Fourth Edition

LINEAR ALGEBRA AND ITS APPLICATIONS

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Chapter 2

Vector Spaces

2.1 Vector Spaces and Subspaces

Elimination can simplify, one entry at a time, the linear system $Ax = b$. Fortunately it also simplifies the theory. The basic questions of *existence* and *uniqueness*—Is there one solution, or no solution, or an infinity of solutions?—are much easier to answer after elimination, We need to devote one more section to those questions, to find every solution for an m by n system. Then that circle of ideas will be complete.

But elimination produces only one kind of understanding of $Ax = b$. Our chief object is to achieve a different and deeper understanding. This chapter may be more difficult than the first one. It goes to the heart of linear algebra.

For the concept of a ***vector space***, we start immediately with the most important spaces. They are denoted by $\mathbf{R}^1, \mathbf{R}^2, \mathbf{R}^3, \ldots$; the space $\mathbf{R}^n$ consists of *all column vectors with n components*. (We write $\mathbf{R}$ because the components are real numbers.) $\mathbf{R}^2$ is represented by the usual x-y plane; the two components of the vector become the x and y coordinates of the corresponding point. The three components of a vector in $\mathbf{R}^3$ give a point in three-dimensional space. The one-dimensional space $\mathbf{R}^1$ is a line.

The valuable thing for linear algebra is that the extension to n dimensions is so straightforward. For a vector in $\mathbf{R}^7$ we just need the seven components, even if the geometry is hard to visualize. Within all vector spaces, two operations are possible:

We can add any two vectors, and we can multiply all vectors by scalars.   **In other words, we can take linear combinations.**

Addition obeys the commutative law $x + y = y + x$; there is a "zero vector" satisfying $0 + x = x$; and there is a vector "$-x$" satisfying $-x + x = 0$. Eight properties (including those three) are fundamental; the full list is given in Problem 5 at the end of this section. ***A real vector space is a set of*** *vectors* ***together with rules for vector addition and multiplication by real numbers***. Addition and multiplication must produce vectors in the space, and they must satisfy the eight conditions.

Normally our vectors belong to one of the spaces $\mathbf{R}^n$; they are ordinary column vectors. If $x = (1,0,0,3)$, then $2x$ (and also $x+x$) has components 2, 0, 0, 6. The formal definition allows other things to be "vectors"-provided that addition and scalar multiplication are all right. We give three examples:

1. *The infinite-dimensional space* $\mathbf{R}^\infty$. Its vectors have infinitely many components, as in $x = (1,2,1,2,\ldots)$. The laws for $x+y$ and cx stay unchanged.

2. *The space of 3 by 2 matrices*. In this case the "vectors" are matrices! We can add two matrices, and $A+B = B+A$, and there is a zero matrix, and so on. This space is almost the same as $\mathbf{R}^6$. (The six components are arranged in a rectangle instead of a column.) Any choice of m and n would give, as a similar example, the vector space of all m by n matrices.

3. *The space of functions* $f(x)$. Here we admit all functions f that are defined on a fixed interval, say $0 \le x \le 1$. The space includes $f(x) = x^2$, $g(x) = \sin x$, their sum $(f+g)(x) = x^2 + \sin x$, and all multiples like $3x^2$ and $-\sin x$. The vectors are functions, and the dimension is somehow a larger infinity than for $\mathbf{R}^\infty$.

Other examples are given in the exercises, but the vector spaces we need most are somewhere else—***they are inside the standard spaces*** $\mathbf{R}^n$. We want to describe them and explain why they are important. Geometrically, think of the usual three-dimensional $\mathbf{R}^3$ and choose any plane through the origin. ***That plane is a vector space in its own right***. If we multiply a vector in the plane by 3, or -3, or any other scalar, we get a vector in the same plane. If we add two vectors in the plane, their sum stays in the plane. This plane through $(0,0,0)$ illustrates one of the most fundamental ideas in linear algebra; it is a ***subspace*** of the original space $\mathbf{R}^3$.

Definition. A ***subspace*** of a vector space is a nonempty subset that satisfies the requirements for a vector space: ***Linear combinations stay in the subspace***.

(i) If we add any vectors x and y in the subspace, $x+y$ is *in the subspace*.

(ii) If we multiply any vector x in the subspace by any scalar c, cx is *in the subspace*.

Notice our emphasis on the word ***space***. A sub*space* is a subset that is "closed" under addition and scalar multiplication. Those operations follow the rules of the host space, keeping us ***inside the subspace***. The eight required properties are satisfied in the larger space and will automatically be satisfied in every subspace. Notice in particular that ***the zero vector will belong to every subspace***. That comes from rule (ii): Choose the scalar to be $c = 0$.

The smallest subspace $\mathbf{Z}$ contains only one vector, the zero vector. It is a "zero-dimensional space," containing only the point at the origin. Rules (i) and (ii) are satisfied, since the sum $0+0$ is in this one-point space, and so are all multiples $c0$. *This is the smallest possible vector space*: the empty set is not allowed. At the other extreme. the largest subspace is the whole of the original space. If the original space is $\mathbf{R}^3$, then the possible subspaces are easy to describe: $\mathbf{R}^3$ itself, any plane through the origin, any line through the origin, or the origin (the zero vector) alone.

The distinction between a subset and a subspace is made clear by examples. In each case, can you add vectors and multiply by scalars, without leaving the space?

Example 1. Consider all vectors in $\mathbf{R}^2$ whose components are positive or zero. This subset is the first quadrant of the x-y plane; the coordinates satisfy $x \geq 0$ and $y \geq 0$. It is *not a subspace*, even though it contains zero and addition does leave us within the subset. Rule (ii) is violated, since if the scalar is -1 and the vector is $[1 \;\; 1]$, the multiple $cx = [-1 \;\; -1]$ is in the third quadrant instead of the first.

If we include the third quadrant along with the first, scalar multiplication is all right. Every multiple cx will stay in this subset. However, rule (i) is now violated, since adding $[1 \;\; 2] + [-2 \;\; -1]$ gives $[-1 \;\; 1]$, which is not in either quadrant. The smallest subspace containing the first quadrant is the whole space $\mathbf{R}^2$.

Example 2. Start from the vector space of 3 by 3 matrices. One possible subspace is the set of *lower triangular matrices*. Another is the set of *symmetric matrices*. $A+B$ and cA are lower triangular if A and B are lower triangular, and they are symmetric if A and B are symmetric. Of course, the zero matrix is in both subspaces.

The Column Space of A

We now come to the key examples, the **column space** and the **nullspace** of a matrix A. ***The column space contains all linear combinations of the columns of*** A. It is a subspace of $\mathbf{R}^m$. We illustrate by a system of $m = 3$ equations in $n = 2$ unknowns:

Combination of columns equals b
$$\begin{bmatrix} 1 & 0 \\ 5 & 4 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}. \tag{1}$$

With $m > n$ we have more equations than unknowns—and *usually there will be no solution*. The system will be solvable only for a very "thin" subset of all possible b's. One way of describing this thin subset is so simple that it is easy to overlook.

> **2A** The system $Ax = b$ is solvable if and only if the vector b can be expressed as a combination of the columns of A. Then b is in the column space.

This description involves nothing more than a restatement of $Ax = b$, *by columns*:

Combination of columns
$$u \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix} + v \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}. \tag{2}$$

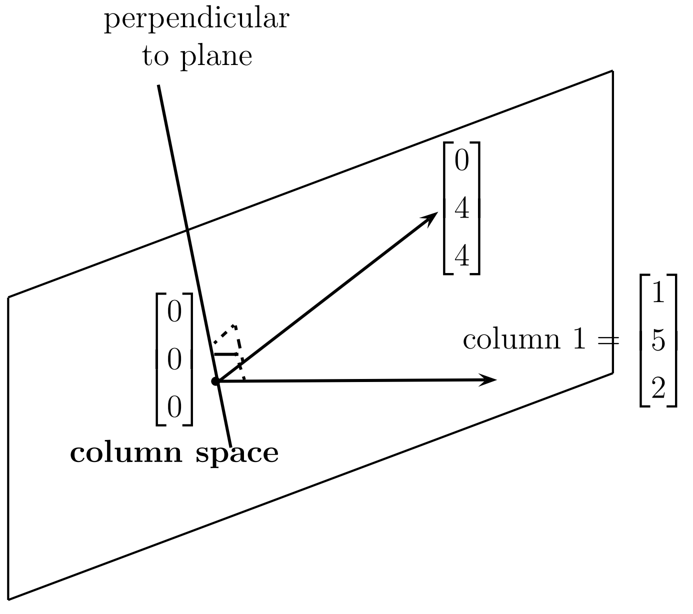

Figure 2.1: The column space $C(A)$, a plane in three-dimensional space.

These are the same three equations in two unknowns. Now the problem is: Find numbers u and v that multiply the first and second columns to produce b. The system is solvable exactly when such coefficients exist, and the vector (u,v) is the solution x.

We are saying that the attainable right-hand sides b are *all combinations of the columns of A*. One possible right-hand side is the first column itself; the weights are $u=1$ and $v=0$. Another possibility is the second column: $u=0$ and $v=1$. A third is the right-hand side $b=0$. With $u=0$ and $v=0$, the vector $b=0$ will always be attainable.

We can describe *all combinations* of the two columns geometrically: *$Ax=b$ can be solved if and only if b lies in the* ***plane*** *that is spanned by the two column vectors* (Figure 2.1). This is the thin set of attainable b. If b lies off the plane, then it is not a combination of the two columns. In that case $Ax=b$ has no solution.

What is important is that this plane is not just a subset of $\mathbf{R}^3$ it is a subspace. It is the ***column space*** of A, consisting of ***all combinations of the columns***. It is denoted by $C(A)$. Requirements (i) and (ii) for a subspace of $\mathbf{R}^m$ are easy to check:

(i) Suppose b and b' lie in the column space, so that $Ax=b$ for some x and $Ax'=b'$ for some x'. Then $A(x+x')=b+b'$, so that $b+b'$ is also a combination of the columns. The column space of all attainable vectors b is closed under addition.

(ii) If b is in the column space $C(A)$, so is any multiple cb. If some combination of columns produces b (say $Ax=b$), then multiplying that combination by c will produce cb. In other words, $A(cx)=cb$.

For another matrix A, the dimensions in Figure 2.1 may be very different. The smallest possible column space (one vector only) comes from the zero matrix $A=0$. The only combination of the columns is $b = 0$. At the other extreme, suppose A is the 5 by 5 identity matrix. Then $\boldsymbol{C}(I)$ is the whole of $\mathbf{R}^5$; the five columns of I can combine to produce any five-dimensional vector b. This is not at all special to the identity matrix. *Any 5 by 5 matrix that is nonsingular will have the whole of* $\mathbf{R}^5$ *as its column space*. For such a matrix we can solve $Ax = b$ by Gaussian elimination; there are five pivots. Therefore every b is in $\boldsymbol{C}(A)$ for a nonsingular matrix.

You can see how Chapter 1 is contained in this chapter. There we studied n by n matrices whose column space is $\mathbf{R}^n$. Now we allow singular matrices, and rectangular matrices of any shape. Then $\boldsymbol{C}(A)$ can be somewhere between the zero space and the whole space $\mathbf{R}^m$. Together with its perpendicular space, it gives one of our two approaches to understanding $Ax = b$.

The Nullspace of A

The second approach to $Ax = b$ is "dual" to the first. We are concerned not only with attainable right-hand sides b, but also with the solutions x that attain them. The right-hand side $b = 0$ always allows the solution $x = 0$, but there may be infinitely many other solutions. (There always are, if there are more unknowns than equations, $n > m$.) ***The solutions to*** $Ax = 0$ ***form a vector space—the nullspace of*** A.

> The ***nullspace*** of a matrix consists of all vectors x such that $Ax = 0$. It is denoted by $\boldsymbol{N}(A)$. It is a subspace of $\mathbf{R}^n$, just as the column space was a subspace of $\mathbf{R}^m$.

Requirement (i) holds: If $Ax = 0$ and $Ax' = 0$, then $A(x + x') = 0$. Requirement (ii) also holds: If $Ax = 0$ then $A(cx) = 0$. Both requirements fail if the right-hand side is not zero! Only the solutions to a *homogeneous* equation ($b = 0$) form a subspace. The nullspace is easy to find for the example given above; it is as small as possible:

$$\begin{bmatrix} 1 & 0 \\ 5 & 4 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The first equation gives $u = 0$, and the second equation then forces $v = 0$. The nullspace contains only the vector $(0, 0)$. This matrix has "independent columns"—a key idea that comes soon.

The situation is changed when a third column is a combination of the first two:

$$\textbf{Larger nullspace} \qquad B = \begin{bmatrix} 1 & 0 & 1 \\ 5 & 4 & 9 \\ 2 & 4 & 6 \end{bmatrix}.$$

B has the same column space as A. The new column lies in the plane of Figure 2.1; it is the sum of the two column vectors we started with. But the nullspace of B contains the vector $(1,1,-1)$ and automatically contains any multiple $(c,c,-c)$:

$$\textbf{Nullspace is a line} \qquad \begin{bmatrix} 1 & 0 & 1 \\ 5 & 4 & 9 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} c & c & -c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The nullspace of B is the line of all points $x = c$, $y = c$, $z = -c$. (The line goes through the origin, as any subspace must.) We want to be able, for any system $Ax = b$, to find $\boldsymbol{C}(A)$ and $\boldsymbol{N}(A)$: all attainable right-hand sides b and all solutions to $Ax = 0$.

The vectors b are in the column space and the vectors x are in the nullspace. We shall compute the dimensions of those subspaces and a convenient set of vectors to generate them. We hope to end up by understanding all *four* of the subspaces that are intimately related to each other and to A—the column space of A, the nullspace of A, and their two perpendicular spaces.

Problem Set 2.1

1. Construct a subset of the x-y plane $\mathbf{R}^2$ that is

(a) closed under vector addition and subtraction, but not scalar multiplication.

(b) closed under scalar multiplication but not under vector addition.

Hint: Starting with u and v, add and subtract for (a). Try cu and cv for (b).

2. Which of the following subsets of $\mathbf{R}^3$ are actually subspaces?

(a) The plane of vectors (b_1, b_2, b_3) with first component $b_1 = 0$.

(b) The plane of vectors b with $b_1 = 1$.

(c) The vectors b with $b_2 b_3 = 0$ (this is the union of two subspaces, the plane $b_2 = 0$ and the plane $b_3 = 0$).

(d) All combinations of two given vectors $(1,1,0)$ and $(2,0,1)$.

(e) The plane of vectors (b_1, b_2, b_3) that satisfy $b_3 - b_2 + 3b_1 = 0$.

3. Describe the column space and the nullspace of the matrices

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 3 \\ 1 & 2 & 3 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

4. What is the smallest subspace of 3 by 3 matrices that contains all symmetric matrices *and* all lower triangular matrices? What is the largest subspace that is contained in both of those subspaces?

5. Addition and scalar multiplication are required to satisfy these eight rules:

1. $x+y=y+x$.
2. $x+(y+z)=(x+y)+z$.
3. There is a unique "zero vector" such that $x+0=x$ for all x.
4. For each x there is a unique vector $-x$ such that $x+(-x)=0$.
5. $1x=x$.
6. $(c_1c_2)x=c_1(c_2x)$.
7. $c(x+y)=cx+cy$.
8. $(c_1+c_2)x=c_1x+c_2x$.

(a) Suppose addition in $\mathbf{R}^2$ adds an extra 1 to each component, so that $(3,1)+(5,0)$ equals $(9,2)$ instead of $(8,1)$. With scalar multiplication unchanged, which rules are broken?

(b) Show that the set of all positive real numbers, with $x+y$ and cx redefined to equal the usual xy and x^c, is a vector space. What is the "zero vector"?

(c) Suppose $(x_1,x_2)+(y_1,y_2)$ is defined to be (x_1+y_2,x_2+y_1). With the usual $cx=(cx_1,cx_2)$, which of the eight conditions are not satisfied?

6. Let $\mathbf{P}$ be the plane in 3-space with equation $x+2y+z=6$. What is the equation of the plane $\mathbf{P}_0$ through the origin parallel to $\mathbf{P}$? Are $\mathbf{P}$ and $\mathbf{P}_0$ subspaces of $\mathbf{R}^3$?

7. Which of the following are subspaces of $\mathbf{R}^\infty$?

(a) All sequences like $(1,0,1,0,\ldots)$ that include infinitely many zeros.

(b) All sequences $(x1,x2,\ldots)$ with $x_j=0$ from some point onward.

(c) All decreasing sequences: $x_{j+1}\le x_j$ for each j.

(d) All convergent sequences: the x_j have a limit as $j\to\infty$.

(e) All arithmetic progressions: $x_{j+1}-x_j$ is the same for all j.

(f) All geometric progressions $(x_1,kx_1,k^2x_1,\ldots)$ allowing all k and x_1.

8. Which of the following descriptions are correct? The solutions x of

$$Ax=\begin{bmatrix}1&1&1\\1&0&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

form

(a) a plane.

(b) a line.

(c) a point.

(d) a subspace.

(e) the nullspace of A.

(f) the column space of A.

9. Show that the set of nonsingular 2 by 2 matrices is not a vector space. Show also that the set of *singular* 2 by 2 matrices is not a vector space.

10. The matrix $A = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}$ is a "vector" in the space $\mathbf{M}$ of all 2 by 2 matrices. Write the zero vector in this space, the vector $\frac{1}{2}A$, and the vector $-A$. What matrices are in the smallest subspace containing A?

11. (a) Describe a subspace of $\mathbf{M}$ that contains $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ but not $B = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$.

(b) If a subspace of $\mathbf{M}$ contains A and B, must it contain I?

(c) Describe a subspace of $\mathbf{M}$ that contains no nonzero diagonal matrices.

12. The functions $f(x) = x^2$ and $g(x) = 5x$ are "vectors" in the vector space $\mathbf{F}$ of all real functions. The combination $3f(x) - 4g(x)$ is the function $h(x) =$ ____. Which rule is broken if multiplying $f(x)$ by c gives the function $f(cx)$?

13. If the sum of the "vectors" $f(x)$ and $g(x)$ in $\mathbf{F}$ is defined to be $f(g(x))$, then the "zero vector" is $g(x) = x$. Keep the usual scalar multiplication $cf(x)$, and find two rules that are broken.

14. Describe the smallest subspace of the 2 by 2 matrix space $\mathbf{M}$ that contains

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

(b) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(c) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

(d) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$.

15. Let $\mathbf{P}$ be the plane in $\mathbf{R}^3$ with equation $x + y - 2z = 4$. The origin $(0,0,0)$ is not in $\mathbf{P}$! Find two vectors in $\mathbf{P}$ and check that their sum is not in $\mathbf{P}$.

16. $\mathbf{P}_0$ is the plane through $(0,0,0)$ parallel to the plane $\mathbf{P}$ in Problem 15. What is the equation for $\mathbf{P}_0$? Find two vectors in $\mathbf{P}_0$ and check that their sum is in $\mathbf{P}_0$.

17. The four types of subspaces of $\mathbf{R}^3$ are planes, lines, R^3 itself, or $\mathbf{Z}$ containing only $(0,0,0)$.

(a) Describe the three types of subspaces of $\mathbf{R}^2$.

(b) Describe the five types of subspaces of $\mathbf{R}^4$.

18. (a) The intersection of two planes through $(0,0,0)$ is probably a ____ but it could be a ____. It can't be the zero vector $\mathbf{Z}$!

(b) The intersection of a plane through $(0,0,0)$ with a line through $(0,0,0)$ is probably a ____ but it could be a ____.

(c) If $\mathbf{S}$ and $\mathbf{T}$ are subspaces of $\mathbf{R}^5$, their intersection $\mathbf{S} \cap \mathbf{T}$ (vectors in both subspaces) is a subspace of $\mathbf{R}^5$. *Check the requirements on $x+y$ and cx.*

19. Suppose $\mathbf{P}$ is a plane through $(0,0,0)$ and $\mathbf{L}$ is a line through $(0,0,0)$. The smallest vector space containing both $\mathbf{P}$ and $\mathbf{L}$ is either ____ or ____.

20. True or false for $\mathbf{M}$ = all 3 by 3 matrices (check addition using an example)?

(a) The skew-symmetric matrices in $\mathbf{M}$ (with $A^{\mathrm{T}} = -A$) form a subspace.

(b) The unsymmetric matrices in $\mathbf{M}$ (with $A^{\mathrm{T}} \neq A$) form a subspace.

(c) The matrices that have $(1,1,1)$ in their nullspace form a subspace.

Problems 21–30 are about column spaces $C(A)$ and the equation $Ax = b$.

21. Describe the column spaces (lines or planes) of these particular matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

22. For which right-hand sides (find a condition on b_1, b_2, b_3) are these systems solvable?

$$\text{(a)} \quad \begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}. \qquad \text{(b)} \quad \begin{bmatrix} 1 & 4 \\ 2 & 9 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

23. Adding row 1 of A to row 2 produces B. Adding column 1 to column 2 produces C. A combination of the columns of ____ is also a combination of the columns of A. Which two matrices have the same column ____?

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}.$$

24. For which vectors (b_1, b_2, b_3) do these systems have a solution?

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

25. (Recommended) If we add an extra column b to a matrix A, then the column space gets larger unless ____. Give an example in which the column space gets larger and an example in which it doesn't. Why is $Ax = b$ solvable exactly when the column space *doesn't* get larger by including b?

26. The columns of AB are combinations of the columns of A. This means: *The column space of AB is contained in* (possibly equal to) *the column space of A.* Give an example where the column spaces of A and AB are not equal.

27. If A is any 8 by 8 invertible matrix, then its column space is ____. Why?

28. True or false (with a counterexample if false)?

(a) The vectors b that are not in the column space $\boldsymbol{C}(A)$ form a subspace.

(b) If $\boldsymbol{C}(A)$ contains only the zero vector, then A is the zero matrix.

(c) The column space of $2A$ equals the column space of A.

(d) The column space of $A - I$ equals the column space of A.

29. Construct a 3 by 3 matrix whose column space contains $(1,1,0)$ and $(1,0,1)$ but not $(1,1,1)$. Construct a 3 by 3 matrix whose column space is only a line.

30. If the 9 by 12 system $Ax = b$ is solvable for every b, then $\boldsymbol{C}(A) =$ ____.

31. Why isn't $\mathbf{R}^2$ a subspace of $\mathbf{R}^3$?

2.2 Solving $Ax = 0$ and $Ax = b$

Chapter 1 concentrated on square invertible matrices. There was one solution to $Ax = b$ and it was $x = -A^{-1}b$. That solution was found by elimination (not by computing A^{-1}). A rectangular matrix brings new possibilities—U may not have a full set of pivots. This section goes onward from U to a reduced form R—**the simplest matrix that elimination can give**. R reveals all solutions immediately.

For an invertible matrix, the nullspace contains only $x = 0$ (multiply $Ax = 0$ by A^{-1}). The column space is the whole space ($Ax = b$ has a solution for every b). The new questions appear when the nullspace contains *more than the zero vector* and/or the column space contains *less than all vectors*:

1. Any vector x_n in the nullspace can be added to a particular solution x_p. The solutions to all linear equations have this form, $x = x_p + x_n$:

$$\textbf{Complete solution} \qquad Ax_p = b \quad \textbf{and} \quad Ax_n = 0 \qquad \textbf{produce} \quad A(x_p + x_n) = b.$$

2. When the column space doesn't contain every b in $\mathbf{R}^m$, we need the conditions on b that make $Ax = b$ solvable.

A 3 by 4 example will be a good size. We will write down all solutions to $Ax = 0$. We will find the conditions for b to lie in the column space (so that $Ax = b$ is solvable). The 1 by 1 system $0x = b$, one equation and one unknown, shows two possibilities:

$0x = b$ has *no solution* unless $b = 0$. The column space of the 1 by 1 zero matrix contains only $b = 0$.

$0x = 0$ has *infinitely many solutions*. The nullspace contains *all* x. A particular solution is $x_p = 0$, and the complete solution is $x = x_p + x_n = 0 + (\text{any } x)$.

Simple, I admit. If you move up to 2 by 2, it's more interesting. The matrix $\begin{bmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{2} & \mathbf{2} \end{bmatrix}$ is not invertible: $y+z=b_1$ and $2y+2z=b_2$ usually have no solution.

> There is ***no solution*** unless $b_2 = 2b_1$. The column space of A contains only those b's, the multiples of $(1,2)$.
>
> When $b_2 = 2b_1$ there are ***infinitely many solutions***. A particular solution to $y+z=2$ and $2y+2z=4$ is $x_p = (1,1)$. The nullspace of A in Figure 2.2 contains $(-1,1)$ and all its multiples $x_n = (-c,c)$:

$$\textbf{Complete solution} \quad \begin{array}{rcrcl} y & + & z & = & 2 \\ 2y & + & 2z & = & 4 \end{array} \quad \text{is solved by} \quad x_p + x_n = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1-c \\ 1+c \end{bmatrix}.$$

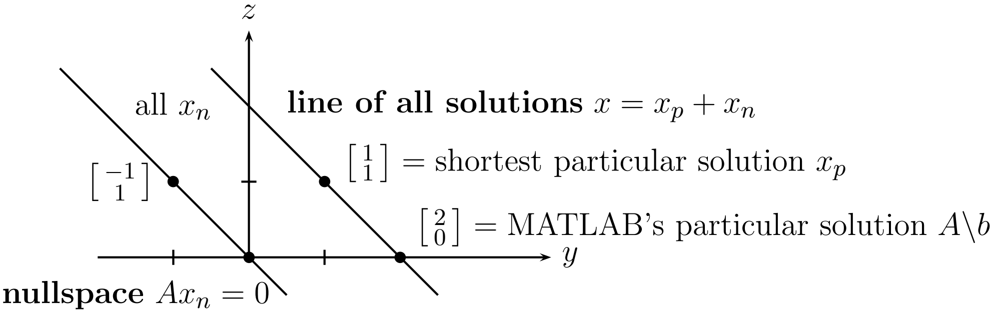

Figure 2.2: The parallel lines of solutions to $Ax_n = 0$ and $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$.

Echelon Form U and Row Reduced Form R

We start by simplifying this 3 by 4 matrix, first to U and then further to R:

$$\textbf{Basic example} \qquad A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}.$$

The pivot $a_{11} = 1$ is nonzero. The usual elementary operations will produce zeros in the first column below this pivot. The bad news appears in column 2:

$$\textbf{No pivot in column 2} \qquad A \to \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & \mathbf{0} & 3 & 3 \\ 0 & \mathbf{0} & 6 & 6 \end{bmatrix}.$$

The candidate for the second pivot has become zero: *unacceptable*. We look below that zero for a nonzero entry—intending to carry out a row exchange. In this case the *entry below it is also zero*. If A were square, this would signal that the matrix was singular. With a rectangular matrix, we must expect trouble anyway, and there is no reason to stop.

All we can do is to *go on to the next column*, where the pivot entry is 3. Subtracting twice the second row from the third, we arrive at U:

$$\textbf{Echelon matrix } U \qquad U = \begin{bmatrix} \mathbf{1} & 3 & 3 & 2 \\ 0 & 0 & \mathbf{3} & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Strictly speaking, we proceed to the fourth column. A zero is in the third pivot position, and nothing can be done. U is upper triangular, but its pivots are not on the main diagonal. The nonzero entries of U have a "staircase pattern," or ***echelon form***. For the 5 by 8 case in Figure 2.3, the starred entries may or may not be zero.

$$U = \begin{bmatrix} \bullet & * & * & * & * & * & * & * \\ 0 & \bullet & * & * & * & * & * & * \\ 0 & 0 & 0 & \bullet & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bullet \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad R = \begin{bmatrix} \mathbf{1} & \mathbf{0} & * & \mathbf{0} & * & * & * & \mathbf{0} \\ 0 & \mathbf{1} & * & \mathbf{0} & * & * & * & \mathbf{0} \\ 0 & 0 & 0 & \mathbf{1} & * & * & * & \mathbf{0} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Figure 2.3: The entries of a 5 by 8 echelon matrix U and its reduced form R.

We can always reach this echelon form U, with zeros below the pivots:

1. The pivots are the first nonzero entries in their rows.
2. Below each pivot is a column of zeros, obtained by elimination.
3. Each pivot lies to the right of the pivot in the row above. This produces the staircase pattern, and zero rows come last.

Since we started with A and ended with U, the reader is certain to ask: Do we have $A = LU$ as before? There is no reason why not, since the elimination steps have not changed. Each step still subtracts a multiple of one row from a row beneath it. The inverse of each step adds back the multiple that was subtracted. These inverses come in the right order to put the multipliers directly into L:

$$\textbf{Lower triangular} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix} \quad \text{and} \quad A = LU.$$

Note that L is square. It has the same number of rows as A and U.

The only operation not required by our example, but needed in general, is row exchange by a permutation matrix P. Since we keep going to the next column when no pivots are available, there is no need to assume that A is nonsingular. Here is $PA = LU$ for all matrices:

> **2B** For any m by n matrix A there is a permutation P, a lower triangular L with unit diagonal, and an m by n echelon matrix U, such that $PA = LU$.

Now comes R. We can go further than U, to make the matrix even simpler. Divide the second row by its pivot 3, so that ***all pivots are*** 1. Then use the pivot row to produce ***zero above the pivot***. This time we subtract a row from a *higher row*. The final result (the best form we can get) is the ***reduced row echelon form*** R:

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & \mathbf{1} & \mathbf{1} \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} \mathbf{1} & \mathbf{3} & \mathbf{0} & \mathbf{-1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} = R.$$

This matrix R is the final result of elimination on A. MATLAB would use the command R = rref(A). Of course rref(R) would give R again!

What is the row reduced form of a square invertible matrix? In that case R is the *identity matrix*. There is a full set of pivots, all equal to 1, with zeros above and below. So rref(A) = I, when A is invertible.

For a 5 by 8 matrix with four pivots, Figure 2.3 shows the reduced form R. **It still contains an identity matrix, in the four pivot rows and four pivot columns**. From R we will quickly find the nullspace of A. $Rx = 0$ has the same solutions as $Ux = 0$ and $Ax = 0$.

Pivot Variables and Free Variables

Our goal is to read off all the solutions to $Rx = 0$. The pivots are crucial:

Nullspace of R (pivot columns in boldface)

$$Rx = \begin{bmatrix} \mathbf{1} & 3 & \mathbf{0} & -1 \\ \mathbf{0} & 0 & \mathbf{1} & 1 \\ \mathbf{0} & 0 & \mathbf{0} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ v \\ \boldsymbol{w} \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The unknowns u, v, w, y go into two groups. One group contains the ***pivot variables***, those that correspond to ***columns with pivots***. The first and third columns contain the pivots, so $\boldsymbol{u}$ and $\boldsymbol{w}$ are the pivot variables. The other group is made up of the ***free variables***, corresponding to ***columns without pivots***. These are the second and fourth columns, so v and y are free variables.

To find the most general solution to $Rx = 0$ (or, equivalently, to $Ax = 0$) we may assign arbitrary values to the free variables. Suppose we call these values simply v and y. The pivot variables are completely determined in terms of v and y:

$$Rx = 0 \qquad \begin{aligned} \boldsymbol{u} + 3v - y &= 0 \quad \text{yields} \quad \boldsymbol{u} = -3v + y \\ \boldsymbol{w} + y &= 0 \quad \text{yields} \quad \boldsymbol{w} = -y \end{aligned} \tag{1}$$

There is a "double infinity" of solutions, with v and y free and independent. The complete solution is a combination of two **special solutions**:

Nullspace contains all combinations of special solutions
$$x = \begin{bmatrix} -3v+y \\ v \\ -y \\ y \end{bmatrix} = v\begin{bmatrix} \mathbf{-3} \\ \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} + y\begin{bmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{-1} \\ \mathbf{1} \end{bmatrix}. \tag{2}$$

Please look again at this complete solution to $Rx = 0$ and $Ax = 0$. The special solution $(-3,1,0,0)$ has free variables $v = 1$, $y = 0$. The other special solution $(1,0,-1,1)$ has $v = 0$ and $y = 1$. *All solutions are linear combinations of these two*. The best way to find all solutions to $Ax = 0$ is from the special solutions:

1. After reaching $Rx = 0$, identify the pivot variables and free variables.
2. Give one free variable the value 1, set the other free variables to 0, and solve $Rx = 0$ for the pivot variables. This x is a special solution.
3. Every free variable produces its own "special solution" by step 2. The combinations of special solutions form the nullspace—all solutions to $Ax = 0$.

Within the four-dimensional space of all possible vectors x, the solutions to $Ax = 0$ form a ***two-dimensional subspace***—the nullspace of A, In the example, $\boldsymbol{N}(A)$ is generated by the special vectors $(-3,1,0,0)$ and $(1,0,-1,1)$. The combinations of these two vectors produce the whole nullspace.

Here is a little trick. The special solutions are especially easy from R. The numbers 3 and 0 and -1 and 1 lie in the "nonpivot columns" of R. **Reverse their signs to find the pivot variables** (not free) **in the special solutions**. I will put the two special solutions from equation (2) into a nullspace matrix N, so you see this neat pattern:

Nullspace matrix (columns are special solutions)
$$N = \begin{array}{cl} \begin{bmatrix} \mathbf{-3} & \mathbf{1} \\ 1 & 0 \\ \mathbf{0} & \mathbf{-1} \\ 0 & 1 \end{bmatrix} & \begin{array}{l} \text{not free} \\ \text{free} \\ \text{not free} \\ \text{free} \end{array} \end{array}$$

The free variables have values 1 and 0. When the free columns moved to the right-hand side of equation (2), their coefficients 3 and 0 and -1 and 1 switched sign. That determined the pivot variables in the special solutions (the columns of N).

This is the place to recognize one extremely important theorem. Suppose a matrix has more columns than rows, $n > m$. Since m rows can hold at most m pivots, ***there must be at least*** $n - m$ ***free variables***. There will be even more free variables if some rows of R reduce to zero; but no matter what, at least one variable must be free. This free variable can be assigned any value, leading to the following conclusion:

2C If $Ax = 0$ has more unknowns than equations ($n > m$), it has at least one special solution: There are more solutions than the trivial $x = 0$.

There must be infinitely many solutions, since any multiple cx will also satisfy $A(cx) = 0$. The nullspace contains the line through x. And if there are additional free variables, the nullspace becomes more than just a line in n-dimensional space. *The nullspace has the same "dimension" as the number of free variables and special solutions.*

This central idea—the ***dimension*** of a subspace—is made precise in the next section. We count the free variables for the nullspace. We count the pivot variables for the column space!

Solving $Ax = b$, $Ux = c$, and $Rx = d$

The case $b \neq 0$ is quite different from $b = 0$. The row operations on A must act also on the right-hand side (on b). We begin with letters (b_1, b_2, b_3) to find the solvability condition—for b to lie in the column space. Then we choose $b = (1, 5, 5)$ and find all solutions x.

For the original example $Ax = b = (b_1, b_2, b_3)$, apply to both sides the operations that led from A to U. The result is an upper triangular system $Ux = c$:

$$Ux = c \qquad \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - 2b_1 \\ b_3 - 2b_2 + 5b_1 \end{bmatrix}. \tag{3}$$

The vector c on the right-hand side, which appeared after the forward elimination steps, is just $L^{-1}b$ as in the previous chapter. Start now with $Ux = c$.

It is not clear that these equations have a solution. The third equation is very much in doubt, because its left-hand side is zero. ***The equations are inconsistent unless*** $b_3 - 2b_2 + 5b_1 = 0$. Even though there are more unknowns than equations, there may be no solution. We know another way of answering the same question: $Ax = b$ can be solved if and only if b lies in the column space of A. This subspace comes from the four columns of A (not of U!):

$$\textbf{Columns of } A \text{ "span" the column space} \qquad \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 6 \\ -3 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 9 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix}.$$

Even though there are four vectors, their combinations only fill out a plane in three-dimensional space. Column 2 is three times column 1. The fourth column equals the third minus the first. *These dependent columns, the second and fourth, are exactly the ones without pivots.*

The column space $\boldsymbol{C}(A)$ can be described in two different ways. On the one hand, it is *the plane generated by columns 1 and 3*. The other columns lie in that plane, and contribute nothing new. Equivalently, it is the plane of all vectors b that satisfy $b_3 - 2b_2 + 5b_1 = 0$; this is the constraint if the system is to be solvable. ***Every column***

satisfies this constraint, so it is forced on b***!*** Geometrically, we shall see that the vector $(5,-2,1)$ is perpendicular to each column.

If b belongs to the column space, the solutions of $Ax=b$ are easy to find. The last equation in $Ux=c$ is $0=0$. To the free variables v and y, we may assign any values, as before. The pivot variables u and w are still determined by back-substitution. For a specific example with $b_3-2b_2+5b_1=0$, choose $b=(1,5,5)$:

$$Ax=b \qquad \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix}.$$

Forward elimination produces U on the left and c on the right:

$$Ux=c \qquad \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}.$$

The last equation is $0=0$, as expected. Back-substitution gives

$$\begin{aligned} 3w+3y=3 &\qquad \text{or} \qquad w=1-y \\ u+3v+3w+2y=1 &\qquad \text{or} \qquad u=-2-3v+y. \end{aligned}$$

Again there is a double infinity of solutions: v and y are free, u and w are not:

$$\begin{array}{c} \textbf{Complete solution} \\ x=x_p+x_n \end{array} \qquad x=\begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + v\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}. \tag{4}$$

This has all solutions to $Ax=0$, plus the new $x_p=(-2,0,1,0)$. That x_p is ***a particular solution*** to $Ax=b$. The last two terms with v and y yield more solutions (because they satisfy $Ax=0$). ***Every solution to*** $Ax=b$ ***is the sum of one particular solution and a solution to*** $Ax=0$:

$$x_{\textbf{complete}} = x_{\textbf{particular}} + x_{\textbf{nullspace}}$$

The particular solution in equation (4) comes from solving the equation *with all free variables set to zero*. That is the only new part, since the nullspace is already computed. When you multiply the highlighted equation by A, you get $Ax_{\text{complete}}=b+0$.

Geometrically, the solutions again fill a two-dimensional surface—but it is not a subspace. It does not contain $x=0$. It is *parallel* to the nullspace we had before, shifted by the particular solution x_p as in Figure 2.2. Equation (4) is a good way to write the answer:

1. Reduce $Ax=b$ to $Ux=c$.

2. With free variables $= 0$, find a particular solution to $Ax_p = b$ and $Ux_p = c$.

3. Find the special solutions to $Ax = 0$ (or $Ux = 0$ or $Rx = 0$). Each free variable, in turn, is 1. Then $x = x_p +$ (any combination x_n of special solutions).

When the equation was $Ax = 0$, the particular solution was the zero vector! It fits the pattern, but $x_{\text{particular}} = 0$ was not written in equation (2). Now x_p is added to the nullspace solutions, as in equation (4).

Question: How does the reduced form R make this solution even clearer? You will see it in our example. Subtract equation 2 from equation 1, and then divide equation 2 by its pivot. On the left-hand side, this produces R, as before. On the right-hand side, these operations change $c = (1, 3, 0)$ to a new vector $d = (-2, 1, 0)$:

Reduced equation $Rx = d$

$$\begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}. \tag{5}$$

Our particular solution x_p, (one choice out of many) has free variables $v = y = 0$. Columns 2 and 4 can be ignored. Then we immediately have $u = -2$ and $w = 1$, exactly as in equation (4). **The entries of d go directly into x_p.** This is because the identity matrix is sitting in the pivot columns of R!

Let me summarize this section, before working a new example. Elimination reveals the pivot variables and free variables. ***If there are r pivots, there are r pivot variables and $n - r$ free variables***. That important number r will be given a name—it is the ***rank of the matrix***.

2D Suppose elimination reduces $Ax = b$ to $Ux = c$ and $Rx = d$, with r pivot rows and r pivot columns. **The rank of those matrices is r.** The last $m - r$ rows of U and R are zero, so there is a solution only if the last $m - r$ entries of c and d are also zero.

The complete solution is $x = x_p + x_n$. One particular solution x_p has all free variables zero. Its pivot variables are the first r entries of d, so $Rx_p = d$.

The nullspace solutions x_n are combinations of $n - r$ special solutions, with one free variable equal to 1. The pivot variables in that special solution can be found in the corresponding column of R (with sign reversed).

You see how the rank r is crucial. It counts the pivot rows in the "row space" and the pivot columns in the column space. There are $n - r$ special solutions in the nullspace. There are $m - r$ solvability conditions on b or c or d.

Another Worked Example

The full picture uses elimination and pivot columns to find the column space, nullspace, and rank. The 3 by 4 matrix A has rank 2:

$$Ax = b \quad \text{is} \quad \begin{array}{rcrcrcrcl} \mathbf{1}x_1 & + & \mathbf{2}x_2 & + & \mathbf{3}x_3 & + & \mathbf{5}x_4 & = & b_1 \\ \mathbf{2}x_1 & + & \mathbf{4}x_2 & + & \mathbf{8}x_3 & + & \mathbf{12}x_4 & = & b_2 \\ \mathbf{3}x_1 & + & \mathbf{6}x_2 & + & \mathbf{7}x_3 & + & \mathbf{13}x_4 & = & b_3 \end{array} \tag{6}$$

1. Reduce $[A \;\; b]$ to $[U \;\; c]$, to reach a triangular system $Ux = c$.
2. Find the condition on b_1, b_2, b_3 to have a solution.
3. Describe the column space of A: Which plane in $\mathbf{R}^3$?
4. Describe the nullspace of A: Which special solutions in $\mathbf{R}^4$?
5. Find a particular solution to $Ax = (0, 6, -6)$ and the complete $x_p + x_n$.
6. Reduce $[U \;\; c]$ to $[R \;\; d]$: Special solutions from R and x_p from d.

Solution. (Notice how the right-hand side is included as an extra column!)

1. The multipliers in elimination are 2 and 3 and -1, taking $[A \;\; b]$ to $[U \;\; c]$.

$$\begin{bmatrix} 1 & 2 & 3 & 5 & \boldsymbol{b_1} \\ 2 & 4 & 8 & 12 & \boldsymbol{b_2} \\ 3 & 6 & 7 & 13 & \boldsymbol{b_3} \end{bmatrix} \rightarrow \left[\begin{array}{cccc|l} 1 & 2 & 3 & 5 & \boldsymbol{b_1} \\ 0 & 0 & 2 & 2 & \boldsymbol{b_2 - 2b_1} \\ 0 & 0 & -2 & -2 & \boldsymbol{b_3 - 3b_1} \end{array}\right] \rightarrow \left[\begin{array}{cccc|l} 1 & 2 & 3 & 5 & \boldsymbol{b_1} \\ 0 & 0 & 2 & 2 & \boldsymbol{b_2 - 2b_1} \\ 0 & 0 & 0 & 0 & \boldsymbol{b_3 + b_2 - 5b_1} \end{array}\right].$$

2. The last equation shows the solvability condition $b_3 + b_2 - 5b_1 = 0$. Then $0 = 0$.
3. The column space of A is the plane containing all combinations of the pivot columns $(1, 2, 3)$ and $(3, 8, 7)$.

   Second description: The column space contains all vectors with $b_3 + b_2 - 5b_1 = 0$. That makes $Ax = b$ solvable, so b is in the column space. *All columns of A pass this test $b_3 + b_2 - 5b_1 = 0$. This is the equation for the plane* (*in the first description of the column space*).
4. The special solutions in N have free variables $x_2 = 1$, $x_4 = 0$ and $x_2 = 0$, $x_4 = 1$:

$$\begin{array}{l} \textbf{Nullspace matrix} \\ \textbf{Special solutions to } Ax = 0 \\ \textbf{Back-substitution in } Ux = 0 \\ \textbf{Just switch signs in } Rx = 0 \end{array} \qquad N = \begin{bmatrix} -2 & -2 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}.$$

5. Choose $b = (0, 6, -6)$, which has $b_3 + b_2 - 5b_1 = 0$. Elimination takes $Ax = b$ to $Ux = c = (0, 6, 0)$. Back-substitute with free variables $= 0$:

$$\textbf{Particular solution to } Ax_p = (0,6,-6) \qquad x_p = \begin{bmatrix} -9 \\ 0 \\ 3 \\ 0 \end{bmatrix} \begin{matrix} \\ \text{free} \\ \\ \text{free} \end{matrix}$$

The complete solution to $Ax = (0, 6, -6)$ is (this x_p) + (all x_n).

6. In the reduced R, the third column changes from $(3, 2, 0)$ to $(0, 1, 0)$. The right-hand side $c = (0, 6, 0)$ becomes $d = (-9, 3, 0)$. Then -9 and 3 go into x_p:

$$\begin{bmatrix} U & c \end{bmatrix} = \left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & 0 \\ 0 & 0 & 2 & 2 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \begin{bmatrix} R & d \end{bmatrix} = \left[\begin{array}{cccc|c} \mathbf{1} & 2 & 0 & 2 & \mathbf{-9} \\ 0 & 0 & \mathbf{1} & 1 & \mathbf{3} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

That final matrix $[R \;\; d]$ **is** $\mathsf{rref}([A \;\; b]) = \mathsf{rref}([U \;\; c])$. The numbers 2 and 0 and 2 and 1 in the free columns of R have opposite sign in the special solutions (the nullspace matrix N). Everything is revealed by $Rx = d$.

Problem Set 2.2

1. Construct a system with more unknowns than equations, but no solution. Change the right-hand side to zero and find all solutions x_n.

2. Reduce A and B to echelon form, to find their ranks. Which variables are free?

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

Find the special solutions to $Ax = 0$ and $Bx = 0$. Find all solutions.

3. Find the echelon form U, the free variables, and the special solutions:

$$A = \begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 2 & 0 & 6 \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

$Ax = b$ is consistent (has a solution) when b satisfies $b_2 =$ ____. Find the complete solution in the same form as equation (4).

4. Carry out the same steps as in the previous problem to find the complete solution of $Mx = b$:

$$M = \begin{bmatrix} 0 & 0 \\ 1 & 2 \\ 0 & 0 \\ 3 & 6 \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}.$$

5. Write the complete solutions $x = x_p + x_n$ to these systems, as in equation (4):

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 5 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

6. Describe the set of attainable right-hand sides b (in the column space) for

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix},$$

by finding the constraints on b that turn the third equation into $0 = 0$ (after elimination). What is the rank, and a particular solution?

7. Find the value of c that makes it possible to solve $Ax = b$, and solve it:

$$\begin{aligned} u + v + 2w &= 2 \\ 2u + 3v - w &= 5 \\ 3u + 4v + w &= c. \end{aligned}$$

8. Under what conditions on b_1 and b_2 (if any) does $Ax = b$ have a solution?

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 4 & 0 & 7 \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

Find two vectors in the nullspace of A, and the complete solution to $Ax = b$.

9. (a) Find the special solutions to $Ux = 0$. Reduce U to R and repeat:

$$Ux = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

(b) If the right-hand side is changed from $(0,0,0)$ to $(a,b,0)$, what are all solutions?

10. Find a 2 by 3 system $Ax = b$ whose complete solution is

$$x = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + w \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}.$$

Find a 3 by 3 system with these solutions exactly when $b_1 + b_2 = b_3$.

11. Write a 2 by 2 system $Ax = b$ with many solutions x_n but no solution x_p. (Therefore the system has no solution.) Which b's allow an x_p?

12. Which of these rules give a correct definition of the *rank* of A?

(a) The number of nonzero rows in R.

(b) The number of columns minus the total number of rows.

(c) The number of columns minus the number of free columns.

(d) The number of 1s in R.

13. Find the reduced row echelon forms R and the rank of these matrices:

(a) The 3 by 4 matrix of all 1s.

(b) The 4 by 4 matrix with $a_{ij} = (-1)^{ij}$.

(c) The 3 by 4 matrix with $a_{ij} = (-1)^j$.

14. Find R for each of these (block) matrices, and the special solutions:

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 3 \\ 2 & 4 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} A & A \end{bmatrix} \qquad C = \begin{bmatrix} A & A \\ A & 0 \end{bmatrix}.$$

15. If the r pivot variables come first, the reduced R must look like

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \qquad \begin{array}{l} I \text{ is } r \text{ by } r \\ F \text{ is } r \text{ by } n-r \end{array}$$

What is the nullspace matrix N containing the special solutions?

16. Suppose all r pivot variables come *last*. Describe the four blocks in the m by n reduced echelon form (the block B should be r by r):

$$R = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

What is the nullspace matrix N of special solutions? What is its shape?

17. (Silly problem) Describe all 2 by 3 matrices A_1 and A_2 with row echelon forms R_1 and R_2, such that $R_1 + R_2$ is the row echelon form of $A_1 + A_2$. Is it true that $R_1 = A_1$ and $R_2 = A_2$ in this case?

18. If A has r pivot columns, then A^{T} has r pivot columns. Give a 3 by 3 example for which the column numbers are different for A and A^{T}.

19. What are the special solutions to $Rx = 0$ and $R^{\mathrm{T}}y = 0$ for these R?

$$R = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

20. *If A has rank r, then it has an r by r submatrix S that is invertible*. Find that submatrix S from the pivot rows and pivot columns of each A:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

21. Explain why the pivot rows and pivot columns of A (not R) always give an r by r invertible submatrix of A.

22. Find the ranks of AB and AM (rank 1 matrix times rank 1 matrix):

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 1.5 & 6 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} 1 & b \\ c & bc \end{bmatrix}.$$

23. Multiplying the rank 1 matrices $A = uv^{\mathrm{T}}$ and $B = wz^{\mathrm{T}}$ gives uz^{T} times the number ____. AB has rank 1 unless ____ $= 0$.

24. Every column of AB is a combination of the columns of A. Then the dimensions of the column spaces give $\mathrm{rank}(AB) \le \mathrm{rank}(A)$.

Problem: Prove also that $\mathrm{rank}(AB) \le \mathrm{rank}(B)$.

25. (Important) Suppose A and B are n by n matrices, and $AB = I$. Prove from $\mathrm{rank}(AB) \le \mathrm{rank}(A)$ that the rank of A is n. So A is invertible and B must be its two-sided inverse. Therefore $BA = I$ (*which is not so obvious!*).

26. If A is 2 by 3 and C is 3 by 2, show from its rank that $CA \ne I$. Give an example in which $AC = I$. For $m < n$, a right inverse is not a left inverse.

27. Suppose A and B have the *same* reduced-row echelon form R. Explain how to change A to B by elementary row operations. So B equals an ____ matrix times A.

28. Every m by n matrix of rank r reduces to (m by r) times (r by n):

$$A = (\text{pivot columns of } A)(\text{first } r \text{ rows of } R) = (\mathbf{COL})(\mathbf{ROW}).$$

Write the 3 by 4 matrix A at the start of this section as the product of the 3 by 2 matrix from the pivot columns and the 2 by 4 matrix from R:

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}$$

29. Suppose A is an m by n matrix of rank r. Its reduced echelon form is R. Describe exactly the *reduced row echelon form of* R^{T} (not A^{T}).

30. (Recommended) Execute the six steps following equation (6) to find the column space and nullspace of A and the solution to $Ax = b$:

$$A = \begin{bmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}.$$

31. For every c, find R and the special solutions to $Ax = 0$:

$$A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 4 & 4 \\ 1 & c & 2 & 2 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1-c & 2 \\ 0 & 2-c \end{bmatrix}.$$

32. What is the nullspace matrix N (of special solutions) for A, B, C?

$$A = \begin{bmatrix} I & I \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} I & I \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} I & I & I \end{bmatrix}.$$

Problems 33–36 are about the solution of $Ax = b$. Follow the steps in the text to x_p and x_n. Reduce the augmented matrix $[A \;\; b]$.

33. Find the complete solutions of

$$\begin{aligned} x + 3y + 3z &= 1 \\ 2x + 6y + 9z &= 5 \\ -x - 3y + 3z &= 5 \end{aligned} \qquad \text{and} \qquad \begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}.$$

34. Under what condition on b_1, b_2, b_3 is the following system solvable? Include b as a fourth column in $[A \;\; b]$. Find all solutions when that condition holds:

$$\begin{aligned} x + 2y - 2z &= b_1 \\ 2x + 5y - 4z &= b_2 \\ 4x + 9y - 8z &= b_3. \end{aligned}$$

35. What conditions on b_1, b_2, b_3, b_4 make each system solvable? *Solve for* x:

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 2 & 5 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 5 & 7 \\ 3 & 9 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}.$$

36. Which vectors (b_1, b_2, b_3) are in the column space of A? Which combinations of the rows of A give zero?

$$\text{(a)} \quad A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 6 & 3 \\ 0 & 2 & 5 \end{bmatrix} \qquad \text{(b)} \quad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix}.$$

37. Why can't a 1 by 3 system have $x_p = (2, 4, 0)$ and $x_n =$ any multiple of $(1, 1, 1)$?

38. (a) If $Ax = b$ has two solutions x_1 and x_2, find two solutions to $Ax = 0$.

(b) Then find another solution to $Ax = b$.

39. Explain why all these statements are false:

(a) The complete solution is any linear combination of x_p and x_n.

(b) A system $Ax = b$ has at most one particular solution.

(c) The solution x_p with all free variables zero is the shortest solution (minimum length $\|x\|$). (Find a 2 by 2 counterexample.)

(d) If A is invertible there is no solution x_n in the nullspace.

40. Suppose column 5 of U has no pivot. Then x_5 is a ____ variable. The zero vector (is) (is not) the only solution to $Ax = 0$. If $Ax = b$ has a solution, then it has ____ solutions.

41. If you know x_p (free variables $= 0$) and all special solutions for $Ax = b$, find x_p and all special solutions for these systems:

$$Ax = 2b \qquad \begin{bmatrix} A & A \end{bmatrix} \begin{bmatrix} x \\ X \end{bmatrix} = b \qquad \begin{bmatrix} A \\ A \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} b \\ b \end{bmatrix}.$$

42. If $Ax = b$ has infinitely many solutions, why is it impossible for $Ax = B$ (new right-hand side) to have only one solution? Could $Ax = B$ have no solution?

43. Choose the number q so that (if possible) the ranks are (a) 1, (b) 2, (c) 3:

$$A = \begin{bmatrix} 6 & 4 & 2 \\ -3 & -2 & -1 \\ 9 & 6 & q \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 3 & 1 & 3 \\ q & 2 & q \end{bmatrix}.$$

44. Give examples of matrices A for which the number of solutions to $Ax = b$ is

(a) 0 or 1, depending on b.

(b) ∞, regardless of b.

(c) 0 or ∞, depending on b.

(d) 1, regardless of b.

45. Write all known relations between r and m and n if $Ax = b$ has

(a) no solution for some b.

(b) infinitely many solutions for every b.

(c) exactly one solution for some b, no solution for other b.

(d) exactly one solution for every b.

46. Apply Gauss-Jordan elimination (right-hand side becomes extra column) to $Ux = 0$ and $Ux = c$. Reach $Rx = 0$ and $Rx = d$:

$$\begin{bmatrix} U & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & \mathbf{0} \\ 0 & 0 & 4 & \mathbf{0} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} U & c \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & \mathbf{5} \\ 0 & 0 & 4 & \mathbf{8} \end{bmatrix}.$$

Solve $Rx = 0$ to find x_n (its free variable is $x_2 = 1$). Solve $Rx = d$ to find x_p (its free variable is $x_2 = 0$).

47. Apply elimination with the extra column to reach $Rx = 0$ and $Rx = d$:

$$\begin{bmatrix} U & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 6 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} U & c \end{bmatrix} = \begin{bmatrix} 3 & 0 & 6 & \mathbf{9} \\ 0 & 0 & 2 & \mathbf{4} \\ 0 & 0 & 0 & \mathbf{5} \end{bmatrix}.$$

Solve $Rx = 0$ (free variable $= 1$). What are the solutions to $Rx = d$?

48. Reduce to $Ux = c$ (Gaussian elimination) and then $Rx = d$:

$$Ax = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 1 & 3 & 2 & 0 \\ 2 & 0 & 4 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 10 \end{bmatrix} = b.$$

Find a particular solution x_p and all nullspace solutions x_n.

49. Find A and B with the given property or explain why you can't.

(a) The only solution to $Ax = \left[\begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}\right]$ is $x = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$.

(b) The only solution to $Bx = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$ is $x = \left[\begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}\right]$.

50. The complete solution to $Ax = \left[\begin{smallmatrix} 1 \\ 3 \end{smallmatrix}\right]$ is $x = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] + c\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$. Find A.

51. The nullspace of a 3 by 4 matrix A is the line through $(2, 3, 1, 0)$.

(a) What is the *rank* of A and the complete solution to $Ax = 0$?

(b) What is the exact row reduced echelon form R of A?

52. Reduce these matrices A and B to their ordinary echelon forms U:

$$\text{(a)}\quad A = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix} \qquad \text{(b)}\quad B = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 8 & 8 \end{bmatrix}.$$

Find a special solution for each free variable and describe every solution to $Ax = 0$ and $Bx = 0$. Reduce the echelon forms U to R, and draw a box around the identity matrix in the pivot rows and pivot columns.

53. True or False? (Give reason if true, or counterexample to show it is false.)

(a) A square matrix has no free variables.

(b) An invertible matrix has no free variables.

(c) An m by n matrix has no more than n pivot variables.

(d) An m by n matrix has no more than m pivot variables.

54. Is there a 3 by 3 matrix with no zero entries for which $U = R = I$?

55. Put as many 1s as possible in a 4 by 7 echelon matrix U and in a *reduced* form R whose pivot columns are 2, 4, 5.

56. Suppose column 4 of a 3 by 5 matrix is all 0s. Then x_4 is certainly a ____ variable. The special solution for this variable is the vector $x =$ ____.

57. Suppose the first and last columns of a 3 by 5 matrix are the same (nonzero). Then ____ is a free variable. Find the special solution for this variable.

58. The equation $x - 3y - z = 0$ determines a plane in $\mathbf{R}^3$. What is the matrix A in this equation? Which are the free variables? The special solutions are $(3, 1, 0)$ and ____. The parallel plane $x - 3y - z = 12$ contains the particular point $(12, 0, 0)$. All points on this plane have the following form (fill in the first components):

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} \\ 0 \\ 1 \end{bmatrix}.$$

59. Suppose column 1 + column 3 + column 5 = 0 in a 4 by 5 matrix with four pivots. Which column is sure to have no pivot (and which variable is free)? What is the special solution? What is the nullspace?

Problems 60–66 ask for matrices (if possible) with specific properties.

60. Construct a matrix whose nullspace consists of all combinations of $(2, 2, 1, 0)$ and $(3, 1, 0, 1)$.

61. Construct a matrix whose nullspace consists of all multiples of $(4, 3, 2, 1)$.

62. Construct a matrix whose column space contains $(1,1,5)$ and $(0,3.1)$ and whose nullspace contains $(1,1,2)$.

63. Construct a matrix whose column space contains $(1,1,0)$ and $(0,1,1)$ and whose nullspace contains $(1,0,1)$ and $(0,0,1)$.

64. Construct a matrix whose column space contains $(1,1,1)$ and whose nullspace is the line of multiples of $(1,1,1,1)$.

65. Construct a 2 by 2 matrix whose nullspace equals its column space.

66. Why does no 3 by 3 matrix have a nullspace that equals its column space?

67. The reduced form R of a 3 by 3 matrix with randomly chosen entries is almost sure to be ____. What R is virtually certain if the random A is 4 by 3?

68. Show by example that these three statements are generally false:

(a) A and A^{T} have the same nullspace.

(b) A and A^{T} have the same free variables.

(c) If R is the reduced form $\mathsf{rref}(A)$ then R^{T} is $\mathsf{rref}(A^{\mathrm{T}})$.

69. If the special solutions to $Rx = 0$ are in the columns of these N, go backward to find the nonzero rows of the reduced matrices R:

$$N = \begin{bmatrix} 2 & 3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} \ \\ \ \\ \ \end{bmatrix} \quad \text{(empty 3 by 1).}$$

70. Explain why A and $-A$ always have the same reduced echelon form R.

2.3 Linear Independence, Basis, and Dimension

By themselves, the numbers m and n give an incomplete picture of the true size of a linear system. The matrix in our example had three rows and four columns, but the third row was only a combination of the first two. After elimination it became a zero row, It had no effect on the homogeneous problem $Ax = 0$. The four columns also failed to be independent, and the column space degenerated into a two-dimensional plane.

The important number that is beginning to emerge (the true size) is the ***rank*** r. The rank was introduced as the *number of pivots* in the elimination process. Equivalently, the final matrix U has r nonzero rows. This definition could be given to a computer. But it would be wrong to leave it there because the rank has a simple and intuitive meaning: *The rank counts the number of genuinely independent rows in the matrix A*. We want definitions that are mathematical rather than computational.

The goal of this section is to explain and use four ideas: