Fourth Edition

LINEAR ALGEBRA AND ITS APPLICATIONS

Gilbert Strang

(b) If the third equation has zero as its second coefficient (it contains $0v$) then no multiple of equation 2 will be subtracted from equation 3.

(c) If the third equation contains $0u$ *and* $0v$, then no multiple of equation 1 or equation 2 will be subtracted from equation 3.

29. (Very optional) Normally the multiplication of two complex numbers

$$(a+ib)(c+id) = (ac-bd) + i(bc+ad)$$

involves the four separate multiplications ac, bd, be, ad. Ignoring i, can you compute $ac-bd$ and $bc+ad$ with only three multiplications? (You may do additions, such as forming $a+b$ before multiplying, without any penalty.)

30. Use elimination to solve

$$\begin{array}{rcrcrcr} u & + & v & + & w & = & 6 \\ u & + & 2v & + & 2w & = & 11 \\ 2u & + & 3v & - & 4w & = & 3 \end{array} \qquad \text{and} \qquad \begin{array}{rcrcrcr} u & + & v & + & w & = & 7 \\ u & + & 2v & + & 2w & = & 10 \\ 2u & + & 3v & - & 4w & = & 3. \end{array}$$

31. For which three numbers a will elimination fail to give three pivots?

$$\begin{aligned} ax+2y+3z &= b_1 \\ ax+ay+4z &= b_2 \\ ax+ay+az &= b_3. \end{aligned}$$

32. Find experimentally the average size (absolute value) of the first and second and third pivots for MATLAB's $\mathsf{lu}(\mathsf{rand}(3,3))$. The average of the first pivot from $\mathsf{abs}(\mathsf{A}(1,1))$ should be 0.5.

1.4 Matrix Notation and Matrix Multiplication

With our 3 by 3 example, we are able to write out all the equations in full. We can list the elimination steps, which subtract a multiple of one equation from another and reach a triangular matrix. For a large system, this way of keeping track of elimination would be hopeless; a much more concise record is needed.

We now introduce **matrix notation** to describe the original system, and **matrix multiplication** to describe the operations that make it simpler. Notice that three different types of quantities appear in our example:

$$\begin{array}{lrcrcrcr} \textbf{Nine coefficients} & 2u & + & v & + & w & = & 5 \\ \textbf{Three unknowns} & 4u & - & 6v & & & = & -2 \\ \textbf{Three right-hand sides} & -2u & + & 7v & + & 2w & = & 9 \end{array} \tag{1}$$

On the right-hand side is the column vector b. On the left-hand side are the unknowns u, v, w. Also on the left-hand side are nine coefficients (one of which happens to be zero). It is natural to represent the three unknowns by a vector:

$$\text{The unknown is } x = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \qquad \text{The solution is } x = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

The nine coefficients fall into three rows and three columns, producing a 3 ***by*** 3 ***matrix***:

$$\textbf{Coefficient matrix} \qquad A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}.$$

A is a *square* matrix, because the number of equations equals the number of unknowns. If there are n equations in n unknowns, we have a square n by n matrix. More generally, we might have m equations and n unknowns. Then A is *rectangular*, with m rows and n columns. It will be an "m by n matrix."

Matrices are added to each other, or multiplied by numerical constants, exactly as vectors are—one entry at a time. In fact we may regard vectors as special cases of matrices; *they are matrices with only one column*. As with vectors, two matrices can be added only if they have the same shape:

$$\begin{array}{l} \textbf{Addition } A+B \\ \textbf{Multiplication } 2A \end{array} \qquad \begin{bmatrix} 2 & 1 \\ 3 & 0 \\ 0 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ -3 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 0 & 1 \\ 1 & 6 \end{bmatrix} \qquad 2\begin{bmatrix} 2 & 1 \\ 3 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 6 & 0 \\ 0 & 8 \end{bmatrix}.$$

Multiplication of a Matrix and a Vector

We want to rewrite the three equations with three unknowns u, v, w in the simplified matrix form $Ax = b$. Written out in full, matrix times vector equals vector:

$$\textbf{Matrix form } Ax = b \qquad \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}. \qquad (2)$$

The right-hand side b is the column vector of "inhomogeneous terms." **The left-hand side is A times x.** This multiplication will be defined *exactly so as to reproduce the original system*. The first component of Ax comes from "multiplying" the first row of A into the column vector x:

$$\textbf{Row times column} \qquad \begin{bmatrix} 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 2u + v + w \end{bmatrix} = \begin{bmatrix} 5 \end{bmatrix}. \qquad (3)$$

The second component of the product Ax is $4u - 6v + 0w$, from the second row of A. The matrix equation $Ax = b$ is equivalent to the three simultaneous equations in equation (1).

Row times column is fundamental to all matrix multiplications. From two vectors it produces a single number. This number is called the ***inner product*** of the two vectors. In other words, the product of a 1 by n matrix (a *row vector*) and an n by 1 matrix (a *column vector*) is a 1 by 1 matrix:

$$\textbf{Inner product} \qquad \begin{bmatrix} 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2\cdot 1 + 1\cdot 1 + 1\cdot 2 \end{bmatrix} = \begin{bmatrix} 5 \end{bmatrix}.$$

This confirms that the proposed solution $x = (1,1,2)$ does satisfy the first equation.

There are two ways to multiply a matrix A and a vector x. One way is *a row at a time*, Each row of A combines with x to give a component of Ax. There are three inner products when A has three rows:

$$Ax \textbf{ by rows} \qquad \begin{bmatrix} 1 & 1 & 6 \\ 3 & 0 & 1 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 1\cdot 2 + 1\cdot 5 + 6\cdot 0 \\ 3\cdot 2 + 0\cdot 5 + 3\cdot 0 \\ 1\cdot 2 + 1\cdot 5 + 4\cdot 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \\ 7 \end{bmatrix}. \tag{4}$$

That is how Ax is usually explained, but the second way is equally important. In fact it is more important! It does the multiplication *a column at a time*. The product Ax is found all at once, as ***a combination of the three columns of A***:

$$Ax \textbf{ by columns} \qquad 2 \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + 5 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 6 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \\ 7 \end{bmatrix}. \tag{5}$$

The answer is twice column 1 plus 5 times column 2. It corresponds to the "column picture" of linear equations. If the right-hand side b has components 7, 6, 7, then the solution has components 2, 5, 0. Of course the row picture agrees with that (and we eventually have to do the same multiplications).

The column rule will be used over and over, and we repeat it for emphasis:

> **1A** Every product Ax can be found using whole columns as in equation (5). Therefore Ax is ***a combination of the columns of A***. The coefficients are the components of x.

To multiply A times x in n dimensions, we need a notation for the individual entries in A. *The entry in the ith row and jth column is always denoted by a_{ij}.* The first subscript gives the row number, and the second subscript indicates the column. (In equation (4), a_{21} is 3 and a_{13} is 6.) If A is an m by n matrix, then the index i goes from 1 to m—there are m rows—and the index j goes from 1 to n. Altogether the matrix has mn entries, and a_{mn} is in the lower right corner.

One subscript is enough for a vector. The jth component of x is denoted by x_j. (The multiplication above had $x_1 = 2$, $x_2 = 5$, $x_3 = 0$.) Normally x is written as a column vector—like an n by 1 matrix. But sometimes it is printed on a line, as in $x = (2,5,0)$. The parentheses and commas emphasize that it is not a 1 by 3 matrix. It is a column vector, and it is just temporarily lying down.

To describe the product Ax, we use the "*sigma*" symbol Σ for summation:

$$\textbf{Sigma notation} \qquad \text{The } i\text{th component of } Ax \text{ is } \sum_{j=1}^{n} a_{ij}x_j.$$

This sum takes us along the ith row of A. The column index j takes each value from 1 to n and we add up the results—the sum is $a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n$.

We see again that the length of the rows (the number of columns in A) must match the length of x. **An m by n matrix multiplies an n-dimensional vector** (and produces an m-dimensional vector). Summations are simpler than writing everything out in full, but matrix notation is better. (Einstein used "tensor notation," in which a repeated index automatically means summation. He wrote $a_{ij}x_j$ or even $a_i^j x_j$, without the Σ. Not being Einstein, we keep the Σ.)

The Matrix Form of One Elimination Step

So far we have a convenient shorthand $Ax = b$ for the original system of equations. What about the operations that are carried out during elimination? In our example, the first step subtracted 2 times the first equation from the second. On the right-hand side, 2 times the first component of b was subtracted from the second component. *The same result is achieved if we multiply b by this elementary matrix* (or *elimination matrix*):

$$\textbf{Elementary matrix} \qquad E = \begin{bmatrix} 1 & 0 & 0 \\ \mathbf{-2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This is verified just by obeying the rule for multiplying a matrix and a vector:

$$Eb = \begin{bmatrix} 1 & 0 & 0 \\ \mathbf{-2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix} = \begin{bmatrix} 5 \\ \mathbf{-12} \\ 9 \end{bmatrix}.$$

The components 5 and 9 stay the same (because of the 1, 0, 0 and 0, 0, 1 in the rows of E). The new second component -12 appeared after the first elimination step.

It is easy to describe the matrices like E, which carry out the separate elimination steps. We also notice the "identity matrix," which does nothing at all.

1B The ***identity matrix*** I, with 1s on the diagonal and 0s everywhere else, leaves every vector unchanged. The ***elementary matrix*** E_{ij} subtracts ℓ times row j from row i. This E_{ij} includes $-\ell$ in row i, column j.

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ has } Ib = b \qquad E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\ell & 0 & 1 \end{bmatrix} \text{ has } E_{31}b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 - \ell b_1 \end{bmatrix}.$$

$Ib = b$ is the matrix analogue of multiplying by 1. A typical elimination step multiplies by E_{31}. The important question is: What happens to A on the left-hand side?

To maintain equality, we must apply the same operation to both sides of $Ax = b$. In other words, we must also multiply the vector Ax by the matrix E. Our original matrix E subtracts 2 times the first component from the second, After this step the new and simpler system (equivalent to the old) is just $E(Ax) = Eb$. It is simpler because of the zero that was created below the first pivot. It is equivalent because we can recover the original system (by adding 2 times the first equation back to the second). So the two systems have exactly the same solution x.

Matrix Multiplication

Now we come to the most important question: *How do we multiply two matrices*? There is a partial clue from Gaussian elimination: We know the original coefficient matrix A, we know the elimination matrix E, and we know the result EA after the elimination step. We hope and expect that

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ times } A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \text{ gives } \quad EA = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ -2 & 7 & 2 \end{bmatrix}.$$

Twice the first row of A has been subtracted from the second row. Matrix multiplication is consistent with the row operations of elimination. We can write the result either as $E(Ax) = Eb$, applying E to both sides of our equation, or as $(EA)x = Eb$. The matrix EA is constructed exactly so that these equations agree, and we don't need parentheses:

Matrix multiplication (EA times x) equals (E times Ax). We just write EAx.

This is the whole point of an "associative law" like $2 \times (3 \times 4) = (2 \times 3) \times 4$. The law seems so obvious that it is hard to imagine it could be false. But the same could be said of the "commutative law" $2 \times 3 = 3 \times 2$—and for matrices EA is not AE.

There is another requirement on matrix multiplication. We know how to multiply Ax, a matrix and a vector. The new definition should be consistent with that one. When a matrix B contains only a single column x, the matrix-matrix product AB should be identical with the matrix-vector product Ax. *More than that*: When B contains several columns b_1, b_2, b_3, the columns of AB should be Ab_1, Ab_2, Ab_3!

$$\textbf{Multiplication by columns} \qquad AB = A\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} Ab_1 \\ Ab_2 \\ Ab_3 \end{bmatrix}.$$

Our first requirement had to do with rows, and this one is concerned with columns. A third approach is to describe each individual entry in AB and hope for the best. In fact, there is only one possible rule, and I am not sure who discovered it. It makes everything work. It does not allow us to multiply every pair of matrices. If they are square, they must have the same size. If they are rectangular, they must *not* have the same shape; ***the number of columns in*** A ***has to equal the number of rows in*** B. Then A can be multiplied into each column of B.

If A is m by n, and B is n by p, then multiplication is possible. *The product* AB *will be* m *by* p. We now find the entry in row i and column j of AB.

1C The i, j entry of AB is the inner product of the ith row of A and the jth column of B. In Figure 1.7, the 3, 2 entry of AB comes from row 3 and column 2:

$$(AB)_{32} = a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} + a_{34}b_{42}. \tag{6}$$

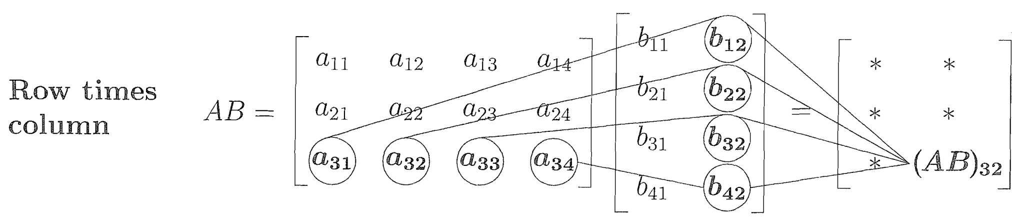

Figure 1.7: A 3 by 4 matrix A times a 4 by 2 matrix B is a 3 by 2 matrix AB.

Note. We write AB when the matrices have nothing special to do with elimination. Our earlier example was EA, because of the elementary matrix E. Later we have PA, or LU, or even LDU. The rule for matrix multiplication stays the same.

Example 1.

$$AB = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 5 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 17 & 1 & 0 \\ 4 & 8 & 0 \end{bmatrix}.$$

The entry 17 is $(2)(1)+(3)(5)$, the inner product of the first row of A and first column of B. The entry 8 is $(4)(2)+(0)(-1)$, from the second row and second column.

The third column is zero in B, so it is zero in AB. B consists of three columns side by side, and A multiplies each column separately. ***Every column of*** AB ***is a combination of the columns of*** A. Just as in a matrix-vector multiplication, the columns of A are multiplied by the entries in B.

Example 2.

$$\textbf{Row exchange matrix} \qquad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ 2 & 3 \end{bmatrix}.$$

Example 3. The 1s in the identity matrix I leave every matrix unchanged:

$$\textbf{Identity matrix} \qquad IA = A \quad \text{and} \quad BI = B.$$

Important: The multiplication AB can also be done *a row at a time*. In Example 1, the first row of AB uses the numbers 2 and 3 from the first row of A. Those numbers give $2[\text{row } 1] + 3[\text{row } 2] = [17 \;\; 1 \;\; 0]$. Exactly as in elimination, where all this started, each row of AB is a ***combination of the rows of*** B.

We summarize these three different ways to look at matrix multiplication.

1D

(i) Each entry of AB is the product of a ***row*** and a ***column***:

$$(AB)_{ij} = (\text{row } i \text{ of } A) \text{ times } (\text{column } j \text{ of } B)$$

(ii) Each column of AB is the product of a matrix and a column:

$$\text{column } j \text{ of } AB = A \text{ times } (\text{column } j \text{ of } B)$$

(iii) Each row of AB is the product of a row and a matrix:

$$\text{row } i \text{ of } AB = (\text{row } i \text{ of } A) \text{ times } B.$$

This leads hack to a key property of matrix multiplication. Suppose the shapes of three matrices A, B, C (possibly rectangular) permit them to be multiplied. The rows in A and B multiply the columns in B and C. Then the key property is this:

1E Matrix multiplication is associative: $(AB)C = A(BC)$. Just write ABC.

AB times C equals A times BC. If C happens to be just a vector (a matrix with only one column) this is the requirement $(EA)x = E(Ax)$ mentioned earlier. It is the whole basis for the laws of matrix multiplication. And if C has several columns, we have only to think of them placed side by side, and apply the same rule several times. Parentheses are not needed when we multiply several matrices.

There are two more properties to mention—one property that matrix multiplication has, and another which it *does not have*. The property that it does possess is:

1F Matrix operations are distributive:

$$A(B+C) = AB + AC \quad \text{and} \quad (B+C)D = BD + CD.$$

Of course the shapes of these matrices must match properly—B and C have the same shape, so they can be added, and A and D are the right size for premultiplication and postmultiplication. The proof of this law is too boring for words.

The property that fails to hold is a little more interesting:

1G Matrix multiplication is not commutative: Usually $FE \neq EF$.

Example 4. Suppose E subtracts twice the first equation from the second. Suppose F is the matrix for the next step, *to add row* 1 *to row* 3:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ \mathbf{-2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \mathbf{1} & 0 & 1 \end{bmatrix}.$$

These two matrices do commute and the product does both steps at once:

$$EF = \begin{bmatrix} 1 & 0 & 0 \\ \mathbf{-2} & 1 & 0 \\ \mathbf{1} & 0 & 1 \end{bmatrix} = FE.$$

In either order, EF or FE, this changes rows 2 and 3 using row 1.

Example 5. Suppose E is the same but G *adds row* 2 *to row* 3. Now the order makes a difference. When we apply E and then G, the second row is altered *before* it affects the third. If E comes *after* G, then the third equation feels no effect from the first. You will see a zero in the $(3,1)$ entry of EG, where there is a -2 in GE:

$$GE = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \mathbf{1} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ \mathbf{-2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ \mathbf{-2} & 1 & 1 \end{bmatrix} \quad \text{but} \quad EG = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ \mathbf{0} & 1 & 1 \end{bmatrix}.$$

Thus $EG \neq GE$. A random example would show the same thing—most matrices don't commute. Here the matrices have meaning. There was a reason for $EF = FE$, and a reason for $EG \neq GE$. It is worth taking one more step, to see what happens with *all three elimination matrices at once*:

$$GFE = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ \mathbf{-1} & 1 & 1 \end{bmatrix} \quad \text{and} \quad EFG = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ \mathbf{-1} & 1 & 1 \end{bmatrix}.$$

The product GFE is the true order of elimination. *It is the matrix that takes the original A to the upper triangular U.* We will see it again in the next section.

The other matrix EFG is nicer. In that order, the numbers -2 from E and 1 from F and G were not disturbed. They went straight into the product. It is the wrong order for elimination. But fortunately *it is the right order for reversing the elimination steps*—which also comes in the next section.

Notice that the product of lower triangular matrices is again lower triangular.

Problem Set 1.4

1. Compute the products

$$\begin{bmatrix} 4 & 0 & 1 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -2 \\ 3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

For the third one, draw the column vectors $(2,1)$ and $(0,3)$. Multiplying by $(1,1)$ just adds the vectors (do it graphically).

2. Working a column at a time, compute the products

$$\begin{bmatrix} 4 & 1 \\ 5 & 1 \\ 6 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 4 & 3 \\ 6 & 6 \\ 8 & 9 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \end{bmatrix}.$$

3. Find two inner products and a matrix product:

$$\begin{bmatrix} 1 & -2 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & -2 & 7 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix} \begin{bmatrix} 3 & 5 & 1 \end{bmatrix}.$$

The first gives the length of the vector (squared).

4. If an m by n matrix A multiplies an n-dimensional vector x, how many separate multiplications are involved? What if A multiplies an n by p matrix B?

5. Multiply Ax to find a solution vector x to the system $Ax =$ *zero vector*. Can you find more solutions to $Ax = 0$?

$$Ax = \begin{bmatrix} 3 & -6 & 0 \\ 0 & 2 & -2 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}.$$

6. Write down the 2 by 2 matrices A and B that have entries $a_{ij} = i + j$ and $b_{ij} = (-1)^{i+j}$. Multiply them to find AB and BA.

7. Give 3 by 3 examples (not just the zero matrix) of

(a) a diagonal matrix: $a_{ij} = 0$ if $i \neq j$.

(b) a symmetric matrix: $a_{ij} = a_{ji}$ for all i and j.

(c) an upper triangular matrix: $a_{ij} = 0$ if $i > j$.

(d) a skew-symmetric matrix: $a_{ij} = -a_{ji}$ for all i and j.

8. Do these subroutines multiply Ax *by rows or columns*? Start with $B(I) = 0$:

```
   DO 10 I = 1, N                      DO 10 J = 1, N
   DO 10 J = 1, N                      DO 10 I = 1, N
10 B(I) = B(I) + A(I,J) * X(J)      10 B(I) = B(I) + A(I,J) * X(J)
```

The outputs $Bx = Ax$ are the same. The second code is slightly more efficient in FORTRAN and much more efficient on a vector machine (the first changes single entries $B(I)$, the second can update whole vectors).

9. If the entries of A are a_{ij}, use subscript notation to write

(a) the first pivot.

(b) the multiplier ℓ_{i1} of row 1 to be subtracted from row i.

(c) the new entry that replaces a_{ij} after that subtraction.

(d) the second pivot.

10. True or false? Give a specific counterexample when false.

(a) If columns 1 and 3 of B are the same, so are columns 1 and 3 of AB.

(b) If rows 1 and 3 of B are the same, so are rows 1 and 3 of AB.

(c) If rows 1 and 3 of A are the same, so are rows 1 and 3 of AB.

(d) $(AB)^2 = A^2B^2$.

11. The first row of AB is a linear combination of all the rows of B. What are the coefficients in this combination, and what is the first row of AB, if

$$A = \begin{bmatrix} 2 & 1 & 4 \\ 0 & -1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}?$$

12. The product of two lower triangular matrices is again lower triangular (all its entries above the main diagonal are zero). Confirm this with a 3 by 3 example, and then explain how it follows from the laws of matrix multiplication.

13. By trial and error find examples of 2 by 2 matrices such that

(a) $A^2 = -I$, A having only real entries.

(b) $B^2 = 0$, although $B \neq 0$.

(c) $CD = -DC$, not allowing the case $CD = 0$.

(d) $EF = 0$, although no entries of E or F are zero.

14. Describe the rows of EA and the *columns* of AE if

$$E = \begin{bmatrix} 1 & 7 \\ 0 & 1 \end{bmatrix}.$$

15. Suppose A commutes with every 2 by 2 matrix ($AB = BA$), and in particular

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{commutes with } B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } B_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Show that $a = d$ and $b = c = 0$. If $AB = BA$ for all matrices B, then A is a multiple of the identity.

16. Let x be the column vector $(1, 0, \ldots, 0)$. Show that the rule $(AB)x = A(Bx)$ forces the first column of AB to equal A times the first column of B.

17. Which of the following matrices are guaranteed to equal $(A+B)^2$?

$$A^2 + 2AB + B^2, \quad A(A+B) + B(A+B), \quad (A+B)(B+A), \quad A^2 + AB + BA + B^2.$$

18. If A and B are n by n matrices with all entries equal to 1, find $(AB)_{ij}$. Summation notation turns the product AB, and the law $(AB)C = A(BC)$, into

$$(AB)_{ij} = \sum_k a_{ik} b_{kj} \qquad \sum_j \left(\sum_k a_{ik} b_{kj} \right) c_{jl} = \sum_k a_{ik} \left(\sum_j b_{kj} c_{jl} \right).$$

Compute both sides if C is also n by n, with every $c_{jl} = 2$.

19. *A fourth way to multiply matrices is* ***columns of*** *A* ***times rows of*** *B*:

$$AB = (\text{column } 1)(\text{row } 1) + \cdots + (\text{column } n)(\text{row } n) = \text{sum of simple matrices.}$$

Give a 2 by 2 example of this important rule for matrix multiplication.

20. The matrix that rotates the x-y plane by an angle θ is

$$A(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Verify that $A(\theta_1)A(\theta_2) = A(\theta_1 + \theta_2)$ from the identities for $\cos(\theta_1 + \theta_2)$ and $\sin(\theta_1 + \theta_2)$. What is $A(\theta)$ times $A(-\theta)$?

21. Find the powers A^2, A^3 (A^2 times A), and B^2, B^3, C^2, C^3. What are A^k, B^k, and C^k?

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad C = AB = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

Problems 22–31 are about elimination matrices.

22. Write down the 3 by 3 matrices that produce these elimination steps:

(a) E_{21} subtracts 5 times row 1 from row 2.

(b) E_{32} subtracts -7 times row 2 from row 3.

(c) P exchanges rows 1 and 2, then rows 2 and 3.

23. In Problem 22, applying E_{21} and then E_{32} to the column $b = (1,0,0)$ gives $E_{32}E_{21}b =$ ____. Applying E_{32} before E_{21} gives $E_{21}E_{32}b =$ ____. When E_{32} comes first, row ____ feels no effect from row ____.

24. Which three matrices E_{21}, E_{31}, E_{32} put A into triangular form U?

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{bmatrix} \quad \text{and} \quad E_{32}E_{31}E_{21}A = U.$$

Multiply those E's to get one matrix M that does elimination: $MA = U$.

25. Suppose $a_{33} = 7$ and the third pivot is 5. If you change a_{33} to 11, the third pivot is ____. If you change a_{33} to ____, there is zero in the pivot position.

26. If every column of A is a multiple of $(1,1,1)$, then Ax is always a multiple of $(1,1,1)$. Do a 3 by 3 example. How many pivots are produced by elimination?

27. What matrix E_{31} subtracts 7 times row 1 from row 3? To reverse that step, R_{31} should ____ 7 times row ____ to row ____. Multiply E_{31} by R_{31}.

28. (a) E_{21} subtracts row 1 from row 2 and then P_{23} exchanges rows 2 and 3. What matrix $M = P_{23}E_{21}$ does both steps at once?

(b) P_{23} exchanges rows 2 and 3 and then E_{31} subtracts row I from row 3. What matrix $M = E_{31}P_{23}$ does both steps at once? Explain why the M's are the same but the E's are different.

29. (a) What 3 by 3 matrix E_{13} will add row 3 to row 1?

(b) What matrix adds row 1 to row 3 and *at the same time* adds row 3 to row 1?

(c) What matrix adds row 1 to row 3 and *then* adds row 3 to row 1?

30. Multiply these matrices:

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 1 & 4 & 0 \end{bmatrix}.$$

31. This 4 by 4 matrix needs which elimination matrices E_{21} and E_{32} and E_{43}?

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}.$$

Problems 32–44 are about creating and multiplying matrices

32. Write these ancient problems in a 2 by 2 matrix form $Ax = b$ and solve them:

(a) X is twice as old as Y and their ages add to 39,

(b) $(x, y) = (2, 5)$ and $(3, 7)$ lie on the line $y = mx + c$. Find m and c.

33. The parabola $y = a + bx + cx^2$ goes through the points $(x, y) = (1, 4)$ and $(2, 8)$ and $(3, 14)$. Find and solve a matrix equation for the unknowns (a, b, c).

34. Multiply these matrices in the orders EF and FE and E^2:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{bmatrix} \qquad F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{bmatrix}.$$

35. (a) Suppose all columns of B are the same. Then all columns of EB are the same, because each one is E times ____.

(b) Suppose all rows of B are $[1\ \ 2\ \ 4]$. Show by example that all rows of EB are *not* $[1\ \ 2\ \ 4]$. It is true that those rows are ____.

36. If E adds row 1 to row 2 and F adds row 2 to row 1, does EF equal FE?

37. The first component of Ax is $\sum a_{1j}x_j = a_{11}x_1 + \cdots + a_{1n}x_n$. Write formulas for the third component of Ax and the $(1, 1)$ entry of A^2.

38. If $AB = I$ and $BC = I$, use the associative law to prove $A = C$.

39. A is 3 by 5, B is 5 by 3, C is 5 by 1, and D is 3 by 1. *All entries are* 1. Which of these matrix operations are allowed, and what are the results?

$$BA \qquad AB \qquad ABD \qquad DBA \qquad A(B + C).$$

40. What rows or columns or matrices do you multiply to find

(a) the third column of AB?

(b) the first row of AB?

(c) the entry in row 3, column 4 of AB?

(d) the entry in row 1, column 1 of CDE?

41. (3 by 3 matrices) Choose the only B so that for every matrix A,

(a) $BA = 4A$.

(b) $BA = 4B$.

(c) BA has rows 1 and 3 of A reversed and row 2 unchanged.

(d) All rows of BA are the same as row 1 of A.

42. True or false?

(a) If A^2 is defined then A is necessarily square.

(b) If AB and BA are defined then A and B are square.

(c) If AB and BA are defined then AB and BA are square.

(d) If $AB = B$ then $A = I$.

43. If A is m by n, how many separate multiplications are involved when

(a) A multiplies a vector x with n components?

(b) A multiplies an n by p matrix B? Then AB is m by p.

(c) A multiplies itself to produce A^2? Here $m = n$.

44. To prove that $(AB)C = A(BC)$, use the column vectors $b_1, \dots, b_n$ of B. First suppose that C has only one column c with entries $c_1, \dots, c_n$:

AB has columns $Ab_1, \dots, Ab_n$, and Bc has one column $c_1b_1 + \cdots + c_nb_n$.

Then $(AB)c = c_1Ab_1 + \cdots + c_nAb_n$, equals $A(c_1b_1 + \cdots + c_nb_n) = A(Bc)$.

Linearity gives equality of those two sums, and $(AB)c = A(Bc)$. The same is true for all other ____ of C. Therefore $(AB)C = A(BC)$.

Problems 45–49 use column-row multiplication and block multiplication.

45. Multiply AB using columns times rows:

$$AB = \begin{bmatrix} 1 & 0 \\ 2 & 4 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 3 & 0 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \begin{bmatrix} 3 & 3 & 0 \end{bmatrix} + \underline{\quad\quad} = \underline{\quad\quad}.$$

46. ***Block multiplication*** separates matrices into blocks (submatrices). If their shapes make block multiplication possible, then it is allowed. Replace these x's by numbers and confirm that block multiplication succeeds.

$$\begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} AC + BD \end{bmatrix} \quad \text{and} \quad \left[\begin{array}{cc|c} x & x & x \\ x & x & x \\ \hline x & x & x \end{array}\right] \left[\begin{array}{cc|c} x & x & x \\ x & x & x \\ \hline x & x & x \end{array}\right].$$

47. Draw the cuts in A and B and AB to show how each of the four multiplication rules is really a block multiplication to find AB:

(a) Matrix A times columns of B.

(b) Rows of A times matrix B.

(c) Rows of A times columns of B.

(d) Columns of A times rows of B.

48. Block multiplication says that elimination on column 1 produces

$$EA = \begin{bmatrix} 1 & \mathbf{0} \\ -\boldsymbol{c}/a & I \end{bmatrix} \begin{bmatrix} a & \boldsymbol{b} \\ \boldsymbol{c} & D \end{bmatrix} = \begin{bmatrix} a & \boldsymbol{b} \\ \mathbf{0} & \_\_ \end{bmatrix}.$$

49. *Elimination for a* 2 *by* 2 *block matrix*: When $A^{-1}A = I$, multiply the first block row by CA^{-1} and subtract from the second row, to find the "*Schur complement*" S:

$$\begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & S \end{bmatrix}.$$

50. With $i^2 = -1$, the product $(A + iB)(x + iy)$ is $Ax + iBx + iAy - By$. Use blocks to separate the real part from the imaginary part that multiplies i:

$$\begin{bmatrix} A & -B \\ ? & ? \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} Ax - By \\ ? \end{bmatrix} \begin{matrix} \text{real part} \\ \text{imaginary part} \end{matrix}$$

51. Suppose you solve $Ax = b$ for three special right-hand sides b:

$$Ax_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad Ax_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad Ax_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

If the solutions x_1, x_2, x_3 are the columns of a matrix X, what is AX?

52. If the three solutions in Question 51 are $x_1 = (1,1,1)$ and $x_2 = (0,1,1)$ and $x_3 = (0,0,1)$, solve $Ax = b$ when $b = (3,5,8)$. Challenge problem: What is A?

53. Find all matrices

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{that satisfy} \quad A \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} A.$$

54. If you multiply a *northwest matrix* A and a *southeast matrix* B, what type of matrices are AB and BA? "Northwest" and "southeast" mean zeros below and above the antidiagonal going from $(1,n)$ to $(n,1)$.

55. Write $2x + 3y + z + 5t = 8$ as a matrix A (how many rows?) multiplying the column vector (x,y,z,t) to produce b. The solutions fill a plane in four-dimensional space. *The plane is three-dimensional with no* 4D *volume*.

56. What 2 by 2 matrix P_1 projects the vector (x,y) onto the x axis to produce $(x,0)$? What matrix P_2 projects onto the y axis to produce $(0,y)$? If you multiply $(5,7)$ by P_1 and then multiply by P_2, you get (____) and (____).

57. Write the inner product of $(1,4,5)$ and (x,y,z) as a matrix multiplication Ax. A has one row. The solutions to $Ax = 0$ lie on a ____ perpendicular to the vector ____. The columns of A are only in ____-dimensional space.

58. In MATLAB notation, write the commands that define the matrix A and the column vectors x and b. What command would test whether or not $Ax = b$?

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad x = \begin{bmatrix} 5 \\ -2 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

59. The MATLAB commands A = eye(3) and v = [3:5]' produce the 3 by 3 identity matrix and the column vector $(3,4,5)$. What are the outputs from A $*$ v and v' $*$ v? (Computer not needed!) If you ask for v $*$ A, what happens?

60. If you multiply the 4 by 4 all-ones matrix A = ones(4,4) and the column v = ones(4,1), what is A $*$ v? (Computer not needed.) If you multiply B = eye(4) + ones(4,4) times w = zeros(4,1) + 2 $*$ ones(4,1), what is B $*$ w?

61. Invent a 3 by 3 **magic matrix** M with entries $1, 2, \ldots, 9$. All rows and columns and diagonals add to 15. The first row could be 8, 3, 4. What is M times $(1,1,1)$? What is the row vector $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ times M?

1.5 Triangular Factors and Row Exchanges

We want to look again at elimination, to see what it means in terms of matrices. The starting point was the model system $Ax = b$:

$$Ax = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix} = b. \tag{1}$$

Then there were three elimination steps, with multipliers $2, -1, -1$:

Step 1. Subtract 2 times the first equation from the second;

Step 2. Subtract -1 times the first equation from the third;

Step 3. Subtract -1 times the second equation from the third.

The result was an equivalent system $Ux = c$, with a new coefficient matrix U:

$$\textbf{Upper triangular} \qquad Ux = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ -12 \\ 2 \end{bmatrix} = c. \tag{2}$$

This matrix U is ***upper triangular***—all entries below the diagonal are zero.

The new right side c was derived from the original vector b by the same steps that took A into U. *Forward elimination* amounted to three row operations:

Start with A and b;

Apply steps 1, 2, 3 in that order;

End with U and c.

$Ux = c$ is solved by back-substitution. Here we concentrate on connecting A to U.

The matrices E for step 1, F for step 2, and G for step 3 were introduced in the previous section. They are called ***elementary matrices***, and it is easy to see how they work. To subtract a multiple ℓ of equation j from equation i, *put the number* $-\ell$ *into the* (i, j) *position*. Otherwise keep the identity matrix, with 1s on the diagonal and 0s elsewhere. Then matrix multiplication executes the row operation.

The result of all three steps is $GFEA = U$. Note that E is the first to multiply A, then F, then G. We could multiply GFE together to find the single matrix that takes A to U (and also takes b to c). It is lower triangular (zeros are omitted):

$$\textbf{From } A \textbf{ to } U \qquad GFE = \begin{bmatrix} 1 & & \\ & 1 & \\ & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ 1 & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ -2 & 1 & \\ & & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ -2 & 1 & \\ -1 & 1 & 1 \end{bmatrix}. \tag{3}$$

This is good, but the most important question is exactly the opposite: How would we get from U back to A? ***How can we undo the steps of Gaussian elimination?***

To undo step 1 is not hard. Instead of subtracting, we *add* twice the first row to the second. (Not twice the second row to the first!) The result of doing both the subtraction and the addition is to bring back the identity matrix:

$$\begin{array}{l}\textbf{Inverse of}\\ \textbf{subtraction}\\ \textbf{is addition}\end{array} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{4}$$

One operation cancels the other. In matrix terms, one matrix is the ***inverse*** of the other. If the elementary matrix E has the number $-\ell$ in the (i, j) position, then its inverse E^{-1} has $+\ell$ in that position. Thus $E^{-1}E = I$, which is equation (4).

We can invert each step of elimination, by using E^{-1} and F^{-1} and G^{-1}. I think it's not bad to see these inverses now, before the next section. The final problem is to undo the whole process at once, and see what matrix takes U back to A.

Since step 3 was last in going from A ***to*** U***, its matrix*** G ***must be the first to be inverted in the reverse direction***. Inverses come in the opposite order! The second reverse step is F^{-1} and the last is E^{-1}:

$$\textbf{From } U \textbf{ back to } A \qquad E^{-1}F^{-1}G^{-1}U = A \text{ is } LU = A. \tag{5}$$

You can substitute $GFEA$ for U, to see how the inverses knock out the original steps.

Now we recognize the matrix L that takes U back to A. It is called L, because it is *lower triangular*. And it has a special property that can be seen only by multiplying the three inverse matrices in the right order:

$$E^{-1}F^{-1}G^{-1} = \begin{bmatrix} 1 & & \\ \mathbf{2} & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ \mathbf{-1} & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & \mathbf{-1} & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ \mathbf{2} & 1 & \\ \mathbf{-1} & \mathbf{-1} & 1 \end{bmatrix} = L. \tag{6}$$

The special thing is that ***the entries below the diagonal are the multipliers*** $\ell = 2, -1$, and -1. When matrices are multiplied, there is usually no direct way to read off the answer. Here the matrices come in just the right order so that their product can be written down immediately. If the computer stores each multiplier ℓ_{ij}—the number that multiplies the pivot row j when it is subtracted from row i, and produces a zero in the i, j position—then these multipliers give a complete record of elimination.

The numbers ℓ_{ij} fit right into the matrix L that takes U back to A.

> **1H** ***Triangular factorization*** $A = LU$ ***with no exchanges of rows***. L is lower triangular, with 1s on the diagonal. The multipliers ℓ_{ij} (taken from elimination) are below the diagonal. U is the upper triangular matrix which appears after forward elimination, The diagonal entries of U are the pivots.

Example 1.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix} \text{ goes to } U = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \text{ with } L = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}. \quad \text{Then } LU = A.$$

Example 2. (which needs a row exchange)

$$A = \begin{bmatrix} 0 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{cannot be factored into } A = LU.$$

Example 3. (with all pivots and multipliers equal to 1)

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = LU.$$

From A to U there are subtractions of rows. From U to A there are additions of rows.

Example 4. (when U is the identity and L is the same as A)

$$\textbf{Lower triangular case} \qquad A = \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix}.$$

The elimination steps on this A are easy: (i) E subtracts ℓ_{21} times row 1 from row 2, (ii) F subtracts ℓ_{31} times row 1 from row 3, and (iii) G subtracts ℓ_{32} times row 2 from row 3. The result is the identity matrix $U = I$. The inverses of E, F, and G will bring back A:

E^{-1} applied to F^{-1} applied to G^{-1} applied to I produces A.

$$\begin{bmatrix} 1 & & \\ \ell_{21} & 1 & \\ & & 1 \end{bmatrix} \text{ times } \begin{bmatrix} 1 & & \\ & 1 & \\ \ell_{31} & & 1 \end{bmatrix} \text{ times } \begin{bmatrix} 1 & & \\ & 1 & \\ & \ell_{32} & 1 \end{bmatrix} \text{ equals } \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix}.$$

The order is right for the ℓ's to fall into position. This always happens! Note that parentheses in $E^{-1}F^{-1}G^{-1}$ were not necessary because of the associative law.

$A = LU$: The n by n case

The factorization $A = LU$ is so important that we must say more. It used to be missing in linear algebra courses when they concentrated on the abstract side. Or maybe it was thought to be too hard—but you have got it. If the last Example 4 allows any U instead of the particular $U = I$, we can see how the rule works in general. ***The matrix L, applied to U, brings back*** A:

$$A = LU \qquad \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix} \begin{bmatrix} \text{row 1 of } U \\ \text{row 2 of } U \\ \text{row 3 of } U \end{bmatrix} = \text{original } A. \tag{7}$$

The proof is to *apply the steps of elimination*. On the right-hand side they take A to U. On the left-hand side they reduce L to I, as in Example 4. (The first step subtracts ℓ_{21} times $(1,0,0)$ from the second row, which removes ℓ_{21}.) Both sides of (7) end up equal to the same matrix U, and the steps to get there are all reversible. Therefore (7) is correct and $A = LU$.

$A = LU$ is so crucial, and so beautiful, that Problem 8 at the end of this section suggests a second approach. We are writing down 3 by 3 matrices, but you can see how the arguments apply to larger matrices. Here we give one more example, and then put $A = LU$ to use.

Example 5. ($A = LU$, with zeros in the empty spaces)

$$A = \begin{bmatrix} 1 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & -1 & 1 & \\ & & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & & \\ & 1 & -1 & \\ & & 1 & -1 \\ & & & 1 \end{bmatrix}.$$

That shows how a matrix A with three diagonals has factors L and U with two diagonals. This example comes from an important problem in differential equations (Section 1.7). The second difference in A is a backward difference L times a forward difference U.

One Linear System = Two Triangular Systems

There is a serious practical point about $A = LU$. It is more than just a record of elimination steps; L and U are the right matrices to solve $Ax = b$. In fact A could be thrown away! We go from b to c by forward elimination (this uses L) and we go from c to x by back-substitution (that uses U). We can and should do it without A:

$$\textbf{Splitting of } Ax = b \qquad \textit{First} \quad Lc = b \quad \textit{and then} \quad Ux = c. \tag{8}$$

Multiply the second equation by L to give $LUx = Lc$, which is $Ax = b$. Each triangular system is quickly solved. That is exactly what a good elimination code will do:

1. ***Factor*** (from A find its factors L and U).
2. ***Solve*** (from L and U and b find the solution x).

The separation into ***Factor*** and ***Solve*** means that a series of b's can be processed. The ***Solve*** subroutine obeys equation (8): two triangular systems in $n^2/2$ steps each. **The solution for any new right-hand side b can be found in only n^2 operations**. That is far below the $n^3/3$ steps needed to factor A on the left-hand side.

Example 6. This is the previous matrix A with a right-hand side $b = (1,1,1,1)$.

$$Ax = b \qquad \begin{array}{rcrcrcrcl} x_1 & - & x_2 & & & & & = & 1 \\ -x_1 & + & 2x_2 & - & x_3 & & & = & 1 \\ & - & x_2 & + & 2x_3 & - & x_4 & = & 1 \\ & & & - & x_3 & + & 2x_4 & = & 1 \end{array} \qquad \text{splits into } Lc = b \text{ and } Ux = c.$$

$$Lc = b \qquad \begin{array}{rcrcrcrcl} c_1 & & & & & & & = & 1 \\ -c_1 & + & c_2 & & & & & = & 1 \\ & - & c_2 & + & c_3 & & & = & 1 \\ & & & - & c_3 & + & c_4 & = & 1 \end{array} \qquad \text{gives} \quad \boldsymbol{c} = \begin{bmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} \end{bmatrix}.$$

$$Ux = c \qquad \begin{array}{rcrcrcrcl} x_1 & - & x_2 & & & & & = & 1 \\ & & x_2 & - & x_3 & & & = & 2 \\ & & & & x_3 & - & x_4 & = & 3 \\ & & & & & & x_4 & = & 4 \end{array} \qquad \text{gives} \quad \boldsymbol{x} = \begin{bmatrix} \mathbf{10} \\ \mathbf{9} \\ \mathbf{7} \\ \mathbf{4} \end{bmatrix}.$$

For these special "tridiagonal matrices," the operation count drops from n^2 to $2n$. You see how $Lc = b$ is solved *forward* (c_1 comes before c_2). This is precisely what happens during forward elimination. Then $Ux = c$ is solved backward (x_4 before x_3).

***Remark* 1.** The LU form is "unsymmetric" on the diagonal: L has 1s where U has the pivots. This is easy to correct. **Divide out of U a diagonal pivot matrix D**:

$$\textbf{Factor out } D \qquad U = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix} \begin{bmatrix} 1 & u_{12}/d_1 & u_{13}/d_1 & \vdots \\ & 1 & u_{23}/d_2 & \vdots \\ & & \ddots & \vdots \\ & & & 1 \end{bmatrix}. \tag{9}$$

In the last example all pivots were $d_i = 1$. In that case $D = I$. But that was very exceptional, and normally LU is different from LDU (also written LDV).

> ***The triangular factorization can be written $A = LDU$, where L and U have 1s on the diagonal and D is the diagonal matrix of pivots.***

Whenever you see LDU or LDV, it is understood that U or V has is on the diagonal—each row was divided by the pivot in D. Then L and U are treated evenly. An example of LU splitting into LDU is

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ & -2 \end{bmatrix} = \begin{bmatrix} 1 & \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ & 1 \end{bmatrix} = LDU.$$

That has the 1s on the diagonals of L and U, and the pivots 1 and -2 in D.

***Remark* 2.** We may have given the impression in describing each elimination step, that the calculations must be done in that order. This is wrong. There is *some* freedom, and there is a "Crout algorithm" that arranges the calculations in a slightly different way. *There is no freedom in the final L, D, and U*. That is our main point:

> **1I** If $A = L_1D_1U_1$ and also $A = L_2D_2U_2$, where the L's are lower triangular with unit diagonal, the U's are upper triangular with unit diagonal, and the D's are diagonal matrices with no zeros on the diagonal, then $L_1 = L_2$, $D_1 = D_2$, $U_1 = U_2$. The LDU factorization and the LU factorization are uniquely determined by A.

The proof is a good exercise with inverse matrices in the next section.

Row Exchanges and Permutation Matrices

We now have to face a problem that has so far been avoided: The number we expect to use as a pivot might be zero. This could occur in the middle of a calculation. It will happen at the very beginning if $a_{11} = 0$. A simple example is

$$\textbf{Zero in the pivot position} \qquad \begin{bmatrix} \mathbf{0} & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

The difficulty is clear; no multiple of the first equation will remove the coefficient 3.

The remedy is equally clear. ***Exchange the two equations***, moving the entry 3 up into the pivot. In this example the matrix would become upper triangular:

$$\textbf{Exchange rows} \qquad \begin{aligned} 3u + 4v &= b_2 \\ 2v &= b_1 \end{aligned}$$

To express this in matrix terms, we need the ***permutation matrix*** P that produces the row exchange. It comes from exchanging the rows of I:

$$\textbf{Permutation} \qquad P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } PA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix}.$$

P has the same effect on b, exchanging b_1 and b_2. *The new system is* $PAx = Pb$. The unknowns u and v are *not* reversed in a row exchange.

A permutation matrix P ***has the same rows as the identity*** (in some order). There is a single "1" in every row and column. The most common permutation matrix is $P = I$ (it exchanges nothing). The product of two permutation matrices is another permutation—the rows of I get reordered twice.

After $P = I$, the simplest permutations exchange two rows. Other permutations exchange more rows. **There are** $n! = (n)(n-1)\cdots(1)$ **permutations of size** n. Row 1 has n choices, then row 2 has $n-1$ choices, and finally the last row has only one choice. We can display all 3 by 3 permutations (there are $3! = (3)(2)(1) = 6$ matrices):

$$I = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \qquad P_{21} = \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix} \qquad P_{32}P_{21} = \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix}$$

$$P_{31} = \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} \qquad P_{32} = \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix} \qquad P_{21}P_{32} = \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}.$$

There will be 24 permutation matrices of order $n = 4$. There are only two permutation matrices of order 2, namely

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

When we know about inverses and transposes (the next section defines A^{-1} and A^{T}), we discover an important fact: P^{-1} ***is always the same as*** P^{T}.

A zero in the pivot location raises two possibilities: ***The trouble may be easy to fix, or it may be serious***. This is decided by looking *below the zero*. If there is a nonzero entry lower down in the same column, then a row exchange is carried out. The nonzero entry becomes the needed pivot, and elimination can get going again:

$$A = \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ d & e & f \end{bmatrix} \qquad \begin{aligned} d = 0 &\implies \text{no first pivot} \\ a = 0 &\implies \text{no second pivot} \\ c = 0 &\implies \text{no third pivot.} \end{aligned}$$

If $d = 0$, the problem is incurable and this matrix is ***singular***. There is no hope for a unique solution to $Ax = b$. If d is *not* zero, an exchange P_{13} of rows 1 and 3 will move d into the pivot. However the next pivot position also contains a zero. The number a is now below it (the e above it is useless). If a is not zero then another row exchange P_{23} is called for:

$$P_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad P_{23}P_{13}A = \begin{bmatrix} \boldsymbol{d} & e & f \\ 0 & \boldsymbol{a} & b \\ 0 & 0 & \boldsymbol{c} \end{bmatrix}$$

One more point: The permutation $P_{23}P_{13}$ will do both row exchanges at once:

$$P_{13}\ \textbf{acts first} \qquad P_{23}P_{13} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = P.$$

If we had known, we could have multiplied A by P in the first place. With the rows in the right order PA, any nonsingular matrix is ready for elimination.

Elimination in a Nutshell: $PA = LU$

The main point is this: If elimination can be completed with the help of row exchanges, then we can imagine that those exchanges are done first (by P). *The matrix PA will not need row exchanges*. In other words, PA allows the standard factorization into L times U. The theory of Gaussian elimination can be summarized in a few lines:

1J In the ***nonsingular*** case, there is a permutation matrix P that reorders the rows of A to avoid zeros in the pivot positions. Then $Ax = b$ has a *unique solution*:

With the rows reordered in advance, PA can be factored into LU.

In the ***singular*** case, no P can produce a full set of pivots: elimination fails.

In practice, we also consider a row exchange when the original pivot is *near* zero—even if it is not exactly zero. Choosing a larger pivot reduces the roundoff error.

You have to be careful with L. Suppose elimination subtracts row 1 from row 2, creating $\ell_{21} = 1$. Then suppose it exchanges rows 2 and 3. If that exchange is done in advance, the multiplier will change to $\ell_{31} = 1$ in $PA = LU$.

Example 7.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 2 & 5 & 8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & \mathbf{0} & 2 \\ 0 & 3 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{bmatrix} = U. \tag{10}$$

That row exchange recovers LU—but now $\ell_{31} = 1$ and $\ell_{21} = 2$:

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad PA = LU. \tag{11}$$

In MATLAB, A([r k] :) exchanges row k with row r below it (where the kth pivot has been found). We update the matrices L and P the same way. At the start, $P = I$ and sign $= +1$:

```
A([r k], :) = A([k r], :);
L([r k], 1:k-1) = L([k r], 1:k-1);
P([r k], :) = P([k r], :);
sign = -sign
```

The "**sign**" of P tells whether the number of row exchanges is even (sign $= +1$) or odd (sign $= -1$). A row exchange reverses sign. The final value of sign is the **determinant of** P and it does not depend on the order of the row exchanges.

To summarize: A good elimination code saves L and U and P. Those matrices carry the information that originally came in A—and they carry it in a more usable form. $Ax = b$ reduces to two triangular systems. This is the practical equivalent of the calculation we do next—*to find the inverse matrix A^{-1} and the solution $x = A^{-1}b$.*

Problem Set 1.5

1. When is an upper triangular matrix nonsingular (a full set of pivots)?

2. What multiple ℓ_{32} of row 2 of A will elimination subtract from row 3 of A? Use the factored form

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 4 & 1 \end{bmatrix} \begin{bmatrix} 5 & 7 & 8 \\ 0 & 2 & 3 \\ 0 & 0 & 6 \end{bmatrix}.$$

What will be the pivots? Will a row exchange be required?

3. Multiply the matrix $L = E^{-1}F^{-1}G^{-1}$ in equation (6) by GFE in equation (3):

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \quad \text{times} \quad \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}.$$

Multiply also in the opposite order. *Why are the answers what they are?*

4. Apply elimination to produce the factors L and U for

$$A = \begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 4 \\ 1 & 4 & 8 \end{bmatrix}.$$

5. Factor A into LU, and write down the upper triangular system $Ux = c$ which appears after elimination, for

$$Ax = \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}.$$

6. Find E^2 and E^8 and E^{-1} if

$$E = \begin{bmatrix} 1 & 0 \\ 6 & 1 \end{bmatrix}.$$

7. Find the products FGH and HGF if (with upper triangular zeros omitted)

$$F = \begin{bmatrix} 1 & & & \\ 2 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad G = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 2 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad H = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & 2 & 1 \end{bmatrix}.$$

8. (***Second proof of*** $A = LU$) The third row of U comes from the third row of A by subtracting multiples of rows 1 and 2 (*of* U!):

$$\text{row 3 of } U = \text{row 3 of } A - \ell_{31}(\text{row 1 of } U) - \ell_{32}(\text{row 2 of } U).$$

(a) Why are rows of U subtracted off and not rows of A? Answer: Because by the time a pivot row is used, ____.

(b) The equation above is the same as

$$\text{row 3 of } A = \ell_{31}(\text{row 1 of } U) + \ell_{32}(\text{row 2 of } U) + 1(\text{row 3 of } U).$$

Which rule for matrix multiplication makes this row 3 of L times U?

The other rows of LU agree similarly with the rows of A.

9. (a) Under what conditions is the following product nonsingular?

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) Solve the system $Ax = b$ starting with $Lc = b$:

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = b.$$

10. (a) Why does it take approximately $n^2/2$ multiplication-subtraction steps to solve each of $Lc = b$ and $Ux = c$?

(b) How many steps does elimination use in solving 10 systems with the same 60 by 60 coefficient matrix A?

11. Solve as two triangular systems, without multiplying LU to find A:

$$LUx = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}.$$

12. How could you factor A into a product UL, upper triangular times lower triangular? Would they be the same factors as in $A = LU$?

13. Solve by elimination, exchanging rows when necessary:

$$\begin{aligned} u + 4v + 2w &= -2 \\ -2u - 8v + 3w &= 32 \\ v + w &= 1 \end{aligned} \qquad \text{and} \qquad \begin{aligned} v + w &= 0 \\ u + v &= 0 \\ u + v + w &= 1. \end{aligned}$$

Which permutation matrices are required?

14. Write down all six of the 3 by 3 permutation matrices, including $P = I$. Identify their inverses, which are also permutation matrices. The inverses satisfy $PP^{-1} = I$ and are on the same list.

15. Find the $PA = LDU$ factorizations (and check them) for

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 1 \end{bmatrix}.$$

16. Find a 4 by 4 permutation matrix that requires three row exchanges to reach the end of elimination (which is $U = I$).

17. The less familiar form $A = LPU$ exchanges rows only at the end:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 2 & 5 & 8 \end{bmatrix} \to L^{-1}A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 3 & 6 \end{bmatrix} = PU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{bmatrix}.$$

What is L is this case? Comparing with $PA = LU$ in Box 1J, the multipliers now stay in place (ℓ_{21} is 1 and ℓ_{31} is 2 when $A = LPU$).

18. Decide whether the following systems are singular or nonsingular, and whether they have no solution, one solution, or infinitely many solutions:

$$\begin{aligned} v - w &= 2 \\ u - v &= 2 \\ u - w &= 2 \end{aligned} \quad \text{and} \quad \begin{aligned} v - w &= 0 \\ u - v &= 0 \\ u - w &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} v + w &= 1 \\ u + v &= 1 \\ u + w &= 1. \end{aligned}$$

19. Which numbers a, b, c lead to row exchanges? Which make the matrix singular?

$$A = \begin{bmatrix} 1 & 2 & 0 \\ a & 8 & 3 \\ 0 & b & 5 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} c & 2 \\ 6 & 4 \end{bmatrix}.$$

Problems 20–31 compute the factorization $A = LU$ (and also $A = LDU$).

20. Forward elimination changes $\left[\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right] x = b$ to a triangular $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right] x = c$:

$$\begin{aligned} x + y &= 5 \\ x + 2y &= 7 \end{aligned} \;\longrightarrow\; \begin{aligned} x + y &= 5 \\ y &= 2 \end{aligned} \qquad \begin{bmatrix} 1 & 1 & \mathbf{5} \\ 1 & 2 & \mathbf{7} \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & \mathbf{5} \\ 0 & 1 & \mathbf{2} \end{bmatrix}.$$

That step subtracted $\ell_{21} =$ ____ times row 1 from row 2. The reverse step *adds* ℓ_{21} times row 1 to row 2. The matrix for that reverse step is $L =$ ____. Multiply this L times the triangular system $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right] x = \left[\begin{smallmatrix} 5 \\ 2 \end{smallmatrix}\right]$ to get ____ $=$ ____. In letters, L multiplies $Ux = c$ to give ____.

21. (Move to 3 by 3) Forward elimination changes $Ax = b$ to a triangular $Ux = c$:

$$\begin{aligned} x+y+z &= 5 \\ x+2y+3z &= 7 \\ x+3y+6z &= 11 \end{aligned} \qquad \begin{aligned} x+y+z &= 5 \\ y+2z &= 2 \\ 2y+5z &= 6 \end{aligned} \qquad \begin{aligned} x+y+z &= 5 \\ y+2z &= 2 \\ z &= 2. \end{aligned}$$

The equation $z = 2$ in $Ux = c$ comes from the original $x + 3y + 6z = 11$ in $Ax = b$ by subtracting $\ell_{31} =$ ____ times equation 1 and $\ell_{32} =$ ____ times the *final* equation 2. Reverse that to recover $[1\ 3\ 6\ 11]$ in $[A\ \ b]$ from the final $[1\ 1\ 1\ 5]$ and $[0\ 1\ 2\ 2]$ and $[0\ 0\ 1\ 2]$ in $[U\ \ c]$:

$$\text{Row 3 of } \begin{bmatrix} A & b \end{bmatrix} = (\ell_{31} \text{ Row } 1 + \ell_{32} \text{ Row } 2 + 1 \text{ Row } 3) \text{ of } \begin{bmatrix} U & c \end{bmatrix}.$$

In matrix notation this is multiplication by L. So $A = LU$ and $b = Lc$.

22. What are the 3 by 3 triangular systems $Lc = b$ and $Ux = c$ from Problem 21? Check that $c = (5,2,2)$ solves the first one. Which x solves the second one?

23. What two elimination matrices E_{21} and E_{32} put A into upper triangular form $E_{32}E_{21}A = U$? Multiply by E_{31}^{-1} and E_{21}^{-1} to factor A into $LU = E_{21}^{-1}E_{32}^{-1}U$:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 5 \\ 0 & 4 & 0 \end{bmatrix}.$$

24. What three elimination matrices E_{21}, E_{31}, E_{32} put A into upper triangular form $E_{32}E_{31}E_{21}A = U$? Multiply by E_{32}^{-1}, E_{31}^{-1} and E_{21}^{-1} to factor A into LU where $L = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}$. Find L and U:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{bmatrix}.$$

25. When zero appears in a pivot position, $A = LU$ *is not possible*! (We need nonzero pivots d, f, i in U.) Show directly why these are both impossible:

$$\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \ell & 1 \end{bmatrix} \begin{bmatrix} d & e \\ 0 & f \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ \ell & 1 & \\ m & n & 1 \end{bmatrix} \begin{bmatrix} d & e & g \\ & f & h \\ & & i \end{bmatrix}.$$

26. Which number c leads to zero in the second pivot position? A row exchange is needed and $A = LU$ is not possible. Which c produces zero in the third pivot position? Then a row exchange can't help and elimination fails:

$$A = \begin{bmatrix} 1 & c & 0 \\ 2 & 4 & 1 \\ 3 & 5 & 1 \end{bmatrix}.$$

27. What are L and D for this matrix A? What is U in $A = LU$ and what is the new U in $A = LDU$?

$$A = \begin{bmatrix} 2 & 4 & 8 \\ 0 & 3 & 9 \\ 0 & 0 & 7 \end{bmatrix}.$$

28. A and B are symmetric across the diagonal (because $4 = 4$). Find their triple factorizations LDU and say how U is related to L for these symmetric matrices:

$$A = \begin{bmatrix} 2 & 4 \\ 4 & 11 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 4 & 0 \\ 4 & 12 & 4 \\ 0 & 4 & 0 \end{bmatrix}.$$

29. (Recommended) Compute L and U for the symmetric matrix

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}.$$

Find four conditions on a, b, c, d to get $A = LU$ with four pivots.

30. Find L and U for the nonsymmetric matrix

$$A = \begin{bmatrix} a & r & r & r \\ a & b & s & s \\ a & b & c & t \\ a & b & c & d \end{bmatrix}.$$

Find the four conditions on a, b, c, d, r, s, t to get $A = LU$ with four pivots.

31. *Tridiagonal matrices* have zero entries except on the main diagonal and the two adjacent diagonals. Factor these into $A = LU$ and $A = LDV$:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} a & a & 0 \\ a & a+b & b \\ 0 & b & b+c \end{bmatrix}.$$

32. Solve the triangular system $Lc = b$ to find c. Then solve $Ux = c$ to find x:

$$L = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 2 \\ 11 \end{bmatrix}.$$

For safety find $A = LU$ and solve $Ax = b$ as usual. Circle c when you see it.

33. Solve $Lc = b$ to find c. Then solve $Ux = c$ to find x. What was A?

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}.$$

34. If A and B have nonzeros in the positions marked by x, which zeros are still zero in their factors L and U?

$$A = \begin{bmatrix} x & x & x & x \\ x & x & x & 0 \\ 0 & x & x & x \\ 0 & 0 & x & x \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} x & x & x & 0 \\ x & x & 0 & x \\ x & 0 & x & x \\ 0 & x & x & x \end{bmatrix}.$$

35. (Important) If A has pivots 2, 7, 6 with no row exchanges, what are the pivots for the upper left 2 by 2 submatrix B (without row 3 and column 3)? Explain why.

36. Starting from a 3 by 3 matrix A with pivots 2, 7, 6, add a fourth row and column to produce M. What are the first three pivots for M, and why? What fourth row and column are sure to produce 9 as the fourth pivot?

37. Use chol(pascal(5)) to find the triangular factors of MATLAB's pascal(5). Row exchanges in [L, U] = lu(pascal(5)) spoil Pascal's pattern!

38. (Review) For which numbers c is $A = LU$ impossible—with three pivots?

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & c & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

39. Estimate the time difference for each new right-hand side b when $n = 800$. Create A = rand(800) and b = rand(800,1) and B = rand(800,9). Compare the times from tic; A\b; toc and tic; A\B; toc (which solves for 9 right sides).

Problems 40–48 are about permutation matrices.

40. There are 12 "*even*" permutations of $(1,2,3,4)$, with an *even number of exchanges*. Two of them are $(1,2,3,4)$ with no exchanges and $(4,3,2,1)$ with two exchanges. List the other ten. Instead of writing each 4 by 4 matrix, use the numbers 4, 3, 2, 1 to give the position of the 1 in each row.

41. How many exchanges will permute $(5,4,3,2,1)$ back to $(1,2,3,4,5)$? How many exchanges to change $(6,5,4,3,2,1)$ to $(1,2,3,4,5,6)$? One is even and the other is odd. For $(n,\dots,1)$ to $(1,\dots,n)$, show that $n = 100$ and 101 are even, $n = 102$ and 103 are odd.

42. If P_1 and P_2 are permutation matrices, so is P_1P_2. This still has the rows of I in some order. Give examples with $P_1P_2 \neq P_2P_1$ and $P_3P_4 = P_4P_3$.

43. (Try this question.) Which permutation makes PA upper triangular? Which permutations make P_1AP_2 lower triangular? ***Multiplying** A **on the** **right by** P_2 **exchanges the** ____ **of** A.*

$$A = \begin{bmatrix} 0 & 0 & 6 \\ 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix}$$

44. Find a 3 by 3 permutation matrix with $P^3 = I$ (but not $P = I$). Find a 4 by 4 permutation $\widehat{P}$ with $\widehat{P}^4 \neq I$.

45. If you take powers of a permutation, why is some P^k eventually equal to I? Find a 5 by 5 permutation P so that the smallest power to equal I is P^6. (This is a challenge question. Combine a 2 by 2 block with a 3 by 3 block.)

46. The matrix P that multiplies (x,y,z) to give (z,x,y) is also a rotation matrix. Find P and P^3. The rotation axis $a = (1,1,1)$ doesn't move, it equals Pa. What is the angle of rotation from $v = (2,3,-5)$ to $Pv = (-5,2,3)$?

47. If P is any permutation matrix, find a nonzero vector x so that $(I-P)x = 0$. (This will mean that $I-P$ has no inverse, and has determinant zero.)

48. If P has 1s on the antidiagonal from $(1,n)$ to $(n,1)$, describe PAP.

1.6 Inverses and Transposes

The inverse of an n by n matrix is another n by n matrix. The inverse of A is written A^{-1} (and pronounced "A inverse"). The fundamental property is simple: *If you multiply by A and then multiply by A^{-1}, you are back where you started*:

$$\textbf{Inverse matrix} \qquad \text{If} \quad b = Ax \quad \text{then} \quad A^{-1}b = x.$$

Thus $A^{-1}Ax = x$. The matrix A^{-1} times A is the identity matrix. ***Not all matrices have inverses. An inverse is impossible when*** Ax ***is zero and*** x ***is nonzero***. Then A^{-1} would have to get back from $Ax = 0$ to x. No matrix can multiply that zero vector Ax and produce a nonzero vector x.

Our goals are to define the inverse matrix and compute it and use it, when A^{-1} exists—and then to understand which matrices don't have inverses.

> **1K** The **inverse** of A is a matrix B such that $BA = I$ and $AB = I$. There is at most one such B, and it is denoted by A^{-1}:
>
> $$A^{-1}A = I \quad \text{and} \quad AA^{-1} = I. \tag{1}$$

Note* 1.** ***The inverse exists if and only if elimination produces n ***pivots*** (row exchanges allowed). Elimination solves $Ax = b$ without explicitly finding A^{-1}.

***Note* 2.** The matrix A cannot have two different inverses, Suppose $BA = I$ and also $AC = I$. Then $B = C$, according to this "proof by parentheses":

$$B(AC) = (BA)C \quad \text{gives} \quad BI = IC \quad \text{which is} \quad B = C. \tag{2}$$

This shows that a *left-inverse* B (multiplying from the left) and a *right-inverse* C (multiplying A from the right to give $AC = I$) must be the *same matrix*.

***Note* 3.** If A is invertible, the one and only solution to $Ax = b$ is $x = A^{-1}b$:

$$\textbf{\textit{Multiply}} \quad Ax = b \quad \textbf{\textit{by}} \quad A^{-1}. \quad \textbf{\textit{Then}} \quad x = A^{-1}Ax = A^{-1}b.$$

Note* 4.** (Important) ***Suppose there is a nonzero vector x ***such that*** $Ax = 0$. *Then* A *cannot have an inverse*. To repeat: No matrix can bring 0 back to x.

If A is invertible, then $Ax = 0$ can only have the zero solution $x = 0$.

***Note* 5.** A 2 by 2 matrix is invertible if and only if $ad - bc$ is not zero:

$$\textbf{2 by 2 inverse} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \tag{3}$$

This number $ad - bc$ is the *determinant* of A. A matrix is invertible if its determinant is not zero (Chapter 4). In MATLAB, the invertibility test is *to find n nonzero pivots*. Elimination produces those pivots before the determinant appears.

***Note* 6.** A diagonal matrix has an inverse provided no diagonal entries are zero:

$$\text{If} \quad A = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix} \quad \text{then} \quad A^{-1} = \begin{bmatrix} 1/d_1 & & \\ & \ddots & \\ & & 1/d_n \end{bmatrix} \quad \text{and} \quad AA^{-1} = I.$$

When two matrices are involved, not much can be done about the inverse of $A + B$. The sum might or might not be invertible. Instead, it is the inverse of their *product*

AB which is the key formula in matrix computations. Ordinary numbers are the same: $(a+b)^{-1}$ is hard to simplify, while $1/ab$ splits into $1/a$ times $1/b$. But for matrices *the order of multiplication must be correct*—if $ABx = y$ then $Bx = A^{-1}y$ and $x = B^{-1}A^{-1}y$. **The inverses come in reverse order**.

1L A product AB of invertible matrices is inverted by $B^{-1}A^{-1}$:

$$\textbf{Inverse of } AB \qquad (AB)^{-1} = B^{-1}A^{-1}. \tag{4}$$

Proof. To show that $B^{-1}A^{-1}$ is the inverse of AB, we multiply them and use the associative law to remove parentheses. Notice how B sits next to B^{-1}:

$$(AB)(B^{-1}A^{-1}) = ABB^{-1}A^{-1} = AIA^{-1} = AA^{-1} = I$$

$$(B^{-1}A^{-1})(AB) = B^{-1}A^{-1}AB = B^{-1}IB = B^{-1}B = I.$$

□

A similar rule holds with three or more matrices:

$$\textbf{Inverse of } ABC \qquad (ABC)^{-1} = C^{-1}B^{-1}A^{-1}.$$

We saw this change of order when the elimination matrices E, F, G were inverted to come back from U to A. In the forward direction, $GFEA$ was U. In the backward direction, $L = E^{-1}F^{-1}G^{-1}$ was the product of the inverses. *Since G came last, G^{-1} comes first*. Please check that A^{-1} would be $U^{-1}GFE$.

The Calculation of A^{-1}: The Gauss-Jordan Method

Consider the equation $AA^{-1} = I$. If it is taken ***a column at a time***, that equation determines each column of A^{-1}. The first column of A^{-1} is multiplied by A, to yield the first column of the identity: $Ax_1 = e_1$. Similarly $Ax_2 = e_2$ and $Ax_3 = e_3$ the e's are the columns of I. In a 3 by 3 example, A times A^{-1} is I:

$$Ax_i = e_i \qquad \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} = \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{5}$$

Thus we have three systems of equations (or n systems). They all have the same coefficient matrix A. The right-hand sides e_1, e_2, e_3 are different, but elimination is possible *on all systems simultaneously*. This is the ***Gauss-Jordan method***. Instead of stopping at U and switching to back-substitution, it continues by subtracting multiples of a row *from the rows above*. This produces zeros above the diagonal as well as below. When it reaches the identity matrix we have found A^{-1}.

The example keeps all three columns e_1, e_2, e_3, and operates on rows of length six:

Example 1. Using the Gauss-Jordan Method to Find A^{-1}

$$\begin{bmatrix} A & e_1 & e_2 & e_3 \end{bmatrix} = \begin{bmatrix} \mathbf{2} & \mathbf{1} & \mathbf{1} & 1 & 0 & 0 \\ \mathbf{4} & \mathbf{-6} & \mathbf{0} & 0 & 1 & 0 \\ \mathbf{-2} & \mathbf{7} & \mathbf{2} & 0 & 0 & 1 \end{bmatrix}$$

$$\textbf{pivot} = \mathbf{2} \to \begin{bmatrix} \mathbf{2} & \mathbf{1} & \mathbf{1} & 1 & 0 & 0 \\ \mathbf{0} & \mathbf{-8} & \mathbf{-2} & -2 & 1 & 0 \\ \mathbf{0} & \mathbf{8} & \mathbf{3} & 1 & 0 & 1 \end{bmatrix}$$

$$\textbf{pivot} = \mathbf{-8} \to \begin{bmatrix} \mathbf{2} & \mathbf{1} & \mathbf{1} & 1 & 0 & 0 \\ \mathbf{0} & \mathbf{-8} & \mathbf{-2} & -2 & 1 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} U & L^{-1} \end{bmatrix}.$$

This completes the first half—forward elimination. The upper triangular U appears in the first three columns. The other three columns are the same as L^{-1}. (This is the effect of applying the elementary operations GFE to the identity matrix.) Now the second half will go from U to I (multiplying by U^{-1}). That takes L^{-1} to $U^{-1}L^{-1}$ which is A^{-1}. **Creating zeros *above* the pivots, we reach** A^{-1}:

$$\textbf{Second half} \quad \begin{bmatrix} U & L^{-1} \end{bmatrix} \to \begin{bmatrix} 2 & 1 & \mathbf{0} & 2 & -1 & -1 \\ 0 & -8 & \mathbf{0} & -4 & 3 & 2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix}$$

$$\textbf{zeros above pivots} \to \begin{bmatrix} 2 & \mathbf{0} & \mathbf{0} & \frac{12}{8} & -\frac{5}{8} & -\frac{6}{8} \\ 0 & -8 & \mathbf{0} & -4 & 3 & 2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix}$$

$$\textbf{divide by pivots} \to \begin{bmatrix} 1 & 0 & 0 & \mathbf{\frac{12}{16}} & -\mathbf{\frac{5}{16}} & -\mathbf{\frac{6}{16}} \\ 0 & 1 & 0 & \mathbf{\frac{4}{8}} & -\mathbf{\frac{3}{8}} & -\mathbf{\frac{2}{8}} \\ 0 & 0 & 1 & \mathbf{-1} & \mathbf{1} & \mathbf{1} \end{bmatrix} = \begin{bmatrix} I & \boldsymbol{A^{-1}} \end{bmatrix}.$$

At the last step, we divided the rows by their pivots 2 and -8 and 1. The coefficient matrix in the left-hand half became the identity. Since A went to I, the same operations on the right-hand half must have carried I into A^{-1}. Therefore we have computed the inverse.

A note for the future: You can see the determinant -16 appearing in the denominators of A^{-1}. **The determinant is the product of the pivots (2)($-$8)(1)**. It enters at the end when the rows are divided by the pivots.

***Remark* 1.** In spite of this brilliant success in computing A^{-1}, I don't recommend it, I admit that A^{-1} solves $Ax = b$ in one step. Two triangular steps are better:

$$x = A^{-1}b \quad \text{separates into} \quad Lc = b \quad \text{and} \quad Ux = c.$$

We could write $c = L^{-1}b$ and then $x = U^{-1}c = U^{-1}L^{-1}b$. But note that we did not explicitly form, and in actual computation *should not form*, these matrices L^{-1} and U^{-1}.

It would be a waste of time, since we only need back-substitution for x (and forward substitution produced c).

A similar remark applies to A^{-1}; the multiplication $A^{-1}b$ would still take n^2 steps. ***It is the solution that we want, and not all the entries in the inverse***.

***Remark* 2.** Purely out of curiosity, we might count the number of operations required to find A^{-1}. The normal count for each new right-hand side is n^2, half in the forward direction and half in back-substitution. With n right-hand sides $e_1, \dots, e_n$ this makes n^3. After including the $n^3/3$ operations on A itself, the total seems to be $4n^3/3$.

This result is a little too high because of the zeros in the e_j. Forward elimination changes only the zeros below the 1. This part has only $n-j$ components, so the count for e_j is effectively changed to $(n-j)^2/2$. Summing over all j, the total for forward elimination is $n^3/6$. This is to be combined with the usual $n^3/3$ operations that are applied to A, and the $n(n^2/2)$ back-substitution steps that finally produce the columns x_j of A^{-1}. *The final count of multiplications for computing A^{-1} is n^3*:

$$\textbf{Operation count} \qquad \frac{n^3}{6} + \frac{n^3}{3} + n\left(\frac{n^2}{2}\right) = n^3.$$

This count is remarkably low. Since matrix multiplication already takes n^3 steps, it requires as many operations to compute A^2 as it does to compute A^{-1}! That fact seems almost unbelievable (and computing A^3 requires twice as many, as far as we can see). Nevertheless, if A^{-1} is not needed, it should not be computed.

***Remark* 3.** In the Gauss-Jordan calculation we went all the way forward to U, before starting backward to produce zeros above the pivots. That is like Gaussian elimination, but other orders are possible. We could have used the second pivot when we were there earlier, to create a zero above it as well as below it. This is not smart. At that time the second row is virtually full, whereas near the end it has zeros from the upward row operations that have already taken place.

Invertible = Nonsingular (n pivots)

Ultimately we want to know which matrices are invertible and which are not. This question is so important that it has many answers. *See the last page of the book!*

Each of the first five chapters will give a different (but equivalent) test for invertibility. Sometimes the tests extend to rectangular matrices and one-sided inverses: Chapter 2 looks for independent rows and independent columns, Chapter 3 inverts AA^{T} or $A^{\mathrm{T}}A$. The other chapters look for ***nonzero determinants*** or ***nonzero eigenvalues*** or ***nonzero pivots***. This last test is the one we meet through Gaussian elimination. We want to show (in a few theoretical paragraphs) that the pivot test succeeds.

Suppose A has a full set of n pivots. $AA^{-1} = I$ gives n separate systems $Ax_i = e_i$ for the columns of A^{-1}. They can be solved by elimination or by Gauss-Jordan. Row exchanges may be needed, but the columns of A^{-1} are determined.

Strictly speaking, we have to show that the matrix A^{-1} with those columns is also a *left*-inverse. Solving $AA^{-1} = I$ has at the same time solved $A^{-1}A = I$, but why? **A 1-sided inverse of a square matrix is automatically a 2-sided inverse**. To see why, notice that *every Gauss-Jordan step is a multiplication on the left by an elementary matrix*. We are allowing three types of elementary matrices:

1. E_{ij} to subtract a multiple ℓ of row j from row i
2. P_{ij} to exchange rows i and j
3. D (or D^{-1}) to divide all rows by their pivots.

The Gauss-Jordan process is really a giant sequence of matrix multiplications:

$$(D^{-1} \cdots E \cdots P \cdots E)A = I. \tag{6}$$

That matrix in parentheses, to the left of A, is evidently a left-inverse! It exists, it equals the right-inverse by Note 2, so ***every nonsingular matrix is invertible***.

The converse is also true: ***If A is invertible, it has n pivots***. In an extreme case that is clear: A cannot have a whole column of zeros. The inverse could never multiply a column of zeros to produce a column of I. In a less extreme case, suppose elimination starts on an invertible matrix A but breaks down at column 3:

Breakdown
No pivot in column 3
$$A' = \begin{bmatrix} d_1 & x & x & x \\ 0 & d_2 & x & x \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & x \end{bmatrix}.$$

This matrix cannot have an inverse, no matter what the x's are. One proof is to use column operations (for the first time?) to make the whole third column zero. By subtracting multiples of column 2 and then of column 1, we reach a matrix that is certainly not invertible. Therefore the original A was not invertible. Elimination gives a complete test: *An n by n matrix is invertible if and only if it has n pivots*.

The Transpose Matrix

We need one more matrix, and fortunately it is much simpler than the inverse. The ***transpose*** of A is denoted by A^{T}. Its columns are taken directly from the rows of A—the ith row of A becomes the ith column of A^{T}:

Transpose
$$\text{If} \quad A = \begin{bmatrix} 2 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix} \quad \text{then} \quad A^{\mathrm{T}} = \begin{bmatrix} 2 & 0 \\ 1 & 0 \\ 4 & 3 \end{bmatrix}.$$

At the same time the columns of A become the rows of A^{T}, If A is an m by n matrix, then A^{T} is n by m. The final effect is to flip the matrix across its main diagonal, and the entry in row i, column j of A^{T} comes from row j, column i of A:

$$\textbf{Entries of } A^{\mathrm{T}} \qquad (A^{\mathrm{T}})_{ij} = A_{ji}. \tag{7}$$

The transpose of a lower triangular matrix is upper triangular. The transpose of A^{T} brings us back to A.

If we add two matrices and then transpose, the result is the same as first transposing and then adding: $(A+B)^{\mathrm{T}}$ is the same as $A^{\mathrm{T}}+B^{\mathrm{T}}$. But what is the transpose of a product AB or an inverse A^{-1}? Those are the essential formulas of this section:

1M

(i) The transpose of AB is $(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}$,

(ii) The transpose of A^{-1} is $(A^{-1})^{\mathrm{T}} = (A^{\mathrm{T}})^{-1}$.

Notice how the formula for $(AB)^{\mathrm{T}}$ resembles the one for $(AB)^{-1}$. In both cases we reverse the order, giving $B^{\mathrm{T}}A^{\mathrm{T}}$ and $B^{-1}A^{-1}$. The proof for the inverse was easy, but this one requires an unnatural patience with matrix multiplication. The first row of $(AB)^{\mathrm{T}}$ is the first column of AB. So the columns of A are weighted by the first column of B. This amounts to the rows of A^{T} weighted by the first row of B^{T}. That is exactly the first row of $B^{\mathrm{T}}A^{\mathrm{T}}$. The other rows of $(AB)^{\mathrm{T}}$ and $B^{\mathrm{T}}A^{\mathrm{T}}$ also agree.

$$\textbf{Start from} \qquad AB = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 3 & 3 \\ 2 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 3 \\ 5 & 5 & 5 \end{bmatrix}$$

$$\textbf{Transpose to} \qquad B^{\mathrm{T}}A^{\mathrm{T}} = \begin{bmatrix} 3 & 2 \\ 3 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 3 & 5 \\ 3 & 5 \end{bmatrix}.$$

To establish the formula for $(A^{-1})^{\mathrm{T}}$, start from $AA^{-1} = I$ and $A^{-1}A = I$ and take transposes. On one side, $I^{\mathrm{T}} = I$. On the other side, we know from part (i) the transpose of a product. You see how $(A^{-1})^{\mathrm{T}}$ is the inverse of A^{T}, proving (ii):

$$\textbf{Inverse of } A^{\mathrm{T}} = \textbf{Transpose of } A^{-1} \qquad (A^{-1})^{\mathrm{T}}A^{\mathrm{T}} = I. \tag{8}$$

Symmetric Matrices

With these rules established, we can introduce a special class of matrices, probably the most important class of all. ***A symmetric matrix is a matrix that equals its own transpose:*** $A^{\mathrm{T}} = A$. The matrix is necessarily square. Each entry on one side of the diagonal equals its "mirror image" on the other side: $a_{ij} = a_{ji}$. Two simple examples are A and D (and also A^{-1}):

$$\textbf{Symmetric matrices} \qquad A = \begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \quad \text{and} \quad A^{-1} = \frac{1}{4}\begin{bmatrix} 8 & -2 \\ -2 & 1 \end{bmatrix}.$$

A symmetric matrix need not be invertible; it could even be a matrix of zeros. *But if A^{-1} exists it is also symmetric*. From formula (ii) above, the transpose of A^{-1} always equals $(A^{\mathrm{T}})^{-1}$; for a symmetric matrix this is just A^{-1}. A^{-1} equals its own transpose; it is symmetric whenever A is. Now we show that ***multiplying any matrix R by R^{T} gives a symmetric matrix***.

Symmetric Products $R^{\mathrm{T}}R$, RR^{T}, and LDL^{T}

Choose any matrix R, probably rectangular. Multiply R^{T} times R. Then the product $R^{\mathrm{T}}R$ is automatically a square symmetric matrix:

$$\boxed{\textbf{\textit{The transpose of}}\quad R^{\mathrm{T}}R \quad \textbf{\textit{is}} \quad R^{\mathrm{T}}(R^{\mathrm{T}})^{\mathrm{T}}, \quad \textbf{\textit{which is}} \quad R^{\mathrm{T}}R. \qquad (9)}$$

That is a quick proof of symmetry for $R^{\mathrm{T}}R$. Its i, j entry is the inner product of row i of R^{T} (column i of R) with column j of R. The (j,i) entry is the same inner product, column j with column i. So $R^{\mathrm{T}}R$ is symmetric.

RR^{T} is also symmetric, but it is different from $R^{\mathrm{T}}R$. In my experience, most scientific problems that start with a rectangular matrix R end up with $R^{\mathrm{T}}R$ or RR^{T} or both.

Example 2. $R = [1\ \ 2]$ and $R^{\mathrm{T}} = \left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]$ produce $R^{\mathrm{T}}R = \left[\begin{smallmatrix}1&2\\2&4\end{smallmatrix}\right]$ and $RR^{\mathrm{T}} = [5]$.
The product $R^{\mathrm{T}}R$ is n by n. In the opposite order, RR^{T} is m by m. Even if $m = n$, it is not very likely that $R^{\mathrm{T}}R = RR^{\mathrm{T}}$. Equality can happen, but it's not normal.

Symmetric matrices appear in every subject whose laws are fair. “Each action has an equal and opposite reaction.” The entry a_{ij} that gives the action of i onto j is matched by a_{ji}. We will see this symmetry in the next section, for differential equations. Here, LU misses the symmetry but LDL^{T} captures it perfectly.

> **1N** Suppose $A = A^{\mathrm{T}}$ can be factored into $A = LDU$ without row exchanges. Then U is the transpose of L. ***The symmetric factorization becomes*** $A = LDL^{\mathrm{T}}$.

The transpose of $A = LDU$ gives $A^{\mathrm{T}} = U^{\mathrm{T}}D^{\mathrm{T}}L^{\mathrm{T}}$. Since $A = A^{\mathrm{T}}$, we now have two factorizations of A into lower triangular times diagonal times upper triangular. (L^{T} is upper triangular with ones on the diagonal, exactly like U.) Since the factorization is unique (see Problem 17), L^{T} must be identical to U.

$$L^{\mathrm{T}} = U \textbf{ and } A = LDL^{\mathrm{T}} \qquad \begin{bmatrix}1&2\\2&8\end{bmatrix} = \begin{bmatrix}\mathbf{1}&\mathbf{0}\\\mathbf{2}&\mathbf{1}\end{bmatrix}\begin{bmatrix}1&0\\0&4\end{bmatrix}\begin{bmatrix}\mathbf{1}&\mathbf{2}\\\mathbf{0}&\mathbf{1}\end{bmatrix} = LDL^{\mathrm{T}}.$$

When elimination is applied to a symmetric matrix, $A^{\mathrm{T}} = A$ is an advantage. The smaller matrices stay symmetric as elimination proceeds, and we can work with half the matrix! The lower right-hand corner remains symmetric:

$$\begin{bmatrix}a&b&c\\b&d&e\\c&e&f\end{bmatrix} \to \begin{bmatrix}a&b&c\\0&d-\frac{b^2}{a}&e-\frac{\boldsymbol{bc}}{\boldsymbol{a}}\\0&e-\frac{\boldsymbol{bc}}{\boldsymbol{a}}&f-\frac{c^2}{a}\end{bmatrix}.$$

The work of elimination is reduced from $n^3/3$ to $n^3/6$. There is no need to store entries from both sides of the diagonal, or to store both L and U.

Problem Set 1.6

1. Find the inverses (no special system required) of

$$A_1 = \begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 2 & 0 \\ 4 & 2 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

2. (a) Find the inverses of the permutation matrices

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

(b) Explain for permutations why P^{-1} *is always the same as* P^{T}. Show that the 1s are in the right places to give $PP^{\mathrm{T}} = I$.

3. From $AB = C$ find a formula for A^{-1}. Also find A^{-1} from $PA = LU$.

4. (a) If A is invertible and $AB = AC$, prove quickly that $B = C$.

(b) If $A = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$, find an example with $AB = AC$ but $B \neq C$.

5. If the inverse of A^2 is B, show that the inverse of A is AB. (Thus A is invertible whenever A^2 is invertible.)

6. Use the Gauss-Jordan method to invert

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

7. Find three 2 by 2 matrices, other than $A = I$ and $A = -I$, that are their own inverses: $A^2 = I$.

8. Show that $A = \left[\begin{smallmatrix} 1 & 1 \\ 3 & 3 \end{smallmatrix}\right]$ has no inverse by solving $Ax = 0$, and by failing to solve

$$\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

9. Suppose elimination fails because there is no pivot in column 3:

$$\textbf{Missing pivot} \qquad A = \begin{bmatrix} 2 & 1 & 4 & 6 \\ 0 & 3 & 8 & 5 \\ 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 9 \end{bmatrix}.$$

Show that A cannot be invertible. The third row of A^{-1}, multiplying A, should give the third row $[0\ 0\ 1\ 0]$ of $A^{-1}A = I$. Why is this impossible?

10. Find the inverses (in any legal way) of

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 0 & -\frac{2}{3} & 1 & 0 \\ 0 & 0 & -\frac{3}{4} & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix}.$$

11. Give examples of A and B such that

(a) $A+B$ is not invertible although A and B are invertible.

(b) $A+B$ is invertible although A and B are not invertible.

(c) all of A, B, and $A+B$ are invertible.

(d) In the last case use $A^{-1}(A+B)B^{-1} = B^{-1}+A^{-1}$ to show that $C = B^{-1}+A^{-1}$ is also invertible—and find a formula for C^{-1}.

12. If A is invertible, which properties of A remain true for A^{-1}?
(a) A is triangular. (b) A is symmetric. (c) A is tridiagonal. (d) All entries are whole numbers. (e) All entries are fractions (including numbers like $\frac{3}{1}$).

13. If $A = \left[\begin{smallmatrix} 3 \\ 1 \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} 2 \\ 2 \end{smallmatrix}\right]$, compute $A^{\mathrm{T}}B$, $B^{\mathrm{T}}A$, AB^{T}, and BA^{T}.

14. If B is square, show that $A = B+B^{\mathrm{T}}$ is always symmetric and $K = B-B^{\mathrm{T}}$ is always *skew-symmetric*—which means that $K^{\mathrm{T}} = -K$. Find these matrices A and K when $B = \left[\begin{smallmatrix} 1 & 3 \\ 1 & 1 \end{smallmatrix}\right]$, and write B as the sum of a symmetric matrix and a skew-symmetric matrix.

15. (a) How many entries can be chosen independently in a symmetric matrix of order n?

(b) How many entries can be chosen independently in a skew-symmetric matrix ($K^{\mathrm{T}} = -K$) of order n? The diagonal of K is zero!

16. (a) If $A = LDU$, with 1s on the diagonals of L and U, what is the corresponding factorization of A^{T}? Note that A and A^{T} (square matrices with no row exchanges) share the same pivots.

(b) What triangular systems will give the solution to $A^{\mathrm{T}}y = b$?

17. If $A = L_1D_1U_1$ and $A = L_2D_2U_2$, prove that $L_1 = L_2$, $D_1 = D_2$, and $U_1 = U_2$. If A is invertible, the factorization is unique.

(a) Derive the equation $L_1^{-1}L_2D_2 = D_1U_1U_2^{-1}$, and explain why one side is lower triangular and the other side is upper triangular.

(b) Compare the main diagonals and then compare the off-diagonals.

18. Under what conditions on their entries are A and B invertible?

$$A = \begin{bmatrix} a & b & c \\ d & e & 0 \\ f & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{bmatrix}.$$

19. Compute the symmetric LDL^{T} factorization of

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 12 & 18 \\ 5 & 18 & 30 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}.$$

20. Find the inverse of

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{4} & 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}.$$

21. (Remarkable) If A and B are square matrices, show that $I - BA$ is invertible if $I - AB$ is invertible. Start from $B(I - AB) = (1 - BA)B$.

22. Find the inverses (directly or from the 2 by 2 formula) of A, B, C:

$$A = \begin{bmatrix} 0 & 3 \\ 4 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} a & b \\ b & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}.$$

23. Solve for the columns of $A^{-1} = \begin{bmatrix} x & t \\ y & z \end{bmatrix}$:

$$\begin{bmatrix} 10 & 20 \\ 20 & 50 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 10 & 20 \\ 20 & 50 \end{bmatrix} \begin{bmatrix} t \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

24. Show that $\left[\begin{smallmatrix} 1 & 2 \\ 3 & 6 \end{smallmatrix}\right]$ has no inverse by trying to solve for the column (x, y):

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x & t \\ y & z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{must include} \quad \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

25. (Important) If A has row 1 + row 2 = row 3, show that A is not invertible:

(a) Explain why $Ax = (1, 0, 0)$ cannot have a solution.

(b) Which right-hand sides (b_1, b_2, b_3) might allow a solution to $Ax = b$?

(c) What happens to row 3 in elimination?

26. If A has column 1 + column 2 = column 3, show that A is not invertible:

(a) Find a nonzero solution x to $Ax = 0$. The matrix is 3 by 3.

(b) Elimination keeps column 1 + column 2 = column 3. Explain why there is no third pivot.

27. Suppose A is invertible and you exchange its first two rows to reach B. Is the new matrix B invertible? How would you find B^{-1} from A^{-1}?

28. If the product $M = ABC$ of three square matrices is invertible, then A, B, C are invertible. Find a formula for B^{-1} that involves M^{-1} and A and C.

29. Prove that a matrix with a column of zeros cannot have an inverse.

30. Multiply $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ times $\left[\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right]$. What is the inverse of each matrix if $ad \neq bc$?

31. (a) What matrix E has the same effect as these three steps? Subtract row 1 from row 2, subtract row 1 from row 3, then subtract row 2 from row 3.

(b) What single matrix L has the same effect as these three reverse steps? Add row 2 to row 3, add row 1 to row 3, then add row 1 to row 2.

32. Find the numbers a and b that give the inverse of `5 * eye(4) – ones(4,4)`:

$$\begin{bmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} a & b & b & b \\ b & a & b & b \\ b & b & a & b \\ b & b & b & a \end{bmatrix}.$$

What are a and b in the inverse of `6 * eye(5) – ones(5,5)`?

33. Show that `A = 4 * eye(4) – ones(4,4)` is *not* invertible: Multiply `A * ones(4,1)`.

34. There are sixteen 2 by 2 matrices whose entries are 1s and 0s. How many of them are invertible?

Problems 35–39 are about the Gauss-Jordan method for calculating A^{-1}.

35. Change I into A^{-1} as you reduce A to I (by row operations):

$$\begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 1 & 4 & 1 & 0 \\ 3 & 9 & 0 & 1 \end{bmatrix}.$$

36. Follow the 3 by 3 text example but with plus signs in A. Eliminate above and below the pivots to reduce $[A \;\; I]$ to $[I \;\; A^{-1}]$:

$$\begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} \mathbf{2} & \mathbf{1} & \mathbf{0} & 1 & 0 & 0 \\ \mathbf{1} & \mathbf{2} & \mathbf{1} & 0 & 1 & 0 \\ \mathbf{0} & \mathbf{1} & \mathbf{2} & 0 & 0 & 1 \end{bmatrix}.$$

37. Use Gauss-Jordan elimination on $[A \;\; I]$ to solve $AA^{-1} = I$:

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \boldsymbol{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

38. Invert these matrices A by the Gauss-Jordan method starting with $[A \;\; I]$:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

39. Exchange rows and continue with Gauss-Jordan to find A^{-1}:

$$\begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{bmatrix}.$$

40. True or false (with a counterexample if false and a reason if true):

(a) A 4 by 4 matrix with a row of zeros is not invertible.

(b) A matrix with Is down the main diagonal is invertible.

(c) If A is invertible then A^{-1} is invertible.

(d) If A^{T} is invertible then A is invertible.

41. For which three numbers c is this matrix not invertible, and why not?

$$A = \begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix}.$$

42. Prove that A is invertible if $a \neq 0$ and $a \neq b$ (find the pivots and A^{-1}):

$$A = \begin{bmatrix} a & b & b \\ a & a & b \\ a & a & a \end{bmatrix}.$$

43. This matrix has a remarkable inverse. Find A^{-1} by elimination on $[A \;\; I]$. Extend to a 5 by 5 “alternating matrix” and guess its inverse:

$$A = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

44. If B has the columns of A in reverse order, solve $(A-B)x=0$ to show that $A-B$ is not invertible. An example will lead you to x.

45. Find and check the inverses (assuming they exist) of these block matrices:

$$\begin{bmatrix} I & 0 \\ C & I \end{bmatrix} \qquad \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} \qquad \begin{bmatrix} 0 & I \\ I & D \end{bmatrix}.$$

46. Use inv(S) to invert MATLAB's 4 by 4 symmetric matrix S = pascal(4). Create Pascal's lower triangular A = abs(pascal(4,1)) and test inv(S) = inv(A') * inv(A).

47. If A = ones(4,4) and b = rand(4,1), how does MATLAB tell you that $Ax=b$ has no solution? If b = ones(4,1), which solution to $Ax=b$ is found by $A\backslash b$?

48. M^{-1} shows the change in A^{-1} (useful to know) when a matrix is subtracted from A. Check part 3 by carefully multiplying MM^{-1} to get I:

1. $M = I - uv^{\mathrm{T}}$ and $M^{-1} = I + uv^{\mathrm{T}}/(1 - v^{\mathrm{T}}u)$.
2. $M = A - uv^{\mathrm{T}}$ and $M^{-1} = A^{-1} + A^{-1}uv^{\mathrm{T}}A^{-1}/(1 - v^{\mathrm{T}}A^{-1}u)$.
3. $M = I - UV$ and $M^{-1} = I_n + U(I_m - VU)^{-1}V$.
4. $M = A - UW^{-1}V$ and $M^{-1} = A^{-1} + A^{-1}U(W - VA^{-1}U)^{-1}VA^{-1}$.

The four identities come from the 1, 1 block when inverting these matrices:

$$\begin{bmatrix} I & u \\ v^{\mathrm{T}} & 1 \end{bmatrix} \qquad \begin{bmatrix} A & u \\ v^{\mathrm{T}} & 1 \end{bmatrix} \qquad \begin{bmatrix} I_n & U \\ V & I_m \end{bmatrix} \qquad \begin{bmatrix} A & U \\ V & W \end{bmatrix}.$$

Problems 49–55 are about the rules for transpose matrices.

49. Find A^{T} and A^{-1} and $(A^{-1})^{\mathrm{T}}$ and $(A^{\mathrm{T}})^{-1}$ for

$$A = \begin{bmatrix} 1 & 0 \\ 9 & 3 \end{bmatrix} \quad \text{and also} \quad A = \begin{bmatrix} 1 & c \\ c & 0 \end{bmatrix}.$$

50. Verify that $(AB)^{\mathrm{T}}$ equals $B^{\mathrm{T}}A^{\mathrm{T}}$ but those are different from $A^{\mathrm{T}}B^{\mathrm{T}}$:

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \qquad AB = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}.$$

In case $AB = BA$ (not generally true!), how do you prove that $B^{\mathrm{T}}A^{\mathrm{T}} = A^{\mathrm{T}}B^{\mathrm{T}}$?

51. (a) The matrix $\left((AB)^{-1}\right)^{\mathrm{T}}$ comes from $(A^{-1})^{\mathrm{T}}$ and $(B^{-1})^{\mathrm{T}}$. *In what order*?

(b) If U is upper triangular then $(U^{-1})^{\mathrm{T}}$ is ____ triangular.

52. Show that $A^2 = 0$ is possible but $A^{\mathrm{T}}A = 0$ is not possible (unless A = zero matrix).

53. (a) The row vector x^{T} times A times the column y produces what number?

$$x^{\mathrm{T}}Ay = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \underline{\qquad}.$$

(b) This is the row $x^{\mathrm{T}}A =$ ____ times the column $y = (0,1,0)$.

(c) This is the row $x^{\mathrm{T}} = [0\ \ 1]$ times the column $Ay =$ ____.

54. When you transpose a block matrix $M = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ the result is $M^{\mathrm{T}} =$ ____. Test it. Under what conditions on A, B, C, D is the block matrix symmetric?

55. Explain why the inner product of x and y equals the inner product of Px and Py. Then $(Px)^{\mathrm{T}}(Py) = x^{\mathrm{T}}y$ says that $P^{\mathrm{T}}P = I$ for any permutation. With $x = (1,2,3)$ and $y = (1,4,2)$, choose P to show that $(Px)^{\mathrm{T}}y$ is not always equal to $x^{\mathrm{T}}(P^{\mathrm{T}}y)$.

Problems 56–60 are about symmetric matrices and their factorizations.

56. If $A = A^{\mathrm{T}}$ and $B = B^{\mathrm{T}}$, which of these matrices are certainly symmetric?
(a) $A^2 - B^2$ (b) $(A+B)(A-B)$ (c) ABA (d) $ABAB$.

57. If $A = A^{\mathrm{T}}$ needs a row exchange, then it also needs a column exchange to stay symmetric. In matrix language, PA loses the symmetry of A but ____ recovers the symmetry.

58. (a) How many entries of A can be chosen independently, if $A = A^{\mathrm{T}}$ is 5 by 5?

(b) How do L and D (5 by 5) give the same number of choices in LDL^{T}?

59. Suppose R is rectangular (m by n) and A is symmetric (m by m).

(a) Transpose $R^{\mathrm{T}}AR$ to show its symmetry. What shape is this matrix?

(b) Show why $R^{\mathrm{T}}R$ has no negative numbers on its diagonal.

60. Factor these symmetric matrices into $A = LDL^{\mathrm{T}}$. The matrix D is diagonal:

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & b \\ b & c \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

The next three problems are about applications of $(Ax)^{\mathrm{T}}y = x^{\mathrm{T}}(A^{\mathrm{T}}y)$.

61. Wires go between Boston, Chicago, and Seattle. Those cities are at voltages x_{B}, x_{C}, x_{S}. With unit resistances between cities, the three currents are in y:

$$y = Ax \quad \text{is} \quad \begin{bmatrix} y_{\mathrm{BC}} \\ y_{\mathrm{CS}} \\ y_{\mathrm{BS}} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_{\mathrm{B}} \\ x_{\mathrm{C}} \\ x_{\mathrm{S}} \end{bmatrix}.$$

(a) Find the total currents $A^{\mathrm{T}}y$ out of the three cities.

(b) Verify that $(Ax)^{\mathrm{T}}y$ agrees with $x^{\mathrm{T}}(A^{\mathrm{T}}y)$—six terms in both.

62. Producing x_1 trucks and x_2 planes requires $x_1 + 50x_2$ tons of steel, $40x_1 + 1000x_2$ pounds of rubber, and $2x_1 + 50x_2$ months of labor. If the unit costs y_1, y_2, y_3 are \$700 per ton, \$3 per pound, and \$3000 per month, what are the values of one truck and one plane? Those are the components of $A^{\mathrm{T}}y$.

63. Ax gives the amounts of steel, rubber, and labor to produce x in Problem 62. Find A. Then $(Ax)^{\mathrm{T}}y$ is the ____ of inputs while $x^{\mathrm{T}}(A^{\mathrm{T}}y)$ is the value of ____.

64. Here is a new factorization of A into *triangular times symmetric*:

Start from $A = LDU$. Then A equals $L(U^{\mathrm{T}})^{-1}$ times $U^{\mathrm{T}}DU$.

Why is $L(U^{\mathrm{T}})^{-1}$ triangular? Its diagonal is all 1s. Why is $U^{\mathrm{T}}DU$ symmetric?

65. A *group* of matrices includes AB and A^{-1} if it includes A and B. "Products and inverses stay in the group." Which of these sets are groups? Lower triangularmatrices L with is on the diagonal, symmetric matrices S, positive matrices M, diagonal invertible matrices D, permutation matrices P. Invent two more matrix groups.

66. If every row of a 4 by 4 matrix contains the numbers 0, 1, 2, 3 in some order, can the matrix be symmetric? Can it be invertible?

67. Prove that no reordering of rows and reordering of columns can transpose a typical matrix.

68. A square ***northwest matrix*** B is zero in the southeast corner, below the antidiagonal that connects $(1,n)$ to $(n,1)$. Will B^{T} and B^2 be northwest matrices? Will B^{-1} be northwest or southeast? What is the shape of $BC =$ ***northwest times southeast***? You are allowed to combine permutations with the usual L and U (southwest and northeast).

69. Compare tic; inv(A); toc for A = rand(500) and A = rand(1000). The n^3 count says that computing time (measured by tic; toc) should multiply by 8 when n is doubled. Do you expect these random A to be invertible?

70. $I =$ eye(1000); $A =$ rand(1000); $B =$ triu(A); produces a random *triangular* matrix B. Compare the times for inv(B) and $B\backslash I$. Backslash is engineered to use the zeros in B, while inv uses the zeros in I when reducing $[B\ \ I]$ by Gauss-Jordan. (Compare also with inv(A) and $A\backslash I$ for the full matrix A.)

71. Show that L^{-1} has entries j/i for $i \le j$ (the -1, 2, -1 matrix has this L):

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 0 & -\frac{2}{3} & 1 & 0 \\ 0 & 0 & -\frac{3}{4} & 1 \end{bmatrix} \quad \text{and} \quad L^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 1 & 0 \\ \frac{1}{4} & \frac{2}{4} & \frac{3}{4} & 1 \end{bmatrix}.$$