Linear Algebra
Theorem on invertible matrices

We sketch here a proof of the theorem on invertible matrices. Recall first the definition:

Definition 1. A square $n \times n$ matrix A is called *invertible* if there is a matrix B such that $AB = BA = I_n$; B is then called the *inverse* of A and denoted A^{-1}.

Remark 2. (a) It is possible to relax the above definition by assuming existence of two $n \times n$ matrices B and C satisfying $BA = AC = I_n$. Indeed, then $B = B(AC) = (BA)C = C$, and we are back to Definition 1. In other words, matrices that are both *left invertible* and *right invertible* are *invertible*; moreover, the left inverse B and the right inverse C are equal and coincide with the inverse matrix A^{-1}.

(b) Theorem 4 also states that a *left invertible* (or *right invertible*) *square* matrix is necessarily invertible.

Problem 3. *Find a* right invertible *matrix B that is not invertible. How to derive from that B an example of a left invertible matrix C that is not invertible?*

Theorem 4 (On invertible matrices)**.** For an $n \times n$ matrix A, the following statements are equivalent:

(a) A is invertible;

(b) A has n pivot positions in its row echelon form;

(c) the reduced row echelon form of A is I_n

(d) A is expressible as a product of elementary matrices;

(e) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution;

(f) $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b} \in \mathbb{R}^n$;

(g) there is an $n \times n$ matrix C s.t. $AC = I$;

(h) there is an $n \times n$ matrix B s.t. $BA = I$;

(i) $A^\top$ is invertible;

(j) the columns of A span $\mathbb{R}^n$;

(k) the columns of A are linearly independent;

(l) the rows of A span $\mathbb{R}^n$;

(m) the rows of A are linearly independent.

Proof. In Lecture 1, we understood the following equivalences:

$$(f) \iff (b) \iff (e).$$

Indeed, consistency of $A\mathbf{x} = \mathbf{b}$ for every $\mathbf{b} \in \mathbb{R}^n$ holds iff there is no zero row in the row echelon form of A, while uniqueness of solutions to $A\mathbf{x} = \mathbf{0}$ holds iff there is no free variables; both statements say there are n pivot entries, one in each row and each column.

$(b) \implies (c)$: (b) means all pivot entries of the row echelon form A_0 of A are on the main diagonal; the Jordan–Gauss backward substitution then makes the upper-triangular part of A_0 zero and diagonal of ones

$(c) \implies (d)$: We transform A to its reduced row echelon form by left-multiplying by elementary matrices E_j. Since each of E_j is invertible, their product $E = E_k \cdots E_1$ is invertible as well. By (c), $EA = I_n$, so that $A = E_1^{-1} \cdots E_k^{-1}$, where each factor E_j^{-1} is an elementary matrix.

$(d) \implies (a)$: Product of invertible matrices is invertible, whence the claim

$(a) \implies (h)$: See Definition 1

$(h) \implies (e)$: Assume B satisfies $BA = I$; if $A\mathbf{x} = \mathbf{0}$, then $\mathbf{x} = (BA)\mathbf{x} = B(A\mathbf{x}) = \mathbf{0}$

$(e) \iff (b)$: was already explained

Therefore, we have proved that

$$(a) \iff (b) \iff (c) \iff (d) \iff (e) \iff (f) \iff (h) \tag{0.1}$$

$(g) \implies (f)$: Given a C satisfying $AC = I_n$ and any $\mathbf{b} \in \mathbb{R}^n$, we find that $\mathbf{x} := C\mathbf{b}$ satisfies $A\mathbf{x} = A(C\mathbf{b}) = (AC)\mathbf{b} = \mathbf{b}$

$(a) \implies (g)$: See Definition 1

Therefore,

$$(g) \implies (f) \iff (a) \implies (g)$$

so that (g) is equivalent to every statement in (0.1)

$(i) \iff (g)$: Transposed (h) for A is (g) for $A^\top$

$(j) \iff (f)$: Because $A\mathbf{x}$ is a linear combination of columns of A

$(k) \iff (e)$: Because $A\mathbf{x}$ is a linear combination of columns of A

$(l) \iff (i)$: (l) for A is (j) for $A^\top$

$(m) \iff (i)$: (m) for A is (k) for $A^\top$

$\square$