Linear Algebra
A short recap on lines and planes in $\mathbb{R}^2$ and $\mathbb{R}^3$

1 Lines in $\mathbb{R}^2$

Any line in $\mathbb{R}^2$ is given by an equation of the form

$$ax_1 + bx_2 = c,$$

with $|a| + |b| > 0$. If $b = 0$, then this is a vertical line $x_1 = c/a$; otherwise, the equation can be written in a more usual form (with $x = x_1$ and $y = x_2$)

$$y = kx + m$$

with *slope* $k := -a/b$ and *intercept* $m := c/b$.

The *normal* vector $\mathbf{n} := (a, b)$ is orthogonal to the *direction* vector $\mathbf{d} = (b, -a)$. The equation of ℓ can alternatively be written in a *parametrized* form

$$\mathbf{x} = \mathbf{x}_0 + t\mathbf{d}, \qquad t \in \mathbb{R},$$

with $\mathbf{x}_0 = (p_1, p_2)$ being any fixed point on ℓ. Written coordinate-wise, we get

$$\begin{aligned} x_1 &= p_1 + tb; \\ x_2 &= p_2 - ta. \end{aligned}$$

Comparing the two equations (parametric and non-parametric), we can easily transform from one to another (look at normal and direction vectors!).

2 Lines in $\mathbb{R}^3$

Parametric equation for a line in $\mathbb{R}^3$,

$$\mathbf{x} = \mathbf{x}_0 + t\mathbf{d}, \quad t \in \mathbb{R},$$

describes the line ℓ in $\mathbb{R}^3$ containing a fixed point $\mathbf{x}_0 = (p_1, p_2, p_3)$ as the set of all points $\mathbf{x}$ such that the vector $\overrightarrow{\mathbf{x}_0\mathbf{x}} = \mathbf{x} - \mathbf{x}_0$ is collinear to the *direction* vector $\mathbf{d} := (d_1, d_2, d_3)$. Writing this system of three equations component-wise and eliminating t, we arrive at the standard non-parametric equation of ℓ,

$$\frac{x_1 - p_1}{d_1} = \frac{x_2 - p_2}{d_2} = \frac{x_3 - p_3}{d_3}.$$

Likewise, denoting the common value of the above three fractions by $t \in \mathbb{R}$, one gets three parametric equations for x_1, x_2, and x_3.

In both $\mathbb{R}^2$ and $\mathbb{R}^3$, if we know two points P_1 and P_2 of the line, we immediately get a direction vector $\mathbf{d} = \overrightarrow{P_1P_2}$ and thus the parametric equation of that line.

3 Planes in $\mathbb{R}^3$

Any plane π in $\mathbb{R}^3$ can be described by a normal equation

$$ax_1 + bx_2 + cx_3 = d, \tag{1}$$

with $\mathbf{n} := (a, b, c)$ being a non-zero *normal* vector. Every vector in π (ie., a vector joining any two points on π) is orthogonal to $\mathbf{n}$. If we have any point $P_0 = (p_1, p_2, p_3) = \mathbf{x}_0$ on π and two *non-collinear* vectors $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ belonging to π (in other words, two *direction* vectors of π), then any point on π can be written as

$$\mathbf{x} = \mathbf{x}_0 + s\mathbf{u} + t\mathbf{v}$$

for a suitable $s, t \in \mathbb{R}$. This expresses the fact that the vector joining $\mathbf{x}_0$ and $\mathbf{x}$ can be written as a linear combination of $\mathbf{u}$ and $\mathbf{v}$. The above is a *parametric* equation of the plane π, and in coordinate form it reads

$$\begin{aligned} x_1 &= p_1 + su_1 + tv_1; \\ x_2 &= p_2 + su_2 + tv_2; \\ x_3 &= p_3 + su_3 + tv_3. \end{aligned} \tag{2}$$

If (1) is given, then to find the parametric equation, we should

1. find any point P_0 on the plane
2. find any two non-collinear direction vectors $\mathbf{u}$ and $\mathbf{v}$ (they must be orthogonal to the normal vector $\mathbf{n}$).

Conversely, equations (2) identify the direction vectors $\mathbf{u}$ and $\mathbf{v}$, and then a normal vector $\mathbf{n}$ (note it is determined only up to a scalar). The constant d is then determined from the knowledge of the point P_0