Homework 1

1. Peak of the CMB Powerspectrum

An isotropic and homogeneous universe is described by the Friedman-Robertson-Walker (FWR) metric. In this case, the line element ds can which can be written as

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right] \tag{1}$$

where $a(t)$ is a time dependent scale factor describing the expansion of the universe and k describes the curvature. The Hubble rate of expansion is defined as $H(t) = \dot{a}(t)/a(t)$. In the usual convention we set $a(\text{today}) = 1$ such that the scale factor is related to redshift z by $a(t) = (1+z)^{-1}$.
The dynamics of the universe are described by Friedman equations. Using the dimensionless density parameters Ω, we can write the first Friedman equation as

$$H^2(t) = \left(\frac{\dot{a}}{a}\right)^2 = H_0^2 \times \left(\Omega_{0,r}a^{-4} + \Omega_{0,m}a^{-3} + \Omega_{0,k}a^{-2} + \Omega_{0,\Lambda}\right). \tag{2}$$

Here H_0 is the Hubble parameter today while $\Omega_{0,r}$, $\Omega_{0,m}$, $\Omega_{0,k}$, and $\Omega_{0,\Lambda}$ are the dimensionless densities for radiation, matter, curvature and dark energy. Measurements have determined that $\Omega_{0,r} \sim 10^{-4}$, $\Omega_{0,m} \sim 0.3$, $\Omega_{0,k} \sim 0$ and $\Omega_{0,\Lambda} \sim 0.7$.

(a) Using the Friedman equation, obtain an expression for $(\dot{a}a)$ as a function of a.

(b) The co-moving distance d_c is defined as the radial distance between us (at $r = 0$) and an object (at $r > 0$ for $a < 0$). It is therefore given by $d_c = \int_0^r \frac{dr}{\sqrt{1-kr^2}}$. Show that one can also write it as $d_c = \int_a^1 da(\dot{a}a)^{-1}$.

We know from atomic physics considerations that the photons decouples from the matter (mostly of neural atoms) at a temperature of $T_{dec} = 0.26$ eV. The sound horizon r_s is distance a sound wave with speed c_s can travel before decoupling: we can define it analogously as $r_s = \int_0^{a_{dec}} da(\dot{a}a)^{-1}$. The angular size of the sound horizon at the time of decoupling is then given by $\theta = r_s/d_c$.

(c) Using the measured CMB temperature, determine the scale factor at the time of decoupling a_{dec}. .

(d) Let us assume, for simplicity, that the universe is matter dominated, so $\Omega_{0,m} = 1$ and $\Omega_{0,m} = \Omega_{0,k} = \Omega_{0,\Lambda} = \Omega_{0,r} = 0$. Calculate the angular size of the sound horizon θ in this case analytically. (Hint: since baryons couple to the relativistic photon gas, the speed of sound is $c_s = 1/\sqrt{3}$).

(e) Let us now assume the actual cosmology. Calculate the angular size of the sound horizon θ numerically with a tool of your choice (such as C++, Python, Mathematica). Provide the code and its output.

2. Formation of Structure

Let us consider the universe filled with a matter fluid in the Newtonian limit. The corresponding system is characterized by its density ρ, its velocity field $\vec{v}$, its pressure P and the Newton potential Φ. The evolution of the system is governed by the Euler equations:

$$\begin{aligned} &\partial\rho/\partial t + \vec{\nabla}(\rho\vec{v}) = 0 && \text{(Continuity Equation)} \\ &\partial\vec{v}/\partial t + (\vec{v}\cdot\vec{\nabla})\vec{v} = -(\vec{\nabla}P)/\rho - \vec{\nabla}\Phi && \text{(Euler Equation)} \\ &\nabla^2\Phi = 4\pi G\rho && \text{(Poisson Equation)} \end{aligned} \tag{3}$$

Let us first consider a static (non-expanding) quasi-homogeneous universe.

(a) Let us assume that a background solution to the Euler equations is given by $\rho_0, \vec{v}_0, P_0$ and Φ_0. Let us introduce small perturbations around the background solution as $\rho = \rho_0 + \rho_1$, $\vec{v} = \vec{v}_0 + \vec{v}_1$, $P = P_0 + P_1$, $\Phi = \Phi_0 + \Phi_1$. Let us further define $c_s^2 = \partial P/\partial\rho$ which will turn out to be the speed of sound. Obtain the set of Euler equations for the perturbed quantifies $\rho_1, \vec{v}_1$ and Φ_1 in first order perturbation theory.

(b) Derive the Jeans Equation $\partial^2\rho_1/\partial t^2 - c_s^2\nabla^2\rho_1 = 4\pi G\rho_0\rho_1$.

(c) Let us define the Jean Length as $\lambda_J^2 = c_s^2/(4\pi G\rho_0)$. Show that perturbations on large scales $\lambda > \lambda_J$ will collapse, while perturbations on small scales $\lambda < \lambda_J$ will be supported by pressure. (Hint: Use the ansatz $\rho_1 \sim e^{i(\vec{k}\vec{x}-\omega t)}$.)

Let us now turn to an expanding universe.

(d) Show that $\rho(t) = \rho_0/a^3(t)$, $v_0 = H(t)\vec{r}$, $\vec{\nabla}P = 0$ and $\vec{\nabla}\Phi = (4\pi G\rho/3)\vec{r}$ is a solution to the Euler equations. (Hint: The second Friedmann equation is $(\ddot{a}/a) = -(4\pi G/3)(\rho + 3P)$.)

(e) As before, let us introduce small perturbations around the background solution as $\rho = \rho_0 + \rho_1$, $\vec{v} = \vec{v}_0 + \vec{v}_1$, $P = P_0 + P_1$, $\Phi = \Phi_0 + \Phi_1$. Obtain the set of Euler equations for the perturbed quantifies $\rho_1, \vec{v}_1$ and Φ_1 in first order perturbation theory for an expanding universe.

(f) Let us now introduce the density contrast $\delta = \rho_1/\rho_0$. Let us further move to co-moving coordinates $\vec{x} = a^{-1}\vec{r}$. Show that the Euler equations can be written as

$$\partial\delta/\partial t + a^{-1}\vec{\nabla}_x\vec{v}_1 = 0, \quad \partial\vec{v}_1/\partial t + H\vec{v}_1 + a^{-1}\vec{\nabla}_x(c_s^2\delta + \Phi_1) = 0, \quad \nabla^2\Phi_1 = 4\pi G a^2\delta. \tag{4}$$

(Hint: Note that the gradient transforms as $\vec{\nabla}_x = a\vec{\nabla}$ and the time derivative as $(\partial/\partial t)_x = (\partial/\partial t) + H(\vec{r}\cdot\vec{\nabla})$).

(g) Derive the Jeans equation $\partial^2\delta/\partial t^2 + 2H(\partial\delta/\partial t) = c_s^2 a^{-2}\nabla_x^2\delta + 4\pi G\rho_0\delta$ for the density contrast δ in an expanding universe.

Expanding the density contrast in Fourier modes $\delta \sim \int d^3k\, \delta_k\, e^{-i\vec{k}\vec{x}}$, we can further rewrite the Jeans equation as

$$\frac{\partial^2 \delta}{\partial t^2} + 2H\frac{\partial \delta}{\partial t} + \left(\frac{c_s^2 k^2}{a^2} - 4\pi G \rho_0\right)\delta = 0.$$

We know from CMB fluctuations that, at the time of decoupling, the density perturbations of baryon matter coupled to photon gas were $\delta\rho/\rho \sim \delta T/T \sim 10^{-5}$. We now want to show that these initial baryonic density perturbations will not grow sufficiently enough to lead the structure we see today.

(h) Show that, in a matter dominated universe with $\Omega_{0,m} = 1$ and $\Omega_{0,r} = \Omega_{0,k} = \Omega_{0,\Lambda} = 0$, the scale factor is $a(t) = (\frac{3}{2}H_0 t)^{2/3}$. Also determine the time dependence of the Hubble parameter $H(t)$.

(i) Let us first consider the baryonic matter. It is tightly coupled to the photon gas and therefore has a relativistic speed of sound $c_s = 1/\sqrt{3}$. This means that the large pressure provided by photons dominates over the gravitational force such that we can ignore the gravitational term. Show that the Jeans equation is solved by a damped oscillator: $\delta = [c_1 \cos(ckt^{1/3}) + c_2 \sin(ckt^{1/3})]/(kt^{1/3})$ where c_1, c_2, c are constants. This explains why baryonic matter density perturbations are initially small. It also describes the baryonic acoustic oscillations features in the CBM and matter power spectrum.

(j) Let us now consider the late universe in which the baryonic matter is decoupled from the photon gas. In this case $c_s \sim 0$. Show that the solution to the Jeans equation is given by $\delta = c_1 t^{2/3} + c_2 t^{-1}$ (Reminder: The first Friedman equation can also be written as $H^2 = \frac{8\pi G}{3}\rho$).

(k) By how much would density perturbations in baryonic matter grow between the decoupling from at photon gas and today? Can this explain the observed density perturbations $\delta \gg 1$? How could dark matter solve the problem.

Homework 2

1. WIMP Freeze-out

The Boltzmann equation for a DM species is given by

$$\frac{dn}{dt} + 3Hn = -\langle \sigma v \rangle \left(n^2 - n_{eq}^2 \right) \tag{5}$$

In class, we have solved it using an analytic approximation and found that we can obtain the observed DM fraction $\Omega \sim 0.25$ for an annihilation cross section $\langle \sigma v \rangle \sim 10^{-9}$ GeV^{-2}. In this exercise, we will solve the Boltzmann equation numerically.

(a) We define $x = m/T$ and $Y = n/s$. Show that the Boltzmann equation can be written as $\frac{dY}{dx} = -\frac{s\langle \sigma v \rangle}{xH} \left(Y^2 - Y_{eq}^2 \right)$. [Hint: You can ignore that g_{s*} changes with temperature.]

(b) The equilibrium number density, the entropy and the Hubble parameter are given by

$$n_{eq} = g \left(\frac{mT}{2\pi} \right)^{3/2} e^{-m/T}, \qquad s = \frac{2\pi}{45} g_{*s}(T) T^3, \qquad H = \left(\frac{\pi^2 g_*(T)}{90} \right)^{1/2} \frac{T^2}{M_{pl}}. \tag{6}$$

Obtain Y_{eq}, s and H in terms of x.

(c) For the numerical solution, we will need $g_*(T)$. You can digitize the data (for example from Fig. 5 in arXiv:1904.07915) and interpolate it. Plot $g_*(T)$ vs. T for 100 keV $< T <$ 1 TeV. [Hints: We can use that $g_* = g_{s*}$ for the temperatures of interest].

(d) Solve the Boltzmann equation numerically for DM particle with $g = 2$, mass $m =$ 100 GeV and annihilation cross section $\langle \sigma v \rangle = 10^{-9}$ GeV^{-2} using tool of your choice (such as C++, Python, Mathematica). Plot Y vs. x for $5 < x < 100$. What is the freeze-out abundance Y_{fo}? How does it compare to the analytic estimate obtained in class $Y_{fo} \sim 10^{-9}$ GeV$/m$? [Hint: The problem is to solve an ODE: $dY/dx = f(x, Y)$. Most tools also already have efficient ODE solvers, such as `odeint` in Python, which one can use for this purpose. No need to reinvent the wheel.]

(e) Using the numerical simulation, for each mass in the range 1 GeV $< m <$ 1 TeV find the value of $\langle \sigma v \rangle$ that leads to the observed abundance $Y = \frac{3.6\text{ eV}}{m}\Omega$. Plot the result.

2. Millicharged Particle Freeze-out

As we have learned in the lecture, millicharged particle χ is a hypothetical Dirac fermion with a mass m_χ and fractional charge $q_\chi < 1$. We want to determine the masses and couplings for which millicharged particles can reproduce the observed relic abundance using feeze-out.

(a) In QFT, we have derived the $ee \to \mu\mu$ scattering cross section via an off-shell photon is given by

$$\sigma(\mu\mu \to ee) = \frac{4\pi}{3} \frac{\alpha^2}{s} \left(\frac{1 - 4m_e^2/s}{1 - 4m_\mu^2/s} \right)^{1/2} \left(1 + \frac{2m_\mu^2}{s} \right) \left(1 + \frac{2m_e^2}{s} \right). \tag{7}$$

Use this result to obtain the $\chi\chi \to ee$ annihilation cross section.

(b) Freeze-out occurs at non-relativistic velocities. In this limit, we can write the center-of-mass energy as $s \sim 4m_\chi^2$ and the velocity in the center of mass frame as $v_{cm} = \frac{p}{E}$. We further use that the relative velocity is given by $v_{rel} = 2v_{cm}$. Show that the thermally averaged annihilation cross section $\langle \sigma\, v \rangle$ is given by.

$$\langle \sigma v_{rel} \rangle_{\chi\chi \to ee} \frac{\pi q_\chi^2 \alpha^2}{m_\chi^2} \left(1 - \frac{4m_e^2}{s}\right)^{1/2} \left(1 + \frac{2m_e^2}{s}\right) \tag{8}$$

(c) In practice, the DM can annihilate into all kinematically accessible Standard Model particles. This includes leptons and hadrons. While the hadronic cross section is hard to estimate from first principles, we can use that the ratio $R = \sigma(ee \to \text{hadrons})/\sigma(ee \to \mu\mu)$ has been measured in electron scattering measurements. We can therefore write

$$\langle \sigma v \rangle_{\chi\chi \to \text{SM}} = \frac{\pi q_\chi^2 \alpha^2}{m_\chi^2} \left[\sum_{\ell = e, \mu \tau} \left(1 - \frac{4m_\ell^2}{s}\right)^{1/2} \left(1 + \frac{2m_\ell^2}{s}\right) + R(s) \left(1 - \frac{4m_\mu^2}{s}\right)^{1/2} \left(1 + \frac{2m_\mu^2}{s}\right) \right]. \tag{9}$$

For masses in the range 10 MeV $< m_\chi <$ 10 GeV estimate the charge that an MCPs needs to be produced with the right relic abundance through thermal freeze-out. [Hint: you need to digitize and interpolate the R-ratio values. You can obtain them from the Particle Data Group (PDG).]

(d) You should see a variety of features in the relic target line, coming from features in $R(s)$. Can you identity some of them? What is their origin? What's their impact?