② $\overline{HP} \le_m INFIN$ (INFIN is not r.e)

$A = \overline{HP}$ $B = INFIN$

$f(\langle M \rangle, x) \longmapsto \langle M' \rangle$ s.t

- if M does not halt on x then $L(M')$ is infinite
- if M halts on x, then $L(M)$ is finite

require this from the reduction

$M'(y)$: Simulate M on x for $|y|$ steps
if M has not halted, accept y

① If M does not halt on x, then $\forall y$
M does not halt on x in $|y|$ steps
$\Rightarrow y \in L(M') \Rightarrow L(M') = \Sigma^* \Rightarrow \langle M' \rangle \in INFIN$

② If M reaches a halt state within k steps on input $x \Rightarrow$ if $|y| > k$, then $y \notin L(M')$
$\Rightarrow L(M') = \{y \mid |y| \le k\} \Rightarrow \langle M' \rangle \notin INFIN$

(3) $\overline{HP} \leq_m FIN$ (INFIN is not co-r.e)

$FIN = \overline{INFIN} = \{ \langle M \rangle \mid L(M) \text{ is finite} \}$

$A = \overline{HP}$, $B = FIN$

$f(\langle M \rangle, x) \longmapsto \langle M' \rangle$

- If M does not halt on x, then $L(M')$ is finite
- If M halts on x, then $L(M')$ is infinite

$M'(y)$: Simulate M on x
If M enters a halting state on x
then accept y

- If M' does not halt on x, then $L(M') = \emptyset$
- If M' halts on x, then $L(M') = \Sigma^*$

Rice's theorem

- Checking if a language L has a property
- Language is represented by a TM that accepts it

A property P of r.e languages is a function f: set of r.e languages $\to \{T, \perp\}$
(subset of languages)

examples: ① language is regular
② language contains ε

Theorem: Let P be a non-trivial property of r.e languages. Then $L = \{\langle M \rangle \mid L(M) \in P\}$ is undecidable

non-trivial: $\exists$ at least one language with the property, & one language without the property

Property of the languages, not of the TMs that accept them

- $\forall M$ s.t $L(M) \in P$, we have $\langle M \rangle \in L$

① $L = \{ \langle M \rangle \mid M \text{ halts on input } \varepsilon \}$

- undecidable

  $HP \leq_m L$

- cannot apply Rice's theorem

  consider $A = \Sigma^* - \{\varepsilon\}$

  $M(y)$: if $y = \varepsilon$ reject & halt
  else accept & halt

  $M'(y)$: if $y = \varepsilon$, loop
  else accept & halt

  $\langle M \rangle \in L$ & $\langle M' \rangle \notin L$

  But $L(M) = L(M')$

② $L = \{ \langle M \rangle \mid M \text{ accepts } \varepsilon \}$

if $L(M') = L(M)$ then

$\varepsilon \in L(M)$ iff $\varepsilon \in L(M') \Rightarrow \langle M \rangle \in L$ iff $\langle M' \rangle \in L$