1. For each of the languages below, find the largest distinguishable set. If it is finite, draw the DFA with the corresponding number of states.

   For the problems below, I have written the distinguishable set. Except for Part (d), I haven't written the explicit proof of why it is distinguishable. Try to do that on your own - if you have doubts regarding the same, please post on Ed.

   For Part (b), I have given a proof showing why the set given is the largest. You can do this in another way as well - construct the minimal DFA and show that the number of states in the minimal DFA is equal to the size of the largest distinguishable set.

   For infinitely large distinguishable sets, all sets are countably infinite and hence will have the same cardinality.

   (a) $L = \{w \mid \text{ ends with } 1111\}$.

   Solution: Consider the set $S = \{0, 1, 11, 111, 1111\}$.

   (b) $L = \{1^k x \mid x \in \{0,1\}^* \text{ and } \#1(x) \geq k, k \geq 1\}$.

   Solution: Consider the set $S = \{\varepsilon, 0, 1, 11\}$. Let's try to show that this is indeed the largest distinguishable set. For a contradiction, assume that there is another string w that can be added to it. We look at possible cases for w.

   - If w starts with 0, then for every z, $wz \notin L$. Hence w and 0 are indistinguishable, and cannot be added to S.
   - If w starts with 10, then for any z, $wz \in L$ iff z contains at least one more 1. Thus, for any z, $1z \in L$ iff $wz \in L$ and hence w and 1 are indistinguishable.
   - If w starts with 11, then for every z, wz and $11z$ are in L, and hence w and 11 are indistinguishable.

   Hence we cannot add any string to S and still make it distinguishable, and hence S must be the largest distinguishable set.

   (c) $L = \{1^k x \mid x \in \{0,1\}^* \text{ and } \#1(x) \leq k, k \geq 1\}$.

   Solution: Consider the set $S = \{1^i 0 \mid i \geq 0\}$.

   (d) $L = \{w \mid \exists\, x, y \in \{0,1\}^* - \varepsilon \text{ such that } w = xyx\}$.

   Solution: Consider the set $S = \{0^i 1^i 01 \mid i \geq 1\}$. Given $w = 0^i 1^i 01$ and $w' = 0^j 1^j 01$. Assume that $i > j$. Choose $z = 0^i 1^i$. Clearly $wz \in L$. Now $w'z = 0^j 1^j 010^i 1^i$. Since $i > j$, there is no way to split this as xyx.

   (e) $L = \{a^i b^j c^k \mid i, j, k \geq 0 \text{ and } i = 1 \Rightarrow j = k\}$.

Solution: Consider the set $S = \{ab^j \mid j \geq 0\}$.

(f) $L = \{w \mid w \neq w^R\}$.

Solution: Consider the set $S = \Sigma^*$.

(g) $L = \{cabc \mid a, b, c$ are non-empty strings$\}$.

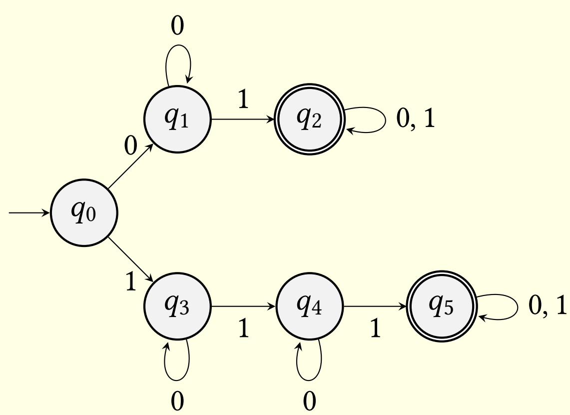
2. Consider the following language that we discussed in detail in class.

$$L = \{w \mid \exists x, y \in \{0,1\}^* - \varepsilon \text{ s.t } w = xy, \text{ and } \#1(y) > \#1(x)\}.$$

Draw two non-isomorphic DFAs with the same number of states for L, and write down the equivalence classes of the Myhill-Nerode relation corresponding to the DFAs.

Solution:

Consider the following DFAs.



We want to show that there is no isomorphism between the two DFAs. Clearly q_0 should be mapped to itself in any isomorphism. Similarly, q_3 must be mapped to itself - this is because we saw in class that there are no states equivalent to q_3 for this DFA. Now, suppose that q_1 is mapped to iself in an isomorphism f. Then f should map q_4 to itself as well. But then, $f(\delta(q_1, 0)) = q_1$, whereas $\delta'(f(q_1), 0) = \delta'(q_1, 0) = q_4$. Thus, the isomorphism cannot map q_1 to itself.

The only other possibility is that the isomorphism f maps q_1 to q_4 and q_4 to q_1. But then, $f(\delta(q_3, 1)) = f(q_4) = q_1$, and $\delta'(f(q_3), 1) = \delta'(q_3, 1) = q_4$.

Thus, there cannot exist an isomorphism between the two DFAs.

3. Given a DFA $M = (Q, \Sigma, \delta, q_0, F)$ accepting L, recall the quotient automata $M/\approx$ defined in class. Verify the following properties mentioned in class.

(a) Show that $L(M/\approx) = L(M)$.

Solution: Let $M/\approx = (Q', \Sigma, \delta', q_0', F')$. Consider a $w \in L(M/\approx)$. Then, we can say that

$$\begin{aligned} w \in L(M/\approx) &\Leftrightarrow \widehat{\delta'}(q_0', w) \in F' \\ &\Leftrightarrow \widehat{\delta'}([q_0], w) \in F' \\ &\Leftrightarrow [\widehat{\delta}(q_0, w)] \in F' \\ &\Leftrightarrow \widehat{\delta}(q_0, w) \in F \Leftrightarrow w \in L. \end{aligned}$$

(b) Show that $\equiv_L \subseteq \equiv_{M/\approx}$, and hence conclude that $M/\approx$ is the minimal automata.

Solution: To prove the statement given, we have to show that if $x \equiv_L y$, then $x \equiv_{M/\approx} y$. Suppose $x \equiv_L y$. Then, for every z, $xz \in L$ iff $yz \in L$. Consider $\widehat{\delta'}(q_0', x) = [\widehat{\delta}(q_0, x)]$, and $\widehat{\delta'}(q_0', y) = [\widehat{\delta}(q_0, y)]$. Now, for any z, we have

$$\begin{aligned} \widehat{\delta'}(q_0', xz) &= \widehat{\delta'}(\widehat{\delta'}(q_0', x), z) \\ &= \widehat{\delta'}([\widehat{\delta}(q_0, x)], z) \end{aligned}$$

Similarly, $\widehat{\delta'}(q_0', yz) = \widehat{\delta'}([\widehat{\delta}(q_0, x)], z)$. Thus, we have

$$\widehat{\delta'}([\widehat{\delta}(q_0, x)], z) \in F \text{ iff } \widehat{\delta'}([\widehat{\delta}(q_0, y)], z) \in F'.$$

Since $M/\approx$ is the quotient automata, this means that $[\widehat{\delta}(q_0, x)] = [\widehat{\delta}(q_0, y)]$, and hence we have $\widehat{\delta'}(q_0', x) = \widehat{\delta'}(q_0', y)$. Therefore, $x \equiv_{M/\approx} y$.

4. Let $S, L \subseteq \Sigma^*$ be two infinite sets. Suppose that for any two strings $x, y \in S$, there exist strings $w, z \in \Sigma^*$ such that $wxz \in L$ and $wyz \notin L$. Show that L is not regular.

Solution: Suppose that L is regular and $M = (Q, \Sigma, \delta, q_0, F)$ is the DFA that accepts L. Let $\mathcal{F} = \{f : Q \to Q\}$ be the set of all functions that map Q to Q. Since Q is finite, this set is definitely finite. Now, each $x \in \Sigma^*$ defines a function $f_x : Q \to Q$ as follows: $f_x(q) = \widehat{\delta}(q, x)$. Since S is infinite, there must exists two strings x, y such that f_x and f_y is the same function mapping Q to Q. We also know that there exists $w, z \in \Sigma^*$ such that $wxz \in L$ and $wyz \notin L$. Let $\widehat{\delta}(q_0, w) = q$. Since f_x and f_y are the same function, we have $\widehat{\delta}(q, x) = \widehat{\delta}(q, y) = q'$. But, this implies that $\widehat{\delta}(q, xz) = \widehat{\delta}(q, yz)$, and this contradicts the fact that $wxz \in L$ and $wyz \notin L$.