Limitations of DFAs

- Finite memory, but strings of unbounded length!
- Cannot count (with some caveats)
  ↳ formalize this idea

$L = \{0^n 1^n \mid n \geq 0\}$
$= \{\varepsilon, 01, 0011, 000111, \ldots\}$

<u>Thm</u>: L is not regular

(There does not exist $M = (Q, \Sigma, \delta, q_0, F)$ such that $L(M) = L$)

$M = (Q, \Sigma, \delta, q_0, F)$ & $|Q| = k$

$w = 0^n 1^n$ when $n \gg k$

$\exists\, i$ s.t $\hat{\delta}(q_0, 0^i) = q$
$\hat{\delta}(q, 0^j) = q$
$\hat{\delta}(q, 0^{n-i-j} 1^n) \in F$

$$\left.\begin{array}{l} \hat{\delta}(q_0, 0^i) = q \\ \hat{\delta}(q, 0^{2j}) = q \\ \hat{\delta}(q, 0^{n-i-j} 1^n) \in F \end{array}\right\} \begin{array}{l} \Rightarrow 0^{i+2j+n-i-j} 1^n \in L \\ \Rightarrow \underbrace{0^{j+n} 1^n \in L}_{\text{but this is a contradiction}} \end{array}$$

Thm: Let $L = \{ 0^{2^n} \mid n \geq 0 \}$

Then L is not-regular

$n \gg k \qquad w = 0^{2^n}$

$2^n = i + j + l \quad$ where $j \leq n$

$$\left.\begin{array}{l} \hat{\delta}(q_0, 0^i) = q \\ \hat{\delta}(q, 0^j) = q \\ \hat{\delta}(q, 0^l) \in F \end{array}\right\} \Rightarrow \begin{array}{l} \hat{\delta}(q_0, 0^i) = q \\ \hat{\delta}(q_0, 0^{2j}) = q \\ \hat{\delta}(q, 0^l) \in F \end{array}$$

$\Rightarrow 0^{i+2j+l} \in L$, but $2^n < i+2j+l < 2^{n+1}$

Formalizing the proof: The Pumping Lemma

Let L be a regular language

$\exists$ integer $k > 0$ s.t $\forall x, y, z$ s.t $xyz \in L$

and $|y| \geq k$, $\exists u, v, w$ s.t $y = uvw$

$v \neq \varepsilon$, & $\forall i \geq 0 \quad x u v^i w z \in L$

- the size of the DFA accepting L

for all strings in L, and long substrings of those substrings

the part of the substring that loops and can be pumped