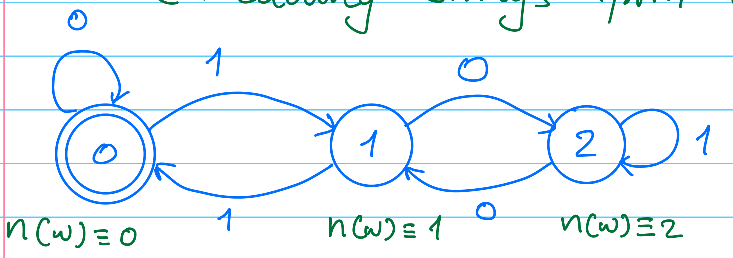
Set of binary strings that are the binary representations of numbers divisible by 3

$000 \in L \quad 001 \notin L \quad 00011 \in L \quad \dots$

(Reading strings from MSB)



$n(w) =$ # corr. to w

$n(w0) = 2 \cdot n(w)$

$n(w1) = 2 \cdot n(w) + 1$

$\delta(q, \sigma) = 2q + \sigma \pmod 3$ → stronger induction hypothesis

$$\hat{\delta}(0, w) \in L \iff \delta(\hat{\delta}(0, w'), \sigma) \in L$$

$$\hat{\delta}(0, w) = \delta(\hat{\delta}(0, w'), \sigma) \quad \text{where } w = w'\sigma$$

$$= \delta(n(w') \bmod 3, \sigma)$$

$$= 2(n(w') \bmod 3) + \sigma \pmod 3$$

$$= 2\,n(w') + \sigma \pmod 3$$

$$= n(w) \pmod 3$$

Closure properties of languages

- $L_1 \cup L_2$ - set union
- $L_1 \cap L_2$ - set intersection
- $L_1 \cdot L_2$ - concatenation

$$= \{ w \cdot w' \mid w \in L_1,\ w' \in L_2 \}$$

Eg ① $L_1, L_2 \subseteq \{0,1,2,..,9\}^*$

$$L_1 = \{ n \in \mathbb{N} \mid n \text{ is even} \}$$

$$L_2 = \{ n \in \mathbb{N} \mid n \text{ is odd} \}$$

Qn: $L_1 \cdot L_2 = \{ n \in \mathbb{N} \mid n \text{ contains } 0,2,4,6,8 \text{ \& ends in } 1,3,5,7,9 \}$

② $L_1 = \{ w \in \{0,1\}^* \mid w \text{ contains} \geq 2 \text{ zeroes} \}$

$L_2 = \{ w \in \{0,1\}^* \mid w \text{ contains} \geq 3 \text{ zeroes} \}$

$L_1 \cdot L_2 = \{ w \in \{0,1\}^* \mid w \text{ contains} \geq 5 \text{ zeroes} \}$

③ $L_1 = \{ w \in \{0,1\}^* \mid w \text{ contains} \geq 2 \text{ zeroes} \}$

$L_2 = \{ w \in \{0,1\}^* \mid w \text{ contains} \leq 3 \text{ zeroes} \}$

$L_1 \cdot L_2 = L_1$

$$L^i = L \cdot L^{i-1} \quad ; \quad L^0 = \{\epsilon\}$$

- Asterate : $L^* = \bigcup_{i \geq 0} L^i$

eg: $L_{odd} = \{ w \in \{0,1\}^* \mid |w| \equiv 1 \pmod 2 \}$

$L_{even} = \{ w \in \{0,1\}^* \mid |w| \equiv 0 \pmod 2 \}$

$$L_{even}^* = L_{even}$$

$$L_{odd}^* = \{0,1\}^*$$

Product construction

Theorem: If $L_1, L_2 \subseteq \Sigma^*$ are regular, then

(i) $L_1 \cup L_2$ is regular

(ii) $\overline{L_1}$ is regular

(iii) $L_1 \cap L_2$ is regular

Proof: (i) $\exists$ $M_1 = (Q_1, \Sigma, \delta_1, s_1, F_1)$ &

$M_2 = (Q_2, \Sigma, \delta_2, s_2, F_2)$ s.t

$L(M_1) = L_1$ & $L(M_2) = L_2$

Construct M s.t $L(M) = L_1 \cup L_2$

- Simulate M_1 & M_2 simultaneously

$Q = Q_1 \times Q_2 \longrightarrow$ to keep track of M_1 & M_2

$\delta : Q_1 \times Q_2 \times \Sigma \rightarrow Q_1 \times Q_2$

$$\delta((q_1, q_2), \sigma) = \underbrace{(\delta_1(q_1, \sigma), \delta_2(q_2, \sigma))}$$

simulation of 1 step of M_1 & M_2

$s = (s_1, s_2)$ → both M_1 & M_2 start from their respective start states

$F = (Q_1 \times F_2) \cup (F_1 \times Q_2)$

→ at least one of M_1 & M_2 should land in an accepting state