Fourth Edition

LINEAR ALGEBRA
AND ITS APPLICATIONS

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Chapter 3

Orthogonality

3.1 Orthogonal Vectors and Subspaces

A basis is a set of independent vectors that span a space. Geometrically, it is a set of coordinate axes. A vector space is defined without those axes, but every time I think of the x-y plane or three-dimensional space or $\mathbf{R}^n$, the axes are there. They are usually perpendicular! *The coordinate axes that the imagination constructs are practically always orthogonal*. In choosing a basis, we tend to choose an orthogonal basis.

The idea of an orthogonal basis is one of the foundations of linear algebra. We need a basis to convert geometric constructions into algebraic calculations, and we need an orthogonal basis to make those calculations simple. A further specialization makes the basis just about optimal: The vectors should have *length* 1. For an ***orthonormal basis*** (orthogonal unit vectors), we will find

1. the length $\|x\|$ of a vector;
2. the test $x^{\mathrm{T}}y = 0$ for perpendicular vectors; and
3. how to create perpendicular vectors from linearly independent vectors.

More than just vectors, *subspaces* can also be perpendicular. We will discover, so beautifully and simply that it will be a delight to see, that ***the fundamental subspaces meet at right angles***. Those four subspaces are perpendicular in pairs, two in $\mathbf{R}^m$ and two in $\mathbf{R}^n$. That will complete the fundamental theorem of linear algebra.

The first step is to find the ***length of a vector***. It is denoted by $\|x\|$, and in two dimensions it comes from the hypotenuse of a right triangle (Figure 3.1a). The square of the length was given a long time ago by Pythagoras: $\|x\|^2 = x_1^2 + x_2^2$.

In three-dimensional space, $x = (x_1, x_2, x_3)$ is the diagonal of a box (Figure 3.1b). Its length comes from *two* applications of the Pythagorean formula. The two-dimensional case takes care of $(x_1, x_2, 0) = (1, 2, 0)$ across the base. This forms a right angle with the vertical side $(0, 0, x_3) = (0, 0, 3)$. The hypotenuse of the bold triangle (Pythagoras again)

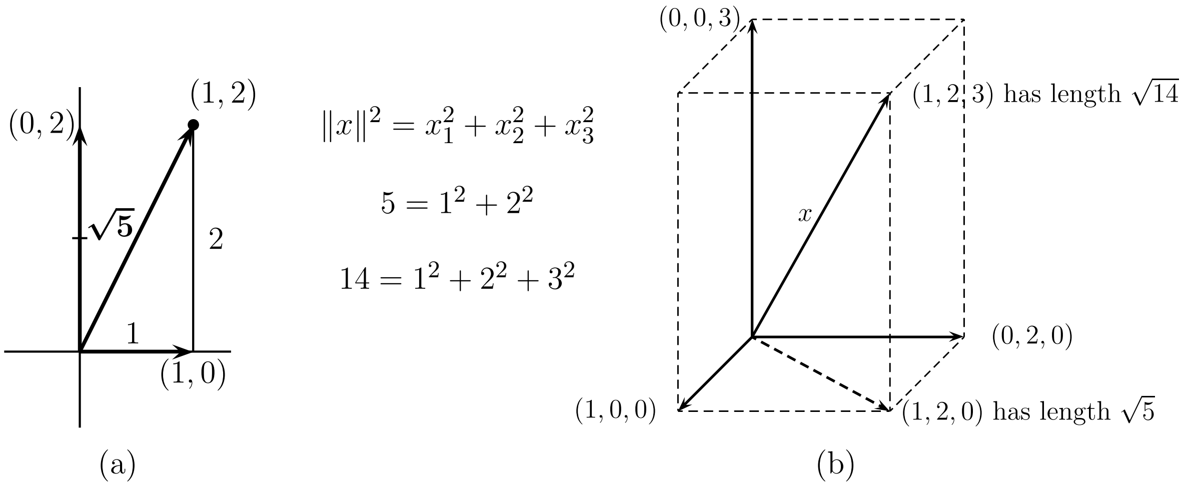


Figure 3.1: The length of vectors (x_1, x_2) and (x_1, x_2, x_3).

is the length $\|x\|$ we want:

$$\textbf{Length in 3D} \qquad \|x\|^2 = 1^2 + 2^2 + 3^2 \quad \text{and} \quad \|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}.$$

The extension to $x = (x_1, \ldots, x_n)$ in n dimensions is immediate. By Pythagoras $n-1$ times, ***the length*** $\|x\|$ ***in*** $\mathbf{R}^n$ ***is the positive square root of*** $x^{\mathrm{T}}x$:

$$\textbf{Length squared} \qquad \|x\|^2 = x_1^2 + x_2^2 + \cdots + x_n^2 = x^{\mathrm{T}}x. \tag{1}$$

The sum of squares matches $x^{\mathrm{T}}x$—and the length of $x = (1, 2, -3)$ is $\sqrt{14}$:

$$x^{\mathrm{T}}x = \begin{bmatrix} 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} = 1^2 + 2^2 + (-3)^2 = 14.$$

Orthogonal Vectors

How can we decide whether two vectors x and y are perpendicular? What is the test for orthogonality in Figure 3.2? In the plane spanned by x and y, those vectors are orthogonal provided they form a *right triangle*. We go back to $a^2 + b^2 = c^2$:

$$\textbf{Sides of a right triangle} \qquad \|x\|^2 + \|y\|^2 = \|x - y\|^2. \tag{2}$$

Applying the length formula (1), this test for orthogonality in $\mathbf{R}^n$ becomes

$$\left(x_1^2 + \cdots + x_n^2\right) + \left(y_1^2 + \cdots + y_n^2\right) = (x_1 - y_1)^2 + \cdots + (x_n - y_n)^2.$$

The right-hand side has an extra $-2x_iy_i$ from each $(x_i - y_i)^2$:

$$\text{right-hand side} = \left(x_1^2 + \cdots + x_n^2\right) - 2(x_1y_1 + \cdots + x_ny_n) + \left(y_1^2 + \cdots + y_n^2\right).$$

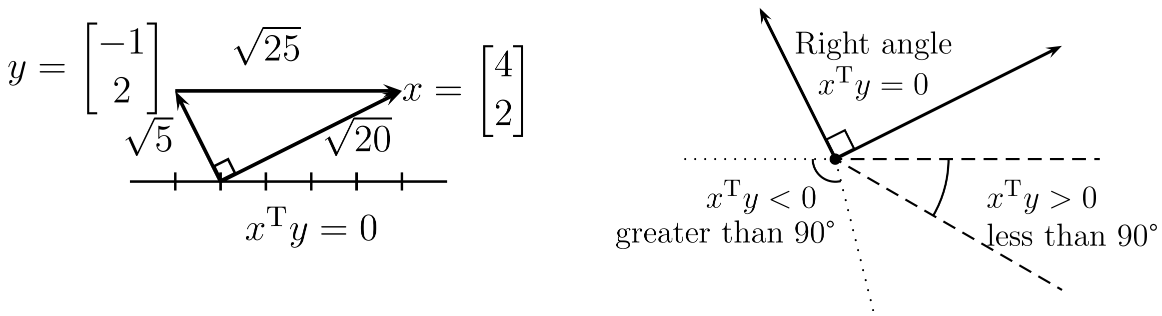


Figure 3.2: A right triangle with $5+20=25$. Dotted angle 100°, dashed angle 30°.

We have a right triangle when that sum of cross-product terms x_iy_i is zero:

$$\textbf{Orthogonal vectors} \qquad x^Ty = x_1y_1 + \cdots + x_ny_n = 0. \tag{3}$$

This sum is $x^Ty = \sum x_iy_i = y^Tx$, the row vector x^T times the column vector y:

$$\textbf{Inner product} \qquad x^Ty = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = x_1y_1 + \cdots + x_ny_n. \tag{4}$$

This number is sometimes called the scalar product or dot product, and denoted by (x,y) or $x \cdot y$. We will use the name ***inner product*** and keep the notation x^Ty.

> **3A** The inner product x^Ty is zero if and only if x and y are orthogonal vectors. If $x^Ty > 0$, their angle is less than 90°. If $x^Ty < 0$, their angle is greater than 90°.

The length squared is the inner product of x with itself: $x^Tx = x_1^2 + \cdots + x_n^2 = \|x\|^2$. The only vector with length zero—the only vector orthogonal to itself—is the zero vector. This vector $x = 0$ is orthogonal to every vector in $\mathbf{R}^n$.

Example 1. $(2,2,-1)$ is orthogonal to $(-1,2,2)$. Both have length $\sqrt{4+4+1} = 3$.

Useful fact: **If nonzero vectors** $v_1,\ldots,v_k$ **are mutually orthogonal** (every vector is perpendicular to every other), **then those vectors are linearly independent**.

Proof. Suppose $c_1v_1 + \cdots + c_kv_k = 0$. To show that c_1 must be zero, take the inner product of both sides with v_1. Orthogonality of the v's leaves only one term:

$$v_1^T(c_1v_1 + \cdots + c_kv_k) = c_1v_1^Tv_1 = 0. \tag{5}$$

The vectors are nonzero, so $v_1^Tv_1 \neq 0$ and therefore $c_1 = 0$. The same is true of every c_i. The only combination of the v's producing zero has all $c_i = 0$: *independence*! □

The coordinate vectors $e_1,\ldots,e_n$ in $\mathbf{R}^n$ are the most important orthogonal vectors. Those are the columns of the identity matrix. They form the simplest basis for $\mathbf{R}^n$, and they are *unit vectors*—each has length $\|e_i\| = 1$. They point along the coordinate axes. If these axes are rotated, the result is a new **orthonormal basis**: a new system of *mutually orthogonal unit vectors*. In $\mathbf{R}^2$ we have $\cos^2\theta + \sin^2\theta = 1$:

Orthonormal vectors in $\mathbf{R}^2$ $\quad v_1 = (\cos\theta, \sin\theta) \quad \text{and} \quad v_2 = (-\sin\theta, \cos\theta).$

Orthogonal Subspaces

We come to the orthogonality of two subspaces. **Every vector** *in one subspace must be orthogonal to* **every vector** *in the other subspace*. Subspaces of $\mathbf{R}^3$ can have dimension 0, 1, 2, or 3. The subspaces are represented by lines or planes through the origin—and in the extreme cases, by the origin alone or the whole space. The subspace $\{0\}$ is orthogonal to all subspaces. A line can be orthogonal to another line, or it can be orthogonal to a plane, but *a plane cannot be orthogonal to a plane*.

I have to admit that the front wall and side wall of a room look like perpendicular planes in $\mathbf{R}^3$. But by our definition, that is not so! There are lines v and w in the front and side walls that do not meet at a right angle. The line along the corner is in *both* walls, and it is certainly not orthogonal to itself.

> **3B** Two subspaces $\mathbf{V}$ and $\mathbf{W}$ of the same space $\mathbf{R}^n$ are *orthogonal* if every vector v in $\mathbf{V}$ is orthogonal to every vector w in $\mathbf{W}$: $v^{\mathrm{T}}w = 0$ for all v and w.

Example 2. Suppose $\mathbf{V}$ is the plane spanned by $v_1 = (1,0,0,0)$ and $v_2 = (1,1,0,0)$. If $\mathbf{W}$ is the line spanned by $w = (0,0,4,5)$, then w is orthogonal to both v's. The line $\mathbf{W}$ will be orthogonal to the whole plane $\mathbf{V}$.

In this case, with subspaces of dimension 2 and 1 in $\mathbf{R}^4$, there is room for a third subspace. The line $\mathbf{L}$ through $z = (0,0,5,-4)$ is perpendicular to $\mathbf{V}$ and $\mathbf{W}$. Then the dimensions add to $2+1+1 = 4$. What space is perpendicular to all of $\mathbf{V}$, $\mathbf{W}$, and $\mathbf{L}$?

The important orthogonal subspaces don't come by accident, and they come two at a time. In fact orthogonal subspaces are unavoidable: ***They are the fundamental subspaces!*** The first pair is the *nullspace* and *row space*. Those are subspaces of $\mathbf{R}^n$—the rows have n components and so does the vector x in $Ax = 0$. We have to show, using $Ax = 0$, that ***the rows of*** A ***are orthogonal to the nullspace vector*** x.

> **3C** **Fundamental theorem of orthogonality** The row space is orthogonal to the nullspace (in $\mathbf{R}^n$). The column space is orthogonal to the left nullspace (in $\mathbf{R}^m$).

First Proof. Suppose x is a vector in the nullspace. Then $Ax = 0$, and this system of m

equations can be written out as rows of A multiplying x:

$$\textbf{Every row is orthogonal to } x \qquad Ax = \begin{bmatrix} \cdots & \textbf{row 1} & \cdots \\ \cdots & \text{row } 2 & \cdots \\ \vdots & \vdots & \vdots \\ \cdots & \text{row } m & \cdots \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{6}$$

The main point is already in the first equation: *row* 1 *is orthogonal to* x. Their inner product is zero; that is equation 1. Every right-hand side is zero, so x is orthogonal to every row. Therefore x is orthogonal to every *combination* of the rows. Each x in the nullspace is orthogonal to each vector in the row space, so $\boldsymbol{N}(A) \perp \boldsymbol{C}(A^{\mathrm{T}})$.

The other pair of orthogonal subspaces comes from $A^{\mathrm{T}}y = 0$, or $y^{\mathrm{T}}A = 0$:

$$y^{\mathrm{T}}A = \begin{bmatrix} y_1 & \cdots & y_m \end{bmatrix} \begin{bmatrix} \mathbf{c} & & \mathrm{c} \\ \mathbf{o} & & \mathrm{o} \\ \mathbf{l} & & \mathrm{l} \\ \mathbf{u} & \cdots & \mathrm{u} \\ \mathbf{m} & & \mathrm{m} \\ \mathbf{n} & & \mathrm{n} \\ & & \\ \mathbf{1} & & n \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \cdots & 0 \end{bmatrix}. \tag{7}$$

The vector y is orthogonal to every column. The equation says so, from the zeros on the right-hand side. Therefore y is orthogonal to every combination of the columns. It is orthogonal to the column space, and it is a typical vector in the left nullspace: $\boldsymbol{N}(A^{\mathrm{T}}) \perp \boldsymbol{C}(A)$. This is the same as the first half of the theorem, with A replaced by A^{T}. □

Second Proof. The contrast with this "coordinate-free proof" should be useful to the reader. It shows a more "abstract" method of reasoning. I wish I knew which proof is clearer, and more permanently understood.

If x is in the nullspace then $Ax = 0$. If v is in the row space, it is a combination of the rows: $v = A^{\mathrm{T}}z$ for some vector z. Now, in one line:

$$\textbf{Nullspace} \perp \textbf{Row space} \qquad v^{\mathrm{T}}x = (A^{\mathrm{T}}z)^{\mathrm{T}}x = z^{\mathrm{T}}Ax = z^{\mathrm{T}}0 = 0. \tag{8}$$

□

Example 3. Suppose A has rank 1, so its row space and column space are lines:

$$\textbf{Rank-1 matrix} \qquad A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \\ 3 & 9 \end{bmatrix}.$$

The rows are multiples of $(1,3)$. The nullspace contains $x = (-3,1)$, which is orthogonal to all the rows. The nullspace and row space are perpendicular lines in $\mathbf{R}^2$:

$$\begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = 0 \quad \text{and} \quad \begin{bmatrix} 2 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = 0 \quad \text{and} \quad \begin{bmatrix} 3 & 9 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = 0.$$

In contrast, the other two subspaces are in $\mathbf{R}^3$. The column space is the line through $(1,2,3)$. The left nullspace must be the *perpendicular plane* $y_1+2y_2+3y_3=0$. That equation is exactly the content of $y^{\mathrm{T}}A=0$.

The first two subspaces (the two lines) had dimensions $1+1=2$ in the space $\mathbf{R}^2$. The second pair (line and plane) had dimensions $1+2=3$ in the space $\mathbf{R}^3$. In general, *the row space and nullspace have dimensions that add to* $r+(n-r)=n$. The other pair adds to $r+(m-r)=m$. Something more than orthogonality is occurring, and I have to ask your patience about that one further point: **the dimensions**.

It is certainly true that the null space is perpendicular to the row space—but it is not the whole truth. $\boldsymbol{N}(A)$ *contains every vector orthogonal to the row space*. The nullspace was formed from *all* solutions to $Ax=0$.

Definition. Given a subspace $\mathbf{V}$ of $\mathbf{R}^n$, the space of *all* vectors orthogonal to $\mathbf{V}$ is called the **orthogonal complement** of $\mathbf{V}$. It is denoted by $\mathbf{V}^{\perp}=$ "$\mathbf{V}$ perp."

Using this terminology, the nullspace is the orthogonal complement of the row space: $\boldsymbol{N}(A)=(\boldsymbol{C}(A^{\mathrm{T}}))^{\perp}$. At the same time, the row space contains all vectors that are orthogonal to the nullspace. A vector z can't be orthogonal to the nullspace but outside the row space. Adding z as an extra row of A would enlarge the row space, but we know that there is a fixed formula $r+(n-r)=n$:

Dimension formula $\qquad$ dim(row space) + dim(nullspace) = number of columns.

Every vector orthogonal to the nullspace is in the row space: $\boldsymbol{C}(A^{\mathrm{T}})=(\boldsymbol{N}(A))^{\perp}$.

The same reasoning applied to A^{T} produces the dual result: *The left nullspace* $\boldsymbol{N}(A^{\mathrm{T}})$ *and the column space* $\boldsymbol{C}(A)$ *are orthogonal complements*. Their dimensions add up to $(m-r)+r=m$, This completes the second half of the fundamental theorem of linear algebra. The first half gave the dimensions of the four subspaces. including the fact that row rank = column rank. Now we know that those subspaces are perpendicular. More than that, the subspaces are orthogonal complements.

3D Fundamental Theorem of Linear Algebra, Part II

The nullspace is the *orthogonal complement* of the row space in $\mathbf{R}^n$.

The left nullspace is the *orthogonal complement* of the column space in $\mathbf{R}^m$.

To repeat, the row space contains everything orthogonal to the nullspace. The column space contains everything orthogonal to the left nullspace. That is just a sentence, hidden in the middle of the book, but *it decides exactly which equations can be solved*! Looked at directly, $Ax=b$ requires b to be in the column space. Looked at indirectly. $Ax=b$ ***requires*** b ***to be perpendicular to the left nullspace***.

3E $Ax=b$ is solvable if and only if $y^{\mathrm{T}}b=0$ whenever $y^{\mathrm{T}}A=0$.

The direct approach was "b must be a combination of the columns." The indirect approach is "b must be orthogonal to every vector that is orthogonal to the columns." That doesn't sound like an improvement (to put it mildly). But if only one or two vectors are orthogonal to the columns. it is much easier to check those one or two conditions $y^{\mathrm{T}}b = 0$. A good example is Kirchhoff's Voltage Law in Section 2.5. Testing for zero around loops is much easier than recognizing combinations of the columns.

When the left-hand sides of $Ax = b$ ***add to zero, the right-hand sides must, too:***

$$\begin{aligned} x_1 - x_2 &= b_1 \\ x_2 - x_3 &= b_2 \\ x_3 - x_1 &= b_3 \end{aligned} \quad \text{is solvable if and only if } b_1 + b_2 + b_3 = 0. \quad \text{Here} \quad A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}.$$

This test $b_1 + b_2 + b_3 = 0$ makes b orthogonal to $y = (1, 1, 1)$ in the left nullspace. By the Fundamental Theorem, b is a combination of the columns!

The Matrix and the Subspaces

We emphasize that $\mathbf{V}$ and $\mathbf{W}$ can be orthogonal without being complements. Their dimensions can be too small. The line $\mathbf{V}$ spanned by $(0, 1, 0)$ is orthogonal to the line $\mathbf{W}$ spanned by $(0, 0, 1)$, but $\mathbf{V}$ is not $\mathbf{W}^{\perp}$. The orthogonal complement of $\mathbf{W}$ is a two-dimensional plane, and the line is only part of $\mathbf{W}^{\perp}$. When the dimensions are right, orthogonal subspaces *are* necessarily orthogonal complements:

$$\text{If} \quad \mathbf{W} = \mathbf{V}^{\perp} \quad \text{then} \quad \mathbf{V} = \mathbf{W}^{\perp} \quad \text{and} \quad \dim \mathbf{V} + \dim \mathbf{W} = n.$$

In other words $\mathbf{V}^{\perp\perp} = \mathbf{V}$. The dimensions of $\mathbf{V}$ and $\mathbf{W}$ are right, and the whole space $\mathbf{R}^n$ is being decomposed into two perpendicular parts (Figure 3.3).

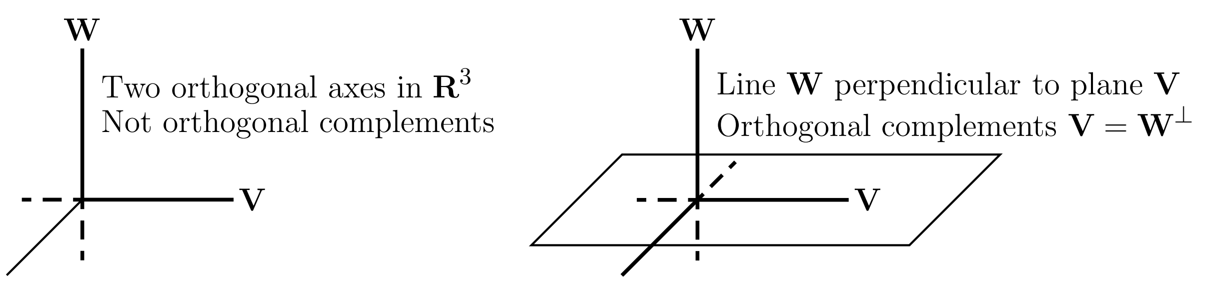


Figure 3.3: Orthogonal complements in $\mathbf{R}^3$: a plane and a line (not two lines).

Splitting $\mathbf{R}^n$ into orthogonal parts will split every vector into $x = v + w$. The vector v is the projection onto the subspace $\mathbf{V}$. The orthogonal component w is the projection of x onto $\mathbf{W}$. The next sections show how to find those projections of x. They lead to what is probably the most important figure in the book (Figure 3.4).

Figure 3.4 summarizes the fundamental theorem of linear algebra. It illustrates the true effect of a matrix—what is happening inside the multiplication Ax. The nullspace

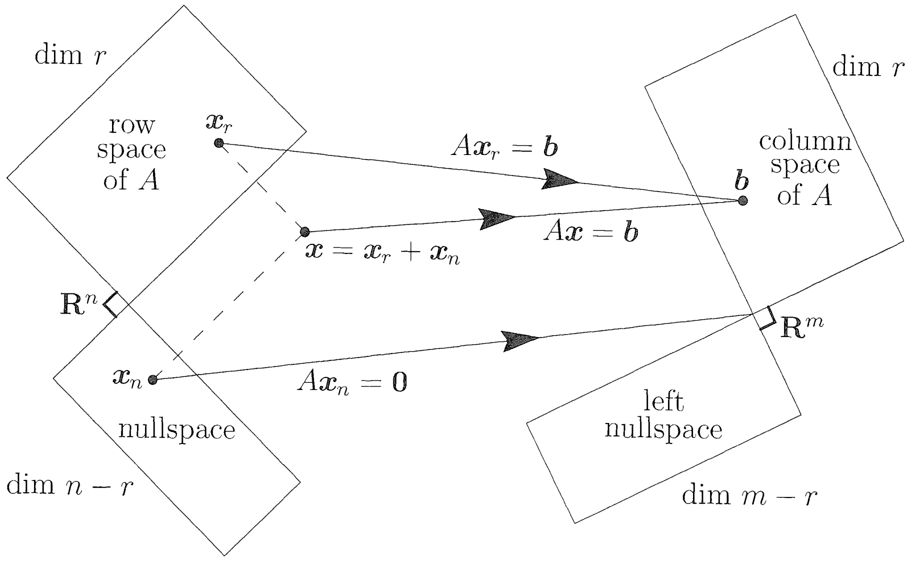


Figure 3.4: The true action $Ax = A(x_{\text{row}} + x_{\text{null}})$ of any m by n matrix.

is carried to the zero vector. Every Ax is in the column space. Nothing is carried to the left nullspace. *The real action is between the row space and column space*, and you see it by looking at a typical vector x. It has a "row space component" and a "nullspace component," with $x = x_r + x_n$. When multiplied by A, this is $Ax = Ax_r + Ax_n$:

The nullspace component goes to zero: $Ax_n = 0$.

The row space component goes to the column space: $Ax_r = Ax$.

Of course everything goes to the column space—the matrix cannot do anything else. I tried to make the row and column spaces the same size, with equal dimension r.

3F From the row space to the column space, A is actually invertible. Every vector b in the column space comes from exactly one vector x_r in the row space.

Proof. Every b in the column space is a combination Ax of the columns. In fact, b is Ax_r, with x_r in the row space, since the nullspace component gives $Ax_n = 0$, If another vector x_r' in the row space gives $Ax_r' = b$, then $A(x_r - x_r') = b - b = 0$. This puts $x_r - x_r'$ in the nullspace and the row space, which makes it orthogonal to itself. Therefore it is zero, and $x_r - x_r'$. Exactly one vector in the row space is carried to b. □

Every matrix transforms its row space onto its column space.

On those r-dimensional spaces A is invertible. On its nullspace A is zero. When A is diagonal, you see the invertible submatrix holding the r nonzeros.

A^{T} goes in the opposite direction, from $\mathbf{R}^m$ to $\mathbf{R}^n$ and from $\boldsymbol{C}(A)$ back to $\boldsymbol{C}(A^{\mathrm{T}})$. Of course the transpose is not the inverse! A^{T} moves the spaces correctly, but not the individual vectors. That honor belongs to A^{-1} if it exists—and it only exists if $r=m=n$. We cannot ask A^{-1} to bring back a whole nullspace out of the zero vector.

When A^{-1} fails to exist, the best substitute is the ***pseudoinverse*** A^+. This inverts A where that is possible: $A^+Ax=x$ for x in the row space. On the left nullspace, nothing can be done: $A^+y=0$. Thus A^+ inverts A where it is invertible, and has the same rank r. One formula for A^+ depends on the ***singular value decomposition***—for which we first need to know about eigenvalues.

Problem Set 3.1

1. Find the lengths and the inner product of $x=(1,4,0,2)$ and $y=(2,-2,1,3)$.

2. Give an example in $\mathbf{R}^2$ of linearly independent vectors that are not orthogonal. Also, give an example of orthogonal vectors that are not independent.

3. Two lines in the plane are perpendicular when the product of their slopes is -1. Apply this to the vectors $x=(x_1,x_2)$ and $y=(y_1,y_2)$, whose slopes are x_2/x_1 and y_2/y_1, to derive again the orthogonality condition $x^{\mathrm{T}}y=0$.

4. How do we know that the ith row of an invertible matrix B is orthogonal to the jth column of B^{-1}, if $i \neq j$?

5. Which pairs are orthogonal among the vectors v_1, v_2, v_3, v_4?

$$v_1=\begin{bmatrix}1\\2\\-2\\1\end{bmatrix},\qquad v_2=\begin{bmatrix}4\\0\\4\\0\end{bmatrix},\qquad v_3=\begin{bmatrix}1\\-1\\-1\\-1\end{bmatrix},\qquad v_4=\begin{bmatrix}1\\1\\1\\1\end{bmatrix}.$$

6. Find all vectors in $\mathbf{R}^3$ that are orthogonal to $(1,1,1)$ and $(1,-1,0)$. Produce an orthonormal basis from these vectors (mutually orthogonal unit vectors).

7. Find a vector x orthogonal to the row space of A, and a vector y orthogonal to the column space, and a vector z orthogonal to the nullspace:

$$A=\begin{bmatrix}1&2&1\\2&4&3\\3&6&4\end{bmatrix}.$$

8. If $\mathbf{V}$ and $\mathbf{W}$ are orthogonal subspaces, show that the only vector they have in common is the zero vector: $\mathbf{V}\cap\mathbf{W}=\{0\}$.

9. Find the orthogonal complement of the plane spanned by the vectors $(1,1,2)$ and $(1,2,3)$, by taking these to be the rows of A and solving $Ax=0$. Remember that the complement is a whole line.

10. Construct a homogeneous equation in three unknowns whose solutions are the linear combinations of the vectors $(1,1,2)$ and $(1,2,3)$. This is the reverse of the previous exercise, but the two problems are really the same.

11. The fundamental theorem is often stated in the form of *Fredholm's alternative*: For any A and b, one and only one of the following systems has a solution:

(i) $Ax = b$.

(ii) $A^{\mathrm{T}}y = 0$, $y^{\mathrm{T}}b \neq 0$.

Either b is in the column space $\boldsymbol{C}(A)$ or there is a y in $\boldsymbol{N}(A^{\mathrm{T}})$ such that $y^{\mathrm{T}}b \neq 0$. Show that it is contradictory for (i) and (ii) both to have solutions.

12. Find a basis for the orthogonal complement of the row space of A:

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \end{bmatrix}.$$

Split $x = (3,3,3)$ into a row space component x_r and a nullspace component x_n.

13. Illustrate the action of A^{T} by a picture corresponding to Figure 3.4, sending $\boldsymbol{C}(A)$ back to the row space and the left nullspace to zero.

14. Show that $x - y$ is orthogonal to $x + y$ if and only if $\|x\| = \|y\|$.

15. Find a matrix whose row space contains $(1,2,1)$ and whose nullspace contains $(1,-2,1)$, or prove that there is no such matrix.

16. Find all vectors that are perpendicular to $(1,4,4,1)$ and $(2,9,8,2)$.

17. If $\mathbf{V}$ is the orthogonal complement of $\mathbf{W}$ in $\mathbf{R}^n$, is there a matrix with row space $\mathbf{V}$ and nullspace $\mathbf{W}$? Starting with a basis for $\mathbf{V}$, construct such a matrix.

18. If $\mathbf{S} = \{0\}$ is the subspace of $\mathbf{R}^4$ containing only the zero vector, what is $\mathbf{S}^{\perp}$? If $\mathbf{S}$ is spanned by $(0,0,0,1)$, what is $\mathbf{S}^{\perp}$? What is $(\mathbf{S}^{\perp})^{\perp}$?

19. *Why are these statements false*?

(a) If $\mathbf{V}$ is orthogonal to $\mathbf{W}$, then $\mathbf{V}^{\perp}$ is orthogonal to $\mathbf{W}^{\perp}$.

(b) $\mathbf{V}$ orthogonal to $\mathbf{W}$ and $\mathbf{W}$ orthogonal to $\mathbf{Z}$ makes $\mathbf{V}$ orthogonal to $\mathbf{Z}$.

20. Let $\mathbf{S}$ be a subspace of $\mathbf{R}^n$. Explain what $(\mathbf{S}^{\perp})^{\perp} = \mathbf{S}$ means and why it is true.

21. Let $\mathbf{P}$ be the plane in $\mathbf{R}^2$ with equation $x + 2y - z = 0$. Find a vector perpendicular to $\mathbf{P}$. What matrix has the plane $\mathbf{P}$ as its nullspace, and what matrix has $\mathbf{P}$ as its row space?

22. Let $\mathbf{S}$ be the subspace of $\mathbf{R}^4$ containing all vectors with $x_1 + x_2 + x_3 + x_4 = 0$. Find a basis for the space $\mathbf{S}^{\perp}$, containing all vectors orthogonal to $\mathbf{S}$.

23. Construct an unsymmetric 2 by 2 matrix of rank 1. Copy Figure 3.4 and put one vector in each subspace. Which vectors are orthogonal?

24. Redraw Figure 3.4 for a 3 by 2 matrix of rank $r = 2$. Which subspace is $\mathbf{Z}$ (zero vector only)? The nullspace part of any vector x in $\mathbf{R}^2$ is $x_n =$ ____.

25. Construct a matrix with the required property or say why that is impossible.

(a) Column space contains $\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$, nullspace contains $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(b) Row space contains $\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$, nullspace contains $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(c) $Ax = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ has a solution and $A^{\mathrm{T}} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

(d) Every row is orthogonal to every column (A is not the zero matrix).

(e) The columns add up to a column of 0s, the rows add to a row of 1s.

26. If $AB = 0$ then the columns of B are in the ____ of A. The rows of A are in the ____ of B. Why can't A and B be 3 by 3 matrices of rank 2?

27. (a) If $Ax = b$ has a solution and $A^{\mathrm{T}}y = 0$, then y is perpendicular to ____.

(b) If $A^{\mathrm{T}}y = c$ has a solution and $Ax = 0$, then x is perpendicular to ____.

28. This is a system of equations $Ax = b$ with *no solution*:

$$\begin{aligned} x + 2y + 2z &= 5 \\ 2x + 2y + 3z &= 5 \\ 3x + 4y + 5z &= 9. \end{aligned}$$

Find numbers y_1, y_2, y_3 to multiply the equations so they add to $0 = 1$. You have found a vector y in which subspace? The inner product $y^{\mathrm{T}}b$ is 1.

29. In Figure 3.4, how do we know that Ax_r is equal to Ax? How do we know that this vector is in the column space? If $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ what is x_r?

30. If Ax is in the nullspace of A^{T} then $Ax = 0$. Reason: Ax is also in the ____ of A and the spaces are ____. *Conclusion: $A^{\mathrm{T}}A$ has the same nullspace as A.*

31. Suppose A is a symmetric matrix ($A^{\mathrm{T}} = A$).

(a) Why is its column space perpendicular to its nullspace?

(b) If $Ax = 0$ and $Az = 5z$, which subspaces contain these "eigenvectors" x and z? **Symmetric matrices have perpendicular eigenvectors** (see Section 5.5).

32. (Recommended) Draw Figure 3.4 to show each subspace for

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}.$$

33. Find the pieces x_r and x_n, and draw Figure 3.4 properly, if

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad x = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

Problems 34–44 are about orthogonal subspaces.

34. Put bases for the orthogonal subspaces $\mathbf{V}$ and $\mathbf{W}$ into the columns of matrices V and W. *Why does* $V^{\mathrm{T}}W =$ *zero matrix*? This matches $v^{\mathrm{T}}w = 0$ for vectors.

35. The floor and the wall are not orthogonal subspaces because they share a nonzero vector (along the line where they meet). Two planes in $\mathbf{R}^3$ cannot be orthogonal! Find a vector in both column spaces $\boldsymbol{C}(A)$ and $\boldsymbol{C}(B)$:

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 2 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 5 & 4 \\ 6 & 3 \\ 5 & 1 \end{bmatrix}.$$

This will be a vector Ax and also $B\widehat{x}$. Think 3 by 4 with the matrix $[A\ \ B]$.

36. Extend Problem 35 to a p-dimensional subspace $\mathbf{V}$ and a q-dimensional subspace $\mathbf{W}$ of $\mathbf{R}^n$. What inequality on $p+q$ guarantees that $\mathbf{V}$ intersects $\mathbf{W}$ in a nonzero vector? These subspaces cannot be orthogonal.

37. Prove that every y in $\boldsymbol{N}(A^{\mathrm{T}})$ is perpendicular to every Ax in the column space, using the matrix shorthand of equation (8). Start from $A^{\mathrm{T}}y = 0$.

38. If $\mathbf{S}$ is the subspace of $\mathbf{R}^3$ containing only the zero vector, what is $\mathbf{S}^{\perp}$? If $\mathbf{S}$ is spanned by $(1,1,1)$, what is $\mathbf{S}^{\perp}$? If $\mathbf{S}$ is spanned by $(2,0,0)$ and $(0,0,3)$, what is $\mathbf{S}^{\perp}$?

39. Suppose $\mathbf{S}$ only contains $(1,5,1)$ and $(2,2,2)$ (not a subspace). Then $\mathbf{S}^{\perp}$ is the nullspace of the matrix $A =$ ____. $\mathbf{S}^{\perp}$ is a subspace even if $\mathbf{S}$ is not.

40. Suppose $\mathbf{L}$ is a one-dimensional subspace (a line) in $\mathbf{R}^3$. Its orthogonal complement $\mathbf{L}^{\perp}$ is the ____ perpendicular to $\mathbf{L}$. Then $(\mathbf{L}^{\perp})^{\perp}$ is a ____ perpendicular to $\mathbf{L}^{\perp}$. In fact $(\mathbf{L}^{\perp})^{\perp}$ is the same as ____.

41. Suppose $\mathbf{V}$ is the whole space $\mathbf{R}^4$. Then $\mathbf{V}^{\perp}$ contains only the vector ____. Then $(\mathbf{V}^{\perp})^{\perp}$ is ____. So $(\mathbf{V}^{\perp})^{\perp}$ is the same as ____.

42. Suppose $\mathbf{S}$ is spanned by the vectors $(1,2,2,3)$ and $(1,3,3,2)$. Find two vectors that span $\mathbf{S}^{\perp}$. This is the same as solving $Ax = 0$ for which A?

43. If $\mathbf{P}$ is the plane of vectors in $\mathbf{R}^4$ satisfying $x_1 + x_2 + x_3 + x_4 = 0$, write a basis for $\mathbf{P}^{\perp}$. Construct a matrix that has $\mathbf{P}$ as its nullspace.

44. If a subspace $\mathbf{S}$ is contained in a subspace $\mathbf{V}$, prove that $\mathbf{S}^{\perp}$ contains $\mathbf{V}^{\perp}$.

Problems 45–50 are about perpendicular columns and rows.

45. Suppose an n by n matrix is invertible: $AA^{-1} = I$. Then the first column of A^{-1} is orthogonal to the space spanned by which rows of A?

46. Find $A^{\mathrm{T}}A$ if the columns of A are unit vectors, all mutually perpendicular.

47. Construct a 3 by 3 matrix A with no zero entries whose columns are mutually perpendicular. Compute $A^{\mathrm{T}}A$. Why is it a diagonal matrix?

48. The lines $3x+y=b_1$ and $6x+2y=b_2$ are ____. They are the same line if ____. In that case (b_1,b_2) is perpendicular to the vector ____. The nullspace of the matrix is the line $3x+y=$ ____. One particular vector in that nullspace is ____.

49. Why is each of these statements false?

(a) $(1,1,1)$ is perpendicular to $(1,1,-2)$, so the planes $x+y+z=0$ and $x+y-2z=0$ are orthogonal subspaces.

(b) The subspace spanned by $(1,1,0,0,0)$ and $(0,0,0,1,1)$ is the orthogonal complement of the subspace spanned by $(1,-1,0,0,0)$ and $(2,-2,3,4,-4)$.

(c) Two subspaces that meet only in the zero vector are orthogonal.

50. Find a matrix with $v=(1,2,3)$ in the row space and column space. Find another matrix with v in the nullspace and column space. Which pairs of subspaces can v *not* be in?

51. Suppose A is 3 by 4, B is 4 by 5, and $AB=0$. Prove $\mathrm{rank}(A)+\mathrm{rank}(B)\leq 4$.

52. The command N = null(A) will produce a basis for the nullspace of A. Then the command B = null(N') will produce a basis for the ____ of A.

3.2 Cosines and Projections onto Lines

Vectors with $x^{\mathrm{T}}y=0$ are orthogonal. Now we allow inner products that are ***not zero***, and angles that are ***not right angles***. We want to connect inner products to angles, and also to transposes. In Chapter 1 the transpose was constructed by flipping over a matrix as if it were some kind of pancake. We have to do better than that.

One fact is unavoidable: *The orthogonal case is the most important*. Suppose we want to find the distance from a point b to the line in the direction of the vector a. We are looking along that line for the point p closest to b. The key is in the geometry: ***The line connecting** b **to** p* (the dotted line in Figure 3.5) ***is perpendicular to** a. This fact will allow us to find the projection p.* Even though a and b are not orthogonal, the distance problem automatically brings in orthogonality.

The situation is the same when we are given a plane (or any subspace $\mathbf{S}$) instead of a line. Again the problem is to find the point p on that subspace that is closest to b. ***This***

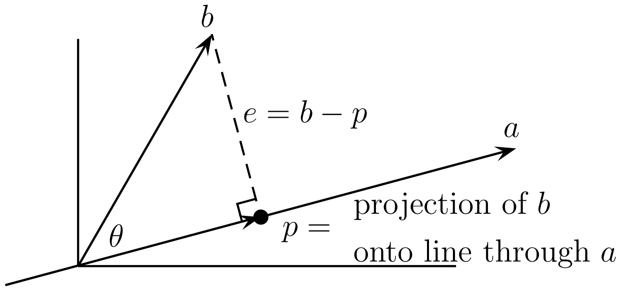


Figure 3.5: The projection p is the point (on the line through a) closest to b.

point** p **is the projection of** b **onto the subspace. A perpendicular line from b to **S** meets the subspace at p. Geometrically, that gives the distance between points b and subspaces **S**. But there are two questions that need to be asked:

1. Does this projection actually arise in practical applications?

2. If we have a basis for the subspace **S**, is there a formula for the projection p?

The answers are certainly yes. This is exactly the problem of the ***least-squares solution to an overdetermined system***. The vector b represents the data from experiments or questionnaires, and it contains too many errors to be found in the subspace **S**. When we try to write b as a combination of the basis vectors for **S**, it cannot be done—the equations are inconsistent, and $Ax = b$ has no solution.

The least-squares method selects p as the best choice to replace b. There can be no doubt of the importance of this application. In economics and statistics, least squares enters *regression analysis*. In geodesy, the U.S. mapping survey tackled 2.5 million equations in 400,000 unknowns.

A formula for p is easy when the subspace is a line. We will project b onto a in several different ways, and relate the projection p to inner products and angles. Projection onto a higher dimensional subspace is by far the most important case; it corresponds to a least-squares problem with several parameters, and it is solved in Section 3.3. The formulas are even simpler when we produce an orthogonal basis for **S**.

inner products and cosines

We pick up the discussion of inner products and angles. You will soon see that it is not the angle, but ***the cosine of the angle***, that is directly related to inner products. We look back to trigonometry in the two-dimensional case to find that relationship. Suppose the vectors a and b make angles α and β with the x-axis (Figure 3.6). The length $\|a\|$ is the hypotenuse in the triangle OaQ. So the sine and cosine of α are

$$\sin\alpha = \frac{a_2}{\|a\|}, \qquad \cos\alpha = \frac{a_1}{\|a\|}.$$

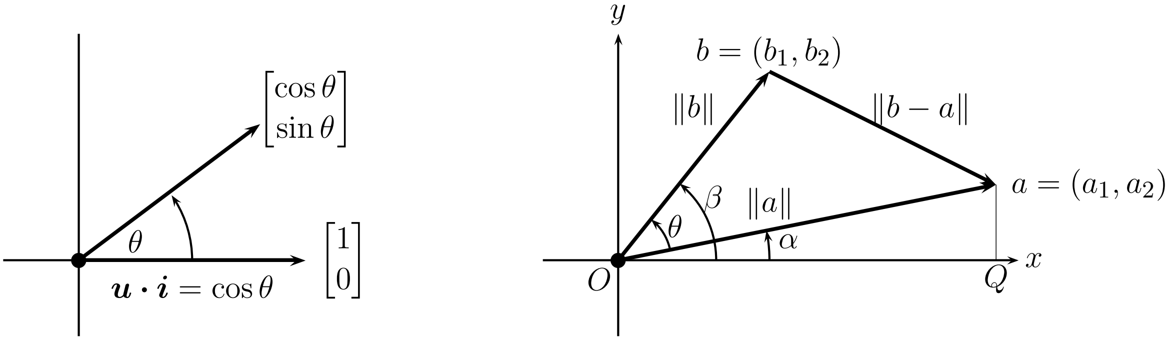


Figure 3.6: The cosine of the angle $\theta=\beta-\alpha$ using inner products.

For the angle β, the sine is $b_2/\|b\|$ and the cosine is $b_1/\|b\|$. The cosine of $\theta=\beta-\alpha$ comes from an identity that no one could forget:

$$\textbf{Cosine formula}\qquad \cos\theta=\cos\beta\cos\alpha+\sin\beta\sin\alpha=\frac{a_1b_1+a_2b_2}{\|a\|\|b\|}. \qquad (1)$$

The numerator in this formula is exactly the inner product of a and b. It gives the relationship between $a^{\mathrm{T}}b$ and $\cos\theta$:

3G The cosine of the angle between any nonzero vectors a and b is

$$\textbf{Cosine of } \theta \qquad \cos\theta=\frac{a^{\mathrm{T}}b}{\|a\|\|b\|}. \qquad (2)$$

This formula is dimensionally correct; if we double the length of b, then both numerator and denominator are doubled, and the cosine is unchanged. Reversing the sign of b, on the other hand, reverses the sign of $\cos\theta$—and changes the angle by 180°.

There is another law of trigonometry that leads directly to the same result. It is not so unforgettable as the formula in equation (1), but it relates the lengths of the sides of any triangle:

$$\textbf{Law of Cosines}\qquad \|b-a\|^2=\|b\|^2+\|a\|^2-2\|b\|\|a\|\cos\theta. \qquad (3)$$

When θ is a right angle, we are back to Pythagoras: $\|b-a\|^2=\|b\|^2+\|a\|^2$. For any angle θ, the expression $\|b-a\|^2$ is $(b-a)^{\mathrm{T}}(b-a)$, and equation (3) becomes

$$b^{\mathrm{T}}b-2a^{\mathrm{T}}b+a^{\mathrm{T}}a=b^{\mathrm{T}}b+a^{\mathrm{T}}a-2\|b\|\|a\|\cos\theta.$$

Canceling $b^{\mathrm{T}}b$ and $a^{\mathrm{T}}a$ on both sides of this equation, you recognize formula (2) for the cosine: $a^{\mathrm{T}}b=\|a\|\|b\|\cos\theta$. In fact, this proves the cosine formula in n dimensions, since we only have to worry about the plane triangle Oab.

Projection onto a Line

Now we want to find the projection point p. This point must be some multiple $p=\widehat{x}a$ of the given vector a—every point on the line is a multiple of a. The problem is to compute

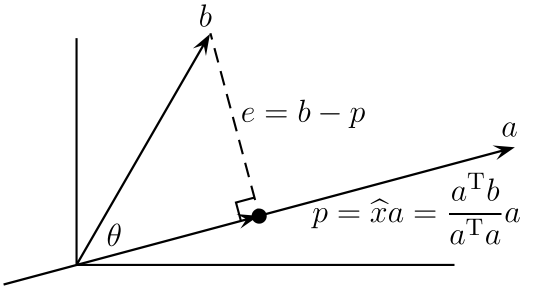


Figure 3.7: The projection p of b onto a, with $\cos\theta = \dfrac{Op}{Ob} = \dfrac{a^{\mathrm{T}}b}{\|a\|\|b\|}$.

the coefficient $\widehat{x}$. All we need is the geometrical fact that ***the line from*** b ***to the closest point*** $p = \widehat{x}a$ ***is perpendicular to the vector*** a:

$$(b - \widehat{a}) \perp a, \quad \text{or} \quad a^{\mathrm{T}}(b - \widehat{a}) = 0, \quad \text{or} \quad \widehat{x} = \frac{a^{\mathrm{T}}b}{a^{\mathrm{T}}a}. \tag{4}$$

That gives the formula for the number $\widehat{x}$ and the projection p:

3H The projection of the vector b onto the line in the direction of a is $p = \widehat{x}a$:

$$\textbf{Projection onto a line} \qquad p = \widehat{x}a = \frac{a^{\mathrm{T}}b}{a^{\mathrm{T}}a}a. \tag{5}$$

This allows us to redraw Figure 3.5 with a correct formula for p (Figure 3.7).

This leads to the Schwarz inequality in equation (6), which is the most important inequality in mathematics. A special case is the fact that arithmetic means $\frac{1}{2}(x+y)$ are larger than geometric means $\sqrt{xy}$. (It is also equivalent—see Problem 1 at the end of this section—to the triangle inequality for vectors.) The Schwarz inequality seems to come almost accidentally from the statement that $\|e\|^2 = \|b-p\|^2$ in Figure 3.7 cannot be negative:

$$\left\| b - \frac{a^{\mathrm{T}}b}{a^{\mathrm{T}}a}a \right\|^2 = b^{\mathrm{T}}b - 2\frac{(a^{\mathrm{T}}b)^2}{a^{\mathrm{T}}a} + \left(\frac{a^{\mathrm{T}}b}{a^{\mathrm{T}}a}\right)^2 a^{\mathrm{T}}a = \frac{(b^{\mathrm{T}}b)(a^{\mathrm{T}}a) - (a^{\mathrm{T}}b)^2}{(a^{\mathrm{T}}a)} \geq 0.$$

This tells us that $(b^{\mathrm{T}}b)(a^{\mathrm{T}}a) \geq (a^{\mathrm{T}}b)^2$—and then we take square roots:

3I All vectors a and b satisfy the ***Schwarz inequality***, which is $|\cos\theta| \leq 1$ in $\mathbf{R}^n$:

$$\textbf{Schwarz inequality} \qquad |a^{\mathrm{T}}b| \leq \|a\|\|b\|. \tag{6}$$

According to formula (2), the ratio between $a^{\mathrm{T}}b$ and $\|a\|\|b\|$ is exactly $|\cos\theta|$. Since all cosines lie in the interval $-1 \leq \cos\theta \leq 1$, this gives another proof of equation (6): *the Schwarz inequality is the same as* $|\cos\theta| \leq 1$. In some ways that is a more easily understood proof, because cosines are so familiar. Either proof is all right in $\mathbf{R}^n$, but notice that ours came directly from the calculation of $\|b-p\|^2$. This stays nonnegative when we introduce new possibilities for the lengths and inner products. The name of Cauchy is also attached to this inequality $|a^{\mathrm{T}}b| \leq \|a\|\|b\|$, and the Russians refer to it as the Cauchy-Schwarz-Buniakowsky inequality! Mathematical historians seem to agree that Buniakowsky's claim is genuine.

One final observation about $|a^{\mathrm{T}}b| \leq \|a\|\|b\|$. *Equality holds if and only if b is a multiple of a.* The angle is $\theta = 0°$ or $\theta = 180°$ and the cosine is 1 or -1. In this case b is identical with its projection p, and the distance between b and the line is zero.

Example 1. Project $b = (1,2,3)$ onto the line through $a = (1,1,1)$ to get $\widehat{x}$ and p:

$$\widehat{x} = \frac{a^{\mathrm{T}}b}{a^{\mathrm{T}}a} = \frac{6}{3} = 2.$$

The projection is $p = \widehat{x}a = (2,2,2)$. The angle between a and b has

$$\cos\theta = \frac{\|p\|}{\|b\|} = \frac{\sqrt{12}}{\sqrt{14}} \qquad \text{and also} \qquad \cos\theta = \frac{a^{\mathrm{T}}b}{\|a\|\|b\|} = \frac{6}{\sqrt{3}\sqrt{14}}.$$

The Schwarz inequality $|a^{\mathrm{T}}b| \leq \|a\|\|b\|$ is $6 \leq \sqrt{3}\sqrt{14}$. If we write 6 as $\sqrt{36}$, that is the same as $\sqrt{36} \leq \sqrt{42}$. The cosine is less than 1, because b is not parallel to a.

Projection Matrix of Rank 1

The projection of b onto the line through a lies at $p = a(a^{\mathrm{T}}b/a^{\mathrm{T}}a)$. That is our formula $p = \widehat{x}a$, but it is written with a slight twist: The vector a is put before the number $\widehat{x} = a^{\mathrm{T}}b/a^{\mathrm{T}}a$. There is a reason behind that apparently trivial change. Projection onto a line is carried out by a ***projection matrix*** P, and written in this new order we can see what it is. *P is the matrix that multiplies b and produces p*:

$$P = a\frac{a^{\mathrm{T}}b}{a^{\mathrm{T}}a} \quad \text{so the projection matrix is} \quad P = \frac{aa^{\mathrm{T}}}{a^{\mathrm{T}}a}. \tag{7}$$

That is a column times a row—a square matrix—divided by the number $a^{\mathrm{T}}a$.

Example 2. The matrix that projects onto the line through $a = (1,1,1)$ is

$$P = \frac{aa^{\mathrm{T}}}{a^{\mathrm{T}}a} = \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$

This matrix has two properties that we will see as typical of projections:

1. P is a symmetric matrix.

2. Its square is itself: $P^2 = P$.

P^2b is the projection of Pb—and Pb is already on the line! So $P^2b = Pb$. This matrix P also gives a great example of the four fundamental subspaces:

The column space consists of the line through $a = (1,1,1)$.

The nullspace consists of the plane perpendicular to a.

The rank is $r = 1$.

Every column is a multiple of a, and so is $Pb = \widehat{x}a$. The vectors that project to $p = 0$ are especially important. They satisfy $a^{\mathrm{T}}b = 0$—they are perpendicular to a and their component along the line is zero. They lie in the nullspace = perpendicular plane.

Actually that example is too perfect. It has the nullspace orthogonal to the column space, which is haywire. The nullspace should be orthogonal to the *row space*. But because P is symmetric, its row and column spaces are the same.

Remark on scaling The projection matrix $aa^{\mathrm{T}}/a^{\mathrm{T}}a$ is the same if a is doubled:

$$a = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \quad \text{gives} \quad P = \frac{1}{12}\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}\begin{bmatrix} 2 & 2 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \quad \text{as before.}$$

The line through a is the same, and that's all the projection matrix cares about. If a has unit length, the denominator is $a^{\mathrm{T}}a = 1$ and the matrix is just $P = aa^{\mathrm{T}}$.

Example 3. Project onto the "θ-direction" in the x-y plane. The line goes through $a = (\cos\theta, \sin\theta)$ and the matrix is symmetric with $P^2 = P$:

$$P = \frac{aa^{\mathrm{T}}}{a^{\mathrm{T}}a} = \frac{\begin{bmatrix} c \\ s \end{bmatrix}\begin{bmatrix} c & s \end{bmatrix}}{\begin{bmatrix} c & s \end{bmatrix}\begin{bmatrix} c \\ s \end{bmatrix}} = \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix}.$$

Here c is $\cos\theta$, s is $\sin\theta$, and $c^2 + s^2 = 1$ in the denominator. This matrix P was discovered in Section 2.6 on linear transformations. Now we know P in any number of dimensions. We emphasize that it produces the projection p:

> ***To project** b **onto** a, **multiply by the projection matrix** P: $p = Pb$.*

Transposes from Inner Products

Finally we connect inner products to A^{T}. Up to now, A^{T} is simply the reflection of A across its main diagonal; the rows of A become the columns of A^{T}, and vice versa. The entry in row i, column j of A^{T} is the (j,i) entry of A:

$$\textbf{Transpose by reflection} \qquad A^{\mathrm{T}}_{ij} = (A)_{ji}.$$

There is a deeper significance to A^{T}, Its close connection to inner products gives a new and much more "abstract" definition of the transpose:

3J The transpose A^{T} can be defined by the following property: The inner product of Ax with y equals the inner product of x with $A^{\mathrm{T}}y$. Formally, this simply means that

$$(Ax)^{\mathrm{T}}y = x^{\mathrm{T}}A^{\mathrm{T}}y = x^{\mathrm{T}}(A^{\mathrm{T}}y). \tag{8}$$

This definition gives us another (better) way to verify the formula $(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}$, Use equation (8) twice:

$$\textbf{Move } A \textbf{ then move } B \qquad (ABx)^{\mathrm{T}}y = (Bx)^{\mathrm{T}}(A^{\mathrm{T}}Y) = x^{\mathrm{T}}(B^{\mathrm{T}}A^{\mathrm{T}}y).$$

The transposes turn up in reverse order on the right side, just as the inverses do in the formula $(AB)^{-1} = B^{-1}A^{-1}$. We mention again that these two formulas meet to give the remarkable combination $(A^{-1})^{\mathrm{T}} = (A^{\mathrm{T}})^{-1}$.

Problem Set 3.2

1. (a) Given any two positive numbers x and y, choose the vector b equal to $(\sqrt{x}, \sqrt{y})$, and choose $a = (\sqrt{y}, \sqrt{x})$. Apply the Schwarz inequality to compare the arithmetic mean $\frac{1}{2}(x+y)$ with the geometric mean $\sqrt{xy}$.

(b) Suppose we start with a vector from the origin to the point x, and then add a vector of length $\|y\|$ connecting x to $x+y$. The third side of the triangle goes from the origin to $x+y$. *The triangle inequality asserts that this distance cannot be greater than the sum of the first two*:

$$\|x+y\| \le \|x\| + \|y\|.$$

After squaring both sides, and expanding $(x+y)^{\mathrm{T}}(x+y)$, reduce this to the Schwarz inequality.

2. Verify that the length of the projection in Figure 3.7 is $\|p\| = \|b\|\cos\theta$, using formula (5).

3. What multiple of $a = (1,1,1)$ is closest to the point $b = (2,4,4)$? Find also the point closest to a on the line through b.

4. Explain why the Schwarz inequality becomes an equality in the case that a and b lie on the same line through the origin, and only in that case. What if they lie on opposite sides of the origin?

5. In n dimensions, what angle does the vector $(1,1,\dots,1)$ make with the coordinate axes? What is the projection matrix P onto that vector?

6. The Schwarz inequality has a one-line proof if a and b are normalized ahead of time to be unit vectors:

$$|a^{\mathrm{T}}b| = \left|\sum a_jb_j\right| \le \sum |a_j||b_j| \le \sum \frac{|a_j|^2+|b_j|^2}{2} = \frac{1}{2}+\frac{1}{2} = \|a\|\|b\|.$$

Which previous problem justifies the middle step?

7. By choosing the correct vector b in the Schwarz inequality, prove that

$$(a_1+\cdots+a_n)^2 \le n(a_1^2+\cdots+a_n^2).$$

When does equality hold?

8. The methane molecule CH_4 is arranged as if the carbon atom were at the center of a regular tetrahedron with four hydrogen atoms at the vertices. If vertices are placed at $(0,0,0)$, $(1,1,0)$, $(1,0,1)$, and $(0,1,1)$—note that all six edges have length $\sqrt{2}$, so the tetrahedron is regular—what is the cosine of the angle between the rays going from the center $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$ to the vertices? (The bond angle itself is about 109.5°, an old friend of chemists.)

9. Square the matrix $P = aa^{\mathrm{T}}/a^{\mathrm{T}}a$, which projects onto a line, and show that $P^2 = P$. (Note the number $a^{\mathrm{T}}a$ in the middle of the matrix $aa^{\mathrm{T}}aa^{\mathrm{T}}$!)

10. Is the projection matrix P invertible? Why or why not?

11. (a) Find the projection matrix P_1 onto the line through $a = \begin{bmatrix}1\\3\end{bmatrix}$ and also the matrix P_2 that projects onto the line perpendicular to a.

(b) Compute $P_1 + P_2$ and P_1P_2 and explain.

12. Find the matrix that projects every point in the plane onto the line $x + 2y = 0$.

13. Prove that the *trace* of $P = aa^{\mathrm{T}}/a^{\mathrm{T}}a$—which is the sum of its diagonal entries—always equals 1.

14. What matrix P projects every point in $\mathbf{R}^3$ onto the line of intersection of the planes $x+y+t=0$ and $x-t=0$?

15. Show that the length of Ax equals the length of $A^{\mathrm{T}}x$ if $AA^{\mathrm{T}} = A^{\mathrm{T}}A$.

16. Suppose P is the projection matrix onto the line through a.

(a) Why is the inner product of x with Py equal to the inner product of Px with y?

(b) Are the two angles the same? Find their cosines if $a = (1,1,-1)$, $x = (2,0,1)$, $y = (2,1,2)$.

(c) Why is the inner product of Px with Py again the same? What is the angle between those two?

Problems 17–26 ask for projections onto lines. Also errors $e = b - p$ and matrices P.

17. Project the vector b onto the line through a. Check that e is perpendicular to a:

$$\text{(a)}\quad b=\begin{bmatrix}1\\2\\2\end{bmatrix}\quad\text{and}\quad a=\begin{bmatrix}1\\1\\1\end{bmatrix}.\qquad\text{(b)}\quad b=\begin{bmatrix}1\\3\\1\end{bmatrix}\quad\text{and}\quad a=\begin{bmatrix}-1\\-3\\-1\end{bmatrix}.$$

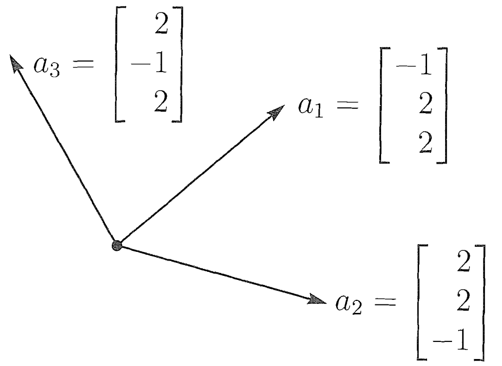
18. *Draw* the projection of b onto a and also compute it from $p=\widehat{x}a$:

$$\text{(a)}\quad b=\begin{bmatrix}\cos\theta\\\sin\theta\end{bmatrix}\quad\text{and}\quad a=\begin{bmatrix}1\\0\end{bmatrix}.\qquad\text{(b)}\quad b=\begin{bmatrix}1\\1\end{bmatrix}\quad\text{and}\quad a=\begin{bmatrix}1\\-1\end{bmatrix}.$$

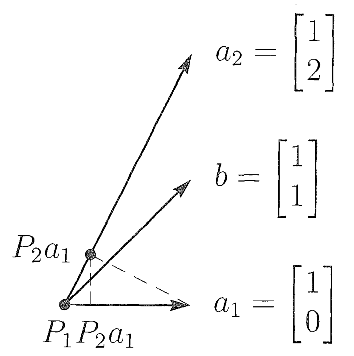
19. In Problem 17, find the projection matrix $P=aa^{\mathrm{T}}/a^{\mathrm{T}}a$ onto the line through each vector a. Verify in both cases that $P^2=P$. Multiply Pb in each case to compute the projection p.

20. Construct the projection matrices P_1 and P_2 onto the lines through the a's in Problem 18. Is it true that $(P_1+P_2)^2=P_1+P_2$? This *would* be true if $P_1P_2=0$.

For Problems 21–26, consult the accompanying figures.



Problems 21–23



Problems 24–26

21. Compute the projection matrices $aa^{\mathrm{T}}/a^{\mathrm{T}}a$ onto the lines through $a_1=(-1,2,2)$ and $a_2=(2,2,-1)$, Multiply those projection matrices and explain why their product P_1P_2 is what it is.

22. Project $b=(1,0,0)$ onto the lines through a_1 and a_2 in Problem 21 and also onto $a_3=(2,-1,2)$. Add the three projections $p_1+p_2+p_3$.

23. Continuing Problems 21–22, find the projection matrix P_3 onto $a_3=(2,-1,2)$. Verify that $P_1+P_2+P_3=I$. The basis a_1, a_2, a_3 is orthogonal!

24. Project the vector $b=(1,1)$ onto the lines through $a_1=(1,0)$ and $a_2=(1,2)$. Draw the projections p_1 and p_2 and add p_1+p_2. The projections do not add to b because the a's are not orthogonal.

25. In Problem 24, the projection of b onto the *plane* of a_1 and a_2 will equal b. Find $P=A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$ for $A=[a_1\ \ a_2]\left[\begin{smallmatrix}1&1\\0&2\end{smallmatrix}\right]$.

26. Project $a_1 = (1,0)$ onto $a_2 = (1,2)$. Then project the result back onto a_1. Draw these projections and multiply the projection matrices P_1P_2: Is this a projection?

3.3 Projections and Least Squares

Up to this point, $Ax = b$ either has a solution or not. If b is not in the column space $\boldsymbol{C}(A)$, the system is inconsistent and Gaussian elimination fails. This failure is almost certain when there are several equations and only one unknown:

$$\begin{array}{c} \textbf{More equations} \\ \textbf{than unknowns—} \\ \textbf{no solution?} \end{array} \qquad \begin{aligned} 2x &= b_1 \\ 3x &= b_2 \\ 4x &= b_3. \end{aligned}$$

This is solvable when b_1, b_2, b_3 are in the ratio 2:3:4. The solution x will exist only if b is on the same line as the column $a = (2,3,4)$.

In spite of their unsolvability, inconsistent equations arise all the time in practice. They have to be solved! One possibility is to determine x from part of the system, and ignore the rest; this is hard to justify if all m equations come from the same source. Rather than expecting no error in some equations and large errors in the others, it is much better *to choose the x that minimizes an average error E in the m equations.*

The most convenient "average" comes from the *sum of squares*:

$$\textbf{Squared error} \qquad E^2 = (2x - b_1)^2 + (3x - b_2)^2 + (4x - b_3)^2.$$

If there is an exact solution, the minimum error is $E = 0$. In the more likely case that b is not proportional to a, the graph of E^2 will be a parabola. The minimum error is at the lowest point, where the derivative is zero:

$$\frac{dE^2}{dx} = 2\big[(2x - b_1)2 + (3x - b_2)3 + (4x - b_3)4\big] = 0.$$

Solving for x, the least-squares solution of this model system $ax = b$ is denoted by $\widehat{x}$:

$$\textbf{Leastsquares solution} \qquad \widetilde{x} = \frac{2b_1 + 3b_2 + 4b_3}{2^2 + 3^2 + 4^2} = \frac{a^{\mathrm{T}}b}{a^{\mathrm{T}}a}.$$

You recognize $a^{\mathrm{T}}b$ in the numerator and $a^{\mathrm{T}}a$ in the denominator.

The general case is the same. We "solve" $ax = b$ by minimizing

$$E^2 = \|ax - b\|^2 = (a_1x - b_1)^2 + \cdots + (a_mx - b_m)^2.$$

The derivative of E^2 is zero at the point $\widehat{x}$, if

$$(a_1\widehat{x} - b_1)a_1 + \cdots + (a_m\widehat{x} - b_m)a_m = 0.$$

We are minimizing the distance from b to the line through a, and calculus gives the same answer, $\widehat{x} = (a_1b_1 + \cdots + a_mb_m)/(a_1^2 + \cdots + a_m^2)$, that geometry did earlier: