Effective Computability

- what can be solved algorithmically?

Hibert's Formalist Program

- Axiomatize mathematics
- Entscheidungsproblem - Given a mathematical statement, check if it is true
  (Algorithm for this)

Completeness theorem: FOL is axiomatizable
(Gödel)

$$\underbrace{\forall x \forall y \exists z \quad x < z < y \text{ or } y < z < x \text{ or } x = y = z}_{\text{not valid since the truth value depends on the interpretation of } x,y,z}$$

Valid formula: $\forall x \forall y (x < y \Rightarrow x < y)$

Multiple notions of effective computability

* λ- Calculus (Alonzo Church) $\to$ Functional Program
* μ- recursive functions (Kurt Gödel)
* Turing machines - intuitively very clear as to why this is the right defn.

All the various definitions compute the same class of functions

Church-Turing thesis: Any physically realizable computationally device can be simulated by a Turing machine

Turing Machines

$$M = (Q, \Sigma, \Gamma, \vdash, \sqcup, \delta, s, t, r)$$

blank symbol, start state, reject state

input alphabet, tape alphabet, left end of the tape, transition function, accept state

symbol to be written in the current cell

$$\delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R\}$$

current state, symbol under the tape head, new state, direction in which the tape head should move

Language of M $L(M) \subseteq \Sigma^*$

- one-way infinite tape

$\vdash \quad w_1 \quad w_2 \quad w_3 \quad w_4 \quad \sqcup \quad \sqcup \quad \sqcup \quad \sqcup \quad \sqcup \quad \sqcup \quad \sqcup \quad \ldots$

Constraints: $\forall p \in Q, \exists q \in Q$ s.t

$$\delta(p, \vdash) = (q, \vdash, R)$$

$$\delta(t, b) = (t, b', L/R)$$

$$\delta(r, b) = (r, b', L/R)$$

* $L = \{ a^n b^n c^n \mid n \geq 0 \}$

variable-cont # of a's b's & c's

$$\vdash \ \overset{\sqcup}{\cancel{a}} \ \overset{\sqcup}{\cancel{a}} \ \overset{\sqcup}{\cancel{a}} \ \overset{\sqcup}{\cancel{b}} \ \overset{\sqcup}{\cancel{b}} \ \overset{\sqcup}{\cancel{b}} \ \overset{\sqcup}{\cancel{c}} \ \overset{\sqcup}{\cancel{c}} \ \overset{\sqcup}{\cancel{c}} \ \dashv \ \sqcup$$

Configuration of a TM : $Q \times \Gamma^* \times \mathbb{N}$

current state

contents of the tape

position of the tape head

Initial configuration: $(s, \underbrace{\vdash w}, 0)$

after $|w|$, there are only $\sqcup$ symbols.

* A TM M accepts a string $x \in \Sigma^*$ if

$$(s, \vdash x, 0) \xrightarrow{*} (t, y, n)$$

y: contents of the tape

n: head position

* A TM M rejects a string $x \in \Sigma^*$ if

$$(s, \vdash x, 0) \xrightarrow{*} (r, y, n)$$