1. **[2 marks]** Consider the following statement about a language $L \subseteq \Sigma^*$.

   If L is regular, then $\exists$ a DFA $M = (Q, \Sigma, \delta, q_0, F)$ such that $|F| = 1$ and $L(M) = L$.

   Is the statement true? Justify your answer.

   Solution: This statement is incorrect.

   Consider the language $L = \{0, 00\}$, where $\Sigma = \{0\}$. Suppose that there is a DFA for L with a single final state. Then $\widehat{\delta}(q_0, 0) = \widehat{\delta}(q_0, 00)$. But then, $\widehat{\delta}(q_0, 00) = \widehat{\delta}(q_0, 000)$ which cannot happen since $00 \in L$ and $000 \notin L$.

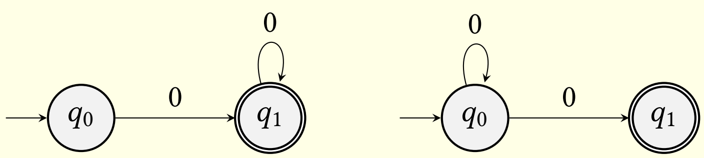
2. **[2 marks]** Consider the following statement.

   Let N_1 and N_2 be two NFAs with the minimum number of states accepting a language L. Then, $N_1 \cong N_2$.

   Is the statement true? Justify your answer.

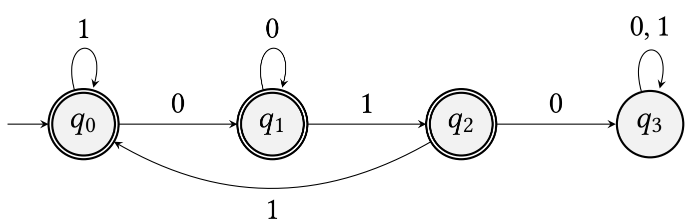
   Solution: This statement is incorrect.

   Consider $L = \{0^i \mid i \geq 1\}$. We have the following NFAs (one of which is a DFA).

   The two NFAs are not isomorphic. They are minimal as well since if there is an NFA with just a single state, then either its language must be empty, or it will accept ε.

3. **[3 marks]** Write down the regular expression corresponding to L given by the DFA below. Explain your answer clearly.

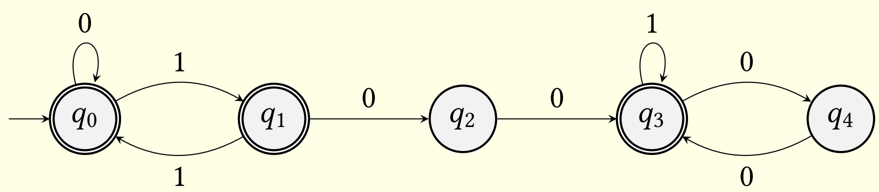
   Solution: We can do state elimination here. We add a new start state s with ε-transition to q_0, and a new final state f with ε-transitions from q_0, q_1, and q_2 to f. One order of elimination is q_3, q_2, q_1, and finally q_0. This gives the final regex as $(1 + 00^*11)^*(\varepsilon + 00^*(1 + \varepsilon))$.

4. **[4 marks]** Draw a DFA corresponding to the language given by the regular expression R below. Explain your answer clearly.

$$R = (0 + 11)^*(1 + 00)^*$$

Solution: We can describe this language as follows: $L(R)$ contains all string w such that w can be written as xy, where $\#1(x)$ is even and no odd occurrence of 1 in x is followed by a 0, and $\#0(y)$ is even and no odd occurrence of 0 in y is followed by a 1.

Thus, we can draw the DFA as follows.



Notice that the only incoming transition to q_1 is from q_0, and the only incoming transition to q_0 is from q_1. Hence the set of strings that reach q_0 are precisely $(0 + 11)^*$, and the set of strings that reach q_1 is $(0 + 11)^*1$. Hence the strings that reach q_3 is $(0 + 11)^*100(1 + 00)^*$. Thus the language accepted by the DFA is $R' = (0+11)^* + (0+11)^*1 + (0+11)^*100(1+00)^*$. Now, $L(R') \subseteq L(R)$ from the way R' is written. For any $w \in L(R)$, if no odd occurrence of a 1 is followed by a 0, then $w \in (0 + 11)^*$. Otherwise $w \in (0 + 11)^*100(1 + 00)$. Hence $L(R) \subseteq L(R')$.

5. **[4 marks]** For a regular expression R, $\mathrm{rev}(R)$ is defined as follows.

- If $R = \varepsilon$, then $\mathrm{rev}(R) = \varepsilon$,
- If $R = \sigma$, for $\sigma \in \Sigma$, then $\mathrm{rev}(R) = \sigma$,
- If $R = R_1 \cdot R_2$, then $\mathrm{rev}(R) = \mathrm{rev}(R_2) \cdot \mathrm{rev}(R_1)$,
- If $R = R_1 + R_2$, then $\mathrm{rev}(R) = \mathrm{rev}(R_1) + \mathrm{rev}(R_2)$, and
- If $R = R_1^*$, then $\mathrm{rev}(R) = \mathrm{rev}(R_1)^*$.

Consider the following statement about R and $\mathrm{rev}(R)$.

If $L = L(R)$, then $L(\mathrm{rev}(R)) = \{w \mid \text{ reverse of } w \text{ is in } L\}$.

Is the statement true? Justify your answer.

Solution: This statement is correct.

Let $R' = \mathrm{rev}(R)$. We do this inductively on the length of R.

- If $R = \varepsilon$, then $R' = \varepsilon$, and $L(R') = L(R)$ and $\varepsilon = \text{rev}(\varepsilon)$. Similarly for $R = \sigma$ for $\sigma \in \Sigma$. These are the simple base cases.

- If $R = R_1 + R_2$, then $R' = \text{rev}(R_1) + \text{rev}(R_2)$. Now, we have

$$\begin{aligned} w \in L(R') &\Leftrightarrow w \in L(\text{rev}(R_1)) \text{ or } w \in L(\text{rev}(R_2)) \\ &\Leftrightarrow \text{rev}(w) \in L(R_1) \text{ or } \text{rev}(w) \in L(R_2) \\ &\Leftrightarrow \text{rev}(w) \in L(R_1 + R_2) = L(R) \end{aligned}$$

- If $R = R_1 \cdot R_2$, then $R' = \text{rev}(R_2) \cdot \text{rev}(R_1)$. Now, we have

$$\begin{aligned} w \in L(R') &\Leftrightarrow w \in L(\text{rev}(R_2)) \cdot L(\text{rev}(R_1)) \\ &\Leftrightarrow \exists w_1, w_2, \text{ s.t } w = w_2 w_1 \text{ and } w_2 \in L(\text{rev}(R_2)) \text{ and } w_1 \in L(\text{rev}(R_1)) \\ &\Leftrightarrow \text{rev}(w_2) \in L(R_2) \text{ and } \text{rev}(w_1) \in L(R_1) \\ &\Leftrightarrow \text{rev}(w_1) \cdot \text{rev}(w_2) \in L(R_1 \cdot R_2) \\ &\Leftrightarrow \text{rev}(w_2 w_1) \in L(R) \end{aligned}$$

- If $R = R_1^*$, then $R' = \text{rev}(R_1)^*$. Now, we have

$$\begin{aligned} w \in L(R') &\Leftrightarrow \exists w_1, w_2, \ldots, w_k \in L(\text{rev}(R_1)) \text{ s.t } w = w_1 \cdot w_2 \cdots w_k \\ &\Leftrightarrow \text{rev}(w_i) \in L(R_1) \text{ and } \text{rev}(w) = \text{rev}(w_k) \cdot \text{rev}(w_{k-1}) \cdots \text{rev}(w_1) \\ &\Leftrightarrow \text{rev}(w) \in L(R_1)^* = L(R_1^*) = L(R) \end{aligned}$$