Proving correctness of constructions

L = set of balanced paranthesis

$S \to [S] \mid SS \mid \varepsilon \quad : \quad G$

Thm: $L(G) = L$

(i) $l(w) = r(w)$
(ii) $\forall$ prefix y, $l(y) \geq r(y)$

Proof: (i) $L(G) \subseteq L$

for any $\beta \in (S \cup \{[,]\})^*$ s.t $S \xrightarrow{*} \beta$

β satisfies (i) & (ii)

Base case: $S \xrightarrow{0} \beta$. Then $w = S$

and the condition is satisfied

Induction step: $S \xrightarrow{n+1} \beta$

$\exists\, \alpha$ s.t $\underbrace{S \xrightarrow{n} \alpha}_{\text{Induction Hypothesis}} \to \beta$

α satisfies (i)&(ii)

$\alpha = \alpha_1 S \alpha_2 \longrightarrow \beta$

(i) $\alpha_1 S \alpha_2 \longrightarrow \alpha_1 \alpha_2$ (deleted S - does not contain [or])

(ii) $\alpha_1 S \alpha_2 \to \alpha_1 SS \alpha_2$ (added SS - does not contain [or])

(iii) $\alpha_1 S \alpha_2 \to \alpha_1 [S] \alpha_2$ (try out all prefixes)

(ii) $L \subseteq L(G)$: induction on the length of $w \in L$

Base case: $w = \varepsilon$ $\quad S \rightarrow \varepsilon$

Induction step $\quad w = w_1 w_2 \dots w_{n+1}$

find the smallest j s.t. $\quad w = xy$, $x = w_1 \dots w_j$

satisfies (i) & (ii)

* $j \neq n+1 \implies y$ satisfies (i) & (ii)

  - $\#1(x) + \#1(y) = \#0(x) + \#0(y)$

    $\implies \#1(y) = \#0(y)$

  - for a prefix z of y

    $\#1(x) + \#1(z) \geq \#0(x) + \#0(z)$

    $\implies \#1(z) \geq \#0(z)$

  By I.H $\quad S \xrightarrow{*} x$, $\quad S \xrightarrow{*} y$

  $\implies S \rightarrow SS \xrightarrow{*} xS \xrightarrow{*} xy$

* $j = n+1 \implies w = [x]$ & $x \in L$ (why?)

  $S \rightarrow [S] \xrightarrow{*} [x]$

Regular languages are context-Free

if L is regular, then $\exists$ CFG G such that

$L(G) = L$

Let $M = (Q, \Sigma, \delta, q_0, F)$ be the DFA such that

$L(M) = L$

Constructing the grammar G:

N = correspond to the states of M

S = correspond to q_0

$N = \{ S_i \mid q_i \in Q \}$ $\quad S = S_0$

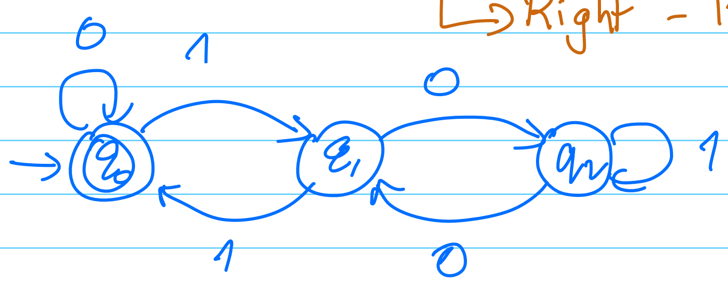
if $\delta(q_i, \sigma) = q_j$ add a production rule

$$S_i \to \sigma S_j$$

For each $q_i \in F$, add $S_i \to \varepsilon$

$\hookrightarrow$ Right-linear grammar

Eg:



$S_0 \to 0S_0 \mid 1S_1 \mid \varepsilon$

$S_1 \to 0S_2 \mid 1S_0$

$S_2 \to 0S_1 \mid 1S_2$

Proof: (i) $L(M) \subseteq L(G)$

* if $\hat{\delta}(q_i, w) = q_j$ then $S_i \xrightarrow{*} w S_j$

Base case: $\hat{\delta}(q_i, \varepsilon) = q_0$ $\quad S_i \xrightarrow{0} S_i$

$\hat{\delta}(q_i, \sigma) = q_j$; then $\exists$ production

$$S_i \rightarrow \sigma S_j$$

Induction step: $\hat{\delta}(q_0, w\sigma) = \delta(\hat{\delta}(q_0, w), \sigma)$

$$= \delta(q_i, \sigma)$$

I.H. $\quad S_0 \xrightarrow{*} w S_i \longrightarrow w \sigma S_j$

(ii) $L(G) \subseteq L(M)$

* if $S_i \xrightarrow{*} w S_j$ then $\hat{\delta}(q_i, w) = q_j$

Induction on the length of the derivation