(1) Let M be any automata, then $\equiv_M \subseteq \equiv_L$

Thus the automat $M_\equiv$ corr. to $\equiv_L$ is a minimal (in terms of # states) automata for L.

Qn: Can there be multiple minimal automata?

Isomorphism: Let $M = (Q, \Sigma, \delta, s, F)$ and $M' = (Q', \Sigma, \delta', s', F')$ be two DFAs

M is said to be isomorphic to M' ($M \cong M'$) if $\exists$ a bijection $f: Q \to Q'$ s.t the following hold.

(i) $s' = f(s)$

(ii) $q \in F \iff f(q) \in F'$

(iii) $f(\delta(q,\sigma)) = \delta'(f(q), \sigma)$ $\forall q \in Q$, $\sigma \in \Sigma$

Thm: If M and M' are two minimal automata accepting the same language, then $M \cong M'$.

Proof: Consider $\equiv_M$ of a minimal DFA M

$\equiv_L$ is a MH relation, & refines $\equiv_M$

# of equivalence classes of $\equiv_M$ = that of $\equiv_L$

$\Longrightarrow \equiv_M = \equiv_L$

$M \longrightarrow \equiv_M \longrightarrow M_{\equiv}$

claim: $M \cong M_{\equiv}$

Proof: $M = (Q, \Sigma, \delta, s, F)$

$x \equiv y$ if $\hat{\delta}(q_0, x) = \hat{\delta}(q_0, y)$

$M_{\equiv} = (Q', \Sigma', \delta', s', F')$

$Q' = \{ [x] \mid x \in \Sigma^* \}$

$f : Q' \to Q \;:\; f([x]) = \hat{\delta}(s, x)$

$f([\varepsilon]) = \hat{\delta}(s, \varepsilon) = s$

$[x] \in F' \iff x \in L$

$\iff \hat{\delta}(q_0, x) \in F$

$\iff f([x]) \in F$

$f(\delta'([x], \sigma)) = f([x\sigma]) = \hat{\delta}(s, x\sigma)$

$= \delta(f([x]), \sigma)$

Minimization algorithm

Defn: Let M be a DFA accepting L

Two states $q, q' \in M$ are equivalent if $\forall x \in \Sigma^* \quad \hat{\delta}(q, x) \in F \iff \hat{\delta}(q', x) \in F$

$q \approx q'$ if $\forall x \in \Sigma^* \quad \hat{\delta}(q, x) \in F \iff \hat{\delta}(q', x) \in F$

$\left.\begin{array}{l}\text{- Reflexive} \\ \text{- Symmetric} \\ \text{- Transitive}\end{array}\right\}$ $\approx$ is an equivalence relation

$M/\approx$:

$$Q' = \{ [q] \mid q \in Q \}$$

$$q_0' = [q_0]$$

$$F' = \{ [q] \mid q \in F \}$$

$$\delta'([q], \sigma) = [\delta(q, \sigma)]$$

Verify: Show that $\equiv_L \subseteq \equiv_{M/\approx}$

& hence conclude that $M/\approx$ is the minimal automata.