Row Vector

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & -1 & 4 \end{bmatrix}$$

$r_1 = [2, 1, 0]$

$r_2 = [3, -1, 4]$

Column Vector

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & -1 & 4 \end{bmatrix}$$

$c_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ $c_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $c_3 = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$

Row Space

Spanned by the row vectors

Column Space

Spanned by the columns vectors

Null Space

Solution space of homogenus system

$Ax = 0$

finding bases for:

row and column space

1) row echolon form
* rows with leading one
* columns with leading one

null space

"solve the homogenous system with parametric equations"

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for the row space:
1) row echolon form
2) take the nonzero rows only
3) type them as a vector $\{(x, y, z), (a, b, c)\}$

for the column space:
1) row echolon form
2) take the column with the leading one only of the origenal matrix
3) type them as a vector

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the basis for the null space

The dimension of the nullspace of A is called the **nullity** of A.

example:

Find the nullspace of the matrix.

$$A = \begin{bmatrix} 1 & 2 & -2 & 1 \\ 3 & 6 & -5 & 4 \\ 1 & 2 & 0 & 3 \end{bmatrix}$$

Step 1

$$A = \begin{bmatrix} 1 & 2 & -2 & 1 \\ 3 & 6 & -5 & 4 \\ 1 & 2 & 0 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 2

$$\begin{aligned} x_1 + 2x_2 \quad + 3x_4 &= 0 \\ x_3 + x_4 &= 0. \end{aligned}$$

Step 3

Choose x_2 and x_4 as free variables to represent the solutions in this parametric form.

$x_1 = -2s - 3t, \quad x_2 = s, \quad x_3 = -t, \quad x_4 = t$

Step 4

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2s - 3t \\ s \\ -t \\ t \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

Step 5

A basis for the nullspace of A consists of the vectors

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

Rank Of Matrix

dim(row Space) = dim(Column Space)
rank(A) = dim(row or Column Space)
nullity(A) = dim(null Space)

How to get the rank:

1) RREF
2) rank = number of leading ones

EXAMPLE 5 Finding the Rank of a Matrix

Find the rank of the matrix

$$A = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 2 & 1 & 5 & -3 \\ 0 & 1 & 3 & 5 \end{bmatrix}.$$

SOLUTION Convert to row-echelon form as follows.

$$A = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 2 & 1 & 5 & -3 \\ 0 & 1 & 3 & 5 \end{bmatrix} \longrightarrow B = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Because B has three nonzero rows, the rank of A is 3.

How to get the nullity:

1) RREF
2) nullity = number of Columns which don't Contain leading ones

Dimension of the Solution Space

If A is an $m \times n$ matrix of rank r, then the dimension of the solution space of $A\mathbf{x} = \mathbf{0}$ is $n - r$. That is,

$$\underbrace{n}_{\text{number of Columns}} = \text{rank}(A) + \text{nullity}(A).$$

/ rank(A) = rank(A^T) / rank(A^T) + nullity(A^T) = m → num of rows

Rank and Nullity of a Matrix

example:

Let B be the reduced row-echelon form of A.

$$A = \begin{array}{c} \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ -2 & -1 & 1 & -1 & 3 \\ 0 & 3 & 9 & 0 & -12 \end{bmatrix} \\ \begin{array}{ccccc} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \end{array} \end{array} \longrightarrow B = \begin{array}{c} \begin{bmatrix} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & 3 & 0 & -4 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccccc} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \mathbf{b}_4 & \mathbf{b}_5 \end{array} \end{array}$$

(a) Find the rank and nullity of A.
(b) Find a subset of the column vectors of A that forms a basis for the column space of A.
(c) If possible, write the third column of A as a linear combination of the first two columns.

(a) Because B has three nonzero rows, the rank of A is 3. Also, the number of columns of A is $n = 5$, which implies that the nullity of A is n − rank = 5 − 3 = 2.

(b) Because the first, second, and fourth column vectors of B are linearly independent, the corresponding column vectors of A,

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 0 \\ -1 \\ -1 \\ 3 \end{bmatrix}, \quad \text{and} \quad \mathbf{a}_4 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix},$$

form a basis for the column space of A.

(c) The third column of B is a linear combination of the first two columns: $\mathbf{b}_3 = -2\mathbf{b}_1 + 3\mathbf{b}_2$. The same dependency relationship holds for the corresponding columns of matrix A.

$$\mathbf{a}_3 = \begin{bmatrix} -2 \\ -3 \\ 1 \\ 9 \end{bmatrix} = -2\begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ -1 \\ -1 \\ 3 \end{bmatrix} = -2\mathbf{a}_1 + 3\mathbf{a}_2$$

Finding the Solution Set of a Nonhomogeneous System

Find the set of all solution vectors of the system of linear equations.

$$\begin{aligned} x_1 \phantom{{}+x_2} - 2x_3 + x_4 &= 5 \\ 3x_1 + x_2 - 5x_3 \phantom{{}+x_4} &= 8 \\ x_1 + 2x_2 \phantom{{}-2x_3} - 5x_4 &= -9 \end{aligned}$$

LUTION The augmented matrix for the system $A\mathbf{x} = \mathbf{b}$ reduces as follows.

$$\begin{bmatrix} 1 & 0 & -2 & 1 & 5 \\ 3 & 1 & -5 & 0 & 8 \\ 1 & 2 & 0 & -5 & -9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 & 1 & 5 \\ 0 & 1 & 1 & -3 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The system of linear equations corresponding to the reduced row-echelon matrix is

$$\begin{aligned} x_1 \phantom{{}+x_2} - 2x_3 + x_4 &= 5 \\ x_2 + x_3 - 3x_4 &= -7. \end{aligned}$$

Letting $x_3 = s$ and $x_4 = t$, you can write a representative solution vector of $A\mathbf{x} = \mathbf{b}$ as follows.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2s - t + 5 \\ -s + 3t - 7 \\ s + 0t + 0 \\ 0s + t + 0 \end{bmatrix} = s\begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 3 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 5 \\ -7 \\ 0 \\ 0 \end{bmatrix}$$

$$= s\mathbf{u}_1 + t\mathbf{u}_2 + \mathbf{x}_p$$

Solutions of a System of Linear Equations

The system of linear equations $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is in the column space of A.

Consistency of a System of Linear Equations

Consider the system of linear equations

$$\begin{aligned} x_1 + x_2 - x_3 &= -1 \\ x_1 \phantom{{}+x_2} + x_3 &= 3 \\ 3x_1 + 2x_2 - x_3 &= 1. \end{aligned}$$

The rank of the coefficient matrix is equal to the rank of the augmented matrix.

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 3 & 2 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[A \ \vdots \ \mathbf{b}] = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 0 & 1 & 3 \\ 3 & 2 & -1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

As shown above, $\mathbf{b}$ is in the column space of A, and the system of linear equations is consistent.

If A is an $n \times n$ matrix, then the following conditions are equivalent.

1. A is invertible.
2. $A\mathbf{x} = \mathbf{b}$ has a unique solution for any $n \times 1$ matrix $\mathbf{b}$.
3. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
4. A is row-equivalent to I_n.
5. $|A| \neq 0$
6. $\text{Rank}(A) = n$
7. The n row vectors of A are linearly independent. and span R^n
8. The n column vectors of A are linearly independent. and span R^n
9. A has nullity 0.
10. A has rank n.

$\rightarrow$ rank + nullity = 0 if rank = n $\rightarrow$ nullity = 0