# Undecidable problems for CFLs

Theorem: $L = \{ G \mid L(G) = \Sigma^* \}$
is undecidable, where $G$ is a CFG.

- We have only seen problems related to TMs as undecidable

## Valid Computation Histories

Given $M, x$ a valid computation history is a string $\# \alpha_1 \# \alpha_2 \# \alpha_3 \# \cdots \# \alpha_k \#$
s.t
1. $\alpha_1$ is an initial configuration
2. $\alpha_i \xrightarrow{1} \alpha_{i+1}$
3. for some $j$, $\alpha_j$ is a halting configuration

Encoding configurations: $\vdash \; w_1 \; w_2 \; w_3 \; w_4$

$- \; - \; q \; - \; -$

position of tape-head

VALCOMPS $(M, x)$ = set of all valid computation histories of $M$ on $x$

$$- \alpha_1 = \begin{array}{ccccccccc} \vdash & x_1 & x_2 & x_3 & . & . & . & x_n \\ s & - & - & - & - & - & - & - \end{array}$$

$$- \text{if } \delta(s, \vdash) = (q, \vdash, R)$$

$$\alpha_2 = \begin{array}{ccccccccc} \vdash & x_1 & x_2 & x_3 & . & . & . & x_n \\ - & q & - & - & - & - & - & - \end{array}$$

$$\text{if } \delta(q, x_1) = (q', y_1, L)$$

$$\alpha_3 = \begin{array}{ccccccccc} \vdash & y_1 & x_2 & x_3 & . & . & . & x_n \\ q' & - & - & - & - & - & - & - \end{array}$$

Claim: VALCOMPS $(M, x) \neq \phi$ iff $M$ halts on $x$

or VALCOMPS $(M, x) = \phi$ iff $M$ does not halt on $x$

or $\overline{\text{VALCOMPS}}(M, x) = \Sigma^*$ iff $(\langle M \rangle, x) \in \overline{HP}$

Theorem: $\overline{\text{VALCOMPS}}(M, x)$ is context-free

When is $\# \alpha_1 \# \alpha_2 \# \alpha_3 \# \ldots \# \alpha_n \#$ in VALCOMPS$(M, x)$ ?

$L_1$ ① $\alpha_1$ is the initial configuration of $M$ on $x$

$L_2$ ② Each $\alpha_i$ is a string of symbols of the form $\underset{-}{\sigma_i}$ or $\underset{q}{\sigma_i}$

and exactly one symbol $\sigma_i$ has a state below it, & only the leftmost symbol has $\vdash$ on the top

$L_3$ ③ starts & ends with $\#$

$L_4$ ④ $t$ or $r$ appears in one of the $\alpha_i$s

$L_5$ ⑤ $\alpha_i \xrightarrow{1} \alpha_{i+1}$ $\forall i$

A string $\#\alpha_1\#\alpha_2\#\cdots\#\alpha_n$ is not in $\overline{\text{VALCOMPS}}(M, x)$ iff

The string is in $\overline{L_1} \cup \overline{L_2} \cup \overline{L_3} \cup \overline{L_4} \cup \overline{L_5}$

Observation: $L_1, L_2, L_3, L_4$ are all regular

$$\delta(q, a) = (q', b, L)$$

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| # | $\vdash$ | a | b | a | a | b | b | # | $\vdash$ | a | b | a | b | b | b | # |
| – | – | – | – | – | $q$ | – | – | – | – | – | – | $q'$ | – | – | – | – |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| a | a | a | a | a | a | b | b | b |
| – | – | – | – | – | $q$ | – | – | – |

$$\delta(q, b) = (q', a, L)$$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| a | a | a | a | a | a | b | b | b |
| – | – | – | $q'$ | – | – | – | – | – |

$$\delta(q, a) = (q', b, R)$$