Fourth Edition

LINEAR ALGEBRA

AND ITS APPLICATIONS

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38. (a) Find all functions that satisfy $\frac{dy}{dx} = 0$.

(b) Choose a particular function that satisfies $\frac{dy}{dx} = 3$.

(c) Find all functions that satisfy $\frac{dy}{dx} = 3$.

39. The cosine space $\mathbf{F}_3$ contains all combinations $y(x) = A\cos x + B\cos 2x + C\cos 3x$. Find a basis for the subspace that has $y(0) = 0$.

40. Find a basis for the space of functions that satisfy

(a) $\dfrac{dy}{dx} - 2y = 0$.

(b) $\dfrac{dy}{dx} - \dfrac{y}{x} = 0$.

41. Suppose $y_1(x)$, $y_2(x)$, $y_3(x)$ are three different functions of x. The vector space they span could have dimension 1, 2, or 3. Give an example of y_1, y_2, y_3 to show each possibility.

42. Find a basis for the space of polynomials $p(x)$ of degree ≤ 3. Find a basis for the subspace with $p(1) = 0$.

43. Write the 3 by 3 identity matrix as a combination of the other five permutation matrices! Then show that those five matrices are linearly independent. (Assume a combi nation gives zero, and check entries to prove each term is zero.) The five permutations are a basis for the subspace of 3 by 3 matrices with row and column sums all equal.

44. *Review*: Which of the following are bases for $\mathbf{R}^3$?

(a) $(1,2,0)$ and $(0,1,-1)$.

(b) $(1,1,-1)$, $(2,3,4)$, $(4,1,-1)$, $(0,1,-1)$.

(c) $(1,2,2)$, $(-1,2,1)$, $(0,8,0)$.

(d) $(1,2,2)$, $(-1,2,1)$, $(0,8,6)$.

45. *Review*: Suppose A is 5 by 4 with rank 4. Show that $Ax = b$ has no solution when the 5 by 5 matrix $[A \;\; b]$ is invertible. Show that $Ax = b$ is solvable when $[A \;\; b]$ is singular.

2.4 The Four Fundamental Subspaces

The previous section dealt with definitions rather than constructions. We know what a basis is, but not how to find one. Now, starting from an explicit description of a subspace, we would like to compute an explicit basis.

Subspaces can be described in two ways. First, we may be given a set of vectors that span the space. (*Example*: The columns span the column space.) Second, we may be told which conditions the vectors in the space must satisfy. (*Example*: The nullspace consists of all vectors that satisfy $Ax = 0$.)

The first description may include useless vectors (dependent columns). The second description may include repeated conditions (dependent rows). We can't write a basis by inspection, and a systematic procedure is necessary.

The reader can guess what that procedure will be. When elimination on A produces an echelon matrix U or a reduced R, we will find a basis for each of the subspaces associated with A. Then we have to look at the extreme case of **full rank**:

> *When the rank is as large as possible, $r = n$ or $r = m$ or $r = m = n$, the matrix has a* **left-inverse** *B or a* **right-inverse** *C or a* **two-sided** *A^{-1}.*

To organize the whole discussion, we take each of the four subspaces in turn. Two of them are familiar and two are new.

1. The ***column space*** of A is denoted by $\boldsymbol{C}(A)$. Its dimension is the rank r.
2. The ***nullspace*** of A is denoted by $\boldsymbol{N}(A)$. Its dimension is $n - r$.
3. The ***row space*** of A is the *column space* of A^{T}. It is $\boldsymbol{C}(A^{\mathrm{T}})$, and it is spanned by the rows of A. Its dimension is also r.
4. The ***left nullspace*** of A is the *nullspace* of A^{T}. It contains all vectors y such that $A^{\mathrm{T}}y = 0$, and it is written $\boldsymbol{N}(A^{\mathrm{T}})$. Its dimension is ____.

The point about the last two subspaces is that *they come from* A^{T}. If A is an m by n matrix, you can see which "host" spaces contain the four subspaces by looking at the number of components:

> The nullspace $\boldsymbol{N}(A)$ and row space $\boldsymbol{C}(A^{\mathrm{T}})$ are subspaces of $\mathbf{R}^n$.
> The left nullspace $\boldsymbol{N}(A^{\mathrm{T}})$ and column space $\boldsymbol{C}(A)$ are subspaces of $\mathbf{R}^m$.

The rows have n components and the columns have m. For a simple matrix like

$$A = U = R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

the column space is the line through $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$. The row space is the line through $[1 \;\; 0 \;\; 0]^{\mathrm{T}}$. It is in $\mathbf{R}^3$. The nullspace is a plane in $\mathbf{R}^3$ and the left nullspace is a line in $\mathbf{R}^2$:

$$\boldsymbol{N}(A) \quad \text{contains} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \boldsymbol{N}(A^{\mathrm{T}}) \quad \text{contains} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Note that all vectors are column vectors. Even the rows are transposed, and the row space of A is the *column* space of A^{T}, Our problem will be to connect the four spaces for U (after elimination) to the four spaces for A:

Basic example $$U = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{came from} \quad A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}.$$

For novelty, we take the four subspaces in a more interesting order.

3. The row space of A For an echelon matrix like U, the row space is clear. It contains all combinations of the rows, as every row space does—but here the third row contributes nothing. The first two rows are a basis for the row space. A similar rule applies to every echelon matrix U or R, with r pivots and r nonzero rows: ***The nonzero rows are a basis, and the row space has dimension*** r. That makes it easy to deal with the original matrix A.

> **2M** The row space of A has the same dimension r as the row space of U, and it has the same bases, because ***the row spaces of*** A ***and*** U ***(and*** R***) are the same***.

The reason is that each elementary operation leaves the row space unchanged. The rows in U are combinations of the original rows in A. Therefore the row space of U contains nothing new. At the same time, because every step can be reversed, nothing is lost; the rows of A can be recovered from U. It is true that A and U have different rows, but the *combinations* of the rows are identical: *same space*!

Note that we did not start with the m rows of A, which span the row space, and discard $m-r$ of them to end up with a basis. According to 2L, we could have done so. But it might be hard to decide which rows to keep and which to discard, so it was easier just to take the nonzero rows of U.

2. The nullspace of A Elimination simplifies a system of linear equations without changing the solutions. The system $Ax = 0$ is reduced to $Ux = 0$, and this process is reversible. ***The nullspace of*** A ***is the same as the nullspace of*** U ***and*** R. Only r of the equations $Ax = 0$ are independent. Choosing the $n-r$ "special solutions" to $Ax = 0$ provides a definite basis for the nullspace:

> **2N** The nullspace $\boldsymbol{N}(A)$ has dimension $n-r$. The "special solutions" are a basis—each free variable is given the value 1, while the other free variables are 0. Then $Ax = 0$ or $Ux = 0$ or $Rx = 0$ gives the pivot variables by back-substitution.

This is exactly the way we have been solving $Ux = 0$. The basic example above has pivots in columns 1 and 3. Therefore its free variables are the second and fourth v and y.

The basis for the nullspace is

$$\textbf{Special solutions}\qquad \begin{matrix} v &=& 1 \\ y &=& 0 \end{matrix} \quad x_1 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}; \qquad \begin{matrix} v &=& 0 \\ y &=& 1 \end{matrix} \quad x_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

Any combination $c_1x_1 + c_2x_2$ has c_1 as its v component, and c_2 as its y component. The only way to have $c_1x_1 + c_2x_2 = 0$ is to have $c_1 = c_2 = 0$, so these vectors are independent. They also span the nullspace; the complete solution is $vx_1 + yx_2$. Thus the $n - r = 4 - 2$ vectors are a basis.

The nullspace is also called the *kernel* of A, and its dimension $n - r$ is the *nullity*.

1. The column space of A The column space is sometimes called the **range**. This is consistent with the usual idea of the range, as the set of all possible values $f(x)$; x is in the domain and $f(x)$ is in the range. In our case the function is $f(x) = Ax$. Its domain consists of all x in $\mathbf{R}^n$; its range is all possible vectors Ax, which is the column space. (In an earlier edition of this book we called it $R(A)$.)

Our problem is to find bases for the column spaces of U and A. ***Those spaces are different*** (just look at the matrices!) but their dimensions are the same.

The first and third columns of U are a basis for its column space. They are the ***columns with pivots***. Every other column is a combination of those two. Furthermore, the same is true of the original A—even though its columns are different. ***The pivot columns of*** A ***are a basis for its column space***. The second column is three times the first, just as in U. The fourth column equals (column 3) − (column 1). The same nullspace is telling us those dependencies.

The reason is this: $Ax = 0$ *exactly when* $Ux = 0$. The two systems are equivalent and have the same solutions. The fourth column of U was also (column 3) − (column 1). Every linear dependence $Ax = 0$ among the columns of A is matched by a dependence $Ux = 0$ among the columns of U, with exactly the same coefficients. *If a set of columns of A is independent, then so are the corresponding columns of U, and vice versa.*

To find a basis for the column space $\boldsymbol{C}(A)$, we use what is already done for U. The r columns containing pivots are a basis for the column space of U. We will pick those same r columns in A:

> **2O** The dimension of the column space $\boldsymbol{C}(A)$ equals the rank r, which also equals the dimension of the row space: ***The number of independent columns equals the number of independent rows***. A basis for $\boldsymbol{C}(A)$ is formed by the r columns of A that correspond, in U, to the columns containing pivots.

The row space and the column space have the same dimension r! This is one of the most important theorems in linear algebra. It is often abbreviated as "***row rank*** = ***column rank***." It expresses a result that, for a random 10 by 12 matrix, is not at all obvious. It also says something about square matrices: *If the rows of a square matrix are linearly independent, then so are the columns* (and vice versa). Again, that does not seem self-evident (at least, not to the author).

To see once more that both the row and column spaces of U have dimension r, consider a typical situation with rank $r = 3$. The echelon matrix U certainly has three independent rows:

$$U = \begin{bmatrix} d_1 & * & * & * & * & * \\ 0 & 0 & 0 & d_2 & * & * \\ 0 & 0 & 0 & 0 & 0 & d_3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We claim that U also has three independent columns, and no more, The columns have only three nonzero components. If we can show that the pivot columns—the first, fourth, and sixth—are linearly independent, they must be a basis (for the column space of U, not A!). Suppose a combination of these pivot columns produced zero:

$$c_1 \begin{bmatrix} d_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} * \\ d_2 \\ 0 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} * \\ * \\ d_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Working upward in the usual way, c_3 must be zero because the pivot $d_3 \neq 0$, then c_2 must be zero because $d_2 \neq 0$, and finally $c_1 = 0$. This establishes independence and completes the proof. Since $Ax = 0$ if and only if $Ux = 0$, the first, fourth, and sixth columns of A—whatever the original matrix A was, which we do not even know in this example—are a basis for $\boldsymbol{C}(A)$.

The row space and column space both became clear after elimination on A. Now comes the fourth fundamental subspace, which has been keeping quietly out of sight. Since the first three spaces were $\boldsymbol{C}(A)$, $\boldsymbol{N}(A)$, and $\boldsymbol{C}(A^{\mathrm{T}})$, the fourth space must be $\boldsymbol{N}(A^{\mathrm{T}})$, It is the nullspace of the transpose, or the ***left nullspace*** of A. $A^{\mathrm{T}}y = 0$ means $y^{\mathrm{T}}A = 0$, and the vector appears on the left-hand side of A.

4. The left nullspace of A (= the nullspace of A^{T}) If A is an m by n matrix, then A^{T} is n by m. Its nullspace is a subspace of $\mathbf{R}^m$; the vector y has m components. Written as $y^{\mathrm{T}}A = 0$, those components multiply the *rows* of A to produce the zero row:

$$y^{\mathrm{T}}A = \begin{bmatrix} y_1 & \cdots & y_m \end{bmatrix} \begin{bmatrix} & A & \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 \end{bmatrix}.$$

The dimension of this nullspace $\boldsymbol{N}(A^{\mathrm{T}})$ is easy to find, For *any* matrix, ***the number of pivot variables plus the number of free variables must match the total number of columns***. For A, that was $r + (n - r) = n$. In other words, rank plus nullity equals n:

$$\text{dimension of } \boldsymbol{C}(A) + \text{dimension of } \boldsymbol{N}(A) = \text{number of columns}.$$

This law applies equally to A^{T}, which has m columns. A^{T} is just as good a matrix as A. But the dimension of its column space is also r, so

$$r + \text{dimension } \left(\boldsymbol{N}(A^{\mathrm{T}})\right) = m. \tag{1}$$

2P The left nullspace $\boldsymbol{N}(A^{\mathrm{T}})$ has dimension $m-r$.

The $m-r$ solutions to $y^{\mathrm{T}}A = 0$ are hiding somewhere in elimination. The rows of A combine to produce the $m-r$ *zero rows* of U. Start from $PA = LU$, or $L^{-1}PA = U$. The last $m-r$ rows of the invertible matrix $L^{-1}P$ must be a basis of y's in the left nullspace—because they multiply A to give the zero rows in U.

In our 3 by 4 example, the zero row was row 3 − 2(row 2) + 5(row 1). Therefore the components of y are 5, −2, 1. This is the same combination as in $b_3 - 2b_2 + 5b_1$ on the right-hand side, leading to $0 = 0$ as the final equation. That vector y is a basis for the left nullspace, which has dimension $m - r = 3 - 2 = 1$. It is the last row of $L^{-1}P$, and produces the zero row in U—and we can often see it without computing L^{-1}. When desperate, it is always possible just to solve $A^{\mathrm{T}}y = 0$.

I realize that so far in this book we have given no reason to care about $\boldsymbol{N}(A^{\mathrm{T}})$. It is correct but not convincing if I write in italics that *the left nullspace is also important*. The next section does better by finding a physical meaning for y from Kirchhoff's Current Law.

Now we know the dimensions of the four spaces. We can summarize them in a table, and it even seems fair to advertise them as the

Fundamental Theorem of Linear Algebra, Part I

1. $\boldsymbol{C}(A)$ = column space of A; dimension r.
2. $\boldsymbol{N}(A)$ = nullspace of A; dimension $n-r$.
3. $\boldsymbol{C}(A^{\mathrm{T}})$ = row space of A; dimension r.
4. $\boldsymbol{N}(A^{\mathrm{T}})$ = left nullspace of A; dimension $m-r$.

Example 1. $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ has $m = n = 2$, and rank $r = 1$.

1. The ***column space*** contains all multiples of $\left[\begin{smallmatrix} 1 \\ 3 \end{smallmatrix}\right]$. The second column is in the same direction and contributes nothing new.
2. The ***nullspace*** contains all multiples of $\left[\begin{smallmatrix} -2 \\ 1 \end{smallmatrix}\right]$. This vector satisfies $Ax = 0$.
3. The ***row space*** contains all multiples of $\left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right]$. I write it as a column vector, since strictly speaking it is in the column space of A^{T}.
4. The ***left nullspace*** contains all multiples of $y = \left[\begin{smallmatrix} -3 \\ 1 \end{smallmatrix}\right]$. The rows of A with coefficients −3 and 1 add to zero, so $A^{\mathrm{T}}y = 0$.

In this example *all four subspaces are lines*. That is an accident, coming from $r = 1$ and $n - r = 1$ and $m - r = 1$. Figure 2.5 shows that two pairs of lines are perpendicular. That is no accident!

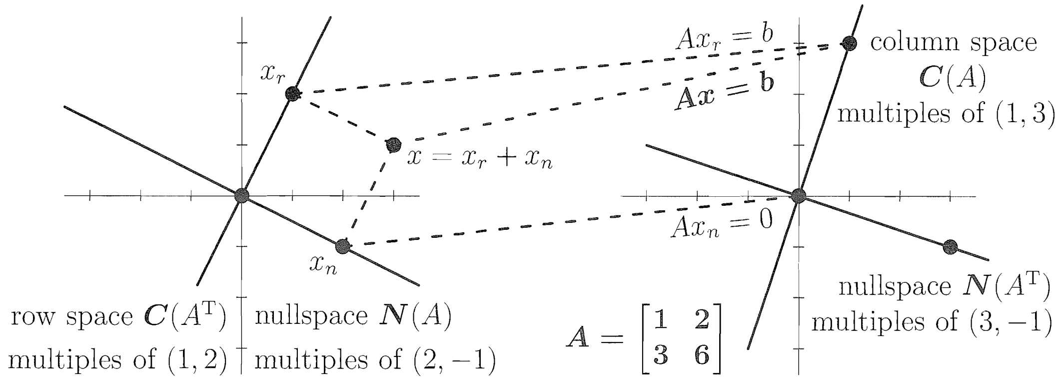

Figure 2.5: The four fundamental subspaces (lines) for the singular matrix A.

If you change the last entry of A from 6 to 7, all the dimensions are different. The column space and row space have dimension $r = 2$. The nullspace and left nullspace contain only the vectors $x = 0$ and $y = 0$. *The matrix is invertible.*

Existence of Inverses

We know that if A has a left-inverse ($BA = I$) and a right-inverse ($AC = I$), then the two inverses are equal: $B = B(AC) = (BA)C = C$. Now, from the rank of a matrix, it is easy to decide which matrices actually have these inverses. Roughly speaking, ***an inverse exists only when the rank is as large as possible***.

The rank always satisfies $r \le m$ and also $r \le n$. An m by n matrix cannot have more than m independent rows or n independent columns. There is not space for more than m pivots, or more than n. We want to prove that when $r = m$ there is a right-inverse, and $Ax = b$ always has a solution. When $r = n$ there is a left-inverse, and the solution (*if it exists*) is unique.

Only a square matrix can have both $r = m$ and $r = n$, and therefore only a square matrix can achieve both existence and uniqueness. Only a square matrix has a two-sided inverse.

2Q EXISTENCE: Full row rank $r = m$. $Ax = b$ has ***at least*** one solution x for every b if and only if the columns span $\mathbf{R}^m$. Then A has a ***right-inverse*** C such that $AC = I_m$ (m by m). This is possible only if $m \le n$.

UNIQUENESS: Full column rank $r = n$. $Ax = b$ has ***at most*** one solution x for every b if and only if the columns are linearly independent. Then A has an n by m **left-inverse** B such that $BA = I_n$. This is possible only if $m \ge n$.

In the existence case, one possible solution is $x = Cb$, since then $Ax = ACb = b$. But there will be other solutions if there are other right-inverses. The number of solutions when the columns span $\mathbf{R}^m$ is 1 or ∞.

In the uniqueness case, if there is a solution to $Ax = b$, it has to be $x = BAx = Bb$. But there may be no solution. The number of solutions is 0 or 1.

There are simple formulas for the best left and right inverses, if they exist:

$$\textbf{One-sided inverses} \qquad B = (A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}} \quad \text{and} \quad C = A^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}.$$

Certainly $BA = I$ and $AC = I$. What is not so certain is that $A^{\mathrm{T}}A$ and AA^{T} are actually invertible. We show in Chapter 3 that $A^{\mathrm{T}}A$ does have an inverse if the rank is n, and AA^{T} has an inverse when the rank is m. Thus the formulas make sense exactly when the rank is as large as possible, and the one-sided inverses are found.

Example 2. Consider a simple 2 by 3 matrix of rank 2:

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix}.$$

Since $r = m = 2$, the theorem guarantees a right-inverse C:

$$AC = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{5} \\ c_{31} & c_{32} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

There are many right-inverses because the last row of C is completely arbitrary. This is a case of existence but not uniqueness. The matrix A has no left-inverse because the last column of BA is certain to be zero. The specific right-inverse $C = A^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}$ chooses c_{31} and c_{32} to be zero:

$$\textbf{Best right-inverse} \qquad A^{\mathrm{T}}(AA^{\mathrm{T}})^{-1} = \begin{bmatrix} 4 & 0 \\ 0 & 5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{16} & 0 \\ 0 & \frac{1}{25} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{5} \\ 0 & 0 \end{bmatrix} = C.$$

This is the *pseudoinverse*—a way of choosing the best C in Section 6.3. The transpose of A yields an example with infinitely many *left*-inverses:

$$BA^{\mathrm{T}} = \begin{bmatrix} \frac{1}{4} & 0 & b_{13} \\ 0 & \frac{1}{5} & b_{23} \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Now it is the last column of B that is completely arbitrary. The best left-inverse (also the pseudoinverse) has $b_{13} = b_{23} = 0$. This is a "uniqueness case," when the rank is $r = n$. There are no free variables, since $n - r = 0$. If there is a solution it will be the only one. You can see when this example has one solution or no solution:

$$\begin{bmatrix} 4 & 0 \\ 0 & 5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \qquad \text{is solvable exactly when} \quad b_3 = 0.$$

A rectangular matrix cannot have both existence and uniqueness. If m is different from n, we cannot have $r = m$ and $r = n$.

A square matrix is the opposite. If $m = n$, we cannot have one property *without* the other. A square matrix has a left-inverse if and only if it has a right-inverse. There is only one inverse, namely $B = C = A^{-1}$. *Existence implies uniqueness and uniqueness implies existence, when the matrix is square*. The condition for invertibility is **full rank**: $r = m = n$. Each of these conditions is a necessary and sufficient test:

1. The columns span $\mathbf{R}^n$, so $Ax = b$ has at least one solution for every b.

2. The columns are independent, so $Ax = 0$ has only the solution $x = 0$.

This list can be made much longer, especially if we look ahead to later chapters. Every condition is equivalent to every other, and ensures that A is invertible.

3. The rows of A span $\mathbf{R}^n$.

4. The rows are linearly independent.

5. Elimination can be completed: $PA = LDU$, with all n pivots.

6. The determinant of A is not zero.

7. Zero is not an eigenvalue of A.

8. $A^{\mathrm{T}}A$ is positive definite.

Here is a typical application to polynomials $P(t)$ of degree $n-1$. The only such polynomial that vanishes at $t_1, \ldots, t_n$ is $P(t) \equiv 0$. No other polynomial of degree $n-1$ can have n roots. This is uniqueness, and it implies existence: Given any values $b_1, \ldots, b_n$, there *exists* a polynomial of degree $n-1$ interpolating these values: $P(t_i) = b_i$. The point is that we are dealing with a square matrix; the number n of coefficients in $P(t) = x_1 + x_2 t + \cdots + x_n t^{n-1}$ matches the number of equations:

Interpolation $P(t_i) = b_i$
$$\begin{bmatrix} 1 & t_1 & t_1^2 & \cdots & t_1^{n-1} \\ 1 & t_2 & t_2^2 & \cdots & t_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & t_n & t_n^2 & \cdots & t_n^{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

That *Vandermonde matrix* is n by n and full rank. $Ax = b$ always has a solution—a polynomial can be passed through any b_i at distinct points t_i. Later we shall actually find the determinant of A; it is not zero.

Matrices of Rank 1

Finally comes the easiest case, when the rank is as *small* as possible (except for the zero matrix with rank 0), One basic theme of mathematics is, given something complicated, to show how it can be broken into simple pieces. For linear algebra, the simple pieces are matrices of ***rank* 1**:

$$\textbf{Rank 1} \qquad A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \\ 8 & 4 & 4 \\ -2 & -1 & -1 \end{bmatrix} \quad \text{has} \quad r = 1.$$

Every row is a multiple of the first row, so the row space is one-dimensional. In fact, we can write the whole matrix *as the product of a column vector and a row vector*:

$$A = (\textbf{column})(\textbf{row}) \qquad \begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \\ 8 & 4 & 4 \\ -2 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \\ -1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \end{bmatrix}.$$

The product of a 4 by 1 matrix and a 1 by 3 matrix is a 4 by 3 matrix. *This product has rank* 1. At the same time, the columns are all multiples of the same column vector; the column space shares the dimension $r = 1$ and reduces to a line.

Every matrix of rank* 1 *has the simple form* $A = uv^{\mathrm{T}}$ = *column times row.

The rows are all multiples of the same vector v^{T}, and the columns are all multiples of u. The row space and column space are lines—the easiest case.

Problem Set 2.4

1. True or false: If $m = n$, then the row space of A equals the column space. If $m < n$, then the nullspace has a larger dimension than ____.

2. Find the dimension and construct a basis for the four subspaces associated with each of the matrices

$$A = \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 2 & 8 & 0 \end{bmatrix} \qquad \text{and} \qquad U = \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

3. Find the dimension and a basis for the four fundamental subspaces for

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad U = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

4. Describe the four subspaces in three-dimensional space associated with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

5. If the product AB is the zero matrix, $AB = 0$, show that the column space of B is contained in the nullspace of A. (Also the row space of A is in the left nullspace of B, since each row of A multiplies B to give a zero row.)

6. Suppose A is an m by n matrix of rank r. Under what conditions on those numbers does

(a) A have a two-sided inverse: $AA^{-1} = A^{-1}A = I$?

(b) $Ax = b$ have *infinitely many solutions* for *every* b?

7. Why is there no matrix whose row space and nullspace both contain $(1,1,1)$?

8. Suppose the only solution to $Ax = 0$ (m equations in n unknowns) is $x = 0$. What is the rank and why? The columns of A are linearly ____.

9. Find a 1 by 3 matrix whose nullspace consists of all vectors in $\mathbf{R}^3$ such that $x_1 + 2x_2 + 4x_3 = 0$. Find a 3 by 3 matrix with that same nullspace.

10. If $Ax = b$ always has at least one solution, show that the only solution to $A^{\mathrm{T}}y = 0$ is $y = 0$. *Hint*: What is the rank?

11. If $Ax = 0$ has a nonzero solution, show that $A^{\mathrm{T}}y = f$ fails to be solvable for some right-hand sides f. Construct an example of A and f.

12. Find the rank of A and write the matrix as $A = uv^{\mathrm{T}}$:

$$A = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 6 \end{bmatrix} \qquad \text{and} \qquad A = \begin{bmatrix} 2 & -2 \\ 6 & -6 \end{bmatrix}.$$

13. If a, b, c are given with $a \neq 0$, choose d so that

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = uv^{\mathrm{T}}$$

has rank 1. What are the pivots?

14. Find a left-inverse and/or a right-inverse (when they exist) for

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}.$$

15. If the columns of A are linearly independent (A is m by n), then the rank is ____, the nullspace is ____, the row space is ____, and there exists a ____-inverse.

16. (*A paradox*) Suppose A has a right-inverse B. Then $AB = I$ leads to $A^{\mathrm{T}}AB = A^{\mathrm{T}}$ or $B(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$. But that satisfies $BA = I$; it is a *left*-inverse. Which step is not justified?

17. Find a matrix A that has $\mathbf{V}$ as its row space, and a matrix B that has $\mathbf{V}$ as its nullspace, if $\mathbf{V}$ is the subspace spanned by

$$\begin{bmatrix}1\\1\\0\end{bmatrix}, \quad \begin{bmatrix}1\\2\\0\end{bmatrix}, \quad \begin{bmatrix}1\\5\\0\end{bmatrix}.$$

18. Find a basis for each of the four subspaces of

$$A = \begin{bmatrix}0&1&2&3&4\\0&1&2&4&6\\0&0&0&1&2\end{bmatrix} = \begin{bmatrix}1&0&0\\1&1&0\\0&1&1\end{bmatrix}\begin{bmatrix}0&1&2&3&4\\0&0&0&1&2\\0&0&0&0&0\end{bmatrix}.$$

19. If A has the same four fundamental subspaces as B, does $A = cB$?

20. (a) If a 7 by 9 matrix has rank 5, what are the dimensions of the four subspaces? What is the sum of all four dimensions?

(b) If a 3 by 4 matrix has rank 3, what are its column space and left nullspace?

21. Construct a matrix with the required property, or explain why you can't.

(a) Column space contains $\left[\begin{smallmatrix}1\\1\\0\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right]$, row space contains $\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}2\\5\end{smallmatrix}\right]$.

(b) Column space has basis $\left[\begin{smallmatrix}1\\2\\3\end{smallmatrix}\right]$, nullspace has basis $\left[\begin{smallmatrix}3\\2\\1\end{smallmatrix}\right]$.

(c) Dimension of nullspace $= 1+$ dimension of left nullspace.

(d) Left nullspace contains $\left[\begin{smallmatrix}1\\3\end{smallmatrix}\right]$, row space contains $\left[\begin{smallmatrix}3\\1\end{smallmatrix}\right]$.

(e) Row space $=$ column space, nullspace $\neq$ left nullspace.

22. Without elimination, find dimensions and bases for the four subspaces for

$$A = \begin{bmatrix}0&3&3&3\\0&0&0&0\\0&1&0&1\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}1&1\\4&4\\5&5\end{bmatrix}.$$

23. Suppose the 3 by 3 matrix A is invertible. Write bases for the four subspaces for A, and also for the 3 by 6 matrix $B = [A \;\; A]$.

24. What are the dimensions of the four subspaces for A, B, and C, if I is the 3 by 3 identity matrix and 0 is the 3 by 2 zero matrix?

$$A = \begin{bmatrix}I & 0\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}I & I\\0^{\mathrm{T}} & 0^{\mathrm{T}}\end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix}0\end{bmatrix}.$$

25. Which subspaces are the same for these matrices of different sizes?

$$\text{(a)}\quad \begin{bmatrix} A \end{bmatrix} \quad\text{and}\quad \begin{bmatrix} A \\ A \end{bmatrix}. \qquad\qquad \text{(b)}\quad \begin{bmatrix} A \\ A \end{bmatrix} \quad\text{and}\quad \begin{bmatrix} A & A \\ A & A \end{bmatrix}.$$

Prove that all three matrices have the same rank r.

26. If the entries of a 3 by 3 matrix are chosen randomly between 0 and 1, what are the most likely dimensions of the four subspaces? What if the matrix is 3 by 5?

27. (Important) A is an m by n matrix of rank r. Suppose there are right-hand sides b for which $Ax = b$ has *no solution*.

(a) What inequalities ($<$ or $\leq$) must be true between m, n, and r?

(b) How do you know that $A^{\mathrm{T}}y = 0$ has a nonzero solution?

28. Construct a matrix with $(1,0,1)$ and $(1,2,0)$ as a basis for its row space and its column space. Why can't this be a basis for the row space and nullspace?

29. Without computing A, find bases for the four fundamental subspaces:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 9 & 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

30. If you exchange the first two rows of a matrix A, which of the four subspaces stay the same? If $y = (1,2,3,4)$ is in the left nullspace of A, write down a vector in the left nullspace of the new matrix.

31. Explain why $v = (1,0,-1)$ cannot be a row of A and also be in the nullspace.

32. Describe the four subspaces of $\mathbf{R}^3$ associated with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad\text{and}\quad I + A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

33. (Left nullspace) Add the extra column b and reduce A to echelon form:

$$\begin{bmatrix} A & b \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 4 & 5 & 6 & b_2 \\ 7 & 8 & 9 & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 0 & -3 & -6 & b_2 - 4b_1 \\ 0 & 0 & 0 & b_3 - 2b_2 + b_1 \end{bmatrix}.$$

A combination of the rows of A has produced the zero row. What combination is it? (Look at $b_3 - 2b_2 + b_1$ on the right-hand side.) Which vectors are in the nullspace of A^{T} and which are in the nullspace of A?

34. Following the method of Problem 33, reduce A to echelon form and look at zero rows. The b column tells which combinations you have taken of the rows:

$$\text{(a)} \quad \begin{bmatrix} 1 & 2 & b_1 \\ 3 & 4 & b_2 \\ 4 & 6 & b_3 \end{bmatrix}. \qquad\qquad \text{(b)} \quad \begin{bmatrix} 1 & 2 & b_1 \\ 2 & 3 & b_2 \\ 2 & 4 & b_3 \\ 2 & 5 & b_4 \end{bmatrix}.$$

From the b column after elimination, read off $m-r$ basis vectors in the left nullspace of A (combinations of rows that give zero).

35. Suppose A is the sum of two matrices of rank one: $A = uv^{\mathrm{T}} + wz^{\mathrm{T}}$.

(a) Which vectors span the column space of A?

(b) Which vectors span the row space of A?

(c) The rank is less than 2 if ____ or if ____.

(d) Compute A and its rank if $u = z = (1,0,0)$ and $v = w = (0,0,1)$.

36. Without multiplying matrices, find bases for the row and column spaces of A:

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \end{bmatrix}.$$

How do you know from these shapes that A is not invertible?

37. True or false (with a reason or a counterexample)?

(a) A and A^{T} have the same number of pivots.

(b) A and A^{T} have the same left nullspace.

(c) If the row space equals the column space then $A^{\mathrm{T}} = A$.

(d) If $A^{\mathrm{T}} = -A$ then the row space of A equals the column space.

38. If $AB = 0$, the columns of B are in the nullspace of A. If those vectors are in $\mathbf{R}^n$, prove that $\operatorname{rank}(A) + \operatorname{rank}(B) \leq n$.

39. Can tic-tac-toe be completed (5 ones and 4 zeros in A) so that $\operatorname{rank}(A) = 2$ but neither side passed up a winning move?

40. Construct any 2 by 3 matrix of rank 1. Copy Figure 2.5 and put one vector in each subspace (two in the nullspace). Which vectors are orthogonal?

41. Redraw Figure 2.5 for a 3 by 2 matrix of rank $r = 2$. Which subspace is $\boldsymbol{Z}$ (zero vector only)? The nullspace part of any vector x in $\mathbf{R}^2$ is $x_n =$ ____.

16. If there is an edge between every pair of nodes (a complete graph), how many edges are there? The graph has n nodes, and edges from a node to itself are not allowed.

17. For both graphs drawn below, verify *Euler's formula*:

$$\textbf{(\# of nodes)} - \textbf{(\# of edges)} + \textbf{(\# of loops)} = \mathbf{1}.$$

18. Multiply matrices to find $A^{\mathrm{T}}A$, and guess how its entries come from the graph:

(a) The diagonal of $A^{\mathrm{T}}A$ tells how many ____ into each node.

(b) The off-diagonals -1 or 0 tell which pairs of nodes are ____.

19. Why does the nullspace of $A^{\mathrm{T}}A$ contain $(1,1,1,1)$? What is its rank?

20. Why does a complete graph with $n = 6$ nodes have $m = 15$ edges? A spanning tree connecting all six nodes has ____ edges. There are $n^{n-2} = 6^4$ spanning trees!

21. The *adjacency matrix* of a graph has $M_{ij} = 1$ if nodes i and j are connected by an edge (otherwise $M_{ij} = 0$). For the graph in Problem 6 with 6 nodes and 4 edges, write down M and also M^2. Why does $(M^2)_{ij}$ count the number of 2-*step paths* from node i to node j?

2.6 Linear Transformations

We know how a matrix moves subspaces around when we multiply by A. The nullspace goes into the zero vector. All vectors go into the column space, since Ax is always a combination of the columns. You will soon see something beautiful—that A takes its row space into its column space, and on those spaces of dimension r it is 100 percent invertible. That is the real action of A. It is partly hidden by nullspaces and left nullspaces, which lie at right angles and go their own way (toward zero).

What matters now is what happens *inside* the space—which means inside n-dimensional space, if A is n by n. That demands a closer look.

Suppose x is an n-dimensional vector. When A multiplies x, it ***transforms*** that vector into a new vector Ax. This happens at every point x of the n-dimensional space $\mathbf{R}^n$. The whole space is transformed, or "mapped into itself," by the matrix A. Figure 2.9 illustrates four transformations that come from matrices:

$$A = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$$

1. A multiple of the identity matrix, $A = cI$, ***stretches*** every vector by the same factor c. The whole space expands or contracts (or somehow goes through the origin and out the opposite side, when c is negative).

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

2. A ***rotation*** matrix turns the whole space around the origin. This example turns all vectors through 90°, transforming every point (x, y) to $(-y, x)$.

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

3. A ***reflection*** matrix transforms every vector into its image on the opposite side of a mirror. In this example the mirror is the 45° line $y = x$, and a point like $(2,2)$ is unchanged. A point like $(2,-2)$ is reversed to $(-2,2)$. On a combination like $v = (2,2) + (2,-2) = (4,0)$, the matrix leaves one part and reverses the other part. The output is $Av = (2,2) + (-2,2) = (0,4)$
That reflection matrix is also a permutation matrix! It is algebraically so simple, sending (x,y) to (y,x), that the geometric picture was concealed.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

4. A ***projection*** matrix takes the whole space onto a lower-dimensional subspace (not invertible). The example transforms each vector (x,y) in the plane to the nearest point $(x,0)$ on the horizontal axis. That axis is the column space of A. The y-axis that projects to $(0,0)$ is the nullspace.

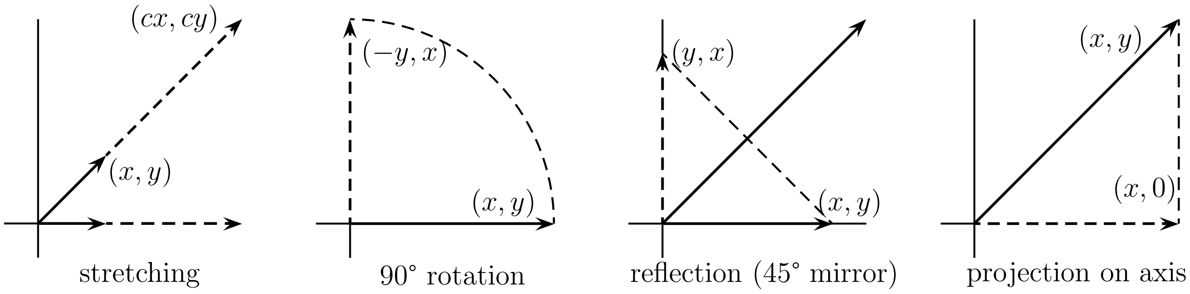

Figure 2.9: Transformations of the plane by four matrices.

Those examples could be lifted into three dimensions. There are matrices to stretch the earth or spin it or reflect it across the plane of the equator (forth pole transforming to south pole). There is a matrix that projects everything onto that plane (both poles to the center). It is also important to recognize that matrices cannot do everything, and some transformations $T(x)$ are *not possible* with Ax:

(i) It is impossible to move the origin, since $A0 = 0$ for every matrix.

(ii) If the vector x goes to x', then $2x$ must go to $2x'$. in general cx must go to cx', since $A(cx) = c(Ax)$.

(iii) If the vectors x and y go to x' and y', then their sum $x+y$ must go to $x'+y'$—since $A(x+y) = Ax + Ay$.

Matrix multiplication imposes those rules on the transformation. The second rule contains the first (take $c = 0$ to get $A0 = 0$). We saw rule (iii) in action when $(4,0)$ was reflected across the 45° line. It was split into $(2,2)+(2,-2)$ and the two parts were reflected separately. The same could be done for projections: split, project separately, and add the projections. These rules apply to ***any transformation that comes from a matrix***.

Their importance has earned them a name: Transformations that obey rules (i)–(iii) are called ***linear transformations***. The rules can be combined into one requirement:

> **2T** For all numbers c and d and all vectors x and y, matrix multiplication satisfies the rule of linearity:
>
> $$A(cx+dy) = c(Ax)+d(Ay). \tag{1}$$
>
> Every transformation $T(x)$ that meets this requirement is a ***linear transformation***.

Any matrix leads immediately to a linear transformation. The more interesting question is in the opposite direction: *Does every linear transformation lead to a matrix*? The object of this section is to find the answer, yes. This is the foundation of an approach to linear algebra—starting with property (1) and developing its consequences—that is much more abstract than the main approach in this book. We preferred to begin directly with matrices, and now we see how they represent linear transformations.

A transformation need not go from $\mathbf{R}^n$ to the same space $\mathbf{R}^n$. It is absolutely permitted to transform vectors in $\mathbf{R}^n$ to vectors in a different space $\mathbf{R}^m$. That is exactly what is done by an m by n matrix! The original vector x has n components, and the transformed vector Ax has m components. The rule of linearity is equally satisfied by rectangular matrices, so they also produce linear transformations.

Having gone that far, there is no reason to stop. The operations in the linearity condition (1) are addition and scalar multiplication, but x and y need not be column vectors in $\mathbf{R}^n$. Those are not the only spaces. By definition, *any vector space allows the combinations $cx+dy$*—the "vectors" are x and y, but they may actually be polynomials or matrices or functions $x(t)$ and $y(t)$. As long as the transformation satisfies equation (1), it is linear.

We take as examples the spaces $\mathbf{P}_n$, in which the vectors are polynomials $p(t)$ of degree n. They look like $p = a_0 + a_1 t + \cdots + a_n t^n$, and the dimension of the vector space is $n+1$ (because with the constant term, there are $n+1$ coefficients).

Example 1. The operation of ***differentiation***, $A = d/dt$, is linear:

$$Ap(t) = \frac{d}{dt}(a_0 + a_1 t + \cdots + a_n t^n) = a_1 + \cdots + n a_n t^{n-1}. \tag{2}$$

The nullspace of this A is the one-dimensional space of constants: $da_0/dt = 0$. The column space is the n-dimensional space $\mathbf{P}_{n-1}$; the right-hand side of equation (2) is always in that space. The sum of nullity ($=1$) and rank ($=n$) is the dimension of the original space $\mathbf{P}_n$.

Example 2. ***Integration*** from 0 to t is also linear (*it takes* $\boldsymbol{P}_n$ *to* $\boldsymbol{P}_{n+1}$):

$$Ap(t) = \int_0^t (a_0 + \cdots + a_n t^n)\,dt = a_0 t + \cdots + \frac{a_n}{n+1} t^{n+1}. \tag{3}$$

This time there is no nullspace (except for the zero vector, as always!) but integration does not produce all polynomials in $\mathbf{P}_{n+1}$. The right side of equation (3) has no constant term. Probably the constant polynomials will be the left nullspace.

Example 3. ***Multiplication*** by a fixed polynomial like $2+3t$ is linear:

$$Ap(t) = (2+3t)(a_0 + \cdots + a_n t^n) = 2a_0 + \cdots + 3a_n t^{n+1}.$$

Again this transforms $\mathbf{P}_n$ to $\mathbf{P}_{n+1}$, with no nullspace except $p = 0$.

In these examples (and in almost all examples), linearity is not difficult to verify. It hardly even seems interesting. If it is there, it is practically impossible to miss. Nevertheless, it is the most important property a transformation can have[1]. Of course most transformations are not linear—for example, to square the polynomial ($Ap = p^2$), or to add 1 ($Ap = p+1$), or to keep the positive coefficients ($A(t-t^2) = t$). It will be linear transformations, and *only those*, that lead us back to matrices.

Transformations Represented by Matrices

Linearity has a crucial consequence: ***If we know*** Ax ***for each vector in a basis, then we know*** Ax ***for each vector in the entire space***. Suppose the basis consists of the n vectors $x_1, \ldots, x_n$. Every other vector x is a combination of those particular vectors (they span the space). Then linearity determines Ax:

$$\textbf{Linearity} \qquad \text{If} \quad x = c_1 x_1 + \cdots + c_n x_n \quad \text{then} \quad Ax = c_1(Ax_1) + \cdots + c_n(Ax_n). \tag{4}$$

The transformation $T(x) = Ax$ has no freedom left, after it has decided what to do with the basis vectors. The rest is determined by linearity. The requirement (1) for two vectors x and y leads to condition (4) for n vectors $x_1, \ldots, x_n$. The transformation does have a free hand with the vectors in the basis (they are independent). When those are settled, the transformation of every vector is settled.

Example 4. What linear transformation takes x_1 and x_2 to Ax_1 and Ax_2?

$$x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{goes to} \quad Ax_1 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}; \quad x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{goes to} \quad Ax_2 = \begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix}.$$

It must be multiplication $T(x) = Ax$ by the matrix

$$A = \begin{bmatrix} 2 & 4 \\ 3 & 6 \\ 4 & 8 \end{bmatrix}.$$

[1] Invertibility is perhaps in second place as an important property.

Starting with a different basis $(1,1)$ and $(2,-1)$, this same A is also the only linear transformation with

$$A\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}6\\9\\12\end{bmatrix} \qquad \text{and} \qquad A\begin{bmatrix}2\\-1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}.$$

Next we find matrices that represent differentiation and integration. ***First we must decide on a basis***. For the polynomials of degree 3 there is a natural choice for the four basis vectors:

$$\textbf{Basis for } \mathbf{P}_3 \qquad p_1=1, \quad p_2=t, \quad p_3=t^2, \quad p_4=t^3.$$

That basis is not unique (it never is), but some choice is necessary and this is the most convenient. The derivatives of those four basis vectors are 0, 1, $2t$, $3t^2$:

$$\textbf{Action of } d/dt \qquad Ap_1=0, \quad Ap_2=p_1, \quad Ap_3=2p_2, \quad Ap_4=3p_3. \tag{5}$$

"d/dt" is acting exactly like a matrix, but which matrix? Suppose we were in the usual four-dimensional space with the usual basis—the coordinate vectors $p_1=(1,0,0,0)$, $p_2=(0,1,0,0)$, $p_3=(0,0,1,0)$, $p_4=(0,0,0,1)$. The matrix is decided by equation (5):

$$\textbf{Differentiation matrix} \qquad A_{\text{diff}}=\begin{bmatrix}0&1&0&0\\0&0&2&0\\0&0&0&3\\0&0&0&0\end{bmatrix}.$$

Ap_1 is its first column, which is zero. Ap_2 is the second column, which is p_1. Ap_3 is $2p_2$ and Ap_4 is $3p_3$. The nullspace contains p_1 (the derivative of a constant is zero). The column space contains p_1, p_2, p_3 (the derivative of a cubic is a quadratic). The derivative of a combination like $p=2+t-t^2-t^3$ is decided by linearity, and there is nothing new about that—it is the way we all differentiate. It would be crazy to memorize the derivative of every polynomial.

The matrix can differentiate that $p(t)$, because matrices build in linearity!

$$\frac{dp}{dt}=Ap \longrightarrow \begin{bmatrix}0&1&0&0\\0&0&2&0\\0&0&0&3\\0&0&0&0\end{bmatrix}\begin{bmatrix}2\\1\\-1\\-1\end{bmatrix}=\begin{bmatrix}1\\-2\\-3\\0\end{bmatrix} \longrightarrow 1-2t-3t^2.$$

In short, *the matrix carries all the essential information*. If the basis is known, and the matrix is known, then the transformation of every vector is known.

The coding of the information is simple. To transform a space to itself, one basis is enough. A transformation from one space to another requires a basis for each.

2U Suppose the vectors $x_1,\dots,x_n$ are a basis for the space $\mathbf{V}$, and vectors $y_1,\dots,y_m$ are a basis for $\mathbf{W}$. Each linear transformation T from $\mathbf{V}$ to $\mathbf{W}$ is represented by a matrix A. The jth column is found by applying T to the jth basis vector x_j, and writing $T(x_j)$ as a combination of the y's:

$$\textbf{Column } j \textbf{ of } A \qquad T(x_j) = Ax_j = a_{1j}y_1 + a_{2j}y_2 + \cdots + a_{mj}y_m. \qquad (6)$$

For the differentiation matrix, column 1 came from the first basis vector $p_1 = 1$. Its derivative is zero, so column 1 is zero. The last column came from $(d/dt)t^3 = 3t^2$. Since $3t^2 = 0p_1 + 0p_2 + 3p_3 + 0p_4$, the last column contained 0, 0, 3. 0. The rule (6) constructs the matrix, a column at a time.

We do the same for integration. That goes from cubics to quartics, transforming $\mathbf{V} = \mathbf{P}_3$ into $\mathbf{W} = \mathbf{P}_4$, so we need a basis for $\mathbf{W}$. The natural choice is $y_1 = 1$, $y_2 = t$, $y_3 = t^2$, $y_4 = t^3$, $y_5 = t^4$, spanning the polynomials of degree 4. The matrix A will be m by n, or 5 by 4. It comes from applying integration to each basis vector of $\mathbf{V}$:

$$\int_0^t 1\,dt = t \quad \text{or} \quad Ax_1 = y_2, \quad \dots, \quad \int_0^t t^3\,dt = \frac{1}{4}t^4 \quad \text{or} \quad Ax_4 = \frac{1}{4}y_5.$$

$$\textbf{Integration matrix} \qquad A_{\text{int}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}.$$

Differentiation and integration are *inverse operations*. Or at least integration *followed* by differentiation brings back the original function. To make that happen for matrices, we need the differentiation matrix from quartics down to cubics, which is 4 by 5:

$$A_{\text{diff}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix} \qquad \text{and} \qquad A_{\text{diff}}A_{\text{int}} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

Differentiation is a ***left-inverse*** of integration. Rectangular matrices cannot have two-sided inverses! In the opposite order, $A_{\text{int}}A_{\text{diff}} = I$ cannot be true. The 5 by 5 product has zeros in column 1. The derivative of a constant is zero. In the other columns $A_{\text{int}}A_{\text{diff}}$ is the identity and the integral of the derivative of t^n is t^n.

Rotations Q, Projections P, and Reflections H

This section began with 90° rotations, projections onto the x-axis, and reflections through the 45° line. Their matrices were especially simple:

$$\underset{\text{(rotation)}}{Q=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}} \qquad \underset{\text{(projection)}}{P=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}} \qquad \underset{\text{(reflection)}}{H=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}}.$$

The underlying linear transformations of the x-y plane are also simple. But rotations through other angles, projections onto other lines, and reflections in other mirrors are almost as easy to visualize, They are still linear transformations, provided that the origin is fixed: $A0=0$. They *must* be represented by matrices. Using the natural basis $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$, we want to discover those matrices.

1. **Rotation** Figure 2.10 shows rotation through an angle θ. It also shows the effect on the two basis vectors. The first one goes to $(\cos\theta,\sin\theta)$, whose length is still 1; it lies on the "θ-line." The second basis vector $(0,1)$ rotates into $(-\sin\theta,\cos\theta)$. By rule (6), those numbers go into the columns of the matrix (we use c and s for $\cos\theta$ and $\sin\theta$). This family of rotations Q_θ is a perfect chance to test the correspondence between transformations and matrices:

   *Does the **inverse** of Q_θ equal $Q_{-\theta}$* (rotation backward through θ)? *Yes.*

$$Q_\theta Q_{-\theta}=\begin{bmatrix}c & -s\\ s & c\end{bmatrix}\begin{bmatrix}c & s\\ -s & c\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}.$$

   *Does the **square** of Q_θ equal $Q_{2\theta}$* (rotation through a double angle)? *Yes.*

$$Q_\theta^2=\begin{bmatrix}c & -s\\ s & c\end{bmatrix}\begin{bmatrix}c & -s\\ s & c\end{bmatrix}=\begin{bmatrix}c^2-s^2 & -2cs\\ 2cs & c^2-s^2\end{bmatrix}=\begin{bmatrix}\cos 2\theta & -\sin 2\theta\\ \sin 2\theta & \cos 2\theta\end{bmatrix}.$$

   *Does the **product** of Q_θ and Q_φ equal $Q_{\theta+\varphi}$* (rotation through θ then φ)? *Yes.*

$$Q_\theta Q_\varphi=\begin{bmatrix}\cos\theta\cos\varphi-\sin\theta\sin\varphi & \cdots\\ \sin\theta\cos\varphi+\cos\theta\sin\varphi & \cdots\end{bmatrix}=\begin{bmatrix}\cos(\theta+\varphi) & \cdots\\ \sin(\theta+\varphi) & \cdots\end{bmatrix}.$$

   The last case contains the first two. The inverse appears when φ is $-\theta$, and the square appears when φ is $+\theta$. All three questions were decided by trigonometric identities (and they give a new way to remember those identities). It was no accident that all the answers were yes. *Matrix multiplication is defined exactly so that **the product of the matrices corresponds to the product of the transformations**.*

   2V Suppose A and B are linear transformations from **V** to **W** and from **U** to **V**. Their product AB starts with a vector u in **U**, goes to Bu in **V**, and

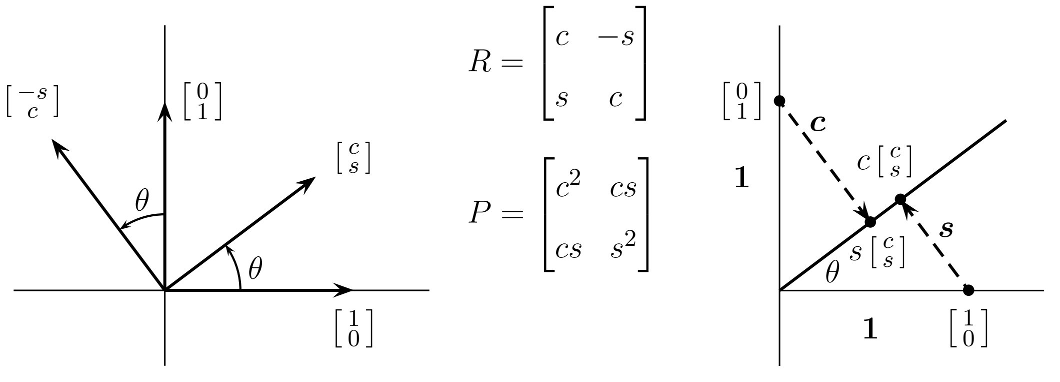

Figure 2.10: Rotation through θ (left). Projection onto the θ-line (right).

finishes with ABu in $\mathbf{W}$. This "composition" AB is again a linear transformation (from $\mathbf{U}$ to $\mathbf{W}$). Its matrix is the product of the individual matrices representing A and B.

For $A_{\text{diff}}A_{\text{int}}$, the composite transformation was the identity (and $A_{\text{int}}A_{\text{diff}}$ annihilated all constants). For rotations, the order of multiplication does not matter. Then $\mathbf{U} = \mathbf{V} = \mathbf{W}$ is the x-y plane, and $Q_\theta Q_\varphi$ is the same as $Q_\varphi Q_\theta$. For a rotation and a reflection, the order makes a difference.

Technical note: To construct the matrices, we need bases for $\mathbf{V}$ and $\mathbf{W}$, and then for $\mathbf{U}$ and $\mathbf{V}$. By keeping the same basis for $\mathbf{V}$, the product matrix goes correctly from the basis in $\mathbf{U}$ to the basis in $\mathbf{W}$. If we distinguish the transformation A from its matrix (call that $[A]$), then the product rule $2V$ becomes extremely concise: $[AB] = [A][B]$. The rule for multiplying matrices in Chapter 1 was totally determined by this requirement—it must match the product of linear transformations.

2. Projection Figure 2.10 also shows the projection of $(1,0)$ onto the θ-line. The length of the projection is $c = \cos\theta$. Notice that the *point* of projection is not (c,s), as I mistakenly thought; that vector has length 1 (it is the rotation), so we must multiply by c. Similarly the projection of $(0,1)$ has length s, and falls at $s(c,s) = (cs, s^2)$, That gives the second column of the projection matrix P:

$$\textbf{Projection onto } \boldsymbol{\theta}\textbf{-line} \qquad P = \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix}.$$

This matrix has no inverse, because the transformation has no inverse. Points on the perpendicular line are projected onto the origin; that line is the nullspace of P. Points on the θ-line are projected to themselves! Projecting twice is the same as projecting once, and $P^2 = P$:

$$P^2 = \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix}^2 = \begin{bmatrix} c^2(c^2+s^2) & cs(c^2+s^2) \\ cs(c^2+s^2) & s^2(c^2+s^2) \end{bmatrix} = P.$$

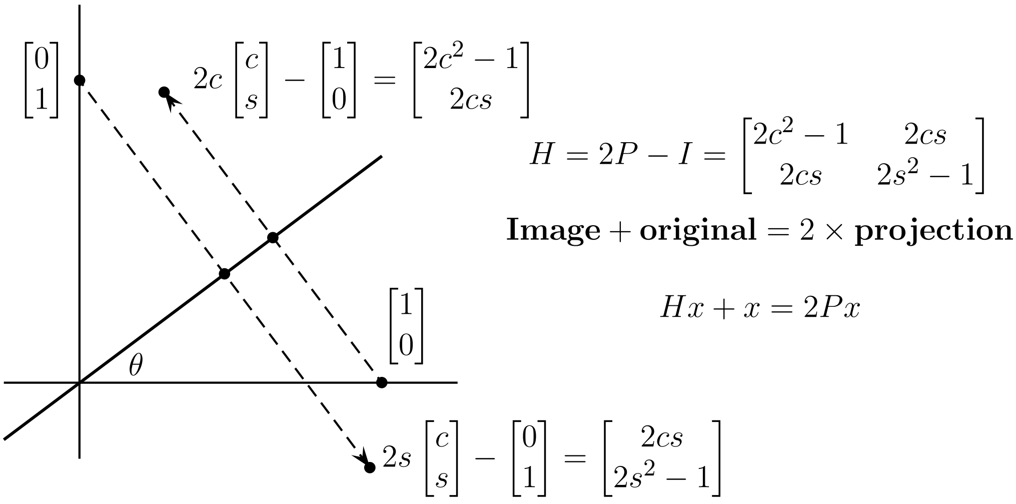

Figure 2.11: Reflection through the θ-line: the geometry and the matrix.

Of course $c^2+s^2=\cos^2\theta+\sin^2\theta=1$. ***A projection matrix equals its own square***.

3. **Reflection** Figure 2.11 shows the reflection of $(1,0)$ in the θ-line. The length of the reflection equals the length of the original, as it did after rotation—but here the θ-line stays where it is. The perpendicular line reverses direction; all points go straight through the mirror, Linearity decides the rest.

$$\textbf{Reflection matrix} \qquad H=\begin{bmatrix}2c^2-1 & 2cs\\2cs & 2s^2-1\end{bmatrix}.$$

This matrix H has the remarkable property $H^2=I$. ***Two reflections bring back the original***. A reflection is its own inverse, $H=H^{-1}$, which is clear from the geometry but less clear from the matrix. One approach is through the relationship of reflections to projections: $H=2P-I$. This means that $Hx+x=2Px$—the image plus the original equals twice the projection. It also confirms that $H^2=I$:

$$H^2=(2P-I)^2=4P^2-4P+I=I, \qquad \text{since} \quad P^2=P.$$

Other transformations Ax can increase the length of x; stretching and shearing are in the exercises. Each example has a matrix to represent it—which is the main point of this section. But there is also the question of choosing a basis, and we emphasize that *the matrix depends on the choice of basis*. Suppose the first basis vector is ***on the θ-line*** and the second basis vector is ***perpendicular***:

(i) The projection matrix is back to $P=\left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right]$. This matrix is constructed as always: its first column comes from the first basis vector (projected to itself). The second column comes from the basis vector that is projected to zero.

(ii) For reflections, that same basis gives $H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. The second basis vector is reflected onto its negative, to produce this second column. The matrix H is still $2P-I$ when the same basis is used for H and P.

(iii) For rotations, the matrix is not changed. Those lines are still rotated through θ, and $Q = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$ as before.

The whole question of choosing the best basis is absolutely central, and we come back to it in Chapter 5. The goal is to make the matrix diagonal, as achieved for P and H. To make Q diagonal requires complex vectors, since all real vectors are rotated.

We mention here the effect on the matrix of a change of basis, while the linear transformation stays the same. ***The matrix*** A (or Q or P or H) ***is altered to*** $S^{-1}AS$. Thus a single transformation is represented by different matrices (via different bases, accounted for by S). The theory of eigenvectors will lead to this formula $S^{-1}AS$, and to the best basis.

Problem Set 2.6

1. What matrix has the effect of rotating every vector through 90° and then projecting the result onto the x-axis? What matrix represents projection onto the x-axis followed by projection onto the y-axis?

2. Does the product of 5 reflections and 8 rotations of the x-y plane produce a rotation or a reflection?

3. The matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ produces a ***stretching*** in the x-direction. Draw the circle $x^2 + y^2 = 1$ and sketch around it the points $(2x, y)$ that result from multiplication by A. What shape is that curve?

4. Every straight line remains straight after a linear transformation. If z is halfway between x and y, show that Az is halfway between Ax and Ay.

5. The matrix $A = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$ yields a ***shearing*** transformation, which leaves the y-axis unchanged. Sketch its effect on the x-axis, by indicating what happens to $(1,0)$ and $(2,0)$ and $(-1,0)$—and how the whole axis is transformed.

6. What 3 by 3 matrices represent the transformations that

(a) project every vector onto the x-y plane?

(b) reflect every vector through the x-y plane?

(c) rotate the x-y plane through 90°, leaving the z-axis alone?

(d) rotate the x-y plane, then x-z, then y-z, through 90°?

(e) carry out the same three rotations, but each one through 180°?

7. On the space $\mathbf{P}_3$ of cubic polynomials, what matrix represents d^2/dt^2? Construct the 4 by 4 matrix from the standard basis 1, t, t^2, t^3. Find its nullspace and column space. What do they mean in terms of polynomials?

8. From the cubics $\mathbf{P}_3$ to the fourth-degree polynomials $\mathbf{P}_4$, what matrix represents multiplication by $2+3t$? The columns of the 5 by 4 matrix A come from applying the transformation to $1, t, t^2, t^3$.

9. The solutions to the linear differential equation $d^2u/dt^2 = u$ form a vector space (since combinations of solutions are still solutions). Find two independent solutions, to give a basis for that solution space.

10. With initial values $u = x$ and $du/dt = y$ at $t = 0$, what combination of basis vectors in Problem 9 solves $u'' = u$? This transformation from initial values to solution is linear. What is its 2 by 2 matrix (using $x = 1$, $y = 0$ and $x = 0$, $y = 1$ as basis for $\mathbf{V}$, and your basis for $\mathbf{W}$)?

11. Verify directly from $c^2 + s^2 = 1$ that reflection matrices satisfy $H^2 = 1$.

12. Suppose A is a linear transformation from the x-y plane to itself. Why does $A^{-1}(x + y) = A^{-1}x + A^{-1}y$? If A is represented by the matrix M, explain why A^{-1} is represented by M^{-1}.

13. The product $(AB)C$ of linear transformations starts with a vector x and produces $u = Cx$. Then rule 2V applies AB to u and reaches $(AB)Cx$.

(a) Is this result the same as separately applying C then B then A?

(b) Is the result the same as applying BC followed by A? Parentheses are unnecessary and the associative law $(AB)C = A(BC)$ holds for linear transformations. This is the best proof of the same law for matrices.

14. Prove that T^2 is a linear transformation if T is linear (from $\mathbf{R}^3$ to $\mathbf{R}^3$).

15. The space of all 2 by 2 matrices has the four basis "vectors"

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

For the linear transformation of *transposing*, find its matrix A with respect to this basis. Why is $A^2 = I$?

16. Find the 4 by 4 cyclic permutation matrix: (x_1, x_2, x_3, x_4) is transformed to $Ax = (x_2, x_3, x_4, x_1)$. What is the effect of A^2? Show that $A^3 = A^{-1}$.

17. Find the 4 by 3 matrix A that represents a *right shift*: (x_1, x_2, x_3) is transformed to $(0, x_1, x_2, x_3)$. Find also the *left shift* matrix B from $\mathbf{R}^4$ back to $\mathbf{R}^3$, transforming (x_1, x_2, x_3, x_4) to (x_2, x_3, x_4). What are the products AB and BA?

18. In the vector space P_3 of all $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$, let $\mathbf{S}$ be the subset of polynomials with $\int_0^1 p(x)dx = 0$. Verify that $\mathbf{S}$ is a subspace and find a basis.

19. A *nonlinear* transformation is invertible if $T(x) = b$ has exactly one solution for every b. The example $T(x) = x^2$ is not invertible because $x^2 = b$ has two solutions for positive b and no solution for negative b. Which of the following transformations (from the real numbers $\mathbf{R}^1$ to the real numbers $\mathbf{R}^1$) are invertible? None are linear, not even (c).

(a) $T(x) = x^3$. (b) $T(x) = e^x$.
(c) $T(x) = x + 11$. (d) $T(x) = \cos x$.

20. What is the axis and the rotation angle for the transformation that takes (x_1, x_2, x_3) into (x_2, x_3, x_1)?

21. A linear transformation must leave the zero vector fixed: $T(0) = 0$. Prove this from $T(v + w) = T(v) + T(w)$ by choosing $w =$ ____. Prove it also from the requirement $T(cv) = cT(v)$ by choosing $c =$ ____

22. Which of these transformations is not linear? The input is $v = (v_1, v_2)$.

(a) $T(v) = (v_2, v_1)$. (b) $T(v) = (v_1, v_1)$.
(c) $T(v) = (0, v_1)$. (d) $T(v) = (0, 1)$.

23. If S and T are linear with $S(v) = T(v) = v$, then $S(T(v)) = v$ or v^2?

24. Suppose $T(v) = v$, except that $T(0, v_2) = (0, 0)$. Show that this transformation satisfies $T(cv) = cT(v)$ but not $T(v + w) = T(v) + T(w)$.

25. Which of these transformations satisfy $T(v + w) = T(v) + T(w)$, and which satisfy $T(cv) = cT(v)$?

(a) $T(v) = v/\|v\|$. (b) $T(v) = v_1 + v_2 + v_3$.
(c) $T(v) = (v_1, 2v_2, 3v_3)$. (d) $T(v) =$ largest component of v.

26. For these transformations of $\mathbf{V} = \mathbf{R}^2$ to $\mathbf{W} = \mathbf{R}^2$, find $T(T(v))$.

(a) $T(v) = -v$.

(b) $T(v) = v + (1, 1)$.

(c) $T(v) = 90^\circ$ rotation $= (-v_2, v_1)$.

(d) $T(v) =$ projection $= \left(\dfrac{v_1 + v_2}{2}, \dfrac{v_1 + v_2}{2}\right)$.

27. The "cyclic" transformation T is defined by $T(v_1, v_2, v_3) = (v_2, v_3, v_1)$. What is $T(T(T(v)))$? What is $T^{100}(v)$?

28. Find the *range* and *kernel* (those are new words for the column space and nullspace) of T.

(a) $T(v_1,v_2)=(v_2,v_1)$. (b) $T(v_1,v_2,v_3)=(v_1,v_2)$.
(c) $T(v_1,v_2)=(0,0)$. (d) $T(v_1,v_2)=(v_1,v_1)$.

29. A linear transformation from $\mathbf{V}$ to $\mathbf{W}$ has an *inverse* from $\mathbf{W}$ to $\mathbf{V}$ when the range is all of $\mathbf{W}$ and the kernel contains only $v=0$. Why are these transformations not invertible?

(a) $T(v_1,v_2)=(v_2,v_2)$ $\quad \mathbf{W}=\mathbf{R}^2$.

(b) $T(v_1,v_2)=(v_1,v_2,v_1+v_2)$ $\quad \mathbf{W}=\mathbf{R}^3$.

(c) $T(v_1,v_2)=v_1$ $\quad \mathbf{W}=\mathbf{R}^1$.

30. Suppose a linear T transforms $(1,1)$ to $(2,2)$ and $(2,0)$ to $(0,0)$. Find $T(v)$ when

(a) $v=(2,2)$. (b) $v=(3,1)$. (c) $v=(-1,1)$. (d) $v=(a,b)$.

Problems 31–35 may be harder. The input space V contains all 2 by 2 matrices M.

31. M is any 2 by 2 matrix and $A=\left[\begin{smallmatrix}1&2\\3&4\end{smallmatrix}\right]$. The linear transformation T is defined by $T(M)=AM$. What rules of matrix multiplication show that T is linear?

32. Suppose $A=\left[\begin{smallmatrix}1&2\\3&6\end{smallmatrix}\right]$. Show that the identity matrix I is not in the range of T. Find a nonzero matrix M such that $T(M)=AM$ is zero.

33. Suppose T transposes every matrix M. Try to find a matrix A that gives $AM=M^{\mathrm{T}}$ for every M. Show that no matrix A will do it. *To professors:* Is this a linear transformation that doesn't come from a matrix?

34. The transformation T that transposes every matrix is definitely linear. Which of these extra properties are true?

(a) $T^2=$ identity transformation.

(b) The kernel of T is the zero matrix.

(c) Every matrix is in the range of T.

(d) $T(M)=-M$ is impossible.

35. Suppose $T(M)=\left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right][M]\left[\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right]$. Find a matrix with $T(M)\neq 0$. Describe all matrices with $T(M)=0$ (the kernel of T) and all output matrices $T(M)$ (the range of T).

Problems 36–40 are about changing the basis

36. (a) What matrix transforms $(1,0)$ into $(2,5)$ and transforms $(0,1)$ to $(1,3)$?

(b) What matrix transforms $(2,5)$ to $(1,0)$ and $(1,3)$ to $(0,1)$?

(c) Why does no matrix transform $(2,6)$ to $(1,0)$ and $(1,3)$ to $(0,1)$?

37. (a) What matrix M transforms $(1,0)$ and $(0,1)$ to (r,t) and (s,u)?
(b) What matrix N transforms (a,c) and (b,d) to $(1,0)$ and $(0,1)$?
(c) What condition on a, b, c, d will make part (b) impossible?

38. (a) How do M and N in Problem 37 yield the matrix that transforms (a,c) to (r,t) and (b,d) to (s,u)?
(b) What matrix transforms $(2,5)$ to $(1,1)$ and $(1,3)$ to $(0,2)$?

39. If you keep the same basis vectors but put them in a different order, the change-of-basis matrix M is a ____ matrix. If you keep the basis vectors in order but change their lengths, M is a ____ matrix.

40. The matrix that transforms $(1,0)$ and $(0,1)$ to $(1,4)$ and $(1,5)$ is $M =$ ____. The combination $a(1,4)+b(1,5)$ that equals $(1,0)$ has $(a,b) = (\quad,\quad)$. How are those new coordinates of $(1,0)$ related to M or M^{-1}?

41. What are the three equations for A, B, C if the parabola $Y = A+Bx+Cx^2$ equals 4 at $x=a$, 5 at $x=b$, and 6 at $x=c$? Find the determinant of the 3 by 3 matrix. For which numbers a, b, c will it be impossible to find this parabola Y?

42. Suppose v_1, v_2, v_3 are eigenvectors for T. This means $T(v_i) = \lambda_i v_i$ for $i = 1,2,3$. What is the matrix for T when the input and output bases are the v's?

43. Every invertible linear transformation can have I as its matrix. For the output basis just choose $w_i = T(v_i)$. Why must T be invertible?

44. Suppose T is reflection across the x-axis and S is reflection across the y-axis. The domain $\mathbf{V}$ is the x-y plane. If $v = (x,y)$ what is $S(T(v))$? Find a simpler description of the product ST.

45. Suppose T is reflection across the 45° line, and S is reflection across the y-axis, If $v = (2,1)$ then $T(v) = (1,2)$. Find $S(T(v))$ and $T(S(v))$. This shows that generally $ST \neq TS$.

46. Show that the product ST of two reflections is a rotation. Multiply these reflection matrices to find the rotation angle:

$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \qquad \begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{bmatrix}.$$

47. The 4 by 4 *Hadamard matrix* is entirely $+1$ and -1:

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

Find H^{-1} and write $v = (7,5,3,1)$ as a combination of the columns of H.

48. Suppose we have two bases $v_1,\dots,v_n$ and $w_1,\dots,w_n$ for $\mathbf{R}^n$. If a vector has coefficients b_i in one basis and c_i in the other basis, what is the change-of-basis matrix in $b = Mc$? Start from

$$b_1v_1 + \cdots + b_nv_n = Vb = c_1w_1 + \cdots + c_nw_n = Wc.$$

Your answer represents $T(v) = v$ with input basis of v's and output basis of w's. Because of different bases, the matrix is not I.

49. True or false: If we know $T(v)$ for n different nonzero vectors in $\mathbf{R}^2$, then we know $T(v)$ for every vector in $\mathbf{R}^n$.

50. (Recommended) Suppose all vectors x in the unit square $0 \le x_1 \le 1$, $0 \le x_2 \le 1$ are transformed to Ax (A is 2 by 2).

(a) What is the shape of the transformed region (all Ax)?

(b) For which matrices A is that region a square?

(c) For which A is it a line?

(d) For which A is the new area still 1?

Review Exercises

1.1 Find a basis for the following subspaces of $\mathbf{R}^4$:

(a) The vectors for which $x1 = 2x_4$.

(b) The vectors for which $x_1 + x_2 + x_3 = 0$ and $x_3 + x_4 = 0$.

(c) The subspace spanned by $(1,1,1,1)$, $(1,2,3,4)$, and $(2,3,4,5)$.

1.2 By giving a basis, describe a two-dimensional subspace of $\mathbf{R}^3$ that contains none of the coordinate vectors $(1,0,0)$, $(0,1,0)$, $(0,0,1)$.

1.3 True or false, with counterexample if false:

(a) If the vectors $x_1,\dots,x_m$ span a subspace S, then $\dim S = m$.

(b) The intersection of two subspaces of a vector space cannot be empty.

(c) If $Ax = Ay$, then $x = y$.

(d) The row space of A has a unique basis that can be computed by reducing A to echelon form.

(e) If a square matrix A has independent columns, so does A^2.