Equivalence of NFAs and DFAs

$N = (Q, \Sigma, \Delta, q_0, F)$

$\Big\downarrow$ simulate the NFA

$M = (\mathcal{P}(Q), \Sigma, \delta, \{q_0\}, F')$

$$\delta(Q', \sigma) = \bigcup_{q \in Q'} \Delta(q, \sigma) = \hat{\Delta}(Q', \sigma)$$

$Q' \subseteq Q$

$$F' = \{ Q' \mid F \cap Q' \neq \emptyset \}$$

Proof of the equivalence

for any ω : $\hat{\Delta}(q_0, \omega) \cap F \neq \emptyset$

iff

$\hat{\delta}(\{q_0\}, \omega) \in F'$

Induction on $|\omega|$

Base case: $|\omega| = 1$: $\omega = \sigma$

$$\hat{\Delta}(q_0, \sigma) = \Delta(q_0, \sigma) = \delta(\{q_0\}, \sigma)$$

Induction step

$$\hat{\delta}(Q, \omega\sigma) = \delta(\hat{\delta}(Q, \omega), \sigma)$$

$$= \delta(\hat{\Delta}(Q, w), \sigma)$$

$$= \hat{\Delta}(\hat{\Delta}(Q, w), \sigma) = \hat{\Delta}(Q, w\sigma)$$

Closure properties

① If L is regular, then

$L^R = \{ w^R \mid w \in L \}$ is regular

- Make all the final states as Start states
- Make the start state a final state

② If L is regular, then

$\sqrt{L} = \{ w \mid w^2 \in L \}$ is regular

- Guess the state q s.t $\hat{\delta}(q_0, w) = q$
- Verify that $\hat{\delta}(q, w) \in F$

$M = (Q, \Sigma, \delta, q_0, F)$ $\quad L(M) = L$

$N = (Q', \Sigma, \Delta, S, F')$

$Q' = Q \times Q \times Q$ → state where we reach starting from the guessed state on reading w

first Q: store the guessed state

second Q: state where w is taking M

$$\Delta((g, q, q'), \sigma) = (g, \delta(q, \sigma), \delta(q', \sigma))$$

$$S = \{ (g, q_0, g) \mid g \in Q \}$$

$$F' = \{ (g, g, f) \mid f \in F \}$$

$$w^2 \in L \iff \hat{\delta}(q_0, w^2) \in F$$

$$\iff \hat{\delta}(\hat{\delta}(q_0, w), w) \in F$$

Let $g = \hat{\delta}(q_0, w)$

then $w^2 \in L \iff \hat{\delta}(g, w) \in F$

Now

from the defn of Δ and a simple induction $\longrightarrow$ $\hat{\Delta}((g, q_0, g), w) = (g, \hat{\delta}(q_0, w), \hat{\delta}(g, w))$

$= (g, g, f)$ where $f \in F$

for $w \in \sqrt{L} \iff \exists\, g$ s.t.

$$\hat{\Delta}((g, q_0, g), w) \cap F \neq \phi$$

for this g $\hat{\Delta}((g, q_0, g), w) = (g, \hat{\delta}(q_0, w), \hat{\delta}(g, w))$

$\iff$ and hence $(g, \hat{\delta}(q_0, w), \hat{\delta}(g, w)) \cap F' \neq \phi$

$$\iff \hat{\delta}(q_0, w) = g \text{ and } \hat{\delta}(g, w) \cap F \neq \phi$$

$$\iff \hat{\delta}(\hat{\delta}(q_0, w), w) \cap F \neq 0$$

$$\iff w^2 \in L$$