PageWork

PS2

2. a) Show every infinite regular language L contains an infinite proper subset L'.
$L' \subset L$ and L' is regular.

Remove some finite string from L. L' you got, L' is regular.
→ Observe that L MUST have a finite length string BECAUSE
DFA of L is finite. Choose shortest path from start to final state
↳ This string MUST be finite.

b) Show that
every infinite regular language L contains infinite proper subset $L' \subset L$ s.t both L' and $L-L'$ are infinite and regular.

→ ~~For any string $s \in L$ without repeating states in DFA start to final, then $s \in L'$~~

→ Choose a string $s \in L$ s.t $|s| > |Q|$. Q is the set of states of the DFA of L.
→ $\exists x, y, z$ where $y \neq \varepsilon$ such that $xyz = s$ and $xy^i z \in L$.
[Note: We can also derive this from Pumping Lemma]
→ Let ~~L be~~ $L_1 = \{x\}$, $L_2 = \{y\}$, $L_3 = \{yy\}$, $L_4 = \{z\}$. All these languages are regular.
Let $L' = L_1 \cdot L_3^* \cdot L_4$, $L_5 = L_1 \cdot L_2 \cdot L_3^* \cdot L_4$ → Both regular
Note: $L' \cap L_5 = \emptyset$
→ Note that if L_a, L_b are regular then $L_a - L_b$ is regular because $L_a - L_b = L_a \cap \sim L_b$
→ ~~Hence~~ Note that both L' and L_5 are infinite.
→ Note that all elements of L', L_5 are in L.
→ Hence $L - L'$ has every element of L_5 ⇒ $L - L'$ is infinite and regular
→ Also L' is infinite and regular

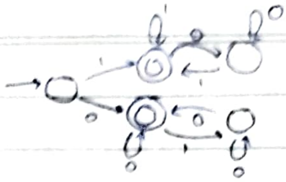
4. a) Can you prove $\sqrt[k]{L}$ is regular. (When L is regular)
Let $M = (Q, \Sigma, \delta, q_0, F)$ where $L(M) = L$.
$N = (Q', \Sigma, \Delta, S, F')$
$Q' = Q \times Q \times Q$ —— $2k-1$ times
$\Delta: Q \times Q \times Q \dots \times \Sigma \to \{Q \times Q \times Q \dots\}$
$\Delta((q_1, q_2 \dots q_{2k-1}), \sigma) = (\delta(q_1), q_2, \delta(q_3), q_4 \dots \delta(q_{2k-1}))$
$S = \{(q_1, q_2 \dots q_{2k-1}) \mid q_1 = q_0,\ q_{2i} = q_{2i+1} \in Q \text{ for all } i \in \{1, 2 \dots k-1\}\}$
$F' = \{(q_1, q_2 \dots q_{2k-1}) \mid q_{2k-1} \in F \text{ and } q_{2i-1} = q_{2i} \text{ for all } i \in \{1, 2, \dots k-1\}\}$
→ This construction accepts $\sqrt[k]{L}$.

b) $L_{\frac{1}{2}-} = \{x \mid \exists y \text{ s.t } |x| = |y|,\ xy \in L\}$ where L is regular.
Show $L_{\frac{1}{2}-}$ is regular.
$M = (Q, \Sigma, \delta, q_0, F)$ where $L(M) = L$
$N = (Q', \Sigma, \Delta, S, F')$
$Q' = Q \times Q \times Q$

1.2

1.1

$\Delta((q_1, q_2, q_3), \sigma) = \{(\delta(q_1), q_2, \delta(q_3, a)) \mid \forall a \in \Sigma\}$
$S = \{(q_1, q_2, q_3) \mid q_2 = q_3 \in Q,\ q_1 = q_0\}$, $F' = \{(q_1, q_2, q_3) \mid q_3 \in F,\ q_1 = q_2\}$

L_4  Accepts any string starting and ending same char.

L_5: Accepts any string ≥ 3 length.

$L_4 \cap L_5 = L_2 = L_3$ ✓ Hence Regular.

c) Let M be DFA of L_1 whose final state is F.

2.1

Create M_1 ~~DFA whose~~ NFA whose final states are set F_1 s.t $\forall f \in F_1$

$\exists\, y \in L_2$ s.t $\hat{\delta}(f, y) = F$.

→ Observe M_1 accepts L_1/L_2. Why?

M_1 accepts string $x \iff \hat{\delta}(q_0, x) \in F_1$

2.2

$\iff \hat{\delta}(\hat{\delta}(q_0, x), y) = F$ for some $y \in L_2$

$\iff \hat{\delta}(q_0, xy) = F$ for some $y \in L_2$

This is the definition of L_1/L_2. i.e M_1 accepts string x iff M accepts xy for some $y \in L_2$. //

5. a) $L_1 = \{1^k x \mid x \in \{0,1\}^* \text{ and } \#1(x) \geq k, k \geq 1\}$

IMP

Define $L_3 = \{1y \mid \#1(y) \geq 1\}$

→ Observe any string $s \in L_1$ can be written as $1y$ $\#1(y) \geq 1$ hence if $s \in L_1 \Rightarrow s \in L_3$

→ Observe any $s \in L_1$ can be written as $1^1 x$ $\#1(x) \geq 1$ hence if $s \in L_3 \Rightarrow s \in L_1$

$L_1 = L_3$. → L_3 is regular (easily can create DFA) ✓ Regular

$L_2 = \{1^k x \mid x \in \{0,1\}^* \text{ and } \#1(x) \leq k, k \geq 1\}$

→ Take Pumping lemma choices $x = 1^k 0$ $y = 1^k$ $z = \epsilon$.

→ $u = 1^l$ $l \geq 0$, $v = 1^m$ $m > 0$, $w = 1^n$ $n \geq 0$.

→ Pump v. We have $1^k 0 1^c$ where $c \gg k$. Observe this string is not in L_2. (Proof can be done, no choice of x has less ones than start contiguous ones)

Irregular//

c) $L = \{a^i b^j c^k \mid i = 1 \Rightarrow j = k \quad i,j,k \geq 0\}$

$x = a$, $y = b^{k+1}$, $z = c^{k+1}$. Pump v. You get $ab^c c^{k+1}$, $c \gg k$.

$ab^c c^{k+1} \notin L$ clearly. Irregular lang.

d) $L = \{w \mid \exists w \in \{0,1\}^* \text{ s.t } w \neq w^R\}$

Let $L_1 = \{w \mid w = w^R, w \in \{0,1\}^*\}$. If L were regular then L_1 is regular. As $L_1 = \sim L$.

→ Let us prove L_1 is irregular. Take $x = 1^{k+1} 0$ $y = 1^{k+1}$ $z = \epsilon$.

→ We have $v = 1^l$, $l > 0$. Pump v.

→ We will get string $1^{k+1} 0 1^c$ where $c \gg k$. Observe this string not in L_1. Hence irregular.

→ Hence by contradiction L is also irregular. (Because if L were regular L_1 would have to be regular).

c) Define quotient $L_1/L_2 = \{x \mid \exists y \in L_2 \text{ such that } xy \in L_1\}$

Show that if L_1 is regular then L_1/L_2 is regular.

d) $rot(L) = \{xy \mid yx \in L\}$. Show $rot(L)$ is regular if L is regular.

Intuition: $a_0\, a_1\, a_2$ / a_7 a_3 / $a_6\, a_5\, a_4$ — If we break it anywhere it should be in $rot(L)$.

Construct ~~N~~FA as follows:

Suppose $M = (Q, \Sigma, \delta, s_0, F)$ is DFA for L.

$N = (Q', \Sigma, \Delta, s, F')$ is εNFA for $rot(L)$

$Q' = Q \times Q$

$s = \{(q,q) \mid q \in Q\}$

$\Delta : Q \times Q \times \Sigma \to P(Q \times Q)$

$\Delta((q_1, q_2), \sigma) = \cancel{\{(q_1, q_2)\}}\ \{(q_1, \delta(q_2))\}$, ~~if $q_2 \in F$ then $\{(q_1$~~

And $\Delta((q_1, F), \varepsilon) = \{(q_1, q_0)\}$.

$F' = \{(q_1, q_2) \mid \cancel{\delta(q_1 ...)}\ q_2 = q_1\}$

→ This ε-NFA keeps a guess start state q_1 and moves q_2 according to input string. If Final state reached it can go to q_0 also. (i.e. indicating crossing ~~state~~ end → start of string).

~~Hence~~ Finally, if ~~$\delta(q_2)$~~ $q_2 = q_1$, then Final state, because that means that the string rotated back to initial position.

5. b) $L_1 = \{w \mid \exists x, y \in \{0,1\}^* - \varepsilon \text{ s.t. } xyx = w\}$

→ For some $k > 0$ take the string $10^{k+1}10^{k+1}$.

→ Take $y = 0^{k+1}$, $x = 10^{k+1}1$, $z = \varepsilon$ (Pumping Lemma Variables)

→ We will get $v = 0^l$ where $l > 0$.

Then we have that for some $i > 0$ $10^{k+1}1uv^iv = 10^{k+1}10^c$

Where $c > 4k$.

→ Observe that $10^{k+1}10^c$, $c > 4k$ cannot be part of L.

[$10^{k+1}10^c = aba$, Note a contains no ones by last a

Note a contains a one by first a]

IMP

$L_2 = \{w \mid \exists x, y \in \{0,1\}^* - \varepsilon \text{ s.t. } xyx^R = w\}$.

Let $L_3 = \{w \mid \exists y \in \{0,1\}^* - \varepsilon \text{ s.t. } w = 1y1 \text{ or } w = 0y0\}$

→ If I take any string from L_3, $s = aya$ It must be in L_2 also, $x = a$

→ If I take any string from L_2, $s = xyx^R$, It must be in L_3.

Take first char of x be a.

a is first char. of s, and last char of s ⇒ In L_3

→ Any string of L_2 in L_3 and vice versa. $L_2 = L_3$.

Index of comments

1.1 This should be \delta(q_1,\sigma)

1.2 Please write a proof of correctness for the construction. Refer to the \sqrt{L} construction given in the notes.

2.1 This is incorrect - but almost correct. Please think carefully.

2.2 This two-way implication looks incorrect.

3.1 This construction is almost correct. Just that after doing the epsilon-transition to the start state, you need to remember that you are reading y now.