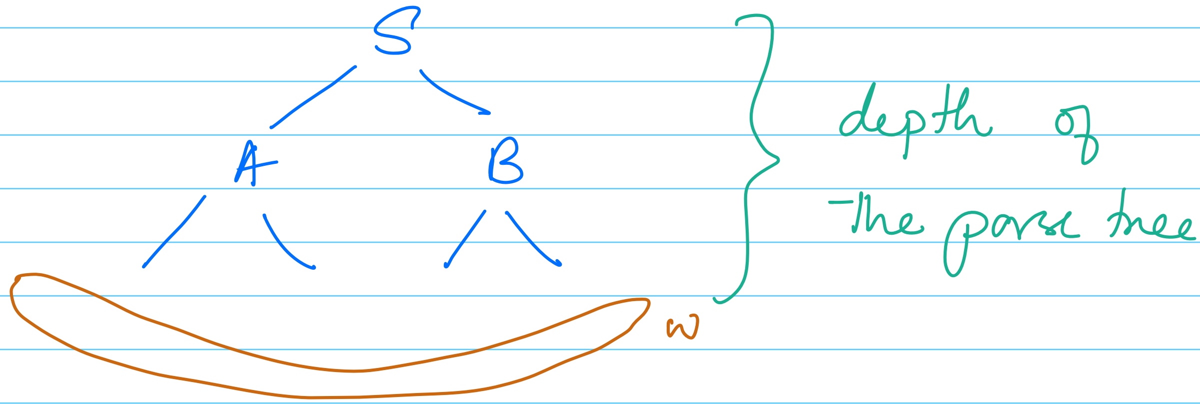
Pumping lemma

Consider G in CNF

let $w \in L(G)$, and consider the parse tree for w



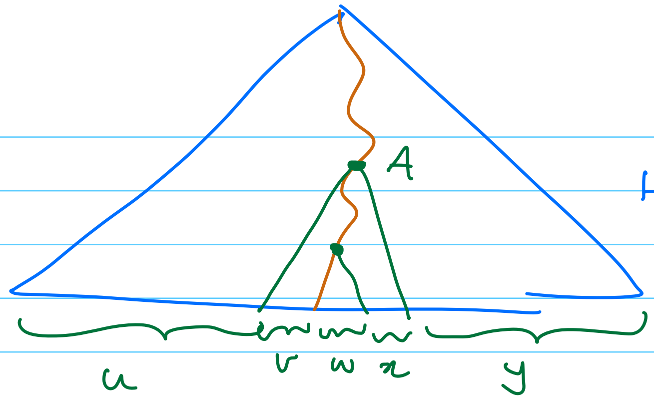
* If G is in CNF, then the parse tree for $w \in L(G)$ is a binary tree.

$G = (N, \Sigma, P, S)$ and let $|N| = n$

Suppose $w \in L(G)$ s.t $k = |w| > 2^{n+1}$

↳ depth of the parse tree $> n+1$

In the longest path from root to leaf, $\exists$ non-terminal $A \in N$ that occurs at least twice



$L = \{0^n 1^n \mid n \geq 0\}$ $\qquad$ $S \to 0S1 \mid \varepsilon$

$\downarrow$

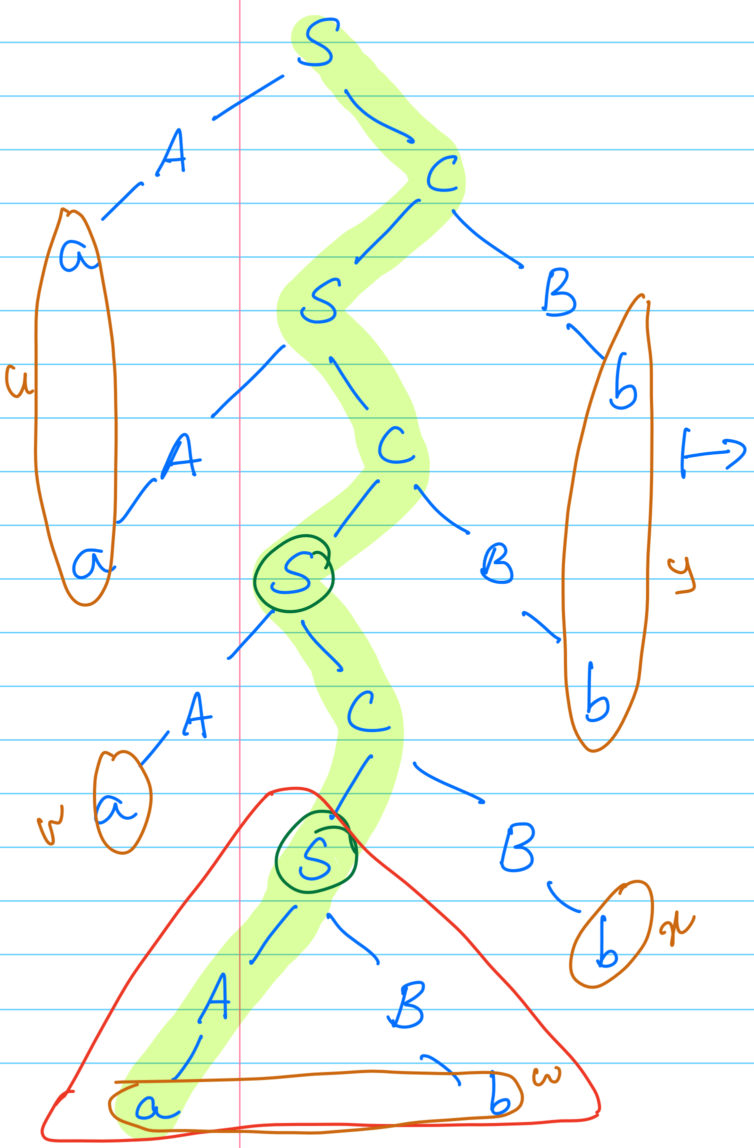
$S \to 01 \mid 0S1$ $\qquad$ $S \to AB \mid AC$

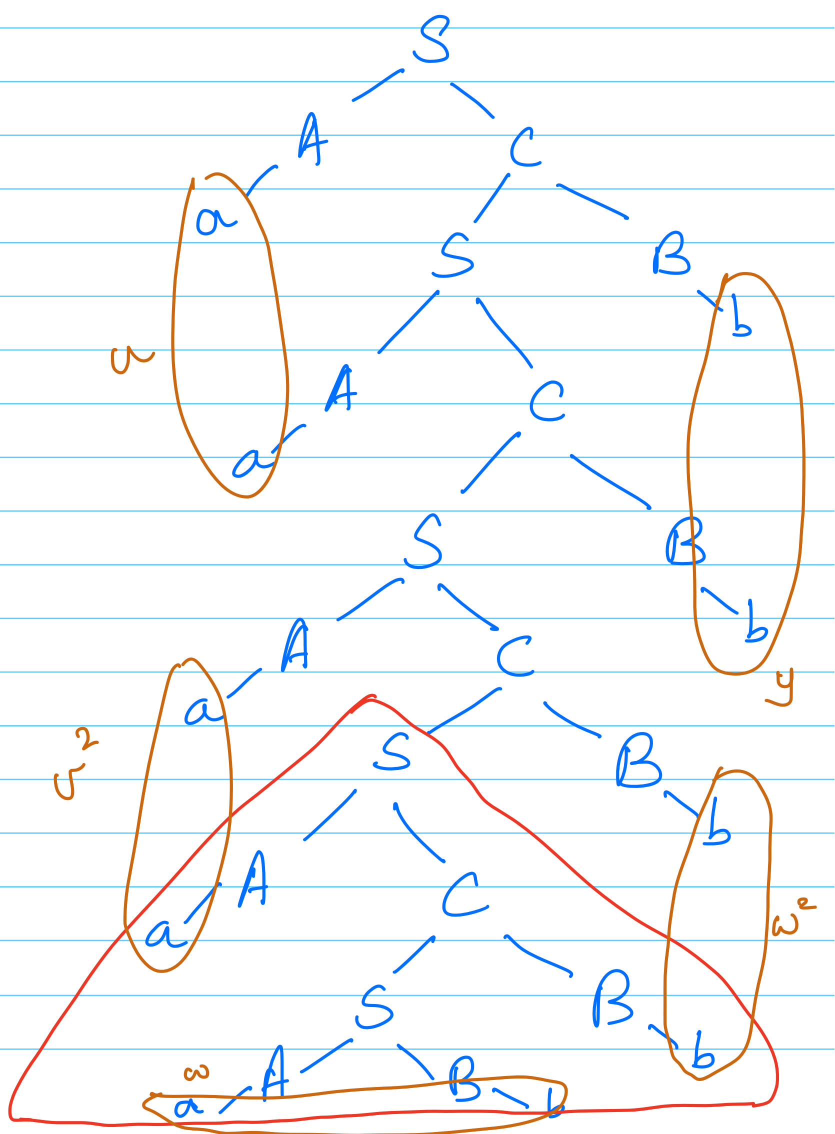
$A \to 0$

$B = 1$

$C \to SB$

$w = 0^4 1^b$





Statement of the pumping lemma

For every CFL L, $\exists$ $k > 0$ s.t
$\forall z \in L$ s.t $|z| \geq k$, $\exists$ u, v, w, x, y s.t
$z = uvwxy$ with $vx \neq \varepsilon$ & $|vwx| \leq k$
s.t $\forall i \geq 0$ $uv^i w x^i y \in L$

Contrapositive form

If $\forall k > 0$ $\exists z \in L$ with $|z| \geq k$ s.t
$\forall u, v, w, x, y$ with $vx \neq \varepsilon$ & $|vwx| \leq k$ s.t
$z = uvwxy$, $\exists i \geq 0$ s.t $uv^i w x^i y \notin L$
then L is not context free

Game between Prover & Spoiler

Prover picks $k > 0$
Spoiler choose $z \in L$ s.t $|z| \geq k$
Prover chooses u, v, w, x, y s.t

- $vx \neq \varepsilon$, $|vwx| \leq k$
- $z = uvwxy$

Spoiler tries to find $i \geq 0$ s.t
$uv^i w x^i y \notin L$

Examples

(i) $L = \{a^n b^n c^n \mid n \geq 0\}$

- Prover chooses $k > 0$
- Spoiler chooses $z = a^k b^k c^k$
- Prover chooses u, v, w, x, y

$vx \neq \varepsilon, \quad |vwx| \leq k$

Multiple options available for the prover

Can prover choose s.t vwx contains a, b, c?

For all these cases, choose $i = 0$

- if v contains a, then x does not contain c
- if x contains c, then v does not contain a
- v, w, x could all be inside b^k

* CFLs are not closed under intersection

$$\{a^n b^n c^n \mid n \geq 0\} = \{a^i b^j c^k \mid i = j\} \cap \{a^i b^j c^k \mid j = k\}$$

$$\downarrow$$

$$S \to PQ$$
$$Q \Rightarrow cQ \mid \varepsilon$$
$$P \to aPb \mid \varepsilon$$