1. Consider the following constrained programming paradigm. You are allowed to access an input stream via a function next() that returns the next character of the input. If the input stream has ended the function returns a special character $\perp$. Essentially, you can read the input once from left to right. Furthermore, you are allowed to use variables that can only store a finite set of values that is independent of the actual length of the input. For instance, you cannot have a counter len that increments whenever you see a new character other than $\perp$. On the other hand, you can have a variable var that can store a number between 1 and 1000000000000.

   You also have access to a function write(c) that writes the character c to an output tape. Once again, characters once written cannot be overwritten, and neither can you go back and insert characters in-between ones already written. The goal of this problem is to understand what can be done by such a constrained programming paradigm.

   (a) Write pseudocode to check if the input stream starts with the symbols int.

   (b) Suppose that the input stream is an alphanumeric string. Write pseudocode to check if this string contains the substring 2024.

   (c) Suppose that the input stream is a number in decimal representation starting from the most-significant digit. Write pseudocode to check if the number is divisible by 11.

   (d) The input stream consists of tuples of digits $(0-9)$ that denote two non-negative integers from their least significant digit. For instance 12 and 123 will be represented by the input stream as $(2,3),(1,2),(0,1),\perp$. Write pseudocode to output the sum of the two numbers starting from the least significant digit. In this example, you should output 5 3 1. *Do you think you can also compute the product of the two numbers?*

   (e) The input stream consists of an alphanumeric string. Write pseudocode to check if every occurence of the character i has an occurrence of e after it. For instance aaaiiiie and aaiieeeee are valid. But, aaaaiiieiabb is not valid.

   Think of the string as a program and the characters i and e as corresponding to if and else. In this constrained programming paradigm, do you think you can check if there are dangling else statements?

   For all the problems given above, check the number of variables that you used and the range of values that the variables can hold. Can you optimize these quantities?

   Solution: These are essentially DFAs, but written differently. Try constructing DFAs for these problems, except Part (d) which asks for a DFA that also prints an output.

2. Let $\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{R}$ denote the set of natural numbers, rational numbers, and real numbers respectively. Let $\mathcal{P}(\mathbb{N})$ denote the set of all subsets of $\mathbb{N}$, and let $\mathcal{P}_f(\mathbb{N})$ denote the set of all *finite* subsets of $\mathbb{N}$.

   (a) Show that there is a bijection $f:\mathbb{Q}\to\mathbb{N}$.

Solution: We will actually give a bijection $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. Since $\mathbb{Q}^+ \subseteq \mathbb{N} \times \mathbb{N}$, and the subset of every countable set is countable, we can conclude that there is a bijection from $\mathbb{Q}^+$ to $\mathbb{N}$ (here $\mathbb{Q}^+$ is set of positive rational numbers). The union of two countable sets is countable, and hence $\mathbb{Q}$ is countable, and in bijection with $\mathbb{N}$.

Think of the set $\mathbb{N}\times\mathbb{N}$ being represented by a table where the rows and columns are both indexed by $\mathbb{N}$. One way to enumerate it is to go in an off-diagonal way as follows: $(1, 1)$ is the first diagonal, followed by the diagonal consisting of $(2, 1)$ and $(1, 2)$, followed by the diagonal cnsisting of $(3, 1), (2, 2), (1, 3)$ and so on. With a little bit of calculation, you can verify that this corresponds to the following map.

$$f(p, q) = \frac{(p + q - 2)(p + q - 1)}{2} + q, \text{ for } p, q \in \mathbb{N}.$$

(b) Show that there exists a bijection $g : \mathbb{N} \to \mathcal{P}_f(\mathbb{N})$.

Solution: We shall first construct a g that maps to a finite set of natural numbers (i.e. an element of $P_f(\mathbb{N})$). Consider the binary representation of any natural number n denoted by $\mathrm{bin}(n)$. Let $\chi(A)$ denote the characteristic vector of any set $A \subseteq \mathbb{N}$. A characteristic vector of a set A is a sequence of bits say $(b_k, \ldots b_2, b_1)$ such that $b_i = 1$ if and only if $i \in A$.

We define g as follows: $g(n) = \chi^{-1}(\mathrm{bin}(n))$. Clearly, every natural number n can be encoded uniquely into a corresponding binary representation. Furthermore, given a characteristic vector, we can easily obtain the set of natural numbers it represents.

Now suppose two natural numbers m and n map to the same set A. This means that $g(m) = g(n) = A$. But then $\mathrm{bin}(m) = \mathrm{bin}(n)$. Thus, $m = n$ and the function g is a bijection.

(c) Show that there exists a bijection $h : \mathbb{R} \to \mathcal{P}(\mathbb{N})$.

Solution: Instead of constructing a direct bijection, we will instead use the following theorem (without proof) to simplify our solution.

Theorem 1 (Cantor-Bernstein-Schröder)**.** *Let A and B be any two sets. If there is an injective map $f : A \to B$ and an injective map $g : B \to A$, then there is a bijection $h : A \to B$.*

Most of the technicalities in the proof arises because the numbers $0.0\overline{1}$ and 0.1 in binary represent the same real number. Hence the simple mapping of the binary representation as a characteristic vector can create issues.

To apply the Cantor-Bernstein-Schröder theorem, we need an injective $f : \mathbb{R} \to \mathcal{P}(N)$ and $g : \mathcal{P}(N) \to \mathbb{R}$. We will use the (non-trivial) fact that there exists a rational number between any two real numbers. Now define $f(r)$ as the set of rational numbers less than r. Thus, this is an injective map to the set of subsets of rational numbers, and from the previous question, this is in bijection with the power set of natural numbers.

For g, we will do the following: map every finite subset to the corresponding natural number via the binary expansion. For an infinite subset, map this to a number between $(0, 1)$ again assuming that it is the binary expansion of the number. This would also be an injective map.

3. There is an island upon which a tribe resides. The tribe consists of 1000 people, with various eye colours. Yet, their religion forbids them to know their own eye color, or even to discuss the topic; thus, each resident can (and does) see the eye colors of all other residents, but has no way of discovering his or her own (there are no reflective surfaces). If a tribesperson does discover his or her own eye color, then their religion compels them to commit ritual suicide at noon the following day in the village square for all to witness. All the tribespeople are highly logical and devout, and they all know that the others are also highly logical and devout (and they all know that they all know that the others are highly logical and devout, and so forth).

   For the purposes of this logic puzzle, "highly logical" means that any conclusion that can logically deduced from the information and observations available to an islander, will automatically be known to that islander.

   Of the 1000 islanders, it turns out that 100 of them have blue eyes and 900 of them have brown eyes, although the islanders are not initially aware of these statistics (each of them can of course only see 999 of the 1000 tribespeople).

   One day, a blue-eyed foreigner visits the island and wins the complete trust of the tribe.

   One evening, he addresses the entire tribe to thank them for their hospitality.

   However, not knowing the customs, the foreigner makes the mistake of mentioning eye color in his address, remarking "how unusual it is to see another blue-eyed person like myself in this region of the world".

   What effect, if anything, does this faux pas have on the tribe?

   Outcome 1: The foreigner's words have no effect, because his comments do not tell the tribe anything that they do not already know (everyone in the tribe can already see that there are several blue-eyed people in their tribe).

   Outcome 2: 100 days after the address, all the blue eyed people commit suicide together.

   Which of the two outcomes will happen, and why?

   Hint: Use induction. What are the base cases, and how will you reason about them?

   Solution: Read this blog post, and the comments therein.