Formalizing the proof: The Pumping Lemma

Let L be a regular language

$\exists$ integer $k > 0$ s.t $\forall x, y, z$ s.t $xyz \in L$ and $|y| > k$, $\exists u, v, w$ s.t $y = uvw$, $v \neq \varepsilon$, & $\forall i \geq 0$ $xuv^iwz \in L$

- the size of the DFA accepting L

for all strings in L, and long substrings of those substrings

the part of the substring that loops and can be pumped

<u>Proof:</u> L is regular. $\exists M = (Q, \Sigma, \delta, q_0, F)$ s.t $L(M) = L$. Let $|Q| = k$

Let $x, y, z \in \Sigma^*$ s.t $xyz \in L$ & $|y| > k$

Suppose $\hat{\delta}(q_0, x) = q$

Let $y = y_1 y_2 \cdots y_r$ $\quad r > k$

$q_l = \hat{\delta}(q, y_1 y_2 \ldots y_l)$ By PHP $\exists s, t$ s.t

$q_s = q_t$ in $\{q_1, q_2, \ldots, q_l\}$

$\Rightarrow \hat{\delta}(q, y_1 \ldots y_s) = q'$

& $\hat{\delta}(q', y_{s+1} \ldots y_t) = q'$

Choose $u = y_1 \ldots y_s$
$v = y_{s+1} \ldots y_t$
$w = y_{t+1} \ldots y_l$

$\hat{\delta}(q_0, x) = q$, $\hat{\delta}(q, y_1 \ldots y_s) = q'$ $\hat{\delta}(q', y_{s+1} \ldots y_t) = q'$

$\hat{\delta}(q', y_{t+1} \ldots y_r z) \in F$

Contrapositive form:

If

$\forall\, k > 0$, $\exists\, x, y, z$ s.t $xyz \in L$ and $|y| > k$

s.t $\forall\, u, v, w$ s.t $y = uvw$, $v \neq \varepsilon$, $\exists\, i \geq 0$

s.t $xuv^i wz \notin L$

then L is not regular.

Pumping lemma as a game

Prover: wants to prove that L is regular

Spoiler: wants to show that the prover is wrong

Prover		Spoiler
choose k	$\longrightarrow$	
	$\longleftarrow$	Find x, y, z s.t $xyz \in L$ and $|y| > k$
Find u, v, w s.t $y = uvw$, $|v| \neq 0$	$\longrightarrow$	
		Find $i \geq 0$ s.t $xuv^i wz \notin L$

If the spoiler has a winning strategy, then the language is not regular

Examples using the Pumping lemma

(1) $L = \{0^n 1^n \mid n \geq 0\}$

- Prover chooses $k > 0$
- what should the spoiler choose for x, y, z s.t $xyz \in L$ & $|y| > k$

  $x = \varepsilon, \; y = 0^{k+1}, \; z = 1^{k+1}$

- Prover choose u, v, w s.t $y = uvw$ and $v \neq \varepsilon$
  - Have to think about all possible values: Since $y = 0^{k+1}$

    u, v, w must all be $0^l, 0^m, 0^n$

    for some l, m, n

- what should the spoiler choose for i?

  $i = 0$ would do

$xuv^0wz = \varepsilon . 0^l 0^n 1^{k+1} = 0^{l+n} 1^{k+1}$

$l + n < k+1 \Rightarrow xuv^0wz \notin L$

Using closure properties

$L = \{ w \in \{0,1\}^* \mid \#1(w) = \#0(w) \}$

$L' = \{ 0^n 1^n \mid n \geq 0 \}$ $\quad \underbrace{L'' = \{ 0^i 1^j \mid i,j \geq 0 \}}_{\text{regular}}$

$\underbrace{L'}_{\text{not regular}} = L \cap \underbrace{L''}_{\text{regular}} \implies$ if L is regular, then it leads to a contradiction