular. If we are asking the columns to be a basis for the whole space $\mathbf{R}^n$, then the matrix must be square and invertible.

Dimension of a Vector Space

A space has infinitely many different bases, but there is something common to all of these choices. The ***number of basis vectors*** is a property of the space itself:

> **2J** Any two bases for a vector space $\mathbf{V}$ contain the same number of vectors. This number, which is shared by all bases and expresses the number of "degrees of freedom" of the space, is the ***dimension*** of $\mathbf{V}$.

We have to prove this fact: All possible bases contain the same number of vectors. The x-y plane in Figure 2.4 has two vectors in every basis; its dimension is 2. In three dimensions we need three vectors, along the x-y-z axes or in three other (linearly independent!) directions. ***The dimension of the space*** $\mathbf{R}^n$ ***is*** n. The column space of U in Example 9 had dimension 2; it was a "two-dimensional subspace of $\mathbf{R}^3$." The zero matrix is rather exceptional, because its column space contains only the zero vector. By convention, the empty set is a basis for that space, and its dimension is zero.

Here is our first big theorem in linear algebra:

> **2K** If $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$ are both bases for the same vector space, then $m = n$. The number of vectors is the same.

Proof. Suppose there are more w's than v's ($n > m$). We will arrive at a contradiction. Since the v's form a basis, they must span the space. *Every w_j can be written as a combination of the v's*: If $w_1 = a_{11}v_1 + \cdots + a_{m1}v_m$, this is the first column of a matrix multiplication VA:

$$W = \begin{bmatrix} w_1 & w_2 & \cdots & w_n \end{bmatrix} = \begin{bmatrix} v_1 & \cdots & v_m \end{bmatrix} \begin{bmatrix} a_{11} & & \\ \vdots & & \\ a_{m1} & & \end{bmatrix} = VA.$$

We don't know each a_{ij}, but we know the shape of A (it is m by n). The second vector w_2 is also a combination of the v's. The coefficients in that combination fill the second column of A. The key is that A has a row for every v and a column for every w. A is a short, wide matrix, since $n > m$. ***There is a nonzero solution to*** $Ax = 0$. Then $VAx = 0$ which is $Wx = 0$. *A combination of the w's gives zero!* The w's could not be a basis—so we cannot have $n > m$.

If $m > n$ we exchange the v's and w's and repeat the same steps. The only way to avoid a contradiction is to have $m = n$. This completes the proof that $m = n$. To repeat: The ***dimension of a space*** is the number of vectors in every basis. □

This proof was used earlier to show that every set of $m+1$ vectors in $\mathbf{R}^m$ must be dependent. The v's and w's need not be column vectors—the proof was all about the matrix A of coefficients. In fact we can see this general result: *In a subspace of dimension k, no set of more than k vectors can be independent, and no set of more than k vectors can. span the space.*

There are other "dual" theorems, of which we mention only one. We can start with a set of vectors that is too small or too big, and end up with a basis:

> **2L** Any linearly independent set in $\mathbf{V}$ can be extended to a basis, by adding more vectors if necessary.
>
> Any spanning set in $\mathbf{V}$ can be reduced to a basis, by discarding vectors if necessary.

The point is that a basis is a ***maximal independent set***. It cannot be made larger without losing independence. A basis is also a ***minimal spanning set***. It cannot be made smaller and still span the space.

You must notice that the word "dimensional" is used in two different ways. We speak about a four-dimensional ***vector***, meaning a vector in $\mathbf{R}^4$. Now we have defined a four-dimensional ***subspace***; an example is the set of vectors in $\mathbf{R}^6$ whose first and last components are zero. The members of this four-dimensional subspace are six-dimensional vectors like $(0,5,1,3,4,0)$.

One final note about the language of linear algebra. We never use the terms "basis of a matrix" or "rank of a space" or "dimension of a basis." These phrases have no meaning. It is *the dimension of the column space that equals the rank of the matrix*, as we prove in the coming section.

Problem Set 2.3

Problems 1–10 are about linear independence and linear dependence.

1. Show that v_1, v_2, v_3 are independent but v_1, v_2, v_3, v_4 are dependent:

$$v_1 = \begin{bmatrix}1\\0\\0\end{bmatrix} \quad v_2 = \begin{bmatrix}1\\1\\0\end{bmatrix} \quad v_3 = \begin{bmatrix}1\\1\\1\end{bmatrix} \quad v_4 = \begin{bmatrix}2\\3\\4\end{bmatrix}.$$

Solve $c_1v_1 + \cdots + c_4v_4 = 0$ or $Ac = 0$. The v's go in the columns of A.

2. Find the largest possible number of independent vectors among

$$v_1 = \begin{bmatrix}1\\-1\\0\\0\end{bmatrix} \quad v_2 = \begin{bmatrix}1\\0\\-1\\0\end{bmatrix} \quad v_3 = \begin{bmatrix}1\\0\\0\\-1\end{bmatrix} \quad v_4 = \begin{bmatrix}0\\1\\-1\\0\end{bmatrix} \quad v_5 = \begin{bmatrix}0\\1\\0\\-1\end{bmatrix} \quad v_6 = \begin{bmatrix}0\\0\\1\\-1\end{bmatrix}.$$

This number is the ____ of the space spanned by the v's.

3. Prove that if $a = 0$, $d = 0$, or $f = 0$ (3 cases), the columns of U are dependent:

$$U = \begin{bmatrix}a & b & c\\0 & d & e\\0 & 0 & f\end{bmatrix}.$$

4. If a, d, f in Problem 3 are all nonzero, show that the only solution to $Ux = 0$ is $x = 0$. Then U has independent columns.

5. Decide the dependence or independence of

(a) the vectors $(1,3,2)$, $(2,1,3)$, and $(3.2,1)$.

(b) the vectors $(1,-3,2)$, $(2,1,-3)$, and $(-3,2,1)$.

6. Choose three independent columns of U. Then make two other choices. Do the same for A. You have found bases for which spaces?

$$U = \begin{bmatrix} 2 & 3 & 4 & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 2 & 3 & 4 & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 4 & 6 & 8 & 2 \end{bmatrix}.$$

7. If w_1, w_2, w_3 are independent vectors, show that the differences $v_1 = w_2 - w_3$, $v_2 = w_1 - w_3$, and $v_3 = w_1 - w_2$ are *dependent*. Find a combination of the v's that gives zero.

8. If w_1, w_2, w_3 are independent vectors, show that the sums $v_1 = w_2 + w_3$, $v_2 = w_1 + w_3$, and $v_3 = w_1 + w_2$ are *independent*. (Write $c_1v_1 + c_2v_2 + c_3v_3 = 0$ in terms of the w's. Find and solve equations for the c's.)

9. Suppose v_1, v_2, v_3, v_4 are vectors in $\mathbf{R}^3$.

(a) These four vectors are dependent because ____.

(b) The two vectors v_1 and v_2 will be dependent if ____.

(c) The vectors v_1 and $(0,0,0)$ are dependent because ____.

10. Find two independent vectors on the plane $x + 2y - 3z - t = 0$ in $\mathbf{R}^4$. Then find three independent vectors. Why not four? This plane is the nullspace of what matrix?

Problems 11–18 are about the space *spanned* by a set of vectors. Take all linear combinations of the vectors

11. Describe the subspace of $\mathbf{R}^3$ (is it a line or a plane or $\mathbf{R}^3$?) spanned by

(a) the two vectors $(1,1,-1)$ and $(-1,-1,1)$.

(b) the three vectors $(0,1,1)$ and $(1,1,0)$ and $(0,0,0)$.

(c) the columns of a 3 by 5 echelon matrix with 2 pivots.

(d) all vectors with positive components.

12. The vector b is in the subspace spanned by the columns of A when there is a solution to ____. The vector c is in the row space of A when there is a solution to ____. *True or false*: If the zero vector is in the row space, the rows are dependent.

13. Find the dimensions of (a) the column space of A, (b) the column space of U, (c) the row space of A, (d) the row space of U. Which two of the spaces are the same?

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 3 & 1 & -1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

14. Choose $x = (x_1, x_2, x_3, x_4)$ in $\mathbf{R}^4$. It has 24 rearrangements like (x_2, x_1, x_3, x_4) and (x_4, x_3, x_1, x_2). Those 24 vectors, including x itself, span a subspace $\mathbf{S}$. Find specific vectors x so that the dimension of $\mathbf{S}$ is: (a) 0, (b) 1, (c) 3, (d) 4.

15. $v+w$ and $v-w$ are combinations of v and w. Write v and w as combinations of $v+w$ and $v-w$. The two pairs of vectors ____ the same space. When are they a basis for the same space?

16. Decide whether or not the following vectors are linearly independent, by solving $c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = 0$:

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \qquad v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \qquad v_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

Decide also if they span $\mathbf{R}^4$, by trying to solve $c_1v_1 + \cdots + c_4v_4 = (0,0,0,1)$.

17. Suppose the vectors to be tested for independence are placed into the rows instead of the columns of A, How does the elimination process from A to U decide for or against independence?

18. To decide whether b is in the sub space spanned by $w_1, \ldots, w_n$, let the vectors w be the columns of A and try to solve $Ax = b$. What is the result for

(a) $w_1 = (1,1,0)$, $w_2 = (2,2,1)$, $w_3 = (0,0,2)$, $b = (3,4,5)$?

(b) $w_1 = (1,2,0)$, $w_2 = (2,5,0)$, $w_3 = (0,0,2)$, $w_4 = (0,0,0)$, and any b?

Problems 19–37 are about the requirements for a basis.

19. If $v_1, \ldots, v_n$ are linearly independent, the space they span has dimension ____. These vectors are a ____ for that space. If the vectors are the columns of an m by n matrix, then m is ____ than n.

20. Find a basis for each of these subspaces of $\mathbf{R}^4$:

(a) All vectors whose components are equal.

(b) All vectors whose components add to zero.

(c) All vectors that are perpendicular to $(1,1,0,0)$ and $(1,0,1,1)$.

(d) The column space (in $\mathbf{R}^2$) and nullspace (in $\mathbf{R}^5$) of $U = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$.

21. Find three different bases for the column space of U above. Then find two different bases for the row space of U.

22. Suppose $v_1, v_2, \ldots, v_6$ are six vectors in $\mathbf{R}^4$.

(a) Those vectors (do)(do not)(might not) span $\mathbf{R}^4$.

(b) Those vectors (are)(are not)(might be) linearly independent.

(c) Any four of those vectors (are)(are not)(might be) a basis for $\mathbf{R}^4$.

(d) If those vectors are the columns of A, then $Ax = b$ (has) (does not have) (might not have) a solution.

23. The columns of A are n vectors from $\mathbf{R}^m$. If they are linearly independent, what is the rank of A? If they span $\mathbf{R}^m$, what is the rank? If they are a basis for $\mathbf{R}^m$, what then?

24. Find a basis for the plane $x - 2y + 3z = 0$ in $\mathbf{R}^3$. Then find a basis for the intersection of that plane with the xy-plane. Then find a basis for all vectors perpendicular to the plane.

25. Suppose the columns of a 5 by 5 matrix A are a basis for $\mathbf{R}^5$.

(a) The equation $Ax = 0$ has only the solution $x = 0$ because ____.

(b) If b is in $\mathbf{R}^5$ then $Ax = b$ is solvable because ____.

Conclusion: A is invertible. Its rank is 5.

26. Suppose $\mathbf{S}$ is a five-dimensional subspace of $\mathbf{R}^6$. True or false?

(a) Every basis for $\mathbf{S}$ can be extended to a basis for $\mathbf{R}^6$ by adding one more vector.

(b) Every basis for $\mathbf{R}^6$ can be reduced to a basis for $\mathbf{S}$ by removing one vector.

27. U comes from A by subtracting row 1 from row 3:

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Find bases for the two column spaces. Find bases for the two row spaces. Find bases for the two nullspace.

28. True or false (give a good reason)?

(a) If the columns of a matrix are dependent, so are the rows.

(b) The column space of a 2 by 2 matrix is the same as its row space.

(c) The column space of a 2 by 2 matrix has the same dimension as its row space.

(d) The columns of a matrix are a basis for the column space.

29. For which numbers c and d do these matrices have rank 2?

$$A = \begin{bmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & c & 2 & 2 \\ 0 & 0 & 0 & d & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} c & d \\ d & c \end{bmatrix}.$$

30. By locating the pivots, find a basis for the column space of

$$U = \begin{bmatrix} 0 & 5 & 4 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Express each column that is not in the basis as a combination of the basic columns, Find also a matrix A with this echelon form U, but a different column space.

31. Find a counterexample to the following statement: If v_1, v_2, v_3, v_4 is a basis for the vector space $\mathbf{R}^4$, and if $\mathbf{W}$ is a subspace, then some subset of the v's is a basis for $\mathbf{W}$.

32. Find the dimensions of these vector spaces:

(a) The space of all vectors in $\mathbf{R}^4$ whose components add to zero.

(b) The nullspace of the 4 by 4 identity matrix.

(c) The space of all 4 by 4 matrices.

33. Suppose $\mathbf{V}$ is known to have dimension k. Prove that

(a) any k independent vectors in $\mathbf{V}$ form a basis;

(b) any k vectors that span $\mathbf{V}$ form a basis.

In other words, if the number of vectors is known to be correct, either of the two properties of a basis implies the other.

34. Prove that if $\mathbf{V}$ and $\mathbf{W}$ are three-dimensional subspaces of $\mathbf{R}^5$, then $\mathbf{V}$ and $\mathbf{W}$ must have a nonzero vector in common. *Hint*: Start with bases for the two subspaces, making six vectors in all.

35. *True or false?*

(a) If the columns of A are linearly independent, then $Ax = b$ has exactly one solution for every b.

(b) A 5 by 7 matrix never has linearly independent columns,

36. If A is a 64 by 17 matrix of rank 11, how many independent vectors satisfy $Ax = 0$? How many independent vectors satisfy $A^{\mathrm{T}}y = 0$?

37. Find a basis for each of these subspaces of 3 by 3 matrices:

(a) All diagonal matrices.

(b) All symmetric matrices ($A^{\mathrm{T}} = A$).

(c) All skew-symmetric matrices ($A^{\mathrm{T}} = -A$).

Problems 38–42 are about spaces in which the "vectors" are functions.

38. (a) Find all functions that satisfy $\frac{dy}{dx} = 0$.

(b) Choose a particular function that satisfies $\frac{dy}{dx} = 3$.

(c) Find all functions that satisfy $\frac{dy}{dx} = 3$.

39. The cosine space $\mathbf{F}_3$ contains all combinations $y(x) = A\cos x + B\cos 2x + C\cos 3x$. Find a basis for the subspace that has $y(0) = 0$.

40. Find a basis for the space of functions that satisfy

(a) $\dfrac{dy}{dx} - 2y = 0$.

(b) $\dfrac{dy}{dx} - \dfrac{y}{x} = 0$.

41. Suppose $y_1(x)$, $y_2(x)$, $y_3(x)$ are three different functions of x. The vector space they span could have dimension 1, 2, or 3. Give an example of y_1, y_2, y_3 to show each possibility.

42. Find a basis for the space of polynomials $p(x)$ of degree ≤ 3. Find a basis for the subspace with $p(1) = 0$.

43. Write the 3 by 3 identity matrix as a combination of the other five permutation matrices! Then show that those five matrices are linearly independent. (Assume a combi nation gives zero, and check entries to prove each term is zero.) The five permutations are a basis for the subspace of 3 by 3 matrices with row and column sums all equal.

44. *Review*: Which of the following are bases for $\mathbf{R}^3$?

(a) $(1,2,0)$ and $(0,1,-1)$.

(b) $(1,1,-1)$, $(2,3,4)$, $(4,1,-1)$, $(0,1,-1)$.

(c) $(1,2,2)$, $(-1,2,1)$, $(0,8,0)$.

(d) $(1,2,2)$, $(-1,2,1)$, $(0,8,6)$.

45. *Review*: Suppose A is 5 by 4 with rank 4. Show that $Ax = b$ has no solution when the 5 by 5 matrix $[A\ \ b]$ is invertible. Show that $Ax = b$ is solvable when $[A\ \ b]$ is singular.

2.4 The Four Fundamental Subspaces

The previous section dealt with definitions rather than constructions. We know what a basis is, but not how to find one. Now, starting from an explicit description of a subspace, we would like to compute an explicit basis.

Subspaces can be described in two ways. First, we may be given a set of vectors that span the space. (*Example*: The columns span the column space.) Second, we may be told which conditions the vectors in the space must satisfy. (*Example*: The nullspace consists of all vectors that satisfy $Ax = 0$.)

The first description may include useless vectors (dependent columns). The second description may include repeated conditions (dependent rows). We can't write a basis by inspection, and a systematic procedure is necessary.

The reader can guess what that procedure will be. When elimination on A produces an echelon matrix U or a reduced R, we will find a basis for each of the subspaces associated with A. Then we have to look at the extreme case of **full rank**:

> *When the rank is as large as possible*, $r = n$ *or* $r = m$ *or* $r = m = n$, *the matrix has a* **left-inverse** B *or a* **right-inverse** C *or a* **two-sided** A^{-1}.

To organize the whole discussion, we take each of the four subspaces in turn. Two of them are familiar and two are new.

1. The ***column space*** of A is denoted by $\boldsymbol{C}(A)$. Its dimension is the rank r.
2. The ***nullspace*** of A is denoted by $\boldsymbol{N}(A)$. Its dimension is $n - r$.
3. The ***row space*** of A is the *column space* of A^{T}. It is $\boldsymbol{C}(A^{\mathrm{T}})$, and it is spanned by the rows of A. Its dimension is also r.
4. The ***left nullspace*** of A is the *nullspace* of A^{T}. It contains all vectors y such that $A^{\mathrm{T}}y = 0$, and it is written $\boldsymbol{N}(A^{\mathrm{T}})$. Its dimension is ____.

The point about the last two subspaces is that *they come from* A^{T}. If A is an m by n matrix, you can see which "host" spaces contain the four subspaces by looking at the number of components:

> The nullspace $\boldsymbol{N}(A)$ and row space $\boldsymbol{C}(A^{\mathrm{T}})$ are subspaces of $\mathbf{R}^n$.
> The left nullspace $\boldsymbol{N}(A^{\mathrm{T}})$ and column space $\boldsymbol{C}(A)$ are subspaces of $\mathbf{R}^m$.

The rows have n components and the columns have m. For a simple matrix like

$$A = U = R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

the column space is the line through $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$. The row space is the line through $[1 \ \ 0 \ \ 0]^{\mathrm{T}}$. It is in $\mathbf{R}^3$. The nullspace is a plane in $\mathbf{R}^3$ and the left nullspace is a line in $\mathbf{R}^2$:

$$\boldsymbol{N}(A) \quad \text{contains} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \boldsymbol{N}(A^{\mathrm{T}}) \quad \text{contains} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Note that all vectors are column vectors. Even the rows are transposed, and the row space of A is the *column* space of A^{T}, Our problem will be to connect the four spaces for U (after elimination) to the four spaces for A:

Basic example $$U = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{came from} \quad A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}.$$

For novelty, we take the four subspaces in a more interesting order.

3. The row space of A For an echelon matrix like U, the row space is clear. It contains all combinations of the rows, as every row space does—but here the third row contributes nothing. The first two rows are a basis for the row space. A similar rule applies to every echelon matrix U or R, with r pivots and r nonzero rows: ***The nonzero rows are a basis, and the row space has dimension*** r. That makes it easy to deal with the original matrix A.

> **2M** The row space of A has the same dimension r as the row space of U, and it has the same bases, because ***the row spaces of*** A ***and*** U ***(and*** R***) are the same***.

The reason is that each elementary operation leaves the row space unchanged. The rows in U are combinations of the original rows in A. Therefore the row space of U contains nothing new. At the same time, because every step can be reversed, nothing is lost; the rows of A can be recovered from U. It is true that A and U have different rows, but the *combinations* of the rows are identical: *same space*!

Note that we did not start with the m rows of A, which span the row space, and discard $m-r$ of them to end up with a basis. According to 2L, we could have done so. But it might be hard to decide which rows to keep and which to discard, so it was easier just to take the nonzero rows of U.

2. The nullspace of A Elimination simplifies a system of linear equations without changing the solutions. The system $Ax = 0$ is reduced to $Ux = 0$, and this process is reversible. ***The nullspace of*** A ***is the same as the nullspace of*** U ***and*** R. Only r of the equations $Ax = 0$ are independent. Choosing the $n-r$ "special solutions" to $Ax = 0$ provides a definite basis for the nullspace:

> **2N** The nullspace $\boldsymbol{N}(A)$ has dimension $n-r$. The "special solutions" are a basis—each free variable is given the value 1, while the other free variables are 0. Then $Ax = 0$ or $Ux = 0$ or $Rx = 0$ gives the pivot variables by back-substitution.

This is exactly the way we have been solving $Ux = 0$. The basic example above has pivots in columns 1 and 3. Therefore its free variables are the second and fourth v and y.

The basis for the nullspace is

$$\textbf{Special solutions}\quad \begin{array}{rcl} v&=&1\\ y&=&0 \end{array}\quad x_1=\begin{bmatrix}-3\\1\\0\\0\end{bmatrix};\qquad \begin{array}{rcl} v&=&0\\ y&=&1 \end{array}\quad x_2=\begin{bmatrix}1\\0\\-1\\1\end{bmatrix}.$$

Any combination $c_1x_1+c_2x_2$ has c_1 as its v component, and c_2 as its y component. The only way to have $c_1x_1+c_2x_2=0$ is to have $c_1=c_2=0$, so these vectors are independent. They also span the nullspace; the complete solution is vx_1+yx_2. Thus the $n-r=4-2$ vectors are a basis.

The nullspace is also called the *kernel* of A, and its dimension $n-r$ is the *nullity*.

1. The column space of A The column space is sometimes called the **range**. This is consistent with the usual idea of the range, as the set of all possible values $f(x)$; x is in the domain and $f(x)$ is in the range. In our case the function is $f(x)=Ax$. Its domain consists of all x in $\mathbf{R}^n$; its range is all possible vectors Ax, which is the column space. (In an earlier edition of this book we called it $R(A)$.)

Our problem is to find bases for the column spaces of U and A. ***Those spaces are different*** (just look at the matrices!) but their dimensions are the same.

The first and third columns of U are a basis for its column space. They are the ***columns with pivots***. Every other column is a combination of those two. Furthermore, the same is true of the original A—even though its columns are different. ***The pivot columns of*** A ***are a basis for its column space***. The second column is three times the first, just as in U. The fourth column equals (column 3) − (column 1). The same nullspace is telling us those dependencies.

The reason is this: $Ax=0$ *exactly when* $Ux=0$. The two systems are equivalent and have the same solutions. The fourth column of U was also (column 3) − (column 1). Every linear dependence $Ax=0$ among the columns of A is matched by a dependence $Ux=0$ among the columns of U, with exactly the same coefficients. *If a set of columns of A is independent, then so are the corresponding columns of U, and vice versa.*

To find a basis for the column space $\boldsymbol{C}(A)$, we use what is already done for U. The r columns containing pivots are a basis for the column space of U. We will pick those same r columns in A:

> **2O** The dimension of the column space $\boldsymbol{C}(A)$ equals the rank r, which also equals the dimension of the row space: ***The number of independent columns equals the number of independent rows***. A basis for $\boldsymbol{C}(A)$ is formed by the r columns of A that correspond, in U, to the columns containing pivots.

The row space and the column space have the same dimension r! This is one of the most important theorems in linear algebra. It is often abbreviated as "***row rank*** = ***column rank***." It expresses a result that, for a random 10 by 12 matrix, is not at all obvious. It also says something about square matrices: *If the rows of a square matrix are linearly independent, then so are the columns* (and vice versa). Again, that does not seem self-evident (at least, not to the author).

To see once more that both the row and column spaces of U have dimension r, consider a typical situation with rank $r = 3$. The echelon matrix U certainly has three independent rows:

$$U = \begin{bmatrix} d_1 & * & * & * & * & * \\ 0 & 0 & 0 & d_2 & * & * \\ 0 & 0 & 0 & 0 & 0 & d_3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We claim that U also has three independent columns, and no more, The columns have only three nonzero components. If we can show that the pivot columns—the first, fourth, and sixth—are linearly independent, they must be a basis (for the column space of U, not A!). Suppose a combination of these pivot columns produced zero:

$$c_1 \begin{bmatrix} d_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} * \\ d_2 \\ 0 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} * \\ * \\ d_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Working upward in the usual way, c_3 must be zero because the pivot $d_3 \neq 0$, then c_2 must be zero because $d_2 \neq 0$, and finally $c_1 = 0$. This establishes independence and completes the proof. Since $Ax = 0$ if and only if $Ux = 0$, the first, fourth, and sixth columns of A—whatever the original matrix A was, which we do not even know in this example—are a basis for $\boldsymbol{C}(A)$.

The row space and column space both became clear after elimination on A. Now comes the fourth fundamental subspace, which has been keeping quietly out of sight. Since the first three spaces were $\boldsymbol{C}(A)$, $\boldsymbol{N}(A)$, and $\boldsymbol{C}(A^{\mathrm{T}})$, the fourth space must be $\boldsymbol{N}(A^{\mathrm{T}})$, It is the nullspace of the transpose, or the ***left nullspace*** of A. $A^{\mathrm{T}}y = 0$ means $y^{\mathrm{T}}A = 0$, and the vector appears on the left-hand side of A.

4. The left nullspace of A (= the nullspace of A^{T}) If A is an m by n matrix, then A^{T} is n by m. Its nullspace is a subspace of $\mathbf{R}^m$; the vector y has m components. Written as $y^{\mathrm{T}}A = 0$, those components multiply the *rows* of A to produce the zero row:

$$y^{\mathrm{T}}A = \begin{bmatrix} y_1 & \cdots & y_m \end{bmatrix} \begin{bmatrix} & A & \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 \end{bmatrix}.$$

The dimension of this nullspace $\boldsymbol{N}(A^{\mathrm{T}})$ is easy to find, For *any* matrix, ***the number of pivot variables plus the number of free variables must match the total number of columns***. For A, that was $r + (n - r) = n$. In other words, rank plus nullity equals n:

$$\text{dimension of } \boldsymbol{C}(A) + \text{dimension of } \boldsymbol{N}(A) = \text{number of columns.}$$

This law applies equally to A^{T}, which has m columns. A^{T} is just as good a matrix as A. But the dimension of its column space is also r, so

$$r + \text{dimension } \left(\boldsymbol{N}(A^{\mathrm{T}})\right) = m. \tag{1}$$

2P The left nullspace $\boldsymbol{N}(A^{\mathrm{T}})$ has dimension $m - r$.

The $m - r$ solutions to $y^{\mathrm{T}}A = 0$ are hiding somewhere in elimination. The rows of A combine to produce the $m - r$ *zero rows* of U. Start from $PA = LU$, or $L^{-1}PA = U$. The last $m - r$ rows of the invertible matrix $L^{-1}P$ must be a basis of y's in the left nullspace—because they multiply A to give the zero rows in U.

In our 3 by 4 example, the zero row was row 3 − 2(row 2) + 5(row 1). Therefore the components of y are 5, −2, 1. This is the same combination as in $b_3 - 2b_2 + 5b_1$ on the right-hand side, leading to $0 = 0$ as the final equation. That vector y is a basis for the left nullspace, which has dimension $m - r = 3 - 2 = 1$. It is the last row of $L^{-1}P$, and produces the zero row in U—and we can often see it without computing L^{-1}. When desperate, it is always possible just to solve $A^{\mathrm{T}}y = 0$.

I realize that so far in this book we have given no reason to care about $\boldsymbol{N}(A^{\mathrm{T}})$. It is correct but not convincing if I write in italics that *the left nullspace is also important*. The next section does better by finding a physical meaning for y from Kirchhoff's Current Law.

Now we know the dimensions of the four spaces. We can summarize them in a table, and it even seems fair to advertise them as the

Fundamental Theorem of Linear Algebra, Part I

1. $\boldsymbol{C}(A)$ = column space of A; dimension r.

2. $\boldsymbol{N}(A)$ = nullspace of A; dimension $n - r$.

3. $\boldsymbol{C}(A^{\mathrm{T}})$ = row space of A; dimension r.

4. $\boldsymbol{N}(A^{\mathrm{T}})$ = left nullspace of A; dimension $m - r$.

Example 1. $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ has $m = n = 2$, and rank $r = 1$.

1. The ***column space*** contains all multiples of $\left[\begin{smallmatrix} 1 \\ 3 \end{smallmatrix}\right]$. The second column is in the same direction and contributes nothing new.
2. The ***nullspace*** contains all multiples of $\left[\begin{smallmatrix} -2 \\ 1 \end{smallmatrix}\right]$. This vector satisfies $Ax = 0$.
3. The ***row space*** contains all multiples of $\left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right]$. I write it as a column vector, since strictly speaking it is in the column space of A^{T}.
4. The ***left nullspace*** contains all multiples of $y = \left[\begin{smallmatrix} -3 \\ 1 \end{smallmatrix}\right]$. The rows of A with coefficients −3 and 1 add to zero, so $A^{\mathrm{T}}y = 0$.

In this example *all four subspaces are lines*. That is an accident, coming from $r = 1$ and $n - r = 1$ and $m - r = 1$. Figure 2.5 shows that two pairs of lines are perpendicular. That is no accident!

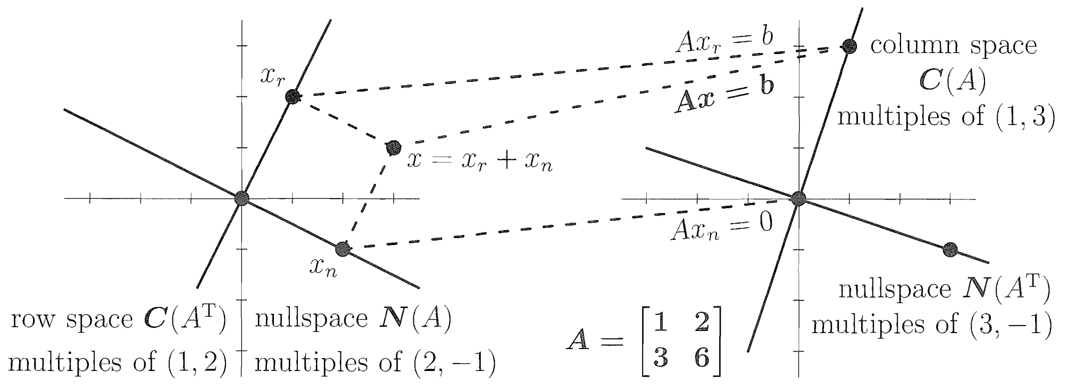

Figure 2.5: The four fundamental subspaces (lines) for the singular matrix A.

If you change the last entry of A from 6 to 7, all the dimensions are different. The column space and row space have dimension $r = 2$. The nullspace and left nullspace contain only the vectors $x = 0$ and $y = 0$. *The matrix is invertible.*

Existence of Inverses

We know that if A has a left-inverse ($BA = I$) and a right-inverse ($AC = I$), then the two inverses are equal: $B = B(AC) = (BA)C = C$. Now, from the rank of a matrix, it is easy to decide which matrices actually have these inverses. Roughly speaking, ***an inverse exists only when the rank is as large as possible***.

The rank always satisfies $r \le m$ and also $r \le n$. An m by n matrix cannot have more than m independent rows or n independent columns. There is not space for more than m pivots, or more than n. We want to prove that when $r = m$ there is a right-inverse, and $Ax = b$ always has a solution. When $r = n$ there is a left-inverse, and the solution (*if it exists*) is unique.

Only a square matrix can have both $r = m$ and $r = n$, and therefore only a square matrix can achieve both existence and uniqueness. Only a square matrix has a two-sided inverse.

2Q EXISTENCE: Full row rank $r = m$. $Ax = b$ has ***at least*** one solution x for every b if and only if the columns span $\mathbf{R}^m$. Then A has a ***right-inverse*** C such that $AC = I_m$ (m by m). This is possible only if $m \le n$.

UNIQUENESS: Full column rank $r = n$. $Ax = b$ has ***at most*** one solution x for every b if and only if the columns are linearly independent. Then A has an n by m **left-inverse** B such that $BA = I_n$. This is possible only if $m \ge n$.

In the existence case, one possible solution is $x = Cb$, since then $Ax = ACb = b$. But there will be other solutions if there are other right-inverses. The number of solutions when the columns span $\mathbf{R}^m$ is 1 or ∞.

In the uniqueness case, if there is a solution to $Ax = b$, it has to be $x = BAx = Bb$. But there may be no solution. The number of solutions is 0 or 1.

There are simple formulas for the best left and right inverses, if they exist:

$$\textbf{One-sided inverses} \qquad B = (A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}} \quad \text{and} \quad C = A^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}.$$

Certainly $BA = I$ and $AC = I$. What is not so certain is that $A^{\mathrm{T}}A$ and AA^{T} are actually invertible. We show in Chapter 3 that $A^{\mathrm{T}}A$ does have an inverse if the rank is n, and AA^{T} has an inverse when the rank is m. Thus the formulas make sense exactly when the rank is as large as possible, and the one-sided inverses are found.

Example 2. Consider a simple 2 by 3 matrix of rank 2:

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix}.$$

Since $r = m = 2$, the theorem guarantees a right-inverse C:

$$AC = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{5} \\ c_{31} & c_{32} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

There are many right-inverses because the last row of C is completely arbitrary. This is a case of existence but not uniqueness. The matrix A has no left-inverse because the last column of BA is certain to be zero. The specific right-inverse $C = A^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}$ chooses c_{31} and c_{32} to be zero:

$$\textbf{Best right-inverse} \qquad A^{\mathrm{T}}(AA^{\mathrm{T}})^{-1} = \begin{bmatrix} 4 & 0 \\ 0 & 5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{16} & 0 \\ 0 & \frac{1}{25} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{5} \\ 0 & 0 \end{bmatrix} = C.$$

This is the *pseudoinverse*—a way of choosing the best C in Section 6.3. The transpose of A yields an example with infinitely many *left*-inverses:

$$BA^{\mathrm{T}} = \begin{bmatrix} \frac{1}{4} & 0 & b_{13} \\ 0 & \frac{1}{5} & b_{23} \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Now it is the last column of B that is completely arbitrary. The best left-inverse (also the pseudoinverse) has $b_{13} = b_{23} = 0$. This is a "uniqueness case," when the rank is $r = n$. There are no free variables, since $n - r = 0$. If there is a solution it will be the only one. You can see when this example has one solution or no solution:

$$\begin{bmatrix} 4 & 0 \\ 0 & 5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \qquad \text{is solvable exactly when} \quad b_3 = 0.$$

A rectangular matrix cannot have both existence and uniqueness. If m is different from n, we cannot have $r = m$ and $r = n$.

A square matrix is the opposite. If $m = n$, we cannot have one property *without* the other. A square matrix has a left-inverse if and only if it has a right-inverse. There is only one inverse, namely $B = C = A^{-1}$. *Existence implies uniqueness and uniqueness implies existence, when the matrix is square*. The condition for invertibility is **full rank**: $r = m = n$. Each of these conditions is a necessary and sufficient test:

1. The columns span $\mathbf{R}^n$, so $Ax = b$ has at least one solution for every b.

2. The columns are independent, so $Ax = 0$ has only the solution $x = 0$.

This list can be made much longer, especially if we look ahead to later chapters. Every condition is equivalent to every other, and ensures that A is invertible.

3. The rows of A span $\mathbf{R}^n$.

4. The rows are linearly independent.

5. Elimination can be completed: $PA = LDU$, with all n pivots.

6. The determinant of A is not zero.

7. Zero is not an eigenvalue of A.

8. $A^{\mathrm{T}}A$ is positive definite.

Here is a typical application to polynomials $P(t)$ of degree $n-1$. The only such polynomial that vanishes at $t_1, \ldots, t_n$ is $P(t) \equiv 0$. No other polynomial of degree $n-1$ can have n roots. This is uniqueness, and it implies existence: Given any values $b_1, \ldots, b_n$, there *exists* a polynomial of degree $n-1$ interpolating these values: $P(t_i) = b_i$. The point is that we are dealing with a square matrix; the number n of coefficients in $P(t) = x_1 + x_2 t + \cdots + x_n t^{n-1}$ matches the number of equations:

$$\textbf{Interpolation} \quad P(t_i) = b_i \qquad \begin{bmatrix} 1 & t_1 & t_1^2 & \cdots & t_1^{n-1} \\ 1 & t_2 & t_2^2 & \cdots & t_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & t_n & t_n^2 & \cdots & t_n^{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

That *Vandermonde matrix* is n by n and full rank. $Ax = b$ always has a solution—a polynomial can be passed through any b_i at distinct points t_i. Later we shall actually find the determinant of A; it is not zero.

Matrices of Rank 1

Finally comes the easiest case, when the rank is as *small* as possible (except for the zero matrix with rank 0), One basic theme of mathematics is, given something complicated, to show how it can be broken into simple pieces. For linear algebra, the simple pieces are matrices of ***rank* 1**:

$$\textbf{Rank 1} \qquad A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \\ 8 & 4 & 4 \\ -2 & -1 & -1 \end{bmatrix} \quad \text{has} \quad r = 1.$$

Every row is a multiple of the first row, so the row space is one-dimensional. In fact, we can write the whole matrix *as the product of a column vector and a row vector*:

$$A = (\textbf{column})(\textbf{row}) \qquad \begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \\ 8 & 4 & 4 \\ -2 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \\ -1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \end{bmatrix}.$$

The product of a 4 by 1 matrix and a 1 by 3 matrix is a 4 by 3 matrix. *This product has rank* 1. At the same time, the columns are all multiples of the same column vector; the column space shares the dimension $r = 1$ and reduces to a line.

Every matrix of rank* 1 *has the simple form* $A = uv^{\mathrm{T}}$ = *column times row.

The rows are all multiples of the same vector v^{T}, and the columns are all multiples of u. The row space and column space are lines—the easiest case.

Problem Set 2.4

1. True or false: If $m = n$, then the row space of A equals the column space. If $m < n$, then the nullspace has a larger dimension than ____.

2. Find the dimension and construct a basis for the four subspaces associated with each of the matrices

$$A = \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 2 & 8 & 0 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

3. Find the dimension and a basis for the four fundamental subspaces for

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

4. Describe the four subspaces in three-dimensional space associated with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

5. If the product AB is the zero matrix, $AB = 0$, show that the column space of B is contained in the nullspace of A. (Also the row space of A is in the left nullspace of B, since each row of A multiplies B to give a zero row.)

6. Suppose A is an m by n matrix of rank r. Under what conditions on those numbers does

(a) A have a two-sided inverse: $AA^{-1} = A^{-1}A = I$?

(b) $Ax = b$ have *infinitely many solutions* for *every* b?

7. Why is there no matrix whose row space and nullspace both contain $(1,1,1)$?

8. Suppose the only solution to $Ax = 0$ (m equations in n unknowns) is $x = 0$. What is the rank and why? The columns of A are linearly ____.

9. Find a 1 by 3 matrix whose nullspace consists of all vectors in $\mathbf{R}^3$ such that $x_1 + 2x_2 + 4x_3 = 0$. Find a 3 by 3 matrix with that same nullspace.

10. If $Ax = b$ always has at least one solution, show that the only solution to $A^{\mathrm{T}}y = 0$ is $y = 0$. *Hint*: What is the rank?

11. If $Ax = 0$ has a nonzero solution, show that $A^{\mathrm{T}}y = f$ fails to be solvable for some right-hand sides f. Construct an example of A and f.

12. Find the rank of A and write the matrix as $A = uv^{\mathrm{T}}$:

$$A = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 6 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 2 & -2 \\ 6 & -6 \end{bmatrix}.$$

13. If a, b, c are given with $a \neq 0$, choose d so that

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = uv^{\mathrm{T}}$$

has rank 1. What are the pivots?

14. Find a left-inverse and/or a right-inverse (when they exist) for

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}.$$

15. If the columns of A are linearly independent (A is m by n), then the rank is ____, the nullspace is ____, the row space is ____, and there exists a ____-inverse.

16. (*A paradox*) Suppose A has a right-inverse B. Then $AB = I$ leads to $A^{\mathrm{T}}AB = A^{\mathrm{T}}$ or $B(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$. But that satisfies $BA = I$; it is a *left*-inverse. Which step is not justified?

17. Find a matrix A that has $\mathbf{V}$ as its row space, and a matrix B that has $\mathbf{V}$ as its nullspace, if $\mathbf{V}$ is the subspace spanned by

$$\begin{bmatrix}1\\1\\0\end{bmatrix}, \quad \begin{bmatrix}1\\2\\0\end{bmatrix}, \quad \begin{bmatrix}1\\5\\0\end{bmatrix}.$$

18. Find a basis for each of the four subspaces of

$$A = \begin{bmatrix}0&1&2&3&4\\0&1&2&4&6\\0&0&0&1&2\end{bmatrix} = \begin{bmatrix}1&0&0\\1&1&0\\0&1&1\end{bmatrix}\begin{bmatrix}0&1&2&3&4\\0&0&0&1&2\\0&0&0&0&0\end{bmatrix}.$$

19. If A has the same four fundamental subspaces as B, does $A = cB$?

20. (a) If a 7 by 9 matrix has rank 5, what are the dimensions of the four subspaces? What is the sum of all four dimensions?

(b) If a 3 by 4 matrix has rank 3, what are its column space and left nullspace?

21. Construct a matrix with the required property, or explain why you can't.

(a) Column space contains $\left[\begin{smallmatrix}1\\1\\0\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right]$, row space contains $\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}2\\5\end{smallmatrix}\right]$.

(b) Column space has basis $\left[\begin{smallmatrix}1\\2\\3\end{smallmatrix}\right]$, nullspace has basis $\left[\begin{smallmatrix}3\\2\\1\end{smallmatrix}\right]$.

(c) Dimension of nullspace $= 1+$ dimension of left nullspace.

(d) Left nullspace contains $\left[\begin{smallmatrix}1\\3\end{smallmatrix}\right]$, row space contains $\left[\begin{smallmatrix}3\\1\end{smallmatrix}\right]$.

(e) Row space $=$ column space, nullspace $\neq$ left nullspace.

22. Without elimination, find dimensions and bases for the four subspaces for

$$A = \begin{bmatrix}0&3&3&3\\0&0&0&0\\0&1&0&1\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}1&1\\4&4\\5&5\end{bmatrix}.$$

23. Suppose the 3 by 3 matrix A is invertible. Write bases for the four subspaces for A, and also for the 3 by 6 matrix $B = [A \ \ A]$.

24. What are the dimensions of the four subspaces for A, B, and C, if I is the 3 by 3 identity matrix and 0 is the 3 by 2 zero matrix?

$$A = \begin{bmatrix}I & 0\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}I & I\\ 0^{\mathrm{T}} & 0^{\mathrm{T}}\end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix}0\end{bmatrix}.$$

25. Which subspaces are the same for these matrices of different sizes?

$$\text{(a)}\quad \begin{bmatrix} A \end{bmatrix} \quad\text{and}\quad \begin{bmatrix} A \\ A \end{bmatrix}. \qquad\qquad \text{(b)}\quad \begin{bmatrix} A \\ A \end{bmatrix} \quad\text{and}\quad \begin{bmatrix} A & A \\ A & A \end{bmatrix}.$$

Prove that all three matrices have the same rank r.

26. If the entries of a 3 by 3 matrix are chosen randomly between 0 and 1, what are the most likely dimensions of the four subspaces? What if the matrix is 3 by 5?

27. (Important) A is an m by n matrix of rank r. Suppose there are right-hand sides b for which $Ax = b$ has *no solution*.

(a) What inequalities ($<$ or $\leq$) must be true between m, n, and r?

(b) How do you know that $A^{\mathrm{T}}y = 0$ has a nonzero solution?

28. Construct a matrix with $(1,0,1)$ and $(1,2,0)$ as a basis for its row space and its column space. Why can't this be a basis for the row space and nullspace?

29. Without computing A, find bases for the four fundamental subspaces:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 9 & 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

30. If you exchange the first two rows of a matrix A, which of the four subspaces stay the same? If $y = (1,2,3,4)$ is in the left nullspace of A, write down a vector in the left nullspace of the new matrix.

31. Explain why $v = (1,0,-1)$ cannot be a row of A and also be in the nullspace.

32. Describe the four subspaces of $\mathbf{R}^3$ associated with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad\text{and}\quad I + A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

33. (Left nullspace) Add the extra column b and reduce A to echelon form:

$$\begin{bmatrix} A & b \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 4 & 5 & 6 & b_2 \\ 7 & 8 & 9 & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 0 & -3 & -6 & b_2 - 4b_1 \\ 0 & 0 & 0 & b_3 - 2b_2 + b_1 \end{bmatrix}.$$

A combination of the rows of A has produced the zero row. What combination is it? (Look at $b_3 - 2b_2 + b_1$ on the right-hand side.) Which vectors are in the nullspace of A^{T} and which are in the nullspace of A?

34. Following the method of Problem 33, reduce A to echelon form and look at zero rows. The b column tells which combinations you have taken of the rows:

$$\text{(a)} \quad \begin{bmatrix} 1 & 2 & b_1 \\ 3 & 4 & b_2 \\ 4 & 6 & b_3 \end{bmatrix}. \qquad\qquad \text{(b)} \quad \begin{bmatrix} 1 & 2 & b_1 \\ 2 & 3 & b_2 \\ 2 & 4 & b_3 \\ 2 & 5 & b_4 \end{bmatrix}.$$

From the b column after elimination, read off $m-r$ basis vectors in the left nullspace of A (combinations of rows that give zero).

35. Suppose A is the sum of two matrices of rank one: $A = uv^{\mathrm{T}} + wz^{\mathrm{T}}$.

(a) Which vectors span the column space of A?

(b) Which vectors span the row space of A?

(c) The rank is less than 2 if ____ or if ____.

(d) Compute A and its rank if $u = z = (1,0,0)$ and $v = w = (0,0,1)$.

36. Without multiplying matrices, find bases for the row and column spaces of A:

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \end{bmatrix}.$$

How do you know from these shapes that A is not invertible?

37. True or false (with a reason or a counterexample)?

(a) A and A^{T} have the same number of pivots.

(b) A and A^{T} have the same left nullspace.

(c) If the row space equals the column space then $A^{\mathrm{T}} = A$.

(d) If $A^{\mathrm{T}} = -A$ then the row space of A equals the column space.

38. If $AB = 0$, the columns of B are in the nullspace of A. If those vectors are in $\mathbf{R}^n$, prove that $\operatorname{rank}(A) + \operatorname{rank}(B) \leq n$.

39. Can tic-tac-toe be completed (5 ones and 4 zeros in A) so that $\operatorname{rank}(A) = 2$ but neither side passed up a winning move?

40. Construct any 2 by 3 matrix of rank 1. Copy Figure 2.5 and put one vector in each subspace (two in the nullspace). Which vectors are orthogonal?

41. Redraw Figure 2.5 for a 3 by 2 matrix of rank $r = 2$. Which subspace is $\boldsymbol{Z}$ (zero vector only)? The nullspace part of any vector x in $\mathbf{R}^2$ is $x_n =$ ____.