

Matrices and Gaussian Elimination

1.1 Introduction

This book begins with the central problem of linear algebra: *solving linear equations*. The most important case, and the simplest, is when the number of unknowns equals the number of equations. We have **n equations in n unknowns**, starting with $n = 2$:

$$\begin{array}{ll} \textbf{Two equations} & 1x + 2y = 3 \\ \textbf{Two unknowns} & 4x + 5y = 6. \end{array} \tag{1}$$

The unknowns are x and y . I want to describe two ways, *elimination* and *determinants*, to solve these equations. Certainly x and y are determined by the numbers 1, 2, 3, 4, 5, 6. The question is how to use those six numbers to solve the system.

1. **Elimination** Subtract 4 times the first equation from the second equation. This eliminates x from the second equation. and it leaves one equation for y :

$$(\textbf{equation 2}) - 4(\textbf{equation 1}) \quad -3y = -6. \tag{2}$$

Immediately we know $y = 2$. Then x comes from the first equation $1x + 2y = 3$:

$$\textbf{Back-substitution} \quad 1x + 2(2) = 3 \quad \text{gives} \quad x = -1. \tag{3}$$

Proceeding carefully, we check that x and y also solve the second equation. This should work and it does: 4 times ($x = -1$) plus 5 times ($y = 2$) equals 6.

2. **Determinants** The solution $y = 2$ depends completely on those six numbers in the equations. There must be a formula for y (and also x) It is a “ratio of determinants” and I hope you will allow me to write it down directly:

$$y = \frac{\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}} = \frac{1 \cdot 6 - 3 \cdot 4}{1 \cdot 5 - 2 \cdot 4} = \frac{-6}{-3} = 2. \tag{4}$$

That could seem a little mysterious, unless you already know about 2 by 2 determinants. They gave the same answer $y = 2$, coming from the same ratio of -6 to -3 . If we stay with determinants (which we don't plan to do), there will be a similar formula to compute the other unknown, x :

$$x = \frac{\begin{vmatrix} 3 & 2 \\ 6 & 5 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}} = \frac{3 \cdot 5 - 2 \cdot 6}{1 \cdot 5 - 2 \cdot 4} = \frac{3}{-3} = -1. \quad (5)$$

Let me compare those two approaches, looking ahead to real problems when n is much larger ($n = 1000$ is a very moderate size in scientific computing). The truth is that direct use of the determinant formula for 1000 equations would be a total disaster. It would use the million numbers on the left sides correctly, but not efficiently. We will find that formula (Cramer's Rule) in Chapter 4, but we want a good method to solve 1000 equations in Chapter 1.

That good method is *Gaussian Elimination*. This is the algorithm that is constantly used to solve large systems of equations. From the examples in a textbook ($n = 3$ is close to the upper limit on the patience of the author and reader) too might not see much difference. Equations (2) and (4) used essentially the same steps to find $y = 2$. Certainly x came faster by the back-substitution in equation (3) than the ratio in (5). For larger n there is absolutely no question. Elimination wins (and this is even the best way to compute determinants).

The idea of elimination is deceptively simple—you will master it after a few examples. It will become the basis for half of this book, simplifying a matrix so that we can understand it. Together with the mechanics of the algorithm, we want to explain four deeper aspects in this chapter. They are:

1. Linear equations lead to ***geometry of planes***. It is not easy to visualize a nine-dimensional plane in ten-dimensional space. It is harder to see ten of those planes, intersecting at the solution to ten equations—but somehow this is almost possible. Our example has two lines in Figure 1.1, meeting at the point $(x, y) = (-1, 2)$. Linear algebra moves that picture into ten dimensions, where the intuition has to imagine the geometry (and gets it right)
2. We move to ***matrix notation***, writing the n unknowns as a vector x and the n equations as $Ax = b$. We multiply A by “elimination matrices” to reach an upper triangular matrix U . Those steps factor A into L times U , where L is lower triangular. I will write down A and its factors for our example, and explain them at the right time:

$$\textbf{Factorization} \quad A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix} = L \textbf{ times } U. \quad (6)$$

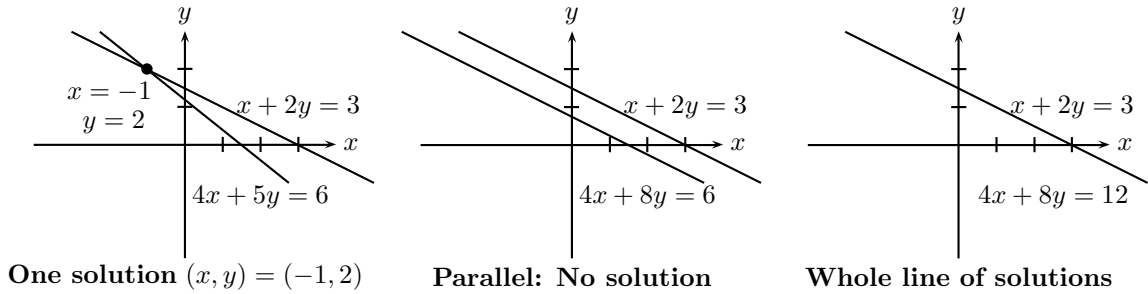


Figure 1.1: The example has one solution. Singular cases have none or too many.

First we have to introduce matrices and vectors and the rules for multiplication. Every matrix has a **transpose** A^T . This matrix has an **inverse** A^{-1} .

3. In most cases elimination goes forward without difficulties. The matrix has an inverse and the system $Ax = b$ has one solution. In exceptional cases the method will *break down*—either the equations were written in the wrong order, which is easily fixed by exchanging them, or the equations don't have a unique solution.

That **singular case** will appear if 8 replaces 5 in our example:

$$\begin{array}{ll} \text{Singular case} & 1x + 2y = 3 \\ \text{Two parallel lines} & 4x + 8y = 6. \end{array} \quad (7)$$

Elimination still innocently subtracts 4 times the first equation from the second. But look at the result!

$$(\text{equation 2}) - 4(\text{equation 1}) \quad 0 = -6.$$

This singular case has **no solution**. Other singular cases have **infinitely many solutions**. (Change 6 to 12 in the example, and elimination will lead to $0 = 0$. Now y can have *any value*.) When elimination breaks down, we want to find every possible solution.

4. We need a rough count of the **number of elimination steps** required to solve a system of size n . The computing cost often determines the accuracy in the model. A hundred equations require a third of a million steps (multiplications and subtractions). The computer can do those quickly, but not many trillions. And already after a million steps, roundoff error could be significant. (Some problems are sensitive; others are not.) Without trying for full detail, we want to see large systems that arise in practice, and how they are actually solved.

The final result of this chapter will be an elimination algorithm that is about as efficient as possible. It is essentially the algorithm that is in constant use in a tremendous variety of applications. And at the same time, understanding it in terms of *matrices*—the coefficient matrix A , the matrices E for elimination and P for row exchanges, and the

final factors L and U —is an essential foundation for the theory. I hope you will enjoy this book and this course.

1.2 The Geometry of Linear Equations

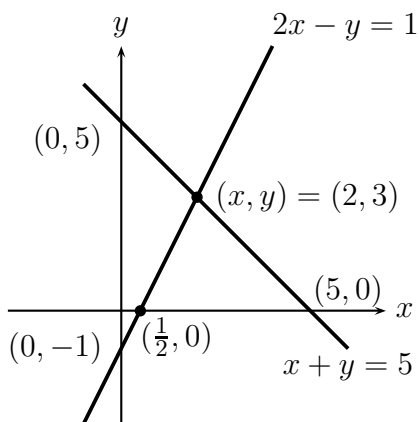
The way to understand this subject is by example. We begin with two extremely humble equations, recognizing that you could solve them without a course in linear algebra. Nevertheless I hope you will give Gauss a chance:

$$\begin{aligned} 2x - y &= 1 \\ x + y &= 5. \end{aligned}$$

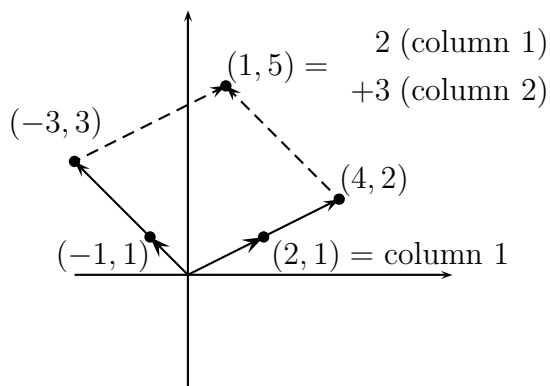
We can look at that system *by rows* or *by columns*. We want to see them both.

The first approach concentrates on the separate equations (the **rows**). That is the most familiar, and in two dimensions we can do it quickly. The equation $2x - y = 1$ is represented by a *straight line* in the x - y plane. The line goes through the points $x = 1$, $y = 1$ and $x = \frac{1}{2}$, $y = 0$ (and also through $(2, 3)$ and all intermediate points). The second equation $x + y = 5$ produces a second line (Figure 1.2a). Its slope is $dy/dx = -1$ and it crosses the first line at the solution.

The point of intersection lies on both lines. It is the only solution to both equations. That point $x = 2$ and $y = 3$ will soon be found by “elimination.”



(a) Lines meet at $x = 2$, $y = 3$



(b) Columns combine with 2 and 3

Figure 1.2: Row picture (two lines) and column picture (combine columns).

The second approach looks at the **columns** of the linear system. The two separate equations are really **one vector equation**:

$$\text{Column form} \quad x \begin{bmatrix} 2 \\ 1 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}.$$

The problem is *to find the combination of the column vectors on the left side that produces the vector on the right side*. Those vectors $(2, 1)$ and $(-1, 1)$ are represented by the bold lines in Figure 1.2b. The unknowns are the numbers x and y that multiply the column vectors. The whole idea can be seen in that figure, where 2 times column 1 is added to 3 times column 2. Geometrically this produces a famous parallelogram. Algebraically it produces the correct vector $(1, 5)$, on the right side of our equations. The column picture confirms that $x = 2$ and $y = 3$.

More time could be spent on that example, but I would rather move forward to $n = 3$. Three equations are still manageable, and they have much more variety:

$$\begin{array}{rclcl} & 2u & + & v & + & w & = & 5 \\ \text{Three planes} & 4u & - & 6v & & & = & -2 \\ & -2u & + & 7v & + & 2w & = & 9. \end{array} \quad (1)$$

Again we can study the rows or the columns, and we start with the rows. Each equation describes a *plane* in three dimensions. The first plane is $2u + v + w = 5$, and it is sketched in Figure 1.3. It contains the points $(\frac{5}{2}, 0, 0)$ and $(0, 5, 0)$ and $(0, 0, 5)$. It is determined by any three of its points—provided they do not lie on a line.

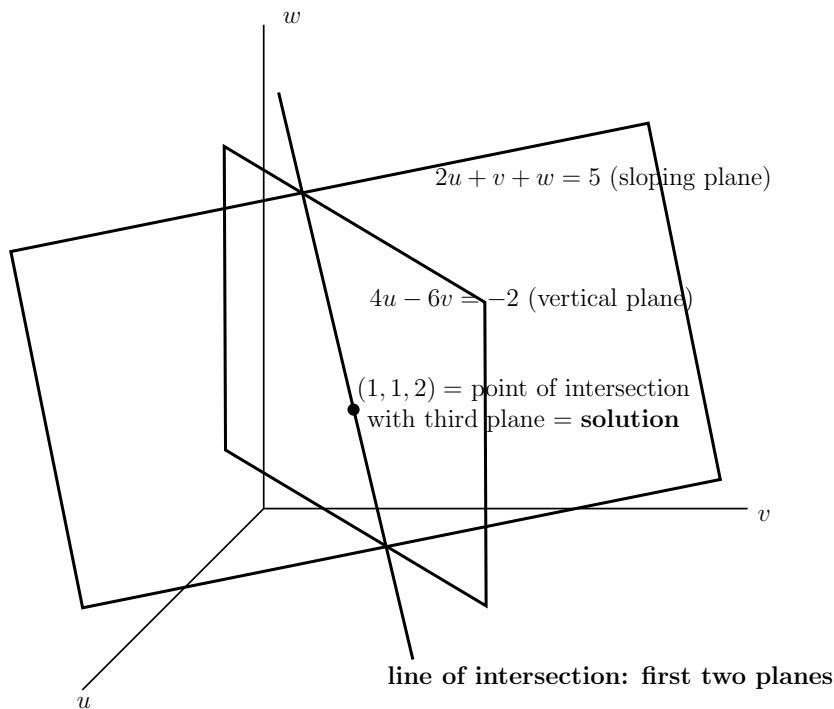


Figure 1.3: The row picture: three intersecting planes from three linear equations.

Changing 5 to 10, the plane $2u + v + w = 10$ would be parallel to this one. It contains $(5, 0, 0)$ and $(0, 10, 0)$ and $(0, 0, 10)$, twice as far from the origin—which is the center point $u = 0, v = 0, w = 0$. Changing the right side moves the plane parallel to itself, and the plane $2u + v + w = 0$ goes through the origin.

The second plane is $4u - 6v = -2$. It is drawn vertically, because w can take any value. The coefficient of w is zero, but this remains a plane in 3-space. (The equation $4u = 3$, or even the extreme case $u = 0$, would still describe a plane.) The figure shows the intersection of the second plane with the first. That intersection is a line. *In three dimensions a line requires two equations*; in n dimensions it will require $n - 1$.

Finally the third plane intersects this line in a point. The plane (not drawn) represents the third equation $-2u + 7v + 2w = 9$, and it crosses the line at $u = 1, v = 1, w = 2$. That triple intersection point $(1, 1, 2)$ solves the linear system.

How does this row picture extend into n dimensions? The n equations will contain n unknowns. The first equation still determines a “plane.” It is no longer a two-dimensional plane in 3-space; somehow it has “dimension” $n - 1$. It must be flat and extremely thin within n -dimensional space, although it would look solid to us.

If time is the fourth dimension, then the plane $t = 0$ cuts through four-dimensional space and produces the three-dimensional universe we live in (or rather, the universe as it was at $t = 0$). Another plane is $z = 0$, which is also three-dimensional; it is the ordinary x - y plane taken over all time. Those three-dimensional planes will intersect! They share the ordinary x - y plane at $t = 0$. We are down to two dimensions, and the next plane leaves a line. Finally a fourth plane leaves a single point. It is the intersection point of 4 planes in 4 dimensions, and it solves the 4 underlying equations.

I will be in trouble if that example from relativity goes any further. The point is that linear algebra can operate with any number of equations. The first equation produces an $(n - 1)$ -dimensional plane in n dimensions. The second plane intersects it (we hope) in a smaller set of “dimension $n - 2$.” Assuming all goes well, every new plane (every new equation) reduces the dimension by one. At the end, when all n planes are accounted for, the intersection has dimension zero. It is a *point*, it lies on all the planes, and its coordinates satisfy all n equations. It is the solution!

Column Vectors and Linear Combinations

We turn to the columns. This time the vector equation (the same equation as (1)) is

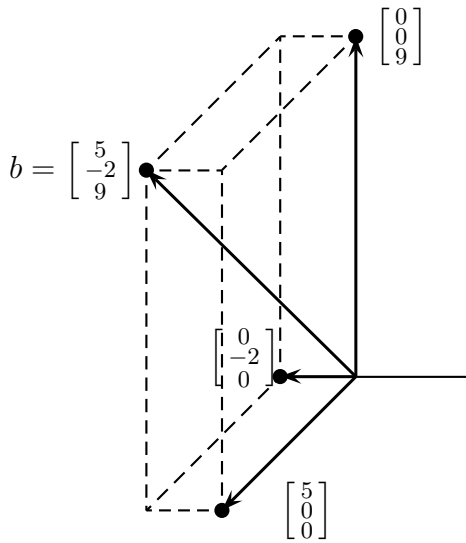
$$\text{Column form} \quad u \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} + v \begin{bmatrix} 1 \\ -6 \\ 7 \end{bmatrix} + w \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix} = b. \quad (2)$$

Those are *three-dimensional column vectors*. The vector b is identified with the point whose coordinates are 5, -2 , 9. Every point in three-dimensional space is matched to a vector, and vice versa. That was the idea of Descartes, who turned geometry into algebra by working with the coordinates of the point. We can write the vector in a column, or we can list its components as $b = (5, -2, 9)$, or we can represent it geometrically by an arrow from the origin. You can choose *the arrow*, or *the point*, or *the three numbers*. In six dimensions it is probably easiest to choose the six numbers.

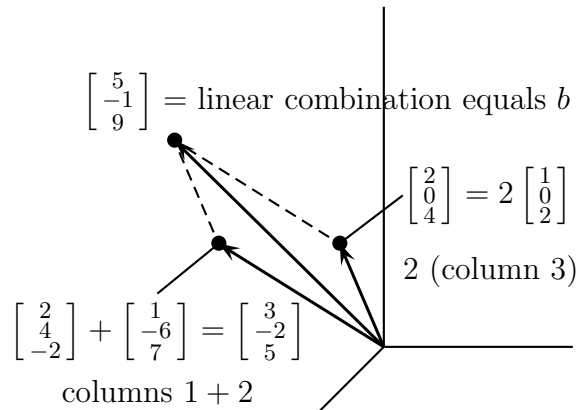
We use parentheses and commas when the components are listed horizontally, and square brackets (with no commas) when a column vector is printed vertically. What really matters is **addition of vectors** and **multiplication by a scalar** (a number). In Figure 1.4a you see a vector addition, component by component:

Vector addition
$$\begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 9 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}.$$

In the right-hand figure there is a multiplication by 2 (and if it had been -2 the vector



(a) Add vectors along axes



(b) Add columns $1 + 2 + (3 + 3)$

Figure 1.4: The column picture: linear combination of columns equals b .

would have gone in the reverse direction):

Multiplication by scalars
$$2 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}, \quad -2 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ -4 \end{bmatrix}.$$

Also in the right-hand figure is one of the central ideas of linear algebra. It uses *both* of the basic operations; vectors are *multiplied by numbers and then added*. The result is called a **linear combination**, and this combination solves our equation:

Linear combination
$$1 \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ -6 \\ 7 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}.$$

Equation (2) asked for multipliers u, v, w that produce the right side b . Those numbers are $u = 1, v = 1, w = 2$. They give the correct combination of the columns. They also gave the point $(1, 1, 2)$ in the row picture (where the three planes intersect).

Our true goal is to look beyond two or three dimensions into n dimensions. With n equations in n unknowns, there are n planes in the row picture. There are n vectors in the column picture, plus a vector b on the right side. The equations ask for a **linear combination of the n columns that equals b** . For certain equations that will be impossible. Paradoxically, the way to understand the good case is to study the bad one. Therefore we look at the geometry exactly when it breaks down, in the **singular case**.

Row picture: Intersection of planes

Column picture: Combination of columns

The Singular Case

Suppose we are again in three dimensions, and the three planes in the row picture *do not intersect*. What can go wrong? One possibility is that two planes may be parallel. The equations $2u + v + w = 5$ and $4u + 2v + 2w = 11$ are inconsistent—and parallel planes give no solution (Figure 1.5a shows an end view). In two dimensions, parallel lines are the only possibility for breakdown. But three planes in three dimensions can be in trouble without being parallel.

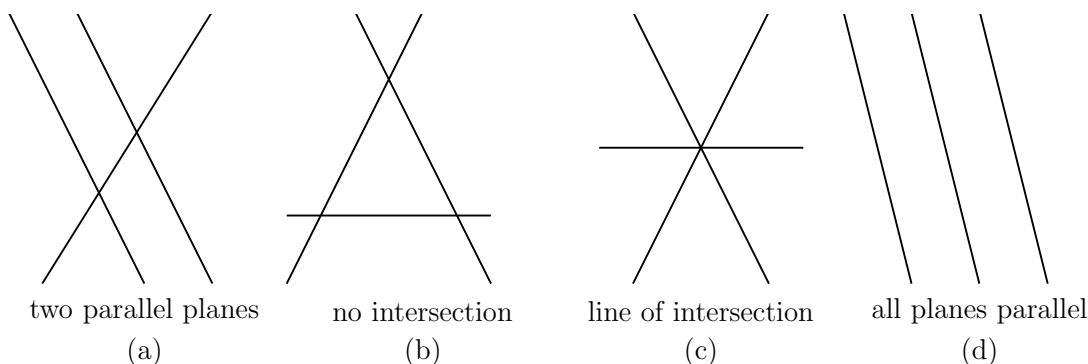


Figure 1.5: Singular cases: no solution for (a), (b), or (d), an infinity of solutions for (c).

The most common difficulty is shown in Figure 1.5b. From the end view the planes form a triangle. Every pair of planes intersects in a line, and those lines are parallel. The third plane is not parallel to the other planes, but it is parallel to their line of intersection. This corresponds to a singular system with $b = (2, 5, 6)$:

$$\begin{array}{rcll}
 & u & + & v & + & w & = & 2 \\
 \text{No solution, as in Figure 1.5b} & 2u & & & + & 3w & = & 5 \\
 & 3u & + & v & + & 4w & = & 6.
 \end{array} \tag{3}$$

The first two left sides add up to the third. On the right side that fails: $2 + 5 \neq 6$. Equation 1 plus equation 2 minus equation 3 is the impossible statement $0 = 1$. Thus the equations are **inconsistent**, as Gaussian elimination will systematically discover.

Another singular system, close to this one, has an **infinity of solutions**. When the 6 in the last equation becomes 7, the three equations combine to give $0 = 0$. Now the third equation is the sum of the first two. In that case the three planes have a whole *line in common* (Figure 1.5c). Changing the right sides will move the planes in Figure 1.5b parallel to themselves, and for $b = (2, 5, 7)$ the figure is suddenly different. The lowest plane moved up to meet the others, and there is a line of solutions. Problem 1.5c is still singular, but now it suffers from *too many solutions* instead of too few.

The extreme case is three parallel planes. For most right sides there is no solution (Figure 1.5d). For special right sides (like $b = (0, 0, 0)$!) there is a whole plane of solutions—because the three parallel planes move over to become the same.

What happens to the **column picture** when the system is singular? it has to go wrong; the question is how. There are still three columns on the left side of the equations, and we try to combine them to produce b . Stay with equation (3):

Singular case: Column picture
Three columns in the same plane
Solvable only for b in that plane

$$u \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + v \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + w \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = b. \quad (4)$$

For $b = (2, 5, 7)$ this was possible; for $b = (2, 5, 6)$ it was not. The reason is that *those three columns lie in a plane*. Then every combination is also in the plane (which goes through the origin). If the vector b is not in that plane, no solution is possible (Figure 1.6). That is by far the most likely event; a singular system generally has no solution. But there is a chance that b *does* lie in the plane of the columns. In that case there are too many solutions; the three columns can be combined in *infinitely many ways* to produce b . That column picture in Figure 1.6b corresponds to the row picture in Figure 1.5c.

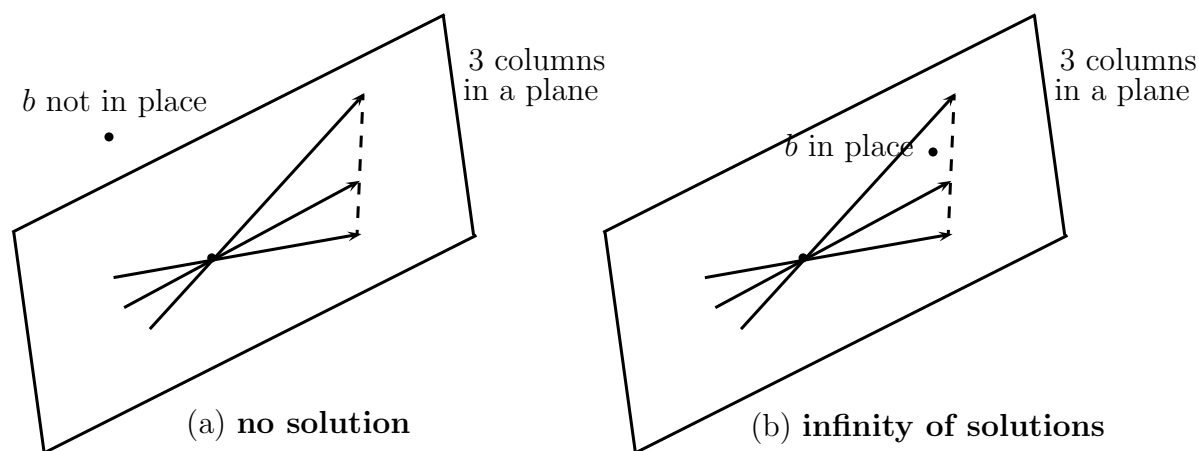


Figure 1.6: Singular cases: b outside or inside the plane with all three columns.

How do we know that the three columns lie in the same plane? One answer is to find a combination of the columns that adds to zero. After some calculation, it is $u = 3$, $v = 1$, $w = -2$. Three times column 1 equals column 2 plus twice column 3. Column 1 is in

the plane of columns 2 and 3. Only two columns are independent.

The vector $b = (2, 5, 7)$ is in that plane of the columns—it is column 1 plus column 3—so $(1, 0, 1)$ is a solution. *We can add an multiple of the combination $(3, -1, -2)$ that gives $b = 0$.* So there is a whole line of solutions—as we know from the row picture.

The truth is that we *knew* the columns would combine to give zero, because the rows did. That is a fact of mathematics, not of computation—and it remains true in dimension n . ***If the n planes have no point in common, or infinitely many points, then the n columns lie in the same plane.***

If the row picture breaks down, so does the column picture. That brings out the difference between Chapter 1 and Chapter 2. This chapter studies the most important problem—the *nonsingular* case—where there is one solution and it has to be found. Chapter 2 studies the general case, where there may be many solutions or none. In both cases we cannot continue without a decent notation (*matrix notation*) and a decent algorithm (*elimination*). After the exercises, we start with elimination.

Problem Set 1.2

- For the equations $x + y = 4$, $2x - 2y = 4$, draw the row picture (two intersecting lines) and the column picture (combination of two columns equal to the column vector $(4, 4)$ on the right side).
- Solve to find a combination of the columns that equals b :

$$\begin{array}{rcll} & u & - & v & - & w & = & b_1 \\ \text{Triangular system} & & & v & + & w & = & b_2 \\ & & & & & w & = & b_3. \end{array}$$

- (Recommended) Describe the intersection of the three planes $u + v + w + z = 6$ and $u + w + z = 4$ and $u + w = 2$ (all in four-dimensional space). Is it a line or a point or an empty set? What is the intersection if the fourth plane $u = -1$ is included? Find a fourth equation that leaves us with no solution.
- Sketch these three lines and decide if the equations are solvable:

$$\begin{array}{rcll} & x & + & 2y & = & 2 \\ \text{3 by 2 system} & & & x & - & y & = & 2 \\ & & & & & y & = & 1. \end{array}$$

What happens if all right-hand sides are zero? Is there any nonzero choice of right-hand sides that allows the three lines to intersect at the same point?

- Find two points on the line of intersection of the three planes $t = 0$ and $z = 0$ and $x + y + z + t = 1$ in four-dimensional space.

- $$\begin{array}{rclcl} u & + & v & + & w & = & 2 \\ u & + & 2v & + & 3w & = & 1 \\ & & v & + & 2w & = & 0 \end{array}$$

9. The column picture for the previous exercise (singular system) is

$$u \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + v \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + w \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = b.$$

- 10.** (Recommended) Under what condition on y_1, y_2, y_3 do the points $(0, y_1), (1, y_2), (2, y_3)$ lie on a straight line?
- 11.** These equations are certain to have the solution $x = y = 0$. For which values of a is there a whole line of solutions?

$$\begin{array}{rcl} ax & + & 2y = 0 \\ 2x & + & ay = 0 \end{array}$$

- Problems 13–15 are a review of the row and column pictures.**

- 13.** Draw the two pictures in two planes for the equations $x - 2y = 0$, $x + y = 6$.
- 14.** For two linear equations in three unknowns x, y, z , the row picture will show (2 or 3) (lines or planes) in (two or three)-dimensional space. The column picture is in (two or three)-dimensional space. The solutions normally lie on a _____.

15. For four linear equations in two unknowns x and y , the row picture shows four _____. The column picture is in _____-dimensional space. The equations have no solution unless the vector on the right-hand side is a combination of _____.
16. Find a point with $z = 2$ on the intersection line of the planes $x + y + 3z = 6$ and $x - y + z = 4$. Find the point with $z = 0$ and a third point halfway between.
17. The first of these equations plus the second equals the third:

$$\begin{array}{rclcl} x & + & y & + & z & = & 2 \\ x & + & 2y & + & z & = & 3 \\ 2x & + & 3y & + & 2z & = & 5. \end{array}$$

The first two planes meet along a line. The third plane contains that line, because if x, y, z satisfy the first two equations then they also _____. The equations have infinitely many solutions (the whole line **L**). Find three solutions.

18. Move the third plane in Problem 17 to a parallel plane $2x + 3y + 2z = 9$. Now the three equations have no solution—*why not*? The first two planes meet along the line **L**, but the third plane doesn't _____ that line.
19. In Problem 17 the columns are $(1, 1, 2)$ and $(1, 2, 3)$ and $(1, 1, 2)$. This is a “singular case” because the third column is _____. Find two combinations of the columns that give $b = (2, 3, 5)$. This is only possible for $b = (4, 6, c)$ if $c = \rule{1cm}{0.4pt}$.
20. Normally 4 “planes” in four-dimensional space meet at a _____. Normally 4 column vectors in four-dimensional space can combine to produce b . What combination of $(1, 0, 0, 0)$, $(1, 1, 0, 0)$, $(1, 1, 1, 0)$, $(1, 1, 1, 1)$ produces $b = (3, 3, 3, 2)$? What 4 equations for x, y, z, t are you solving?
21. When equation 1 is added to equation 2, which of these are changed: the planes in the row picture, the column picture, the coefficient matrix, the solution?
22. If (a, b) is a multiple of (c, d) with $abcd \neq 0$, show that (a, c) is a multiple of (b, d) . This is surprisingly important: call it a challenge question. You could use numbers first to see how a, b, c , and d are related. The question will lead to:
If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has dependent rows then it has dependent columns.
23. In these equations, the third column (multiplying w) is the *same* as the right side b . The column form of the equations *immediately* gives what solution for (u, v, w) ?

$$\begin{array}{rclcl} 6u & + & 7v & + & 8w & = & 8 \\ 4u & + & 5v & + & 9w & = & 9 \\ 2u & - & 2v & + & 7w & = & 7. \end{array}$$

1.3 An Example of Gaussian Elimination

The way to understand elimination is by example. We begin in three dimensions:

$$\begin{array}{rclcl} & 2u & + & v & + & w & = & 5 \\ \text{Original system} & 4u & - & 6v & & & = & -2 \\ & -2u & + & 7v & + & 2w & = & 9. \end{array} \quad (1)$$

The problem is to find the unknown values of u , v , and w , and we shall apply Gaussian elimination. (Gauss is recognized as the greatest of all mathematicians, but certainly not because of this invention, which probably took him ten minutes. Ironically, it is the most frequently used of all the ideas that bear his name.) The method starts by **subtracting multiples of the first equation from the other equations**. The goal is to eliminate u from the last two equations. This requires that we

(a) subtract 2 times the first equation from the second

(b) subtract -1 times the first equation from the third.

$$\begin{array}{rclcl} & 2u & + & v & + & w & = & 5 \\ \text{Equivalent system} & & - & 8v & - & 2w & = & -12 \\ & & & 8v & + & 3w & = & 14. \end{array} \quad (2)$$

The coefficient 2 is the **first pivot**. Elimination is constantly dividing the pivot into the numbers underneath it, to find out the right multipliers.

The pivot for the **second stage of elimination** is -8 . We now ignore the first equation. A multiple of the second equation will be subtracted from the remaining equations (in this case there is only the third one) so as to eliminate v . We add the second equation to the third or, in other words, we

(c) subtract -1 times the second equation from the third.

The elimination process is now complete, at least in the “forward” direction:

$$\begin{array}{rclcl} & 2u & + & v & + & w & = & 5 \\ \text{Triangular system} & & - & 8v & - & 2w & = & -12 \\ & & & & & 1w & = & 2. \end{array} \quad (3)$$

This system is solved backward, bottom to top. The last equation gives $w = 2$. Substituting into the second equation, we find $v = 1$. Then the first equation gives $u = 1$. This process is called **back-substitution**.

To repeat: Forward elimination produced the pivots 2, -8 , 1. It subtracted multiples of each row from the rows beneath, It reached the “triangular” system (3), which is solved in reverse order: Substitute each newly computed value into the equations that are waiting.

Remark. One good way to write down the forward elimination steps is to include the right-hand side as an extra column. There is no need to copy u and v and w and $=$ at every step, so we are left with the bare minimum:

$$\begin{bmatrix} 2 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 8 & 3 & 14 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

At the end is the triangular system, ready for back-substitution. You may prefer this arrangement, which guarantees that operations on the left-hand side of the equations are also done on the right-hand side—because *both sides are there together*.

In a larger problem, forward elimination takes most of the effort. We use multiples of the first equation to produce zeros below the first pivot. Then the second column is cleared out below the second pivot. The forward step is finished when the system is triangular; equation n contains only the last unknown multiplied by the last pivot. Back-substitution yields the complete solution in the opposite order—beginning with the last unknown, then solving for the next to last, and eventually for the first.

By definition, **pivots cannot be zero**. We need to divide by them.

The Breakdown of Elimination

Under what circumstances could the process break down? Something must go wrong in the singular case, and something might go wrong in the nonsingular case. This may seem a little premature—after all, we have barely got the algorithm working. But the possibility of breakdown sheds light on the method itself.

The answer is: With a full set of n pivots, there is only one solution. The system is non singular, and it is solved by forward elimination and back-substitution. But **if a zero appears** in a pivot position, elimination has to stop—either temporarily or permanently. The system might or might not be singular.

If the first coefficient is zero, in the upper left corner, the elimination of u from the other equations will be impossible. The same is true at every intermediate stage. Notice that a zero can appear in a pivot position, even if the original coefficient in that place was not zero. Roughly speaking, **we do not know whether a zero will appear until we try**, by actually going through the elimination process.

In many cases this problem can be cured, and elimination can proceed. Such a system still counts as nonsingular; it is only the algorithm that needs repair. In other cases a breakdown is unavoidable. Those incurable systems are singular, they have no solution or else infinitely many, and a full set of pivots cannot be found.

Example 1. Nonsingular (cured by exchanging equations 2 and 3)

$$\begin{array}{rcl}
 u + v + w = ___ & u + v + w = ___ & u + v + w = ___ \\
 2u + 2v + 5w = ___ \rightarrow & 3w = ___ \rightarrow & 2v + 4w = ___ \\
 4u + 6v + 8w = ___ & 2v + 4w = ___ & 3w = ___
 \end{array}$$

The system is now triangular, and back-substitution will solve it.

Example 2. Singular (incurable)

$$\begin{array}{rcl}
 u + v + w = ______ & u + v + w = ______ \\
 2u + 2v + 5w = ______ \longrightarrow & 3w = ______ \\
 4u + 4v + 8w = ______ & 4w = ______
 \end{array}$$

There is no exchange of equations that can avoid zero in the second pivot position. The equations themselves may be solvable or unsolvable. If the last two equations are $3w = 6$ and $4w = 7$, there is no solution. If those two equations happen to be consistent—as in $3w = 6$ and $4w = 8$ —then this singular case has an infinity of solutions. We know that $w = 2$, but the first equation cannot decide both u and v .

Section 1.5 will discuss row exchanges when the system is not singular. Then the exchanges produce a full set of pivots. Chapter 2 admits the singular case, and limps forward with elimination. The $3w$ can still eliminate the $4w$, and we will call 3 the second pivot. (There won't be a third pivot.) For the present we trust all n pivot entries to be nonzero, without changing the order of the equations. That is the best case, with which we continue.

The Cost of Elimination

Our other question is very practical. *How many separate arithmetical operations does elimination require, for n equations in n unknowns?* If n is large, a computer is going to take our place in carrying out the elimination. Since all the steps are known, we should be able to predict the number of operations.

For the moment, ignore the right-hand sides of the equations, and count only the operations on the left. These operations are of two kinds. We divide by the pivot to find out what multiple (say ℓ) of the pivot equation is to be subtracted. When we do this subtraction, we continually meet a “multiply-subtract” combination; the terms in the pivot equation are multiplied by ℓ , and then subtracted from another equation.

Suppose we call each division, and each multiplication-subtraction, one operation. In column 1, *it takes n operations for every zero we achieve*—one to find the multiple ℓ , and the other to find the new entries along the row. There are $n - 1$ rows underneath the first one, so the first stage of elimination needs $n(n - 1) = n^2 - n$ operations. (Another approach to $n^2 - n$ is this: *All n^2 entries need to be changed, except the n in the first row.*) Later stages are faster because the equations are shorter.

When the elimination is down to k equations, only $k^2 - k$ operations are needed to clear out the column below the pivot—by the same reasoning that applied to the first stage, when k equaled n . Altogether, the total number of operations is the sum of $k^2 - k$ over all values of k from 1 to n :

$$\begin{aligned} \text{Left side} \quad (1^2 + \cdots + n^2) - (1 + \cdots + n) &= \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} \\ &= \frac{n^3 - n}{3}. \end{aligned}$$

Those are standard formulas for the sums of the first n numbers and the first n squares. Substituting $n = 1$ and $n = 2$ and $n = 100$ into the formula $\frac{1}{3}(n^3 - n)$, forward elimination can take no steps or two steps or about a third of a million steps:

If n is at all large, *a good estimate for the number of operations is $\frac{1}{3}n^3$.*

If the size is doubled, and few of the coefficients are zero, the cost is multiplied by 8.

Back-substitution is considerably faster. The last unknown is found in only one operation (a division by the last pivot). The second to last unknown requires two operations, and so on. Then the total for back-substitution is $1 + 2 + \cdots + n$.

Forward elimination also acts on the right-hand side (subtracting the same multiples as on the left to maintain correct equations). This starts with $n - 1$ subtractions of the first equation. Altogether *the right-hand side is responsible for n^2 operations*—much less than the $n^3/3$ on the left. The total for forward and back is

$$\text{Right side} \quad [(n-1) + (n-2) + \cdots + 1] + [1 + 2 + \cdots + n] = n^2.$$

Thirty years ago, almost every mathematician would have guessed that a general system of order n could not be solved with much fewer than $n^3/3$ multiplications. (There were even theorems to demonstrate it, but they did not allow for all possible methods.) Astonishingly, that guess has been proved wrong. *There now exists a method that requires only $Cn^{\log_2 7}$ multiplications!* It depends on a simple fact: Two combinations of two vectors in two-dimensional space would seem to take 8 multiplications, but they can be done in 7. That lowered the exponent from $\log_2 8$, which is 3, to $\log_2 7 \approx 2.8$. This discovery produced tremendous activity to find the smallest possible power of n . The exponent finally fell (at IBM) below 2.376. Fortunately for elimination, the constant C is so large and the coding is so awkward that the new method is largely (or entirely) of theoretical interest. The newest problem is the cost with *many processors in parallel*.

Problem Set 1.3

Problems 1–9 are about elimination on 2 by 2 systems.

1. What multiple ℓ of equation 1 should be subtracted from equation 2?

$$\begin{aligned} 2x + 3y &= 1 \\ 10x + 9y &= 11. \end{aligned}$$

After this elimination step, write down the upper triangular system and circle the two pivots. The numbers 1 and 11 have no influence on those pivots.

2. Solve the triangular system of Problem 1 by back-substitution, y before x . Verify that x times $(2, 10)$ plus y times $(3, 9)$ equals $(1, 11)$. If the right-hand side changes to $(4, 44)$, what is the new solution?
3. What multiple of equation 2 should be *subtracted* from equation 3?

$$\begin{aligned} 2x - 4y &= 6 \\ -x + 5y &= 0. \end{aligned}$$

After this elimination step, solve the triangular system. If the right-hand side changes to $(-6, 0)$, what is the new solution?

4. What multiple ℓ of equation 1 should be subtracted from equation 2?

$$\begin{aligned} ax + by &= f \\ cx + dy &= g. \end{aligned}$$

The first pivot is a (assumed nonzero). Elimination produces what formula for the second pivot? What is y ? The second pivot is missing when $ad = bc$.

5. Choose a right-hand side which gives no solution and another right-hand side which gives infinitely many solutions. What are two of those solutions?

$$\begin{aligned} 3x + 2y &= 10 \\ 6x + 4y &= \underline{\quad}. \end{aligned}$$

6. Choose a coefficient b that makes this system singular. Then choose a right-hand side g that makes it solvable. Find two solutions in that singular case.

$$\begin{aligned} 2x + by &= 16 \\ 4x + 8y &= g. \end{aligned}$$

7. For which numbers a does elimination break down (a) permanently, and (b) temporarily?

$$\begin{aligned} ax + 3y &= -3 \\ 4x + 6y &= 6. \end{aligned}$$

Solve for x and y after fixing the second breakdown by a row exchange.

8. For which three numbers k does elimination break down? Which is fixed by a row exchange? In each case, is the number of solutions 0 or 1 or ∞ ?

$$\begin{aligned} kx + 3y &= 6 \\ 3x + ky &= -6. \end{aligned}$$

9. What test on b_1 and b_2 decides whether these two equations allow a solution? How many solutions will they have? Draw the column picture.

$$\begin{aligned} 3x - 2y &= b_1 \\ 6x - 4y &= b_2. \end{aligned}$$

Problems 10–19 study elimination on 3 by 3 systems (and possible failure).

10. Reduce this system to upper triangular form by two row operations:

$$\begin{aligned} 2x + 3y + z &= 8 \\ 4x + 7y + 5z &= 20 \\ -2y + 2z &= 0. \end{aligned}$$

Circle the pivots. Solve by back-substitution for z , y , x .

11. Apply elimination (circle the pivots) and back-substitution to solve

$$\begin{aligned} 2x - 3y &= 3 \\ 4x - 5y + z &= 7 \\ 2x - y - 3z &= 5. \end{aligned}$$

List the three row operations: Subtract ____ times row ____ from row ____.

12. Which number d forces a row exchange, and what is the triangular system (not singular) for that d ? Which d makes this system singular (no third pivot)?

$$\begin{aligned} 2x + 5y + z &= 0 \\ 4x + dy + z &= 2 \\ y - z &= 3. \end{aligned}$$

13. Which number b leads later to a row exchange? Which b leads to a missing pivot? In that singular case find a nonzero solution x , y , z .

$$\begin{aligned} x + by &= 0 \\ x - 2y - z &= 0 \\ y + z &= 0. \end{aligned}$$

14. (a) Construct a 3 by 3 system that needs two row exchanges to reach a triangular form and a solution.
(b) Construct a 3 by 3 system that needs a row exchange to keep going, but breaks down later.

15. If rows 1 and 2 are the same, how far can you get with elimination (allowing row exchange)? If columns 1 and 2 are the same, which pivot is missing?

$$\begin{array}{rcl} 2x - y + z & = & 0 \\ 2x - y + z & = & 0 \\ 4x + y + z & = & 2 \end{array} \qquad \begin{array}{rcl} 2x + 2y + z & = & 0 \\ 4x + 4y + z & = & 0 \\ 6x + 6y + z & = & 2. \end{array}$$

16. Construct a 3 by 3 example that has 9 different coefficients on the left-hand side, but rows 2 and 3 become zero in elimination. How many solutions to your system with $b = (1, 10, 100)$ and how many with $b = (0, 0, 0)$?
17. Which number q makes this system singular and which right-hand side t gives it infinitely many solutions? Find the solution that has $z = 1$.

$$\begin{array}{rcl} x + 4y - 2z & = & 1 \\ x + 7y - 6z & = & 6 \\ 3y + qz & = & t. \end{array}$$

18. (Recommended) It is impossible for a system of linear equations to have exactly two solutions. *Explain why.*
- (a) If (x, y, z) and (X, Y, Z) are two solutions, what is another one?
- (b) If 25 planes meet at two points, where else do they meet?
19. Three planes can fail to have an intersection point, when no two planes are parallel. The system is singular if row 3 of A is a ____ of the first two rows. Find a third equation that can't be solved if $x + y + z = 0$ and $x - 2y - z = 1$.

Problems 20–22 move up to 4 by 4 and n by n .

20. Find the pivots and the solution for these four equations:

$$\begin{array}{rcl} 2x + y & & = 0 \\ x + 2y + z & & = 0 \\ y + 2z + t & = & 0 \\ z + 2t & = & 5. \end{array}$$

21. If you extend Problem 20 following the 1, 2, 1 pattern or the $-1, 2, -1$ pattern, what is the fifth pivot? What is the n th pivot?
22. Apply elimination and back-substitution to solve

$$\begin{array}{rcl} 2u + 3v & & = 0 \\ 4u + 5v + w & = & 3 \\ 2u - v - 3w & = & 5. \end{array}$$

What are the pivots? List the three operations in which a multiple of one row is subtracted from another.

23. For the system

$$\begin{aligned} u + v + w &= 2 \\ u + 3v + 3w &= 0 \\ u + 3v + 5w &= 2, \end{aligned}$$

what is the triangular system after forward elimination, and what is the solution?

24. Solve the system and find the pivots when

$$\begin{aligned} 2u - v &= 0 \\ -u + 2v - w &= 0 \\ -v + 2w - z &= 0 \\ -w + 2z &= 5. \end{aligned}$$

You may carry the right-hand side as a fifth column (and omit writing u, v, w, z until the solution at the end).

25. Apply elimination to the system

$$\begin{aligned} u + v + w &= -2 \\ 3u + 3v - w &= 6 \\ u - v + w &= -1. \end{aligned}$$

When a zero arises in the pivot position, exchange that equation for the one below it and proceed. What coefficient of v in the third equation, in place of the present -1 , would make it impossible to proceed—and force elimination to break down?

26. Solve by elimination the system of two equations

$$\begin{aligned} x - y &= 0 \\ 3x + 6y &= 18. \end{aligned}$$

Draw a graph representing each equation as a straight line in the x - y plane; the lines intersect at the solution. Also, add one more line—the graph of the new second equation which arises after elimination.

27. Find three values of a for which elimination breaks down, temporarily or permanently, in

$$\begin{aligned} au + u &= 1 \\ 4u + av &= 2. \end{aligned}$$

Breakdown at the first step can be fixed by exchanging rows—but not breakdown at the last step.

28. True or false:

- (a) If the third equation starts with a zero coefficient (it begins with $0u$) then no multiple of equation 1 will be subtracted from equation 3.

- (b) If the third equation has zero as its second coefficient (it contains $0v$) then no multiple of equation 2 will be subtracted from equation 3.
- (c) If the third equation contains $0u$ and $0v$, then no multiple of equation 1 or equation 2 will be subtracted from equation 3.

29. (Very optional) Normally the multiplication of two complex numbers

$$(a + ib)(c + id) = (ac - bd) + i(bc + ad)$$

involves the four separate multiplications ac, bd, bc, ad . Ignoring i , can you compute $ac - bd$ and $bc + ad$ with only three multiplications? (You may do additions, such as forming $a + b$ before multiplying, without any penalty.)

30. Use elimination to solve

$$\begin{array}{rrrrrrcl} u & + & v & + & w & = & 6 \\ u & + & 2v & + & 2w & = & 11 \\ 2u & + & 3v & - & 4w & = & 3 \end{array}$$

and

$$\begin{array}{rrrrrrcl} u & + & v & + & w & = & 7 \\ u & + & 2v & + & 2w & = & 10 \\ 2u & + & 3v & - & 4w & = & 3. \end{array}$$

31. For which three numbers a will elimination fail to give three pivots?

$$ax + 2y + 3z = b_1$$
$$ax + ay + 4z = b_2$$
$$ax + ay + az = b_3.$$

32. Find experimentally the average size (absolute value) of the first and second and third pivots for MATLAB’s `lu(rand(3,3))`. The average of the first pivot from `abs(A(1,1))` should be 0.5.

1.4 Matrix Notation and Matrix Multiplication

With our 3 by 3 example, we are able to write out all the equations in full. We can list the elimination steps, which subtract a multiple of one equation from another and reach a triangular matrix. For a large system, this way of keeping track of elimination would be hopeless; a much more concise record is needed.

We now introduce **matrix notation** to describe the original system, and **matrix multiplication** to describe the operations that make it simpler. Notice that three different types of quantities appear in our example:

Nine coefficients

Three unknowns

Three right-hand sides

$$\begin{array}{rrrrrrcl} 2u & + & v & + & w & = & 5 \\ 4u & - & 6v & & & = & -2 \\ -2u & + & 7v & + & 2w & = & 9 \end{array}$$

(1)