Product construction

Theorem: If $L_1, L_2 \subseteq \Sigma^*$ are regular, then

(i) $L_1 \cup L_2$ is regular

(ii) $\overline{L_1}$ is regular

(iii) $L_1 \cap L_2$ is regular

Proof: (i) $\exists$ $M_1 = (Q_1, \Sigma, \delta_1, s_1, F_1)$ &

$M_2 = (Q_2, \Sigma, \delta_2, s_2, F_2)$ s.t

$L(M_1) = L_1$ & $L(M_2) = L_2$

Construct M s.t $L(M) = L_1 \cup L_2$

- Simulate M_1 & M_2 simultaneously

$Q = Q_1 \times Q_2 \longrightarrow$ to keep track of M_1 & M_2

$\delta : Q_1 \times Q_2 \times \Sigma \to Q_1 \times Q_2$

$$\delta((q_1, q_2), \sigma) = \underbrace{(\delta_1(q_1, \sigma), \delta_2(q_2, \sigma))}_{\text{simulation of 1 step of } M_1 \text{ \& } M_2}$$

$s = (s_1, s_2)$ ↳ both M_1 & M_2 start from their respective start states

$F = (Q_1 \times F_2) \cup (F_1 \times Q_2)$

↳ at least one of M_1 & M_2 should land in an accepting state

Claim: $L(M) = L_1 \cup L_2$

① $\hat{\delta}((q_1, q_2), w) = (\hat{\delta}_1(q_1, w), \hat{\delta}_2(q_2, w))$

Base case: By definition

Induction step: $w = w'\sigma$

$$\hat{\delta}((q_1, q_2), w\sigma) = \delta(\hat{\delta}((q_1, q_2), w), \sigma)$$
$$= \delta((\hat{\delta}_1(q_1, w), \hat{\delta}_2(q_2, w)), \sigma)$$
$$= (\delta_1(\hat{\delta}_1(q_1, w), \sigma), \delta_2(\hat{\delta}_2(q_2, w), \sigma))$$
$$= (\hat{\delta}_1(q_1, w\sigma), \hat{\delta}_2(q_2, w\sigma))$$

② $\hat{\delta}((s_1, s_2), w) \in (Q_1 \times F_2) \cup (F_1 \times Q_2)$

iff $\hat{\delta}_1(s_1, w) \in F_1$ or $\hat{\delta}_2(s_2, w) \in F_2$

$$\hat{\delta}((s_1, s_2), w) = (\hat{\delta}_1(s_1, w), \hat{\delta}_2(s_2, w))$$
$$\downarrow$$
$$(Q_1 \times F_2) \cup (F_1 \times Q_2)$$

$\iff \hat{\delta}_1(s_1, w) \in F_1$ or $\hat{\delta}_2(s_2, w) \in F_2$

(ii) $\forall w \in \Sigma^* \quad \hat{\delta}(s, w) \in F$ or $\hat{\delta}(s,w) \in Q \setminus F$

$w \notin L \iff \hat{\delta}(s,w) \in Q \setminus F$

$M' = (Q, \Sigma, \delta, s, Q \setminus F)$

$L(M') = \overline{L(M)}$ for $M = (Q, \Sigma, \delta, s, F)$

(iii) $L_1 \cap L_2 = \overline{\overline{L_1 \cap L_2}} = \overline{\overline{L_1} \cup \overline{L_2}}$

Exercise: Try using the product construction

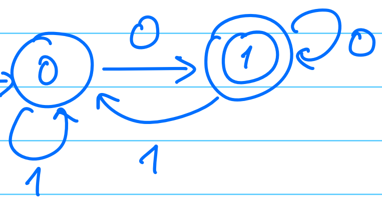
Qn: If L_1 & L_2 are regular, is $L_1 \cdot L_2$ regular?

$L_1 = \{ w \in \{0,1\}^* \mid w \text{ ends with a } 0 \}$

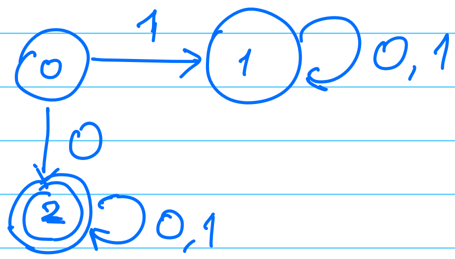
$L_2 = \{ w \in \{0,1\}^* \mid w \text{ starts with a } 0 \}$

$L = L_1 \cdot L_2$

$M_1 =$



M_2:



Can you patch up M_1 and M_2?

- The new DFA should guess when it should stop the simulation of M_1 and start simulating M_2

- This is captured by the idea of non-determinism