PS0

2. a) Let us make bijection from $\mathbb{N}\times\mathbb{N} \to \mathbb{N}$

~~$f(p,q) = \frac{(p+q-1)(p+q)}{2} + p$~~

~~We have that, for every $q \in \mathbb{Q}$, $q = p/q$. p,q coprime~~

~~Hence we have one-one function from~~

$g: \mathbb{Q} \to \mathbb{N}\times\mathbb{N}$. $g(p/q) = (p,q)$. p,q coprime.

$\to$ ~~f~~ $g \circ f: \mathbb{Q} \to \mathbb{N}$ one-one function $|\mathbb{N}| \geq |\mathbb{Q}|$ Hence $|\mathbb{N}| = |\mathbb{Q}|$

$\to h: \mathbb{N} \to \mathbb{Q}$ one-one function $h(n) = \frac{n}{1}$. $|\mathbb{Q}| \geq |\mathbb{N}|$ So there exists bijection between them.

1 (1,1) (1,2) (1,3) (1,4) (1,5)
2 (2,1) (2,2) (2,3)
3 (3,1) (3,2)
4 (4,1)
5
6

b) To show $\exists$ bijection from $g: \mathbb{N} \to \mathcal{P}_f(\mathbb{N})$, set of all finite subsets of $\mathbb{N}$

ϵ
$\{0\}$
$\{1\}$
$\{0,1\}$
$\{2\}$
$\{0,2\}$
$\{1,2\}$
$\{0,1,2\}$

~~$h: \mathcal{P}_f(\mathbb{N}) \to \mathbb{N}$~~

~~$h(S) = 2^{max(S)} + h(S - \{max(S)\})$~~

~~$h(\epsilon) = 1$~~

~~Inductively assume, $h(X) \leq 2^{max(X)+1}$.~~

~~$\Rightarrow h(S) = 2^{max(S)} + h(S - max(S)) \leq 2^{max(S)} + 2^{(max(S)-1)+1} \leq 2^{max(S)}$~~

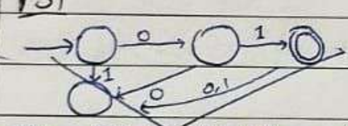
Suppose a_n is ordering of sets for all numbers until max is n.
Then we can construct a_{n+1} also. Duplicate $a_n + a_n$ and add number $n+1$ to second part.

$\to$ We can create ordering until any n. So $f: \mathcal{P}_f(\mathbb{N}) \to \mathbb{N}$. one-one fun.

$\to g: \mathbb{N} \to \mathcal{P}_f(\mathbb{N})$ is $g(n) = \{n\}$. one-one fun. Hence cardinalities same.

PS1

~~a)~~

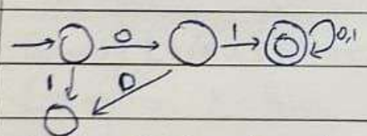


~~b) DFA accepts~~

~~$\to$ Above DFA accepts $L = \{01\}$~~

~~$\to$ Hence $\exists$ DFA that accepts $L + \Sigma^*$~~

a)

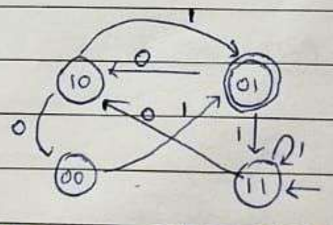


$\to$ For $\epsilon, 0, 1$ DFA does not accept

$\to 01$ is accepting

~~$\to$ Any string of format $01X$ is accepted~~

b)



L_1 (Used in c))

$\to$ DFA always keeps track of last 2 digits After first 2 digits read

$\to$ Induction proof.

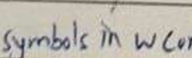
c) Observe that if $L_2 = \{w \mid w$ contains 01 as substring$\}$,

Then $L_2 = L_1 \bullet \Sigma^*$. L_1 from b) part.

If we have a string with 01 as substring we can split it into strings x_1 and x_2 where x_1 ends with 01 and x_2 is remaining. $x_1 \in L_1$, $x_2 \in \Sigma^*$.

d) $L_3 = \{w \in \Sigma^* \mid$ every block of 4 symbols in w contains atleast 2 0s$\}$

~~$L_4 = \{w \mid w$ has no 0s$\}$~~ $\to$



Construct states representing last 4 bits.

Initial state 0000. Final states any with more than 2 0s

Once you reach any state with less than 2 0s, stay there.

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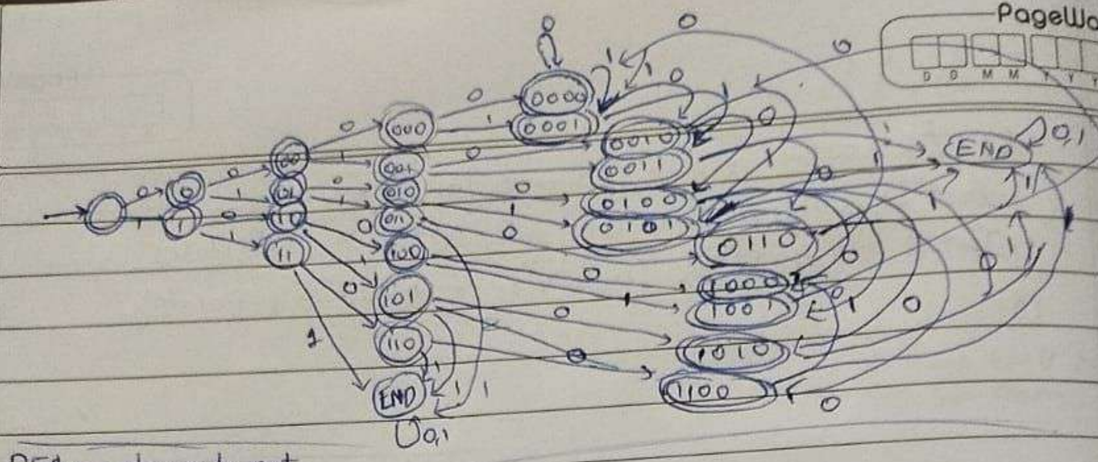
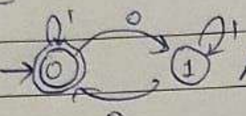
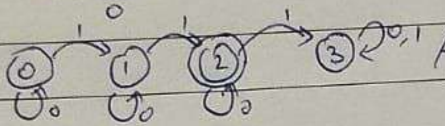
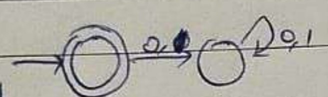


~~DFA can shown to not accept any~~

e) L_1 :  Accepts strings of even number of 0's

L_2 :  Accepts strings of 2 1's.

$L_1 \cup L_2$ is required.

f)  M accepts only ϵ. $L(M) = L$. $\bar{L}$ also regular.

M

2. → If $xy = yx$ then $xxyy = xyxy$. Now take $z = xy$. So such z exists by construction.

→ If $xxyy = z^2$, then do following:-

1) If $|x| = |y|$ then $xx = z = yy$ clearly $\Rightarrow x = y$.
$\Rightarrow xy = yx$

2) If $|x| > |y|$ then:-

Let $|x| = a$, $|y| = b$. As $xxyy = zz$ we can say

$x \cdot x[1:b] = x[b+1:a]yy = z$

$\Rightarrow x[1:b] = y$ for sure, $x = x[b+1:a]y$

$\Rightarrow$ Rearrange to get $x[1:b)\, x[b+1:a]\, y = yx$

$\Rightarrow xy = yx$

3) Similar to 2).