1. **[2 marks]** Let $\mathcal{L}_{\text{reg}}$ denote the set of all regular languages over the alphabet $\{0, 1\}$. Consider the following statement.

   The set $\mathcal{L}_{\text{reg}}$ is uncountable.

   Is the statement true? Justify your answer.

   Solution: We saw in class the set of all C programs is countable. The set $\mathcal{L}_{\text{reg}}$ is clearly a subset of that.

   Another way to see it is that every regular language is accepted by a DFA $M = (Q, \Sigma, \delta, q_0, F)$. Thus every DFA M has a finite description. Hence the set of all DFAs are countable, and consequently, the set of regular languages $\mathcal{L}_{\text{reg}}$ is also countable.

   So, the statement is **false**.

2. **[2 marks]** Suppose that $\{L_i\}_{i\in\mathbb{N}}$ is a countably infinite collection of languages over the alphabet $\{0, 1\}$. Consider the following statement.

   If all the languages L_i are regular, then $\bigcup_{i\geq 1} L_i$ is regular.

   Is the statement true? Justify your answer.

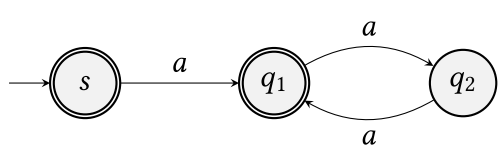
   Solution: We have seen in class that there exists non-regular languages. Let L be such a non-regular language - hence, L is a countable set. Now, L can be written as $\cup_{i\geq 1} L_i$ where each L_i is a singleton set containing the i^{th} string of L. Clearly, each of the L_is are regular, but L is not regular.

   Hence, the statement is **false**.

   It is not sufficient to say that the product construction taught in class will give a DFA with infinitely many states. That only shows that the product construction does not work for infinite union. It still does not justify why the union of infinitely many regular languages is not regular.

3. **[3 marks]** The *index* of a language is defined as the smallest k such that $L^k = L^{k+1}$. If such an integer does not exist, then the index is said to be infinity.

   Consider the following automaton M over the alphabet $\Sigma = \{a\}$.



   What is the index of the language accepted by M? If the index k is finite, clearly describe L^i for each $i \leq k$.

   Solution: Let $L = L(M)$. From the DFA M, we can that $L = \{\varepsilon\} \cup \{a^n | n \text{ is odd}\}$. Thus, $L^2 = L.L$ is the set of all string a^n where $n \geq 0$. This is because every string a^n for n even can be written as $a^{n_1}.a^{n_2}$ where n_1 and n_2 are odd. Every string a^n for n odd can be written as $\varepsilon.a^n$.

   Now, $L^3 = L.L^2 = L^2$ since $L^2 = \{a^n | n \geq 0\}$ and $\varepsilon \in L$.

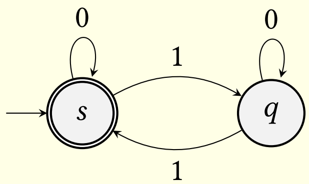
   Thus, the index of M is 2.

4. Let $\Sigma = \{0, 1\}$, and let $\#1(w)$ refer to the number of 1s in a string w. The symbol ε refers to the empty string. Construct DFAs and justify your constructions for the following languages.

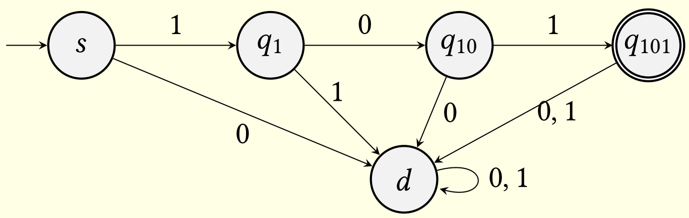
If you are using the closure properties, then you just need to describe the DFAs for the languages you are combining, and explain clearly how the closure properties will be used. **Do not use NFAs**.

(a) **[3 marks]** $L = \{w | w \in \{0,1\}^* - \{101\}, \text{and } \#1(w) \text{is even}\}$.

Solution: The language $L = L_1 - L_2$, where L_1 is the set of all strings w with $\#1(w)$ even, and $L_2 = \{101\}$. A DFA for L_1 would be as follows.



A DFA for L_2 would be as follows.



Now, $L = L_1 - L_2 = L_1 \cap \overline{L_2}$. Use the closure properties.

(b) **[5 marks]** $L = \{w \mid \exists\, x, y \in \{0,1\}^* - \varepsilon \text{ such that } w = xy, \text{ and } \#1(y) > \#1(x)\}$.

Solution: We can show that $L = L_1 \cup L_2$, where

$$\begin{aligned} L_1 &= \{w \mid w \text{ starts with } 0 \text{ and } 0 < \#1(w) \leq 2\}, \\ L_2 &= \{w \mid \#1(w) \geq 3\}. \end{aligned}$$

($\supseteq$) Let $w \in L_1 \cup L_2$. If $w \in L_1$, then $w = 0.y$ and we have $\#1(0) = 0$ and $\#1(y) > 0$. If $w \in L_2$, then let x be the substring of w that ends at the first occurence of 1, and let y be the remainder of x. Clearly $\#1(x) = 1$ and $\#1(y) \geq 2$. Hence, $L_1 \cup L_2 \subseteq L$.

($\subseteq$) We will do a case analysis.

- Suppose that $w \in L$ and $0 < \#1(w) \leq 2$. Then if w starts with 1, then for any x, y such that $|x|, |y| > 1$ and $w = xy$, $\#1(x) \geq 1$ and $\#1(y) \leq 1$. Therefore, w starts with 0 and hence $w \in L_1$.
- If $w \in L$, and $\#1(w) \geq 3$, then $w \in L_2$ (from the definition of L_2).

Hence $L \subseteq L_1 \cup L_2$.

The DFAs for L_1 and L_2 are as follows.